CHAPTER II .

CLASSICAL INVOLUTIVE BANACH-LIE ALGEBRAS AND GROUPS OF BOUNDED AND COMPACT OPERATORS

Let $\underline{g}(\mathcal{H}; C_o)$ be a classical Lie algebra of finite rank operators. Its closure in some uniform crossnorm α will be denoted by $\underline{g}(\mathcal{H}; C_\alpha)$ and its closure in the weak (or, for that matter, strong) topology by $\underline{g}(\mathcal{H}; L)$. The aim of this Chapter is to study the Banach-Lie algebras obtained this way and the corresponding Banach-Lie groups of operators on $\mathcal{H}$. As in Chapter I, we usually assume the complex Hilbert space $\mathcal{H}$ to be separable and infinite dimensional in order to keep reasonably simple notations; but this is again a matter of convenience more than of necessity.

The results of this Chapter rely heavily on those of Chapter I. Numerous properties of operators in $L(\mathcal{H})$ have been used; though occasional, crucial help has been found in Herstein [86] and Johnson-Sinclair [92].

II.1.- Review of Banach-Lie groups and Banach-Lie algebras

The best reference for Banach-Lie groups is Lazard [107]; for shorter introductions, see Eells [54] section 3 and Lazard-Tits [108].

A Banach-Lie group is a (real or complex) Banach manifold G furnished with a group-structure which satisfies natural compatibility conditions. The tangent space $\mathfrak{g}$ at the identity of G is then a Banach space and a Lie algebra with the multiplicity being jointly continuous; so that $\mathfrak{g}$ is, by definition, a Banach-Lie algebra. The norm on $\mathfrak{g}$ is defined by G up to equivalence only, and so involves an arbitrary choice; in all the following examples, the identity

$$\| [X,Y] \|_{\mathfrak{g}} \leqslant 2 \| X \|_{\mathfrak{g}} \| Y \|_{\mathfrak{g}}$$

will be satisfied for all $X,Y \in \mathfrak{g}$.

Standard properties of one parameter subgroups, the exponential map, adjoint representations and derivatives of smooth homomorphisms carry over from the finite dimensional to the Banach case. Similarly, if a Banach-Lie algebra is the algebra of any Banach-Lie group at all - in which case it is said to be enlargable - then it is the Lie algebra of a unique (up to isomorphism) connected and simply connected one.

A sub Banach-Lie group of a Banach-Lie group G is a subgroup H, furnished with a Banach manifold structure which makes the canonical injection of H into G smooth. It must be emphasized that H is not in general a submanifold of G ; indeed, the topology of H defined by its Banach manifold structure (referred to as its "own" topology below) might be strictly finer that its topology induced from G ; "submanifolds" of this type have already been defined by Chevalley [36], Chap. III §VI.

A <u>sub Banach-Lie algebra</u> of a Banach-Lie algebra $\underline{g}$ is a sub Lie algebra $\underline{h}$, furnished with a norm which makes it a Banach-Lie algebra and which makes the canonical injection of $\underline{h}$ into $\underline{g}$ continuous. The definitions are such that, given a Banach-Lie group G with Lie algebra $\underline{g}$, there is a bijective correspondence between sub Banach-Lie groups of G which are connected in their own topology and sub Banach-Lie algebras of $\underline{g}$.

Let G be a Banach-Lie group, let $\underline{g}$ be its Lie algebra, let H be a sub Banach-Lie groups of G and let $\underline{h}$ be the corresponding sub Banach-Lie algebra of $\underline{g}$. Then H is a submanifold in G if and only if

i) the own topology of H coincides with that induced by G

ii) $\underline{h}$ is closed and splits in $\underline{g}$.

If these conditions are fulfilled, H is sometimes said to be a <u>sub Banach-Lie group-manifold</u> of G. For example, any closed sub Banach-Lie group of G with a finite codimensional Lie algebra is a sub Banach-Lie group-manifold of G.

A Banach-Lie algebra is <u>simple</u> if it has no non-trivial closed ideal. It is <u>algebraically simple</u> if it has no non-trivial ideal at all. A Banach-Lie algebra $\underline{g}$ is provisionally said to be <u>semi-simple</u> if it has no closed abelian ideals and if its dimension is at least two (see the critic of this notion in an appendix to this Chapter). An <u>involutive Banach-Lie algebra</u> is a Banach-Lie algebra $\underline{g}$ furnished with an involution $X \mapsto X^*$ (def. I.1.) such that $\| X^* \| = \| X \|$ for all $X \in \underline{g}$.

Let E be a Banach space and let X be an operator on E. Let $\{X\}^+$ be the closed associative subalgebra of L(E) generated by X. Then X is said to be <u>semi-simple</u> if $\{X\}^+$ does not contain any non-zero quasi-nilpotent element. An element X in a Banach-Lie algebra $\underline{g}$ is <u>semi-simple</u> if ad(X) is a semi-simple operator on $\underline{g}$. A <u>c-involutive Banach-Lie algebra</u> is an involutive Banach-Lie algebra in which any normal element is semi-simple. The definition of <u>Cartan subalgebras</u> given in section I.3 carries over to the Banach case without change.

Unless otherwise stated, homomorphisms [resp. *-homomorphisms] between Banach-Lie algebras [resp. involutive Banach-Lie algebras] are not supposed to be necessarily continuous. Hence a slightly different terminology as in Balachandran [12], [14], or as in [76]. Definition I.8 (about conjugations and real forms) and proposition I.11 (about the cohomological formulation of the problem of classifying real forms) carry over to the Banach case with the minor evident modifications. For example, a <u>conjugation</u> of an involutive Banach-Lie algebra $\underline{g}$ is a map $\sigma : \underline{g} \to \underline{g}$ such that the map $X \longmapsto \sigma(-X^*)$ is an involutive *-automorphism of $\underline{g}$; the corresponding <u>real form</u> is given by $\{X \in \underline{g} | \sigma(X) = X\}$.

II.2.- Classical complex Banach-Lie algebras of bounded operators : definitions and ideal-structure

In the theory of associative involutive Banach algebras, one of the first examples usually given is the C^*-algebra $L(\mathcal{H}_{\mathbb{K}})$ of continuous operators in a Hilbert space $\mathcal{H}_{\mathbb{K}}$ over $\mathbb{K}$. In the case $\mathcal{H}_{\mathbb{K}}$ is separable and infinite dimensional, it is well-known that any associative proper ideal of $L(\mathcal{H}_{\mathbb{K}})$ contains the ideal $C_0(\mathcal{H}_{\mathbb{K}})$ of finite rank operators and is contained in the ideal $C(\mathcal{H}_{\mathbb{K}})$ of compact operators (see for example Schatten [151] chap. I th. 11); in particular, $C(\mathcal{H}_{\mathbb{K}})$ is the only non-trivial closed ideal of $L(\mathcal{H}_{\mathbb{K}})$ and the algebra $L(\mathcal{H}_{\mathbb{K}})/C(\mathcal{H}_{\mathbb{K}})$ is algebraically simple. This can be summed up graphically in the following picture:

$$\begin{array}{l} L(\mathcal{H}_{\mathbb{K}}) \\ C(\mathcal{H}_{\mathbb{K}}) \\ \{0\} \end{array}$$

The aim of this section is to point out analogous facts in the Lie case; the results are those of section 5 in [126].

Let $\mathcal{H}$ be a complex Hilbert space, say <u>separable and infinite dimensional</u>. The associative involutive Banach algebra $L(\mathcal{H})$ defines an involutive Banach-Lie algebra which will be denoted by $\underline{gl}(\mathcal{H}; L)$, or sometimes simply by $\underline{gl}(\mathcal{H})$. The following subalgebras are clearly all ideals in $\underline{gl}(\mathcal{H})$:

$\mathbb{C}\, id_{\mathcal{H}}$ = center of $\underline{gl}(\mathcal{H})$.

$\underline{sl}(\mathcal{H}; C_0)$ = subalgebra of finite rank operators with 0-trace.

$\underline{gl}(\mathcal{H}; C_0)$ = subalgebra of all finite rank operators.

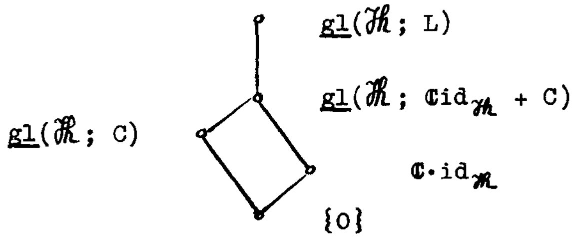
$\underline{gl}(\mathcal{H}; C)$ = subalgebra of all compact operators.

$gl(\mathcal{H}; \mathbb{C}id_{\mathcal{H}} + C) = \{X \in \underline{gl}(\mathcal{H}) \mid X = \lambda\, id_{\mathcal{H}} + Y$ for some $\lambda \in \mathbb{C}$, and some $Y \in C(\mathcal{H})\}$.

<u>Proposition 1A</u>. Let $\underline{a}$ be a non-trivial ideal of $\underline{gl}(\mathcal{H})$ which is distinct from the center. Then

$$\underline{sl}\,(\mathcal{H};\,C_o) \subset \underline{a} \subset \underline{gl}(\mathcal{H};\,\mathbb{C}\mathrm{id}_{\mathcal{H}} + C).$$

In particular, the only non-trivial closed ideals of $\underline{gl}(\mathcal{H})$ are the center, $\underline{gl}(\mathcal{H};\,C)$ and $\underline{gl}(\mathcal{H};\,\mathbb{C}\mathrm{id}_{\mathcal{H}} + C)$; the Banach-Lie algebra $\underline{gl}(\mathcal{H};\,L)\Big/\underline{gl}(\mathcal{H};\,\mathbb{C}\mathrm{id}_{\mathcal{H}} + C)$ is algebraically simple. The picturial summing up for closed ideals is:



<u>Proof</u>.

<u>Step one</u> : $\underline{sl}(\mathcal{H};\,C_o) \subset \underline{a}$.

Choose $X \in \underline{a}$, X not a multiple of the identity. Either X is normal, or $[X,X^*]$ is not a multiple of the identity (Putnam [134] chap. I th. 1.2.1). Hence X can be supposed normal without loss of generality. By Schur's lemma, there exists $Y \in \underline{sl}(\mathcal{H};\,C_o)$ such that $[X,Y] \neq 0$; hence $\underline{a} \cap \underline{sl}(\mathcal{H};\,C_o)$ is a non-zero ideal of $\underline{sl}(\mathcal{H};\,C_o)$. Step one follows then from proposition I.1A.

<u>Step two</u> : the derived ideal of $\underline{gl}(\mathcal{H})$ is $\underline{gl}(\mathcal{H})$ itself.

This is a well-known result due to Halmos; see for example Dieudonné [45] chap. XV §11 exercise 23.

<u>Step three</u> : the center of the simple associative algebra $L(\mathcal{H})\Big/C(\mathcal{H})$ consists exactly of the multiples of the identity. See Calkin [30] th. 2.9.

<u>Step four</u> : let $\mathcal{A}$ be a simple associative algebra, with center $\mathcal{Z}$, over a field of characteristic not 2; let $\underline{a}$ be a Lie ideal in $\mathcal{A}$;

then either $\underline{a} \subset \mathcal{Z}$, or $[\mathcal{A}, \mathcal{A}] \subset \underline{a}$. See Herstein [86], th. 4; or for a cheaper but here clearly sufficient result, Bourbaki [25] §1 exercise 7.

Step five. As a consequence of steps two to four, a Lie ideal of $L(\mathcal{H})/C(\mathcal{H})$ is either trivial or equal to the center. Hence any Lie ideal of $L(\mathcal{H})$ is contained in $\underline{gl}(\mathcal{H};\, \mathbb{C}\mathrm{id}_{\mathcal{H}} + C)$. As $\underline{sl}(\mathcal{H};\, C_o)$ is dense in $\underline{gl}(\mathcal{H};\, C)$ (see the appendix, proposition II.19. iv), and as $\underline{gl}(\mathcal{H};\, C)$ is of codimension one in $\underline{gl}(\mathcal{H};\, \mathbb{C}\mathrm{id}_{\mathcal{H}} + C)$, the proposition follows. ∎

Let now $J_{\mathbb{R}}$ be a conjugation of $\mathcal{H}$. The involutive antiautomorphism of $L(\mathcal{H})$ defined by $X \mapsto J_{\mathbb{R}} X^* J_{\mathbb{R}}$ is again denoted by $\varphi_{\mathbb{R}}$ (see section I.1). The orthogonal complex Banach-Lie algebra corresponding to $\underline{gl}(\mathcal{H})$ is

$$\underline{o}(\mathcal{H}, J_{\mathbb{R}};\, L) = \{X \in \underline{gl}(\mathcal{H};\, L) \mid \varphi_{\mathbb{R}}(X) = -X\}.$$

and will sometimes be simply denoted by $\underline{o}(\mathcal{H}, J_{\mathbb{R}})$. Its center is reduced to zero and the following subalgebras are clarly both ideals in $\underline{o}(\mathcal{H}, J_{\mathbb{R}})$:

$\underline{o}(\mathcal{H}, J_{\mathbb{R}};\, C_o)$ = subalgebra of finite rank operators.

$\underline{o}(\mathcal{H}, J_{\mathbb{R}};\, C)$ = subalgebra of compact operators.

Proposition 1B. Let $\underline{a}$ be a non-trivial ideal of $\underline{o}(\mathcal{H}, J_{\mathbb{R}})$. Then $\underline{o}(\mathcal{H}, J_{\mathbb{R}};\, C_o) \subset \underline{a} \subset \underline{o}(\mathcal{H}, J_{\mathbb{R}};\, C)$.
In particular, the only non-trivial closed ideal of $\underline{o}(\mathcal{H}, J_{\mathbb{R}})$ is $\underline{o}(\mathcal{H}, J_{\mathbb{R}};\, C)$; the Banach-Lie algebra $\underline{o}(\mathcal{H}, J_{\mathbb{R}};\, L)/\underline{o}(\mathcal{H}, J_{\mathbb{R}};\, C)$ is algebraically simple. The pictural summing up for closed ideals is:

$$\begin{array}{l} \underline{o}(\mathcal{H}, J_{\mathbb{R}};\, L) \\ \quad | \\ \underline{o}(\mathcal{H}, J_{\mathbb{R}};\, C) \\ \quad | \\ \{0\} \end{array}$$

Proof.

Step one : $\underline{o}(\mathcal{H},J_{\mathbb{R}};\ C_o) \subset \underline{a}$. This step is proved as the corresponding one for proposition 1A; note that $\underline{o}(\mathcal{H},J_{\mathbb{R}})$ does not contain the identity.

Step two ; let $\mathcal{B}$ be a set of generators in an associative algebra $\mathcal{R}$; then $[\mathcal{B},\mathcal{R}] = [\mathcal{R},\mathcal{R}]$.

For each $n \in N^*$, let $\mathcal{B}^n$ be the vector space generated in $\mathcal{R}$ by products of n elements of $\mathcal{B}$; we first show by induction on n that $[\mathcal{B}^n,\mathcal{R}] \subset [\mathcal{B},\mathcal{R}]$ for all $n \in N^*$. It is trivial when $n = 1$; it follows in the general case from the induction hypothesis via the computation :

$$[X_1X_2\ldots X_n\ ,\ Y] = X_1(X_2\ldots X_nY) - (X_2\ldots X_nY)X_1 + X_2\ldots X_n(YX_1) -$$
$$-(YX_1)X_2\ldots X_n \in [\mathcal{B},\mathcal{R}] + [\mathcal{B}^{n-1},\mathcal{R}] \subset [\mathcal{B},\mathcal{R}]$$

for all $X_1, \ldots, X_n \in \mathcal{B}$, for all $Y \in \mathcal{R}$. Now, to say that $\mathcal{B}$ is a set of generators in $\mathcal{R}$ is to say that $\mathcal{R} = \sum_{n\in N^*} \mathcal{B}^n$; hence :
$[\mathcal{R},\mathcal{R}] \subset \sum_{n\in N^*} [\mathcal{B}^n,\mathcal{R}] \subset [\mathcal{B},\mathcal{R}] \subset [\mathcal{R},\mathcal{R}]$ and step two follows.

Step three : $\underline{o}(\mathcal{H},J_{\mathbb{R}})$ is a set of generators in the associative algebra $L(\mathcal{H})$; compare with Herstein [86]th. 8.

For the proof of this step, let $\mathcal{R}$ denote for short $L(\mathcal{H})$, let $\underline{o}$ denote $\underline{o}(\mathcal{H},J_{\mathbb{R}};\ L)$, let $\mathcal{A}$ be the associative algebra generated by $\underline{o}$ in $\mathcal{R}$, and let $\underline{m} = \{X \in L(\mathcal{H})\ |\ \varphi_{\mathbb{R}}(X) = X\}$, so that $\mathcal{R} = \underline{o} \oplus \underline{m}$.

Let $A,B \in \underline{o}$, and let $P \in \underline{m}$. Thus $PA + AP \in \underline{o}$, and so $(PA + AP)B \in \mathcal{A}$. Similarly $(PB + BP)A \in \mathcal{A}$. By subtraction : $P[A,B] + APB - BPA \in \mathcal{A}$. However, $APB - BPA = APB - \varphi_R(APB)$ is in $\underline{o}$, and so is in $\mathcal{A}$. Consequently $P[A,B] \in \mathcal{A}$, or otherwise said, $\underline{m}[A,B] \subset \mathcal{A}$ for all $A,B \in \underline{o}$. But clearly $\underline{o}[A,B] \subset \mathcal{A}$ for all $A,B \in \underline{o}$, so that $\mathcal{R}[A,B] = (\underline{o} \oplus \underline{m})[A,B] \subset \mathcal{A}$ for all $A,B \in \underline{o}$. Similarly, $[C,D]\mathcal{R} \subset \mathcal{A}$ for all $C,D \in \underline{o}$. Hence $\mathcal{R}[A,B]\,[C,D]\mathcal{R} \subset \mathcal{A}$ for all $A,B,C,D \in \underline{o}$.

$\mathcal{R}[A,B]\,[C,D]\mathcal{R}$ is clearly a two-sided ideal in $\mathcal{R}$. Hence, either it is inside $C(\mathcal{H})$, or it is the whole of $\mathcal{R} = L(\mathcal{H})$. By a convenient choice of A,B,C,D, this ideal can clearly be forced in the second case, so that $\mathcal{R} \subset \mathcal{A}$, which proves step three.

Step four : the derived ideal of $\underline{o}(\mathcal{H},J_{\mathbb{R}})$ is $\underline{o}(\mathcal{H},J_{\mathbb{R}})$ itself.

Let $\underline{o}$ and $\underline{m}$ be as in step three. By step two in the proof of proposition 1A and by steps two and three above :
$\underline{gl}(\mathcal{H}) = [\underline{gl}(\mathcal{H}), \underline{gl}(\mathcal{H})] = [\underline{o}, \underline{o}\oplus\underline{m}] \subset [\underline{o},\underline{o}] + [\underline{o},\underline{m}]$. As $[\underline{o},\underline{o}] \subset \underline{o}$ and as $[\underline{o},\underline{m}] \subset \underline{m}$, the conclusion follows.

Step five : let $\mathcal{R}$ be a simple associative algebra, with center $\mathcal{Z}$, over a field of characteristic not 2, and such that the dimension of $\mathcal{R}$ over $\mathcal{Z}$ is larger than 16 ; let φ be an involutive antiautomorphism of $\mathcal{R}$ and let $\underline{a}$ be a Lie ideal in $\underline{o} = \{X \in \mathcal{R} \mid \varphi(X) = -X\}$; then either $\underline{a} \subset \mathcal{Z}$, or $[\underline{o},\underline{o}] \subset \underline{a}$. See Herstein [86], th. 9.

Step six : As a consequence of step three in the proof of proposition 1A and of steps four and five above, the only Lie ideals of $\underline{o}(\mathcal{H},J_{\mathbb{R}};L)/\underline{o}(\mathcal{H},J_{\mathbb{R}};C)$ are the trivial ones; hence any Lie ideal of $\underline{o}(\mathcal{H},J_{\mathbb{R}})$ is contained in $\underline{o}(\mathcal{H},J_{\mathbb{R}};C)$. As $\underline{o}(\mathcal{H},J_{\mathbb{R}};C_o)$ is dense in $\underline{o}(\mathcal{H},J_{\mathbb{R}};C)$, the proposition follows. ∎

Let now $J_{\mathbb{Q}}$ be an anticonjugation of $\mathcal{H}$. The involutive antiautomorphisms of $L(\mathcal{H})$ defined by $X \mapsto -J_{\mathbb{Q}}X^*J_{\mathbb{Q}}$ is again denoted by $\varphi_{\mathbb{Q}}$. The symplectic complex Banach-Lie algebra corresponding to $\underline{gl}(\mathcal{H})$ is

$$\underline{sp}(\mathcal{H},J_{\mathbb{Q}};\ L) = \{X \in \underline{gl}(\mathcal{H};L) \mid \varphi_{\mathbb{Q}}(X) = -X\},$$

and will sometimes be simply denoted by $\underline{sp}(\mathcal{H},J_{\mathbb{Q}})$. Its center is reduced to zero and the following subalgebras are clearly both ideals in $\underline{sp}(\mathcal{H},J_{\mathbb{Q}})$:

$\underline{sp}(\mathcal{H},J_{\mathbb{Q}};C_o)$ = subalgebra of finite rank operators.

$\underline{sp}(\mathcal{H},J_{\mathbb{Q}};C)$ = subalgebra of compact operators.

Proposition 1C. Let $\underline{a}$ be a non-trivial ideal of $\underline{sp}(\mathcal{H},J_{\mathbb{Q}})$. Then
$\underline{sp}(\mathcal{H},J_{\mathbb{Q}};C_o) \subset \underline{a} \subset \underline{sp}(\mathcal{H},J_{\mathbb{Q}};C)$.

In particular, the only non-trivial closed ideal of $\underline{sp}(\mathcal{H},J_{\mathbb{Q}})$ is $\underline{sp}(\mathcal{H},J_{\mathbb{Q}};\ C)$; the Banach-Lie algebra $\underline{sp}(\mathcal{H},J_{\mathbb{Q}};\ L)/\underline{sp}(\mathcal{H},J_{\mathbb{Q}};\ C)$ is algebraically simple. The pictorial summing up for closed ideals is similar to that given in proposition 1B.

Proof : as for proposition 1B. ■

Definition 1. A classical complex Banach-Lie algebra of bounded operators is one of the involutive Banach-Lie algebras

$$\underline{gl}(\mathcal{H};\ L) \qquad \underline{o}(\mathcal{H},J_{\mathbb{R}};\ L) \qquad \underline{sp}(\mathcal{H},J_{\mathbb{Q}};\ L)$$

as defined above, where $\mathcal{H}$ is a complex Hilbert space and where $J_{\mathbb{R}}$ [resp. $J_{\mathbb{Q}}$] is some conjugation [resp. anticonjugation] of $\mathcal{H}$.

II.3.- Classical complex Banach-Lie algebras of bounded operators : derivations, automorphisms and real forms

The information collected in Chapter I about the classical Lie algebras of finite rank operators can be used in the study of the classical Banach-Lie algebras of bounded operators. The three propositions which follow are respectively corollaries of sections I.2, I.5 and I.6; the analogue of section I.3 on Cartan subalgebras is left to the reader; the consideration of roots (section I.4) does not seem to me to be of any further help for the knowledge of the algebras of the present Chapter.

In this section, $\mathcal{H}$ is an infinite dimensional complex Hilbert space.

Proposition 2.

A.- Let Δ be a derivation of $\underline{gl}(\mathcal{H})$. Then there exists a unique operator D (up to addition of a scalar multiple of the identity) in $\underline{gl}(\mathcal{H})$ such that $\Delta = ad(D)$.

B.- Let Δ be a derivation of $\underline{o}(\mathcal{H}, J_{\mathbb{R}})$. Then there exists a unique operator D in $\underline{o}(\mathcal{H}, J_{\mathbb{R}})$ such that $\Delta = ad(D)$.

C.- Let Δ be a derivation of $\underline{sp}(\mathcal{H}, J_{\mathbb{Q}})$. Then there exists a unique operator D in $\underline{sp}(\mathcal{H}, J_{\mathbb{Q}})$ such that $\Delta = ad(D)$.

Proof (case A). Let Δ be a derivation of $\underline{gl}(\mathcal{H})$.

Step one : definition of D and δ.

As $\underline{sl}(\mathcal{H}; C_0)$ is an ideal in $\underline{gl}(\mathcal{H})$ which is equal to its own derived ideal, it is globally invariant by Δ. According to proposition I.2A, there exists an endomorphism $D \in Lin(\mathcal{H})$ such that $\Delta(Y) = [D,Y]$ for all $Y \in \underline{sl}(\mathcal{H}; C_0)$. Let then δ be the function defined by

$$\begin{cases} \underline{gl}(\mathcal{H}) \longrightarrow Lin(\mathcal{H}) \\ X \longmapsto \Delta(X) - [D,X] \end{cases}.$$

Step two : δ is zero.

Choose $X \in \underline{gl}(\mathcal{H})$; for every $Y \in \underline{sl}(\mathcal{H}\,;\,C_o)$:

$$\Delta([X,Y]) = \Big[D,[X,Y]\Big] = \Big[\Delta(X),Y\Big] + \Big[X,\ \Delta(Y)\Big] =$$

$$= \Big[[D,X],Y\Big] + \Big[\delta(X),Y\Big] + \Big[X,[D,Y]\Big].$$

It follows from Jacobi identity that $\delta(X)$ commutes with all finite rank operators having zero trace, i.e. that $\delta(X)$ is multiple of the identity, say $\delta(X) = \varepsilon(X)\mathrm{id}_{\mathcal{H}}$. As Δ is linear, ε defines a linear form or $\underline{gl}(\mathcal{H})$. From the derivation rule for Δ, it follows easily that ε vanishes on commutators. As $\underline{gl}(\mathcal{H})$ is equal to its derived ideal, it follows that ε vanishes, hence so does δ.

Step three : Δ is continuous.

Steps one and two show that Δ is already a derivation of the associative algebra $L(\mathcal{H})$. The end of the proof follows then from standard properties of derivations in C^*- and von Neumann algebras; see Dixmier [46] chap III §9. The fact that the continuity of Δ implies that of D is left to the reader.

Modifications for cases B and C. Let Δ be a derivation of $\underline{o}(\mathcal{H},J_{\mathbb{R}})$. Steps one and two can be repeated almost without change, so that $\Delta(X) = [D,X]$ for all $X \in \underline{o}(\mathcal{H},J_{\mathbb{R}})$, where D is an ad hoc operator on $\mathcal{H}$. It follows that Δ can be extended to a derivation of $\underline{gl}(\mathcal{H})$, whence the continuity of D, whence that of Δ. This can be repeated for $\underline{sp}(\mathcal{H},J_{\mathbb{Q}})$. ∎

Proposition 2 can be rephrased as follows.

Corollary. Let $\underline{g}$ be a classical complex Banach-Lie algebra of bounded operators. Then any derivation of $\underline{g}$ is continuous and inner.

In particular, any Lie derivation of the C^*-algebra $L(\mathcal{H})$ is already a derivation in the associative sense.

Proposition 3.

A.- Let φ be a $*$-automorphism of $\underline{gl}(\mathcal{H})$ and let $J_{\mathbb{R}}$ be a fixed conjugation on $\mathcal{H}$. Then there exists a unitary operator $V \in U(\mathcal{H})$ such that

$$\text{either } \varphi = \varphi_{VJ} \quad : \quad \begin{cases} \underline{gl}(\mathcal{H}) & \longrightarrow \quad \underline{gl}(\mathcal{H}) \\ X & \longmapsto \quad -VJ_{\mathbb{R}}X^*J_{\mathbb{R}}V^* \end{cases}$$

$$\text{or} \quad \varphi = \varphi_V \quad : \quad \begin{cases} \underline{gl}(\mathcal{H}) & \longrightarrow \quad \underline{gl}(\mathcal{H}) \\ X & \longmapsto \quad VXV^* \end{cases}$$

The two cases exclude each other.

The operator V is uniquely determined by φ, up to multiplication by a complex number of modulus one. Otherwise said, the sequence

$$\{1\} \longrightarrow U(1) \xrightarrow{j} \tilde{U}(\mathcal{H}) \xrightarrow{\pi} \overset{*}{\mathrm{Aut}}(\underline{gl}(\mathcal{H})) \longrightarrow \{1\}$$

is exact (notations as in proposition I.10A).

Similarly and more briefly in the two other cases :

B.- The sequence $\{1\} \to Z_2 \longrightarrow O(\mathcal{H}_{\mathbb{R}}) \longrightarrow \overset{*}{\mathrm{Aut}}(\underline{o}(\mathcal{H},J_{\mathbb{R}})) \longrightarrow \{1\}$ is exact.

C.- The sequence $\{1\} \to Z_2 \longrightarrow Sp(\mathcal{H}_{\mathbb{Q}}) \longrightarrow \overset{*}{\mathrm{Aut}}(\underline{sp}(\mathcal{H},J_{\mathbb{Q}})) \longrightarrow \{1\}$ is exact.

Proof (case A). Let φ be a $*$-automorphism of $\underline{gl}(\mathcal{H})$.

Step one. As $\underline{sl}(\mathcal{H}; C_o)$ is absolutely minimal among the ideals of $\underline{gl}(\mathcal{H})$ having dimension strictly bigger than one (proposition 1), it is globally invariant by φ. According to proposition I.10A, there exists $V \in U(\mathcal{H})$ such that the restriction of φ to $\underline{sl}(\mathcal{H}; C_o)$ is either φ_{VJ} or φ_V.

Step two. Suppose first that $\varphi(Y) = \varphi_V(Y)$ for all $Y \in \underline{sl}(\mathcal{H}; C_o)$. Choose $X \in \underline{gl}(\mathcal{H})$; for any $Y \in \underline{sl}(\mathcal{H}; C_o)$:
$\varphi([X,Y]) = V[X,Y]V^* = [VXV^*,VYV^*] = [\varphi(X), \varphi(Y)] = [\varphi(X),VYV^*]$. It follows from Schur's lemma that $\varphi(X) - VXV^*$ is a multiple of the identity, say $\varphi(X) - VXV^* = \varepsilon(X)\mathrm{id}_{\mathcal{H}}$. As φ is linear, ε defines a

linear form on $\underline{gl}(\mathcal{H})$. From the automorphism rule for φ, it follows easily that ε vanishes on commutators. As $\underline{gl}(\mathcal{H})$ is equal to its derived ideal, it follows that ε vanishes; hence $\varphi = \varphi_V$.

Step three. Suppose now that $\varphi(Y) = \varphi_{VJ}(Y)$ for all $Y \in \underline{sl}(\mathcal{H}; C_o)$. The same argument as in step two shows that $\varphi(X) = \varphi_{VJ}(X)$ for all $X \in \underline{gl}(\mathcal{H})$. ∎

Proposition 3 can be rephrased as follows.

Corollary. Let $\underline{g}$ be a classical complex Banach-Lie algebra of bounded operators. Then any *-automorphism of $\underline{g}$ is isometric and inner.

In particular, any Lie *-automorphism of the C^*-algebra $L(\mathcal{H})$ is already either a *-automorphism or the negative of a *-antiautomorphism in the associative sense.

Remarks.

i) The case A of proposition 2 follows as well from general facts about derivations of von Neumann algebras (Dixmier [46] chap. III §9 th. 1) and about Lie derivations of primitive rings (Martindale III [117] th. 2).

ii) The case A of proposition 3 follows as well from general facts about *-automorphisms of von Neumann algebras (Dixmier [46] chap. III § 3, n° 2) and about Lie automorphisms of prime rings (Martindale [118] th. 13). With the help of other methods, it is possible to show that all (not necessarily *-) automorphisms of $\underline{gl}(\mathcal{H})$ are inner : see for example Arnold [3] th. 4.

iii) I do not know of any published results which would be the analogues of those in [117] and [118] for rings with involution, and of which propositions 2B, 2C, 3B and 3C would be typical examples. However, these propositions are again corollaries of particular results due to Rickart [140].

iv) The word "inner" in the corollary above has a slightly broader sense than in the associative case : in proposition 3A indeed, the group $\tilde{U}(\mathcal{H})$ appears, not only its subgroup $U(\mathcal{H})$.

The considerations of section I.6 can now be repeated with almost no change for classical Banach-Lie algebras of bounded operators. Definition I.8 and proposition I.11 have evident analogues in the present context, and so have each of the conjugations and real forms described in section I.6. For example, the conjugation of $\underline{gl}(\mathcal{H})$ defined by $X \longmapsto J_{\mathbb{R}} X J_{\mathbb{R}}$ is again denoted by σ_{AI} and the corresponding real form is $\underline{gl}(\mathcal{H}_{\mathbb{R}}; L) = \{X \in \underline{gl}(\mathcal{H}; L) \mid \sigma_{AI}(X) = X\}$. The other involutive Banach-Lie algebras appearing in the following definition can be expressed similarly.

Definition 2. A classical real Banach-Lie algebra of bounded operators is one of the involutive Banach-Lie algebras listed below. ($\mathcal{H}$ is assumed to be separable and infinite dimensional for simplicity of the notations, but the list could be made valid for a space of arbitrary dimension with minor modifications only.)

Type AI : $\underline{gl}(\mathcal{H}_{\mathbb{R}}; L)$.

AII : $\underline{u^*}(\mathcal{H}; L)$.

AIII : $\underline{u}(\mathcal{H}, r, \infty; L)$ where $r \in \mathbb{N} \cup \{\infty\}$; the compact form of type A $\underline{u}(\mathcal{H}; L)$ corresponds to $r = 0$.

Type BDI : $\underline{o}(\mathcal{H}, r, \infty; L)$ where $r \in \mathbb{N} \cup \{\infty\}$; the compact form of type BD $\underline{o}(\mathcal{H}_{\mathbb{R}}; L)$ corresponds to $r = 0$.

BDIII : $\underline{o^*}(\mathcal{H}; L)$.

Type CI : $\underline{sp}(\mathcal{H}, \mathbb{R}; L)$.

CII : $\underline{sp}(\mathcal{H}, r, \infty; L)$ where $r \in \mathbb{N} \cup \{\infty\}$; the compact form of type C $\underline{sp}(\mathcal{H}_{\mathbb{Q}}; L)$ corresponds to $r = 0$.

There are analogues to proposition 1 and 2 for each classical real Banach-Lie algebra of bounded operators. Both their statements and their proofs are left to the reader.

Proposition 4. Let $\underline{g}$ be a classical complex Banach-Lie algebra of bounded operators and let $\underline{s}$ be a real form of $\underline{g}$. Then $\underline{s}$ is *-isomorphic to a classical real Banach-Lie algebra of bounded operators, of one of the types A, B and C, as described in definition 2.

Proof : see propositions I.12 and II.3. ∎

II.4.– Classical Banach-Lie groups of bounded operators

Each of the Banach-Lie algebras described in the two preceding sections is the Lie algebra of a Banach-Lie group of operators.

For example $\underline{gl}(\mathcal{H}; L)$ is the Lie algebra of the general linear group $GL(\mathcal{H}; L)$, or for short $GL(\mathcal{H})$, of all invertible operators on $\mathcal{H}$. The subgroup of $GL(\mathcal{H})$ consisting of those elements which leave invariant the bilinear form

$$\begin{cases} \mathcal{H} \times \mathcal{H} \longrightarrow \mathbb{C} \\ (x,y) \longrightarrow \langle x | J_{\mathbb{R}} y \rangle \end{cases}$$

is denoted by $O(\mathcal{H}, J_{\mathbb{R}}; L)$ or by $O(\mathcal{H}, J_{\mathbb{R}})$; it is a sub Banach-Lie group-manifold of $GL(\mathcal{H})$ and its Lie algebra is clearly $\underline{o}(\mathcal{H}, J_{\mathbb{R}})$. Similarly, the subgroup of $GL(\mathcal{H})$ consisting of those elements which leave invariant the bilinear form

$$\begin{cases} \mathcal{H} \times \mathcal{H} \longrightarrow \mathbb{C} \\ (x,y) \longrightarrow \langle x | J_{\mathbb{Q}} y \rangle \end{cases}$$

is denoted by $Sp(\mathcal{H}, J_{\mathbb{Q}}; L)$ or by $Sp(\mathcal{H}, J_{\mathbb{Q}})$; it is again a sub Banach-Lie group-manifold of $GL(\mathcal{H})$ and its Lie algebra is clearly $\underline{sp}(\mathcal{H}, J_{\mathbb{Q}})$.

Definition 3. A classical complex Banach-Lie group of bounded operators is one of the Banach-Lie groups $GL(\mathcal{H}; L)$, $O(\mathcal{H}, J_{\mathbb{R}}; L)$ and $Sp(\mathcal{H}, J_{\mathbb{Q}}; L)$ as defined above, where $\mathcal{H}$ is a complex Hilbert space and where $J_{\mathbb{R}}$ [resp. $J_{\mathbb{Q}}$] is some conjugation [resp. anticonjugation] of $\mathcal{H}$.

The definition and the list of the classical real Banach-Lie groups of bounded operators is left to the reader (see definition 2 section II.3).

In the rest of this section, we review some of the standard facts known about the groups introduced above. The set of selfadjoints [resp. positive] operators in $L(\mathcal{H})$ is denoted by $\mathrm{Sym}(\mathcal{H})$ [resp. $\mathrm{Pos}(\mathcal{H})$].

<u>Proposition 5</u>. Let $G(\mathcal{H})$ be a classical real or complex Banach-Lie group of bounded operators on $\mathcal{H}$ and let $\underline{g}(\mathcal{H})$ be its Lie algebra.

i) The exponential map $\underline{g}(\mathcal{H}) \longrightarrow G(\mathcal{H})$ is given by the traditional power series.

ii) The exponential map provides an analytic morphism, which is a local isomorphism

$$\underline{g}(\mathcal{H}) \cap \underline{u}(\mathcal{H}) \xrightarrow{\quad \exp \quad} G(\mathcal{H}) \cap U(\mathcal{H}).$$

iii) The exponential map provides an analysic isomorphism

$$\underline{g}(\mathcal{H}) \cap \mathrm{Sym}(\mathcal{H}) \text{ —— } \exp \longrightarrow G(\mathcal{H}) \cap \mathrm{Pos}(\mathcal{H}).$$

iv) The polar decomposition provides an analytic isomorphism

$$G(\mathcal{H}) \longrightarrow \Big(G(\mathcal{H}) \cap U(\mathcal{H})\Big) \times \Big(G(\mathcal{H}) \cap \mathrm{Pos}(\mathcal{H})\Big)$$

<u>Proof</u> : for i), see Lazard [107] prop. 10.6 and remarque 21.6; for ii) to iv), see Lang [103] chap. 7 prop. 3, 5 and 6 when $G(\mathcal{H}) = GL(\mathcal{H})$; the other cases can be proved the same way. ∎

<u>Proposition 6</u>.

A.- The exponential map $\underline{u}(\mathcal{H}; L) \longrightarrow U(\mathcal{H}; L)$ is onto.

B.- The image of the exponential map $\underline{o}(\mathcal{H}_{\mathbb{R}}; L) \longrightarrow O(\mathcal{H}_{\mathbb{R}}; L)$ is the set of those operators such that the multiplicity with which -1 appears in their point spectrum is either finite and even (possibly zero) or infinite.

C.- The exponential map $\underline{sp}(\mathcal{H}_{\mathbb{Q}}; L) \longrightarrow Sp(\mathcal{H}_{\mathbb{Q}}; L)$ is onto.

<u>Proof</u>: in Putnam and Winter [136], section 7 to 11; their argument, for $\mathbb{K} = \mathbb{R}$, can be easily adapted (and simplified) for the cases $\mathbb{K} = \mathbb{C}$ and $\mathbb{K} = \mathbb{Q}$. ∎

Proposition 7. Let $\mathcal{H}$ be infinite dimensional. Then the real Banach-Lie groups $U(\mathcal{H})$, $O(\mathcal{H}_{\mathbb{R}})$ and $Sp(\mathcal{H}_{\mathbb{Q}})$ are contractible. Similarly, the complex Banach-Lie groups $GL(\mathcal{H})$, $O(\mathcal{H}, J_{\mathbb{R}})$ and $Sp(\mathcal{H}, J_{\mathbb{Q}})$ are contractible.

Proof : due to Kuiper [101], [89]; for a recent discussion on this result, see Mityagini [123]. ∎

The homotopy type of any other classical Banach-Lie group of operators on $\mathcal{H}$ follows trivially from proposition 7, via proposition II.5. iv.

It is often possible to translate in the context of the classical Banach-Lie groups of bounded operators properties of their Lie algebras as those seen earlier in this chapter.

For example, Brown and Pearcy have shown that any element in $GL(\mathcal{H}; L)$ is a product of two commutators ([134] section I.4).

There are detailed studies, due to Kadison, of the normal closed subgroups of $GL(\mathcal{H})$, $U(\mathcal{H})$, and of the general linear and unitary groups in the other types of factors [93], [94], [95]. According to Rosenberg ([144], p. 279), some of these results have been further improved by Kaplansky (unpublished). Similar problems have been dealt with by Sunouchy [171].

It seems a sound conjecture, and probably even an easy one (though tedious) to check, that analogues of these various results about the derived groups and the normal subgroups of $GL(\mathcal{H})$ and $U(\mathcal{H})$ still hold for the other classical Banach-Lie groups of bounded operators.

II.5.- Classical Banach-Lie algebras of compact operators

In this section, $\mathcal{H}$ is a complex Hilbert space, of infinite dimension if not otherwise stated. Properties of the ideals $C_p(\mathcal{H})$ are recalled in an appendix to this chapter.

Let $p \in \bar{\mathbb{R}}$, $1 \leqslant p \leqslant \infty$. The associative involutive Banach algebra $C_p(\mathcal{H})$ defines an involutive Banach-Lie algebra which will be denoted by $\underline{gl}(\mathcal{H}; C_p)$, or by $\underline{gl}(\mathcal{H}; C)$ when $p = \infty$. When $p = 1$, the closure of its derived ideal is the involutive Banach-Lie algebra consisting of nuclear operators with trace zero, and will be denoted by $\underline{sl}(\mathcal{H}; C_1)$.

Let $J_{\mathbb{R}}$ be a conjugation of $\mathcal{H}$. The involutive antiautomorphism of $C_p(\mathcal{H})$ defined by $X \longmapsto J_{\mathbb{R}}X^*J_{\mathbb{R}}$ is again denoted by $\varphi_{\mathbb{R}}$ (see sections I.1 and II.2). The <u>orthogonal complex Banach-Lie algebra</u> corresponding to $\underline{gl}(\mathcal{H}; C_p)$ is

$$\underline{o}(\mathcal{H}, J_{\mathbb{R}}; C_p) = \{X \in \underline{gl}(\mathcal{H}; C_p) \mid \varphi_{\mathbb{R}}(X) = -X\}.$$

Let $J_{\mathbb{Q}}$ be an anticonjugation of $\mathcal{H}$. The involutive antiautomorphism of $C_p(\mathcal{H})$ defined by $X \mapsto -J_{\mathbb{Q}}X^*J_{\mathbb{Q}}$ is again denoted by $\varphi_{\mathbb{Q}}$. The <u>symplectic complex Banach-Lie algebra</u> corresponding to $\underline{gl}(\mathcal{H}; C_p)$ is

$$\underline{sp}(\mathcal{H}, J_{\mathbb{Q}}; C_p) = \{X \in \underline{gl}(\mathcal{H}; C_p) \mid \varphi_{\mathbb{Q}}(X) = -X\}.$$

<u>Definition 4</u>. A <u>classical complex Banach-Lie algebra of compact operators</u> is one of the involutive Banach-Lie algebras

$$\underline{gl}(\mathcal{H}; C_p) \qquad \underline{sl}(\mathcal{H}; C_1) \qquad \underline{o}(\mathcal{H}, J_{\mathbb{R}}; C_p) \qquad \underline{sp}(\mathcal{H}, J_{\mathbb{Q}}; C_p)$$

as defined above, where $\mathcal{H}$ is a complex Hilbert space, where $J_{\mathbb{R}}$ [resp. $J_{\mathbb{Q}}$] is some conjugation [resp. anticonjugation] of $\mathcal{H}$, and where p is in $\bar{\mathbb{R}}$, with $1 \leqslant p \leqslant \infty$.

Many properties of these algebras follow straightforwardly from those of the Lie algebras studied so far. The end of this section will list some of them; proofs, being evident, will be non-existent or very sketchy.

<u>Proposition 8</u>. Let $\underline{g}(\mathcal{H}; C_p)$ be a classical complex Banach-Lie algebra of compact operator on $\mathcal{H}$. Then the involution defined by $X \longmapsto X^*$ is a c-involution; and any nonzero ideal of $\underline{g}(\mathcal{H}; C_p)$ contains the corresponding classical complex Banach-Lie algebra of finite rank operators. In particular, $\underline{g}(\mathcal{H}; C_p)$ is a (topologically) simple c-involutive Banach-Lie algebra (except $\underline{gl}(\mathcal{H}; C_1)$ which is not simple).

<u>Proof</u> : see propositions I.1 and II.1. ∎

<u>Definition 5</u>. Let $\underline{g}(\mathcal{H}; C_p)$ be a classical complex Banach-Lie algebra of compact operators on $\mathcal{H}$. A derivation Δ of $\underline{g}(\mathcal{H}; C_p)$ is said to be <u>spatial</u> if there exists an operator D in the corresponding classical complex Banach-Lie algebra of bounded operators such that $\Delta(X) = [D,X]$ for all $X \in \underline{g}(\mathcal{H}; C_p)$. A *-automorphism φ of $\underline{g}(\mathcal{H}; C_p)$ is said to be <u>spatial</u> if there exists an operator V in the corresponding classical complex Banach-Lie group of bounded operators such that one of the following holds:

either $\varphi(X) = -VJ_{\mathbb{R}}X^*J_{\mathbb{R}}V^*$ for all $X \in \underline{g}(\mathcal{H}; C_p)$, for some conjugation $J_{\mathbb{R}}$ on $\mathcal{H}$,

or $\varphi(X) = VXV^*$ for all $X \in \underline{g}(\mathcal{H}; C_p)$.

<u>Proposition 9</u>. Any derivation of a classical complex Banach-Lie algebra of compact operators on $\mathcal{H}$ is continuous and spatial.

<u>Proof</u> : see propositions I.2, II.2 and corollary; the reference given in the step three of the proof of propositions II.2 must now be

replaced by Johnson and Sinclair [92]. The situation described by proposition II.2 and II.9 is clearly reminiscent of that known for simple C^*-algebras [146]. ∎

Proposition 10. The Cartan subalgebras of a classical complex Banach-Lie algebra of compact operators are described exactly as in section I.3 : those of $\underline{gl}(\mathcal{H}; C_p)$ and of $\underline{sp}(\mathcal{H}, J_{\mathbb{Q}}; C_p)$ are conjugated by the group of *-automorphisms; those of $\underline{o}(\mathcal{H}, J_{\mathbb{R}}; C_p)$ split into two conjugation-classes under the action of the group of *-automorphism.

Proof : see proposition I.3. ∎ Note that a Cartan subalgebra of a classical complex Banach-Lie algebra of compact operators on $\mathcal{H}$ contains a dense set of regular elements if and only if $\mathcal{H}$ is separable [9]. In all cases, such a subalgebra is equal to its normalizer.

The notions of root, root vector, and type of a Cartan subalgebra of $\underline{o}(\mathcal{H}, J_{\mathbb{R}}; C_p)$, are defined in the evident way.

Proposition 11. Let $\underline{g}$ be a classical complex Banach-Lie algebra of compact operators and let $\underline{h}$ be a Cartan subalgebra of $\underline{g}$. Then the roots of $\underline{g}$ with respect to $\underline{h}$ are given by the formulae (I.6) to (I.19).

Proof : section I.4. ∎

Proposition 12. Any *-automorphism of a classical complex Banach-Lie algebra of compact operators on $\mathcal{H}$ is isometric and spatial.

Proof : propositions I.10 and II.3. ∎

The considerations of section I.6 on real forms can again be repeated with almost no change. The notations should be by now obvious enough for us to give without further comment the following definition.

<u>Definition 6</u>. A <u>classical real Banach-Lie algebra of compact operators</u> is one of the c-involutive Banach-Lie algebras listed below (the list is given for $\mathcal{H}$ separable and infinite dimensional; $p \in \bar{\mathbb{R}}$ with $1 \leq p \leq \infty$).

<u>Type AI</u> : $\underline{gl}(\mathcal{H}_{\mathbb{R}};\ C_p)$ and $\underline{sl}(\mathcal{H}_{\mathbb{R}};\ C_1)$.

AII : $\underline{u^*}(\mathcal{H};\ C_p)$ and $\underline{su}^*(\mathcal{H};\ C_1)$.

AIII : $\underline{u}(\mathcal{H},r,\infty;\ C_p)$ and $\underline{su}(\mathcal{H},r,\infty;\ C_1)$ where $r \in N \cup \{\infty\}$; the compact forms of type A $\underline{u}(\mathcal{H};\ C_p)$ and $\underline{su}(\mathcal{H};\ C_1)$ correspond to $r = 0$.

<u>Type BDI</u> : $\underline{o}(\mathcal{H},r,\infty;\ C_p)$ where $r \in N \cup \{\infty\}$; the compact form of type B $\underline{o}(\mathcal{H}_{\mathbb{R}};\ C_p)$ corresponds to $r = 0$.

BDIII : $\underline{o^*}(\mathcal{H};\ C_p)$.

<u>Type CI</u> : $\underline{sp}(\mathcal{H},\mathbb{R};\ C_p)$.

CII : $\underline{sp}(\mathcal{H},r,\infty;\ C_p)$ where $r \in N \cup \{\infty\}$; the compact form of type C $\underline{sp}(\mathcal{H}_{\mathbb{Q}};\ C_p)$ corresponds to $r = 0$.

<u>Algebras with a complex structure</u> : the algebras $\underline{gl}(\mathcal{H};\ C_p)$, $\underline{sl}(\mathcal{H};\ C_1)$, $\underline{o}(\mathcal{H},J_{\mathbb{R}};\ C_p)$ and $\underline{sp}(\mathcal{H},J_{\mathbb{Q}};\ C_p)$ viewed as real Lie algebras.

<u>Proposition 13</u>. Let $\underline{g}$ be a classical complex Banach-Lie algebra of compact operators and let $\underline{s}$ be a real form of $\underline{g}$. Then $\underline{s}$ is *-isomorphic to a classical real Banach-Lie algebra of compact operators, of one of the types A, B and C, as described in definition 6.

<u>Proof</u> : see section I.6 and proposition II.4. ∎

We end this section with two remarks.

Pearcy and Topping [131] have shown that the derived ideal of $\underline{gl}(\mathcal{H}; C)$ is the whole algebra, and that the derived ideal of $\underline{gl}(\mathcal{H}; C_{2p})$ is exactly $\underline{gl}(\mathcal{H}; C_p)$ for all $p \in \mathbb{R}$ such that $p > 1$; it is not known if the derived ideal of $\underline{gl}(\mathcal{H}; C_2)$ is the whole of $\underline{sl}(\mathcal{H}; C_1)$. Similar results can be conjectured (and are probably easy to prove) for the corresponding orthogonal and symplectic complex Lie algebras.

There is a sort of duality among the Banach-Lie algebras introduced in this Chapter which might suggest a generalization of the notion of L^*-algebra. More precisely, let $\underline{a}$ and $\underline{b}$ be two sub involutive Banach-Lie algebras of $\underline{gl}(\mathcal{H}; L)$; then $\underline{b}$ is said to be <u>dual to $\underline{a}$ in $\underline{g}$</u> if there exists a sesquilinear pairing $\langle\!\langle \,|\, \rangle\!\rangle : \underline{a} \times \underline{b} \longrightarrow \mathbb{C}$ and if the following conditions are satisfied :

i) $[\underline{a},\underline{b}]$ is an ideal both in $\underline{a}$ and in $\underline{b}$;

ii) $\langle\!\langle [A,X] | Y \rangle\!\rangle = \langle\!\langle X | [A^*,Y] \rangle\!\rangle$ for all $A,X \in \underline{a}$, for all $Y \in \underline{b}$;

iii) $\langle\!\langle X | [B,Y] \rangle\!\rangle = \langle\!\langle [B^*,X] | Y \rangle\!\rangle$ for all $X \in \underline{a}$, for all $B,Y \in \underline{b}$;

iv) the map $\left\{ \begin{array}{l} \underline{b} \longrightarrow (\underline{a})^{\text{dual}} \\ B \longmapsto \begin{pmatrix} X & \underline{a} \\ \downarrow & \downarrow \\ \langle\!\langle X | B^* \rangle\!\rangle & \mathbb{C} \end{pmatrix} \end{array} \right.$ defines

an isometry of the Banach space $\underline{b}$ onto the strong dual of the Banach space $\underline{a}$.

Examples of such a situation are given by :

$\underline{a} = \underline{gl}(\mathcal{H}; C)$	$\underline{b} = \underline{gl}(\mathcal{H}; C_1)$
$= \underline{gl}(\mathcal{H}; C_p)$	$= \underline{gl}(\mathcal{H}; C_q)$
$(p \in \mathbb{R},\ 1 < p < \infty)$	$(q \in \mathbb{R},\ \frac{1}{p} + \frac{1}{q} = 1)$
$= \underline{gl}(\mathcal{H}; C_1)$	$= \underline{gl}(\mathcal{H}; L)$

and by the corresponding orthogonal complex and symplectic complex pairs; in all cases, the pairing is given by $\langle\!\langle X|Y \rangle\!\rangle = \text{trace}(XY^*)$. The "duality" carries over to Cartan subalgebras, roots, ... The project might be suggested to try and extend (at least part of) Schue's results [153], [154] to "pairs of involutive Banach-Lie algebras", according to some definition tailored from the above examples. (Schue's L^*-case is given by $p = q = 2$.)

II.6.- Classical Banach-Lie groups of compact operators

This section describes Banach-Lie groups which correspond to the Banach-Lie algebras of section II.5 in the same way as the groups described in section II.4 correspond to the algebras of section II.2 and II.3.

Let $p \in \bar{\mathbb{R}}$, $1 \leqslant p \leqslant \infty$ and let $GL(\mathcal{H}; C_p)$ be the group of those invertible operators on $\mathcal{H}$ which can be written as $id_{\mathcal{H}} + X$, where X is in $C_p(\mathcal{H})$. As there is a distinguished bijection between $GL(\mathcal{H}; C_p)$ and an open subset in $C_p(\mathcal{H})$, the group is in fact a sub Banach-Lie group of $GL(\mathcal{H}; L)$ - not a sub group-manifold (II.3) - and its Lie algebra is $\underline{gl}(\mathcal{H}; C_p)$. When $p = 1$, $SL(\mathcal{H}; C_1)$ is the subgroup of $GL(\mathcal{H}; C_1)$ consisting of those operators having determinant $+1$; it is again a Banach-Lie group, with Lie algebra $\underline{sl}(\mathcal{H}; C_1)$.

Similarly, one defines for each $p \in \bar{\mathbb{R}}$, $1 \leqslant p \leqslant \infty$, the Banach-Lie groups $O(\mathcal{H}, J_{\mathbb{R}}; C_p)$ whose connected component is denoted by $O^+(\mathcal{H}, J_{\mathbb{R}}; C_p)$ and whose Lie algebra is $\underline{o}(\mathcal{H}, J_{\mathbb{R}}; C_p)$; $Sp(\mathcal{H}, J_{\mathbb{Q}}; C_p)$ whose Lie algebra is $\underline{sp}(\mathcal{H}, J_{\mathbb{Q}}; C_p)$; and the groups of operators corresponding to the Lie algebras of definition 6. These groups are respectively called the <u>classical complex Banach-Lie groups of compact operators</u> and the <u>classical real Banach-Lie groups of compact operators</u>.

Unless otherwise stated, we will always consider the <u>connected components</u> of these groups. When explicitely written down, the connected component of a group will be affected by a subscript + ; for example, $O^+(\mathcal{H}_{\mathbb{R}}; C_p)$ is of index 2 in $O(\mathcal{H}_{\mathbb{R}}; C_p)$. (Hence, if $\mathcal{H}$ was finite dimensional, $O^+(\mathcal{H}_{\mathbb{R}})$ would denote the Lie group classically written $SO(\mathcal{H}_{\mathbb{R}})$).

The properties of these groups corresponding to propositions 5 to 7 of section II.4 will now be listed. The space of self-adjoint operators in $C_p(\mathcal{H})$ is denoted by $Sym(\mathcal{H}; C_p)$. The set of those positive operators on $\mathcal{H}$ which can be written as $id_{\mathcal{H}} + X$ with $X \in C_p(\mathcal{H})$ is denoted by $Pos(\mathcal{H};\, C_p)$.

<u>Proposition 14</u>. Let $G(\mathcal{H};\, C_p)$ be a classical real or complex Banach-Lie group of compact operators on $\mathcal{H}$ and let $\underline{g}(\mathcal{H};\, C_p)$ be its Lie algebra.

i) The exponential map $\underline{g}(\mathcal{H};\, C_p) \longrightarrow G(\mathcal{H};\, C_p)$ is given by the traditional power series.

ii) The exponential map provides an analytic morphism, which is a local isomorphism

$$\underline{g}(\mathcal{H};\, C_p) \cap \underline{u}(\mathcal{H};\, C_p) \xrightarrow{\quad \exp \quad} G(\mathcal{H};\, C_p) \cap U(\mathcal{H};\, C_p).$$

iii) The exponential map provides an analytic isomorphism

$$\underline{g}(\mathcal{H};\, C_p) \cap Sym(\mathcal{H};\, C_p) \xrightarrow{\quad \exp \quad} G(\mathcal{H};\, C_p) \cap Pos(\mathcal{H};\, C_p).$$

iv) The polar decomposition provides an analytic isomorphism from $G(\mathcal{H};\, C_p)$ to

$$\Big(G(\mathcal{H};\, C_p) \cap U(\mathcal{H};\, C_p)\Big) \times \Big(G(\mathcal{H};\, C_p) \cap Pos(\mathcal{H};\, C_p)\Big).$$

<u>Proof</u> : see proposition II.5. ∎

<u>Proposition 15</u>

A.- The exponential map $\underline{u}(\mathcal{H};\, C_p) \longrightarrow U(\mathcal{H};\, C_p)$ is onto.

B.- The image of the exponential map $\underline{o}(\mathcal{H}_{\mathbb{R}};\, C_p) \to O(\mathcal{H}_{\mathbb{R}};\, C_p)$ is the set of those operators such that the multiplicity with which -1 appears in their spectrum is (finite and) even (possibly zero).

C.- The exponential map $\underline{sp}(\mathcal{H}_{\mathbb{Q}};\, C_p) \to Sp(\mathcal{H}_{\mathbb{Q}};\, C_p)$ is onto.

<u>Proof</u> : see proposition II.6; case B is easier than the original result of Putnam and Wintner insofar as the spectral theorem for compact operators is good enough; a detailed proof was written up in [75]. ∎

The statement corresponding to proposition II.7 is a corollary of a much more general theorem, due to Geba [64] and extending former results by Elworthy [61], [62], Palais [129] and Svarc [172]. We first give the result proved, though not stated in this generality, by by Geba.

Let E be an infinite dimensional Banach space over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}, \mathbb{Q}\}$. Let $\mathcal{P}(E)$ be a subspace of L(E), furnished with a norm (but not necessarily complete), and enjoying the following properties :

i) If $X \in \mathcal{P}(E)$, $id_{\mathcal{H}} + X$ is a Fredholm operator on E ; equivalently, $\mathcal{P}(E)$ is contained in the set of Riesz operators on E (see Schechter [152]).

ii) If $X \in \mathcal{P}(E)$, then $X + C_o(E) \subset \mathcal{P}(E)$; in particular, $C_o(E) \subset \mathcal{P}(E)$.

iii) The multiplication $C_o(E) \times \mathcal{P}(E) \longrightarrow \mathcal{P}(E)$ is continuous, where $C_o(E)$ is furnished with the uniform norm and $\mathcal{P}(E)$ with its own norm.

Let then $GL(E;\mathcal{P})$ be the subset of GL(E) consisting of all those invertible operators on E which can be written as $id_{\mathcal{H}} + X$, with $X \in \mathcal{P}(E)$; let $GL(E;\mathcal{P})$ be endowed with the topology inherited from the norm given on $\mathcal{P}(E)$. Note that if $\mathcal{P}(E)$ is an ideal in L(E), then $GL(E;\mathcal{P})$ is a group; if $\mathcal{P}(E)$ is a normed algebra, the multiplication is continuous in $GL(E;\mathcal{P})$; if $\mathcal{P}(E)$ is a Banach algebra, $GL(E;\mathcal{P})$ is a Banach-Lie group. Numerous examples of such spaces $\mathcal{P}(E)$ can be found for example in Pietsch [132], [133].

<u>Proposition 16</u>. Let E and $GL(E;\mathcal{P})$ be as above. Then $GL(E;\mathcal{P})$ is homotopically equivalent to the stable general linear group $GL(\infty, \mathbb{K})$.

<u>Proof</u> : see Geba [64]. ∎

Note that, if $e = (e_n)_{n \in N}$ is an orthonormal basis of the Hilbert space $\mathcal{H}_{\mathbb{K}} = E$, if $GL(n, \mathbb{K})$ is identified via e to the evident subgroup of $GL(\mathcal{H}_{\mathbb{K}}; \mathcal{C})$, and if $GL(\infty, \mathbb{K})$ is the inductive limit of the $GL(n, \mathbb{K})$'s, then the inclusion $GL(\infty, \mathbb{K}) \to GL(\mathcal{H}_{\mathbb{K}}; \mathcal{C})$ is itself a homotopy equivalence. The inclusions of the corollary are constructed the same way.

Corollary. Let $\mathcal{H}$ be infinite dimensional, separable and complex; let $p \in \bar{\mathbb{R}}$, $1 \leq p \leq \infty$. Then the homotopy types of the classical Banach-Lie groups of compact operators are given by the following table, where all arrows are homotopy equivalences.

Complex groups.

Type A : $S^1 \times SU(\infty) \sim GL(\infty, \mathbb{C}) \longrightarrow GL(\mathcal{H}; C_p)$.

$SU(\infty) \longrightarrow SL(\infty, \mathbb{C}) \longrightarrow SL(\mathcal{H}; C_1)$.

Type B : $SO(\infty) \longrightarrow SO(\infty, \mathbb{C}) \longrightarrow O^+(\mathcal{H}, J_{\mathbb{R}}; C_p)$.

Type C : $Sp(\infty) \longrightarrow Sp(\infty, \mathbb{C}) \longrightarrow Sp(\mathcal{H}, J_{\mathbb{Q}}; C_p)$.

Real groups.

Type AI : $SO(\infty) \longrightarrow GL^+(\infty, \mathbb{R}) \longrightarrow GL^+(\mathcal{H}_{\mathbb{R}}; C_p)$.

$SO(\infty) \longrightarrow SL(\infty, \mathbb{R}) \longrightarrow SL(\mathcal{H}_{\mathbb{R}}; C_1)$.

AII : $S^1 \times Sp(\infty) \sim U^*(\infty) \longrightarrow U^*(\mathcal{H}; C_p)$.

$Sp(\infty) \longrightarrow SU^*(\infty) \longrightarrow SU^*(\mathcal{H}; C_1)$.

AIII : if $r = 0$ $S^1 \times SU(\infty) \sim U(\mathcal{H}; C_p)$.

$SU(\infty) \longrightarrow SU(\mathcal{H}; C_1)$.

if $r \in N$ $S^1 \times SU(r) \times S^1 \times SU(\infty) \sim U(\mathcal{H}, r, \infty; C_p)$

$SU(r) \times S^1 \times SU(\infty) \sim SU(\mathcal{H}, r, \infty; C_1)$.

if $r = \infty$ $S^1 \times SU(\infty) \times S^1 \times SU(\infty) \sim U(\mathcal{H}, \infty, \infty; C_p)$.

$S(U(\infty) \times U(\infty)) \sim SU(\mathcal{H}, \infty, \infty; C_1)$.

Type BDI : if $r = 0$ $SO(\infty) \longrightarrow O^+(\mathcal{H}_{\mathbb{R}}; C_p)$.

if $r \in N$ $SO(r) \times SO(\infty) \longrightarrow O^+(\mathcal{H}, r, \infty; C_p)$.

	if $r = \infty$	$SO(\infty) \times SO(\infty) \longrightarrow O^{+}(\mathcal{H},\infty,\infty;\, C_p)$.
BDIII	:	$U(\infty) \longrightarrow SO^{*}(\infty) \longrightarrow O^{*}(\mathcal{H};\, C_p)$.
Type CI	:	$S^1 \times SU(\infty) \sim Sp(\infty, \mathbb{R}) \longrightarrow Sp(\mathcal{H},\mathbb{R};\, C_p)$.
CII	: if $r = 0$	$Sp(\infty) \longrightarrow Sp(\mathcal{H}_{\mathbb{Q}};\, C_p)$.
	if $r \in N$	$Sp(r) \times Sp(\infty) \longrightarrow Sp(\mathcal{H},r,\infty;\, C_p)$.
	if $r = \infty$	$Sp(\infty) \times Sp(\infty) \longrightarrow Sp(\mathcal{H},\infty,\infty;\, C_p)$.

Proof : via propositions II.14.iv) and II.16. ■

* * * * *

The "classical Banach-Lie groups of operators" considered so far are by no means the only examples which appear naturally as sub Banach-Lie groups of the general linear group of a Hilbert (or Banach) space. Indeed, the following remarks show that two standard constructions in the theory of Banach spaces provide rich sources of examples.

Groups tied to extention problems

Let F and E be two Banach spaces over $\mathbb{K}$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$) and let $F \xrightarrow{i} E$ be a dense continuous linear injection. Let $\underline{g}$ be the sub Lie algebra of L(F) consisting of those operators X on F which factor through continuous operators from E to F:

$$\begin{tikzcd} F \arrow[r, "i"] \arrow[d, "X"'] & E \arrow[dl, dashed, "\tilde{X}"] \\ F & \end{tikzcd}$$

Then $\underline{g}$ is a Banach-Lie algebra for the norm $|X| = \| \tilde{X} \|_{L(E,F)}$. (One can suppose the norm of i to be bounded by 1 for the identity page II.2 to hold.) Let G be the group of those invertible operators on F of the form $id_F + X$, where X factors through E. Then G has a

Banach-Lie group structure with Lie algebra $\underline{g}$. If i is, for example, compact, then G is a sub Banach-Lie group of the Fredholm group of F; other restrictions on i are of interest (p-summability, radonifying maps). Note that, in general, the ideal of finite rank operators $C_o(F)$ does <u>not</u> belong to $\underline{g}$, so that the condition ii) of page II.28 is not satisfied. However :

<u>Conjecture</u> : If i is compact, then the group G defined above is homotopically equivalent to $GL(K^\infty)$.

To prove the conjecture, one has probably to modify Geba's argument (see proposition II.16) for the situations where pairs of Banach spaces are involved. Another method of attacking the problem would be to try and apply Cerf's result ([34] chap. III §1) to the inclusion of G into the Fredholm group of F.

Another procedure to manufacture normed Lie algebras is to consider extention problems summarized by the diagram

$$\begin{tikzcd} F \arrow[r, "i"] \arrow[d] & E \arrow[d, dotted] \\ F \arrow[r, "i"] & E \end{tikzcd}$$

<u>Groups tied to lifting problems</u>

Let $E \xrightarrow{j} F$ be a dense continuous linear injection between two real or complex Banach spaces and let $\underline{g}$ be the sub Banach-Lie algebra of L(F) consisting of those operators X on F which factor through continuous operators from F to E :

$$\begin{tikzcd} & F \arrow[d, "X"] \arrow[dl, dashed, "\tilde{\tilde{X}}"'] \\ E \arrow[r, "j"] & F \end{tikzcd}$$

Let G be the group of those invertible operators on F of the form $id_F + X$, where X factors through E. Then G has a Banach-Lie group structure with Lie algebra $\underline{g}$. The above conjecture can be repeated in this case when j is compact.

Another procedure to manufacture normed Lie algebras is again to consider lifting problems summarized by the diagram.

$$\begin{tikzcd} E \arrow[r, "j"] \arrow[d, dashed] & F \arrow[d] \\ E \arrow[r, "j"] & F \end{tikzcd}$$

Mixed Problems

We will give one example only of a construction involving both extension and lifting: Let ($\mathcal{H} \xrightarrow{i} E$) be an Abstract Wiener Space (see Gross [68]). The Wiener group defined in the introduction (Section 1.1) can be thought as resulting of a two-stages construction according to the diagram $(\mathcal{H}^* = \mathcal{H})$

$$\begin{tikzcd} E^* \arrow[r, "i^*"] & \mathcal{H}^* \arrow[r, "i"] \arrow[d, "X"] & E \arrow[dl, dashed, "\tilde{X}"'] \arrow[dll, dashed, "\tilde{\tilde{X}}"'] \\ E^* \arrow[r, "i^*"] & \mathcal{H}^* \arrow[r, "i"] & E \end{tikzcd}$$

One more example

Let $\mathcal{H}$ be an infinite dimensional complex Hilbert space and let $GL(\mathcal{H};\ U(L), Pos(C_2))$ be the subgroup of $GL(\mathcal{H})$ consisting of those operators X such that $[X] = (XX^*)^{\frac{1}{2}}$ lies in $Pos(\mathcal{H};\ C_2)$; this Banach-Lie group, of Lie algebra $\underline{u}(\mathcal{H};\ L) \oplus Sym(\mathcal{H};\ C_2)$, happens to play an important role in the theory of boson fields (Shale [159]).

About the need for ad hoc groups in quantum physics, see also Dieudonné-Bleuler [43].

II.7. - Riemannian geometry on Hilbert-Lie groups

As the previous sections have shown, the groups of operators introduced so far share with the finite dimensional classical Lie groups various properties of an algebraic-topological or differential-topological character. In the cases where these groups are moreover modelled (as manifold) on a Hilbert space, one expect the similarity to carry over to Riemannian-geometrical properties; this is shown to be (at least partially) so in this section. Groups corresponding to L*-algebras play clearly a distinguished role here, which they had no reason to do before.

<u>Definition 7</u>. A <u>Riemann-Hilbert-Lie group</u>, of for short a <u>RHL-group</u>, is a real Banach-Lie group modelled on a Hilbert space and whose Lie algebra is given with a distinguished scalar product. A connected RHL-group will always be considered as furnished with the left invariant Riemannian structure defined by the scalar product on its Lie algebra, and with the corresponding Levi-Civitàconnection.

Let G be a connected RHL-group with Lie algebra $\mathfrak{g}$, and let $\langle\!\langle \mid \rangle\!\rangle$ denote the scalar product on $\mathfrak{g}$. Let $X,Z \in \mathfrak{g}$; the relation

$$\langle\!\langle [X,Y] \mid Z \rangle\!\rangle = \langle\!\langle B(Z,X) \mid Y \rangle\!\rangle \text{ for all } Y \in \mathfrak{g}$$

defines a bilinear continuous map $B : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g}$.

Geodesics through the identity of G are not necessarily one parameter subgroups, except when B is skew-symmetric!

<u>Proposition 17</u>. Let G be a connected RHL-group and let B be as above.

i) Let ξ and η be two left-invariant vector fields on G; the covariant derivative of η along ξ is given by

$$2\,\nabla_{\xi}\eta = [\xi,\eta] - B(\xi,\eta) - B(\eta,\xi)$$

ii) The sectional curvature $R_{\xi\eta}$ of G attached to two vector fields ξ and η depends on B only; when B is skew-symmetric:

$$R_{\xi\eta} = \langle\!\langle [\xi,\eta] \,|\, B(\xi,\eta) - \tfrac{3}{4}[\xi,\eta] \rangle\!\rangle$$

<u>Proof</u> : see Arnold [4], [5], or, in the standard case where $B(X,Y) = [X,Y]$, Milnor [120] part IV. ∎

<u>Examples</u>.

i) Any compact Lie group can be given a bi-invariant Riemannian metric, hence can be a fortiori considered as a RHL-group.

ii). More generally, let G be a Banach-Lie group whose Lie algebra is a real L^*-algebra $\underline{g}$. Then G is clearly a RHL-group and the map B is given by $B(X,Y) = [X,-Y^*]$ for all $X,Y \in \underline{g}$. In particular, if $\underline{g}$ is compact (in the sense of definitions I.8 and II.6; $\underline{g}$ might be infinite dimensional), $B(X,Y) = [X,Y]$ for all $X,Y \in \underline{g}$ and the (always positive) sectional curvature is given by the familiar formula $R_{\xi\eta} = \frac{1}{4}\| [\xi,\eta] \|^2$. According to the classification of the separable L^*-algebras, the separable infinite dimensional examples are essentially:

$$U(\mathcal{H};\ C_2) \qquad O^+(\mathcal{H}_{\mathbb{R}};\ C_2) \qquad Sp(\mathcal{H}_{\mathbb{Q}};\ C_2)$$

iii) Let $\underline{g}$ be the Lie algebra of all Hilbert-Schmidt operators on a separable infinite dimensional complex Hilbert space $\mathcal{H}$. The standard norm on $\underline{g}$ is given by the scalar product

$$\langle\!\langle\,|\,\rangle\!\rangle \begin{cases} \underline{g} \times \underline{g} \longrightarrow \mathbb{C} \\ (X,Y) \longmapsto \operatorname{trace}(XY^*) \end{cases} ;$$

furnished with it, $\underline{g}$ was denoted by $\underline{gl}(\mathcal{H};\ C_2)$ from section II.5 onwards; the choice of $\langle\!\langle\,|\,\rangle\!\rangle$ as scalar product on $\underline{g}$ is natural, as proposition I.8 shows.

However, the Lie algebra $\underline{g}$ can be given other scalar products. One example is as follows : let T be a positive norm-increasing bounded operator on $\mathcal{H}$ and define

$\langle\langle \mid \rangle\rangle_T \ : \begin{cases} \underline{g} \times \underline{g} \longrightarrow \mathbb{C} \\ (X,Y) \longmapsto \mathrm{trace}\,(TXY^*) \end{cases}$. In this way, each operator such as T defines a RHL-structure on the group $GL(\mathcal{H}; C_2)$. An intrinsic study of such structures can be carried along, similar to those of Smiley [165] and Saworotnow [149], [150], who consider non-standard structures on the H*-algebra $C_2(\mathcal{H})$.

II.8.- Remarks, projects and questions

8.1.- There are three separable infinite dimensional compact simple real L*-algebras : $\underline{u}(\mathcal{H}; C_2)$, $\underline{o}(\mathcal{H}_{\mathbb{R}}; C_2)$ and $\underline{sp}(\mathcal{H}_{\mathbb{Q}}; C_2)$. From general principles about universal coverings (Lazard [107], 16.5 and 20.7), each of them corresponds to a unique (up to isomorphism) connected and simply connected Hilbert-Lie group. The group $Sp(\mathcal{H}_{\mathbb{Q}}; C_2)$ is simply connected (section II.6). Though we have not yet written it up in all details, the explicit construction of the universal covering of $O^+(\mathcal{H}_{\mathbb{R}}; C_2)$ could be worked out (see number 8.2 below). But what is the group defined by $\underline{u}(\mathcal{H}; C_2)$, namely the (infinite) universal covering of $U(\mathcal{H}; C_2)$???

8.2.- The universal covering of $O^+(\mathcal{H}_{\mathbb{R}}; C_2)$ will be denoted by $Spin(\mathcal{H}_{\mathbb{R}}; C_2)$. Its construction in finite dimensions is standard (for example : Atiyah-Bott-Shapiro [6] and Karoubi [98]) and can be extended to infinite dimensions by using results from the theory of infinite dimensional Clifford algebras and of the canonical anticommutation relations (Bourbaki [24], Shale-Stinespring [160], [161], Guichardet [69], Slawny [163], [164]). Indeed, the abstract group underlying the Banach-Lie group $Spin(\mathcal{H}_{\mathbb{R}}; C_2)$ is directly related to the group of those canonical transformations which are implementable in all Fock representations of the CAR over the complexification of $\mathcal{H}_{\mathbb{R}}$ (terminology in this last phrase as in [163]). The interested reader should however be warned that our note [78] was relying on a paper [21] which contains an incorrect step in the proof of its lemma 6.

From the abstract definition of Spin($\mathcal{H}_{\mathbb{R}}$; C_2) as a covering group, and using standard techniques of algebraic geometry ([88], 3.1 and 2.10.1), one can easily prove results as in Haefliger [71] and Milnor [121], [122] about the existence of <u>spin-structures</u> on ad hoc Hilbert manifolds.

From the concrete construction of Spin($\mathcal{H}_{\mathbb{R}}$; C_2), it will be possible to define an infinite dimensional analogue of the <u>Dirac operator</u>. Motivations for the interest in infinite dimensional elliptic operators can be found in Dalec'kii [40] and Vishik [197].

8.3.- It would be highly interesting to be able and construct on some Banach-Lie groups <u>measures</u> which would be tied in some sense to the group structure. Consider for example the space $L^2(I \times I)$ of square integrable real valued functions on the product of two unit intervals. This space is isomorphic to $C_2(L^2(I))$, as indicated for example in Ambrose [2] section 1 example 2; hence it has the structure of a L*-algebra to which corresponds a group of the type GL($\mathcal{H}_{\mathbb{R}}$; C_2).

<u>Problem</u> : find a subalgebra $\underline{s}$ of $L^2(I \times I)$ such that the inclusion $\underline{s} \longrightarrow L^2(I \times I)$ be an abstract Wiener space, and translate the Gauss measure so defined on $L^2(I \times I)$ onto the group level. (For abstract Wiener spaces and measures on infinite dimensional manifolds, see Gross [68], Eells-Elworthy [57], Eells [59].)

8.4.- From the explicit knowledge of the root structure of the classical complex Banach-Lie algebra of compact operators, it is elementary to deduce an Iwasawa decomposition for each of the classical real Banach-Lie algebras of compact operators. This can also be done in full generality for separable semi-simple L*-algebras (as in Helgason [84] chap. VI, th. 3.4). A natural further step is to obtain the corresponding global statement. The standard proof ([84], chap. VI section 5) depends on finite dimensional arguments. We conjecture however that such <u>global Iwasawa deompositions</u> exist for the classical real Banach-Lie groups of compact operators; the proof of this conjecture is likely to be through a case-by-case checking.

8.5.- One of the outputs of carrying out the programme sketched in 8.4 would be to furnish several examples of Hilbert-Lie groups which would be <u>nilpotent</u> and <u>solvable</u>. More generally, some theory of nilpotent Banach-Lie algebras is yet to be done; it will hopefully extend the theory or arbitrary finite dimensional real and complex Lie algebras in the same sense that L*-algebras generalise reductive finite dimensional real and complex Lie algebras. We feel that the first thing to do in this direction is a detailed study of groups of quasi-nilpotent operators in Hilbert space. The following example is a very first step.

<u>Example</u>. Let $\mathcal{H}$ be a separable Hilbert space over $\mathbb{R}$ or $\mathbb{C}$. Let $e = (e_n)_{n \in N}$ be a fixed orthonormal basis of $\mathcal{H}$. Let $\underline{t} = \underline{t}(\mathcal{H}, e; C_2)$ be the Lie algebra of Hilbert-Schmidt operators on $\mathcal{H}$ whose matrix representation with respect to e is strictly upper triangular. The intersection of all the terms in the lower central series of $\underline{t}$ is reduced to zero, hence the "nilpotency" of $\underline{t}$. Let $T = T(\mathcal{H}, e; C_2)$ be the group of operators on $\mathcal{H}$ of the form $id_{\mathcal{H}} + X$, with X strictly upper triangular with respect to e, and X Hilbert-Schmidt. Clearly, T is a sub Banach-Lie group of $GL(\mathcal{H}; C_2)$, and the exponential map $\underline{t} \longrightarrow T$ is given by the traditional power series.

<u>Lemma</u> : Let X be a Hilbert-Schmidt quasi-nilpotent operator on $\mathcal{H}$; then the estimate $\| X^n \|_2 \leq \left((n-1)!\right)^{-\frac{1}{2}} \| X \|_2{}^n$ holds for all $n \in \mathbb{N}^*$.

The proof of this lemma, non trivial, is due to Ringrose ([142], th. 5). It follows easily that the series defining $\log(\mathrm{id}_{\mathcal{H}} + X)$ converges in the $C_2(\mathcal{H})$-norm for all $X \in \underline{t}$. Hence the

Proposition : The exponential map $\underline{t} \to T$ is a diffeomorphism. (Recall : if G is a connected, simply connected, finite dimensional nilpotent Lie group, then the exponential map is always a diffeomorphism from the Lie algebra of G onto G.)

Corollary : the exponential map induces on T a structure of Fredholm manifold (see Elworthy-Tromba [62], prop. 1.2).

A starting point for the study of more general "nilpotent" Lie algebras of operators on $\mathcal{H}$ might be found in Schue [155]. See as well Vasilescu [180], Limic [112].

8.6.- Let $\mathcal{A}$ be an associative involutive complex Banach algebra. The answers to the following questions seem to depend strongly on each other : Is the norm topology of $\mathcal{A}$ unique? Are the derivations and the (*-)isomorphisms of $\mathcal{A}$ all continuous? spatial? inner? See Loy [113] and Kadison [96] for an introduction to these problems. Similar questions can be asked for semi-simple (in some sense) Banach-Lie algebras; propositions II.2, II.3, II.9 and II.12 suggest a large proportion of positive answers; idem for Banach-Jordan algebras (see for example [17]).

We recall for memory the projects and conjectures stated pages I.24, II.25 and II.31.

Projects 8.1 to 8.3 are, in my view, the most interesting by far.

Appendix : about semi-simplicity of infinite dimensional Lie algebras

The temporary definition adopted page I.10 was : a Lie algebra is semi-simple if it has no non-trivial abelian ideal. More generally, if $\underline{g}$ is any real or complex Lie algebra, one can define a first notion of a radical as follows (and as in Vasilescu [180]): An ideal $\underline{a}$ in $\underline{g}$ is said to be primitive if any ideal $\underline{b}$ of $\underline{g}$ such that $[\underline{b},\underline{b}] \subset \underline{a}$ is moreover contained in $\underline{a}$, namely such that $\underline{b} \subset \underline{a}$; the radical $R_{\underline{g}}$ of $\underline{g}$ is then the intersection of all primitive ideals of $\underline{g}$. Our temporary definition can then be rephrased as follows: a Lie algebra is semi-simple if its radical $R_{\underline{g}}$ is reduced to zero.

However, this notion is in general not satisfactory. Indeed, there exists an infinite dimensional locally finite locally nilpotent real Lie algebra which is semi-simple according to the definition above:

Example (I. Stewart). Levich [110] gives a locally nilpotent torsion-free group with no subnormal abelian subgroup. By Mal'cev correspondence (e.g. Stewart [170]), it implies that there exists a locally nilpotent Lie algebra over the rationals without abelian subideals; hence ditto over $\mathbb{R}$ or $\mathbb{C}$. The existence of even "worse" examples follows from Levich-Tokarenko [111] and Simonjan [162].

Questions : Is it possible to single out a class of Lie algebras for which the definition above is satisfactory? In particular, is the class of c-involutive Lie algebras (definition I.2) convenient for this purpose?

There are naturally many other ways to define the radical of a Lie algebra, hence to define semi-simplicity: Baer radical, Hirsch-Plotkin radical, Fitting radical; see Stewart [169] part two.

Simple Lie algebras, which were the main concern of Chapter I, are semi-simple with respect to any reasonable definition of semi-simplicity.

We have (temporarily) defined the semi-simplicity of a normed Lie algebra in the same way as in Chapter I (page II.3).

Questions : Is that definition satisfactory for (normed or) Banach-Lie algebras in general ? for c-involutive Banach-Lie algebras ? (it is for L*-algebras).

This question is clearly tied to the project 8.5 (page II.37).

Even if the answer to the later question was affirmative, it would be convenient to have a definition of semi-simplicity in terms of representations. It is indeed possible in finite dimensions (Bourbaki [25] §6, th. 2 and remark 1, and ex. 20) and standard for associative Banach algebras (Rickart [141]).

Appendix : review of norm ideals (Schatten)

Let $\mathcal{H}$ be a separable infinite dimensional complex Hilbert space. If X is an operator on $\mathcal{H}$, [X] denotes its absolute value, that is [X] is the positive operator on $\mathcal{H}$ such $[X]^2 = XX^*$.

Let $\mathcal{I}$ be a non-zero two-sided (associative) ideal in $L(\mathcal{H})$ and let $\alpha : \mathcal{I} \longrightarrow \mathbb{R}$ be a norm. Then α is said to be a crossnorm if $\alpha(X) = \| X \|$ for all operators X of rank one; it is unitarily invariant if $\alpha(VXW) = \alpha(X)$ for all $X \in \mathcal{I}$ and for all $V, W \in U(\mathcal{H})$; it is uniform if $\alpha(YXZ) \leq \| Y \| \alpha(X) \| Z \|$ for all $X \in \mathcal{I}$ and for all $Y, Z \in L(\mathcal{H})$. A norm ideal in $L(\mathcal{H})$ is a non-zero two-sided ideal in $L(\mathcal{H})$ together with a uniform crossnorm with respect to which the ideal becomes a Banach space. A minimal norm ideal in $L(\mathcal{H})$ is a norm ideal in $L(\mathcal{H})$ such that none of its non-trivial closed subspace is again a norm ideal.

A crossnorm on $C_o(\mathcal{H})$ is unitarily invariant if and only if it is uniform. If α is a uniform crossnorm, then trace $([X]) \geq \alpha(X) \geq \| X \|$ for all $X \in C_o(\mathcal{H})$. If α is a uniform crossnorm on $C_o(\mathcal{H})$, then the map

$$\alpha' : \begin{cases} C_o(\mathcal{H}) \longrightarrow \mathbb{R} \\ X \longrightarrow \sup\left(\dfrac{|\mathrm{trace}(YX)|}{\alpha(Y)}\right) \end{cases}$$

is again a uniform crossnorm said to be associated with α (sup is taken over all non-zero Y's in $C_o(\mathcal{H})$).

Proposition 18. Let α be a uniform crossnorm on $C_o(\mathcal{H})$. Let $L_\alpha(\mathcal{H}) = \{X \in L(\mathcal{H}) \mid \sup\left(\dfrac{|\mathrm{trace}(YX)|}{\alpha'(Y)}\right) < \infty \}$ and let $C_\alpha(\mathcal{H}) = L_\alpha(\mathcal{H}) \cap C(\mathcal{H})$. Then :

i) $C_o(\mathcal{H}) \subset C_\alpha(\mathcal{H}) \subset L_\alpha(\mathcal{H})$ and $C_\alpha(\mathcal{H})$ and $L_\alpha(\mathcal{H})$ are ideals in $L(\mathcal{H})$. If $X \in C_o(\mathcal{H})$, then $\alpha(X) = \sup\left(\dfrac{|\mathrm{trace}(YX)|}{\alpha'(Y)}\right)$.

ii) The evident norm on $L_\alpha(\mathcal{H})$, which is still denoted by α, makes it a norm ideal in $L(\mathcal{H})$ and an involutive Banach algebra.

iii) The restriction of α to $C_\alpha(\mathcal{H})$, which is still denoted by α, makes it a minimal norm ideal and an involutive Banach algebra; conversely, any minimal norm ideal in $L(\mathcal{H})$ is of this type.

iv) Any norm on $L_\alpha(\mathcal{H})$ [resp. on $C_\alpha(\mathcal{H})$] which makes it a Banach algebra is equivalent to α. $L_\alpha(\mathcal{H})$ [resp. $C_\alpha(\mathcal{H})$] is the maximal [resp. minimal] norm ideal in $L(\mathcal{H})$ with respect to α.

v) Let $X \in C_\alpha(\mathcal{H})$ and $Y \in L_{\alpha'}(\mathcal{H})$; then XY and YX are trace-class and trace(XY) = trace (YX). The map

$$\begin{cases} \left(C_o(\mathcal{H}) \text{ furnished with } \alpha\right) \times L_{\alpha'}(\mathcal{H}) \longrightarrow \mathbb{C} \\ (X,Y) \longmapsto \mathrm{trace}(XY) \end{cases}$$

defines an isometric isomorphism of Banach spaces between $L_{\alpha'}(\mathcal{H})$ and the strong dual of the normed space $\left(C_o(\mathcal{H}) \text{ furnished with } \alpha\right)$. Similarly, the strong dual of $C_\alpha(\mathcal{H})$ can be identified with $L_{\alpha'}(\mathcal{H})$.

vi) The following are equivalent :

- $C_\alpha(\mathcal{H})$ is reflexive
- $C_{\alpha'}(\mathcal{H})$ is reflexive
- $L_\alpha(\mathcal{H})$ and $L_{\alpha'}(\mathcal{H})$ are both minimal.

vii) The Hilbert-Schmidt crossnorm is the only uniform crossnorm which is the same as its associate.

Proofs : see among many other possible places Schatten [151], or Dunford-Schwartz ([50] section XI.9), or Gohberg-Krein ([65]chap. III).∎

Example. Let $p \in \bar{\mathbb{R}}$, $1 \leqslant p \leqslant \infty$; the function defined by

$$\| \ \|_p \begin{cases} C_o(\mathcal{H}) \longrightarrow \mathbb{R} \\ X \longmapsto (\mathrm{trace}[X]^p)^{1/p} \end{cases}$$ when $p < \infty$ and by

$\| \ \|_\infty = \| \ \|$ when $p = \infty$ is a uniform crossnorm on $C_o(\mathcal{H})$. The minimal norm ideal defined by $\| \ \|_p$ will be denoted by $C_p(\mathcal{H})$. The associated crossnorm is given by $\| \ \|_q$ where $\frac{1}{p} + \frac{1}{q} = 1$.

If $1 < p < \infty$, $C_p(\mathcal{H})$ is reflexive;

if $p = 1$, $C_1(\mathcal{H}) = L_{\| \ \|_1}(\mathcal{H})$ is the ideal of <u>nuclear</u> operators;

if $p = 2$, $q = 2$ and $C_2(\mathcal{H}) = L_{\| \ \|_2}(\mathcal{H})$ is the ideal of <u>Hilbert-Schmidt</u> operators;

if $p = \infty$, $C_\infty(\mathcal{H}) = C(\mathcal{H})$ is the ideal of <u>compact</u> operators and $L_{\| \ \|_\infty}(\mathcal{H})$ is the trivial ideal of all operators in $L(\mathcal{H})$.

Proposition 19.

i) Let $p, p' \in \bar{\mathbb{R}}$ with $1 \leqslant p \leqslant p' \leqslant \infty$; then $C_p(\mathcal{H}) \subset C_{p'}(\mathcal{H})$ and the canonical injection $C_p(\mathcal{H}) \longrightarrow C_{p'}(\mathcal{H})$ is continuous.

ii) Let $p \in \bar{\mathbb{R}}$ with $2 \leqslant p \leqslant \infty$, let $(e_n)_{n \in N}$ be an orthonormal basis of $\mathcal{H}$ and let X be a bounded operator on $\mathcal{H}$; then $X \in C_p(\mathcal{H})$ if and only if $\sum |Xe_n|^p < \infty$ (the inequality implies that $X \in C_p(\mathcal{H})$ for $1 \leqslant p < 2$).

iii) On $C_1(\mathcal{H})$, the function which associates to an operator X the sum of its eigenvalues (repeated according to multiplicity) is the only continuous linear functional which coincides with the trace on $C_0(\mathcal{H})$.
iv) Let $p \in \bar{\mathbb{R}}$ with $1 < p$; then the space $\underline{sl}(\mathcal{H};\ C_0)$ of finite rank operators with zero trace is dense in $C_p(\mathcal{H})$.

Proof. The proofs of i) to iii) are standard [151]. Here is the sketch of a proof for iv).

As any operator in $C_p(\mathcal{H})$ is a linear combination of four positive (hence diagonizable) operators in $C_p(\mathcal{H})$, statement iv) clearly follows from the following lemma. Let $\ell^p_{\mathbb{R}}(N^*)$ be the Banach space of p-summable sequences of real numbers, and let $\mathcal{Z}$ be the subspace consisting of those sequences with finitely many non zero terms only and and whose sum do vanish; then $\mathcal{Z}$ is dense in $\ell^p_{\mathbb{R}}(N^*)$ for all $p \in \bar{\mathbb{R}}$ with $1 < p \leqslant \infty$. The lemma itself is a straightforward corollary of Hardy's inequality (see for example Rudin [145] chap. 3, exercise 15, with a hint), and can be found in full details elsewhere ([76], lemma 3.3). ∎

Remark. In sections II.5 and II.6, we deal with Banach-Lie algebras corresponding to the particular ideal $C_p(\mathcal{H})$ instead of those corresponding to arbitrary minimal norm ideals $C_\alpha(\mathcal{H})$. This restriction is made only in order to keep the notations reasonably simple. In fact, the case $C_1(\mathcal{H})$ is characteristic of all the cases for which the closure of $\underline{sl}(\mathcal{H};\ C_0)$ is one codimensional in $C_\alpha(\mathcal{H})$, and the cases $C_p(\mathcal{H})$ for $1 < p \leqslant \infty$ are characteristic of all the cases for which $\underline{sl}(\mathcal{H};\ C_0)$ is dense in $C_\alpha(\mathcal{H})$.