CHAPTER IV.

ON THE COHOMOLOGY OF THE CLASSICAL COMPLEX LIE ALGEBRAS OF COMPACT OPERATORS

Let U be a (finite dimensional, simple, compact) classical group, let G be its complexification, let $\underline{u}$ be the Lie algebra of U and let $\underline{g}$ be that of G. The following facts are well-known (Cartan [31], Chevalley-Eilenberg [38], Koszul [100]) :

The real (or de Rham) cohomology $H^*(U)$ of the group U, the cohomology $H^*(\underline{u})$ of its Lie algebra and the algebra $J^*(\underline{u})$ of invariant cocycles on $\underline{u}$ are isomorphic to each other. Moreover, $J^*(\underline{u})$ is an exterior algebra over the space $P^*(\underline{u})$ of primitive cocycles, which is of dimension ℓ (ℓ = rank of U). The same is true for the complex algebras $H^*(G)$, $H^*(\underline{g})$ and $J^*(\underline{g})$; indeed, the inclusion of U in G is a homotopy equivalence, and the relations $H^*(\underline{g}) = H^*(\underline{u}) \otimes_{\mathbb{R}} \mathbb{C}$ and $J^*(\underline{g}) = J^*(\underline{u}) \otimes_{\mathbb{R}} \mathbb{C}$ are true by general principles.

Let now $I(U)$ be the Z-graded algebra of those polynomial functions on $\underline{u}$ with real values which are invariant by the adjoint action of U on $\underline{u}$. Let $H^*(B_U)$ be the real cohomology of the classifying space B_U for U-principal bundles. Then $I(U)$ and $H^*(B_U)$ are isomorphic to each other. Moreover, $I(U)$ is a polynomial algebra over ℓ generators.

Finally, there exists a canonical linear map T (denoted by ρ in Cartan [31]) from $I(U)$ to $J^*(\underline{u})$ which transforms generators of $I(U)$ into generators of $J^*(\underline{u})$. This map plays a crucial role in the study of transgressions.

The purpose of this chapter is to investigate the analogue propositions when $\underline{g}$ is replaced by a classical complex Lie algebra of compact operators. Indications have been sketched in [82], [83]. In the absence of any general theorem, our only method is to perform explicit computations.

IV.1. - The algebra $J_c^*(\underline{g})$ of invariant cochains

This section is an easy consequence of the (classical) computations written up in [81].

Let $\underline{g}$ be a (possibly infinite dimensional) Lie algebra over $\mathbb{K}$ ($\mathbb{K}$ is $\mathbb{R}$ or $\mathbb{C}$). If $k \in \mathbb{N}^*$, a <u>k-cochain</u> is a multilinear alternating map $\underline{g} \times \dots \times \underline{g}$ (k times) $\longrightarrow \mathbb{K}$. The vector space of all k-cochains on $\underline{g}$ will be denoted by $C^k(\underline{g})$ and the algebra of all cochains on $\underline{g}$ by $C^*(\underline{g}) = \bigoplus_{k\in N} C^k(\underline{g})$; according to the usual convention, $C^0(\underline{g}) = \mathbb{K}$. The standard explicit formulas for the Lie derivative θ^* and for the "exterior" differentiation d carry over to the infinite dimensional case without change, so that we will not define every notion in detail; see for example Koszul [100]. We will denote by $Z^k(\underline{g})$ [resp. $B^k(\underline{g})$, $H^k(\underline{g})$, $J^k(\underline{g})$] the space of the k-cocycles [resp. k-boundaries, k-cohomology classes, k-invariant cocycles] on $\underline{g}$.

Let now $\underline{g}$ be a Banach-Lie algebra over $\mathbb{K}$. The space of <u>continuous k-cochains</u> on $\underline{g}$ will be denoted by $C_c^k(\underline{g})$; similarly for $Z_c^k(\underline{g})$, $B_c^k(\underline{g})$, $H_c^k(\underline{g})$ and $J_c^k(\underline{g})$. We now proceed to describe $J_c^*(\underline{u})$ when $\underline{u}$ is the compact form of a classical complex Lie algebra $\underline{g}$ of compact operators on the <u>infinite dimensional separable complex</u> Hilbert space $\mathcal{H}$.

Consider first the associative algebra $C_0(\mathcal{H})$. For each $k \in \mathbb{N}^*$, define the k-linear alternating map

$$q_k : \begin{cases} C_0(\mathcal{H}) \times \dots \times C_0(\mathcal{H}) \longrightarrow \mathbb{C} \\ (X_1, \dots, X_k) \longmapsto \operatorname{trace}\left(\sum_\sigma sg(\sigma)\, X_{\sigma(1)} \cdots X_{\sigma(k)}\right) \end{cases}$$

where the sum is taken over all permutations of the symmetric group in k variables $\mathfrak{S}_k$.

It is easy to check that q_k vanishes whenever k is even.

Let then $\underline{u}_o$ denote for short the Lie algebra of finite rank operators $\underline{u}(\mathcal{H}; C_o)$ and define for each $k \in N^*$

$$\hat{\rho}_k \qquad \begin{cases} \underline{u}_o \times \dots \times \underline{u}_o \longrightarrow \mathbb{R} \\ (X_1, \dots, X_{2k-1}) \qquad i^{k(2k-1)} q_{2k-1}(X_1, \dots, X_{2k-1}). \end{cases}$$

Proposition 1A. The cochains $(\hat{\rho}_k)_{k \in N^*}$ are primitive cocycles on $\underline{u}_o$. The algebra $J^*(\underline{u}_o)$ is an exterior algebra generated by $(\hat{\rho}_k)_{k \in N^*}$; moreover, these primitive generators are unique up to multiplication by non-zero real numbers.

Proof.

Step one : Let F be an n-dimensional subspace of $\mathcal{H}$ and let $\underline{u}(F)$ be the subalgebra of $\underline{u}_o$ consisting of those operators which map F into itself and its orthogonal complement onto zero. Let $\hat{\rho}_k^F$ be the restriction of $\hat{\rho}_k$ to $\underline{u}(F)$ for all $k \in N^*$. Then $(\hat{\rho}_1^F, \dots, \hat{\rho}_n^F)$ are primitive cocycles on $\underline{u}(F)$, they generate $J^*(\underline{u}(F))$, and they are the unique primitive generators of $J^*(\underline{u}(F))$ up to multiplication by a non-zero real numbers. The proof of this step is classical; the explicit form of the $\hat{\rho}_k^F$'s is apparently due to Dynkin [52]; references and pedestrian computations have been collected in [81].

Step two. Let γ be an invariant ℓ-cochain on $\underline{u}_o$. According to the results recalled under step one, for any subspace F of $\mathcal{H}$ as above, there are constants (a priority depending on F) such that the restriction of γ to $\underline{u}(F)$ can be written as

$$\gamma^F = \sum_{1 \leq k \leq \ell} c_F^k \hat{\rho}_k^F + \sum_{1 \leq k_1 < k_2 \leq \ell} c_F^{k_1, k_2} \hat{\rho}_{k_1}^F \wedge \hat{\rho}_{k_2}^F + \dots$$

(finite sum).

But if $F_1 \subset F_2$, γ^{F_1} is the restriction of γ^{F_2} to $\underline{u}(F_1)$. It follows clearly from the finite dimensional case that the constants c's do not depend on F for $\dim F$ large enough. Hence $J^*(\underline{u}_o)$ is an exterior algebra generated by the $\hat{\rho}_k$'s .

Step three. The primitivity and the unicity of the generators $\hat{\rho}_k$'s can be proved similarly, using finite dimensional subalgebras of $\underline{u}_o$. ∎

Let now $p \in \bar{\mathbb{R}}$ with $1 \leqslant p \leqslant \infty$, and let $\underline{u}_p$ denote for short the Lie algebra of compact operators $\underline{u}(\mathcal{H}\,;\,C_p)$. For each $k \in \mathbb{N}^*$ with $2k-1 \geqslant p$, $\hat{\rho}_k$ extends uniquely to a continuous $(2k-1)$-cochain on $\underline{u}_p$ which will be denoted by the same letter.

Corollary. The cochains $(\hat{\rho}_k)_{\substack{k \in \mathbb{N}^* \\ 2k-1 \geqslant p}}$ are primitive cocycles on $\underline{u}_p$. The algebra $J_c^*(\underline{u}_p)$ is an exterior algebra generated by $(\hat{\rho}_k)_{k \in \mathbb{N}^*,\ 2k-1 \geqslant p}$; moreover, these primitive generators are unique up to multiplication by non-zero real numbers. In particular $J_c^*(u_\infty) = J^o(u_\infty) = \mathbb{R}$.

Proof : clear, as $\underline{u}_o$ is dense in $\underline{u}_p$ and as those $\hat{\rho}_k$'s which are defined must be continuous. ∎

Remarks.

i) Proposition 1A still holds for the algebra $\underline{su}(\mathcal{H}\,;\,C_o)$ and the set of generators $(\hat{\rho}_k)_{k \in \mathbb{N}^*, k \geqslant 2}$; the corollary still holds for $\underline{su}(\mathcal{H}\,;\,C_1)$ in the same way.

ii) The same description as above is also valid for the complex Lie algebras $\underline{gl}(\mathcal{H}\,;\,C_p)$, $p \in \bar{\mathbb{R}}$ with $p = 0$ or $1 \leqslant p \leqslant \infty$, and $\underline{sl}(\mathcal{H}\,;\,C_p)$, $p = 0$ or $p = 1$.

Let now $\mathcal{H}_{\mathbb{R}}$ be a real Hilbert space, which can be considered as the set of fixed points of a conjugation in $\mathcal{H}$. Let $\underline{so}_o$ be the Lie algebra of finite rank operators $\underline{o}(\mathcal{H}_{\mathbb{R}}\,;\,C_o)$. It is easy to check that the restriction of q_k to $\underline{so}_o$ vanishes when $k \equiv 1 \pmod 4$. For each $k \in N^*$, let $\hat{\sigma}_{2k}$ be the restriction of q_{4k-1} to $\underline{so}_o$.

<u>Proposition 1B</u>. The cochains $(\hat{\sigma}_{2k})_{k \in N^*}$ are primitive cocycles on $\underline{so}_o$. The algebra $J^*(\underline{so}_o)$ is an exterior algebra generated by $(\hat{\sigma}_{2k})_{k \in N^*}$; moreover, these primitive generators are unique up to multiplication by non-zero real numbers.

<u>Proof</u> : as for proposition 1A. ∎

Let $p \in \bar{\mathbb{R}}$ with $1 \leqslant p \leqslant \infty$, and let $\underline{o}_p$ denote for short the Lie algebra of compact operators $\underline{o}(\mathcal{H}_{\mathbb{R}}; C_p)$ ($\underline{o}_p$ is also denoted by $\underline{so}_1$ when $p = 1$). For each $k \in N^*$ with $4k-1 \geqslant p$, $\hat{\sigma}_{2k}$ extends uniquely to a continuous $(4k-1)$-cochain on $\underline{o}_p$ which will be denoted by the same letter.

<u>Corollary</u>. Proposition 1B holds if $\underline{so}_o$ is replaced by $\underline{o}_p$, $J^*(\underline{so}_o)$ by $J^*_c(\underline{o}_p)$, and $(\hat{\sigma}_{2k})_{k \in N^*}$ by $(\hat{\sigma}_{2k})_{k \in N^*,\ 4k-1 \geqslant p}$. In particular $J^*_c(\underline{o}_\infty) = \mathbb{R}$.

Let now $\mathcal{H}_{\mathbb{Q}}$ be a quaternionic Hilbert space, which can be considered as $\mathcal{H}$ furnished with an anticonjugation. Let $\underline{sp}_o$ be the Lie algebra of finite rank operators $\underline{sp}(\mathcal{H}_{\mathbb{Q}}; C_o)$. It is again easy to check that the restriction of q_k to $\underline{sp}_o$ vanishes when $k \equiv 1 \pmod 4$. For each $k \in N^*$, let $\hat{\tau}_{2k}$ be the restriction of $2q_{4k-1}$ to $\underline{sp}_o$.

<u>Proposition 1C</u>. The cochains $(\hat{\tau}_{2k})_{k \in N^*}$ are primitive cocycles on $\underline{sp}_o$. The algebra $J^*(\underline{sp}_o)$ is an exterior algebra generated by $(\hat{\tau}_{2k})_{k \in N^*}$; moreover, these primitive generators are unique up to multiplication by non-zero real numbers.

Proof : as for proposition 1A. ∎

Let $p \in \bar{\mathbb{R}}$ with $1 \leqslant p \leqslant \infty$, and let $\underline{sp}_p$ denote for short the Lie algebra of compact operators $\underline{sp}(\mathcal{H}_{\mathbb{Q}};\ C_p)$. For each $k \in \mathbb{N}^*$ with $4k-1 \geqslant p$, $\hat{\tau}_{2k}$ extends uniquely to a continuous $(4k-1)$-cochain on $\underline{sp}_p$ which will be denoted by the same letter.

Corollary. Proposition 1C holds if $\underline{sp}_o$ is replaced by $\underline{sp}_p$, $J^*(\underline{sp}_o)$ by $J^*_c(\underline{sp}_p)$, and $(\hat{\tau}_{2k})_{k \in \mathbb{N}^*}$ by $(\hat{\tau}_{2k})_{k \in \mathbb{N}^*,\ 4k-1 \geqslant p}$. In particular $J^*_c(\underline{sp}_\infty) = \mathbb{R}$.

If $\mathbb{K}$ is now one of $\mathbb{R}$, $\mathbb{C}$, $\mathbb{Q}$, let $\mathrm{Hilb}^+(\mathcal{H}_{\mathbb{K}};\ C_p)$ denote the connected component of the group defined section III.1, and $\underline{\mathrm{hilb}}(\mathcal{H}_{\mathbb{K}};\ C_p)$ its Lie algebra.

Proposition 2. The real cohomology algebra of the Banach-Lie group $\mathrm{Hilb}^+(\mathcal{H}_{\mathbb{K}};\ C_p)$ is isomorphic to the algebra of continuous invariant cochains on the Banach-Lie algebra $\underline{\mathrm{hilb}}(\mathcal{H}_{\mathbb{K}};\ C_p)$ if and only if $p = 1$.

Proof : Proposition 2 follows by comparison between the corollary to proposition II.16 and the well-known results about the cohomology rings of the finite dimensional classical Lie groups on the one hand, and proposition IV.1 on the other hand. ∎

Remark. In the Riemannian case $\mathrm{Hilb}^+(\mathcal{H}_{\mathbb{K}};\ C_2)$, the explicit form of the generators of $J^*_c(\underline{\mathrm{hilb}}(\mathcal{H}_{\mathbb{K}};\ C_2))$ provides the harmonic differential forms explicitly on the group; a tiny part of Hodge's theory can be recuperated in this way. The unitary case $U(\mathcal{H}_{\mathbb{C}};\ C_2)$ is remarkable in so far as there is no harmonic form to generate the first cohomology group.

IV.2. – The cohomology algebra $H_c^*(\underline{g})$

Let $\underline{g}$ be a Banach-Lie algebra. From the usual conventions, it follows that $H_c^0(\underline{g})$ is identified with the base field. It is an immediate consequence of the definitions that $H_c^1(\underline{g})$ is isomorphic to the topological dual of the Banach space $\underline{g}/\overline{[\underline{g},\underline{g}]}$. Hence, as long as scalar-valued cohomology is concerned, real problems start with $H_c^2(\underline{g})$. In this section, $H_c^2(\underline{g})$ is computed when $\underline{g}$ is an infinite dimensional classical complex Lie algebra of compact operators.

Let $\mathcal{H}$ be an infinite dimensional separable complex Hilbert space. Let $p \in \bar{\mathbb{R}}$ with $1 \leqslant p \leqslant \infty$, and let $\underline{a}$ be the Banach-Lie algebra $\underline{gl}(\mathcal{H} ; C_p)$. Let $q \in \bar{\mathbb{R}}$ be defined by $\frac{1}{p} + \frac{1}{q} = 1$, and let $\underline{b}$ be the Banach-Lie algebra $\underline{gl}(\mathcal{H} ; C_p)$ if $q < \infty$, and the Banach-Lie algebra $\underline{gl}(\mathcal{H} ; L)$ if $q = \infty$. Consider as in section II.5 (the last remark) the duality

$$\langle\langle\,|\,\rangle\rangle \quad : \quad \begin{cases} \underline{a} \times \underline{b} \longrightarrow \mathbb{C} \\ (X,Y) \longmapsto \mathrm{trace}(XY^*) \end{cases} .$$

Let now ω be a 2-cochain on $\underline{a}$. According to the duality, ω defines a unique continuous linear operator $\Delta : \underline{a} \longrightarrow \underline{b}$ such that $\omega(X,Y) = \langle\langle X|\Delta(Y^*)\rangle\rangle$ for all $X,Y \in \underline{a}$; naturally, as ω is skew-symmetric, $\langle\langle X|\Delta(Y^*)\rangle\rangle = - \langle\langle Y|\Delta(X^*\rangle\rangle$ for all $X,Y \in \underline{a}$. It follows directly from the definitions that ω is a cocycle [resp.a coboundary] if and only if Δ is a derivation [resp. an inner derivation]. Write $\mathrm{Der}(\underline{a},\underline{b})$ the space of all derivations from $\underline{a}$ to $\underline{b}$ and $\mathrm{Int}(\underline{a},\underline{b})$ the subspace of $\mathrm{Der}(\underline{a},\underline{b})$ containing those derivations Δ for which there exists $D \in \underline{b}$ with $\Delta(X) = [D,X]$ for all $X \in \underline{a}$. Hence $H_c^2(\underline{a}) = \mathrm{Der}(\underline{a},\underline{b})/_{\mathrm{Int}(\underline{a},\underline{b})}$ is explicitly known from section I.2.

Proposition 3A. Let p, q and $\underline{a}$ be as above. Then the vector space $H^2_c(\underline{a})$ is isomorphic to :

$$\begin{cases} \{0\} & \text{when } p = \infty \\ C_r(\mathcal{H})/C_q(\mathcal{H}), \text{ with } \frac{1}{r} + \frac{1}{p} = \frac{1}{q} & \text{when } 2<p\leq\infty \\ (L(\mathcal{H})/\mathbb{C}\mathrm{id}_{\mathcal{H}})/C_q(\mathcal{H}) & \text{when } 1<p\leq 2 \\ \{0\} & \text{when } p = 1 \end{cases}$$

Proof : it is an immediate consequence of the considerations which precede the proposition and of the following lemma. ∎

Lemma. Let D be a continuous operator on $\mathcal{H}$. Let $p \in \bar{\mathbb{R}}$ with $2<p\leq\infty$, and let $q,r \in \bar{\mathbb{R}}$ be such that $\frac{1}{p} + \frac{1}{q} = 1$ and $\frac{1}{r} + \frac{1}{p} = \frac{1}{q}$. Suppose that $[D,X] \in C_q(\mathcal{H})$ for all $X \in C_p(\mathcal{H})$. Then $D \in C_r(\mathcal{H})$.

Proof. Let $\mathcal{I} = \{Y \in L(\mathcal{H}) \mid [Y,X] \in C_q(\mathcal{H})$ for all $X \in C_p(\mathcal{H})\}$; then $\mathcal{I}$ is clearly a non trivial Lie ideal of $L(\mathcal{H})$ which contains D. Hence (see proposition II.1A), D must be compact (up to a scalar multiple of the identity of $\mathcal{H}$). It is then sufficient to prove the lemma when D is a compact positive operator.

Let $(e_n)_{n \in N}$ be an orthonormal basis of $\mathcal{H}$ which diagonalises D and let $\lambda = (\lambda_n)_{n \in N}$ be a decreasing sequence of positive real numbers such that $D = \sum_{n\in N} \lambda_n e_n \otimes \overline{e_n}$. We want to show that $\lambda \in \ell^r$.

Let f be a map from N into itself such that $\lambda_{f(n)} \leq \frac{1}{2}\lambda_n$ for all $n \in N$. For each sequence of real numbers $\alpha = (\alpha_n)_{n \in N} \in \ell^p$, let X_α be the operator $\sum_{n\in N} \alpha_n e_n \otimes \overline{e_{f(n)}}$, which is in $C_p(\mathcal{H})$.

Then $[D,X_\alpha] = \sum_{n\in N} \alpha_n(\lambda_n - \lambda_{f(n)}) e_n \otimes \overline{e_{f(n)}}$ must be in $C_q(\mathcal{H})$, hence the sequence of real numbers $\left(\alpha_n(\lambda_n - \lambda_{f(n)})\right)_{n \in N}$ must be in ℓ^q, and by the choice of f so must be $(\alpha_n \lambda_n)_{n \in N}$. This being true for all $\alpha \in \ell^p$, the sequence λ is in ℓ^r, whence the lemma. ∎

Conjectures

i) $H^*_c(\underline{gl}(\mathcal{H}; C_1))$ is isomorphic to $J^*_c(\underline{gl}(\mathcal{H}; C_1))$.

ii) $H^*_c(\underline{gl}(\mathcal{H}; C_2))$ is generated by the canonical image of $J^*_c(\underline{gl}(\mathcal{H}; C_2))$ and by $H^2_c(\underline{gl}(\mathcal{H}; C_2)) \approx (L(\mathcal{H})/\mathbb{C}id_{\mathcal{H}})/_{C_2(\mathcal{H})}$.

iii) $H^*_c(\underline{gl}(\mathcal{H}; C_\infty)) = H^0(\underline{gl}(\mathcal{H}; C_\infty)) = \mathbb{C}$.

Remarks.

i) If $p \neq 1$, the real cohomology algebra of the Banach-Lie group $U(\mathcal{H}; C_p)$ is not isomorphic to the scalar cohomology $H^*_c(\underline{u}(\mathcal{H}; C_p))$ of its Lie algebra. Indeed, the first Betti number of $U(\mathcal{H}; C_p)$ is then equal to $+1$, while $H^1_c(\underline{u}(\mathcal{H}; C_p)) = \{0\}$; moreover, if $p \neq \infty$, the second Betti number of $U(\mathcal{H}; C_p)$ vanishes, while $\dim H^2_c(\underline{u}(\mathcal{H}; C_p)) = \infty$.

ii) If the conjecture i) above was true, then the real cohomology of $U(\mathcal{H}; C_1)$ would be isomorphic to $H^*_c(\underline{u}(\mathcal{H}; C_1))$; similarly for the group $SU(\mathcal{H}; C_1)$.

iii) In the L*-case $p = 2$, the canonical image of $J^*_c(\underline{gl}(\mathcal{H}; C_2))$ is a proper subalgebra $\mathcal{F}$ in $H^*_c(\underline{gl}(\mathcal{H}; C_2))$ which has the following property: let $\underline{s}$ be a classical simple complex Lie algebra of finite dimension which is a subalgebra of $\underline{gl}(\mathcal{H}; C_2)$; then the restrictions of $\mathcal{F}$ and of $H^*_c(\underline{gl}(\mathcal{H}; C_2))$ to $\underline{s}$ are equal. $\mathcal{F}$ is in this sense a kind of "finite approximation" for $H^*_c(\underline{gl}(\mathcal{H}; C_2))$.

iv) It is known that the first and second scalar cohomology spaces of the associative C*-algebra $C(\mathcal{H})$ do vanish; see Guichardet [69].

v) **Proposition II.10 can also be** expressed in a cohomology formulation about $H^1(\underline{g},\underline{g})$ ($\underline{g}$-valued cohomology of $\underline{g}$).

vi) Any statement and conjecture about $\underline{gl}(\mathcal{H}; C_1)$ has an immediate and equivalent counterpart for $\underline{sl}(\mathcal{H}; C_1)$ which we will not write down explicitly.

Proposition 3B. Let $J_{\mathbb{R}}$ be a conjugation of $\mathcal{H}$; let $p \in \bar{\mathbb{R}}$ with $1 \leqslant p \leqslant \infty$ and let $q \in \bar{\mathbb{R}}$ be defined by $\frac{1}{p} + \frac{1}{q} = 1$. Consider the Banach-Lie algebra $\underline{a} = \underline{o}(\mathcal{H}, J_{\mathbb{R}}; C_p)$. Then the vector space $H_c^2(\underline{a})$ is isomorphic to

$\{0\}$ when $p = \infty$

$\underline{o}(\mathcal{H}, J_{\mathbb{R}}; C_r)/\underline{o}(\mathcal{H}, J_{\mathbb{R}}; C_q)$, with $\frac{1}{r} + \frac{1}{p} = \frac{1}{q}$ when $2 < p \leqslant \infty$

$\underline{o}(\mathcal{H}, J_{\mathbb{R}}; L)/\underline{o}(\mathcal{H}, J_{\mathbb{R}}; C_q)$ when $1 < p \leqslant 2$

$\{0\}$ when $p = 1$.

Proposition 3C. It is obtained by replacing $J_{\mathbb{R}}$ by $J_{\mathbb{Q}}$ and $\underline{o}$ by $\underline{sp}$ in proposition 3B.

Proofs : as for proposition 3A. ∎

Remarks : as those following proposition 3A.

IV. 3. - The algebra $I(G)$ of invariant polynomials

Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{K}$ ($\mathbb{K}$ is $\mathbb{R}$ or $\mathbb{C}$). We will denote by $S_c(\mathfrak{g}) = \bigoplus_{k\in N} S_c^k(\mathfrak{g})$ the Z-graded commutative $\mathbb{K}$-algebra with unit defined by the continuous symmetric multilinear maps from $\mathfrak{g}$ to $\mathbb{K}$, and by $P_c(\mathfrak{g})$ that defined by the continuous polynomial maps on $\mathfrak{g}$. It is well-known that $S_c(\mathfrak{g})$ and $P_c(\mathfrak{g})$ are naturally isomorphic (see for example Douady [48] section 1.1).

Suppose now that $\mathfrak{g}$ is the Lie algebra of a Banach-Lie group G. A function $F \in S_c^k(\mathfrak{g})$ is <u>invariant</u> by G if $F(gX_1, \dots, gX_k) = F(X_1, \dots, X_k)$ for all $X_1, \dots, X_k \in \mathfrak{g}$ and for all $g \in G$, where gX denotes the result of the transform of X by g according to the adjoint action of G in $\mathfrak{g}$. Invariant functions define a subalgebra of $S_c(\mathfrak{g})$ which will be denoted by $S(G) = \bigoplus_{k\in N} S^k(G)$. Similarly, invariant polynomials define a subalgebra $I(G)$ of $P_c(\mathfrak{g})$; the natural isomorphism between $S_c(\mathfrak{g})$ and $P_c(\mathfrak{g})$ induces an isomorphism between $S(G)$ and $I(G)$. In this section, we want to describe $I(G)$ when G is one of the classical Banach-Lie group of compact operators $Hilb(\mathcal{H}_{\mathbb{K}}; C_p)$, with $\mathcal{H}_{\mathbb{K}}$ an infinite dimensional separable Hilbert space over $\mathbb{K}$.

Let first $\mathcal{H}$ be a complex space, let $p \in \bar{\mathbb{R}}$ with $1 \leq p \leq \infty$, and consider the group $U_p = U(\mathcal{H}; C_p)$. To any orthonormal basis $e = (e_n)_{n\in N}$ in $\mathcal{H}$ corresponds a maximal torus T in U_p, and the Lie algebra of T is isomorphic to the space of real sequences ℓ^p (when $p < \infty$) or c_o (when $p = \infty$). The inclusion $T \longrightarrow U_p$ induces a morphism $I(U_p) \longrightarrow I(T)$. Hence, functions in $I(U_p)$ can be looked at as continuous polynomial functions on ℓ^p (or c_o if $p = \infty$) which are invariant with respect to the infinite symmetric group W_A (see proposition I.7A). By following the same method as in section IV.1, and by using the standard Newton theorem on elementary symmetric functions, it is easy to compute explicitly $I(U_p)$. We will give now the result and leave the easy checking to the reader (see as well [83]).

Let $k \in N^*$, $k \geqslant p$. Let ρ_k be the continuous polynomial function defined by $\rho_k(X) = \text{trace}\,\{(-\sqrt{-1}\ X)^k\}$ for all $X \in \underline{u}(\mathcal{H}\ ;\ C_p)$.

Proposition 4A. The algebra $I(U_p)$ is a polynomial algebra generated by the functions $(\rho_k)_{k \in N^*, k \geqslant p}$. In particular $I(U_\infty) = I^o(U_\infty) = \mathbb{R}$.

Remarks.

i) The generators $(\rho_k)_{k \in N^*, k \geqslant p}$ are not unique. Indeed, if $p = 1$, other systems of generators are provided by the invariant Chern functions and the dual invariant Chern functions; see Murakami [125] .

ii) Propositions 4B and 4C are analogous to proposition 4A and are left to the reader. Similarly for $I(SU(\mathcal{H}\ ;\ C_1))$.

$\mathrm{Hilb}^+(\mathcal{H}_{\mathbb{K}};\ C_p)$ being again as in pages III.4 and IV.6, one has :

Proposition 5. The real cohomology algebra of the classifying space of the Banach-Lie group $\mathrm{Hilb}^+(\mathcal{H}_{\mathbb{K}};\ C_p)$ is isomorphic to the algebra $I(\mathrm{Hilb}^+(\mathcal{H}_{\mathbb{K}};\ C_p))$ if and only if $p = 1$.

Proof : proposition 5 follows immediately from propositions III.1 and IV.4. ∎

Projects.

1. - Make explicit the map T from $I(\mathrm{Hilb}^+(\mathcal{H}_{\mathbb{K}};\ C_p))$ to $J^*_c(\underline{\mathrm{hilb}}(\mathcal{H}_{\mathbb{K}};\ C_p))$ which sends ρ_k to $\hat{\rho}_k$ for all $k \in N$, $k \geqslant p$ (see the bottom of page IV.1). The image of T will contain the canonical set of generators in $J^*_c(\underline{\mathrm{hilb}}(\mathcal{H}_{\mathbb{K}};\ C_p))$ <u>if and only if</u> $\underline{1 \leqslant p \leqslant 2 \text{ or } p = \infty}$, as it is immediately seen from the corollary to proposition IV.1 and from proposition IV.4.

2. - Study explicitly the Weil algebras (see Cartan [31], [32]) of the classical complex Banach-Lie algebras of compact operators, especially in the case of nuclear operators ($p = 1$) . This project would best follow some work on conjectures i) and ii) section IV.2.