CHAPTER III.

EXAMPLES OF INFINITE DIMENSIONAL HILBERT SYMMETRIC SPACES

The content of this chapter overlaps substantially with that of a previous note [75]. Hopefully, these *examples* will be followed sometime by a *theory* of E. Cartan's symmetric spaces in the context of Banach (or at least Hilbert) manifolds.

Let G be one of the Banach-Lie groups considered in chapter II and let H be a sub Banach-Lie group-manifold of G. Our examples are particular homogeneous spaces of the type G/H. In order to avoid ambiguity, we first recall a standard result (see for example Lazard [107] 23.10).

Result. Let G be a Banach-Lie group with unit e and let H be a sub Banach-Lie group-manifold of G. Then there exists on the left cosets space G/H a manifold structure, uniquely determined by the condition that the canonical projection $G \xrightarrow{\pi} G/H$ is a submersion. The canonical action $G \times G/H \longrightarrow G/H$ is then smooth.

In particular : if $\underline{g}$ is the Lie algebra of G, if $\underline{h}$ is that of H, and if $\underline{m}$ is a complementary subspace of $\underline{h}$ in $\underline{g}$, then the restriction of $D\pi(e)$ to $\underline{m}$ is an isomorphism between $\underline{m}$ and the tangent space to G/H at $\pi(e)$.

III.1. - A list of examples

The standard reference for finite dimensional symmetric spaces is Helgason [84] .

Among finite dimensional Riemannian manifolds, the class of symmetric spaces plays a considerable role, both for its own sake and as a rich source of examples for more general situations. The theory of the (finite dimensional) Riemannian globally symmetric spaces finds one of its achievements in the famous E.Cartan's classification : any such space (say simply connected) is a product of finitely many irreducible terms, and each term (disregarding Euclidean type) is isometric to a space of a well-known list described as follows.

i) The simply connected compact Lie groups

$SU(n)$, $Spin(n)$ and $Sp(n)$

where n is a positive integer.

ii) The Grassmann manifolds

$G_k(\mathbb{C}^n) = SU(n)/SU(n) \cap [U(k) \times U(n-k)]$ (hermitian)

$SG_k(\mathbb{R}^n) = SO(n)/SO(k) \times SO(n-k)$ (hermitian if $k = 2$)

$G_k(\mathbb{Q}^n) = Sp(n)/Sp(k) \times Sp(n-k)$

where $k,n \in \mathbb{N}$ with $1 \leqslant k \leqslant n/2$ and where $SG_k(\mathbb{R}^n)$ is the simply connected manifold of oriented k-planes in $\mathbb{R}^n$, which double-covers the manifold $G_k(\mathbb{R}^n)$ of k-planes in $\mathbb{R}^n$.

iii) The manifolds

$\mathbb{RC}^+(n) = SU(n)/SO(n)$

$\mathbb{CQ}(n) = Sp(n)/U(n)$

$\mathbb{CR}^+(n) = SO(2n)/U(n)$

$\mathbb{QC}^+(n) = SU(2n)/Sp(n)$

where $n \in \mathbb{N}^*$ and where our notations are justified by the following example : Let $CR(n) = {}^{O(2n)}/_{U(n)}$ be the space of orthogonal complex structures on the Euclidean space $\mathbb{R}^{2n}$; then $CR(n)$ has two connected components, one of which being $CR^+(n)$.

iv) The list given in i), ii) and iii) above needs two modifications : first to delete some redundant terms in the case n is an integer smaller than 5 ; second to add a small number of spaces whose description involves the exceptional simple Lie groups. With these modifications, the list contains exactly all (up to isometry) irreducible simply connected Riemannian symmetric spaces of the <u>compact type</u>.

v) The duals of the spaces already described, namely the irreducible Riemannian symmetric spaces of the <u>non-compact type</u>.

Let now $\mathcal{H}_{\mathbb{K}}$ be a $\mathbb{K}$-Hilbert space, separable and infinite dimensional. The classical Banach-Lie groups of compact operators on $\mathcal{H}_{\mathbb{K}}$ (section II.6) make it possible to consider, by analogy, a new list of infinite dimensional homogeneous Banach manifolds. In order to unify the notations, we will write now $\mathrm{Hilb}(\mathcal{H}_{\mathbb{K}}; C_p)$ the Banach-Lie group written previously $U(\mathcal{H}_{\mathbb{C}}; C_p)$ [resp. $O(\mathcal{H}_{\mathbb{R}}; C_p)$, $Sp(\mathcal{H}_{\mathbb{Q}}; C_p)$] when $\mathbb{K}$ is $\mathbb{C}$ [resp. $\mathbb{R}$, $\mathbb{Q}$].

<u>Grassmann manifolds as Hilbert manifolds, and their real cohomology rings</u>

Let $k \in \mathbb{N}^*$ and let $\mathbb{K}^k$ be identified to some fixed subspace of $\mathcal{H}_{\mathbb{K}}$. The Grassmann manifold of k-planes in $\mathcal{H}_{\mathbb{K}}$ is given by

$$G_k(\mathcal{H}_{\mathbb{K}}) = \mathrm{Hilb}(\mathcal{H}_{\mathbb{K}};\, C_2) \Big/ \mathrm{Hilb}(k) \times \mathrm{Hilb}(\mathcal{H}_{\mathbb{K}} \ominus \mathbb{K}^k\,;\, C_2)\ ;$$

it is clearly a Hilbert manifold (see section II.7). In the real case, the Grassmann manifold of oriented k-planes is

$$SG_k(\mathcal{H}_{\mathbb{K}}) = O^+(\mathcal{H}_{\mathbb{R}};\, C_2) \Big/ SO(k) \times O^+(\mathcal{H}_{\mathbb{R}} \ominus \mathbb{R}^k;\, C_2)\ .$$

On the other hand, let $\mathcal{H}_{\mathbb{K}} = \mathcal{H}^-_{\mathbb{K}} \oplus \mathcal{H}^+_{\mathbb{K}}$ be the Hilbert sum of two infinite dimensional Hilbert spaces. Define then

$$G_\infty(\mathcal{H}_{\mathbb{K}};\, C_2) = \mathrm{Hilb}(\mathcal{H}_{\mathbb{K}};\, C_2) \Big/ \mathrm{Hilb}(\mathcal{H}^-_{\mathbb{K}};\, C_2) \times \mathrm{Hilb}(\mathcal{H}^+_{\mathbb{K}};\, C_2)$$

and similarly for $SG_\infty(\mathcal{H}_{\mathbb{R}};\, C_2)$. Remark that (say when $\mathbb{K} = \mathbb{C}$) the underlying set of $G_\infty(\mathcal{H};\, C_2)$ is <u>not</u> that one of the standard Grassmannian $G_\infty(\mathcal{H};\, L) = U(\mathcal{H}) / U(\mathcal{H}^-) \times U(\mathcal{H}^+)$. Indeed $G_\infty(\mathcal{H};\, L)$ describes all closed subspaces of $\mathcal{H}$ which have both infinite dimension and codimension, while $G_\infty(\mathcal{H};\, C_2)$ describes only those which are the image of $\mathcal{H}^-$ by an element of $U(\mathcal{H};\, C_2)$.

Similarly, from the groups $\mathrm{Hilb}(\mathcal{H}_{\mathbb{K}};\, C_p)$, one defines the manifolds $G_\infty(\mathcal{H}_{\mathbb{K}};\, C_p)$ and $SG_\infty(\mathcal{H}_{\mathbb{R}};\, C_p)$ for any $p \in \bar{\mathbb{R}}$ with $1 \leqslant p \leqslant \infty$.

It is well-known (see for example McAlpin [114], section IID) that $G_k(\mathcal{H}_{\mathbb{K}})$ is a classifying space for Hilb(k)-principal bundles, and that $G_\infty(\mathcal{H}_{\mathbb{K}};\, C_p)$ is a classifying space for $\mathrm{Hilb}(\mathcal{H}_{\mathbb{K}};\, C_p)$-principal bundles, hence for $\mathbb{K}$-vector bundles of unbounded finite dimension. This shows the relevance of the spaces just defined and of the corresponding Stiefel manifolds in algebraic topology. However, we will restrict ourselves to recall only the following results.

Proposition 1.

i) The real cohomology of $G_k(\mathcal{H}_{\mathbb{K}})$ is a polynomial algebra over generators of the following even degrees :

$2,4,6,\dots\dots,2k$ if $\mathbb{K} = \mathbb{C}$

$4,8,12,\dots\dots,4m$ if $\mathbb{K} = \mathbb{R}$ and $k = 2m+1$

$4,8,12,\dots\dots,4k$ if $\mathbb{K} = \mathbb{Q}$

$4,8,\dots,4m-4,2m$ if $\mathbb{K} = \mathbb{R}$ and $k = 2m$.

ii) The real cohomology of $G_\infty(\mathcal{H}_{\mathbb{K}}; C_p)$ is for any p a polynomial algebra over generators of the following degrees :

$2,4,\dots\dots,2k,\dots$ if $\mathbb{K} = \mathbb{C}$

$4,8,\dots\dots,4k,\dots$ if $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{Q}$.

Proof : these results are classical; see for example Borel [22], section 18 and chap. IV. ∎

Four other Hilbert manifolds and their homotopy types

Let $\mathcal{H}$ be a complex Hilbert space, separable and infinite dimensional, and let $J_{\mathbb{R}}$ be a fixed conjugation on $\mathcal{H}$. The set of all conjugations on $\mathcal{H}$ is then clearly described by the homogeneous space $\mathbb{R}C(\mathcal{H}\,;\,L) = U(\mathcal{H})/O(\mathcal{H}_{\mathbb{R}})$, where $\mathcal{H}_{\mathbb{R}}$ is the real Hilbert space defined by the fixed points of $J_{\mathbb{R}}$ in $\mathcal{H}$. Define a new Hilbert manifold by

$$\mathbb{R}C^+(\mathcal{H}\,;\,C_2) = U(\mathcal{H}\,;\,C_2)\Big/O^+(\mathcal{H}_{\mathbb{R}}\,;\,C_2) \quad ;$$

it can be referred to as the connected component of the space of those conjugations on $\mathcal{H}$ which are "Hilbert-Schmidt perturbations" of the given $J_{\mathbb{R}}$. In the same way, one defines

$$\mathbb{C}\mathbb{Q}(\mathcal{H}\,;\,C_2) = {}^{Sp(\mathcal{H}_{\mathbb{Q}};\,C_2)}\big/_{U(\mathcal{H}\,;\,C_2)}$$

$$\mathbb{C}\mathbb{R}^+(\mathcal{H}\,;\,C_2) = {}^{O^+(\mathcal{H}_{\mathbb{R}};\,C_2)}\big/_{U(\mathcal{H}\,;\,C_2)}$$

$$\mathbb{Q}\mathbb{C}(\mathcal{H}\,;\,C_2) = {}^{U(\mathcal{H}\,;\,C_2)}\big/_{Sp(\mathcal{H}_{\mathbb{Q}};\,C_2)} \quad .$$

Similarly for the Banach manifolds $\mathbb{R}\mathbb{C}^+(\mathcal{H}\,;\,C_p)$,..., for any $p \in \bar{\mathbb{R}}$ with $1 \leqslant p \leqslant \infty$.

The homotopy type of these spaces is again well-known. We will treat in detail the first example only; the same method applies to the other cases above, and indeed to many more.

Let $\mathcal{H}$, $J_{\mathbb{R}}$ and $\mathcal{H}_{\mathbb{R}}$ be as above, and let $e = (e_n)_{n \in N^*}$ be an orthonormal basis of $\mathcal{H}_{\mathbb{R}}$; then e is a fortiori an orthonormal basis of $\mathcal{H}$. For each $n \in N^*$, identify $\mathbb{R}^n$ [resp. $\mathbb{C}^n$] to the subspace of $\mathcal{H}_{\mathbb{R}}$ [resp. $\mathcal{H}$] span by the first n basis vectors. The group $SO(n)$ [resp. $U(n)$] is accordingly identified to a subgroup of $O^+(\mathcal{H}_{\mathbb{R}};\,C_p)$ [resp. $U(\mathcal{H}\,;\,C_p)$]. Let $SO(\infty)$ denote the limit of the closed expanding system (see Hansen [72]) defined by the groups $SO(n)$'s, and similarly for $U(\infty) = \varinjlim U(n)$ and $\mathbb{R}\mathbb{C}^+(\infty) = {}^{U(\infty)}/_{SO(\infty)} = \varinjlim {}^{U(n)}/_{SO(n)}$. Consider the diagram

$$\begin{array}{ccccccc}
SO(n) & \longrightarrow & SO(n+1) & \dashrightarrow & SO(\infty) & \xrightarrow{\alpha} & O^+(\mathcal{H}_{\mathbb{R}};\,C_p) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
U(n) & \longrightarrow & U(n+1) & \dashrightarrow & U(\infty) & \xrightarrow{\beta} & U(\mathcal{H}\,;\,C_p) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathbb{R}\mathbb{C}^+(n) & \longrightarrow & \mathbb{R}\mathbb{C}^+(n+1) & \dashrightarrow & \mathbb{R}\mathbb{C}^+(\infty) & \xrightarrow{\gamma} & \mathbb{R}\mathbb{C}^+(\mathcal{H}\,;\,C_p)
\end{array}$$

All the "vertical" triples are Serre fibrations, so that there are homotopy long exact sequences associated to each of them. In particular, for each $i \in N^*$, the two rows of the diagram

$$\begin{array}{ccccccc}
\pi_i(SO(\infty)) & \longrightarrow & \pi_i(U(\infty)) & \longrightarrow & \pi_i(\mathbb{R}\mathbb{C}^+(\infty)) & \longrightarrow \\
\downarrow{\alpha_i} & & \downarrow{\beta_i} & & \downarrow{\gamma_i} & \\
\pi_i(O^+(\mathcal{H}_{\mathbb{R}};C_p)) & \longrightarrow & \pi_i(U(\mathcal{H};C_p)) & \longrightarrow & \pi_i(\mathbb{R}\mathbb{C}^+(\mathcal{H};C_p)) & \longrightarrow
\end{array}$$

$$\begin{array}{ccccc}
\longrightarrow & \pi_{i-1}(SO(\infty)) & \longrightarrow & \pi_{i-1}(U(\infty)) \\
 & \downarrow{\alpha_{i-1}} & & \downarrow{\beta_{i-1}} \\
\longrightarrow & \pi_{i-1}(O^+(\mathcal{H}_{\mathbb{R}};C_p)) & \longrightarrow & \pi_{i-1}(U(\mathcal{H};C_p))
\end{array}$$

are exact. As α and β are homotopy equivalences (corollary to proposition II.16), α_i and β_i are isomorphisms. It follows from the five lemma (see e.g. Cartan-Eilenberg [33] chap. I, §1) that γ_i is a group isomorphism for all $i \in \mathbb{N}$ (the case $i = 0$ is trivial) ; in other words, γ is a weak homotopy equivalence. But $\mathbb{R}\mathbb{C}^+(\infty)$ is homotopically equivalent to an ANR (Hansen [72] cor. 6.4) and the Banach manifold $\mathbb{R}\mathbb{C}^+(\mathcal{H};C_p)$ is an ANR. Hence Whitehead's lemma applies (see e.g. Palais [130] section 6.6) and γ is a homotopy equivalence. We have proved :

<u>Proposition 2</u>. Notations being as above, the four following inclusion maps are homotopy equivalences for all $p \in \bar{\mathbb{R}}$ with $1 \leqslant p \leqslant \infty$:

$$\mathbb{R}\mathbb{C}^+(\infty) \longrightarrow \mathbb{R}\mathbb{C}^+(\mathcal{H};C_p)$$
$$\mathbb{C}\mathbb{Q}(\infty) \longrightarrow \mathbb{C}\mathbb{Q}(\mathcal{H};C_p)$$
$$\mathbb{C}\mathbb{R}^+(\infty) \longrightarrow \mathbb{C}\mathbb{R}^+(\mathcal{H};C_p)$$
$$\mathbb{Q}\mathbb{C}(\infty) \longrightarrow \mathbb{Q}\mathbb{C}(\mathcal{H};C_p)$$

Examples of Hilbert (and Banach) manifolds corresponding to the irreducible Riemannian symmetric spaces of the <u>non-compact type</u> can be obtained the same way.

III.2. - On symmetric spaces

The classical theory of E. Cartan's symmetric spaces takes a constant advantage of the close relationship between the three following notions : orthogonal symmetric Lie algebra, Riemannian symmetric pair, and Riemannian (globally) symmetric space. We check now that these notions still make sense for the manifolds listed in section III.1. The various cases are sufficiently similar to each other for us to deal with one example only. We follow again the terminology of Helgason [84] .

Let $\mathcal{H} = \mathcal{H}^- \oplus \mathcal{H}^+$ be the direct sum of two infinite dimensional separable real Hilbert spaces. Consider the Hilbert Riemannian manifold

$$M = SG_\infty(\mathcal{H} ; C_2) = O^+(\mathcal{H} ; C_2) \Big/ O^+(\mathcal{H}^-; C_2) \times O^+(\mathcal{H}^+; C_2) .$$

Put $G = O^+(\mathcal{H}; C_2)$ and $K = O^+(\mathcal{H}^-; C_2) \times O^+(\mathcal{H}^+; C_2)$; the Lie algebras of these Banach-Lie groups are respectively $\underline{g} = \underline{o}(\mathcal{H}; C_2)$ and $\underline{k} = \underline{o}(\mathcal{H}^-; C_2) \times \underline{o}(\mathcal{H}^+; C_2)$. Let I be the orthogonal operator on $\mathcal{H}$ which is equal to minus the identity on $\mathcal{H}^-$ and to the identity on $\mathcal{H}^+$. Then the map $\sigma\begin{cases} G \longrightarrow G \\ X \longmapsto IXI \end{cases}$ is an involutive analytic automorphism of G ; let $s : \underline{g} \longrightarrow \underline{g}$ be its derivative at the identity e of G . The set of fixed points of s is $\underline{k}$ and its eigenspace for the eigenvalue -1 is $\underline{m} = (C_2(\mathcal{H}^- \rightarrow \mathcal{H}^+) \oplus C_2(\mathcal{H}^+ \rightarrow \mathcal{H}^-)) \cap \underline{g}$; note that $\underline{m}$ is isomorphic as a Hilbert space to $C_2(\mathcal{H})$.

The object $(\underline{g},s)$ should obviously be called an <u>orthogonal symmetric Lie algebra</u>, of the compact type and irreducible ([84] , pages 178, 194 and 306). The pair (G,K) should be called a <u>Hilbert Riemannian symmetric pair</u>, of the compact type and irreducible ([84], pages 174, 195 and 306), which is clearly associated with $(\underline{g}, s)$.

Let now π be the canonical projection $G \longrightarrow M$, let e be the identity in G and let ε be the image $\pi(e)$. Define a map $s_\varepsilon : M \longrightarrow M$ by the condition $s_\varepsilon \circ \pi = \pi \circ \sigma$. Let now $x = bK$ be an arbitrary point in M , where bK is the coset corresponding to some b in G ; let λ_b be the diffeomorphism of M defined by b and put $s_x = \lambda_b \circ s_\varepsilon \circ \lambda_b{-1}$ (s_x depends on the coset bK only, not on b). It is easy to check that s_x is then an involutive isometry of M with x as isolated fixed point. In other words, M is a <u>Hilbert Riemannian globally symmetric space</u>.

Finally, all considerations concerning the <u>duality</u> of M with the manifold

$$O^+(\mathcal{H}, \infty, \infty ; C_2)/O^+(\mathcal{H}^-; C_2) \times O^+(\mathcal{H}^+; C_2)$$

can now be repeated precisely as in Helgason [84], chap. V §2, example I.

Many basic geometrical properties of finite dimensional symmetric spaces carry over to the spaces introduced in section III.1 and to cartesian products of them. For example, furnished with their natural Riemannian metric, these Hilbert manifolds have positive (resp. negative) curvature if they are of the compact (resp. non-compact) type. It follows among other things that, for those of the non-compact type, the exponential map at some point is a diffeomorphism (McAlpin [114], section IH); hence, via the Elworthy-Tromba construction, these spaces have distinguished Fredholm structures ([62] , proposition 1.2).

Instead of dealing further with this kind of general statements, we prefer now to consider in some detail a specific example. It was studied, in finite dimensions, by Mostow [124]; his proves carry over to the Hilbert case with minor modifications only.

Let $\mathcal{H}$ be an infinite dimensional separable real Hilbert space. Put $G = GL(\mathcal{H}\,;\,C_2)$, $K = O(\mathcal{H}\,;\,C_2)$, let $S = Sym(\mathcal{H}\,;\,C_2)$ be the subspace of the self-adjoints operators in $C_2(\mathcal{H})$ and let $P = Pos(\mathcal{H}\,;\,C_2)$ be the set of positive definite operators of the form $id_{\mathcal{H}} + X$ with $X \in C_2(\mathcal{H})$. The set P is furnished with a structure of Hilbert manifold by the obvious bijection between P and the open subset of S containing those X whose spectrum lies in the real interval $]-1,\infty[$.

The group G acts smoothly and transitively on the manifold P by $\begin{cases} G \times P \longrightarrow P \\ (X,A) \longmapsto X^*AX \end{cases}$, and the isotropy subgroup at $id_{\mathcal{H}}$ is clearly K . The induced map $\begin{cases} G \longrightarrow P \\ X \longmapsto X^*X \end{cases}$ is a submersion; indeed, its derivative at some point X in G is $Df(X) \begin{cases} C_2(\mathcal{H}) \longrightarrow S \\ Y \longmapsto X^*Y + Y^*X \end{cases}$, and is the composition of the isomorphism from $C_2(\mathcal{H})$ into itself given by $Y \longmapsto X^*Y$ and of the surjective map with splitting kernel from $C_2(\mathcal{H})$ to S given by $Z \longmapsto Z + Z^*$. It follows that the natural map $G/_K \longrightarrow P$ is an analytic diffeomorphism of Hilbert manifolds (see the result recalled page III.1).

Let $A \in P$ and let S,T be two tangent vectors at P to A. The tangent space at P to A being identified with S , one can define a smooth Riemannian structure on P by

$$\langle S|T\rangle_A = \mathrm{trace}(A^{-1}SA^{-1}T) \quad ;$$

this structure is clearly G-invariant.

<u>Proposition 3.</u>

i) Two points A and B in P can always be joined by a geodesic of minimal length, which is unique and given as follows : let $X \in P$ be such that $X^*X = A$ and let $Y = X^{*-1}BX^{-1}$; then the geodesic is $\begin{cases} [0,1] \longrightarrow P \\ t \longmapsto X^* \exp(t \log Y)\, X \end{cases}$.

ii) The curvature is smaller than or equal to zero at all points of P ; the sum of the angles of a geodesic triangle in P is always smaller than or equal to π .

iii) The exponential map $S \longrightarrow P$ is a diffeomorphism, which is conformal at the origin of S .

<u>Definition</u>. Let $\underline{m}$ be a subspace of a Banach-Lie algebra $\underline{g}$. A <u>Lie triple system</u> in $\underline{m}$ is a closed subspace $\underline{t}$ of $\underline{m}$ such that $[\underline{t}, [\underline{t},\underline{t}]] \subset \underline{t}$.

<u>Proposition 4</u>. Let $\mathcal{E}$ be a closed linear subspace of S and let E be the image of $\mathcal{E}$ by the exponential map; then the following are equivalent :

i) E is a geodesic subspace of P

ii) for all $e,f \in E$, $efe \in E$

iii) for all $S,T \in \mathcal{E}$, $[S, [S,T]] \in \mathcal{E}$

iv) $\mathcal{E}$ is a Lie triple system in S .

<u>Proof</u> of propositions 3 and 4 : as in Mostow [124] . ■

III.3. - Poincaré series

This section describes an apparently eccentric fact. It can be read as a challenge to whoever will find of it a respectable interpretation. It shows an unexpected connection between the cohomology of the spaces of section III.1 and the classical domain of modular functions.

Let M be a topological space (say, up to homotopy, an ANR, in order to avoid complications). For each $k \in N$, let b_k be the k^{th} Betti number of M, and suppose M such that all the b_k's are finite. Then the Poincaré series of M is, by definition, the formal power series $P_M(t) = \sum_{k \in N} b_k t^k$; it clearly depends on the homotopy type of M only. We want now to compute P_M when M is a classical Banach-Lie group of compact operators or when M is one of the spaces introduced in section III.1.

Recall of some formal power series P_M

The spaces we are interested in are all homotopically equivalent to stable spaces, in the sense one speaks of the stable classical groups (results as the corollary to prop. II.16, or as proposition III.2). Their Betti number can easily be found in the literature.

Classical groups. The various ways of computing P_M when M is a finite dimensional classical Lie group are reviewed by Samelson in [148]. What we need can be summed up as follows (see again the corollary to prop. II.16).

The Poincaré series of $U(\infty)$, or of $U(\mathcal{H}; C_p)$, is given by

$$P_U(t) = \prod_{k=1}^{\infty} (1 + t^{2k-1}) \quad .$$

The Poincaré series of $SO(\infty)$, or of $O^+(\mathcal{H}_{\mathbb{R}}; C_p)$, is given by $P_O(t) = \prod_{k=1}^{\infty} (1 + t^{4k-1})$. The Poincaré series of $Sp(\infty)$, or of $Sp(\mathcal{H}_{\mathbb{Q}}; C_p)$, is given by $P_{Sp}(t) = P_O(t)$.

<u>Grassmannian</u>. From the results recalled in proposition III.1, it is clear that P_M is a polynomial whenever M is a Grassmannian $G_K(\mathcal{H}_{\mathbb{K}})$ with k finite, and that one has :

$$\text{for } G_\infty(\mathcal{H} ; C_p) \quad : \quad P_{G\mathbb{C}}(t) = \prod_{k=1}^{\infty} (1 - t^{2k})^{-1}$$

$$\text{for } G_\infty(\mathcal{H}_{\mathbb{R}} ; C_p) \quad : \quad P_{G\mathbb{R}}(t) = \prod_{k=1}^{\infty} (1 - t^{4k})^{-1}$$

$$\text{for } G_\infty(\mathcal{H}_{\mathbb{Q}} ; C_p) \quad : \quad P_{G\mathbb{Q}}(t) = P_{G\mathbb{R}}(t) \quad .$$

<u>Other spaces.</u> The Poincaré polynomials of the finite dimensional manifolds $\mathbb{RC}^+(n)$, $\mathbb{CQ}(n)$, $\mathbb{CR}^+(n)$ and $\mathbb{QC}^+(n)$ can be computed with the aid of Hirsch formula (see Borel [22] section 20, particularly theorem 6a). From that, and using proposition III.2, it follows easily that one has :

$$\text{for } \mathbb{RC}^+(\mathcal{H} ; C_p) \quad : \quad P_{\mathbb{RC}}(t) = \prod_{k=1}^{\infty} (1 + t^{4k+1})$$

$$\text{for } \mathbb{CQ}(\mathcal{H} ; C_p) \quad : \quad P_{\mathbb{CQ}}(t) = \prod_{k=1}^{\infty} (1 + t^{2k})$$

$$\text{for } \mathbb{CR}^+(\mathcal{H} ; C_p) \quad : \quad P_{\mathbb{CR}}(t) = P_{\mathbb{CQ}}(t)$$

$$\text{for } \mathbb{QC}(\mathcal{H} ; C_p) \quad : \quad P_{\mathbb{QC}}(t) = P_{\mathbb{RC}}(t) \quad .$$

The Poincaré series of the other classical Banach-Lie groups of compact operators are polynomials or are similar to one of $P_U(t)$ and $P_O(t)$, according to the corollary to proposition II.16. The Hilbert manifolds corresponding to the Riemannian symmetric space of the non-compact type are of no interest in this section, because these spaces are contractible (see Helgason [84], chap. VI, theorem 1.1.iii).

Recall of infinite products and of Jacobi functions

The infinite products written above have been first introduced by Euler, in the domain of number theory [63]. The relationship between the Poincaré polynomials of, say, the Grassmannians, and combinatorial properties of integer numbers is standard; it is shown by the method of Schubert's cells for the computation of the cohomology rings of these manifolds (see for example Chern [35], chap. IV, section 1).

On the other hand, let $\theta_1, \dots, \theta_4$ be the four Jacobi Theta-functions; notations are as in Whittaker-Watson [184], or Bateman [19]. Let λ be the modular function defined by

$$\lambda(t) = \left[\frac{\theta_2(0,t)}{\theta_3(0,t)}\right]^4 = 1 - \left[\frac{\theta_4(0,t)}{\theta_3(0,t)}\right]^4$$

For the properties of λ, see [19] section 13.24; an elementary introduction is in [147], chap. VII, §11.

We need now the formulas

$$\prod_{k=1}^{\infty} (1 + t^{2k-1})^{24} = 16\, t \left[\frac{\theta_3(0,t)}{\theta_2(0,t)} \; \frac{\theta_3(0,t)}{\theta_4(0,t)}\right]^4$$

and

$$\prod_{k=1}^{\infty} (1 + t^{2k})^{24} = \frac{1}{256 t^2} \left[\frac{\theta_2(0,t)}{\theta_3(0,t)}\right]^8 \left[\frac{\theta_3(0,t)}{\theta_4(0,t)}\right]^4$$

as well as

$$\prod_{k=1}^{\infty} (1 - t^{2k})^{12} = \frac{[\theta_1'(0,t)]^4}{16t}$$

(the dash $'$ in the last formula holds for derivation with respect to the first variable).

Indications for the proofs of these equalities can be found in Whittaker-Watson [184]; see example 10 following section 21.9 for the first two, and section 21.42 for the third.

Poincaré series

It is easy to see that all the infinite products written in this section converge in the unit disc of the complex plane, and that they define in this domain holomorphic functions. By juxtaposition of the facts recalled so far, it is elementary to check that these functions are given as follows (the function defined by the product $P_U(t)$ is still denoted the same way) :

$$[P_U(t)]^{24} = \frac{16t}{\lambda(t)\,[1-\lambda(t)]}$$

$$P_{G\mathbb{C}}(t)^{12} = \frac{16t}{[\theta_1'(0,t)]^4}$$

$$P_{G\mathbb{R}}(t) = P_{G\mathbb{Q}}(t) = P_{G\mathbb{C}}(t^2)$$

$$[P_{\mathbb{C}\mathbb{Q}}(t)]^{24} = \frac{\lambda^2(t)}{256t^2\,[1-\lambda(t)]}$$

$$P_{\mathbb{C}\mathbb{R}}(t) = P_{\mathbb{C}\mathbb{Q}}(t)$$

We have not yet been able to express in terms of standard functions either $P_O = P_{Sp}$ or $P_{\mathbb{R}\mathbb{C}} = P_{\mathbb{Q}\mathbb{C}}$. This reduces clearly to <u>one</u> problem, because of the functional relation given by $(1+t)P_O(t)P_{\mathbb{R}\mathbb{C}}(t) = P_U(t)$, which is trivially checked from the infinite product expansions.

We will sum up this section by a result and a problem.

<u>Result</u>. Let M be a space which is (up to homotopy equivalence) either a stable classical group, or a stable irreducible globally symmetric space. Then the Poincaré series of M defines a holomorphic

complex function (still denoted by P_M) in the open unit disc of the complex plane. These functions are either polynomials, or surprisingly simply related to modular functions.

Remark. The stable spaces defined by the irreducible hermitian symmetric spaces of the compact type are $G_k(\mathbb{C}^\infty) = \varinjlim_n G_k(\mathbb{C}^n)$ for $k \geqslant 1$, $G_2(\mathbb{R}^\infty) = \varinjlim G_2(\mathbb{R}^n)$, $\mathbb{C}\mathbb{Q}(\infty)$ and $\mathbb{C}\mathbb{R}^+(\infty)$. For them and for $U(\infty)$, the above result can be expressed more precisely : if P_M is the function defined by the Poincaré series of one of these spaces, then P_M is the product of a rational function by a modular function.

Problem. Express the function defined by $P_0(t) = \prod_{k=1}^{\infty} (1 + t^{4k-1})$ in terms of known functions.

About particular functions defined by Poincaré series of classifying spaces, see Venkov [181] and Quillen [137].

III.4. - Miscellaneous

Hilbert manifolds of constant curvature (Michal [119], McAlpin [114] end of section IH, Unsain [176])

As in finite dimensions, spaces of constant curvature are symmetric spaces. More precisely, the following holds.

Let M be a simply connected complete Riemannian manifold of constant curvature, modelled on an infinite dimensional separable real Hilbert space $\mathcal{H}$. Then M is isometric to a sphere $\mathcal{H}(r) = \{x \in \mathcal{H} \mid |x|^2 = r^2\}$ if the curvature of M is positive, to $\mathcal{H}$ itself if the curvature of M vanishes, and to a hyperbolic space
$\mathrm{Hyp}(r) = \{\xi \oplus x \in \mathbb{R} \oplus \mathcal{H} \mid -|\xi|^2 + |x|^2 = -r^2\}$ if the curvature of M is negative. In particular, M is diffeomorphic to one of

$$\mathcal{H}(1) = O^+(\mathcal{H};\, C_2) \big/ O^+(\mathcal{H} \oplus \mathbb{R};\, C_2) \qquad \text{(curvature +1)}$$

$$\mathcal{H} \qquad \text{(curvature 0)}$$

$$\mathrm{Hyp}(1) = O^+(\mathcal{H},1,\infty;\, C_2) \big/ O^+(\mathcal{H} \oplus \mathbb{R};\, C_2) \qquad \text{(curvature -1)}$$

furnished with their standard Riemannian structures.

Various examples of Hilbert manifolds are given by the quotients of these three manifolds by the standard discrete subgroups of isometries (see Wolf [185] section 2.7). Each of these quotients is an Eilenberg-McLane space, because their universal coverings written just above are contractible.

An explicit trivialisation of the tangent bundle to the unit sphere in a Hilbert space

The unit sphere $\mathcal{H}(1)$ in the infinite dimensional real Hilbert space $\mathcal{H}$ being contractible, its tangent bundle $T(\mathcal{H}(1))$ is obviously trivial. We indicate below an explicit trivialisation. Let $\mathbb{R}^\infty$ be the inductive limit of the vector spaces R^n's, canonically embedded in each other; consider $\mathbb{R}^\infty$ as a subspace of the Hilbert space $\mathcal{H} = \ell^2$. Zvengrowski ([191], sections 3 and 4) has constructed a norm-preserving multiplication orthogonal to the identity on $\mathbb{R}^\infty$, that is a linear map $\mu: \mathbb{R}^\infty \otimes \mathbb{R}^\infty \longrightarrow \mathbb{R}^\infty$ such that

i) $|\mu(x \otimes y)| = |x||y|$ for all $x, y \in \mathbb{R}^\infty$,

ii) $\mu(e_o \otimes x) = x$ for all $x \in \mathbb{R}^\infty$, where e_o is the first element of the canonical basis of ℓ^2.

iii) $\langle y | \mu(x \otimes y) \rangle = 0$ for all $x, y \in R^\infty$ with $x \perp e_o$.

The unique continuous extention $\bar{\mu}$ of μ to ℓ^2 makes ℓ^2 an absolute valued (not associative) real algebra. For each $x \in \ell^2$, the left-multiplication operator $L_x : y \longmapsto \bar{\mu}(x \otimes y)$ is continuous and invertible, while the right-multiplication operator R_x is not onto in general. Furnished with the multiplication $\bar{\mu}$, the space ℓ^2 could therefore be called a "normal real left-division algebra with a left-unit".

Let now $(e_n)_{n \in N}$ be the canonical basis of ℓ^2, and let μ_j be the vector field on $\mathcal{H}(1)$ defined by $\mu_j(x) = \bar{\mu}(e_j \otimes x)$ for all $x \in \mathcal{H}(1)$, for all $j \in N^*$. Then, for each $x \in \mathcal{H}(1)$, the family $(\mu_j(x))_{j \in N^*}$ is an orthonormal basis of the tangent space to $\mathcal{H}(1)$ at x. This provides an explicit trivialisation of $T(\mathcal{H}(1))$.

The question of the existence of a division algebra structure on ℓ^2 is, to my knowledge, unsolved. The answer has been conjectured to be negative by Wright ([186], page 332). It is known that, if such an algebra existed, it could not be associative (Gel'fand-Mazur theorem), and even not alternative (Bruck and Kleinfeld [29]).

In connection with the study of Fredholm structures on the projective space defined by $\mathcal{H}(1)$, the following question can be asked.

Problem. Does there exist a bilinear map $b : \mathcal{H} \times \mathcal{H} \longrightarrow \mathcal{H}$ having the properties i and ii below?

i) symmetry : $b(x,y) = b(y,x)$ for all $x,y \in \mathcal{H}$.

ii) the left-multiplication by $x : y \longmapsto b(x,y)$ is a Fredholm operator for all $x \in \mathcal{H}$, $x \neq 0$.