# CHAPTER I.

## CLASSICAL INVOLUTIVE LIE ALGEBRAS OF FINITE RANK OPERATORS

Let $\mathcal{H}_{\mathbb{K}}$ be a Hilbert space over $\mathbb{K}$, of infinite dimension if not otherwise stated. The associative algebra of finite rank operators on $\mathcal{H}_{\mathbb{K}}$ defines a Lie algebra which will be denoted by $\underline{gl}(\mathcal{H}_{\mathbb{K}}; C_0)$; it is a real Lie algebra if $\mathbb{K}$ is $\mathbb{R}$ or $\mathbb{Q}$ and a complex one if $\mathbb{K} = \mathbb{C}$. The aim of this first chapter is to study various subalgebras of $\underline{gl}(\mathcal{H}_{\mathbb{K}}; C_0)$ and, when $\mathbb{K} = \mathbb{C}$, their real forms. For part of what follows, the Hilbert space structure on the vector space $\mathcal{H}_{\mathbb{K}}$ is not needed; see for example Stewart [168], section 4.4.

The main references for this chapter are Balachandran [12] and de la Harpe [76], which were devoted to the computations of, respectively, sections 3-4 and sections 5-6. However, the point of view here is quite different, as no topology is introduced on the Lie algebras of this chapter.

I.1.- Classical complex Lie algebras of finite rank operators

In this section, the base field is $\mathbb{C}$ if not otherwise stated.

Let $\underline{sl}(\mathcal{H}; C_o)$ be the complex Lie algebra of finite rank operators with zero trace on the complex Hilbert space $\mathcal{H}$. It is the derived ideal of $\underline{gl}(\mathcal{H}; C_o)$, where it is of codimension 1.

Lemma 1. Any finite subset of $\underline{sl}(\mathcal{H}; C_o)$ is contained in a finite dimensional simple subalgebra of $\underline{sl}(\mathcal{H}; C_o)$.

Proof. Let $\{X_1, \dots, X_n\}$ be a finite subset of $\underline{sl}(\mathcal{H}; C_o)$. For each $j \in \{1, \dots, n\}$, $\ker X_j$ is finite codimensional and $\operatorname{im} X_j$ is finite dimensional in $\mathcal{H}$. Hence there exists a finite dimensional subspace $F$ in $\mathcal{H}$, of dimension at least two, such that $\ker X_j \subset F^{\perp}$ and $\operatorname{im} X_j \subset F$ for all $j \in \{1, \dots, n\}$. Let $\underline{sl}(F)$ be identified with the subalgebra of $\underline{sl}(\mathcal{H}; C_o)$ consisting of those operators on $\mathcal{H}$ which map $F$ into itself and $F^{\perp}$ onto zero. Then $\{X_1, \dots, X_n\}$ is contained in $\underline{sl}(F)$.

The property of the Lie algebra $\underline{sl}(\mathcal{H}; C_o)$ stated in lemma 1 is sometimes called "local simplicity".

Proposition 1A. The Lie algebra $\underline{sl}(\mathcal{H}; C_o)$ is simple.

Proof. Let $\underline{a}$ be a non zero ideal in $\underline{sl}(\mathcal{H}; C_o)$. Suppose that $\underline{a}$ is not trivial, and let $X \in \underline{a}$, $X \neq 0$, $Y \in \underline{sl}(\mathcal{H}; C_o)$, $Y \notin \underline{a}$. According to the lemma, there exists a finite dimensional simple subalgebra of $\underline{sl}(\mathcal{H}; C_o)$, say $\underline{s}$, which contains $X$ and $Y$. Then $\underline{s} \cap \underline{a}$ is a non trivial ideal of $\underline{s}$, which is absurd. Hence $\underline{a} = \underline{sl}(\mathcal{H}; C_o)$. ■

Let $J_{\mathbb{R}}$ be a conjugation on $\mathcal{H}$. Let $\varphi_{\mathbb{R}}$ be the map

$$\begin{cases} C_o(\mathcal{H}) \longrightarrow C_o(\mathcal{H}) \\ X \longmapsto J_{\mathbb{R}} X^* J_{\mathbb{R}} \end{cases}$$

; $\varphi_{\mathbb{R}}$ is an involutive antiautomorphism of the associative algebra $C_o(\mathcal{H})$. The orthogonal complex Lie algebra corresponding to $\underline{sl}(\mathcal{H}; C_o)$ and given by $\varphi_{\mathbb{R}}$ is by definition the Lie algebra

(1) $\underline{o}(\mathcal{H}, J_{\mathbb{R}}; C_o) = \{X \in \underline{sl}(\mathcal{H}; C_o) \mid \varphi_R(X) = -X\}$. If $\mathcal{H}$ is of finite dimension and if $e = (e_1, \ldots, e_n)$ is a $J_{\mathbb{R}}$-basis of type zero in $\mathcal{H}$ (see appendix), operators in $\underline{o}(\mathcal{H}, J_{\mathbb{R}}; C_o)$ are exactly those whose matrix representation with respect to $e$ is skew-symmetric; in this case, the orthogonal complex Lie algebra is denoted usually by $\underline{so}(n, \mathbb{C})$ [84].

Let $J_{\mathbb{Q}}$ be an anticonjugation in $\mathcal{H}$. Let $\varphi_{\mathbb{Q}}$ be the map $X \longmapsto -J_{\mathbb{Q}}X^*J_{\mathbb{Q}}$ ; $\varphi_{\mathbb{Q}}$ is again an involutive antiautomorphism of the associative algebra $C_o(\mathcal{H})$. The <u>symplectic complex Lie algebra</u> corresponding to $\underline{sl}(\mathcal{H}; C_o)$ and given by $\varphi_{\mathbb{Q}}$ is by definition the Lie algebra

(2) $\underline{sp}(\mathcal{H}, J_{\mathbb{Q}}; C_o) = \{X \in \underline{sl}(\mathcal{H}; C_o) \mid \varphi_{\mathbb{Q}}(X) = -X\}$. In the case $\mathcal{H}$ is finite (hence even) dimensional, it is easy to check that $\underline{sp}(\mathcal{H}, J_{\mathbb{Q}}; C_o)$ is the algebra usually denoted by $\underline{sp}(n, \mathbb{C})$.

<u>Proposition 1BC</u>. The Lie algebras $\underline{o}(\mathcal{H}, J_{\mathbb{R}}; C_o)$ and $\underline{sp}(\mathcal{H}, J_{\mathbb{Q}}; C_o)$ are simple.

<u>Proof</u> : as for proposition 1A; in the proof of lemma 1, the space F can now be chosen invariant by $J_{\mathbb{R}}$ or $J_{\mathbb{Q}}$ . ∎

<u>Remarks</u>.

i) Proposition 1 is a particular case of much more general results : it follows from the simplicity of the associative algebra $C_o(\mathcal{H})$, and from known facts about the Lie structure of simple rings; see Herstein [86], principally theorems 4, 8 and 10.

ii) Analogues of proposition 1, for groups of the kind

$$SL(\mathcal{H}; C_o) = \left\{ X \in GL(\mathcal{H}) \;\middle|\; \begin{array}{c} X = \text{identity} + \text{finite rank operator} \\ \text{and} \quad \det(X) = 1 \end{array} \right\}$$

are known in various cases to algebraists; see e.g. [39], [143] and [144].

Several of the notions used later on in this work cannot be defined for arbitrary complex Lie algebras, though they are very useful in numerous examples. The two following definitions will allow us to restrict the class of algebras under consideration.

Definition 1. Let $\underline{g}$ be a Lie algebra over R or C. An involution in $\underline{g}$ is a semi-linear map $X \mapsto X^*$ from $\underline{g}$ to $\underline{g}$ such that $(X^*)^* = X$ for all $X \in \underline{g}$ and such that $[X,Y]^* = [Y^*,X^*]$ for all $X,Y \in \underline{g}$. A Lie algebra furnished with an involution is an involutive Lie algebra. A self-adjoint subset of an involutive algebra is a subset globally invariant by the involution. A normal element in an involutive algebra is an element $X$ such that $[X,X^*] = 0$.

Let $\underline{g}$ be a complex Lie algebra; then $\underline{g}$ admits an involution if and only if $\underline{g}$ has a real form (see e.g. Helgason [84] chap. III §6). There exist finite dimensional complex Lie algebras which do not admit any involution (see Bourbaki [25] §5 exercise 8c).

Definition 2. Let $V$ be a vector space over $\mathbb{R}$ or $\mathbb{C}$ (possibly infinite dimensional). Let $X$ be an endomorphism of $V$ and let $\{X\}^+$ be the associative subalgebra of $\mathrm{End}(V)$ generated by $X$. Then $X$ is said to be semi-simple if $\{X\}^+$ does not contain any non zero nilpotent element.

Let $\underline{g}$ be a Lie algebra over $\mathbb{R}$ or $\mathbb{C}$. An element $X$ in $\underline{g}$ is said to be semi-simple if $\mathrm{ad}(X)$ is semi-simple as endomorphism of $\underline{g}$. A c-involution in $\underline{g}$ is an involution in $\underline{g}$ such that any normal element with respect to it is semi-simple. A Lie algebra furnished with a c-involution is called a c-involutive Lie algebra.

The definition of a semi-simple element coincides with the standard one when $V$ or $\underline{g}$ is finite dimensional.

Example. Let $\underline{g}$ be a semi-simple complex Lie algebra of finite dimension and let $\tau$ be an involution in $\underline{g}$. If $\{X \in \underline{g} \mid \tau(X) = -X\}$ is a compact real form of $\underline{g}$, then $\tau$ is a c-involution.

Indeed, define $\langle\langle \mid \rangle\rangle \begin{cases} \underline{g} \times \underline{g} \longrightarrow \mathbb{C} \\ (X,Y) \mapsto (B(X) \mid \tau(Y)) \end{cases}$, where $B$ is the Killing form of $\underline{g}$. Then $\langle\langle \mid \rangle\rangle$ is a scalar product on $\underline{g}$, and if $X$ is an element of $\underline{g}$, the adjoint of $ad(X)$ (in the Hilbert space sense) is precisely $ad(\tau(X))$. Hence $X$ is normal if and only if $ad(X)$ is a normal operator on the Hilbert space $\underline{g}$, that is if and only if $ad(X)$ is semi-simple.

Example. Let $\underline{g}$ be a semi-simple real Lie algebra of finite dimension and let $\tau$ be an involution in $\underline{g}$; let $k = \{X \in \underline{g} \mid \tau(X) = -X\}$ and $\underline{p} = \{X \in \underline{g} \mid \tau(X) = X\}$. If $\underline{k} \oplus \underline{p}$ is a Cartan decomposition of $\underline{g}$, then $\tau$ is a c-involution on $\underline{g}$.

The proof mimics that of the previous example.

Example. Let $\underline{g}$ be one of the Lie algebras $\underline{sl}(\mathcal{H}; C_o)$, $\underline{o}(\mathcal{H}, J_{\mathbb{R}}; C_o)$ and $\underline{sp}(\mathcal{H}, J_{\mathbb{Q}}; C_o)$, where $\mathcal{H}$ is some complex Hilbert space. Then $\underline{g}$ has a natural involution given by $X \longmapsto X^*$, where $X^*$ is the adjoint of the operator $X$. This involution is clearly a c-involution in $\underline{g}$.

Definition 3. <u>A classical complex Lie algebra of finite rank operators</u> is one of the c-involutive Lie algebras $\underline{sl}(\mathcal{H}; C_o)$, $\underline{o}(\mathcal{H}, J_{\mathbb{R}}; C_o)$ and $\underline{sp}(\mathcal{H}, J_{\mathbb{Q}}; C_o)$, where $\mathcal{H}$ is a complex Hilbert space of arbitrary dimension.

The algebras of definition 3 enlarge somehow the list of the finite dimensional classical complex Lie algebras, and the following sections will show how they share certain properties of these.

## 1.2. - Derivations

In this section, $\mathcal{H}$ is a complex Hilbert space (of infinite dimension). An earlier stage of the results below has been summed up in [80].

Lemma 2. Let $\underline{g}$ be a finite dimensional complex Lie algebra and let $\underline{s}$ be a semi-simple subalgebra of $\underline{g}$. Let $\Delta : \underline{s} \to \underline{g}$ be a derivation. Then there exists $D \in \underline{g}$ such that $\Delta(X) = [D,X]$ for all $X \in \underline{s}$.

N.B. : a derivation from $\underline{s}$ into $\underline{g}$ is, in more sophisticated terms, a skew derivation of type $(j,j)$, where $j$ is the inclusion of $\underline{s}$ in $\underline{g}$ ; see Chevalley [37] chap. I §3.

Proof. Consider $\underline{g}$ as the $\underline{s}$-module defined by $X_{\underline{g}}Z = [X,Z]$ for all $X \in \underline{s}$, $Z \in \underline{g}$. A 1-dimensional $\underline{g}$-cochain on $\underline{s}$ is a linear map $\omega : \underline{s} \to \underline{g}$. It is a cocycle if and only if it is a derivation and it is a coboundary if and only if it is an "inner" derivation, namely if there exists $D \in \underline{g}$ such that $\Delta(X) = [D,X]$ for all $X \in \underline{s}$. Lemma 2 is then a standard and trivial consequence of the first Whitehead lemma; see for example Bourbaki [25] §6 exercice 1a or Jacobson [90] chap. III Lemma 3. ∎

Lemma 3. A. - Let $\Delta$ be a derivation of $\underline{sl}(\mathcal{H} ; C_0)$. For every finite dimensional subspace $E$ of $\mathcal{H}$ of dimension at least 2, there exists an operator $Y_E \in \underline{sl}(\mathcal{H} ; C_0)$ such that

(3) $\Delta(X) = [Y_E,X]$ for all $X \in \underline{sl}(E)$.

Moreover, the restriction of $Y_E$ to $E$ is uniquely defined by these properties, up to the addition of a scalar multiple of the identity.

B. - Let $\Delta$ be a derivation of $\underline{o}(\mathcal{H}, J_{\mathbb{R}}; C_0)$. For every finite dimensional $J_{\mathbb{R}}$-invariant subspace $E$ of $\mathcal{H}$ of dimension at least 5, there exists an operator $Y_E \in \underline{o}(\mathcal{H}, J_{\mathbb{R}}; C_0)$ such that

(4) $\Delta(X) = [Y_E, X]$ for all $X \in \underline{o}(E, J_{\mathbb{R}})$.

Moreover, the restriction of $Y_E$ to $E$ is uniquely defined by these properties.

C. - Let $\Delta$ be a derivation of $\underline{sp}(\mathcal{H}, J_{\mathbb{Q}}; C_o)$. For every finite dimensional $J_{\mathbb{Q}}$-invariant subspace $E$ of $\mathcal{H}$ of dimension at least 2, there exists an operator

$Y_E \in \underline{sp}(\mathcal{H}, J_{\mathbb{Q}}; C_o)$ such that

(5) $\Delta(X) = [Y_E, X]$ for all $X \in \underline{sp}(E, J_{\mathbb{Q}})$.

Moreover, the restriction of $Y_E$ to $E$ is uniquely defined by these properties.

Proof (case A). As the image of $\underline{sl}(E)$ by $\Delta$ is finite dimensional, there exists a finite dimensional subspace $F$ in $\mathcal{H}$, orthogonal to $E$, and such that $\Delta(\underline{sl}(E)) \subset \underline{sl}(E \oplus F)$ (see lemma 1). By lemma 2 and as $\underline{sl}(E)$ is simple as soon as $\dim E \geqslant 2$, there exists $Y_E \in \underline{sl}(E \oplus F)$ such that (3) holds. Let now $Y_E$ and $Y_E'$ be two operators in $\underline{sl}(\mathcal{H}; C_o)$ such that (3) holds for both of them. Let the matrix of $Y_E - Y_E'$ relatively to the decomposition $E \oplus E^{\perp}$ of $\mathcal{H}$ be denoted by

$\begin{pmatrix} a_E & b_E \\ c_E & d_E \end{pmatrix}$. As $\left[ Y_E - Y_E', X \right] = 0$ for all $X \in \underline{sl}(E)$, it follows

that $\begin{cases} [a_E, X] = 0 \\ -Xb_E = 0 \\ c_E X = 0 \end{cases}$ for all $X \in \underline{sl}(E)$. Hence $a_E$ is a scalar multiple of the identity of $E$, and $b_E$ and $c_E$ vanish. This ends the proof of case A.

The cases B and C are proved the same way; note that $\underline{o}(E, J_{\mathbb{R}})$ is simple as soon as $\dim E \geqslant 5$ and that $\underline{sp}(E, J_{\mathbb{Q}})$ is simple as soon as $\dim E \geqslant 2$. ∎

The associative algebra of all linear maps (not necessarily bounded) of $\mathcal{H}$ into itself is denoted by $\mathrm{Lin}(\mathcal{H})$.

Lemma 4. Let $D \in \mathrm{Lin}(\mathcal{H})$; suppose that one of the following holds:

A. - D commutes with all operators in $\underline{sl}(\mathcal{H}\, ; C_o)$.

B. - D commutes with all operators in $\underline{o}(\mathcal{H}, J_{\mathbb{R}}; C_o)$.

C. - D commutes with all operators in $\underline{sp}(\mathcal{H}, J_{\mathbb{Q}}; C_o)$.

Then D is a multiple of the identity of $\mathcal{H}$.

Proof: standard. ∎

Proposition 2.

A. - Let $\Delta$ be a derivation of $\underline{sl}(\mathcal{H}\, ; C_o)$. Then there exists $D \in \mathrm{Lin}(\mathcal{H})$ such that $\Delta(X) = [D,X]$ for all $X \in \underline{sl}(\mathcal{H}\, ; C_o)$. Moreover, D is uniquely defined by these properties, up to addition of a scalar multiple of $\mathrm{id}_{\mathcal{H}}$.

B. - Let $\Delta$ be a derivation of $\underline{o}(\mathcal{H}, J_{\mathbb{R}}; C_o)$. Then there exists $D \in \mathrm{Lin}(\mathcal{H})$ such that $\langle Dx \mid y\rangle = -\langle x \mid J_{\mathbb{R}}DJ_{\mathbb{R}}y\rangle$ for all $x,y \in \mathcal{H}$ and such that $\Delta(X) = [D,X]$ for all $X \in \underline{o}(\mathcal{H}, J_{\mathbb{R}}; C_o)$. Moreover, D is uniquely defined by these properties.

C. - Let $\Delta$ be a derivation of $\underline{sp}(\mathcal{H}, J_{\mathbb{Q}}; C_o)$. Then there exists $D \in \mathrm{Lin}(\mathcal{H})$ such that $\langle Dx \mid y\rangle = +\langle x \mid J_{\mathbb{Q}}DJ_{\mathbb{Q}}y\rangle$ for all $x,y \in \mathcal{H}$ and such that $\Delta(X) = [D,X]$ for all $X \in \underline{sp}(\mathcal{H}, J_{\mathbb{Q}}; C_o)$. Moreover, D is uniquely defined by these properties.

Proof (case A). Let F be an arbitrary 2-dimensional subspace of $\mathcal{H}$ and let f be a non zero vector in F. For any vector $x \in \mathcal{H}$, let E be the space span by F and x and let $Y_E$ be as in lemma 3A, and such that $Y_E f \perp f$ ; such an operator $Y_E$ is uniquely defined (the condition on f is only to get rid of the "up to addition of a multiple of the identity"-uniqueness). Put then $y = Y_E x$. The map D from $\mathcal{H}$ to $\mathcal{H}$ which sends x to y is well defined and fulfills the required conditions. The unicity part of proposition 2 follows trivially from lemma 4. ∎

Let $\underline{gl}(\mathcal{H}\,;\,\mathrm{Lin})$ be the Lie algebra of all linear maps from $\mathcal{H}$ into itself, endowed with the evident product; proposition 2 can then be restated as follows : the Lie algebra of the derivations of $\underline{sl}(\mathcal{H}\,;\,C_o)$ is isomorphic to the quotient of $\underline{gl}(\mathcal{H}\,;\,\mathrm{Lin})$ by its center $\mathbb{C}\mathrm{id}_{\mathcal{H}}$ .

Similarly, the Lie algebra of the derivations of $\underline{o}(\mathcal{H}\,,J_{\mathbb{R}};\,C_o)$ is isomorphic to : $\underline{o}(\mathcal{H},J_{\mathbb{R}};\,\mathrm{Lin}) =$
$= \{D \in \underline{gl}(\mathcal{H}\,;\,\mathrm{Lin}) \mid \langle Dx \mid y\rangle = -\langle x \mid J_{\mathbb{R}}DJ_{\mathbb{R}}y\rangle$ for all $x,y \in \mathcal{H}\}$.
And that of $\underline{sp}(\mathcal{H}\,,J_{\mathbb{Q}};\,C_o)$ to : $\underline{sp}(\mathcal{H}\,,J_{\mathbb{Q}};\,\mathrm{Lin}) =$
$= \{D \in \underline{gl}(\mathcal{H}\,;\,\mathrm{Lin}) \mid \langle Dx \mid y\rangle = +\langle x \mid J_{\mathbb{Q}}DJ_{\mathbb{Q}}y\rangle$ for all $x,y \in \mathcal{H}\}$.

Remarks.

i) Proposition 2A is still true if $\underline{sl}(\mathcal{H}\,;\,C_o)$ is replaced by $\underline{gl}(\mathcal{H}\,;\,C_o)$. In other words, if $\Delta$ is a derivation of $\underline{gl}(\mathcal{H}\,;\,C_o)$, then $\Delta$ is also a derivation of the associative simple algebra $C_o(\mathcal{H})$.

ii) Proposition 2A can alternatively be deduced from general results, due to Martindale [117], about Lie derivations of certain associative rings (see his theorem 2 with $R = C_o(\mathcal{H})$ and $\bar{R} = \mathrm{Lin}(\mathcal{H})$ ).

iii) The people who live in $\underline{o}(\mathcal{H}\,,J_{\mathbb{R}};\,\mathrm{Lin})$ and $\underline{sp}(\mathcal{H}\,,J_{\mathbb{Q}};\,\mathrm{Lin})$ are in fact continuous (because they have adjoints; [45], 12. 16. 7). It is not necessarily the case for those in $\underline{gl}(\mathcal{H}\,;\,\mathrm{Lin})$.

I.3. - Cartan subalgebras

In order to make sense, the definition of Cartan subalgebra given below requires a definition of semi-simplicity for infinite dimensional Lie algebra. (According to tradition, the word "semi-simplicity" is confusingly applied to both Lie algebras and elements inside a Lie algebra.)

For this chapter, a Lie algebra is defined to be semi-simple if it has no non trivial abelian ideals. A critic of this definition will be given in an appendix to chapter II. But certainly, any other reasonable definition of semi-simplicity will imply the absence of non trivial abelian ideals, so that the matter of this chapter will not have to be changed in any respect whatsoever.

Definition 4. Let $\underline{g}$ be a semi-simple c-involutive Lie algebra over $\mathbb{K}$. A Cartan subalgebra of $\underline{g}$ is a subalgebra $\underline{h}$ of $\underline{g}$ which is maximal among the abelian self-adjoint subalgebras of $\underline{g}$.

In particular, a Cartan subalgebra of $\underline{g}$ is maximal among the abelian subalgebras of $\underline{g}$, and all its elements are semi-simple. When $\underline{g}$ is finite dimensional, it follows that Cartan subalgebras of $\underline{g}$ according to definition 4 are Cartan subalgebras of $\underline{g}$ in the usual sense.

Conversely, let $\underline{g}$ be a finite dimensional semi-simple complex Lie algebra and let $\underline{h}$ be a Cartan subalgebra of $\underline{g}$ in the usual sense. Then there exists a c-involution in $\underline{g}$ for which $\underline{h}$ is invariant ([156], exposé 11, théorèmes 2 and 3).

Let $\underline{g}$ be a finite dimensional semi-simple real Lie algebra and let $\underline{h}$ be a Cartan subalgebra of $\underline{g}$ in the usual sense. If $\underline{h}$ is standard (Kostant [99]), there is evidently a c-involution in $\underline{g}$ for which $\underline{h}$ is invariant. (Is this still true when $\underline{h}$ is not assumed to be standard?)

The purpose of the present section is to determine the Cartan subalgebras of the classical complex Lie algebras of finite rank operators in a (complex) Hilbert space $\mathcal{H}$. The analysis follows an argument devised by Balachandran [12] for the study of certain $L^*$-algebras. For simplicity in the notations, $\mathcal{H}$ is supposed to be <u>infinite dimensional and separable</u> in the rest of this section.

<u>Proposition 3A</u>. Let $\underline{h}$ be a Cartan subalgebra in $\underline{sl}(\mathcal{H} ; C_o)$. Then there exists an orthonormal **basis** $e = (e_n)_{n \in N}$ of $\mathcal{H}$ such that $\underline{h}$ consists of those operators in $\underline{sl}(\mathcal{H} ; C_o)$ which are diagonal with respect to $e$. In particular, two Cartan subalgebras of $\underline{sl}(\mathcal{H} ; C_o)$ are conjugated by an element of the full unitary group $U(\mathcal{H})$.

<u>Proof</u>: immediate via the spectral theorem. ∎

A basis such as $e$ in proposition 4A is said to be <u>compatible</u> with $\underline{h}$.

<u>Proposition 3B</u>. Let $\underline{h}$ be a Cartan subalgebra in $\underline{g} = \underline{o}(\mathcal{H}, J_{\mathbb{R}}; C_o)$. Then :

Either there exists a $J_{\mathbb{R}}$-basis of type 1 in $\mathcal{H}$ (see definition in the appendix), say $f = (f_n)_{n \in Z}$, such that $\underline{h}$ consists of those operators in $\underline{g}$ which are diagonal with respect to $f$.

Or there exists a $J_{\mathbb{R}}$-basis of type 2 in $\mathcal{H}$, say $g = (g_n)_{n \in Z^*}$, such that $\underline{h}$ consists of those operators in $\underline{g}$ which are diagonal with respect to $g$.

The two cases exclude each other.

In particular, there are two conjugacy classes of Cartan

subalgebras in $\underline{g}$ under the action of the group
$O(\mathcal{H}_{\mathbb{R}}) = \{X \in U(\mathcal{H}) \mid XJ_{\mathbb{R}} = J_{\mathbb{R}}X\}$ .

Lemma 5B. Let $\underline{g}$ be as in proposition 3B and let $x$ be a vector of norm 1 in $\mathcal{H}$ such that $y = J_{\mathbb{R}}x$ is orthogonal to $x$. Then the self-adjoint operator $T = x \otimes \bar{x} - y \otimes \bar{y}$ belongs to $\underline{g}$. Furthermore, if $e = (e_n)_{n \in N}$ is any orthonormal basis of $\mathcal{H}$ containing $x$, then $T$ commutes with all the operators in $\underline{g}$ which are diagonal with respect to $e$.

Proof of lemma 5B.

Let $z \in \mathcal{H}$ , $z$ orthogonal to both $x$ and $y$; then $J_{\mathbb{R}}z$ is orthogonal to both $x$ and $y$ and $(TJ_{\mathbb{R}} + J_{\mathbb{R}}T^*)z = 0$. Trivially, $(TJ_{\mathbb{R}} + J_{\mathbb{R}}T^*)x = 0$ and $(TJ_{\mathbb{R}} + J_{\mathbb{R}}T^*)y = 0$. Hence $T \in \underline{g}$ .

Let now $e$ be as in the lemma, with $e_o = x$, and let $X$ be an operator in $\underline{g}$ which is diagonal with respect to $e$. Suppose first that $X$ is self-adjoint and let $(\xi_n)_{n \in N}$ be the sequence of real numbers such that $Xe_n = \xi_n e_n$ for all $n \in N$. Now

$XJ_{\mathbb{R}}x = X \sum \langle J_{\mathbb{R}}x \mid e_n \rangle e_n = \sum \xi_n \langle y \mid e_n \rangle e_n$ and

$J_{\mathbb{R}}Xx = \xi_o J_R x = \xi_o \sum \langle y \mid e_n \rangle e_n$ ; hence $\xi_n = - \xi_o$ whenever $\langle y \mid e_n \rangle \neq 0$.

Let $\mathcal{H}_I$ [resp. $\mathcal{H}_{II}$] be the closed span of those $e_n$ for which $\langle y \mid e_n \rangle \neq 0$ [resp. $\langle y \mid e_n \rangle = 0$]. As $\mathcal{H}_I$ is invariant by $T$ and as $X|\mathcal{H}_I$ is a scalar operator, $X|\mathcal{H}_I$ and $T|\mathcal{H}_I$ commute. If $z = \sum z_n e_n \in \mathcal{H}_{II}$ , then $XTz = Xz_o e_o = \xi_o z_o e_o$ and $TXz = \sum \xi_n z_n T(e_n) = \xi_o z_o e_o$ , so that $X|\mathcal{H}_{II}$ and $T|\mathcal{H}_{II}$ commute. Hence $X$ commutes with $T$.

The general case follows clearly from the case where $X$ is self-adjoint. ∎

Proof of proposition 3B.

Let $\underline{h}$ be a Cartan subalgebra in $\underline{g}$. From the spectral theorem, it follows that there exists an orthonormal basis $e = (e_n)_{n \in N}$ in $\mathcal{H}$ such that the operators of $\underline{h}$ are diagonal with respect to $e$. Let $e_I = \{e_n \in e \mid Xe_n = 0$ for all $X \in \underline{h}\}$ and let $e_{II}$ be the set-complement of $e_I$ in $e$; let $\mathcal{H}_I$ [resp. $\mathcal{H}_{II}$] be the closed span of $e_I$ [resp. $e_{II}$] .

Step one : $J_{\mathbb{R}} \mathcal{H}_I = \mathcal{H}_I$ and $J_{\mathbb{R}} \mathcal{H}_{II} = \mathcal{H}_{II}$ .

Let $e_k$ be a vector in $e_I$ and let $X$ be a self-adjoint operator in $\underline{h}$ represented by its diagonal matrix $(\xi_n)_{n \in N}$ . Then $0 = J_{\mathbb{R}}Xe_k = - \sum \langle J_{\mathbb{R}}e_k \mid e_n \rangle \xi_n e_n$ , so that $\xi_n = 0$ if $\langle J_{\mathbb{R}}e_k \mid e_n \rangle \neq 0$ ; hence $J_{\mathbb{R}}e_k = \sum_n \langle J_{\mathbb{R}}e_k \mid e_n \rangle e_n$ where the last sum is only over those $n$ such that $\xi_n = 0$ for all self-adjoint $X \in \underline{h}$ . Hence $J_{\mathbb{R}}e_k \in \mathcal{H}_I$ , that is $J_{\mathbb{R}} \mathcal{H}_I \subset \mathcal{H}_I$ .

Let now $e_\ell$ be a vector in $e_{II}$ and let $X$ be a self-adjoint operator of $\underline{h}$ represented by a diagonal matrix $(\xi_n)_{n \in N}$ with $\xi_\ell \neq 0$. As in the proof of lemma 5B, the equality $XJ_{\mathbb{R}}e = - J_{\mathbb{R}}Xe$ implies that $\xi_n = - \xi_\ell$ whenever $\langle J_{\mathbb{R}}e_\ell \mid e_n \rangle \neq 0$. Hence $J_{\mathbb{R}}e_\ell \in \mathcal{H}_{II}$, that is $J_{\mathbb{R}} \mathcal{H}_{II} \subset \mathcal{H}_{II}$ .

As $J_{\mathbb{R}}^2 = id_{\mathcal{H}}$ , the claim follows. As by product of the proof, one obtains that $J_{\mathbb{R}}e_\ell \perp e_\ell$ for all $e_\ell \in e_{II}$ .

Step two : $\dim \mathcal{H}_I \leqslant 1$.

Suppose that $\dim \mathcal{H}_I \geqslant 2$ ; then there exists a non zero vector $x \in \mathcal{H}_I$ such that $y = J_{\mathbb{R}}x$ is orthogonal to $x$ . The operator $y \otimes \bar{x} - x \otimes \bar{y}$ is a skew-adjoint element of $\underline{g}$ which does not belong to $\underline{h}$ and which commutes with every element of $\underline{h}$ . As this is absurd by definition of $\underline{h}$ , the claim follows.

Step three : the case $\dim \mathcal{H}_I = 1$ .

Assume the base has been re-indexed such that $e = (f_n)_{n \in Z}$ with $\mathcal{H}_I = \mathbf{C}f_o$ , and suppose moreover that $J_{\mathbb{R}}f_o = f_o$ . Let $n \in Z$ and let $T = f_n \otimes \overline{f_n} - J_{\mathbb{R}}f_n \otimes \overline{(J_{\mathbb{R}}f_n)}$ . Suppose first that $T \neq 0$ ;

then $n \neq 0$, $J_{\mathbb{R}} f_n$ is orthogonal to $f_n$, and lemma 5B applies. By maximality of Cartan subalgebras, $T \in \underline{h}$, hence $J_{\mathbb{R}} f_n \otimes \overline{(J_{\mathbb{R}} f_n)}$ is diagonal with respect to e, which implies that $J_{\mathbb{R}} f_n$ is parallel to $f_m$ for some $m \in Z^*$, $m \neq n$. Suppose now that $T = 0$; then obviously $n = 0$. Modulo a second re-indexing of the vectors in e and multiplication of each of them by ad hoc constants, $e = (f_n)_{n \in Z}$ can be made a $J_{\mathbb{R}}$-basis of type I.

Step four: the case $\dim \mathcal{H}_I = 0$. The same argument shows how to obtain from e a $J_{\mathbb{R}}$-basis of type II.

Finally, suppose there exists a Cartan subalgebra $\underline{h}$ in $\underline{g}$, a $J_{\mathbb{R}}$-basis $f = (f_n)_{n \in Z}$ of type one and a $J_{\mathbb{R}}$-basis $g = (g_n)_{n \in Z^*}$ of type two in $\mathcal{H}$, such that the elements of $\underline{h}$ are diagonal with respect to both e and f. Choose $g_k \in g$ such that $\langle f_o \mid g_k \rangle \neq 0$ and let $T = g_k \otimes \overline{g_k} - g_{-k} \otimes \overline{g_{-k}}$. Then $0 = Tf_o = \sum \langle f_o \mid g_n \rangle T(g_n) = \langle f_o \mid g_k \rangle g_k - \langle f_o \mid g_{-k} \rangle g_{-k}$, which is absurd. Proposition 3B follows. ∎

Definition 5. A Cartan subalgebra $\underline{h}$ in $\underline{g} = \underline{o}(\mathcal{H}, J_{\mathbb{R}}; C_o)$ is said to be of type one [resp. of type two] if it consists of all operators of $\underline{g}$ which are diagonal with respect to some $J_{\mathbb{R}}$-basis of type one [resp. of type two] in $\mathcal{H}$. Such a basis is said to be compatible with $\underline{h}$.

Proposition 3B implies that two Cartan subalgebras of $\underline{g}$ are of the same type if and only if they are conjugated by an element of $U(\mathcal{H})$, if and only if they are conjugated by an element of $O(\mathcal{H}_{\mathbb{R}})$.

Proposition 3C. Let $\underline{h}$ be a Cartan subalgebra in $g = \underline{sp}(\mathcal{H}, J_{\mathbb{Q}}; C_o)$. Then there exists a $J_{\mathbb{Q}}$-basis in $\mathcal{H}$, say $e = (e_n)_{n \in Z^*}$, such that $\underline{h}$ consists of those operators in $\underline{g}$ which are diagonal with respect to e. In particular, two Cartan subalgebras of $\underline{g}$ are conjugated by an element of the group $Sp(\mathcal{H}_{\mathbb{Q}}) = \{X \in U(\mathcal{H}) \mid XJ_{\mathbb{Q}} = J_{\mathbb{Q}}X\}$.

<u>Lemma 5C</u>. Let $\underline{g}$ be as in proposition 3C, let $x$ be a vector of norm 1 in $\mathcal{H}$ and let $y = J_{\mathbb{Q}}x$ ($y$ is automatically orthogonal to $x$). Then the self-adjoint operator $T = x \otimes \bar{x} - y \otimes \bar{y}$ belongs to $\underline{g}$. Furthermore, if $e = (e_n)_{n \in N}$ is any orthonormal basis of $\mathcal{H}$ containing $x$, then $T$ commutes with all the operators in $\underline{g}$ which are diagonal with respect to $e$.

Proofs of lemma 5C and of proposition 3C: as for lemma 5B and proposition 3B. ∎

Again, a basis such as $e$ in proposition 4C is said to be <u>compatible</u> with $\underline{h}$.

<u>Corollary to proposition 3</u> : Let $\underline{h}$ be a Cartan subalgebra of a classical complex Lie algebra of finite rank operators. Then $\underline{h}$ is equal to its normalizer.

<u>Proof</u> : immediate. ∎

<u>Remark</u>. Let $\underline{g}$ be one of the infinite dimensional Lie algebras $\underline{sl}(\mathcal{H}; C_o)$, $\underline{o}(\mathcal{H}, J_{\mathbb{R}}; C_o)$ and $\underline{sp}(\mathcal{H}, J_{\mathbb{Q}}; C_o)$. Let $\underline{h}$ be a Cartan subalgebra of $\underline{g}$ and let $H$ be an element of $\underline{h}$. It follows trivially from proposition 3 that the normalizer $N(H) = \{X \in \underline{g} \mid [H,X] = 0\}$ is always <u>strictly</u> larger than $\underline{h}$. In particular, $\underline{h}$ contains <u>no regular elements</u>, unlike to what happens either in the finite dimensional case or in the separable $L^*$-case [9].

## I.4. - Roots

Let $\underline{g}$ be a semi-simple c-involutive complex Lie algebra and let $\underline{h}$ be a Cartan subalgebra of $\underline{g}$. For any linear functional $\alpha : \underline{h} \to \mathbb{C}$, write $\underline{g}_\alpha = \{X \in \underline{g} \mid [H,X] = \alpha(H)X$ for all $H \in \underline{h}\}$. Then $\alpha$ is called a root of $\underline{g}$ with respect to $\underline{h}$, or simply a root, if $\underline{g}_\alpha \neq 0$; if $\alpha$ is a root, $\underline{g}_\alpha$ is its root space and any non zero vector in $\underline{g}_\alpha$ is a root vector of $\alpha$. The zero functional is clearly a root, and $\underline{g}_0 = \underline{h}$. The set of nonzero roots of $\underline{g}$ with respect to $\underline{h}$ will be denoted by $\mathcal{R}$.

Lemma 6. Let $\underline{g}$, $\underline{h}$ and $\mathcal{R}$ be as above. Let $\mathcal{R}' = (\alpha_\iota)_{\iota \in I}$ be a subset of $\mathcal{R}$ and, for each $\iota \in I$, let $X_\iota$ be a root vector of $\alpha_\iota$. Suppose that the following conditions are satisfied :

i) $\alpha_\iota(H^*) = \overline{\alpha_\iota(H)}$ for all $H \in \underline{h}$, for all $\iota \in I$;

ii) there exists a subspace $\underline{m} \subset \underline{g}$ such that $\underline{g} = \underline{h} \oplus \underline{m}$, such that $[\underline{h},\underline{m}] \subset \underline{m}$ and such that $X_\iota \in \underline{m}$ for all $\iota \in I$;

iii) there exists a sesquilinear form $\langle\langle \mid \rangle\rangle : \underline{m} \times \underline{m} \to \mathbb{C}$ such that $\langle\langle [H,X] \mid Y \rangle\rangle = \langle\langle X \mid [H^*,Y] \rangle\rangle$ for all $H \in \underline{h}$ and $X,Y \in \underline{m}$ ;

iv) the set $(X_\iota)_{\iota \in I}$ is total in $\underline{m}$ with respect to $\langle\langle \mid \rangle\rangle$, that is $X \in \underline{m}$ and $\langle\langle X \mid X_\iota \rangle\rangle = 0$ for all $\iota \in I$ implies $X = 0$.

Then $\mathcal{R}' = \mathcal{R}$.

Proof. Let $\beta$ be a non zero root of $\underline{g}$ with respect to $\underline{h}$. By definition of a root, there exists $X \in \underline{g} - \{0\}$ such that $[H,X] = \beta(h)X$ for all $H \in \underline{h}$. As $\underline{g} = \underline{h} \oplus \underline{m}$ and as $[\underline{h},\underline{m}] \subset \underline{m}$, $X$ lies in $\underline{m}$. Then, for all $\iota \in I$ and for all $H \in \underline{h}$ :

$\beta(H) \langle\langle X \mid X_\iota \rangle\rangle = \langle\langle [H,X] \mid X_\iota \rangle\rangle = \langle\langle X \mid [H^*,X_\iota] \rangle\rangle = \alpha_\iota(H) \langle\langle X \mid X_\iota \rangle\rangle$.

As $X \neq 0$, there exists $j \in I$ such that $\langle\langle X \mid X_j \rangle\rangle \neq 0$. Hence

$\beta(H) = \alpha_j(H)$ for all $H \in \underline{h}$, that is $\beta = \alpha_j$, and $\beta \in \mathcal{R}'$ . ■

For simplicity in the notations again, $\mathcal{H}$ is supposed to be an <u>infinite dimensional separable complex</u> Hilbert space in the rest of this section. When $\mathcal{H}$ is furnished with an orthonormal basis $e = (e_n)_{n \in N}$, and if $i$ and $j$ are two natural integers, $E_{i,j}$ is the operator on $\mathcal{H}$ whose matrix representation with respect to $e$ is given by $(E_{i,j})_{m,n} = \delta_{i,m}\,\delta_{j,n}$ for all $m,n \in N$.

<u>Proposition 4A</u>. Let $\underline{g} = \underline{sl}(\mathcal{H}\,;\,C_o)$, let $\underline{h}$ be a Cartan subalgebra of $\underline{g}$ and let $e = (e_n)_{n \in N}$ be a basis of $\mathcal{H}$ compatible with $\underline{h}$. For each $i \in N$, let $\lambda_i : \underline{h} \longrightarrow \mathbb{C}$ be defined by $\lambda_i(H) = \mathrm{trace}(HE_{i,i})$. Then $\mathcal{R} = \{\lambda_i - \lambda_j \in \underline{h}^{\mathrm{dual}} \mid i,j \in N,\ i \neq j\}$ .
All root spaces corresponding to non zero roots are of dimension one, and they are given by

(6) $\quad [H, E_{i,j}] = (\lambda_i - \lambda_j)(H)\, E_{i,j}$ for all $H \in \underline{h}$ .

<u>Proof</u>. The verification of (6) is elementary. The fact that there are no other roots than those written above follows from lemma 6 where $\underline{m}$ can be chosen as the space of all operators in $\underline{sl}(\mathcal{H}\,;\,C_o)$ whose matrix representations with respect to $e$ have no diagonal terms, and where $\langle\langle\ |\ \rangle\rangle$ can be defined by $\langle\langle X \mid Y \rangle\rangle = \mathrm{trace}(XY^*)$ . ■

In the case of proposition 4A, the <u>c-involutive Lie algebra having a Cartan decomposition associated to $\underline{g}$ and $\underline{h}$</u> can be defined to be the derived algebra of the Lie algebra $\underline{h} \oplus (\oplus_{\alpha \in \mathcal{R}} \underline{g}_{\alpha})$; it is clearly equal to $\underline{sl}(\infty, \mathbb{C}) = \bigcup_{n \in N} \underline{sl}(n, \mathbb{C})$, where $\underline{sl}(n, \mathbb{C})$ is the subalgebra of $\underline{sl}(\mathcal{H}\,;\,C_o)$ consisting of those operators which map the span of $(e_o, e_1, \ldots, e_{n-1})$ into itself and its orthogonal complement in $\mathcal{H}$ onto zero. About the relationship between these considerations and $L^*$-algebras, see the project at the end of section 1.4.

Proposition 4B(type one). Let $\underline{g} = \underline{o}(\mathcal{H}, J_{\mathbb{R}}; C_o)$, let $\underline{h}$ be a Cartan subalgebra of $\underline{g}$ of type one, and let $e = (e_n)_{n \in Z}$ be a $J_{\mathbb{R}}$-basis compatible with $\underline{h}$. For each $i \in N$, let $\varepsilon_i = \frac{1}{2}(E_{i,i} - E_{-i,-i})$ and let $\lambda_i : \underline{h} \to \mathbb{C}$ be defined by $\lambda_i(H) = \text{trace}(H\varepsilon_i)$. Then

$$\begin{aligned} \mathcal{R} = & \{\lambda_i - \lambda_j \in \underline{h}^{\text{dual}} \mid i,j \in N^*,\ i \neq j\} \cup \\ & \{\pm(\lambda_i + \lambda_j) \in \underline{h}^{\text{dual}} \mid i,j \in N^*,\ i < j\} \cup \\ & \{\pm \lambda_i \in \underline{h}^{\text{dual}} \mid i \in N^*\} . \end{aligned}$$

All root spaces corresponding to non zero roots are of dimension one, and they are given by

$$\begin{array}{llll}
(7) & [H, E_{i,j} - E_{-j,-i}] = & (\lambda_i - \lambda_j)(H) & (E_{i,j} - E_{-j,-i}) \\
(8) & [H, E_{j,-i} - E_{i,-j}] = & (\lambda_i + \lambda_j)(H) & (E_{j,-i} - E_{i,-j}) \\
(9) & [H, E_{-i,j} - E_{-j,i}] = & -(\lambda_i + \lambda_j)(H) & (E_{-i,j} - E_{-j,i}) \\
(10) & [H, E_{j,o} - E_{o,-j}] = & \lambda_j(H) & (E_{j,o} - E_{o,-j}) \\
(11) & [H, E_{o,j} - E_{-j,o}] = & -\lambda_j(H) & (E_{o,j} - E_{-j,o})
\end{array}$$

for all $H \in \underline{h}$ .

Proof : classical. The c-involutive Lie algebra having a Cartan decomposition associated to $\underline{g}$ and $\underline{h}$ is clearly $\underline{so}(\infty, \mathbb{C})$ . ∎

Proposition 4B(type two). Let $\underline{g} = \underline{o}(\mathcal{H}, J_{\mathbb{R}}; C_o)$ , let $\underline{h}$ be a Cartan subalgebra of $\underline{g}$ of type two, and let $e = (e_n)_{n \in Z^*}$ be a $J_{\mathbb{R}}$-basis compatible with $\underline{h}$. For each $i \in N^*$, let $\varepsilon_i$ and $\lambda_i$ be defined formally as in proposition 4B(type one). Then

$$\begin{aligned} \mathcal{R} = & \{\lambda_i - \lambda_j \in \underline{h}^{\text{dual}} \mid i,j \in N^*,\ i \neq j\} \cup \\ & \{\pm(\lambda_i + \lambda_j)\ \underline{h}^{\text{dual}} \mid i,j \in N^*,\ i < j\} \cup \end{aligned}$$

All roots spaces corresponding to non zero roots are of dimension one, and they are given by

$$(12) \qquad [\, H,\, E_{i,j} - E_{-j,-i}] = (\lambda_i - \lambda_j)(H) \quad (E_{i,j} - E_{-j,-i})$$

$$(13) \qquad [\, H,\, E_{j,-i} - E_{i,-j}] = (\lambda_i + \lambda_j)(H) \quad (E_{j,-i} - E_{i,-j})$$

$$(14) \qquad [\, H,\, E_{-i,j} - E_{-j,i}] = -(\lambda_i + \lambda_j)(H) \quad (E_{-i,j} - E_{-j,i})$$

for all $H \in \underline{h}$ .

Proof : as for proposition 4B(type one). ∎

Proposition 4C. Let $\underline{g} = \underline{sp}(\mathcal{H}, J_{\mathbb{Q}}; C_o)$, let $\underline{h}$ be a Cartan subalgebra of $\underline{g}$ and let $e = (e_n)_{n \in Z^*}$ be a $J_{\mathbb{Q}}$-basis compatible with $\underline{h}$. For each $i \in N^*$ , let $\varepsilon_i$ and $\lambda_i$ be defined formally as in proposition 4B(type one). Then

$$= \{\lambda_i - \lambda_j \in \underline{h}^{dual} \mid i,j \in N^* ,\ i \neq j\} \cup$$
$$\{\pm(\lambda_i + \lambda_j)\ \underline{h}^{dual} \mid i,j \in n^* ,\ i < j\} \cup$$
$$\{\ \pm 2\, \lambda_i\ \ \underline{h}^{dual} \mid i \in N^*\} .$$

All root spaces corresponding to non zero roots are of dimension one, and they are given by

$$(15) \qquad [\, H,\, E_{i,j} - E_{-j,-i}] = (\lambda_i - \lambda_j)(H) \quad (E_{i,j} - E_{-j,-i})$$

$$(16) \qquad [\, H,\, E_{j,-i} + E_{i,-j}] = (\lambda_i + \lambda_j)(H) \quad (E_{j,-i} + E_{i,-j})$$

$$(17) \qquad [\, H,\, E_{-i,j} + E_{j,-i}] = -(\lambda_i + \lambda_j)(H) \quad (E_{-i,j} + E_{j,-i})$$

$$(18) \qquad [\, H\, ,\, E_{i,-i}] = 2\lambda_i(H)\, E_{i,-i}$$

$$(19) \qquad [\, H\, ,\, E_{-i,i}] = -2\lambda_i(H)\, E_{-i,i}$$

Proof : classical. The c-involutive Lie algebra having a Cartan decomposition associated to $\underline{g}$ and $\underline{h}$ is clearly $\underline{sp}(\infty , \mathbb{C})$. ∎

Let $\underline{g}$ be a classical complex Lie algebra of finite rank operators, let $\underline{h}$ be a Cartan subalgebra of $\underline{g}$ and let $\mathcal{R}$ be the set of non zero roots of $\underline{g}$ with respect to $\underline{h}$. Proposition 4 shows in particular that $\mathfrak{R}$ behaves much as for finite dimensional classical algebras. Namely :

i) The set $\mathfrak{R}$ is <u>reduced</u> : if $\alpha \in \mathfrak{R}$ , the only roots proportional to $\alpha$ are $0$ and $\pm\alpha$ .

ii) Let $\alpha,\beta$ be two roots in $\mathfrak{R}$; then there exists two positive integers $p$ and $q$ such that, when $j \in Z$, $\beta + j\alpha$ is a root if and only if $-q \leqslant j \leqslant p$. The set $\{\gamma \in \mathfrak{R} \cup \{0\} \mid \gamma = \beta + j\alpha$ for some $j \in Z\}$ is called the <u>$\alpha$-chain of roots defined by $\beta$</u>, and $q + p$ is the <u>length of the chain</u>. The rational integer $q - p$ will be denoted by $n(\beta,\alpha)$. The quotient $n(\alpha,\beta)/n(\beta,\alpha)$, if it is defined, will be called the <u>ratio of $\alpha$ and $\beta$</u>, and will be denoted by $r(\alpha,\beta)$. Given $\alpha \in \mathfrak{R}$, the maximum of the numbers $\sqrt{r(\alpha,\beta)}$ when $\beta$ runs over all roots in $\mathfrak{R}$ which make them defined will be called the <u>length of the root $\alpha$</u>.

iii) Let $\alpha,\beta$ be two roots in $\mathfrak{R}$. From the definition of $n(\beta,\alpha)$ given above, it is clear that $\beta - n(\beta,\alpha)\alpha$ is again a non zero root. The <u>symmetry of root $\alpha$</u> is by definition the map

$$s_\alpha \begin{cases} \mathfrak{R} \longrightarrow \mathfrak{R} \\ \beta \longmapsto \beta - n(\beta,\alpha)\alpha \ ; \end{cases}$$

it is never the identity (because $n(\alpha,\alpha) = 2$) and $s_\alpha^{\ 2} = id_{\mathfrak{R}}$ (because $n(\beta-j\alpha,\ \alpha) = n(\beta,\alpha) - 2j$). The group of permutations of $\mathfrak{R}$ generated by these symmetries is the <u>Weyl group of $\underline{g}$ with respect to $\underline{h}$</u> .

Those complex c-involutive Lie algebras for which the above definitions make sense will be called admissible. Finite dimensional semi-simple complex Lie algebras are all admissible (see Serre [157] chap. V and VI, and/or Bourbaki [26] chap. VI §1, in particular proposition 9).

<u>Definition 6</u>. A semi-simple c-involutive complex Lie algebra $\underline{g}$ is said to be <u>admissible</u> if the following holds.

Let $\underline{h}$ be any Cartan subalgebra of $\underline{g}$ and let $\mathfrak{R}$ be the set of non zero roots of $\underline{g}$ with respect to $\underline{h}$. Then :

i) $\mathcal{R}$ is reduced.

ii) Let $\alpha, \beta \in \mathcal{R}$ ; then there exists $p, q \in N$ such that, when $j \in Z$, $\beta + j\alpha \in \mathcal{R} \cup \{0\}$ if and only if $-q \leq j \leq p$.

iii) The length of each non zero root is finite.

iv) If the ratio $r(\alpha, \beta)$ as above is defined, then $\sqrt{r(\alpha, \beta)}$ is equal to the quotient of the length of $\alpha$ by that of $\beta$.

In an admissible Lie algebra, the ratio of any two roots can now be defined in the obvious way. In our standard examples, proposition 5 gives the values of the quantities defined above.

Proposition 5.

A. - Let $\underline{g} = \underline{sl}(\mathcal{H}; C_o)$, let $\underline{h}$ be a Cartan subalgebra of $\underline{g}$ and let $\mathcal{R}$ be the set of non zero roots of $\underline{g}$ with respect to $\underline{h}$. Let $\alpha$ and $\beta$ be two roots in $\mathcal{R}$ which are not proportional. Then :

i) $n(\beta, \alpha) \in \{-1, 0, +1\}$ ;

ii) $n(\beta, \alpha) = 0$ if and only if the length of the $\alpha$-chain of roots defined by $\beta$ is zero ;

iii) $r(\alpha, \beta)$ is always equal to $+1$.

B(type one). - Let $\underline{g} = \underline{o}(\mathcal{H}, J_R; C_o)$, let $\underline{h}$ be a Cartan subalgebra of $\underline{g}$ of type one and let $\mathcal{R}$ be the set of non zero roots of $\underline{g}$ with respect to h. Let $\alpha$ and $\beta$ be two roots in which are not proportional. Then :

i) $n(\beta, \alpha) \in \{-2, -1, 0, +1, +2\}$ and the five values do occur ;

ii) $\alpha$ and $\beta$ can be chosen such that $n(\beta, \alpha) = 0$ and such that the length of the $\alpha$-chain of roots defined by $\beta$ is two ;

iii) $r(\alpha, \beta) \in \{\frac{1}{2}, 1, 2\}$ and the three values do occur.

B(type two). – Let $\underline{g} = \underline{o}(\mathcal{H}, J_{\mathbb{R}}; C_o)$, let $\underline{h}$ be a Cartan subalgebra of $\underline{g}$ of type two and let $\mathcal{R}$ be the set of non zero roots of $\underline{g}$ with respect to $\underline{h}$. Let $\alpha$ and $\beta$ be two roots in $\mathcal{R}$ which are not proportional. Then :

i) $n(\beta, \alpha) \in \{-1, 0, +1\}$ ;

ii) $n(\beta, \alpha) = 0$ if and only if the length of the $\alpha$-chain of roots defined by $\beta$ is zero.

iii) $r(\alpha, \beta)$ is always equal to $+1$.

C. – Let $\underline{g} = \underline{sp}(\mathcal{H}, J_{\mathbb{Q}}; C_o)$, let $\underline{h}$ be a Cartan subalgebra of $\underline{g}$ and let $\mathcal{R}$ be the set of non zero roots of $\underline{g}$ with respect to $\underline{h}$. Let $\alpha$ and $\beta$ be two roots in $\mathcal{R}$ which are not proportional. Then :

i) $n(\beta, \alpha) \in \{-2, -1, 0, 1, 2\}$ and the five values do occur ;

ii) $n(\beta, \alpha) = 0$ if and only if the length of the $\alpha$-chain of roots defined by $\beta$ is zero ;

iii) $r(\alpha, \beta) \in \{\frac{1}{2}, 1, 2\}$ and the three values do occur.

Proof : this follows from proposition 4 by a tedious but elementary inspection. ■

Let $\underline{g_1}$ and $\underline{g_2}$ be two semi-simple c-involutive complex Lie algebras, let $\underline{h_1}$ be a Cartan subalgebra of $\underline{g_1}$ and let $\mathcal{R}_1$ be the set of non zero roots of $\underline{g_1}$ with respect to $\underline{h_1}$. Let $\varphi : \underline{g_1} \longrightarrow \underline{g_2}$ be an isomorphism for the structures of involutive complex Lie algebras, let $\underline{h_2} = \varphi(\underline{h_1})$ and let $\mathcal{R}_2 = \{\beta \in \underline{h_2}^{dual} \mid \beta = \alpha.\varphi^{-1}$ for some $\alpha \in R_1\}$. It is clear that $\underline{h_2}$ is a Cartan subalgebra of $\underline{g_2}$ and that $\mathcal{R}_2$ is the set of non zero roots of $\underline{g_2}$ with respect to $\underline{h_2}$ The map from $\mathcal{R}_1$ to $\mathcal{R}_2$ which sends $\alpha$ to $\alpha.\varphi^{-1}$ associates chains of roots to chains of roots ; moreover, $n(\beta \circ \varphi^{-1}, \alpha \circ \varphi^{-1}) = n(\beta, \alpha)$ for all $\alpha, \beta \in \mathcal{R}_1$. In particular, the following proposition is a straightforward corollary of proposition 5.

Proposition 6.

i) The Lie algebras $\underline{sl}(\mathcal{H}; C_o)$, $\underline{o}(\mathcal{H}, J_{\mathbb{R}}; C_o)$ and $\underline{sp}(\mathcal{H}, J_{\mathbb{Q}}; C_o)$ are pairewise non-isomorphic as involutive complex Lie algebras.

ii) Let $\underline{h}$ be a Cartan subalgebra of $\underline{o}(\mathcal{H}, J_{\mathbb{R}}; C_o)$ and let $\varphi$ be an automorphism of $\underline{o}(\mathcal{H}, J_{\mathbb{R}}; C_o)$ which commutes with the involution; then $\underline{h}$ and $\varphi(\underline{h})$ are of the same type.

Definition 7. Let $\underline{g}$ be a semi-simple c-involutive complex Lie algebra, let $\underline{h}$ be a Cartan subalgebra of $\underline{g}$ and let $\mathcal{R}$ be the set of non zero roots of $\underline{g}$ with respect to $\underline{h}$. A simple basis of roots in $\mathcal{R}$ is a subset $S$ of $\mathcal{R}$ such that

i) the roots in $S$ are linearly independent in $\underline{h}^{dual}$,

ii) any root in $\mathcal{R}$ can be written as a linear combination of elements in $S$, with integer coefficients which are all of the same sign.

Let $\underline{g}$, $\underline{h}$ and $\mathcal{R}$ be as in definition 7, let $\underline{k}$ be the subpsace of $\underline{h}^{dual}$ span by $\mathcal{R}$ and let $S$ be a simple basis of roots in $\mathcal{R}$. Positive roots, simple roots and the lexicographic order of $\underline{k}$ given by a total order on $S$ are defined as in the finite dimensional case. Chosen an order, the set of positive [resp. negative] roots in $\mathcal{R}$ will be denoted by $\mathcal{R}^+$ [resp. $\mathcal{R}^-$]. It is not clear (to me) which admissible semi-simple c-involutive complex Lie algebra admit simple basis of roots. Following Balachandran [11], such an algebra is said to be regular if it does admit such a basis of roots for any Cartan subalgebra.

Proposition 7A. With the same notations as in proposition 5A, let $\alpha_i = \lambda_i - \lambda_{i+1}$ and let $s_i$ be the symmetry of root $\alpha_i$, for each $i \in N$. Then

i) $S = (\alpha_i)_{i \in N}$ is a simple basis of roots in $\mathcal{R}$ .

ii) The Weyl group of $\underline{g}$ with respect to $\underline{h}$ , say $W_A$ , is generated by $(s_i)_{i \in N}$ . In other words, $W_A$ is isomorphic to the group of finite permutations of the set $N$ .

Proof : again by inspection, from proposition 4A; the consideration of subsystems of roots and of ad hoc subalgebras (as in Schue [153] section 3) can shorten the inspection needed for ii). ∎

The reader will easily guess by now what propositions 7B(type one), 7B(type two) and 7C are. The only point to stress is that a Weyl group is attached to a Lie algebra given together with a type of Cartan subalgebra of it.

Project : The conditions imposed on the involution and the system of roots in definition 2 and 6 seem very restrictive. As a starting point for a general study, the following can be proposed :

Conjecture : Let $\underline{g}$ be a regular semi-simple c-involutive complex Lie algebra; then there exists a semi-simple complex L*-algebra $\bar{\underline{g}}$ which contains $\underline{g}$ as a self-adjoint dense subalgebra.

If true, this conjecture would be a sort of analogue of the results due to Kaplansky, about the classification of the semi-simple associative algebras known as semi-simple dual Q-rings [97].

## I.5. - Automorphisms

As in sections I.3 and I.4, $\mathcal{H}$ is supposed to be an infinite dimensional separable complex Hilbert space in this section. Up to the notations, results would be the same for arbitrary dimension.

Proposition 8. Let $\underline{g}$ be one of the c-involutive complex Lie algebras $\underline{sl}(\mathcal{H}; C_o)$ , $\underline{o}(\mathcal{H}, J_{\mathbb{R}}; C_o)$ and $\underline{sp}(\mathcal{H}, J_{\mathbb{Q}}; C_o)$ defined in section I.1. Then there exists a hermitian form $B : \underline{g} \times \underline{g} \longrightarrow \mathbb{C}$ such that

(20) $B([X,Y] \mid Z) = B(Y \mid [X^*,Z])$ for all $X,Y,Z \in \underline{g}$ . Any hermitian form with this property is a scalar multiple of the scalar product $\langle\langle \mid \rangle\rangle \begin{cases} \underline{g} \times \underline{g} \longrightarrow \mathbb{C} \\ (X,Y) \longmapsto \mathrm{trace}(XY^*) \end{cases}$ .

Let now $\underline{h}$ be a Cartan subalgebra of $\underline{g}$ and let $\mathcal{R}$ be the set of non zero roots of $\underline{g}$ with respect to $\underline{h}$ . For any $\alpha \in \mathcal{R}$ , there exists a unique vector $H_\alpha \in \underline{h}$ such that $\alpha(H) = \langle\langle H \mid H_\alpha \rangle\rangle$ for all $H \in \underline{h}$ . Moreover,

$$(21) \qquad r(\alpha, \beta) = \frac{\langle\langle H_\alpha \mid H_\alpha \rangle\rangle}{\langle\langle H_\beta \mid H_\beta \rangle\rangle} \qquad \text{for all } \alpha, \beta \in \mathcal{R} .$$

Proof : The sesquilinear form $\langle\langle \mid \rangle\rangle$ defined in the proposition is clearly hermitian definite positive, and it satisfies (20); hence the existence. Let now $B$ be a hermitian form on $\underline{g}$ satisfying (20). For any finite dimensional subspace $E$ of $\mathcal{H}$ , let $B_E$ be the restriction of $B$ to the subalgebra of $\underline{g}$ consisting of those operators which map $E$ into itself and its orthogonal onto zero ($E$ is submitted to the same restrictions as in lemma 3, page I.6). According to well-known results (see for example Koszul [100] théorèmes 11.1 and 11.2), there exists a family $(c_E)$ of complex numbers such that $B_E(X \mid Y) = c_E \mathrm{trace}(XY^*)$ for all $X,Y \in \underline{g} \cap \underline{end}(E)$. As the $B_E$'s are restrictions of each other, the $c_E$'s must be equal to a same constant c. Hence the unicity.

The last part of proposition 8 follows from a contemplation of the facts collected in proposition 4 and 5. ∎

N.B. : We will sometimes write "the root $H_\alpha$" instead of "the vector $H_\alpha$ in $\underline{h}$ which represents the root $\alpha$ of $\underline{g}$ with respect to $\underline{h}$ in the sense of proposition 8".

Proposition 9. Let $\underline{g}$ and $\langle\langle | \rangle\rangle$ be as in proposition 8 and let $\varphi$ be a *-automorphism of $\underline{g}$. Then $\varphi$ is unitary :

(22) $\langle\langle \varphi(X) \mid \varphi(Y) \rangle\rangle = \langle\langle X \mid Y \rangle\rangle$ for all $X,Y \in \underline{g}$.

Proof. It follows trivially from the unicity part of proposition 8 that there exists a non zero constant $c \in \mathbf{C}^*$ such that

(23) $\langle\langle \varphi(X) \mid \varphi(Y) \rangle\rangle = c\langle\langle X \mid Y \rangle\rangle$ for all $X,Y \in \underline{g}$. Let now $\underline{h}$ be a Cartan subalgebra of $\underline{g}$, let $\alpha$ be a non zero root of $\underline{g}$ with respect to $\underline{h}$, let $H_\alpha$ be as in proposition 8, and let $X_\alpha$ be a non zero root vector of $\alpha$, so that

(24) $[H,X_\alpha] = \langle\langle H \mid H_\alpha \rangle\rangle X_\alpha$ for all $H \in \underline{h}$. Let $\underline{k} = \varphi(\underline{h})$; then $\underline{k}$ is a Cartan subalgebra of $\underline{g}$ (of the same type as $\underline{h}$ if $\underline{g} = \underline{o}(\mathcal{H}, J_R; C_o)$ ), $\alpha \circ \varphi^{-1}$ is a non zero root of $\underline{g}$ with respect to $\underline{k}$, and $\varphi(X_\alpha)$ is a non zero root vector of $\alpha \circ \varphi^{-1}$ ; let $K_\alpha$ be the vector of $\underline{k}$ such that $\langle\langle K \mid K_\alpha \rangle\rangle = \alpha \circ \varphi^{-1}(K)$ for all $K \in \underline{k}$. By application of $\varphi$ to the equality (24), and then by use of (23):

$$[K, \varphi(X_\alpha)] = \frac{1}{c} \langle\langle K \mid \varphi(H_\alpha) \rangle\rangle \varphi(X_\alpha) \quad \text{for all } K \in \underline{k} .$$

But on the other hand :

$$[K, \varphi(X_\alpha)] = \langle\langle K \mid K_\alpha \rangle\rangle \varphi(X_\alpha) \quad \text{for all } K \in \underline{k} .$$

It follows that $K_\alpha = \frac{1}{c} \varphi(H_\alpha)$. Now, clearly, $\langle\langle H_\alpha \mid H_\alpha \rangle\rangle =$ $= \langle\langle K_\alpha \mid K_\alpha \rangle\rangle$ : hence, using (23) :

$$\langle\langle H_\alpha \mid H_\alpha \rangle\rangle = \frac{1}{c\bar{c}} \langle\langle \varphi(H_\alpha) \mid \varphi(H_\alpha) \rangle\rangle = \frac{1}{\bar{c}} \langle\langle H_\alpha \mid H_\alpha \rangle\rangle \quad ,$$

so that $c = 1$. ∎

## I.5A. - *-automorphisms of $\underline{g} = \underline{sl}(\mathcal{H}; C_o)$

Let $\varphi$ be a *-automorphism of $\underline{g}$, let $\underline{h}$ be a Cartan subalgebra of $\underline{g}$, let $\underline{k}$ be the Cartan subalgebra $\varphi(\underline{h})$ of $\underline{g}$, and let $e = (e_n)_{n \in N}$ [resp. $f = (f_n)_{n \in N}$] be an orthonormal basis of $\mathcal{H}$ compatible with $\underline{h}$ [resp. with $\underline{k}$] ; let $(E_{i,j})_{i,j \in N}$ [resp. $(F_{i,j})_{i,j \in N}$] be the corresponding operators defined as before proposition 4A; for each $i \in N$, let $E_i$ denote $E_{i,i}$ and let $F_i$ denote $F_{i,i}$ .

Lemma 7A. Let $J_{\mathbb{R}}$ be the conjugation in $\mathcal{H}$ such that $J_{\mathbb{R}} e_n = e_n$ for all $n \in N$. Then there exists a unitary operator $U$ such that $\varphi$ coincides on $\underline{h}$

either with $\varphi_{UJ} : H \longmapsto -UJ_{\mathbb{R}}H^*J_{\mathbb{R}}U^*$

or with $\varphi_U : H \longmapsto UHU^*$ .

Proof.

$\varphi$ induces a bijective map from the roots of $\underline{g}$ with respect to $\underline{h}$ to the roots of $\underline{g}$ with respect to $\underline{k}$, hence from the vectors in $\underline{h}$ representing the roots of $\underline{g}$ with respect to $\underline{h}$ (the $H_\alpha$'s of proposition 8) to the vectors in $\underline{k}$ representing the roots of $\underline{g}$ with respect to $\underline{k}$. Hence, there exist two maps $a,b : N \to N$ such that $a(n) \neq b(n)$ for all $n \in N$ and such that (see proposition 4A)

$$\varphi(E_n - E_{n+1}) = F_{a(n)} - F_{b(n)} \quad \text{for all } n \in N .$$

As $E_o - E_2$ is a root with respect to $\underline{h}$,
$\varphi(E_o - E_2) = F_{a(o)} - F_{b(o)} + F_{a(1)} - F_{b(1)}$ must be a root with

respect to $\underline{k}$ , so that either $F_{a(o)} = F_{b(1)}$ , or $F_{b(o)} = F_{a(1)}$ .

Suppose first that $a(o) = b(1)$ . As $E_1 - E_3$ is a root with respect to $\underline{h}$ , $\varphi(E_1 - E_3) = F_{a(1)} - F_{b(1)} + F_{a(2)} - F_{b(2)}$ is a root with respect to $\underline{k}$ . If one had $F_{b(1)} = F_{a(2)}$ , this would imply the clearly absurd relation $\varphi(E_o-E_3)=F_{b(1)}+F_{a(1)}-F_{b(o)}-F_{b(2)}$ ; hence $F_{a(1)} = F_{b(2)}$ . By iteration of the same argument : $\varphi(E_n - E_{n+1}) = F_{b(n+1)} - F_{b(n)}$ for all $n \in N$ and the map $b$ must be a permutation of $N$ . By reordering the basis $f$ , we can suppose that $\varphi(E_n - E_{n+1}) = F_{n+1} - F_n$ for all $n \in N$ . Let now $U$ be the unitary operator on $\mathcal{H}$ such that $Ue_n = f_n$ for all $n \in N$ and let $\varphi_{UJ}$ be the $*$-automorphism $\begin{cases} \underline{g} \longrightarrow \underline{g} \\ X \longmapsto -UJ_{\mathbb{R}}X^*J_{\mathbb{R}}U^* \end{cases}$ . As $\varphi_{UJ}(E_n) = -F_n$ for all $n \in N$ , $\varphi$ and $\varphi_{UJ}$ coincide on the set $(E_n-E_{n+1})_{n \in N}$ which generate $\underline{h}$ , hence $\varphi$ and $\varphi_{UJ}$ coincide on $\underline{h}$ .

Suppose now that $b(o) = a(1)$ . The same argument implies that modulo a reordering of the basis $f$ , $\varphi(E_n - E_{n+1}) = F_n - F_{n+1}$ for all $n \in N$. Hence $\varphi$ coincides on $\underline{h}$ with the restriction of the $*$-automorphism

$\begin{cases} \underline{g} \longrightarrow \underline{g} \\ X \longmapsto UXU^* \end{cases}$ , where $U$ is the unitary operator on $\mathcal{H}$ such that $Ue_n = f_n$ for all $n \in N$. ∎

<u>Lemma 8A1</u>. Suppose that $\varphi$ coincides on $\underline{h}$ with the $*$-automorphism $\varphi_{UJ}$ . Then there exists a unitary operator $V$ on $\mathcal{H}$ such that $\varphi = \varphi_{VJ}$ .

<u>Proof</u>.

The root vector corresponding to the root $E_p - E_q$ of $\underline{g}$ with respect to $\underline{h}$ is $E_{p,q}$ . Hence $\varphi(E_{p,q})$ must be a root vector corresponding to the root $F_q - F_p$ of $\underline{g}$ with respect to $\underline{k}$ . As $\varphi$

is unitary (proposition I.9), there exists a complex number of modulus one, say $\exp(is_{p,q})$ with $s_{p,q} \in \mathbb{R}$, such that $\varphi(E_{p,q}) = \exp(is_{p,q})F_{q,p}$. This can be done for all $p,q \in N$. As $E_{p,q}^* = E_{q,p}$ one must have $\exp(-is_{p,q}) = \exp(is_{q,p})$ for all $p,q \in N$. As $[E_{p,n}, E_{n,q}] = E_{p,q}$ one must have $\exp(is_{p,n} + is_{n,q}) = -\exp(is_{p,q})$ for all triples $p,n,q$ of distinct positive integers.

Let $S$ be the operator on $\mathcal{H}$ whose representation with respect to the basis $f$ is the matrix

$$\begin{bmatrix} 1 & & & & & \\ & \exp(is_{o,1}) & & O & & \\ & & \exp(is_{o,2}) & & & \\ & & & \exp(is_{o,3}) & & \\ & O & & & \ddots & \\ & & & & & \ddots \end{bmatrix}$$

and let $\varphi_{SUJ}$ be the $*$-automorphism $\begin{cases} g \longrightarrow g \\ X \mapsto -SUJ_{\mathbb{R}}X^*J_{\mathbb{R}}U^*S^* \end{cases}$.

Then $\varphi_{SUJ}(E_n) = +S\ \varphi_{UJ}(E_n)S^* = -F_n = \varphi(E_n)$ for all $n \in N$ and

$\varphi_{SUJ}(E_{p,q}) = +S\ \varphi_{UJ}(E_{p,q})S^* = -\ SF_{q,p}S^* =$

$= -\exp(is_{o,q} - is_{o,p})F_{q,p} = +\exp(is_{p,q})F_{q,p} = \varphi(E_{p,q})$ for all $p,q \in N$ with $p \neq q$. As the $E_n$'s and the $E_{p,q}$'s generate a subspace dense in $g$ (with respect to $\langle\langle\ |\ \rangle\rangle$), $\varphi_{SUJ}$ is identical to $\varphi$, which ends the proof. ■

<u>Lemma 8A2</u>. Suppose that $\varphi$ coincides on <u>h</u> with the $*$-automorphism $\varphi_U$. Then there exists a unitary operator $V$ on $\mathcal{H}$ such that $\varphi = \varphi_V$.

<u>Proof</u> : as for 8A1. ■

<u>Proposition 10A</u>. Let $g = \underline{sl}(\mathcal{H}; C_o)$ and let $J_{\mathbb{R}}$ be a fixed conjugation on $\mathcal{H}$. Let $\varphi$ be a $*$-automorphism of $g$. Then there

exists a unitary operator $V \in U(\mathcal{H})$ such that

either $\varphi = \varphi_{VJ}$ $\quad \begin{cases} \mathfrak{g} \longrightarrow \mathfrak{g} \\ X \longmapsto -VJ_{\mathbb{R}}X^*J_{\mathbb{R}}V^* \end{cases}$

or $\varphi = \varphi_V$ $\quad \begin{cases} \mathfrak{g} \longrightarrow \mathfrak{g} \\ X' \longmapsto VXV^* \end{cases}$ .

The two cases exclude each other.

The operator $V$ is uniquely determined by $\varphi$ , up to multiplication by a complex number of modulus one.

Otherwise said, the sequence

$$\{1\} \longrightarrow U(1) \xrightarrow{\ j\ } \tilde{U}(\mathcal{H}) \xrightarrow{\ \pi\ } \overset{*}{\mathrm{Aut}}(\mathfrak{g}) \longrightarrow \{1\}$$

is exact, where $\tilde{U}(\mathcal{H})$ is the group of all unitary and antiunitary bounded operators on $\mathcal{H}$ , and where $\overset{*}{\mathrm{Aut}}(\mathfrak{g})$ is the group of $*$-automorphisms of $\mathfrak{g}$ ; the map $j$ is given by $j(e^{i\psi}) = e^{i\psi}\, \mathrm{id}_{\mathcal{H}}$ , and the map $\pi$ is given by $\pi(V) = \varphi_V$ [resp. $\pi(VJ_{\mathbb{R}}) = \varphi_{VJ}$] when $V$ is unitary [resp. when $VJ_{\mathbb{R}}$ is antiunitary].

Proof.

The existence of the operator $V$ has been proved in lemmas 7A and 8A.

Suppose now that $\varphi_{V_1} = \varphi_{V_2J}$ for some unitary operators $V_1, V_2$ on $\mathcal{H}$ and let $V = V_2^*V_1$ ; then $VXJ_{\mathbb{R}} = -X^*J_{\mathbb{R}}V$ for all $X \in \mathfrak{g}$ . Let $y$ be a vector of norm 1 in $\mathcal{H}$ ; for each vector $z$ in $\mathcal{H}$ which is of norm 1 and orthogonal to $y$ , $y \otimes \bar{z} \in \mathfrak{g}$ , so that $V(y \otimes \bar{z})J_{\mathbb{R}} = -(z \otimes \bar{y})J_{\mathbb{R}}V$ and $Vy = (V(y \otimes \bar{z})J_{\mathbb{R}})(J_{\mathbb{R}}z) = (-(z \otimes \bar{y})J_{\mathbb{R}}V)(J_{\mathbb{R}}z) = -\langle J_{\mathbb{R}}VJ_{\mathbb{R}}z \mid y\rangle z$. It follows that $Vy = 0$ , which is absurd because $V$ is unitary.

The third part of proposition 10A is a straightforward consequence of Schur's lemma. ■

Remarks.

i) Proposition 10A is still true if $\underline{sl}(\mathcal{H}, C_o)$ is replaced by $\underline{gl}(\mathcal{H}, C_o)$. In other words, if $\varphi$ is a *-automorphism of $\underline{gl}(\mathcal{H}; C_o)$, then $\varphi$ is either a *-automorphism of the associative simple algebra $C_o(\mathcal{H})$, or the negative of a *-antiautomorphism of $C_o(\mathcal{H})$. This is the generalisation of a well-known result about rings of matrices, and should be compared with general results obtained more recently by Martindale [118].

ii) Let D be a bounded skew-adjoint operator on $\mathcal{H}$ which has a purely continuous spectrum and such that $\| D \| < \frac{1}{2}\log 2$. Let $\varphi$ be the *-automorphism of $\underline{sl}(\mathcal{H}; C_o)$ defined by $\varphi(X) = \exp(D)\, X \exp(-D)$. Then $\varphi$ has no non zero fixed point [13]. Such fixed-point-free automorphisms do not occur in finite dimensions (see Borel-Mostow [23]).

## I.5B. – *-automorphisms of $\underline{g} = \underline{o}(\mathcal{H}, J_{\mathbb{R}}; C_o)$

Let $\varphi$ be a *-automorphism of $\underline{g}$, let $\underline{h}$ be a Cartan subalgebra of $\underline{g}$ of type two, let $\underline{k}$ be the Cartan subalgebra $\varphi(\underline{h})$ of $\underline{g}$, which is of type two (proposition I.6), and let $e = (e_n)_{n \in Z^*}$ [resp. $f = (f_n)_{n \in Z^*}$] be an orthonormal basis of $\mathcal{H}$ compatible with $\underline{h}$ [resp. with $\underline{k}$]; let $(E_{i,j})_{i,j \in Z^*}$ [resp. $(F_{i,j})_{i,j \in Z^*}$] be the corresponding elementary operators; for each $i \in N^*$, let $\varepsilon_i$ denote $\frac{1}{2}(E_{i,i} - E_{-i,-i})$ and let $\mathcal{F}_i$ denote $\frac{1}{2}(F_{i,i} - F_{-i,-i})$.

Lemma 7B. There exists a unitary operator U on $\mathcal{H}$ which commutes with $J_{\mathbb{R}}$ and such that $\varphi$ coincides on $\underline{h}$ with $\varphi_U : H \longmapsto UHU^*$.

Proof.

For the same reasons as in lemma 7A, there exist two maps $a,b : N^* \longrightarrow N^*$ such that $a(n) \neq b(n)$ for all $n \in N^*$ and two sequences $c = (c_n)_{n \in N^*}$, $d = (d_n)_{n \in N^*}$ consisting of zeros and ones only, such that

(25) $\varphi(\varepsilon_n - \varepsilon_{n+1}) = (-1)^{c_n} \mathcal{F}_{a(n)} + (-1)^{d_n} \mathcal{F}_{b(n)}$ for all $n \in N^*$.

As $\varepsilon_1 - \varepsilon_3$ is a root with respect to $\underline{h}$, $\varphi(\varepsilon_1 - \varepsilon_3) =$

$= (-1)^{c_1} \mathcal{F}_{a(1)} + (-1)^{d_1} \mathcal{F}_{b(1)} + (-1)^{c_2} \mathcal{F}_{a(2)} + (-1)^{d_2} \mathcal{F}_{b(2)}$

must be a root with respect to $\underline{k}$. It follows that, possibly modulo a re-definition of $a$, $b$, $c$ and $d$, equation (25) can be written for $n = 1$ and for $n = 2$ as

$$\varphi(\varepsilon_1 - \varepsilon_2) = (-1)^{c_1} \mathcal{F}_{a(1)} - (-1)^{c_2} \mathcal{F}_{a(2)}$$
$$\varphi(\varepsilon_2 - \varepsilon_3) = (-1)^{c_2} \mathcal{F}_{a(2)} + (-1)^{d_2} \mathcal{F}_{b(2)} .$$

By iteration of the same procedure, one obtains a map $a : N^* \longrightarrow N^*$ which must be a permutation of $N^*$, and a sequence $c = (c_n)_{n \in N^*}$ consisting of zeros and ones only, such that

(26) $\varphi(\varepsilon_n - \varepsilon_{n+1}) = (-1)^{c_n} \mathcal{F}_{a(n)} - (-1)^{c_{n+1}} \mathcal{F}_{a(n+1)}$ for all $n \in N^*$.

Let now $f' = (f'_n)_{n \in Z^*}$ be the orthonormal basis of defined by

$$\begin{cases} f'_n = f_{n'} \\ f'_{-n} = f_{-n'} \end{cases} \quad \text{where } n' = (-1)^{c_n} a(n) \quad \text{for all } n \in N^*$$

and let $(\mathcal{F}'_n)_{n \in N^*}$ be the corresponding sequence of operators :

$$\mathcal{F}'_n = (-1)^{c_n} \mathcal{F}_{a(n)} \qquad \text{for all } n \in N^* .$$

The basis $f'$ of $\mathcal{H}$ is clearly compatible with $\underline{k}$, and equations (26) can now be written as $\varphi(\varepsilon_n - \varepsilon_{n+1}) = \mathcal{F}'_n - \mathcal{F}'_{n+1}$ for all $n \in N^*$.

It follows that there is no loss of generality in supposing to start with that (26) can be written as

(27) $\varphi(\varepsilon_n - \varepsilon_{n+1}) = \mathcal{F}_n - \mathcal{F}_{n+1}$ for all $n \in N^*$.

The same argument as that used to end the proof of lemma 7A can now be applied. ∎

Lemma 8B. There exists a unitary operator $V$ on $\mathcal{H}$ which commutes with $J_{\mathbb{R}}$ and such that $\varphi = \varphi_V$.

Proof.

For reasons analogous to those in the proof of lemma 8A, there exist three sequences of real numbers $(s_{p,q})_{p,q \in N^*}$, $(t_{p,q})_{p,q \in N^*}$ and $(v_{p,q})_{p,q \in N^*}$, with $t_{p,q} = t_{q,p}$ and $v_{p,q} = v_{q,p}$ for all $p,q \in N^*$, $p > q$, such that :

$$\varphi(E_{p,q} - E_{-q,-p}) = \exp(is_{p,q})\,(F_{p,q} - F_{-q,-p})$$

$$\varphi(E_{q,-p} - E_{p,-q}) = \exp(it_{p,q})\,(F_{q,-p} - F_{p,-q})$$

$$\varphi(E_{-p,q} - E_{-q,p}) = \exp(iv_{p,q})\,(F_{-p,q} - F_{-q,p})$$

for all $p,q \in N^*$, $p \neq q$.

Again as in lemma 8A : $\exp(-is_{p,q}) = \exp(is_{q,p})$ and $\exp(-it_{p,q}) = \exp(iv_{p,q})$ for all $p,q \in N^*$, $p \neq q$ ; $\exp(is_{p,n} + is_{n,q}) = \exp(is_{p,q})$ and $\exp(it_{p,m} + iv_{m,q}) = \exp(is_{p,q})$ for all triple of distinct natural integers $p,m,q$.

Let $S$ be the operator on $\mathcal{H}$ whose representation with respect to the basis $f$ is the matrix

$$\begin{bmatrix} \ddots & & & \vdots & & & \\ & e^{is_{1,3}} & & \vdots & & 0 & \\ & & e^{is_{1,2}} & \vdots & & & \\ & & & 1 & & & \\ \hline & & & & 1 & & \\ & 0 & & \vdots & & e^{-is_{1,2}} & \\ & & & \vdots & & & e^{-is_{1,3}} \\ & & & \vdots & & & & \ddots \end{bmatrix}$$

Then S commutes with $J_{\mathbb{R}}$ , and :

$$\varphi_{SU}(\varepsilon_n) = S\, \mathfrak{F}_n S^* = \mathfrak{F}_n \quad \text{for all} \quad n \in N^*$$

$$\varphi_{SU}(E_{p,q} - E_{-q,-p}) = \exp(-is_{1,p} + is_{1,q})\ (F_{p,q} - F_{-q,-p}) =$$
$$= \varphi(E_{p,q} - E_{-q,-p})$$

$$\varphi_{SU}(E_{q,-p} - E_{p,-q}) = \exp(-is_{1,p} - is_{1,q})\ (F_{q,-p} - F_{p,-q})$$

$$\varphi_{SU}(E_{-p,q} - E_{-q,p}) = \exp(is_{1,p} + is_{1,q})\ (F_{-p,q} - F_{-q,p})$$

for all $p,q \in N^*$ with $p \neq q$ .

If $w_{p,q} = s_{1,p} + s_{1,q} + t_{p,q}$ for all $p,q \in N^*$ with $p \neq q$ , it is easy to check that $\exp(iw_{p,q})$ takes the same value, say $\exp(iw)$ for all $p,q \in N^*$ , $p \neq q$ . It follows that the two last expressions can be written as

$$\varphi_{SU}(E_{q,-p} - E_{p,-q}) = \exp(-iw)\quad (F_{q,-p} - F_{p,-q})$$

$$\varphi_{SU}(E_{-p,q} - E_{-q,p}) = \exp(iw)\quad (F_{-p,q} - F_{-q,p})$$

for all $p,q \in N^*$ with $p \neq q$ .

Let finally T be the operator on $\mathfrak{H}$ which is multiplication by $\exp(iw)$ in the span of $(f_n)_{n \in N^*}$ and multiplication by $\exp(-iw)$ in the span of $(f_{-n})_{n \in N^*}$ . Then TSU commutes with $J_{\mathbb{R}}$ and $\varphi = \varphi_{TSU}$ . ∎

<u>Proposition 10B</u>. Let $\underline{g} = \underline{o}(\mathcal{H}, J_{\mathbb{R}};\ C_o)$ . Let $\varphi$ be a *-automorphism of $\underline{g}$ . Then there exists an operator $V \in O(\mathcal{H}_{\mathbb{R}}) = \{X \in U(\mathcal{H}) \mid XJ_{\mathbb{R}} = J_{\mathbb{R}}X\}$ such that

$$\varphi = \varphi_V \quad : \quad \left\{ \begin{array}{lcl} \underline{g} & \longrightarrow & \underline{g} \\ X & \longmapsto & VXV^* \end{array} \right. .$$

The operator $V$ is uniquely determined by $\varphi$ , up to a sign. Otherwise said, the sequence

$$\{1\} \longrightarrow Z_2 \xrightarrow{\quad j \quad} O(\mathcal{H}_{\mathbb{R}}) \xrightarrow{\quad \pi \quad} \overset{*}{\mathrm{Aut}}(\underline{g}) \longrightarrow \{1\}$$

is exact, where the notations are similar to those defined in proposition 10A.

<u>Proof</u> : via lemmas 7B and 8B, and Schur's lemma. ∎

<u>Remarks</u>.

i) Any *-automorphism of $\underline{o}(\mathcal{H}, J_{\mathbb{R}};\ C_o)$ extends to a *-automorphism of $\underline{sl}(\mathcal{H};\ C_o)$ , and this in two different ways. Indeed, both $\varphi_V$ and $\varphi_{VJ}$ restrict to

$$\left\{ \begin{array}{lcl} \underline{o}(\mathcal{H}, J_{\mathbb{R}};\ C_o) & \longrightarrow & \underline{o}(\mathcal{H}, J_{\mathbb{R}};\ C_o) \\ X & \longmapsto & VXV^* = -VJ_{\mathbb{R}}X^*J_{\mathbb{R}}V^* \end{array} \right. .$$

ii) In finite dimension, it is well known that all automorphisms of $B_n = \underline{so}(2m+1, \mathbb{C})$ are inner. Inner automorphisms form a subgroup of index 2 in the group of automorphisms of $D_n = \underline{so}(2n, \mathbb{C})$ (if $n > 4$).

## I.5C. – *-automorphisms of $\underline{g} = \underline{sp}(\mathcal{H}, J_Q; C_o)$

Let $\varphi$ be a *-automorphism of $\underline{g}$, let $\underline{h}$ be a Cartan subalgebra of $\underline{g}$, let $\underline{k}$ be the Cartan subalgebra $\varphi(\underline{h})$ of $\underline{g}$, and let $e = (e_n)_{n \in Z*}$ [resp. $f = (f_n)_{n \in Z*}$] be an orthonormal basis of $\mathcal{H}$ compatible with $\underline{h}$ [resp. $\underline{k}$]; the notations $E_{i,j}$ $F_{i,j}$ $\varepsilon_i$ $\mathcal{F}_i$ are used as before (page I.31).

Lemma 7C. There exists a unitary operator $U$ on $\mathcal{H}$ which commutes with $J_Q$ and such that $\varphi$ coincides on $\underline{h}$ with $\varphi_U : H \longmapsto UHU^*$.

Proof.

$\varphi$ induces a bijective map from the roots of $\underline{g}$ with respect to $\underline{h}$ which are of length $\sqrt{2}$ to the roots of $\underline{g}$ with respect to $\underline{k}$ which are of length $\sqrt{2}$. Hence there exists a map $a : N^* \longrightarrow N^*$ and a sequence $c = (c_n)_{n \in N^*}$ consisting of zeros and ones only, such that

(28) $\varphi(\varepsilon_n) = (-1)^{c_n} \mathcal{F}_{a(n)}$ for all $n \in N^*$.

Let now $f' = (f'_n)_{n \in Z*}$ be the orthonormal basis of $\mathcal{H}$ defined by

$$\begin{cases} f'_n = (-1)^{c_n} f_{n'} \\ f'_{-n} = \qquad f_{-n'} \end{cases} \quad \text{where } n' = (-1)^{c_n} a(n) \quad \text{for all } n \in N^* .$$

and let $(\mathcal{F}'_n)_{n \in N^*}$ be the corresponding sequence of operators :

$$\mathcal{F}'_n = (-1)^{c_n} \mathcal{F}_{a(n)} \qquad \text{for all } n \in N^* .$$

The basis $f'$ of $\mathcal{H}$ is clearly compatible with $\underline{k}$, and equations (28) can now be written as

$$\varphi(\varepsilon_n) = \mathcal{F}'_n \quad \text{for all } n \in N^* .$$

It follows that there is no loss of generality in supposing to start with that (28) can be written as

(29) $\varphi(\varepsilon_n) = \mathcal{F}_n$ for all $n \in N^*$ .

As for lemmas 7A and 7B, this ends the proof. ∎

Lemma 8C. There exists a unitary operator $V$ on $\mathcal{H}$ which commutes with $J_{\mathbb{Q}}$ and such that $\varphi = \varphi_V$ .

Proof.

For reasons analogous to those in the proofs of lemmas 8A and 8B, there exists three sequences of real numbers

$(r_n)_{n \in N^*}$ , $(s_{p,q})_{p,q \in N^*}$ and $(t_{p,q})_{p,q \in N^*}$ , with $t_{p,q} = t_{q,p}$ for all $p,q \in N^*$ , $p > q$, such that

$$\left.\begin{array}{l} \varphi(E_{n,-n}) = \exp(i2r_n)F_{n,-n} \\ \varphi(E_{-n,n}) = \exp(-i2r_n)F_{-n,n} \end{array}\right\} \quad \text{for all } n \in N^*$$

$$\varphi(E_{p,q} - E_{-q,-p}) = \exp(is_{p,q})\,(F_{p,q} - F_{-q,-p})$$
$$\varphi(E_{q,-p} + E_{p,-q}) = \exp(it_{p,q})\,(F_{q,-p} + F_{p,-q})$$
$$\varphi(E_{-p,q} + E_{-q,p}) = \exp(-it_{p,q})\,(F_{-p,q} + F_{-q,p})$$
$$\text{for all } p,q \in N^* ,\ p \neq q .$$

Again as in lemma 8A : $\exp(-is_{p,q}) = \exp(is_{q,p})$ for all $p,q \in N^*$ , $p \neq q$ ; $\exp(-i2r_p + is_{p,q}) = \exp(-it_{p,q})$ and $\exp(-i2r_q - is_{p,q}) = \exp(-it_{p,q})$ for all $p,q \in N^*$ , $p \neq q$ ; the last two expressions can be written as

$$\left\{\begin{array}{l} \exp(is_{p,q}) = \exp(ir_p - ir_q) \\ \exp(it_{p,q}) = \exp(ir_p + ir_q) \end{array}\right. \quad \text{for all } p,q \in N^* ,\ p \neq q .$$

Let $S$ be the operator on $\mathcal{H}$ whose representation with respect to the basis $f$ is the matrix

$$\left(\begin{array}{ccc|ccc} \ddots & & & & & \\ & e^{-ir_3} & & & 0 & \\ & & e^{-ir_2} & & & \\ & & \quad e^{-ir_1} & & & \\ \hline & & & e^{ir_1} & & \\ & & & & e^{ir_2} & \\ & 0 & & & & e^{ir_3} \\ & & & & & \ddots \end{array}\right)$$

Then $S$ commutes with $J_{\mathbb{Q}}$ and $\varphi = \varphi_{SU}$ . ∎

Proposition 10C. Let $\underline{g} = \underline{sp}(\mathcal{H}, J_{\mathbb{Q}}; C_o)$ . Let $\varphi$ be a $*$-automorphism of $\underline{g}$. Then there exists an operator $V \in Sp(\mathcal{H}_{\mathbb{Q}}) = \{X \in U(\mathcal{H}) \mid XJ_{\mathbb{Q}} = J_{\mathbb{Q}}X\}$ such that

$$\varphi = \varphi_V \ : \quad \left\{\begin{array}{ccc} \underline{g} & \longrightarrow & \underline{g} \\ X & \longmapsto & VXV^* \end{array}\right. .$$

The operator V is uniquely determined by $\varphi$ , up to a sign. Otherwise said, the sequence

$$\{1\} \rightarrow Z_2 \xrightarrow{\ j\ } Sp(\mathcal{H}_{\mathbb{Q}}) \xrightarrow{\ \pi\ } \overset{*}{\mathrm{Aut}}(\underline{g}) \rightarrow \{1\}$$

is exact, where the notations are similar to those defined in proposition 10A.

Proof : via lemmas 7C and 8C, and Schur's lemma. ∎

Remarks.

i) Any $*$-automorphism of $\underline{sp}(\mathcal{H}, J_{\mathbb{Q}}; C_o)$ extends to a $*$-automorphism of $\underline{sl}(\mathcal{H}; C_o)$, and this in two different ways. Indeed, both $\varphi_V$ and $X \longmapsto VJ_{\mathbb{Q}}X^*J_{\mathbb{Q}}V^*$ restrict to

$$\left\{\begin{array}{ccc} \underline{sp}(\mathcal{H}, J_{\mathbb{Q}}; C_o) & \longrightarrow & \underline{sp}(\mathcal{H}, J_{\mathbb{Q}}; C_o) \\ X & \longmapsto & VXV^* = VJ_{\mathbb{Q}}X^*J_{\mathbb{Q}}V^* \end{array}\right.$$

ii) In finite dimension, it is well known that all automorphisms of $C_n = \underline{sp}(n, \mathbb{C})$ are inner $(n > 3)$.

## I.6. - Real forms

Definition 8. Let $\underline{g}$ be a complex involutive Lie algebra, with the involution denoted by $X \longmapsto X^*$. A conjugation of $\underline{g}$ is a semi-linear map $\sigma : \underline{g} \longrightarrow \underline{g}$ such that

$$\sigma^2(X) = X \quad \text{and} \quad \sigma(X^*) = (\sigma(X))^* \quad \text{for all } X \in \underline{g}$$

$$\sigma([X,Y]) = [\sigma(X), \sigma(Y)] \quad \text{for all } X,Y \in \underline{g} .$$

A real form of $\underline{g}$ is a real involutive Lie algebra $\underline{s}$ whose complexification is $*$-isomorphic to $\underline{g}$. The canonical conjugation of $\underline{g}$ is the map $X \longmapsto -X^*$ and its canonical real form, sometimes called the compact form of $\underline{g}$, is the real involutive Lie algebra $\underline{u} = \{X \in \underline{g} \mid X^* = -X\}$.

Standard results (Helgason [84] ch. III th. 7.1) show that a conjugation in the usual sense on a finite dimensional semi-simple Lie algebra is always a conjugation in the sense of definition 8 for a convenient c-involution on $\underline{g}$.

Let $\underline{g}$ and $\underline{u}$ be as above, let $\underline{s}$ be a real form of $\underline{g}$ and let $\varphi : \underline{u} \otimes_{\mathbb{R}} \mathbb{C} \longrightarrow \underline{s} \otimes_{\mathbb{R}} \mathbb{C}$ be a $*$-isomorphism. Let $\sigma$ be the conjugation of $\underline{s} \otimes_{\mathbb{R}} \mathbb{C}$ defined by $\sigma(X + iY) = X - iY$ for all $X,Y \in \underline{s}$. Then the map $\tilde{\Gamma}_{\underline{s},\varphi} = \varphi^{-1}\sigma\varphi$ is a conjugation of $\underline{g}$ and the map

$$\Gamma_{\underline{s},\varphi} : \begin{cases} \underline{g} \longrightarrow \underline{g} \\ X \longmapsto \tilde{\Gamma}_{\underline{s},\varphi}(-X^*) \end{cases}$$

is an involutive $*$-automorphism of $\underline{g}$.

Conversely, let $\Gamma$ be an involutive $*$-automorphism of $\underline{g}$. Then $\underline{s}_{\Gamma} = \{X \in \underline{g} \mid \Gamma(-X^*) = X\}$ is a real involutive Lie algebra and

$$\varphi_{\Gamma} : \begin{cases} \underline{g} \longrightarrow \underline{s}_{\Gamma} \otimes_{\mathbb{R}} \mathbb{C} \\ X \longmapsto \{\tfrac{1}{2}(X + \Gamma(-X^*))\} + i\,\{\tfrac{1}{2}i\,(X - \Gamma(-X^*))\} \end{cases}$$

is a $*$-isomorphism, so that $\underline{s}_{\Gamma}$ is a real form of $\underline{g}$.

In other words, there is a map from the set of all involutive *-automorphism of $\underline{g}$ to the set of *-isomorphism classes of real forms of $\underline{g}$, and this map is onto.

Let now $\Gamma$ be an involutive *-automorphism of $\underline{g}$ and let $\Psi$ be an arbitrary *-automorphism of $\underline{g}$. Then $\Gamma$ and $\Psi^{-1}\,\Gamma\,\Psi$ define clearly *-isomorphic real forms of $\underline{g}$.

Conversely, let $\Gamma_1$ and $\Gamma_2$ be two *-automorphisms of $\underline{g}$ which define *-isomorphic real forms $\underline{s_1}$ and $\underline{s_2}$, let $\psi$ be some *-isomorphism from $\underline{s_1}$ to $\underline{s_2}$ and let $\Psi$ be the complexification of $\psi$. Then $\Gamma_2(-\Psi(Y)^*) = \Psi(Y)$ for all $Y \in \underline{s_1}$, so that $\Gamma_1(\Psi^{-1}\,\Gamma_2\,\Psi)(Y) = Y$ for all $Y \in \underline{s_1}$ hence for all $Y \in \underline{g}$. It follows that $\Gamma_1 = \Psi^{-1}\,\Gamma_2\,\Psi$.

These remarks boil down to the following proposition.

Proposition 11. Let $\underline{g}$ be a complex involutive Lie algebra. Then there is a bijective correspondence, as above, between

i) The set of conjugacy classes of involutive *-automorphisms of $\underline{g}$.

ii) The set of *-isomorphism classes of real forms of $\underline{g}$.

Alternative "proof". The Galois group $Z_2 = \mathrm{Gal}(\mathbb{C}/\mathbb{R})$ acts trivially on the group of *-automorphisms $\overset{*}{\mathrm{Aut}}(\underline{g}) = \overset{*}{\mathrm{Aut}}(\underline{u} \otimes_{\mathbb{R}} \mathbb{C})$, where $\underline{u}$ is the canonical real form of $\underline{g}$. According to general principles, the set of *-isomorphism classes of real forms of $\underline{g}$ is in bijection with the first cohomology $H^1(Z_2; \overset{*}{\mathrm{Aut}}(\underline{u} \otimes_{\mathbb{R}} \mathbb{C}))$. The rest of the proof follows directly from the definitions; see for example Serre [158], annexe to chap. VII and beginning of chap. X (mainly proposition 4). ∎

We now proceed to describe and classify the real forms of the classical complex Lie algebras of finite rank operators. The notations are close to those of Helgason [84]. Up to minor details, the method follows that of the three independent works by Balachandran [15], de la Harpe [76], [77] and Unsain [174], [175], devised for dealing with the $L^*$-version of the same problem. As in previous sections, $\mathcal{H}$ is an <u>infinite dimensional separable complex Hilbert space</u>. Generalisation to arbitrary dimensions is a question of notation only.

## I.6A. - Real forms of $\underline{g} = \underline{sl}(\mathcal{H}, C_o)$

Type AI.

Let $J_{\mathbb{R}}$ be a conjugation of $\mathcal{H}$ and let $\sigma_{AI}$ be the conjugation of $\underline{g}$ defined by $\sigma_{AI}(X) = J_{\mathbb{R}}XJ_{\mathbb{R}}$. The real form corresponding to $\sigma_{AI}$ will be denoted by $\underline{sl}(\mathcal{H}_{\mathbb{R}}; C_o)$ ; it is indeed isomorphic to the involutive real Lie algebra consisting of all finite rank operators with trace zero in the real Hilbert space

$$\mathcal{H}_{\mathbb{R}} = \{x \in \mathcal{H} \mid J_{\mathbb{R}}x = x\} .$$

Type AII.

Let $J_{\mathbb{Q}}$ be an anticonjugation of $\mathcal{H}$ and let $\sigma_{AII}$ be the conjugation of $\underline{g}$ defined by $\sigma_{AII}(X) = -J_{\mathbb{Q}}XJ_{\mathbb{Q}}$. The corresponding real form is denoted by $\underline{su^*}(\mathcal{H}; C_o)$.

Type AIII.

Let $r \in N \cup \{\infty\}$ ; let $F_r$ be a r-dimensional subspace of $\mathcal{H}$ and let $F^r$ be its orthogonal complement; if $r = \infty$, the space $F^r$ is supposed to be infinite dimensional. Let $I_r$ be the operator on $\mathcal{H}$ which coincides with minus the identity on $F_r$ and with the identity on $F^r$. Let $\sigma_{AIIIr}$ be the conjugation of $\underline{g}$ defined by $\sigma_{AIIIr}(X) = -I_r X^* I_r$ . The corresponding real form is denoted by $\underline{su}(\mathcal{H}, r, \infty; C_o)$. When $r = 0$, it is the canonical real form of $\underline{g}$, which is called the compact form of $\underline{g}$ , and denoted by $\underline{su}(\mathcal{H}\,; C_o)$.

<u>Proposition 12A</u>. Let $\underline{s}$ be a real form of $\underline{g}$ . Then $\underline{s}$ is $*$-isomorphic to one of the c-involutive real Lie algebras defined above.

<u>Proof</u>.

Let $\varphi$ be an involutive $*$-automorphism of $\underline{g}$ .By proposition 10A, there exists a unitary operator $V$ on $\mathcal{H}$ such that $\varphi$ is either equal to $\varphi_{VJ}$ or equal to $\varphi_V$ .

Suppose first that $\varphi = \varphi_{VJ}$ . As $\varphi^2 = id_{\underline{g}}$ the operator $VJ_{\mathbb{R}}VJ_{\mathbb{R}}$ commutes with any operator in $\underline{g}$ ; by Schur's lemma, it follows that there exists a complex number $\zeta$ of modulus one such that

(30) $\qquad VJ_{\mathbb{R}}VJ_{\mathbb{R}} = \zeta id_{\mathcal{H}}$ .

Multiplying (30) on the right by $J_{\mathbb{R}}$ gives $\zeta J_{\mathbb{R}} = VJ_{\mathbb{R}}V$ ; multiplying (30) on the left by $V^*$ and on the right by $V$ gives $J_{\mathbb{R}}VJ_{\mathbb{R}}V = \zeta id_{\mathcal{H}}$ . It follows that

$\bar{\zeta} id_{\mathcal{H}} = J_{\mathbb{R}}\zeta J_{\mathbb{R}} = J_{\mathbb{R}}VJ_{\mathbb{R}}V = \zeta id_{\mathcal{H}}$ , so that $\zeta = \pm 1$.

If $\zeta = +1$, $VJ_{\mathbb{R}}$ is a conjugation of $\mathcal{H}$, so that the conjugation in $\underline{g}$ associated to $\varphi$ is as in type AI above. If $\zeta = -1$, $VJ_{\mathbb{R}}$ is an anticonjugation of $\mathcal{H}$, so that the conjugation in $\underline{g}$ associated to $\varphi$ is as in type AII above.

Suppose now that $\varphi = \varphi_V$ . As $\varphi^2 = id_{\underline{g}}$ , $V^2$ is a multiple of the identity of $\mathcal{H}$ and there is no loss of generality in supposing $V^2 = id_{\mathcal{H}}$ (see proposition 10A). Let $F_r$ be the eigenspace of $V$ corresponding to the eigenvalue $-1$ and let $F^r$ be that corresponding to $+1$. Again, there is no loss of generality in supposing $\dim F_r \leqslant \dim F^r$. The conjugation in $\underline{g}$ associated to $\varphi$ is then as one of those in type AIII described above. ∎

Remarks.

i) Modulo an elementary checking, it follows from propositions 10A and 11 that the real forms described above are pairwise not *-isomorphic.

ii) Subsection I.6A could be repeated for $\underline{gl}(\mathcal{H}\,;\, C_o)$ ; see remark i) following proposition 10A.

## I.6B. - Real forms of $\underline{g} = \underline{o}(\mathcal{H}, J_{\mathbb{R}};\, C_o)$

Type BDI.

Let $r \in N \cup \{\infty\}$ and let $F_r$ , $F^r$ and $I_r$ be as above (subsection 6A, type AIII), with moreover $I_r J_{\mathbb{R}} = J_{\mathbb{R}} I_r$ . Let $\sigma_{BDIr}$ be the conjugation of $\underline{g}$ defined by $\sigma_{BDIr}(X) = -I_r X^* I_r$ . The corresponding real form is denoted by $\underline{o}(\mathcal{H}, r, \infty;\, C_o)$ .

Alternatively, let $\mathcal{H}_{\mathbb{R}}$ be the real Hilbert space $\{x \in \mathcal{H} \mid J_{\mathbb{R}} x = x\}$ and let $\tau$ be the conjugation of $\underline{sl}(\mathcal{H}_{\mathbb{R}};\, C_o)$ defined formally as $\sigma_{BDIr}$ . Then the involutive real Lie algebra $\underline{s} = \{X \in \underline{sl}(\mathcal{H}_{\mathbb{R}};\, C_o) \mid \tau(X) = X\}$ is *-isomorphic to $\underline{o}(\mathcal{H}, r, \infty;\, C_o)$; indeed, let T be the unitary operator on $\mathcal{H}$ which coincides with (-i)identity on $F_r$ and with the identity on $F^r$ ; then the map $X \longmapsto TXT^*$ is a *-automorphism of $\underline{sl}(\mathcal{H}\,;\, C_o)$ which sends $\underline{s}$ onto $\underline{o}(\mathcal{H}, r, \infty;\, C_o)$ .

When $r = 0$ , the conjugation $\sigma_{BDIo}$ defines the canonical real form of $\underline{g}$ , which is called the compact form of $\underline{g}$ , and denoted by $\underline{o}(\mathcal{H}_{\mathbb{R}};\, C_o)$.

Type BDIII.

Let $J_{\mathbb{Q}}$ be an anticonjugation of $\mathcal{H}$ which commutes with $J_{\mathbb{R}}$ and let $\sigma_{BDIII}$ be the conjugation of $\underline{g}$ defined by $\sigma_{BDIII}(X) = -J_{\mathbb{Q}}XJ_{\mathbb{Q}}$ $(= +J_{\mathbb{Q}}J_{\mathbb{R}}X^{*}J_{\mathbb{R}}J_{\mathbb{Q}})$ . The corresponding real form is denoted by $\underline{o}^{*}(\mathcal{H}\ ;\ C_{o})$ .

Proposition 12B. Let $\underline{s}$ be a real form of $\underline{g}$ . Then $\underline{s}$ is $*$-isomorphic to one of the c-involutive real Lie algebras defined above.

Proof : as in the case of proposition 12A, the proof follows easily from proposition 10 and from Schur's lemma. ∎

## I.6C. - Real forms of $\underline{g} = \underline{sp}(\mathcal{H}, J_{\mathbb{Q}};\ C_{o})$

Type CI.

Let $J_{\mathbb{R}}$ be a conjugation of $\mathcal{H}$ which commutes with $J_{\mathbb{Q}}$ and let $\sigma_{CI}$ be the conjugation of $\underline{g}$ defined by $\sigma_{CI}(X) = J_{\mathbb{R}}XJ_{\mathbb{R}}$ . The corresponding real form is denoted by $\underline{sp}(\mathcal{H}, \mathbb{R};\ C_{o})$ .

Type CII.

Let $\mathcal{H}_{I}$ and $\mathcal{H}_{II}$ be two infinite dimensional subspaces of $\mathcal{H}$ such that $\mathcal{H} = \mathcal{H}_{I} \oplus \mathcal{H}_{II}$ . Let $r \in \mathbb{N} \cup \{\infty\}$ ; let $F_{Ir}$ [resp. $F_{IIr}$ ] be a r-dimensional subspace of $\mathcal{H}_{I}$ [resp. $\mathcal{H}_{II}$] and let $F_{I}^{r}$ [resp. $F_{II}^{r}$] be its orthogonal complement in $\mathcal{H}_{I}$ [resp. $\mathcal{H}_{II}$] ; if $r = \infty$, the spaces $F_{I}^{r}$ and $F_{II}^{r}$ are supposed to be infinite dimensional. The anticonjugation $J_{\mathbb{Q}}$ is supposed to map semi-isometrically $E_{Ir}$ onto $E_{IIr}$, $E_{I}^{r}$ onto $E_{II}^{r}$ , and consequently $E_{IIr}$ onto $E_{Ir}$ and $E_{II}^{r}$ onto $E_{I}^{r}$ .

Let $K_r$ be the operator on $\mathcal{H}$ which coincides with minus the identity on $E_{Ir} \oplus E_{IIr}$ and with the identity on $E_I^r \oplus E_{II}^r$ (in particular, $K_r$ commutes with $J_{\mathbb{Q}}$).

Let $\sigma_{CIIr}$ be the conjugation of $\underline{g}$ defined by $\sigma_{CIIr}(X) = -K_r X^* K_r$ . The corresponding real form is denoted by $\underline{sp}(\mathcal{H}, r, \infty; C_o)$. When $r = 0$ , it is the canonical real form of $\underline{g}$, which is called the compact form of $\underline{g}$ , and which is denoted by $\underline{sp}(\mathcal{H}_{\mathbb{Q}}; C_o)$ ; it is indeed isomorphic to the involutive real Lie algebra consisting of all skew-adjoint finite rank operators in the quarternionic Hilbert space $\mathcal{H}_{\mathbb{Q}}$ defined by $\mathcal{H}$ and $J_{\mathbb{Q}}$ .

Proposition 12C. Let $\underline{s}$ be a real form of $\underline{g}$. Then $\underline{s}$ is *-isomorphic to one of the c-involutive real Lie algebras defined above.

Proof : see proposition 12B. ■

## I.6R. - Classical real Lie algebras of finite rank operators

Definition 9. A classical real Lie algebra of finite rank operators is one of the c-involutive simple real Lie algebras listed below. ($\mathcal{H}$ is assumed to be separable and infinite dimensional for simplicity of the notations, but the list could be made valid for a space of arbitrary dimension with minor modifications only.)

Type AI : $\underline{sl}(\mathcal{H}_{\mathbb{R}}; C_o)$ .

AII : $\underline{su}^*(\mathcal{H}; C_o)$ .

AIII : $\underline{su}(\mathcal{H}, r, \infty; C_o)$ where $r \in N \cup \{\infty\}$; the compact form of type A $\underline{su}(\mathcal{H}; C_o)$ corresponds to $r = 0$.

Type BDI : $\underline{o}(\mathcal{H}, r, \infty; C_o)$ where $r \in N \cup \{\infty\}$ ; the compact form of type BD $\underline{o}(\mathcal{H}_{\mathbb{R}}; C_o)$ corresponds to $r = 0$ .

BDIII : $\underline{o}^*(\mathcal{H}; C_o)$ .

Type CI : $\underline{sp}(\mathcal{H}, \mathbb{R}; C_o)$ .

CII : $\underline{sp}(\mathcal{H}, r, \infty; C_o)$ where $r \in N \cup \{\infty\}$ ; the compact form of type C $\underline{sp}(\mathcal{H}_{\mathbb{Q}}; C_o)$ corresponds to $r = 0$ .

Algebras with a complex structure : the algebras $\underline{sl}(\mathcal{H}; C_o)$ , $\underline{o}(\mathcal{H}, J_{\mathbb{R}}; C_o)$ and $\underline{sp}(\mathcal{H}, J_{\mathbb{Q}}; C_o)$ viewed as real Lie algebras.

These algebras are clearly all simple. The involution in each of them is defined by taking the adjoint of an operator in $\mathcal{H}$ (or in $\mathcal{H}_{\mathbb{R}}$ , or in $\mathcal{H}_{\mathbb{Q}}$ ), and is a c-involution. The three algebras of the fourth group are the only ones which can be given a complex structure.

The Cartan subalgebras of the real Lie algebras of type A, B and C listed in definition 9 can be classified, using precisely the same method as that devised by Alagia [1] for the $L^*$-case.

## Appendix : conjugations and anticonjugations of a complex Hilbert space

Let $V$ be a complex vector space of arbitrary dimension. The real vector space obtained by restricting the scalars will be denoted by $V^{\mathbb{R}}$. If $\sigma$ is a map from $V$ into itself, the map from $V^{\mathbb{R}}$ into itself induced by $\sigma$ will be denoted by $\sigma^{\mathbb{R}}$. The map from $V^{\mathbb{R}}$ into itself induced by the multiplication by $i$ in $V$ will be denoted by $J$.

Lemma. Let $V$ be a complex vector space and let $\sigma$ be a semi-linear involution in $V$. Let $V_+ = \{x \in V^{\mathbb{R}} \mid \sigma^{\mathbb{R}}x = x\}$ and $V_- = \{x \in V^{\mathbb{R}} \mid \sigma^{\mathbb{R}}x = -x\}$. Then $V_- = JV_+$ and $V^{\mathbb{R}} = V_+ \oplus V_-$.

Proof. Define $\sigma_+ = \frac{1}{2}(\sigma^{\mathbb{R}} + 1)$ and $\sigma_- = \frac{1}{2}(\sigma^{\mathbb{R}} - 1)$. Clearly, $V_+ = \mathrm{im}\, \sigma_+$ and $V_- = \mathrm{im}\, \sigma_-$ ; as $x = \sigma_+(x) + \sigma_-(x)$ for all $x \in V$, the space $V$ is the sum of $V_+$ and $V_-$ ; as $V_+ \cap V_- = \{0\}$, the sum is direct. Finally, let $x \in V$ ; then $x \in J(V_+)$ if and only if $Jx \in V_+$, if and only if $Jx = \sigma_+(Jx)$, if and only if $Jx = -J\,\sigma_-(x)$, if and only if $x = \sigma_-(-x)$, if and only if $x \in V_-$ ; so that $JV_+ = V_-$. ∎

Let now $\mathcal{H}$ be a complex Hilbert space. For simplicity of the notations, $\mathcal{H}$ is supposed to be separable and infinite dimensional. A semi-linear operator $J_{\mathbb{R}}$ on $\mathcal{H}$ is a <u>conjugation</u> if $\langle J_{\mathbb{R}}x | J_{\mathbb{R}}y\rangle = \overline{\langle x|y\rangle}$ for all $x,y \in \mathcal{H}$ and if $J_{\mathbb{R}}^2 = 1$. The set of all fixed points of $J_{\mathbb{R}}$ supports a real Hilbert space $\mathcal{H}_{\mathbb{R}}$. In particular, there exists an orthonormal basis $e = (e_n)_{n \in N}$ in $\mathcal{H}$ such that $J_{\mathbb{R}}(\sum x_n e_n) = \sum \overline{x_n} e_n$ for all $x = \sum x_n e_n \in \mathcal{H}$ ; such a basis of $\mathcal{H}$ is said to be a <u>$J_{\mathbb{R}}$-basis of type zero</u>. Given $J_{\mathbb{R}}$, two $J_{\mathbb{R}}$-basis of type zero are conjugated by an element of the full orthogonal group $O(\mathcal{H}_{\mathbb{R}})$. There exists as well an orthonormal basis $f = (f_n)_{n \in Z}$ such that $J_{\mathbb{R}} f_n = f_{-n}$ for all $n \in Z$. Indeed, let $e$ be a $J_{\mathbb{R}}$-basis of type zero; put $f_o = e_o$, and

$f_{-n} = \frac{1}{\sqrt{2}}(e_{2n-1} + ie_{2n})$, $f_n = \frac{1}{\sqrt{2}}(e_{2n-1} - ie_{2n})$ for all $n \in N^*$.

A basis such as $f$ is said to be a <u>$J_{\mathbb{R}}$-basis of type one</u>. Finally, there exists an orthonormal basis $g = (g_n)_{n \in Z^*}$ such that $J_{\mathbb{R}} g_n = g_{-n}$ for all $n \in Z^*$. Such a basis is said to be a <u>$J_{\mathbb{R}}$-basis of type two</u>.

A semi-linear operator $J_{\mathbb{Q}}$ on $\mathcal{H}$ is an <u>anticonjugation</u> if $\langle J_{\mathbb{Q}} x | J_{\mathbb{Q}} y \rangle = \overline{\langle x|y \rangle}$ for all $x, y \in \mathcal{H}$ and if $J_{\mathbb{Q}}^2 = -1$. The space furnished with the natural structure of quaternionic Hilbert space provided by $J_{\mathbb{Q}}$ will be denoted by $\mathcal{H}_{\mathbb{Q}}$. There exists an orthonormal basis $e = (e_n)_{n \in Z^*}$ in $\mathcal{H}$ such that

$$J_{\mathbb{Q}}\left( \sum_{n \in N^*} x_{-n} e_{-n} + \sum_{n \in N^*} x_n e_n \right) = \sum_{n \in N^*} \overline{x_{-n}} e_n - \sum_{n \in N^*} \overline{x_n} e_{-n}$$

for all $x = \sum_{n \in Z^*} x_n e_n \in \mathcal{H}$ ; such a basis is said to be a $J_{\mathbb{Q}}$-basis. Given $J_{\mathbb{Q}}$, two $J_{\mathbb{Q}}$-basis are conjugated by an element of the full symplectic group $Sp(\mathcal{H}_{\mathbb{Q}})$.