Representations of Lie Groups

Main goal: discuss representations of compact/complex semisimple Lie groups. (At the end we will also briefly discuss infinite-dimensional representations of noncompact groups.)

Note: These groups and their representations (in finite-dimensional case) were completely classified in the early 20th century by work of Killing, E. Cartan and Weyl.

References:

- Adams: "Lectures on Lie Groups"
- Fulton & Harris: "Representation Theory".

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§1: Introduction

Mantra: examples are the key to this subject.

E.g.: S^1, $O(3)$, $SO(3)$. More generally,

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any closed subgroup of $GL(n,\mathbb{R})$.

The Lie algebra of a Lie group G can be defined in various equivalent ways. We take it to be the tangent space at the identity:

$$\mathfrak{g} = T_1 G$$

And define its Lie bracket via the following construction.

Recall the adjoint action $G \curvearrowright G$ (given by conjugation). Then for $g \in G$ we have a map $Ad_g : G \longrightarrow G$.

Since $Ad_g(1) = 1$, we then obtain $d_1 Ad_g : \mathfrak{g} \longrightarrow \mathfrak{g}$, and in fact $d_1 Ad_g \in GL(\mathfrak{g})$. Thus we get a map:

$$G \longrightarrow GL(\mathfrak{g})$$

which is by defn the adjoint action of

G on its Lie algebra. Again this maps $1 \in G$ to $\mathrm{Id} \in GL(\mathfrak{g})$. So differentiating at 1 again gives a map:

$$\mathfrak{g} \longrightarrow \mathrm{Lie}\, GL(\mathfrak{g}) = \mathrm{End}\, \mathfrak{g} = \mathfrak{g} \otimes \mathfrak{g}^*$$

This is equivalent to a map $\mathfrak{g} \otimes \mathfrak{g} \longrightarrow \mathfrak{g}$, which is by defn the Lie bracket of $\mathfrak{g}$.

One can check this gives $\mathfrak{g}$ the structure of a Lie algebra.

Correspondence between Lie groups and Lie algebras

This construction has a (partial) inverse, giving an equivalence of categories:

(1-connected Lie groups) $\longleftrightarrow$ (finite-dimensional Lie algebras)

(where 1-connected means connected and simply-connected).

<u>Note</u>: The universal cover of any connected Lie group has a natural Lie group structure. Thus there is a natural 1-connected Lie group associated to any Lie group G: simply take the universal cover of the identity component of G. Since these are locally isomorphic near 1, their Lie algebras will be the same.

To get an idea of how one might construct a Lie group from a Lie algebra, consider a Lie group G. Given a path $g(t)$ in G, we get two elements of our Lie algebra for each t:

$$\left. \begin{aligned} (dL_{g(t)})^{-1}(\dot{g}(t)) &\in T_1 G = \mathfrak{g} \\ (dR_{g(t)})^{-1}(\dot{g}(t)) &\in T_1 G = \mathfrak{g} \end{aligned} \right\}$$

After making a choice of left or right, we then see that a path $g(t)$ in G is completely determined by $g(0) \in G$ and the path $\xi(t)$ in $\mathfrak{g}$ (given by one of the formulae above): we extend $\xi(t)$ to a time-dependent vector field (left- or right-invariant) and take the flow line passing through $g(0)$ at time 0.

We now consider paths with $g(0) = 1$. Since G is simply connected we have:

elements of G $\longleftrightarrow$ homotopy classes of paths in G with $g(0) = 1$.

$\longleftrightarrow$ equivalence classes of paths in $\mathfrak{g}$.

And we can describe this equivalence relation explicitly.
This gives us an invariant way to construct G from a Lie algebra.

Equations of motion for rigid bodies

Suppose we have a time-dependent family of rotations, $g_t \in SO(3)$.
Then we can interpret:

$g^{-1}\dot{g} := (dL_g)^{-1}\dot{g} \quad \longleftrightarrow \quad$ angular velocity in frame

$\dot{g}g^{-1} := (dR_g)^{-1}\dot{g} \quad \longleftrightarrow \quad$ angular velocity in space.

So the physics leads us to a study of $\mathrm{Lie}\, SO(3) \equiv (\mathbb{R}^3, \times)$.

More examples (classical groups)

We organise the classical groups as follows.

Compact:

$$\begin{tikzcd} SO(n) & \boxed{\begin{matrix} U(n) \\ \\ SU(n) \end{matrix}} \arrow[l, bend right, "\substack{\text{commuting w/} \\ \text{complex conjugation}}"', "\sigma^2 = 1"] \arrow[r, bend left, "\substack{\text{commuting w/} \\ \text{complex structure}}", "J^2 = -1"'] & Sp(n) \end{tikzcd}$$

$$\rightsquigarrow \text{"complexification"}$$

Complex:

$$\begin{tikzcd} SO(n, \mathbb{C}) & \boxed{\begin{matrix} GL(n, \mathbb{C}) \\ \\ SL(n, \mathbb{C}) \end{matrix}} \arrow[l, "\substack{\text{respecting} \\ \text{symmetric} \\ \text{form}}"'] \arrow[r, bend left, "\substack{\text{respecting} \\ \text{skew-symmetric} \\ \text{form}}"] & Sp(n, \mathbb{C}) \end{tikzcd}$$

Most of the groups we will be interested in fall into this picture.

§2: Study of SU(2)

This is a very important example:

$$SU(2) = \{A \in \mathbb{C}^{2\times 2} : A^*A = I, \det A = 1\}$$
$$= \left\{ \begin{pmatrix} z & w \\ -\bar{w} & \bar{z} \end{pmatrix} : |z|^2 + |w|^2 = 1 \right\} \cong S^3$$

(Aside: S^1 and S^3 are the only spheres which admit a Lie group structure.)

Certainly SU(2) is 1-connected. The adjoint action takes the form:

$$SU(2) \longrightarrow GL(3, \mathbb{R})$$

And in fact it takes values in SO(3). This map is actually a double cover, and since $\pi_1 SO(3) = \mathbb{Z}_2$ we have:

$$SO(3) = SU(2)/\mathbb{Z}_2 = SU(2)/\{\pm I\}$$

So that SU(2) is the universal cover of SO(3). Thus, Lie SU(2) = Lie SO(3) = $(\mathbb{R}^3, \times)$.

~~And $\mathrm{Lie}\,SU(2) = \mathrm{Lie}\,SO(3) = \mathbb{R}^3$ equipped w/ cross product.~~
~~(So $SU(2)$ is universal cover of $SO(3)$)~~

Now, we also have $SU(2) \subseteq SL(2,\mathbb{C})$.
Furthermore:

$$\begin{aligned}\mathrm{Lie}\,SL(2,\mathbb{C}) &= \mathrm{Lie}\,SU(2) \otimes_{\mathbb{R}} \mathbb{C} \\ &= \mathrm{Lie}\,SU(2) \oplus i\,\mathrm{Lie}\,SU(2)\end{aligned}$$

We can thus consider:

$$\left.\begin{array}{l}\text{Reps. of } SU(2) \\ \text{Reps. of } SL(2,\mathbb{C}) \\ \text{Reps. of } \mathrm{Lie}\,SU(2) \\ \text{Reps. of } \mathrm{Lie}\,SL(2,\mathbb{C})\end{array}\right\}$$

(All representations are <u>complex</u> in this course.) The point is that these four are equivalent (mostly follows from above discussion).

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Algebraic approach

We look at reps. of Lie $SL(2,\mathbb{C})$. We use the following basis:

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},\quad X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},\quad Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

The structure constants are then:

$$[H,X] = 2X,\quad [H,Y] = -2Y,\quad [X,Y] = H$$

Let V be a fdvs acted on by $sl(2,\mathbb{C})$, irreducible as a module.

Consider $\langle iH \rangle \leq$ Lie $SU(2)$. This is the Lie algebra of:

$$S^1 = \left\{ \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} : \theta \in \mathbb{R} \right\} \leq SU(2).$$

Irreps. of S^1 are 1D, given by $\lambda \mapsto \lambda^?$

So restricting action on V to $\langle iH \rangle$, can decompose V into a direct sum of 1D reps.

$$V = \bigoplus_{\alpha \in S \subseteq \mathbb{Z}} V_\alpha$$ ← as a vector space, not an $SL(2,\mathbb{C})$-module

where $\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}$ acts on V_α as $e^{i\alpha\theta}$, so H acts on V_α as α.

Let $e_\alpha \in V_\alpha$. Then $He_\alpha = \alpha e_\alpha$, and:

$$HXe_\alpha = XHe_\alpha + 2Xe_\alpha$$
$$= (\alpha+2)Xe_\alpha$$

$$\Rightarrow \boxed{Xe_\alpha \in V_{\alpha+2}}$$

By a similar argument:

$$\boxed{Ye_\alpha \in V_{\alpha-2}}$$

Note that if $g = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \in SU(2)$ then:

$$g\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}g^{-1} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\Rightarrow gHg^{-1}(e_\alpha) = -\alpha e_\alpha$$

$\Rightarrow Hg^{-1}($[redacted]$e_\alpha) = -\alpha g^{-1}e_\alpha$

So if α is a weight, so is $-\alpha$, (with [redacted] $V_{-\alpha} = \langle g^{-1}$[redacted]$e_\alpha \rangle$).

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Choose $\alpha_{max} \in S$ maximal and choose $e \in V_{\alpha_{max}}$ (the highest weight vector).

Claim: If V is irreducible then it is generated by $\{Y^p e\}$.

Proof: we must prove $XY^p e$ belongs to this span. (From this it follows that the submodule generated by e is spanned by $\{Y^p e\}$, hence is equal V since V irreducible)
But we know how X and Y change weights. So need $XY^p e$ a multiple of $Y^{p-1}e$, which we can prove easily by induction. $\square$

we conclude that V is completely determined by $\alpha_{max} \in \mathbb{N}_{>0}$.

Thus the irreps. of $\begin{pmatrix} SU(2) \\ SL(2,\mathbb{C}) \end{pmatrix}$ are the $S^d U$ (where $U \equiv \mathbb{C}^2$ w/ tautological rep.).
(cf. Polys. homogeneous).

Geometric approach

If $SL(2,\mathbb{C}) \curvearrowright V$ then it also acts on $\mathbb{P}(V^*)$. Suppose we know that $\exists$ an orbit isomorphic to $\mathbb{CP}^1$ not contained in any proper linear subspace. Call it C. Now $SL(2,\mathbb{C})$ has a standard action on $\mathbb{CP}^1 = S^2$ by Möbius transformations.

Here, $V = H^0(C, L)$ where $L \to C$ is a line bundle (see later). So the classification is that of line bundles over $\mathbb{CP}^1$ with a lift of the action $SL(2,\mathbb{C}) \curvearrowright \mathbb{CP}^1$.

This is well-known from cpx geometry: $L = \mathcal{O}(d)$. Then:

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$V = H^0(C, O(d)) = S^d U$ ← homogeneous polys. of degree d.

<u>Note</u>: $SO(3) = SU(2)/\{\pm I\}$. So every rep. of $SO(3)$ lifts to a rep. of $SU(2)$. Conversely a rep. of $SU(2)$ factors through to one of $SO(3)$ iff $-I$ acts trivially. For irreps $S^d U$ this happens iff d even.

Alternatively look at harmonic polys: those f which $\Delta f = 0$. Then:

$H_1 = \{\text{linear polys.}\}$

$H_2 = \{\text{tracefree } 2\times 2 \text{ symmetric matrices}\}$

$H_k = S^{2k} U$