LECTURE 2. LINEAR SYMPLECTIC GEOMETRY

2.1. **Standard symplectic structure and symplectic group.**

In Chapter 1, we have defined the standard symplectic structure

$$\omega_0 = \sum_{j=1}^{n} dx_j \wedge dy_j$$

in $\mathbb{R}^{2n}$ with the standard metric $\langle \cdot, \cdot \rangle$. If we take ω_0 as a skew bilinear form in $\mathbb{R}^{2n}$, then for any vectors u, v there is

$$\omega_0(u, v) = -u^T J_0 v = \langle J_0 u, v \rangle, \text{ or } \langle u, v \rangle = \omega_0(u, J_0 v).$$

where

$$J_0 = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$$

is the standard complex structure in $\mathbb{R}^{2n}$.

Definition 2.1.1. A symplectomorphism ψ preserving ω_0 is a diffeomorphism $\psi : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ such that

$$\psi^* \omega_0 = \omega_0. \tag{1}$$

A linear symplectormorphism ψ is a linear map $\psi : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ such that (1) holds. In other words, ψ is symplectic matrix in the standard basis such that

$$\psi^T J_0 \psi = J_0.$$

The following proposition shows that the symplectic matrix forms a group

Proposition 2.1.2. *If ϕ and ψ are two symplectic matrices, then $\psi\phi$, ψ^{-1} and ψ^T are also symplectic matrices.*

Proof. It is easy to see that $\psi\phi$ is a symplectic matrix. Now if $\psi^T J_0 \psi = J_0$, then by multiplying the identity right by ψ^{-1} and left by $(\psi^{-1})^T$ we have

$$J_0 = (\psi^{-1})^T J_0 \psi^{-1},$$

which shows that ψ^{-1} is a symplectic matrix. On the other hand, if take the inverse of the above identity, we get

$$J_0^{-1} = \psi J_0^{-1} \psi^T,$$

which shows that ψ^T is a symplectic matrix since $J_0^{-1} = -J_0$. □

Denote by $\mathrm{Sp}(2n) = \mathrm{Sp}(2n, \mathbb{R})$ by the symplectic group. We can also define the complex symplectic group $\mathrm{Sp}(2n, \mathbb{C})$.

Under the following identification

$$X + iY \to \begin{pmatrix} X & -Y \\ Y & X \end{pmatrix}$$

The complex linear group $\mathrm{GL}(n, \mathbb{C})$ can be embedded into $\mathrm{GL}(2n, \mathbb{R})$ as a subgroup.

Lemma 2.1.3.

$$\mathrm{Sp}(2n) \cap O(2n) = \mathrm{Sp}(2n) \cap \mathrm{GL}(n, \mathbb{C}) = O(2n) \cap \mathrm{GL}(n, \mathbb{C}) = U(n).$$

Proof. We only prove the last identity. Suppose that $\psi \in O(2n) \cap \mathrm{GL}(n, \mathbb{C})$. Since $\psi \in \mathrm{GL}(n, \mathbb{C})$, there is the real representation

$$\psi = \begin{pmatrix} X & -Y \\ Y & X \end{pmatrix}$$

Since ψ is orthogonal, the following relations hold

$$X^T Y = Y^T X, X^T X + Y^T Y = I.$$

This is precisely the condition that $\psi = X + iY$ is a unitary matrix □

Lemma 2.1.4. *The following conclusions hold:*

(1) *If $\lambda \in \mathbb{C}$ is an eigenvalue of a symplectic matrix A, then so are $1/\lambda, \bar{\lambda}, 1/\bar{\lambda}$. Furthermore, the multiplicities of λ and λ^{-1} agree.*
(2) *If ± 1 is an eigenvalue of ψ then it occurs with even multiplicity.*
(3) *If $\psi u = \alpha u, \psi v = \beta v, \alpha\beta \neq 1$, then $\omega_0(u, v) = 0$.*

Proof. (i) Since ψ is a real matrix, so if λ is an eigenvalue then the complex conjugate $\bar{\lambda}$ is also an eigenvalue. By the following relation

$$\psi^T = J_0 \psi^{-1} J_0^{-1},$$

ψ^T is similar to ψ^{-1}. So if λ is an eigenvalue, then λ^{-1} is the eigenvalue of ψ and they have the common multiplicity. This proves (1)

(ii) Since the determinant of ψ is 1, if -1 is the eigenvalue then it must appear even time and so does for 1. This proves (2)

(iii) The last statement follows from the identity

$$\alpha\beta\langle u, J_0 v\rangle = \langle \psi u, J_0 \psi v\rangle = \langle u, J_0 v\rangle.$$

□

Lemma 2.1.5. *If $P \in \mathrm{Sp}(2n)$ is positive and symmetric, then all its powers $P^\alpha, \forall \alpha \in \mathbb{R}$, are symplectic.*

Proof. It suffices to know that P^α preserves the symplectic form ω_0. The space $\mathbb{R}^{2n}$ can be decomposed into the direct sum of the eigenspaces V_λ of P w. r. t. the eigenvalue λ. Take $u \in V_\lambda, u' \in V_{\lambda'}$, then by Lemma 2.1.4 if $\lambda\lambda' \neq 1$, then

$$\omega_0(u, u') = 0.$$

Thus

$$\omega_0(P^\alpha u, P^\alpha u') = (\lambda^\alpha \lambda'^\alpha)\omega_0(u, u') = \omega_0(u, u').$$

no matter $\lambda\lambda' = 1$ or $\lambda\lambda' \neq 1$. □

Proposition 2.1.6. *The unitary group $U(n)$ is a deformation retract of* $\mathrm{Sp}(2n)$.

Proof. The proof is similar to the proof of the fact that $\mathrm{GL}(n, \mathbb{R})/O(n)$ is contractible. The proof depends on the polar decomposition for any $A \in \mathrm{GL}(n)$, there exists an orthogonal matrix O such that

$$A = (AA^T)^{1/2} O.$$

In our case, for any symplectic matrix ψ we have the decomposition

$$\psi = PQ,$$

where $P = (\psi\psi^T)^{1/2}$ and $Q = (\psi\psi^T)^{-1/2}\psi$. By Lemma 2.1.5, P is a positive and symmetric matrix and so is symplectic. Now the map

$$\mathrm{Sp}(2n) \times [0,1] \to \mathrm{Sp}(2n) : (\psi, t) \to (\psi\psi^T)^{-t/2}\psi$$

is a deformation retract of $\mathrm{Sp}(2n)$ to $U(n)$. □

2.2. **General linear symplectic vector space.**

We can generalize our discussion of the standard symplectic space $(\mathbb{R}^{2n}, \omega_0)$ to the general symplectic vector space.

Definition 2.2.1. A symplectic vector space is a pair (V, ω) consisting of a finite dimensional real vector space V and a nondegenerate skew-bilinear form $\omega : V \times V \to \mathbb{R}$. Here skew-symmetric means that for all $u, v \in V$, there is

$$\omega(u, v) = -\omega(v, u).$$

nondegeneracy means that for every $v \in V$, if

$$\omega(u, v) = 0,$$

then $u = 0$.

It is easy to see the existence of the non-degenerate skew-bilinear form implies the real dimension of V is even.

Definition 2.2.2. A linear symplectomorphism of the symplectic vector space (V, ω) is a vector space isomorphism $\psi : V \to V$ which preserves the symplectic structure

$$\psi^*\omega = \omega,$$

or in another word,

$$\omega(\psi u, \psi v) = \omega(u, v)$$

for $u, v \in V$.

The linear symplectomorphisms form a group, denoted by $\mathrm{Sp}(V, \omega)$. If ω is the standard symplectic form in $\mathbb{R}^{2n}$, $\mathrm{Sp}(V, \omega) = \mathrm{Sp}(2n)$.

The symplectic complement of a linear subspace $W \subset V$ is defined to be the subspace

$$W^\omega = \{v \in V : \omega(v, w) = 0, \forall w \in W\}.$$

Definition 2.2.3. A subspace W is called

- isotropic if $W \subset W^\omega$;
- coisotropic if $W^\omega \subset W$;
- symplectic if $W \cap W^\omega = \{0\}$;
- Lagrangian if $W = W^\omega$.

It is easy to see that W is isotropic iff ω vanishes on W, and W is symplectic iff $\omega|_W$ is non-degenerate.

Lemma 2.2.4. *For any subspace $W \subset V$,*

$$\dim W + \dim W^\omega = \dim V, (W^\omega)^\omega = W.$$

Proof. Define the map from V to V^*:

$$v \in V \to \iota_v \omega,$$

where ι_v is the contraction. Since ω is non-degenerate, this map is isomorphism and sends W^ω to the annihilator $W^\perp$ of W. So

$$\dim W^\omega + \dim W = \dim W^\perp + W = \dim V.$$

The second identity is obvious. □

The following result asserts that all symplectic vector space of the same dimension are linearly symplectomorphic

Theorem 2.2.5. *Let (V, ω) be a symplectic vector space of dimension $2n$. Then there exists a basis $\{u_1, \cdots, u_n, v_1, \cdots, v_n\}$ such that*

$$\omega_(u_j, u_k) = \omega(v_j, v_k) = 0,\ \omega(u_j, v_k) = \delta_{jk}.$$

Such a basis is called a symplectic basis. Moreover there exists a vector space isomorphism $\psi : \mathbb{R}^{2n} \to V$ such that

$$\psi^*\omega = \omega_0.$$

Proof. Prove by induction. Since ω is non-degenerate there exists vectors $u_1, v_1 \in V$ such that

$$\omega(u_1, v_1) = 1.$$

Hence the subspace spanned by u_1, v_1 is symplectic. Let W denote its symplectic complement. Then (W, ω) is also a symplectic subspace with dimension $2n - 2$. By induction, we can find a basis $\{u_2, \cdots, u_n, v_2, \cdots, v_n\}$ of W satisfying the requirement. Hence $\{u_1, \cdots, u_n, v_1, \cdots, v_n\}$ is the required basis. We call it as the ω-standard bais.

The linear map defined by

$$\psi z = \sum_{j=1}^{n} (x_j u_j + y_j v_j)$$

for $z = (x_1, \cdots, x_n, y_1, \cdots, y_n)$ is the required linear symplectomorphism from $\mathbb{R}^{2n} \to V$. □

Corollary 2.2.6. *Suppose that ω_t is a smooth family of symplectic forms in V, then there exists a smooth family of symplectic matrices $\psi_t : \mathbb{R}^{2n} \to V$ such that $\psi_t^* \omega_t = \omega_0$.*

Proof. Leave as exercise. □

Corollary 2.2.7. *Let V be a real $2n$-dimensional space and ω be a skew-symmetric bilinear form, then ω is non-degenerate iff*

$$\omega^n = \omega \wedge \cdots \wedge \omega \neq 0.$$

Proof. Leave as exercise. □

Proposition 2.2.8. *Any isotropic subspace is contained in some Larangian subspace. Moreover any basis $\{u_1, \cdots, u_n\}$ of a Lagrangian subspace can be extended to a symplectic basis of (V, ω).*

Proof. Let W be an isotropic subspace. We can extend W to a larger space W_1 by adjoining some vector $v \in W^\omega - W$. So the maximal isotropic space must be Lagrangian. On the other hand, if $\Lambda \subset W$ is a Lagrangian space contained in an isotropic subspace, then $W^\omega \subset \Lambda^\omega = \Lambda$, which shows that $W = \Lambda$. Hence the Lagrangian space is the maximal isotropic subspace.

It suffices to prove the second conclusion by assuming $V = \mathbb{R}^{2n}$ with the standard symplectic structure. Let Λ be a Lagrangian subspace, then $J_0\Lambda$ is also a Lagrangian subspace, which can be also viewed as the dual space Λ^* under the isomorphism $\iota_{(\cdot)}\omega_0 : \mathbb{R}^{2n} \to (\mathbb{R}^{2n})^*$. Then we choose the dual basis in $J_0\Lambda$ of basis $\{u_1, \cdots, u_n\}$ in Λ. □

Every coisotropic subspace $W \subset V$ gives rise to a new symplectic vector space obtained by dividing W by its symplectic complement. This construction of a subquotient is called (linear) symplectic reduction.

Lemma 2.2.9. *Let (V, ω) be a symplectic vector space and $W \subset V$ be a coisotropic subspace. Then we have*

(1) *The quotient $V' = W/W^\omega$ carries a natural symplectic structure ω' by ω.*
(2) *If $L \subset V$ is a Lagrangian subspace, then $L' = ((L \cap W) + W^\omega)/W^\omega$ is a Lagrangian subspace of V'.*

Proof. (i) Define $[w] = w + W^\omega$. Let $w_1 \in [w_1], w_2 \in [w_2]$ and define

$$\omega'([w_1], [w_2]) := \omega(w_1, w_2).$$

This definition is independent of the choice of w_1, w_2, since W is a coisotropic subspace and

$$\omega(w, v) = 0, \forall w \in W, v \in W^\omega.$$

If $\omega'([w_0], [w]) = 0$ for any $w \in W$, then we must have $w_0 \in W^\omega$. So $[w_0] = 0$ and ω' is nondegenerate.

(ii) We first show that $\hat{L} = (L \cap W) + W^\omega$ is a Lagrangian subspace of V. We have

$$\begin{aligned} \hat{L}^\omega =& (L \cap W)^\omega \cap W \\ =& (L^\omega + W^\omega) \cap W \\ =& L \cap W + W^\omega \\ =& \hat{L}. \end{aligned}$$

Now let $w \in W$ such that $\omega'([w], [w_1]) = 0$ for all $[w_1] \in L' = \hat{L}/W^\omega$. Then $\omega(w, w_1) = 0$ for any $w_1 \in \hat{L}$ and hence $w \in \hat{L}^\omega = \hat{L}$. This shows that $[w] \in L'$ and L' is a Lagrangian submanifold of V'.

□

2.2.1. Space of symplectic forms .

Let $K(V)$ denote the space of all symplectic forms on the vector space. Consider the action of $\mathrm{GL}(2n, \mathbb{R})$ on $K(V)$ as follows:

$$\omega \mapsto \psi^*\omega.$$

Since $\mathrm{GL}(2n, \mathbb{R})$ gives the transitive transformation between different symplectic basis, the action of $\mathrm{GL}(2n, \mathbb{R})$ on $K(V)$ is transitive. The isotropy group at any ω is $\mathrm{Sp}(V, \omega) \cong \mathrm{Sp}(\mathbb{R}^{2n})$. So

$$K(V) \cong \mathrm{GL}(2n, \mathbb{R})/\,\mathrm{Sp}(2n). \tag{2}$$

Hence $K(V)$ can be equipped with the topology of a noncompact homogeneous space.

Since $O(2n)$ and $U(n)$ are retract kernels of $\mathrm{GL}(2n,\mathbb{R})$ and $\mathrm{Sp}(2n)$ respectively, we have the homotopy equivalence relation:

$$K(V) \sim O(2n)/U(n). \tag{3}$$

2.3. **Complex structure.**

A complex structure on a vector space V is an automorphism $J : V \to V$ such that $J^2 = -I$. Under the following action

$$\mathbb{C} \times V \to V : (s + it, v) \to sv + tJv,$$

V becomes a complex vector space. In particular a space V with a complex structure must be an even dimensional space over $\mathbb{R}$. Denote by $\mathscr{J}(V)$ the space of complex structure. For example, $J_0 \in \mathscr{J}(\mathbb{R}^{2n})$. If we think of $\mathbb{R}^{2n}$ as $\mathbb{C}^n$ by identification $(x, y) \to x + iy$, then the action of J_0 on $\mathbb{R}^{2n}$ is just the multiplication of i on $\mathbb{C}$.

2.3.1. *Space of complex structures.*

Proposition 2.3.1. *The space $\mathscr{J}(\mathbb{R}^{2n})$ is diffeomorphic to the homogeneous space* $\mathrm{GL}(2n,\mathbb{R})/GL(n,\mathbb{C})$. *This space is the disjoint union of two connected components: $\mathscr{J}(\mathbb{R}^{2n}) = \mathscr{J}^+(\mathbb{R}^{2n}) \coprod \mathscr{J}^-(\mathbb{R}^{2n})$. Here $\mathscr{J}^+(\mathbb{R}^{2n})$ is the one containing the standard complex structure J_0. We have*

(1) *$\mathscr{J}^+(\mathbb{R}^{2n})$ is diffeomorphic to the homogeneous space* $\mathrm{GL}^+(2n,\mathbb{R})/\mathrm{GL}(n,\mathbb{C})$
(2) *$\mathscr{J}^+(\mathbb{R}^{2n})$ is homotopy equivalent to $SO(2n)/U(n)$.*
(3) *$\mathscr{J}^+(\mathbb{R}^4) = S^2$.*

Proof. (i) Define the action of $\mathrm{GL}^+(2n,\mathbb{R})$ on $\mathscr{J}(\mathbb{R}^{2n})$ as

$$A \cdot J := A^{-1}JA.$$

Then this action is transitive and the isotropic subgroup at J_0 is $\mathrm{GL}(n,\mathbb{C})$. Hence $\mathscr{J}(\mathbb{R}^{2n})$ can be equipped with the structure of the homogeneous space $\mathrm{GL}^+(2n,\mathbb{R})/\mathrm{GL}(n,\mathbb{C})$. This proves (1).

(ii) (2) is obvious since $\mathrm{SO}(2n)$ and $U(n)$ are the retract kernels of $\mathrm{GL}^+(2n,\mathbb{R})$ and $\mathrm{GL}(n,\mathbb{C})$.

(iii) Let $\{e_1, e_2, e_3, e_4\}$ be the orthonormal basis of $\mathbb{R}^4$. Take $J \in \mathscr{J}(\mathbb{R}^4) \cap SO(4)$. Now we claim that J is uniquely determined by the unit vector Je_1 lying in the 3-dimensional space $\{x_1 = 0\}$. In fact, $\{e_1, Je_1\}$ forms an orthonormal basis of 2 plane $E \subset \mathbb{R}^4$. Now for any unit vector $u \in E^\perp$, we have orthonormal basis $\{u, Ju\}$ of $E^\perp$. the requirement that (e_1, Je_1, u, Ju) forms an oriented orthonomal basis of $\mathbb{R}^4$ uniquely determine J. Hence $\mathscr{J}(\mathbb{R}^{2n})$ is the sphere.

□

2.3.2. *Space of compatible complex structures.*

If V also has a symplectic form ω we say that ω and J are compactible if for all $u, v \in V$,

$$\omega(Ju, Jv) = \omega(u, v),$$

and for all nonzero $u \in V$,

$$\omega(u, Ju) > 0.$$

A compatible pair (ω, J) defines a metric (inner product) g_J by

$$g_J(u, v) = \omega(u, Jv).$$

This metric is symmetric since we have

$$g_J(u, v) = \omega(u, Jv) = \omega(Ju, J^2 v) = -\omega(Ju, v) = \omega(v, Ju) = g_J(v, u).$$

The following conclusion shows that every complex structure is isomorphic to the standard complex structure.

Proposition 2.3.2. *Let V be a real $2n$ dimensional vector space with a complex structure J, then exists a vector space isomorphism $\psi : \mathbb{R}^{2n} \to V$ such that*

$$J\psi = \psi J_0.$$

Proof. Take any inner product g in V and we can define a J-invariant symmetric metric

$$g_J(u, v) := \frac{1}{2}(g_J(u, v) + g_J(Ju, Jv)).$$

Define a bilinear form

$$\omega_{g_J}(u, v) := g_J(Ju, v).$$

Then we have

$$\omega_{g_J}(u, v) = g_J(Ju, v) = g_J(J^2 u, Jv) = -g_J(Jv, u) = -\omega_{g_J}(v, u)$$

and

$$\omega_{g_J}(Ju, Jv) = -g_J(u, Jv) = -\omega_{g_J}(v, u) = \omega_{g_J}(u, v).$$

Also ω_{g_J} is non-degenerate bilinear form. Hence (V, ω_{g_J}) becomes a symplectic vector space and the pair (ω_{g_J}, J) are compatible.

Then there exists a ω_{g_J}-standard basis $\{u_1, \cdots, u_n, v_1, \cdots, v_n$ such that

$$\omega_{g_J}(u_j, u_k) = \omega_{g_J}(v_j, v_k) = 0, \omega_{g_J}(u_j, v_k) = \delta_{jk}.$$

This is equivalent to

$$g_J(Ju_j, u_k) = g_J(Jv_j, v_k) = 0, g_J(Ju_j, v_k) = \delta_{jk}.$$

This shows that $\{u_1, \cdots, u_n, v_1, \cdots, v_n\} = \{u_1, \cdots, u_n, Ju_1, \cdots, Ju_n\}$.

Now we define the linear symplectomorphism $\psi : \mathbb{R}^{2n} \to V$ such that the canonical symplectic basis $\{e_1, \cdots, e_n, J_0 e_1, \cdots, J_0 e_n\}$ is mapped to the basis $\{u_1, \cdots, u_n, Ju_1, \cdots, Ju_n$. This shows that

$$\psi J_0 = J\psi.$$

□

Denote $\mathscr{J}(V, \omega)$ by the set of all the complex structure J compatible with ω. The following proposition shows that $\mathscr{J}(V, \omega)$ is a contractible space

Proposition 2.3.3. *$\mathscr{J}(V, \omega)$ is homeomorphic to the space $\mathbb{P}$ of symmetric positive symplectic matrices. Hence $\mathscr{J}(V, \omega)$ is a contractible space.*

Proof. It suffices to prove the result for $\mathscr{J}(\mathbb{R}^{2n}, \omega_0)$. Let $J \in \mathscr{J}(\mathbb{R}^{2n}, \omega_0)$, then J must satisfies the conditions:

$$J^2 = -I, J^T J_0 J = J_0, \langle v, -J_0 J v\rangle > 0, \forall v \neq 0.$$

The first two identities implies that

$$(J_0 J)^T = -J^T J_0 = J_0 J.$$

Hence $-J_0 J$ is positive definite and symmetric. Conversely, if a matrix P has such properties, then it is easy to check that $J = -J_0^T P \in \mathscr{J}(\mathbb{R}^{2n}, \omega_0)$. By Lemma 2.1.5, $\mathbb{P}$ is contractible and so does $\mathscr{J}$. □

2.3.3. *Space of tamed complex structures.*

In many situations, it is not necessary to consider the compatible complex structure. For example, in order to treat the compactness theorem for pseudo-holomorphic curves, it only needs the condition

$$\omega(v, Jv) > 0, \forall v \neq 0.$$

The complex structure only satisfying the above condition is called a ω-tamed complex structure. Denote the set of the tamed complex structures by $\mathscr{J}_t(V, \omega)$.

The following proposition is due to Gromov [Gr?]:

Proposition 2.3.4. *The space $\mathscr{J}_t(V, \omega)$ is contractible.*

Proof. Let $K = K(V)$ be the space of all symplectic forms on V and $\mathscr{J} = \mathscr{J}(V)$ be the space of all complex structures.

Consider the spaces

$$C_t = \{(\omega, J) \in K \times \mathscr{J} | J \in \mathscr{J}_t(V, \omega)\}$$

and

$$C = \{(\omega, J) \in K \times \mathscr{J} | J \in \mathscr{J}(V, \omega)\}.$$

It is easy to see that all the projections

$$C_t \to K,\ C_t \to \mathscr{J},\ C \to K,\ C \to J$$

are locally trivial fibrations. Since the fiber of $C_t \to \mathscr{J}$ is the set of all ω-tamed complex structure which is convex, hence the total space is homotopic to $\mathscr{J}$. A similar result holds for $C \to \mathscr{J}$. Since the two maps $C_t \to \mathscr{J}$ and $C \to \mathscr{J}$ are homotopic equivalences, the projection $C \to C_t$ is also a homotopy equivalent. This fact together with the conclusion that the projection $C \to K$ has contractible fibres implies that the projection $C_t \to K$ is a homotopy equivalence. So the fibers of $C \to K$ is contractible. $\square$