INTRODUCTION

This work contains three parts of similar lengths.

Chapter I is devoted to the study of some infinite dimensional Lie algebras of linear operators.

Chapter II to that of Banach-Lie algebras and Banach-Lie groups related to them.

And Chapters III and IV to applications : infinite dimensional symmetric spaces, cohomology of the stable classical groups.

The main results are briefly described in sections 3 to 4 of this introduction. The first two sections of the introduction are respectively concerned with various examples of Banach-Lie groups, and with some indications about the general theory of Banach-Lie groups.

I have found it necessary and helpful to attempt and draw together the relevant literature. Some of the references in the bibliography do not appear in the text. Those of particular importance for our purpose are indicated by a * .

0.1.- Banach-Lie groups : examples

Banach-Lie algebras and Banach-Lie groups arise naturally in many different contexts.

The earliest work devoted to them seems to be one by Pérès (1919) a propos of integral equations (Delsarte : introduction to Chapter IV of [42]).

Groups of operators on Hilbert space

The first non trivial examples are provided by the general linear group GL($\mathcal{H}$) of all invertible bounded operators on a Hilbert space $\mathcal{H}$ over $\mathbb{C}$ or $\mathbb{R}$ and by various of its subgroups : unitary, orthogonal, symplectic groups ... More generally, the groups of units and the group of unitary elements in any associative involutive Banach algebra with unit are Banach-Lie groups. An important example is given by the C*-algebra $L(\mathcal{H})/C(\mathcal{H})$, where L($\mathcal{H}$) is the algebra of all bounded operators on $\mathcal{H}$ and where C($\mathcal{H}$) is the ideal of compact operators; both the group of units and the group of unitary elements are then homotopically equivalent to the space of Fredholm operators on $\mathcal{H}$, hence are both classifying spaces for K-theory; for the relevance of these two groups in K-theory, see Atiyah-Singer [7] and Eells [55] .

If $\mathcal{I}(\mathcal{H})$ is a Banach algebra and an ideal in the associative algebra L($\mathcal{H}$), then the subgroup GL($\mathcal{H}$;$\mathcal{I}$) in GL($\mathcal{H}$) consisting of those operators of the form $id_{\mathcal{H}}$ + X with X $\in \mathcal{I}(\mathcal{H})$ is a Banach-Lie group locally diffeomorphic to $\mathcal{I}(\mathcal{H})$. The standard examples are when $\mathcal{I}(\mathcal{H})$ is the ideal of compact operators C($\mathcal{H}$) - in which case GL($\mathcal{H}$; C) is the so-called Fredholm group of $\mathcal{H}$ considered by Delsarte [42] - when $\mathcal{I}(\mathcal{H})$ is the ideal of Hilbert-Schmidt operators $C_2(\mathcal{H})$ -

in which case the Lie algebra of $GL(\mathcal{H}; C_2)$ is one of the simple <u>L*-algebras</u> [153] – and when $\mathcal{J}(\mathcal{H})$ is the ideal of trace class operators $C_1(\mathcal{H})$ – in which case the relationship between cohomologies of the group and of its Lie algebra are particularly interesting (see our Chapter IV).

Here is a somewhat more sophisticated example : Let E be a separable real Banach space furnished with a Gaussian measure γ and let $\tilde{G}$ be the group of those invertible operators X on E such that i) γ and $X_*(\gamma)$ are in the same measure class; ii) the Radon-Nicodym derivative d_X of $X_*(\gamma)$ with respect to γ is continuous on E ; iii) $d_X{}^{-1}$ is continuous on E. Then an explicit formula is know for d_X when X is in the subgroup G of $\tilde{G}$ defined as follows : Let $E^* \to \mathcal{H} \to E$ be the abstract Wiener space defined by (E,γ) and let $\underline{g}$ be the Banach-Lie algebra of those operators Y on $\mathcal{H}$ which extend to an operator $Y' \in L(E,E^*)$ as indicated in the diagram

$$\begin{tikzcd} E^* \arrow[r] & \mathcal{H} \arrow[r] \arrow[d, "Y"] & E \arrow[dll, dashed, "Y'"'] \\ E^* \arrow[r] & \mathcal{H} \arrow[r] & E \end{tikzcd}$$

Then G is the sub Banach-Lie group of $GL(\mathcal{H}; C_1)$ with Lie algebra $\underline{g}$; and G is called the <u>Wiener group</u> of (E,γ). It does not seem to be known whether or not G is a proper subgroup of $\tilde{G}$. My references for this example are J. Radcliff and P. Stefan in [58].

Other examples of Banach-Lie groups acting linearly on vector spaces can be found in Kadison [93], [94], [95], Ouzilou [128], Rickart [138], [139], [140], Sunouchy [171].

<u>Structural groups and manifolds of maps</u>

As in finite dimensions, a large part of the current interest in infinite dimensional manifolds is devoted to extra structures on them, hence to the structural Banach-Lie groups of appropriate bundles. The orthogonal group of a real Hilbert space is the structural group of a Riemannian manifold (Eells [54] section 5). Problems partially motivated by theoretical mechanics have led A. Weinstein to a careful investigation of symplectic manifolds [182], [183]. Manifolds given together with a reduction of their structural group to the Fredholm group, called Fredholm manifolds, arise naturally in many concrete problems of global analysis : degree theory, elliptic problems; see Elworthy [61], Elworthy-Tromba [62], Eells [55], Eells-Elworthy [56]. The Wiener group is a basic ingredient for the theory of Wiener manifolds, recently developed by Eells-Elworthy [57], Eells [59].

Manifolds of maps themselves can be Banach-Lie groups. For example, let S be a compact manifold and let G be a finite dimensional Lie group. The space of those maps from S to G which belong to certain classes (continuous, or Sobolev if S and G have Riemannian structures) are naturally Banach-Lie groups under pointwise multiplication; they have good applications to the algebraic topology of homogeneous spaces of G (Eells [53]).

Automorphism groups of infinite dimensional geometric objects

The (historically) first example in this category is given by Wigner's theorem : Let $\mathcal{A}$ be the group of symmetries of a projective complex Hilbert space; more precisely, if $P(\mathcal{H})$ is the set of lines in the complex Hilbert space $\mathcal{H}$, the transition probability is defined by

$$\begin{cases} P(\mathcal{H}) \times P(\mathcal{H}) \longrightarrow \mathbb{R} \\ (\xi, \eta) \longmapsto \dfrac{|\langle X|Y\rangle|^2}{|X|^2|Y|^2} \end{cases}$$

where X [resp. Y] is any non-zero vector in the complex line ξ [resp. η]; a permutation of $P(\mathcal{H})$ belongs to $\mathcal{A}$ if and only if it preserves the transition probability. Let $\tilde{U}(\mathcal{H})$ be the Banach-Lie group of all unitary and antiunitary operators on $\mathcal{H}$; any element of $\tilde{U}(\mathcal{H})$ induces clearly a symmetry of $P(\mathcal{H})$ and there is a sequence.

$$\{e\} \longrightarrow U(1) \longrightarrow \tilde{U}(\mathcal{H}) \longrightarrow \mathcal{A} \longrightarrow \{e\}.$$

Wigner proved that this sequence is exact; hence $\mathcal{A}$ is a Banach-Lie group whose Lie algebra is the quotient of the Banach-Lie algebra $\underline{u}(\mathcal{H}) = \{X \in L(\mathcal{H}) \mid X^* = -X\}$ by its centre. A similar result holds for quaternionic spaces. Proofs and comments are given in detail by Bargman [18]. The short exact sequence written above is formally the same as that appearing in propositions I.10.A and II.3.A.

Let J be a complex Banach subspace of the complex C*-algebra $L(\mathcal{H})$ which contains X^* and X^2 whenever it contains the operator X, and let J(1) be the open unit ball in J. (In particular, if $J = L(\mathcal{H})$, J(1) is a generalization of the Siegel's generalized unit disc.) Then any holomorphic diffeomorphism of J(1) onto itself is the composition of a Möbius transformation and of a linear isometry; in particular this group of diffeomorphisms of J(1) is a Banach-Lie group. The proof of this statement uses a generalized Schwartz lemma and is due to L. Harris [73], [74].

Greenfield and Wallach [66], [67] have studied infinite dimensional analogues of the classical bounded domains of type I and have computed explicitely their groups of holomorphic diffeomorphisms. These groups are again Banach-Lie groups canonically given as subgroups of $GL(\mathcal{H})$, where $\mathcal{H}$ is a complex Hilbert space.

In a general situation, the following fact has been observed by Eells : Let M be an infinite dimensional Riemannian manifold which is connected and complete. Let G^+ be the connected component of its group of isometries. Then G^+ is a Banach-Lie group whose Lie algebra consists of the Killing vector fields of M. Indeed, an isometry of M is uniquely determined by its value and that of its first derivative at any one point of M; this finiteness condition implies the Banach character of the group G^+.

A result about Banach-Lie groups and finite dimensional manifolds

Almost all the previous examples are Banach-Lie groups which act on infinite dimensional spaces or manifolds. This happens to be a general fact, at least when some (rather weak) semi-simplicity hypothesis is assumed about the group. Indeed, the following result has been proved in [126] as a corollary of the theory of the primitive Lie algebras of E. Cartan.

Theorem 1. Let G be a connected Banach-Lie group and let M be a finite dimensional smooth manifold. Suppose there exists a smooth and effective action of G on M which is ample and primitive. Then G is finite dimensional.

Primitivity in theorem 1 means : If $\underline{g}$ is the Banach-Lie algebra of G, if G_o is the isotropy subgroup at some point of M and if $\underline{g_o}$ is its Lie algebra, then $\underline{g_o}$ is maximal among the proper closed subalgebras of $\underline{g}$. Any transitive action of a second countable Lie group is ample. (Details in [126].) A corollary of the proof is:

Theorem 2. Let G and M be as above and suppose that the Lie algebra of G contains no closed finite codimensional ideal. Then the only smooth action of G on M is the trivial action which associates to each element of G the identity transformation of M.

The groups introduced in Chapter II of the present work satisfy the conditions of theorem 2.

0.2.- Banach-Lie groups : general theory

There is a theory of local Banach-Lie groups (i.e. group-germs) and of Banach-Lie algebras, due to Birkhoff [20] and Dynkin [51], of which the results are essentially as in finite dimensions. The global theory, however, offers many facts without finite dimensional analogues. For instance, Banach-Lie algebras which are not enlargable, that is which are not Lie algebras of any Banach-Lie group, have been constructed by Van Est-Korthagen [177] and Douady-Lazard [49]. In contrast, there are good criteria, one of which is :

Theorem 3. Let $\underline{g}$ be a Banach-Lie algebra whose center is reduced to zero. Then there exists a Banach-Lie group of which $\underline{g}$ is the Lie algebra.

Theorem 3 will be good enough for our purpose; its proof can be found in Lazard [107], section 22. There are other criteria for enlargibility in [177] and Swierczkowski [173].

The best reference about the general theory of Banach-Lie groups is Lazard [107]; some of the technical definitions we need will be recalled in section II.1. The background about Banach manifolds can be found in Bourbaki [27], [28], Dieudonné [44], Lang [103], Lazard [106]. Other references about Banach-Lie groups include : Eells [54] section 3, Laugwitz [104], [105], Leslie [109], Maissen [116]; and about infinite dimensional manifolds : Kuiper [102], Moulis [194].

0.3.- Structure of the classical Banach-Lie algebras and groups of operators in Hilbert space

(Chapters I, II and IV.)

The primary purpose of this work is to study in detail certain classical Banach-Lie algebras and Banach-Lie groups of operators on a Hilbert space.

L*-algebras and classical Lie algebras of operators

The starting point of our work was Schue's classification of complex L*-algebras [153], [154]. By definition, a L*-algebra over $\mathbb{K}$ ($\mathbb{K}$ is $\mathbb{R}$ or $\mathbb{C}$) is both an involutive Lie algebra and a $\mathbb{K}$-Hilbert space such that the following holds : if $X \mapsto X^*$ denotes the involution and $\langle\langle \mid \rangle\rangle$ the scalar product of $\underline{g}$, then $\langle\langle [X,Y] \mid Z \rangle\rangle = \langle\langle Y \mid [X^*,Z] \rangle\rangle$ for all $X,Y,Z \in \underline{g}$; and $\underline{g}$ is said to be semi-simple if moreover its derived ideal is dense : $\overline{[\underline{g},\underline{g}]} = \underline{g}$. It is an easy corollary of theorem 3 that any L*-algebra is enlargable; a L*-group is a Banach-Lie group whose Lie algebra has a structure of L*-algebra.

It is relatively easy to show that a semi-simple L*-algebra is the Hilbert direct sum of its simple closed ideals [153]. Schue gave moreover a complete classification of the separable complex L*-algebras. As in finite dimensions, the classification of the separable simple real L*-algebras can then easily be reduced to the following problem : classify the real forms of a small number of explicitely given complex L*-algebras. We solved this last problem in [76]; the results were found independently by Balachandran [15] and Unsain [174], [175].

The use of Hilbert space techniques is crucial for the general theory of L*-algebras. However, the structure of a L*-algebra is unnecessarily restrictive as soon as one considers problems about explicitely given algebras. This is the reason why the whole of Chapter I is a systematic study of Lie algebras of finite rank operators on a (real or) complex Hilbert space $\mathcal{H}$. In Chapter II, we can then consider various Banach-Lie algebras of bounded operators (sections 2-4) and of compact operators (sections 5-6); these include among others all the separable simple L*-algebras.

Derivations, automorphisms, real forms

Those algebras that we call classical Lie algebras of operators are defined by enumeration on pages I.5 and I.46 (algebras of finite rank operators), pages II.10. and II.15 (algebras of bounded operators), and pages II.20 and II.23 (algebras of compact operators). The classification result then reads :

Theorem 4 (= propositions I.12, II.4 and II.13 of the text). Let $\underline{g}$ be a classical complex Lie algebra of finite rank operators (resp. bounded operators, compact operators)on the complex Hilbert space $\mathcal{H}$ and let $\underline{s}$ be a real form of $\underline{g}$. Then $\underline{s}$ is *-isomorphic to one of the classical real Lie algebras listed page I.46 (resp. II.15, II.23).

Theorem 4 is essentially an extension of the analogous finite dimensional result of E. Cartan (see Helgason [84], chap. IX § 4). It follows from the

Theorem 5 (= proposition I.10, II.3 and II.12). Let $\underline{g}$ be as in theorem 4 and let φ be an automorphism of $\underline{g}$ such that $\varphi(X^*) = \varphi(X)^*$ for all $X \in \underline{g}$. Then there exists either a unitary operator V on $\mathcal{H}$ such that $\varphi(X) = VXV^*$ for all $X \in \underline{g}$, or an antiunitary operator $\tilde{V}$ on $\mathcal{H}$ such that $\varphi(X) = -\tilde{V}X^*\tilde{V}^*$ for all $X \in \underline{g}$. In particular, if $\underline{g}$ is a classical complex Banach-Lie algebra of bounded (resp. compact) operators, then any *-automorphism of $\underline{g}$ is isometric and inner (resp. spatial).

The other results in Chapters I and II are:

Any derivation of a classical complex Lie algebra of operators is spatial; in particular, any derivation of a classical complex Banach-Lie algebra of bounded or compact operators is continuous. (Propositions I.2, II.2 and II.9.) This was partially announced in [80].

A conjugation theorem about Cartan subalgebras (proposition I.3 and remark page I.47), and a description of the associated systems of roots (section I.4). This is due, in the L*-context, to Balachandran [12].

A partial classification of the ideals in the classical complex Banach-Lie algebras of bounded operators (section II.2) which is essentially as section 5 of [126].

Most of the propositionsin Chapters I and II fall into three cases corresponding respectively to the general linear groups (type A), the orthogonal complex groups (type B) and the symplectic complex groups (type C). The results for type A can often be proved from theorems on associative algebras (as in Dixmier [46], chap. III §9) and from theorems on the Lie structure of associative rings (as in Herstein [86], and Martindale [117], [118]); but our methods are original, and apply also to types B and C.

Some other considerations (on systems of roots, on classical groups) are more or less known, but it would be difficult to provide a short list of adequate references, either for the material or for the viewpoint.

Cohomology

The Classical Banach-Lie groups of compact operators and the Grassmann manifolds defined by them provide models for the classifying spaces of finite dimensional vector bundles (section III.1). Hence it is natural to study the relationship between the cohomologies of these groups, of their Lie algebras and of their classifying spaces. The cohomology of a Banach-Lie algebra is defined in the standard way (as in Koszul [100]) with the restriction that the cochains are continuous; we consider scalar cohomology only. In this introduction, we consider algebras of type A only; see chapter IV for analogous results about types B and C.

Let $\mathcal{H}$ be an infinite dimensional complex Hilbert space, let $p \in \bar{\mathbb{R}}$ with $1 \leqslant p \leqslant \infty$, let $C_p(\mathcal{H})$ be one of the Schatten's minimal ideals (see [151] and the second appendix to Chapter II) and let $\underline{gl}(\mathcal{H};\ C_p)$ be the corresponding classical complex Banach-Lie algebra of compact operators.

<u>Theorem 6</u> (= proposition IV.1 and corollaries). Let $J_c^*(\underline{g})$ be the algebra of <u>invariant continuous cochains</u> on the Lie algebra $\underline{g} = \underline{gl}(\mathcal{H};\ C_p)$. Then $J_c^*(\underline{g})$ is an <u>exterior algebra</u> generated by a family $(\hat{\mathfrak{J}}_k)_{k \in N,\ 2k-1 \geqslant p}$ of primitive cocycles of odd degrees, each of them being unique up to multiplication by a scalar. In particular, if $\underline{g}$ is the Lie algebra defined by the minimal ideal of all compact operators (i.e. if $p = \infty$), then $J_c^*(\underline{g}) \approx \mathbb{C}$.

If $\underline{g}$ was a finite dimensional classical Lie algebra, it is well known that $J^*(\underline{g})$ would be isomorphic (as a ring) to the de Rham cohomology of the classical group defined by $\underline{g}$ (Chevalley-Eilenberg [38]). In infinite dimensions, theorem 6 implies the following

<u>Corollary</u> (= proposition IV.2). Let $\underline{g}$ be as in theorem 6 and let G be the corresponding classical Banach-Lie group. Then $J_c^*(\underline{g})$ is isomorphic to the real cohomology of G is and only if $p = 1$.

The computation of $J_c^*(\underline{g})$ is relatively easy. That of the cohomology algebra $H_c^*(\underline{g})$ seems considerably more difficult; however:

<u>Theorem 7</u> (= proposition IV.3). Let $\underline{g}$ be as in theorem 6 and let $b_k(p)$ be the dimension of the <u>cohomology space</u> $H_c^k(\underline{g})$. Then :

$$b_1(p) = \begin{cases} 1 & \text{if } p = 1 \\ 0 & \text{otherwise} \end{cases}$$

$$b_2(p) = \begin{cases} 0 & \text{if } p = 1 \text{ or if } p = \infty \\ \infty & \text{otherwise.} \end{cases}$$

If $\underline{g}$ was a finite dimensional classical Lie algebra, it is well known that $J^*(\underline{g})$ would be isomorphic to $H^*(\underline{g})$. In infinite dimensions, theorem 7 implies the following

<u>Corollary</u>. Let $\underline{g}$ be as in theorem 6 and suppose that p is different from 1. Then $J_c^*(\underline{g})$ and $H_c^*(\underline{g})$ are <u>not</u> isomorphic.

<u>Conjecture</u>. When p = 1, i.e. when $\underline{g}$ is the Banach-Lie algebra of trace class operators $\underline{gl}(\mathcal{H}; C_1)$, then $J_c^*(\underline{g})$ and $H_c^*(\underline{g})$ are isomorphic.

Finally, let $I_c(G)$ be the algebra of those continuous scalar-valued polynomial functions on $\underline{g} = \underline{gl}(\mathcal{H}; C_p)$ which are invariant by the adjoint action of $G = GL(\mathcal{H}; C_p)$.

<u>Theorem 8</u> (= proposition IV.4). Let $G = GL(\mathcal{H}; C_p)$. Then $I_c(G)$ is a <u>polynomial algebra</u> generated by a family $(\mathcal{S}_k)_{k \in N, k \geqslant p}$ of functions of even degrees. In particular, if G is the Fredholm group of $\mathcal{H}$ (i.e. if $p = \infty$), then $I_c(G) \approx \mathbf{C}$.

If G was a finite dimensional classical Lie group, it is well known that I(G) would be isomorphic (as a ring) to the real cohomology of the classifying space B_G. In infinite dimensions, theorem 8 implies the following

<u>Corollary</u>. Let G be as in theorem 8. Then $I_c(G)$ is isomorphic to the real cohomology of the classifying space B_G if and only if p = 1.

Theorems 6, 7 and 8 were partially announced in [82] and [83].

Background

As it must be clear after this section 3 of the introduction, our work relies heavily on both the theory of finite dimensional semi-simple Lie algebras over $\mathbb{C}$ or $\mathbb{R}$ and the theory of associative Banach algebras of operators. We refer for the former essentially to Bourbaki [25], Helgason [84] chap. II and III, Serre [157], and occasionally to Bourbaki [26], Chevalley [36], [37], Jacobson [90], Séminaire Sophus Lie [156]. We refer for the later essentially to Schatten [151], and occasionally to Dieudonné [45] chap. XV, Dixmier [46], Calkin [30], Johnson-Sinclair [92].

0.4.- Homogeneous spaces of the classical Banach-Lie groups, symmetric spaces (Chapter III.)

An analytic Riemannian manifold M is called a Riemannian globally symmetric space if any of its points $m \in M$ is an isolated fixed point of some isometry s_m of M (Helgason [84]). Though this definition looks quite convenient for Hilbert manifolds, little is known about such infinite dimensional spaces in general. However, we seem to be in a good position for further studies because of the following facts:

One of the achievements of the finite dimensional theory of symmetric spaces is the classification of E. Cartan; it makes it possible, among other things, to check conjectures by inspection. Its three main steps are : the classification of complex semi-simple Lie algebras, the classification of the real forms of these and the classification of the symmetric spaces themselves. Suppose such a pattern carries over to the infinite dimensional Hilbert case; then step one is due to Schue and step two is theorem 4 above; so that step three is a natural problem to consider. Short of having solved it, we give in Chapter III the (hopefully complete list of) examples of

<u>irreducible Hilbert Riemannian globally symmetric spaces</u>. Emphasis is put on those which correspond to the finite dimensional ones of the compact type; those of non-compact type can be dealt with the same way. The homotopy type (hence the homology) of these spaces is recalled in propositions III.1 and III.2. (Section III.1.)

We then check that various notions related to that of a symmetric space make sense in infinite dimensions : orthogonal symmetric Lie algebra, Hilbert Riemannian symmetric pair, duality. Geometrical properties of the infinite dimensional spaces are then shown, still on examples, to be much the same as in finite dimensions : curvature, geodesics (proposition III.3), geodesic subspaces (proposition III.4). (Section III.2.)

Section III.3 points to an unexpected fact : Let M be an irreducible Hilbert Riemannian globally symmetric space, or for that matter a "stable" irreducible Riemannian globally symmetric space in the same sense as one speaks of "stable classical groups" (because of homotopy equivalences as in propositions II.16 and III.2). Let $P_M(t)$ be the formal power series whose k^{th} coefficient is the k^{th} Betti number of M. Then $P_M(t)$ defines an holomorphic function in the open unit disc of the complex plane. This function is either a polynomial function, or is suprisingly simply related to modular functions.

The last section of Chapter III is made of two independent remarks about particular symmetric spaces. The first recalls the classification of the (simply connected) spaces of constant curvature in the Hilbert case. The second gives an explicit trivialization of the tangent bundle to the unit sphere of a Hilbert space.

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