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On Accessibility and Normal Accessibility: The Openness of Controllability in the Fine C^0 Topology

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I. INTRODUCTION

For a control system to be an effective device in the modeling of a physical process, it is essential that its properties which have a direct physical interpretation not be unduly sensitive to small variations in the system data. This observation leads to the study of the well-posedness (or structural stability) of a variety of qualitative properties of control systems. Two such properties that will concern us here are those of *accessibility* (the ability to reach a subset of the state space having nonempty interior from every initial state) and *complete controllability* (the ability to reach the entire state space from every initial state). These properties are of particular significance in geometric control theory, where the state space is allowed to be a differentiable manifold and a control system is viewed as a system of vector fields on that manifold.

In [6] H. Sussmann has proved (among other things) that for finite systems of vector fields the accessibility and complete controllability properties are stable under small perturbations in the fine C^1 topology. The key idea in Sussmann's reasoning appears to be his introduction of the related property of *normal accessibility* ([6, p. 296]; see also Section II for a precise definition). The normal accessibility property involves a maximal rank condition on the derivatives of certain C^1 mappings associated to the system of vector fields, and it is natural to expect such a condition to be stable under small perturbations in the fine C^1 topology. While it is readily apparent that the normal accessibility property implies the accessibility property, the rather surprising fact is that the two properties are actually equivalent. Sussmann gives a proof of this equivalence in his paper [6, Theorem 4.1], but there appears to be a gap in the reasoning. We will comment on this in more detail at the beginning of Section III.

The equivalence of accessibility and normal accessibility is fundamental to

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the results derived by Sussmann in [6]. It is the author's opinion that this equivalence is probably fundamental to the study of a wide variety of structural stability questions involving controllability properties of nonlinear systems. For these reasons an alternative proof of the equivalence of accessibility and normal accessibility would seem desirable.

Our aims in this paper are threefold:

(i) to give an alternative proof of the equivalence of the accessibility and normal accessibility properties;

(ii) to prove that complete controllability is stable under small perturbations in the fine C^0 (as opposed to C^1) topology;

(iii) to prove the result stated in (ii) for arbitrary (i.e., possibly infinite) systems of C^1 vector fields.

Along the way, we will obtain another result which is of interest. Namely, if S is a completely controllable system of C^1 vector fields on a manifold M, then there exists a *countable* subset $S_0 \subseteq S$ such that S_0 is completely controllable; furthermore, if M is compact, then S_0 can be chosen to be *finite* (see Corollaries 4.9 and 4.10).

The results quoted in (ii) and (iii) above improve to some extent the earlier work of Sussmann [6]. However, the careful reader will notice that some of our techniques of proof are direct extensions of the techniques employed by Sussmann. Thus we cannot overemphasize the influence of Sussmann's work on this paper. The author also wishes to thank Professor Sussmann for a helpful conversation.

II. Preliminaries

Let M denote a finite-dimensional, second-countable, Hausdorff differentiable manifold of class C^k with $k \geqslant 2$ and set $n = \dim M$. These assumptions imply that M is a metrizable topological space. Let TM denote the tangent bundle of M (TM is a differentiable manifold of class C^{k-1}) and let $\pi\colon TM \to M$ denote the canonical projection. We fix once and for all a metric d on M compatible with the manifold topology and a *Finsler structure* ω on TM [4, Definition 4.6]; the Finsler norm of a tangent vector $v \in TM$ will be denoted by $\|v\|_\omega$.

Recall that a C^1 *vector field* on M is a C^1 mapping $X\colon M \to TM$ such that $\pi \circ X$ is the identity mapping on M. Let $V^1(M)$ denote the set of all C^1 vector fields on M. We equip $V^1(M)$ with the weak C^0 topology, whose definition we now recall. For $X \in V^1(M)$ and $K \subseteq M$ compact we set

$$\|X\|_K = \max\{\|X(x)\|_\omega \mid x \in K\};$$

furthermore, for $X \in V^1(M)$, $K \subseteq M$ compact, and $\delta > 0$ we set

$$\mathscr{U}(X; K, \delta) = \{Y \in V^1(M) \mid \|Y - X\|_K < \delta\}.$$

It is easily seen that the family of sets

$$\{\mathscr{U}(X; K, \delta) \mid X \in V^1(M), K \subseteq M \text{ compact}, \delta > 0\}$$

forms an open basis for a topology on $V^1(M)$, which is called the *weak C^0 topology*. This topology is independent of the choice of the Finsler structure ω and is separable and metrizable [5, p. 35]. It follows that every set of C^1 vector fields on M has a countable dense subset.

For $X \in V^1(M)$ and $x \in M$ we let $t \mapsto X_t(x)$ denote the maximal integral curve of X passing through x at time $t = 0$. The mapping $(t, x) \mapsto X_t(x)$ is called the *global flow* of X; the global flow is defined on an open subset of $\mathbb{R} \times M$ and is of class C^1. We let $\mathscr{D}(X) \subseteq \mathbb{R} \times M$ denote the domain of definition of the global flow of X.

DEFINITION 2.1. Let $S \subseteq V^1(M)$, let S^m denote the m-fold cartesian product of S with itself ($m \in \mathbb{N}$), and let $\mathscr{S} = \bigcup_{m=1}^{\infty} S^m$. Elements of $\mathscr{S}$ are referred to as *finite S-sequences* [6, p. 295]. For $(x, y) \in M \times M$ we say that y is *reachable from x via S* if there exist a finite S-sequence $(X^1,..., X^q)$ and nonnegative real numbers $s_1,..., s_q$ such that the expression $(X^q_{s_q} \circ \cdots \circ X^1_{s_1})(x)$ is defined and equals y. We let $A_S(x)$ denote the set of all points in M that are reachable from x via S. The set $A_S(x)$ is called the *reachable* (or *attainable*) *set* of S from x.

DEFINITION 2.2 [7, p. 109]. A family S of C^1 vector fields on M is said to have the *accessibility property* if for every $x \in M$ the reachable set $A_S(x)$ has nonempty interior.

DEFINITION 2.3. Let $S \subseteq V^1(M)$, let $(x, y) \in M \times M$, and let k be an integer satisfying $0 \leqslant k \leqslant n = \dim M$. We say that y is *normally k-reachable from x via S* if there exist a finite S-sequence $(X^1,..., X^q)$ and positive real numbers $s_1,..., s_q$ such that the expression $(X^q_{s_q} \circ \cdots \circ X^1_{s_1})(x)$ is defined and equals y and the mapping

$$(t_1,..., t_q) \mapsto (X^q_{t_q} \circ \cdots \circ X^1_{s_1})(x),$$

which is defined and C^1 on an open neighborhood of $(s_1,..., s_q)$ in $\mathbb{R}^q$, has rank k at $(s_1,..., s_q)$.

DEFINITION 2.4 [6, p. 296]. A family S of C^1 vector fields on M is said to have the *normal accessibility property* if for every $x \in M$ there exists $y \in A_S(x)$ such that y is normally n-reachable from x via S.

Remarks 2.5 [6, p. 296]. (i) If S has the normal accessibility property, then S has the accessibility property. This is an immediate consequence of the surjective mapping theorem [2, p. 378].

(ii) If w is reachable from y, y is normally n-reachable from x, and x is reachable from z, then w is normally n-reachable from z.

III. Accessibility Implies Normal Accessibility

In this section we will prove that the accessibility property implies the normal accessibility property (hence, by Remark 2.5(i) the two properties are equivalent). Before we begin with the details of the proof, we will first give a brief discussion of Sussmann's original proof of this fact [6, Theorem 4.1] and indicate why we feel that an alternative proof is desirable. In general terms the idea behind our proof is a direct extension of Sussmann's idea, but the specific details of the argument become somewhat involved.

A capsule summary of Sussmann's reasoning goes as follows. Argue by contradiction and assume that S is a family of C^1 vector fields on M that has the accessibility property, but does not have the normal accessibility property. Then for some $x_0 \in M$ there is no point $y \in A_S(x_0)$ such that y is normally n-reachable from x_0 via S. Let k denote the largest integer in $\{1,..., n\}$ for which there exists a point $y \in A_S(x_0)$ such that y is normally k-reachable from x_0 via S. The choice of x_0 implies that $k < n$. Using Zorn's lemma, Sussmann obtains a maximal (with respect to inclusion), connected, k-dimensional, C^1 immersed submanifold N of M such that $N \subseteq A_S(x_0)$ and every vector field of S is "tangent" to N (see Definition 3.1).

Sussmann then attempts to show that N is *forward S-invariant*; i.e., if $x \in N$, $X \in S$, $t \geqslant 0$, and $X_t(x)$ is defined, then $X_t(x) \in N$. If we momentarily grant the truth of this assertion, then Sussmann's argument can be concluded quickly. The forward S-invariance of N clearly implies that $A_S(x) \subseteq N$ for every $x \in N$. But N is a connected submanifold of M of positive codimension. The topological assumptions on M imply that N has empty interior relative to M. Consequently, $\operatorname{int} A_S(x) = \varnothing$ for every $x \in N$, which contradicts the fact that S has the accessibility property.

However, the reasoning employed by Sussmann to prove the forward S-invariance of N would appear to prove the following slightly more general result. Let S be a family of C^1 vector fields on M and let N be a maximal, connected, k-dimensional, C^1 immersed submanifold of M such that every vector field in S is tangent to N; then N is forward S-invariant.

It is the opinion of the author that this last result is not true. For example, it should be possible to construct a family S of vector fields in $\mathbb{R}^3$ such that one of its reachable sets has the form displayed in Fig. 1. This set can be

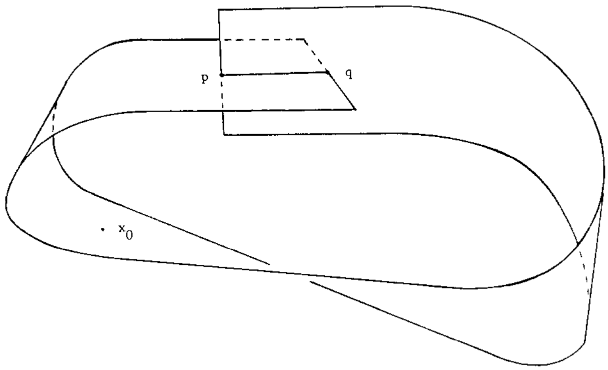

FIG. 1. A reachable set A in $\mathbb{R}^3$ having no forward S-invariant, two-dimensional submanifolds containing the initial point x_0.

obtained from a rectangular strip (Fig. 2) by putting a slit at each end, giving the strip a quarter twist, and joining the ends through the slits. Denoting this reachable set by A, we can arrange things so that the sets $A\backslash\{p\}$ and $A\backslash\{q\}$ are both maximal, connected, two-dimensional, immersed submanifolds of $\mathbb{R}^3$, every vector field of S is tangent to $A\backslash\{p\}$ and $A\backslash\{q\}$, but neither of these sets is forward S-invariant.

Sussmann's basic idea is to show that in the absence of normal accessibility there is a forward S-invariant subset of M having empty interior. This prevents the reachable set from any point of this forward S-invariant subset from having any interior and thus precludes the accessibility property. We propose to follow more-or-less the same idea, but with one modification. Rather than show that there is a forward S-invariant subset of M which consists of a *single* connected submanifold of M of positive codimension, we will show that there is a forward S-invariant subset of M which consists of a *countable union* of connected submanifolds of M of positive codimension. Such a set still has empty interior in M, so the proof can proceed as above.

When the family of vector fields S is countable, it is fairly easy to implement this alternative line of reasoning to prove that accessibility implies normal accessibility. In fact, for S countable this result is an immediate

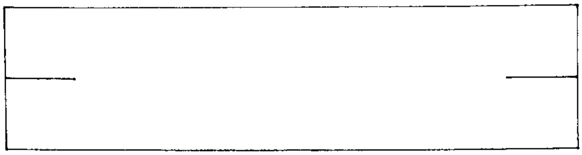

FIG. 2. A rectangular strip used to construct the reachable set in Fig. 1.

consequence of our Proposition 3.10 (one must observe that the set Ω_k in Proposition 3.10 is forward T-invariant). The major difficulty seems to arise when S is uncountable. In this case we will, in very rough terms, choose a countable dense subset S_0 of S (in the weak C^0 topology) and compare the reachable sets of S with the reachable sets of S_0.

We now commence with the details of our argument.

DEFINITION 3.1. Let $N \subseteq M$ be a C^1 immersed submanifold of M and let $i: N \to M$ denote the inclusion mapping. A vector field $X \in V^1(M)$ is said to be *tangent* to N if for every $x \in N$ we have $X(x) \in \text{image}\, di_x$ (di denotes the differential of i). In this case there is an induced C^0 vector field $\mathbf{X}: N \to TN$ which is defined by the relation $di \circ \mathbf{X} = X \circ i$.

PROPOSITION 3.2. *Let $X \in V^1(M)$ be tangent to a C^1 immersed submanifold $N \subseteq M$. Then for every $x_0 \in N$ there exist an $\varepsilon > 0$ and a subset $V_0 \subseteq N$ containing x_0 and open in the topology of N such that $(-\varepsilon, \varepsilon) \times V_0 \subseteq \mathscr{D}(X)$ and*

$$(t, x) \in (-\varepsilon, \varepsilon) \times V_0 \Rightarrow X_t(x) \in N.$$

Proof. Let $\mathbf{X}$ denote the C^0 vector field on N induced by X. Fix $x_0 \in N$, let (φ, U) be a C^1 chart of N at x_0, and let $\mathbf{X}_U: \varphi(U) \to \mathbb{R}^k$ ($k = \dim N$) denote the local representative of $\mathbf{X}$ with respect to (φ, U) given by

$$\mathbf{X}_U(y) = d\varphi_{\varphi^{-1}(y)}(\mathbf{X}(\varphi^{-1}(y))), \qquad y \in \varphi(U).$$

By the Carathéodory existence theorem of ordinary differential equations [3] there exist an $\varepsilon > 0$, an open neighborhood W_0 of $\varphi(x_0)$ with $W_0 \subseteq \varphi(U)$, and a mapping $\alpha: (-\varepsilon, \varepsilon) \times W_0 \to \varphi(U)$ such that for every $y \in W_0$ we have

$$\alpha(0, y) = y,$$

$$\frac{\partial}{\partial t} \alpha(t, y) = \mathbf{X}_U(\alpha(t, y)), \qquad t \in (-\varepsilon, \varepsilon).$$

We appeal to the Carathéodory theorem here because the mapping $\mathbf{X}_U$ may only be continuous. Set $V_0 = \varphi^{-1}(W_0)$ and define $\mu: (-\varepsilon, \varepsilon) \times V_0 \to M$ by

$$\mu(t, x) = (i \circ \varphi^{-1} \circ \alpha)(t, \varphi(x)),$$

where $i: N \to M$ is the inclusion mapping. Clearly the image of μ is contained in N and a routine computation shows that for every $x \in V_0$ the mapping $t \mapsto \mu(t, x)$ of $(-\varepsilon, \varepsilon)$ into M is an integral curve of X passing through x at time $t = 0$. Hence we have $X_t(x) = \mu(t, x) \in N$ for every $(t, x) \in (-\varepsilon, \varepsilon) \times V_0$. This completes the proof. ∎

PROPOSITION 3.3. *Let $X \in V^1(M)$ be tangent to a C^1 immersed submanifold $N \subseteq M$. Let $x_0 \in N$, let $\bar{t} > 0$ be such that $X_t(x_0)$ is defined (in M) for $t \in [0, \bar{t}]$, and let*

$$t^* = \sup\{s \in [0, \bar{t}] \mid X_t(x) \in N \text{ for every } t \in [0, s]\}.$$

Then either $t^* = \bar{t}$ or $X_t(x_0) \in N$ for every $t \in [0, t^*)$ and $X_{t^*}(x_0) \notin N$.

Proof. From Proposition 3.2 it follows that $t^* > 0$. Assume that $t^* < \bar{t}$. By the definition of t^* it is clear that $X_t(x_0) \in N$ for every $t \in [0, t^*)$. If $X_{t^*}(x_0) \in N$, then another application of Proposition 3.2 yields an $\varepsilon > 0$ such that $(t^* - \varepsilon, t^* + \varepsilon) \subseteq [0, \bar{t}]$ and

$$|t| < \varepsilon \Rightarrow X_t(X_{t^*}(x_0)) = X_{t+t^*}(x_0) \in N.$$

We infer that $X_t(x_0) \in N$ for every $t \in [0, t^* + \varepsilon)$, which contradicts the definition of t^*. Hence if $t^* < \bar{t}$, then $X_{t^*}(x_0) \notin N$ and the proof is complete. ∎

We continue to present some results about C^1 vector fields that are tangent to C^1 submanifolds of M, but we will henceforth assume that the submanifolds are embedded rather than immersed. Thus the topology of the submanifold coincides with the topology that it inherits from M as a topological subspace.

PROPOSITION 3.4. *Let $X \in V^1(M)$ be a tangent to a C^1 embedded submanifold $N \subseteq M$. Let $\bar{t} > 0$ and let V_0 be an open subset of N such that $[0, \bar{t}] \times V_0 \subseteq \mathscr{D}(X)$ and*

$$(t, x) \in [0, \bar{t}] \times V_0 \Rightarrow X_t(x) \in N.$$

Then the mapping of $[0, \bar{t}] \times V_0$ into N given by $(t, x) \mapsto X_t(x)$ is continuous in the topology of N.

Proof. This is an immediate consequence of the following elementary fact from topology. Let $\mathfrak{X}$ and $\mathfrak{Y}$ be topological spaces, let $\mathfrak{X}_0 \subseteq \mathfrak{X}$ and $\mathfrak{Y}_0 \subseteq \mathfrak{Y}$ be subspaces, and let $f\colon \mathfrak{X} \to \mathfrak{Y}$ be a continuous mapping such that $f(\mathfrak{X}_0) \subseteq \mathfrak{Y}_0$; then the induced mapping $f_0\colon \mathfrak{X}_0 \to \mathfrak{Y}_0$ is continuous.

Remark 3.5. From properties of submanifolds [8, p. 26] it follows that the mapping $(t, x) \mapsto X_t(x)$ of $[0, \bar{t}] \times V_0$ into N is of class C^1.

PROPOSITION 3.6. *Let $X \in V(M)$, let $x_0 \in M$, and let $[a, b]$ be a compact interval such that $[a, b] \times \{x_0\} \subseteq \mathscr{D}(X)$. Then for every $\varepsilon > 0$ there exist an* *open neighborhood* V_0 *of* x_0, *a compact set* $C \subseteq M$, *and a* $\delta > 0$ *such that* $Y \in V^1(M)$ *and* $\|Y - X\|_C < \delta$ *imply that* $[a, b] \times V_0 \subseteq \mathscr{D}(Y)$ *and*

$$d(Y_t(z), X_t(w)) < \varepsilon \qquad \text{for every} \quad t \in [a, b] \text{ and } z, w \in V_0.$$

Proof. See [4, Proposition 4.8]. ∎

PROPOSITION 3.7. *Let* $X \in V^1(M)$, *let* $[a, b]$ *be a compact interval, and let* $K \subseteq M$ *be a compact set such that* $[a, b] \times K \subseteq \mathscr{D}(X)$. *Then for every* $\varepsilon > 0$ *there exist a compact set* $C \subseteq M$ *and a* $\delta > 0$ *such that* $Y \in V^1(M)$ *and* $\|Y - X\|_C < \delta$ *imply that* $[a, b] \times K \subseteq \mathscr{D}(Y)$ *and*

$$t \in [a, b], \qquad z, w \in K, \qquad \text{and} \qquad d(z, w) < \delta \Rightarrow d(Y_t(z), X_t(w)) < \varepsilon.$$

Proof. This follows by Proposition 3.6 and a routine compactness argument. ∎

The next result refines the preceding one to the case where the vector fields X and Y are tangent to a C^1 embedded submanifold of M. We preface this result with one piece of notation. If $A, B \subseteq M$ are arbitrary nonempty subsets, then

$$\operatorname{dist}[A, B] = \inf\{d(x, y) \mid (x, y) \in A \times B\}.$$

PROPOSITION 3.8. *Let* $X \in V^1(M)$ *be tangent to a* C^1 *embedded submanifold* $N \subseteq M$, *let* $\bar{t} > 0$, *and let* $K \subseteq N$ *be a compact set such that* $[0, \bar{t}] \times K \subseteq \mathscr{D}(X)$ *and*

$$(t, x) \in [0, \bar{t}] \times K \Rightarrow X_t(x) \in N.$$

Then for every $\varepsilon > 0$ *there exist a compact set* $C \subseteq M$ *and a* $\delta > 0$ *such that if* $Y \in V^1(M)$ *is tangent to* N *and satisfies* $\|Y - X\|_C < \delta$, *then* $[0, \bar{t}] \times K \subseteq \mathscr{D}(Y)$ *and*

$$(t, x) \in [0, \bar{t}] \times K \Rightarrow Y_t(x) \in N \qquad \text{and} \qquad d(Y_t(x), X_t(x)) < \varepsilon.$$

Proof. By Proposition 3.4 the mapping $(t, x) \mapsto X_t(x)$ of $[0, \bar{t}] \times K$ into N is continuous, so the set

$$H = \{X_t(x) \mid (t, x) \in [0, \bar{t}] \times K\}$$

is a compact subset of N. Let W be an open subset of N such that $H \subseteq W$ and $\operatorname{cl}_N W$ (the closure of W with respect to the topology of N) is compact in N. The existence of W follows from the local compactness of N. Let

$$\varepsilon_1 = \begin{cases} \varepsilon & \text{if } N \text{ is compact,} \\ \min\{\varepsilon, \operatorname{dist}[H, N \backslash W]\} & \text{if } N \text{ is noncompact.} \end{cases}$$

Observe that $\varepsilon_1 > 0$, since for N noncompact H is a nonempty compact subset of N, $N\backslash W$ is a nonempty (since $N\backslash W \supseteq N\backslash \mathrm{cl}_N W \neq \varnothing$) closed subset of N, and $H \cap (N\backslash W) = \varnothing$.

By Proposition 3.7 there exist a compact set $C \subseteq M$ and a $\delta > 0$ such that if $Y \in V^1(M)$ satisfies $\| Y - X\|_C < \delta$, then $[0, \bar{t}] \times K \subseteq \mathscr{D}(Y)$ and

$$(t, x) \in [0, \bar{t}] \times K \Rightarrow d(Y_t(x), X_t(x)) < \varepsilon_1 \leqslant \varepsilon.$$

It will suffice to show that under the additional assumption that Y is tangent to N we have $Y_t(x) \in N$ for every $(t, x) \in [0, \bar{t}] \times K$.

We use Proposition 3.3 applied to the vector field Y. Fix $x \in K$ and let

$$t^* = \sup\{s \in [0, \bar{t}] \mid Y_t(x) \in N \text{ for every } t \in [0, s]\}.$$

Then $0 < t^* \leqslant \bar{t}$ and for $t \in [0, t^*)$ we have $Y_t(x) \in N$. If N is compact, then the continuity of $Y_t(x)$ in t yields $Y_{t^*}(x) \in N$. If N is noncompact, then for $t \in [0, t^*)$ we have

$$d(Y_t(x), X_t(x)) < \varepsilon_1 \leqslant \mathrm{dist}[H, N\backslash W].$$

Since $X_t(x) \in H$, we obtain

$$\mathrm{dist}[Y_t(x), H] < \mathrm{dist}[H, N\backslash W],$$

which implies that $Y_t(x) \in W \subseteq \mathrm{cl}_N W$. The compactness of $\mathrm{cl}_N W$ and the continuity of $Y_t(x)$ in t yield $Y_{t^*}(x) \in \mathrm{cl}_N W \subseteq N$.

Thus in either case we have $Y_{t^*}(x) \in N$. Proposition 3.3 implies that $t^* = \bar{t}$, so we infer that $Y_t(x) \in N$ for $t \in [0, \bar{t}]$. This completes the proof. ∎

LEMMA 3.9. *Let $W_0 \subseteq \mathbb{R}^q$ be an open set, let $f: W_0 \to M$ be a C^1 mapping, and assume that f has rank k at each point of W_0 ($1 \leqslant k \leqslant \min\{q, n\}$). Then for every $(s_1, ..., s_q) \in W_0$ there exists an open neighborhood W_1 of $(s_1, ..., s_q)$ with $W_1 \subseteq W_0$ such that for every nonempty open subset $W \subseteq W_1$ the set $f(W)$ is a C^1 embedded k-dimensional submanifold of M.*

Proof. This is a consequence of the rank theorem [2, p. 391]. ∎

PROPOSITION 3.10. *Let $T \subseteq V^1(M)$ be a countable subset, let $x_0 \in M$, and for $x \in A_T(x_0)$ let*

$$r(x) = \max\{j \in \{0, 1, ..., n\} \mid x \text{ is normally } j\text{-reachable from } x_0 \text{ via } T\}.$$

Assume that $r(x) < n$ for every $x \in A_T(x_0)$ and let

$$k = \max\{r(x) \mid x \in A_T(x_0)\} < n.$$

Then the set

$$\Omega_k = \{x \in A_T(x_0) \mid r(x) = k\}$$

is first category in M.

Proof. For $m \in \mathbb{N}$ let T^m denote the m-fold cartesian product of T with itself and let $\mathscr{T} = \bigcup_{m=1}^{\infty} T^m$ denote the set of all finite T-sequences. Observe that $\mathscr{T}$ is countable because T is countable. Let $\Delta = (Y^1,..., Y^p) \in \mathscr{T}$ and let $\mathscr{D}_k^+(\Delta)$ denote the set of all points $(s_1,..., s_p) \in \mathbb{R}^p$ such that $s_i > 0$ for $1 \leqslant i \leqslant p$, the expression $(Y_{s_p}^p \circ \cdots \circ Y_{s_1}^1)(x_0)$ is defined, and the C^1 mapping $(t_1,..., t_p) \mapsto (Y_{t_p}^p \circ \cdots \circ Y_{t_1}^1)(x_0)$ has rank k at $(s_1,..., s_p)$. Then $\mathscr{D}_k^+(\Delta)$ is an open (possibly empty) subset of $\mathbb{R}^p$; this follows from the maximality property of k, the fact that the rank of a C^1 mapping is locally nondecreasing, and the openness of the domain of definition of the global flow of a C^1 vector field. Let $f_\Delta: \mathscr{D}_k^+(\Delta) \to M$ denote the C^1 mapping defined by

$$f_\Delta(t_1,..., t_p) = (Y_{t_p}^p \circ \cdots \circ Y_{t_1}^1)(x_0).$$

By Lemma 3.9 we can write $\mathscr{D}_k^+(\Delta) = \bigcup_{i=1}^{\infty} W_i$, where for every $i \in \mathbb{N}$ the set W_i is open and $f_\Delta(W_i)$ is a C^1 embedded k-dimensional submanifold of M. Since $k < n$, it follows that

$$f_\Delta(\mathscr{D}_k^+(\Delta)) = \bigcup_{i=1}^{\infty} f_\Delta(W_i)$$

is a countable union of first category sets and hence is first category in M. Likewise

$$\Omega_k = \bigcup_{\Delta \in \mathscr{T}} f_\Delta(\mathscr{D}_k^+(\Delta))$$

is first category in M. ∎

The following theorem gives a sufficient condition for a compact subset of the image of a continuous mapping to be "stable" under small perturbations of the mapping.

THEOREM 3.11. *Let $\mathfrak{X}$ be a topological space, let $h: \mathfrak{X} \to M$ be a continuous mapping, and let $C \subseteq h(\mathfrak{X})$ be a compact set such that h has a continuous local right inverse at every point of C. Then there exist a compact set $K \subseteq \mathfrak{X}$ and an $\varepsilon > 0$ such that if $\mathbf{h}: K \to M$ is any continuous mapping satisfying $d(\mathbf{h}(x), h(x)) \leqslant \varepsilon$ for every $x \in K$, then $C \subseteq \mathbf{h}(K)$.*

Proof. See [4, Theorem 3.3]. ∎

We now come to the main result of this section.

THEOREM 3.12. *Let S be a family of C^1 vector fields on M. If S has the accessibility property, then S has the normal accessibility property.*

Proof. We argue by contradiction and assume that S has the accessibility property, but does not have the normal accessibility property. Then for some $x_0 \in M$ there is no point $y \in A_S(x_0)$ such that y is normally n-reachable from x_0 via S. Let k denote the largest integer in $\{1,..., n\}$ for which there exists a point $x_1 \in A_S(x_0)$ such that x_1 is normally k-reachable from x_0 via S. By assumption $k < n$ and by definition there exist a finite S-sequence $(X^1,..., X^q)$ and positive real numbers $s_1,..., s_q$ such that $(X^q_{s_q} \circ \cdots \circ X^1_{s_1})(x_0) = x_1$ and the mapping

$$f(t_1,..., t_q) = (X^q_{t_q} \circ \cdots \circ X^1_{t_1})(x_0),$$

which is defined and C^1 in an open neighborhood of $(s_1,..., s_q)$ in $\mathbb{R}^q$, has rank k at $(s_1,..., s_q)$. The maximality property of k implies that there exists an open neighborhood W_0 of $(s_1,..., s_q)$ in $\mathbb{R}^q$ such that f is defined and has rank k on W_0. By Lemma 3.9 there exists an open neighborhood W_1 of $(s_1,..., s_q)$ in $\mathbb{R}^q$ with $W_1 \subseteq W_0$ such that for every nonempty open subset $W \subseteq W_1$ the set $f(W)$ is a C^1 embedded k-dimensional submanifold of M. There is no loss of generality in assuming that all coordinates of all points of W_1 are positive real numbers. Consequently, $f(W_1) \subseteq A_S(x_0)$.

To enhance readability, the remainder of the proof will be presented as a series of claims.

Claim 1. Let $(Y^1,..., Y^p)$ be a finite S-sequence, let $r_1,..., r_p$ be nonnegative real numbers, and let $W_2 \subseteq W_1$ be an open set in $\mathbb{R}^q$ such that $(Y^p_{r_p} \circ \cdots \circ Y^1_{r_1})(x)$ is defined for every $x \in f(W_2)$. Then

$$N = (Y^p_{r_p} \circ \cdots \circ Y^1_{r_1})(f(W_2))$$

is a C^1 embedded k-dimensional submanifold of M and every vector field in S is tangent to N.

Proof of Claim 1. By the way the open set $W_1 \subseteq \mathbb{R}^q$ was chosen, the set $f(W_2)$ is a C^1 embedded k-dimensional submanifold of M. For each $i = 1,..., p$ the mapping $x \mapsto Y^i_{r_i}(x)$ is a C^1 diffeomorphism of one open subset of M onto another. Such a mapping carries C^1 embedded submanifolds onto C^1 embedded submanifolds of the same dimension. Therefore N as defined above is a C^1 embedded k-dimensional submanifold of M.

Let $Y \in S$ be arbitrary and let

$$g(t, t_1,..., t_q) = (Y_t \circ Y^p_{r_p} \circ \cdots \circ Y^1_{r_1} \circ f)(t_1,..., t_q)$$

for $(t_1,\dots,t_q)\in W_2$ and t near 0. Fix $(\bar t_1,\dots,\bar t_q)\in W_2$ and set $z_0=f(\bar t_1,\dots,\bar t_q)$, $z_1=(Y^p_{r_p}\circ\cdots\circ Y^1_{r_1}(z_0))$. An easy computation yields

$$\left.\frac{\partial g}{\partial t}\right|_{(0,\bar t_1,\dots,\bar t_q)}=Y((Y^p_{r_p}\circ\cdots\circ Y^1_{r_1}\circ f)(\bar t_1,\dots,\bar t_q))=Y(z_1) \tag{1}$$

and

$$\left.\frac{\partial g}{\partial t_i}\right|_{(0,\bar t_1,\dots,\bar t_q)}=d(Y^p_{r_p}\circ\cdots\circ Y^1_{r_1})_{z_0}\left(\left.\frac{\partial f}{\partial t_i}\right|_{(\bar t_1,\dots,\bar t_q)}\right),\qquad 1\leqslant i\leqslant q. \tag{2}$$

It is clear that the tangent vectors (2) are tangent to N at the point z_1.

By the maximality property of k and the fact that $g(t,t_1,\dots,t_q)\in A_S(x_0)$ for $(t_1,\dots,t_q)\in W_2$ and $t\geqslant 0$, the mapping g must have rank k at $(0,\bar t_1,\dots,\bar t_q)$. Consequently, the tangent vectors (1) and (2) span a k-dimensional subspace of $T_{z_1}M$. But the mapping f has rank k at $(\bar t_1,\dots,\bar t_q)$, so the tangent vectors

$$\left.\frac{\partial f}{\partial t_i}\right|_{(\bar t_1,\dots,\bar t_q)},\qquad 1\leqslant i\leqslant q,$$

span a k-dimensional subspace of $T_{z_0}M$. Since the differential

$$d(Y^p_{r_p}\circ\cdots\circ Y^1_{r_1})_{z_0}\colon T_{z_0}M\to T_{z_1}M$$

is a linear isomorphism, we infer that the tangent vectors (2) by themselves span a k-dimensional subspace of $T_{z_1}M$. It follows that $Y(z_1)$ is a linear combination of the tangent vectors (2). Since each of the tangent vectors in (2) is tangent to N at z_1, we conclude that Y is tangent to N at z_1. Because z_1 can be any point of N, we have Y tangent to N and Claim 1 is proved.

We now fix a countable dense subset S_0 of S in the weak C^0 topology. The existence of S_0 was commented on in Section II.

Claim 2. Let $(Y^1,\dots,Y^p)$ be a finite S-sequence, let $r_1,\dots,r_p$ be nonnegative real numbers, and let $W_2\subseteq W_1$ be an open set in $\mathbb{R}^q$ such that $(Y^p_{r_p}\circ\cdots\circ Y^1_{r_1})(x)$ is defined for every $x\in f(W_3)$. If

$$y\in(Y^p_{r_p}\circ\cdots\circ Y^1_{r_1})(f(W_2)),$$

then there exist a vector field $Z\in S_0$ and an open set $W_3\subseteq W_2$ such that $(Z_{r_p}\circ Y^{p-1}_{r_{p-1}}\circ\cdots\circ Y^1_{r_1})(x)$ is defined for every $x\in f(W_2)$ and

$$y\in(Z_{r_p}\circ Y^{p-1}_{r_{p-1}}\circ\cdots\circ Y^1_{r_1})(f(W_3)).$$

Proof of Claim 2. By Claim 1 the set

$$N = (Y^{p-1}_{r_{p-1}} \circ \cdots \circ Y^1_{r_1})(f(W_2))$$

is a C^1 embedded k-dimensional submanifold of M and every vector field in S is tangent to N (if $p=1$, then set $N = f(W_2)$). Let $z_0 \in N$ be such that $y = Y^p_{r_p}(z_0)$ and observe that $[0, r_p] \times N \subseteq \mathscr{D}(Y^p)$. By Proposition 3.2 there exist an $\varepsilon > 0$ and an open subset V_0 of N containing z_0 such that $(-\varepsilon, \varepsilon) \times V_0 \subseteq \mathscr{D}(Y^p)$ and

$$(t, x) \in (-\varepsilon, \varepsilon) \times V_0 \Rightarrow Y^p_t(x) \in N.$$

This yields the implication

$$\begin{aligned} &s, t \in [0, r_p], \qquad |t-s| < \varepsilon, \qquad \text{and} \qquad x \in V_0 \\ &\quad \Rightarrow Y^p_t(x) = Y^p_s(Y^p_{t-s}(x)) \in Y^p_s(N). \end{aligned} \tag{3}$$

Let $0 = \alpha_0 < \alpha_1 < \cdots < \alpha_l = r_p$ be a partition of $[0, r_p]$ of mesh less than ε. Then using (3) we see that for $1 \leqslant i \leqslant l$

$$\alpha_{i-1} \leqslant t \leqslant \alpha_i \Rightarrow Y^p_t(V_0) \subseteq Y^p_{\alpha_{i-1}}(N). \tag{4}$$

It is clear that the sets $Y^p_{\alpha_i}(N)$, $0 \leqslant i \leqslant l$, are C^1 embedded k-dimensional submanifolds of M. Statement (4) yields the inclusion

$$Y^p_{\alpha_i}(V_0) \subseteq Y^p_{\alpha_{i-1}}(N) \cap Y^p_{\alpha_i}(N), \qquad 1 \leqslant i \leqslant l. \tag{5}$$

Furthermore, for $1 \leqslant i \leqslant l$ the set $Y^p_{\alpha_i}(V_0)$ is open relative to both of the submanifolds $Y^p_{\alpha_{i-1}}(N)$ and $Y^p_{\alpha_i}(N)$. It is obvious that $Y^p_{\alpha_i}(V_0)$ is open relative to $Y^p_{\alpha_i}(N)$. Since $Y^p_{\alpha_{i-1}}(N)$ is a C^1 k-dimensional manifold and $Y^p_{\alpha_i}(V_0)$ is a C^1 k-dimensional submanifold of $Y^p_{\alpha_{i-1}}(N)$ (note that $Y^p_{\alpha_i}(V_0) =$

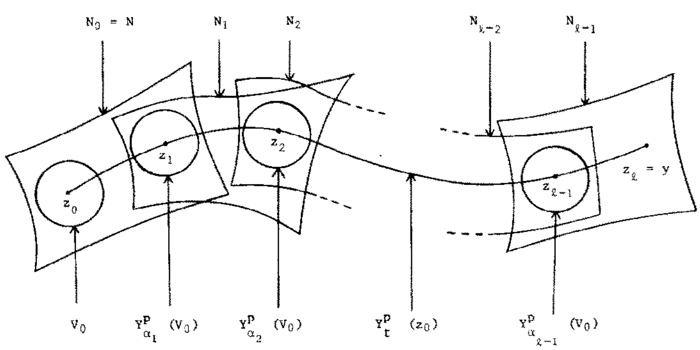

FIG. 3. The chain of k-dimensional submanifolds used in the proof of Claim 2 of Theorem 3.12.

$Y^p_{\alpha_i-\alpha_{i-1}}(Y^p_{\alpha_{i-1}}(V_0))$; cf. Remark 3.5), the inverse function theorem implies that $Y^p_{\alpha_i}(V_0)$ is open relative to $Y^p_{\alpha_{i-1}}(N)$.

For $0 \leqslant i \leqslant l-1$ set $N_i = Y^p_{\alpha_i}(N)$; then N_i is a C^1 embedded k-dimensional submanifold of M and by Claim 1 every vector field in S is tangent to N_i. Note that $N_0 = N$. For $1 \leqslant i \leqslant l$ set $z_i = Y^p_{\alpha_i}(z_0)$ and note that $z_l = y$ (see Fig. 3).

By the local compactness of N_0 there exists an open set $U_0 \subseteq N_0$ such that $z_0 \in U_0 \subseteq \mathrm{cl}_{N_0} U_0 \subseteq V_0$ and $\mathrm{cl}_{N_0} U_0$ is compact. Let $V_1 = Y^p_{\alpha_1}(U_0)$; then $z_1 \in V_1$ and $V_1 \subseteq Y^p_{\alpha_1}(V_0)$ is open in N_0 and N_1. By the local compactness of N_1 there exists an open set $U_1 \subseteq N_1$ such that $z_1 \in U_1 \subseteq \mathrm{cl}_{N_1} U_1 \subseteq V_1$ and $\mathrm{cl}_{N_1} U_1$ is compact. Continuing this process, we obtain sets $U_0, U_1, \ldots, U_{l-1}$ and $V_1, \ldots, V_{l-1}$ such that:

(a) for $0 \leqslant i \leqslant l-1$, $z_i \in U_i \subseteq \mathrm{cl}_{N_i} U_i \subseteq V_i \subseteq N_i$, U_i and V_i are open in N_i, and $\mathrm{cl}_{N_i} U_i$ is compact;

(b) for $1 \leqslant i \leqslant l-1$, $V_i = Y^p_{\alpha_i-\alpha_{i-1}}(U_{i-1})$.

This construction also ensures that:

(c) for $0 \leqslant i \leqslant l-1$, $V_i \subseteq Y^p_{\alpha_i}(V_0)$.

For $0 \leqslant i \leqslant l-1$ define mappings $h_i \colon V_i \to M$ by

$$h_i(x) = Y^p_{\alpha_{i+1}-\alpha_i}(x)$$

and observe that

$$\begin{aligned} h_i(V_i) &= Y^p_{\alpha_{i+1}-\alpha_i}(V_i) \subseteq Y^p_{\alpha_{i+1}-\alpha_i}(Y^p_{\alpha_i}(V_0)) \\ &= Y^p_{\alpha_{i+1}}(V_0) \subseteq Y^p_{\alpha_i}(N) = N_i \end{aligned}$$

by (c) and (5). By Proposition 3.4 we can regard h_i as a continuous mapping of V_i into N_i, $0 \leqslant i \leqslant l-1$.

Let $L_i = h_i(\mathrm{cl}_{N_i} U_i)$ for $0 \leqslant i \leqslant l-1$. Then L_i is a compact subset of the image of h_i and it is easy to check that

$$z_{i+1} \in L_i, \qquad 0 \leqslant i \leqslant l-1,$$

and

$$\begin{aligned} &h_i(V_i) \supseteq L_i \supseteq V_{i+1}, \qquad 0 \leqslant i \leqslant l-2, \\ &h_{l-1}(V_{l-1}) \supseteq L_{l-1}, \qquad y = z_l \in L_{l-1}. \end{aligned} \tag{6}$$

Furthermore, for $0 \leqslant i \leqslant l-1$ the mapping $h_i \colon V_i \to N_i$ has a continuous local right inverse at each point of L_i given by the mapping $x \mapsto Y^p_{\alpha_i-\alpha_{i+1}}(x)$ (suitably restricted to an open neighborhood in N_i of the point in question).

By Theorem 3.11 for each $i = 0, \ldots, l-1$ there exist a compact set $K_i \subseteq V_i$

and an $\varepsilon_i > 0$ such that if $\mathbf{h}_i: K_i \to N_i$ is any continuous mapping satisfying $d(\mathbf{h}_i(x), h_i(x)) \leqslant \varepsilon_i$ for every $x \in K_i$, then $L_i \subseteq \mathbf{h}_i(K_i)$.

Since $[0, r_p] \times N \subseteq \mathscr{D}(Y^p)$, it is easy to see that

$$[0, \alpha_{i+1} - \alpha_i] \times K_i \subseteq \mathscr{D}(Y^p), \qquad 0 \leqslant i \leqslant l-1,$$

and for $0 \leqslant t \leqslant \alpha_{i+1} - \alpha_i$ we have, using (c) and (4),

$$\begin{aligned} Y_t^p(K_i) &\subseteq Y_t^p(V_i) \subseteq Y_t^p(Y_{\alpha_i}^p(V_0)) \\ &= Y_{t+\alpha_i}^p(V_0) \subseteq Y_{\alpha_i}^p(N_0) = N_i. \end{aligned}$$

Thus for each $i = 0, \ldots, l-1$ Proposition 3.8 yields a compact set $C_i \subseteq M$ and a $\delta_i > 0$ such that if $Z \in V^1(M)$ is tangent to N_i and satisfies $\|Z - Y^p\|_{C_i} < \delta_i$, then $[0, \alpha_{i+1} - \alpha_i] \times K_i \subseteq \mathscr{D}(Z)$ and

$$(t, x) \in [0, \alpha_{i+1} - \alpha_i] \times K_i \Rightarrow Z_t(x) \in N_i \qquad \text{and} \qquad d(Z_t(x), Y_t^p(x)) < \varepsilon_i;$$

in particular, $Z_{\alpha_{i+1}-\alpha_i}(x)$ is defined for every $x \in K_i$ and

$$d(Z_{\alpha_{i+1}-\alpha_i}(x), h_i(x)) < \varepsilon_i \qquad \text{for every} \quad x \in K_i.$$

By our choice of the real numbers $\varepsilon_i > 0$ and the compact sets $K_i \subseteq V_i$, we infer that

$$Z_{\alpha_{i+1}-\alpha_i}(K_i) \supseteq L_i, \tag{7}$$

provided that $Z \in V^1(M)$ is tangent to N_i and satisfies $\|Z - Y^p\|_{C_i} < \delta_i$, $0 \leqslant i \leqslant l-1$.

Let $C = \bigcup_{i=0}^{l-1} C_i$ and let $\delta = \min\{\delta_0, \ldots, \delta_{l-1}\}$; then $C \subseteq M$ is compact and δ is positive. Since S_0 is a dense subset of S in the weak C_0 topology, there exists $Z \in S_0$ such that $\|Z - Y^p\|_C < \delta$. Consequently, for $0 \leqslant i \leqslant l-1$ we have $\|Z - Y^p\|_{C_i} < \delta_i$ and Z is tangent to N_i by Claim 1. It follows from (7) that

$$Z_{\alpha_{i+1}-\alpha_i}(K_i) \supseteq L_i, \qquad 0 \leqslant i \leqslant l-1.$$

Hence we obtain

$$y = z_l \in L_{l-1} \subseteq Z_{\alpha_l - \alpha_{l-1}}(K_{l-1}),$$

and for $1 \leqslant i \leqslant l-1$

$$K_i \subseteq V_i = h_{i-1}(U_{i-1}) \subseteq h_{i-1}(\mathrm{cl}_{N_{i-1}} U_{i-1}) = L_{i-1} \subseteq Z_{\alpha_i - \alpha_{i-1}}(K_{i-1}).$$

These relations yield

$$\begin{aligned} y &\in Z_{\alpha_l-\alpha_{l-1}}(K_{l-1}) \\ &\subseteq Z_{\alpha_l-\alpha_{l-1}}(Z_{\alpha_{l-1}-\alpha_{l-2}}(K_{l-2})) \\ &\subseteq \cdots \\ &\subseteq Z_{\alpha_l-\alpha_{l-1}}(Z_{\alpha_{l-1}-\alpha_{l-2}}(\cdots Z_{\alpha_1-\alpha_0}(K_0)\cdots)) \\ &= Z_{\alpha_l-\alpha_0}(K_0) = Z_{r_p}(K_0). \end{aligned}$$

Since

$$K_0 \subseteq N_0 = N = (Y^{p-1}_{r_{p-1}} \circ \cdots \circ Y^1_{r_1})(f(W_2)),$$

there exists $w \in W_2$ such that

$$y = (Z_{r_p} \circ Y^{p-1}_{r_{p-1}} \circ \cdots \circ Y^1_{r_1})(f(w)).$$

By continuity there exists an open set $W_3 \subseteq W_2$ such that $w \in W_3$ and $(Z_{r_p} \circ Y^{p-1}_{r_{p-1}} \circ \cdots \circ Y^1_{r_1})(x)$ is defined for every $x \in f(W_3)$. It is obvious that

$$y \in (Z_{r_p} \circ Y^{p-1}_{r_{p-1}} \circ \cdots \circ Y^1_{r_1})(f(W_3)),$$

so the proof of Claim 2 is complete.

Claim 3. Let $y \in A_S(x_1)$. Then there exist a finite S_0-sequence $(Z^1,..., Z^p)$ and positive real numbers $\bar{t}_1,..., \bar{t}_q, \bar{r}_1,..., \bar{r}_p$ such that

$$y = (Z^p_{\bar{r}_p} \circ \cdots \circ Z^1_{\bar{r}_1} \circ X^q_{\bar{t}_q} \circ \cdots \circ X^1_{\bar{t}_1})(x_0)$$

and the mapping

$$(t_1,..., t_q, r_1,..., r_p) \mapsto (Z^p_{r_p} \circ \cdots \circ Z^1_{r_1} \circ X^q_{t_q} \circ \cdots \circ X^1_{t_1})(x_0) \tag{8}$$

has rank k at $(\bar{t}_1,..., \bar{t}_q, \bar{r}_1,..., \bar{r}_p)$.

Proof of Claim 3. The statement is clear if $y = x_1$. If $y \in A_S(x_1)$ and $y \neq x_1$, then there exist a finite S-sequence $(Y^1,..., Y^p)$ and positive real numbers $\bar{r}_1,..., \bar{r}_p$ such that

$$y = (Y^p_{\bar{r}_p} \circ \cdots \circ Y^1_{\bar{r}_1})(x_1) = (Y^p_{\bar{r}_p} \circ \cdots \circ Y^1_{\bar{r}_1})(f(s_1,..., s_q)).$$

By continuity there exists an open set $W_2 \subseteq W_1$ such that $(s_1,..., s_q) \in W_2$ and $(Y^p_{\bar{r}_p} \circ \cdots \circ Y^1_{\bar{r}_1})(x)$ is defined for every $x \in f(W_2)$. Obviously we have

$$y \in (Y^p_{\bar{r}_p} \circ \cdots \circ Y^1_{\bar{r}_1})(f(W_2)).$$

After p-applications of Claim 2 we obtain vector fields $Z^1,\dots, Z^p \in S_0$ and an open set $W \subseteq W_2$ such that

$$y \in (Z^p_{\bar{r}_p} \circ \cdots \circ Z^1_{\bar{r}_1})(f(W)).$$

Let $(\bar{t}_1,\dots, \bar{t}_q) \in W \subseteq W_1$ be such that

$$\begin{aligned} y &= (Z^p_{\bar{r}_p} \circ \cdots \circ Z^1_{\bar{r}_1})(f(\bar{t}_1,\dots, \bar{t}_q)) \\ &= (Z^p_{\bar{r}_p} \circ \cdots \circ Z^1_{\bar{r}_1} \circ X^q_{\bar{t}_q} \circ \cdots \circ X^1_{\bar{t}_1})(x_0). \end{aligned}$$

By our choice of the open set $W_1 \subseteq \mathbb{R}^q$ the coordinates $\bar{t}_i$ are positive, $1 \leqslant i \leqslant q$, and the mapping

$$(t_1,\dots, t_q) \mapsto (X^q_{t_q} \circ \cdots \circ X^1_{t_1})(x_0)$$

has rank k at $(\bar{t}_1,\dots, \bar{t}_q)$. It follows that the mapping (8) has rank $\geqslant k$ at $(\bar{t}_1,\dots, \bar{t}_q, \bar{r}_1,\dots, \bar{r}_p)$. However, by the maximality property of k the rank is precisely k. This completes the proof of Claim 3.

We now conclude the proof of the theorem. Let

$$T = S_0 \cup \{X^1,\dots, X^q\};$$

then T is a countable family of C^1 vector fields on M. Let

$$\Omega_k = \{x \in A_T(x_0) \mid x \text{ is normally } k\text{-reachable from } x_0 \text{ via } T\}.$$

By the maximality property of k we infer from Proposition 3.10 that Ω_k is first category in M. Since M is locally compact, M is a Baire space and it follows that $\operatorname{int} \Omega_k = \varnothing$. However, Claim 3 implies that $A_S(x_1) \subseteq \Omega_k$, so that $\operatorname{int} A_S(x_1) = \varnothing$. This contradicts the fact that S has the accessibility property and proves the theorem. ∎

IV. The Openness of Controllability in the Fine C^0 Topology

Having established the equivalence of accessibility and normal accessibility, we will now show how this result leads to the openness of complete controllability in the fine C^0 topology for arbitrary systems of C^1 vector fields. One can use similar techniques to prove the openness of the accessibility property in the fine C^0 topology, but we will not do this in detail here (see Remark 4.14). Our presentation has been strongly influenced by Sussmann's paper [6] and especially by Sections 3 and 4 of that paper.

Proposition 4.1. *Let* $\{X^1,\dots, X^q\} \subseteq V^1(M)$, *let* $\rho, s_1,\dots, s_q$ *be positive* *real numbers such that* $0 < \rho < \min\{s_1, ..., s_q\}$, *and let* $K \subseteq M$ *be a compact set such that*

$$x \in K \quad \text{and} \quad |t_i - s_i| \leqslant \rho, \quad 1 \leqslant i \leqslant q,$$
$$\Rightarrow \text{the expression } (X^q_{t_q} \circ \cdots \circ X^1_{t_1})(x) \text{ is defined.}$$

Then for every $\varepsilon > 0$ *there exist a compact set* $C \subseteq M$ *and a* $\delta > 0$ *such that if* $\{Y^1, ..., Y^q\} \subseteq V^1(M)$ *satisfies* $\| Y^i - X^i \|_C < \delta$, $1 \leqslant i \leqslant q$, *then*

$$x \in K \quad \text{and} \quad |t_i - s_i| \leqslant \rho, \quad 1 \leqslant i \leqslant q,$$
$$\Rightarrow \text{the expression } (Y^q_{t_q} \circ \cdots \circ Y^1_{t_1})(x) \text{ is defined}$$

and

$$z, w \in K, \quad d(z, w) < \delta, \quad \text{and} \quad |t_i - s_i| \leqslant \rho, \quad 1 \leqslant i \leqslant q,$$
$$\Rightarrow d((Y^p_{t_q} \circ \cdots \circ Y^1_{t_1})(z), (X^q_{t_q} \circ \cdots \circ X^1_{t_1})(w)) < \varepsilon.$$

Proof. We use induction on q. The case $q = 1$ follows immediately from Proposition 3.7. Let $q > 1$ be given, assume that the proposition holds for $q - 1$, and let $\varepsilon > 0$ be a positive real number. The set

$$K_q = \{(X^{q-1}_{t_{q-1}} \circ \cdots \circ X^1_{t_1})(x) \mid x \in K \text{ and } |t_i - s_i| \leqslant \rho, \ 1 \leqslant i \leqslant q - 1\}$$

is compact in M, and by assumption $[0, s_q + \rho] \times K_q \subseteq \mathscr{D}(X^q)$. Since $\mathscr{D}(X^q)$ is open in $\mathbb{R} \times M$, there exists a compact set $F_q \subseteq M$ such that $K_q \subseteq \operatorname{int} F_q$ and $[0, s_q + \rho] \times F_q \subseteq \mathscr{D}(X^q)$.

By Proposition 3.7 there exist a compact set $C_q \subseteq M$ and a $\delta_q > 0$ such that $Y^q \in V^1(M)$ and $\| Y^q - X^q \|_{C_q} < \delta_q$ imply that $[0, s_q + \rho] \times F_q \subseteq \mathscr{D}(Y^q)$ and

$$t \in [0, s_q + \rho], \quad z, w \in F_q, \quad \text{and} \quad d(z, w) < \delta_q \Rightarrow d(Y^q_t(z), X^q_t(w)) < \varepsilon.$$

Let

$$\boldsymbol{\delta}_q = \begin{cases} \delta_q & \text{if } M \backslash \operatorname{int} F_q = \varnothing, \\ \min\{\delta_q, \operatorname{dist}[K_q, M \backslash \operatorname{int} F_q]\} & \text{if } M \backslash \operatorname{int} F_q \neq \varnothing; \end{cases}$$

note that for M compact we could have $M = F_q = \operatorname{int} F_q$. By the induction assumption there exist a compact set $C_{q-1} \subseteq M$ and a $\delta_{q-1} > 0$ such that if $\{Y^1, ..., Y^{q-1}\} \subseteq V^1(M)$ satisfies $\| Y^i - X^i \|_{C_{q-1}} < \delta_{q-1}$, $1 \leqslant i \leqslant q - 1$, then

$$x \in K \quad \text{and} \quad |t_i - s_i| \leqslant \rho, \quad 1 \leqslant i \leqslant q - 1,$$
$$\Rightarrow \text{the expression } (Y^{q-1}_{t_{q-1}} \circ \cdots \circ Y^1_{t_1})(x) \text{ is defined}$$

and

$$z, w \in K, \qquad d(z, w) < \delta_{q-1}, \qquad \text{and} \qquad |t_i - s_i| \leqslant \rho, \qquad 1 \leqslant i \leqslant q-1,$$
$$\Rightarrow d((Y^{q-1}_{t_{q-1}} \circ \cdots \circ Y^1_{t_1})(z), (X^{q-1}_{t_{q-1}} \circ \cdots \circ X^1_{t_1})(w)) < \boldsymbol{\delta}_q.$$

Then $C = C_{q-1} \cup C_q$ and $\delta = \min\{\delta_{q-1}, \boldsymbol{\delta}_q\}$ are seen to satisfy our requirements. ∎

PROPOSITION 4.2. *Let $S \subseteq V^1(M)$ and let $(x_0, y_0) \in M \times M$ be such that y_0 is normally n-reachable from x_0 via S. Then there exist a finite set $\{X^1,..., X^q\} \subseteq S$, open neighborhoods U of x_0 and V of y_0, a compact set $C \subseteq M$, and a $\delta > 0$ such that if $\{Y^1,..., Y^q\} \subseteq V^1(M)$ satisfies $\| Y^i - X^i \|_C < \delta$, $1 \leqslant i \leqslant q$, then for every $(x, y) \in U \times V$ y is reachable from x via $\{Y^1,..., Y^q\}$.*

Proof. By the definition of normal n-reachability there exist a finite S-sequence $(X^1,..., X^q)$ and positive real numbers $s_1,..., s_q$ such that the mapping

$$f(t_1,..., t_q) = (X^q_{t_q} \circ \cdots \circ X^1_{t_1})(x_0)$$

is defined and C^1 on an open neighborhood of $(s_1,..., s_q)$ in $\mathbb{R}^q$, $f(s_1,..., s_q) = y_0$, and f has rank n at $(s_1,..., s_q)$. Choose a positive real number ρ and an open neighborhood W of x_0 such that $0 < \rho < \min\{s_1,..., s_q\}$, $\bar{W}$ is compact, and $(X^q_{t_q} \circ \cdots \circ X^1_{t_1})(x)$ is defined whenever $x \in \bar{W}$ and $|t_i - s_i| \leqslant \rho$, $1 \leqslant i \leqslant q$. We can regard f as a C^1 mapping of the open set

$$\Sigma = \{(t_1,..., t_q) \in \mathbb{R}^q \mid |t_i - s_i| < \rho, 1 \leqslant i \leqslant q\}$$

into M. Since $f(s_1,..., s_q) = y_0$ and f has rank n at $(s_1,..., s_q)$, the implicit function theorem yields a continuous (in fact C^1) local right inverse of the mapping $f: \Sigma \to M$ at the point y_0. Moreover, by continuity of the derivative of f there exists an open neighborhood V of y_0 in M such that $\bar{V}$ is compact, $\bar{V} \subseteq f(\Sigma)$, and f has a continuous local right inverse at each point of $\bar{V}$ (see [4, Propositions 3.4 and 5.2]).

From Theorem 3.11 we obtain an $\varepsilon > 0$ such that if $\mathbf{f}: \Sigma \to M$ is any continuous mapping satisfying

$$d(\mathbf{f}(t_1,..., t_q), f(t_1,..., t_q)) \leqslant \varepsilon, \qquad (t_1,..., t_q) \in \Sigma,$$

then $\bar{V} \subseteq \mathbf{f}(\Sigma)$. From Proposition 4.1 we obtain a compact set $C \subseteq M$ and a $\delta > 0$ such that if $\{Y^1,..., Y^q\} \subseteq V^1(M)$ satisfies $\| Y^i - X^i \|_C < \delta$, $1 \leqslant i \leqslant q$, then

$$x \in \bar{W} \qquad \text{and} \qquad |t_i - s_i| \leqslant \rho, \qquad 1 \leqslant i \leqslant q,$$
$$\Rightarrow \text{the expression } (Y^q_{t_q} \circ \cdots \circ Y^1_{t_1})(x) \text{ is defined}$$

and

$$z, w \in \bar{W}, \qquad d(z, w) < \delta, \qquad \text{and} \qquad |t_i - s_i| \leqslant \rho, \qquad 1 \leqslant i \leqslant q,$$
$$\Rightarrow d((Y^q_{t_q} \circ \cdots \circ Y^1_{t_1})(z), (X^q_{t_q} \circ \cdots \circ X^1_{t_1})(w)) < \varepsilon. \tag{9}$$

Define U to be the open neighborhood of x_0 in M given by

$$U = W \cap \{x \in M \mid d(x, x_0) < \delta\}.$$

Suppose that $\{Y^1,..., Y^q\} \subseteq V^1(M)$ satisfies $\| Y^i - X^i \|_C < \delta$, $1 \leqslant i \leqslant q$, and let $x \in U$. Then the expression $(Y^q_{t_q} \circ \cdots \circ Y^1_{t_1})(x)$ is defined whenever $|t_i - s_i| \leqslant \rho$, $1 \leqslant i \leqslant q$, so that the mapping $\mathbf{f}\colon \Sigma \to M$ given by

$$\mathbf{f}(t_1,..., t_q) = (Y^q_{t_q} \circ \cdots \circ Y^1_{t_1})(x)$$

is defined and C^1 on Σ. Since $x, x_0 \in W$ and $d(x, x_0) < \delta$, we infer from (9) that

$$d(\mathbf{f}(t_1,..., t_q), f(t_1,..., t_q)) < \varepsilon, \qquad (t_1,..., t_q) \in \Sigma.$$

Our choice of ε implies that $\bar{V} \subseteq \mathbf{f}(\Sigma)$. Consequently, every point of V is reachable from x via $\{Y^1,..., Y^q\}$. Since $x \in U$ was arbitrary, the proof is complete. ∎

Let $S \subseteq V^1(M)$ and let $\Omega \subseteq M$ be an open set. It is clear that Ω is also an n-dimensional manifold of the same differentiability class as M and S induces a family of C^1 vector fields on Ω, which we denote by S_Ω. Following Sussmann [6, p. 296], for $(x, y) \in \Omega \times \Omega$ we say that y is reachable (resp., normally n-reachable) from x via S *within the set* Ω if y is reachable (resp., normally n-reachable) from x via S_Ω in the manifold Ω.

Let $(x, y) \in M \times M$ and suppose that y is reachable (resp., normally n-reachable) from x via S. Then there exist a finite S-sequence $(X^1,..., X^q)$ and nonnegative real numbers $s_1,..., s_q$ such that $(X^q_{s_q} \circ \cdots \circ X^1_{s_1})(x)$ is defined and equals y. We have a corresponding S-*trajectory* η: $[0, s_1 + \cdots + s_q] \to M$ joining x and y given by

$$\eta(t) = \begin{cases} X^1_t(x) & 0 \leqslant t \leqslant s_1, \\ (X^2_{t-s_1} \circ X^1_{s_1})(x) & s_1 \leqslant t \leqslant s_1 + s_2, \\ \quad\vdots & \quad\vdots \\ (X^q_{t-(s_1+\cdots+s_{q-1})} \circ X^{q-1}_{s_{q-1}} \circ \cdots \circ X^1_{s_1})(x) & \\ & s_1 + \cdots + s_{q-1} \leqslant t \leqslant s_1 + \cdots + s_q. \end{cases}$$

Since the image of η is compact and M is locally compact, there exists a relatively compact open set $\Omega \subseteq M$ which contains the image of η. It is evident that y is reachable (resp., normally n-reachable) from x via S within Ω. This observation leads to the following refinement of Proposition 4.2.

COROLLARY 4.3. *Let $S \subseteq V^1(M)$ and let $(x_0, y_0) \in M \times M$ be such that y_0 is normally n-reachable from x_0 via S. Then there exist a relatively compact open set $\Omega \subseteq M$, a finite set $\{X^1,..., X^q\} \subseteq S$, open neighborhoods U of x_0 and V of y_0 with $U, V \subseteq \Omega$, a compact set $C \subseteq \Omega$, and a $\delta > 0$ such that if $\{Y^1,..., Y^q\} \subseteq V^1(M)$ satisfies $\| Y^i - X^i \|_C < \delta$, $1 \leqslant i \leqslant q$, then for every $(x, y) \in U \times V$ y is reachable from x via $\{Y^1,..., Y^q\}$ within Ω.*

Proof. By the remarks preceding the statement of the corollary there exists a relatively compact open set Ω such that y_0 is normally n-reachable from x_0 via S within Ω. The corollary follows by applying Proposition 4.2 to the family of C^1 vector fields S_Ω on the manifold Ω. ∎

PROPOSITION 4.4. *Let $S \subseteq V^1(M)$ and let K, L be compact subsets of M such that for every $(x, y) \in K \times L$ y is normally n-reachable from x via S. Then there exist a relatively compact open set $\Omega \subseteq M$, a finite set $T = \{X^1,..., X^q\} \subseteq S$, a compact set $C \subseteq \Omega$, and a $\delta > 0$ such that if $\{Y^1,..., Y^q\} \subseteq V^1(M)$ satisfies $\| Y^i - X^i \|_C < \delta$, $1 \leqslant i \leqslant q$, then for every $(x, y) \in K \times L$ y is reachable from x via $\{Y^1,..., Y^q\}$ within Ω.*

Proof. By Corollary 4.3 for every $(x, y) \in K \times L$ there exist a relatively compact open set $\Omega(x, y) \subseteq M$, a finite set $T(x, y) = \{Z^1,..., Z^l\} \subseteq S$, open neighborhoods $U(x, y)$ of x and $V(x, y)$ of y, a compact set $C(x, y) \subseteq \Omega(x, y)$, and a $\delta(x, y) > 0$ such that if $\{Y^1,..., Y^l\} \subseteq V^1(M)$ satisfies $\| Y^i - Z^i \|_{C(x,y)} < \delta(x, y)$, $1 \leqslant i \leqslant l$, then for every $(z, w) \in U(x, y) \times V(x, y)$ w is reachable from z via $\{Y^1,..., Y^l\}$ within $\Omega(x, y)$. Since $K \times L$ is compact and

$$K \times L \subseteq \bigcup_{(x,y) \in K \times L} U(x, y) \times V(x, y),$$

there exist $(x_1, y_1),..., (x_p, y_p) \in K \times L$ such that

$$K \times L \subseteq \bigcup_{i=1}^{p} U(x_i, y_i) \times V(x_i, y_i).$$

Then $\Omega = \bigcup_{i=1}^p \Omega(x_i, y_i)$, $T = \bigcup_{i=1}^p T(x_i, y_i)$, $C = \bigcup_{i=1}^p C(x_i, y_i)$, and $\delta = \min\{\delta(x_i, y_i) \mid 1 \leqslant i \leqslant p\}$ are seen to satisfy our requirements. ∎

It will be convenient to state explicitly the following special case of Proposition 4.4.

COROLLARY 4.5. *Let $S \subseteq V^1(M)$ and let $K \subseteq M$ be a compact set such that for every $(x, y) \in K \times K$ y is normally n-reachable from x via S. Then* *there exist a relatively compact open set $\Omega \subseteq M$ and a finite set $T \subseteq S$ such that for every $(x, y) \in K \times K$ y is reachable from x via T within Ω.*

DEFINITION 4.6 [6, p. 296]. A family S of C^1 vector fields on M is said to be *completely controllable* (resp., *normally completely controllable*) if for every $(x, y) \in M \times M$ y is reachable (resp., normally n-reachable) from x via S.

The following theorem is due to Sussmann.

THEOREM 4.7. *Let M be connected and let $S \subseteq V^1(M)$. Then the following statements are equivalent.*

(i) *S is completely controllable.*

(ii) *S is normally completely controllable.*

(iii) *For every $x \in M$ x is normally n-reachable from x via S.*

Proof. See [6, Theorem 4.3]. ∎

Before turning to the proof of the openness of complete controllability in the weak C^0 topology, we first state some interesting consequences of the results derived thus far.

THEOREM 4.8. *Let $S \subseteq V^1(M)$ be completely controllable. Then for every compact set $K \subseteq M$ there exists a finite set $T \subseteq S$ such that for every $(x, y) \in K \times K$ y is reachable from x via T.*

Proof. Theorem 4.7 implies that S is normally completely controllable. Hence for every $(x, y) \in K \times K$ y is normally n-reachable from x via S and the result follows from Corollary 4.5. ∎

COROLLARY 4.9. *If M is compact and $S \subseteq V^1(M)$ is completely controllable, then there exists a finite set $T \subseteq S$ such that T is completely controllable.*

COROLLARY 4.10. *If M is noncompact and $S \subseteq V^1(M)$ is completely controllable, then there exists a countable set $T \subseteq S$ such that T is completely controllable.*

Proof. Let (K_i) be a sequence of compact subsets of M such that $K_i \subseteq K_{i+1}$ for every $i \in \mathbb{N}$ and $M = \bigcup_{i=1}^{\infty} K_i$. By Theorem 4.8 for every $i \in \mathbb{N}$ there exists a finite set $T_i \subseteq S$ such that for every $(x, y) \in K_i \times K_i$ y is reachable from x via T_i. Then $T = \bigcup_{i=1}^{\infty} T_i$ is countable and completely controllable. ∎

We will need two additional propositions for the proof of the main theorem.

PROPOSITION 4.11. *Let $\Omega \subseteq M$ be open, let $T = \{X^1, \dots, X^q\} \subseteq V^1(M)$ be a finite set, let $K \subseteq \Omega$ be compact, let $\Sigma \subseteq \Omega$ be open, and assume that for every $x \in K$ there exists $y \in \Sigma$ such that y is reachable from x via T within Ω. Then there exist a compact set $C \subseteq \Omega$ and a $\delta > 0$ such that if $\{Y^1, \dots, Y^q\} \subseteq V^1(M)$ satisfies $\| Y^i - X^i \|_C < \delta$, $1 \leqslant i \leqslant q$, then for every $x \in K$ there exists $y \in \Sigma$ such that y is reachable from x via $\{Y^1, \dots, Y^q\}$ within Ω.*

Proof. It suffices to consider the special case where $\Omega = M$. The general case of an arbitrary open set $\Omega \subseteq M$ follows from this special case by applying the special case to the manifold Ω and the family of vector fields T_Ω induced on Ω by T.

By assumption for every $x \in K$ there exist a finite T-sequence $(Z^1, \dots, Z^l)$ and positive real numbers $s_1, \dots, s_l$ such that $(Z^l_{s_l} \circ \cdots \circ Z^1_{s_1})(x) \in \Sigma$. It follows by continuity that there exists an open neighborhood U_x of x such that $\bar{U}_x$ is compact and for every $z \in \bar{U}_x$ the expression $(Z^l_{s_l} \circ \cdots \circ Z^1_{s_1})(z)$ is defined and is an element of Σ. Let

$$\varepsilon = \operatorname{dist}[(Z^l_{s_l} \circ \cdots \circ Z^1_{s_1})(\bar{U}_x), M \backslash \Sigma] > 0$$

(the case $M = \Sigma$ is not terribly interesting, so we tacitly assume that $M \backslash \Sigma \neq \varnothing$). Proposition 4.1 yields a compact set $C_x \subseteq M$ and a $\delta_x > 0$ such that if $\{Y^1, \dots, Y^l\} \subseteq V^1(M)$ satisfies $\| Y^i - Z^i \|_{C_x} < \delta_x$, $1 \leqslant i \leqslant l$, then $(Y^l_{s_l} \circ \cdots \circ Y^1_{s_1})(z)$ is defined for every $z \in \bar{U}_x$ and

$$d((Y^l_{s_l} \circ \cdots \circ Y^1_{s_1})(z), (Z^l_{s_l} \circ \cdots \circ Z^1_{s_1})(z)) < \varepsilon;$$

in particular, $(Y^l_{s_l} \circ \cdots \circ Y^1_{s_1})(\bar{U}_x) \subseteq \Sigma$.

Find such sets U_x, C_x and positive real numbers δ_x for every $x \in K$. Since $K \subseteq \bigcup_{x \in K} U_x$ and K is compact, there exist $x_1, \dots, x_p \in K$ such that $K \subseteq \bigcup_{i=1}^p U_{x_i}$. Then the compact set $C = \bigcup_{i=1}^p C_{x_i}$ and the positive real number $\delta = \min\{\delta_{x_i} \mid 1 \leqslant i \leqslant p\}$ are seen to satisfy our requirements. ∎

PROPOSITION 4.12. *Let $\Omega \subseteq M$ be open, let $T = \{X^1, \dots, X^q\} \subseteq V^1(M)$ be a finite set, let $K \subseteq \Omega$ be compact, let $\Sigma \subseteq \Omega$ be open, and assume that for every $x \in K$ there exists $y \in \Sigma$ such that x is reachable from y via T within Ω. Then there exist a compact set $C \subseteq \Omega$ and a $\delta > 0$ such that if $\{Y^1, \dots, Y^q\} \subseteq V^1(M)$ satisfies $\| Y^i - X^i \|_C < \delta$, $1 \leqslant i \leqslant q$, then for every $x \in K$ there exists $y \in \Sigma$ such that x is reachable from y via $\{Y^1, \dots, Y^q\}$ within Ω.*

Proof. This follows by applying Proposition 4.11 to the set $-T = \{X \in V^1(M) \mid -X \in T\}$. ∎

We are now prepared to prove the principal result of this section. Our argument is based on techniques used by Sussmann to prove a related result [6, Lemma 3.9]. We have made appropriate modifications to handle infinite systems of vector fields and to prove openness in the fine C^0 topology.

THEOREM 4.13. *Let $\mathscr{A}$ be an index set and let $S = \{X^\alpha \mid \alpha \in \mathscr{A}\}$ be a family of C^1 vector fields on M indexed by $\mathscr{A}$. If S is completely controllable, then there exists a positive continuous function $\psi: M \to (0, \infty)$ such that if $R = \{Y^\alpha \mid \alpha \in \mathscr{A}\}$ is another family of C^1 vector fields on M indexed by $\mathscr{A}$ which satisfies*

$$\| Y^\alpha(x) - X^\alpha(x)\|_\omega < \psi(x) \qquad \textit{for every } x \in M, \textit{for every } \alpha \in \mathscr{A},$$

then R is completely controllable.

Proof. By Theorem 4.7 S is normally completely controllable. Let (K_i) be a sequence of compact subsets of M such that $K_i \subseteq \operatorname{int} K_{i+1}$ for every $i \in N$ and $M = \bigcup_{i=1}^{\infty} K_i$. We will construct an associated sequence of compact sets (L_i) and a collection $\{T_i \mid i \geqslant 2\}$ of finite subsets of S as follows.

Let $L_1 = K_1$ and $L_2 = K_2$. Since S is normally completely controllable, for every $(x, y) \in L_2 \times L_2$ y is normally n-reachable from x via S. Proposition 4.4 yields a relatively compact open set $\Omega_2 \subseteq M$, a finite set $T_2 = \{X^\alpha \mid \alpha \in \mathscr{A}_2\} \subseteq S$ ($\mathscr{A}_2 \subseteq \mathscr{A}$ finite), a compact set $C_2 \subseteq \Omega_2$, and a $\delta_2 > 0$ such that if $P_2 = \{Y^\alpha \mid \alpha \in \mathscr{A}_2\} \subseteq V^1(M)$ satisfies $\| Y^\alpha - X^\alpha \|_{C_2} < \delta_2$, $\alpha \in \mathscr{A}_2$, then for every $(x, y) \in L_2 \times L_2$ y is reachable from x via P_2 within Ω_2.

For $i \geqslant 3$ the L_i and T_i are constructed inductively. Let $L_3 = K_3 \cup \bar{\Omega}_2$; then L_3 is compact and for every $(x, y) \in L_3 \times L_3$ y is normally n-reachable from x via S. Corollary 4.5 yields a relatively compact open set $\Omega_3 \subseteq M$ and a finite set $T_3 = \{X^\alpha \mid \alpha \in \mathscr{A}_3\} \subseteq S$ ($\mathscr{A}_3 \subseteq \mathscr{A}$ finite) such that for every $(x, y) \in L_3 \times L_3$ y is reachable from x via T_3 within Ω_3.

Let $i \geqslant 3$ and assume that we have chosen a compact set $L_i \subseteq M$ and a finite set $T_i = \{X^\alpha \mid \alpha \in \mathscr{A}_i\} \subseteq S$ ($\mathscr{A}_i \subseteq \mathscr{A}$ finite) such that for $(x, y) \in L_i \times L_i$ y is reachable from x via T_i within a relatively compact open set Ω_i. Let $L_{i+1} = K_{i+1} \cup \bar{\Omega}_i$; then L_{i+1} is compact and for every $(x, y) \in L_{i+1} \times L_{i+1}$ y is normally n-reachable from x via S. Corollary 4.5 yields a relatively compact open set $\Omega_{i+1} \subseteq M$ and a finite set $T_{i+1} = \{X^\alpha \mid \alpha \in \mathscr{A}_{i+1}\} \subseteq S$ ($\mathscr{A}_{i+1} \subseteq \mathscr{A}$ finite) such that for every $(x, y) \in L_{i+1} \times L_{i+1}$ y is reachable from x via T_{i+1} within Ω_{i+1}.

By induction we obtain a sequence of compact sets (L_i) and a collection $\{T_i \mid i \geqslant 2\}$ of finite subsets of S which have the following properties:

(a) for every $i \in \mathbb{N}$ $K_i \subseteq L_i \subseteq \operatorname{int} L_{i+1}$;

(b) $M = \bigcup_{i=1}^{\infty} L_i$;

(c) for every $i \geqslant 2$ and for every $(x, y) \in L_i \times L_i$ y is reachable from x via T_i within $\operatorname{int} L_{i+1}$.

We establish two additional properties.

Claim 1. For every $i \geqslant 3$ and for every x in the compact set $L_i \backslash \operatorname{int} L_{i-1}$ there exists y in the open set $\operatorname{int} L_{i-1} \backslash L_{i-2}$ such that y is reachable from x via T_i within the open set $\operatorname{int} L_{i+1} \backslash L_{i-2}$.

Proof of Claim 1. Let $x \in L_i \backslash \operatorname{int} L_{i-1}$ and let $z \in \operatorname{int} L_{i-1} \backslash L_{i-2}$. By property (c) above z is reachable from x via T_i within the open set $\operatorname{int} L_{i+1}$. Hence there exists a T_i-trajectory η: $[0, \bar{t}] \rightarrow \operatorname{int} L_{i+1}$ such that $\eta(0) = x$ and $\eta(\bar{t}) = z$. If $\eta([0, \bar{t}]) \subseteq \operatorname{int} L_{i+1} \backslash L_{i-2}$, then we are done. Otherwise $\eta([0, \bar{t}]) \cap L_{i-2} \neq \varnothing$ and the set

$$B = \{t \in [0, \bar{t}] \mid \eta(t) \in L_{i-2}\}$$

is nonempty and closed in $[0, \bar{t}]$. Since $0 \notin B$, we have $t^* = \inf B > 0$. It follows that $\eta([0, t^*)) \subseteq \operatorname{int} L_{i+1} \backslash L_{i-2}$ and $\eta(t^*) \in L_{i-2} \subseteq \operatorname{int} L_{i-1}$. The continuity of η yields $s \in [0, t^*)$ such that $\eta(s) \in \operatorname{int} L_{i-1}$. Therefore the T_i-trajectory η satisfies $\eta([0, s]) \subseteq \operatorname{int} L_{i+1} \backslash L_{i-2}$, $\eta(0) = x$, and $\eta(s) \in \operatorname{int} L_{i-1} \backslash L_{i-2}$. This proves Claim 1.

Claim 2. For every $i \geqslant 3$ and for every x in the compact set $L_i \backslash \operatorname{int} L_{i-1}$ there exists y in the open set $\operatorname{int} L_{i-1} \backslash L_{i-2}$ such that x is reachable from y via T_i within the open set $\operatorname{int} L_{i+1} \backslash L_{i-2}$.

Proof of Claim 2. The proof is very similar to the proof of Claim 1 so we omit the details.

Let $i \geqslant 3$. Using Claim 1 and Proposition 4.11 with $\Omega = \operatorname{int} L_{i+1} \backslash L_{i-2}$, we obtain a compact set $D_i \subseteq \operatorname{int} L_{i+1} \backslash L_{i-2}$ and a $\lambda_i > 0$ such that if $P_i = \{Y^\alpha \mid \alpha \in \mathscr{A}_i\} \subseteq V^1(M)$ satisfies $\| Y^\alpha - X^\alpha \|_{D_i} < \lambda_i$, $\alpha \in \mathscr{A}_i$ (recall that $\mathscr{A}_i \subseteq \mathscr{A}$ is the finite index set of $T_i \subseteq S$), then for every $x \in L_i \backslash \operatorname{int} L_{i-1}$ there exists $y \in \operatorname{int} L_{i-1} \backslash L_{i-2}$ such that y is reachable from x via P_i within $\operatorname{int} L_{i+1} \backslash L_{i-2}$.

Using Claim 2 and Proposition 4.12 with $\Omega = \operatorname{int} L_{i+1} \backslash L_{i-2}$, we obtain a compact set $E_i \subseteq \operatorname{int} L_{i+1} \backslash L_{i-2}$ and a $\rho_i > 0$ such that if $P_i = \{Y^\alpha \mid \alpha \in \mathscr{A}_i\} \subseteq V^1(M)$ satisfies $\| Y^\alpha - X^\alpha \|_{E_i} < \rho_i$, $\alpha \in \mathscr{A}_i$, then for every $x \in L_i \backslash \operatorname{int} L_{i-1}$ there exists $y \in \operatorname{int} L_{i-1} \backslash L_{i-2}$ such that x is reachable from y via P_i within $\operatorname{int} L_{i+1} \backslash L_{i-2}$.

For $i \geqslant 3$ we set $C_i = D_i \cup E_i \subseteq \operatorname{int} L_{i+1} \backslash L_{i-2}$ and $\delta_i = \min\{\lambda_i, \rho_i\}$. Let C_2 and δ_2 be the compact set and positive real number determined above and note that $C_2 \subseteq \operatorname{int} L_3$. The family of open sets

$$\{\operatorname{int} L_3\} \cup \{\operatorname{int} L_{i+1} \backslash L_{i-2} \mid i \geqslant 2\}$$

is easily seen to be neighborhood finite. Consequently, the family of compact sets $\{C_i \mid i \geqslant 2\}$ is also neighborhood finite. By a standard argument there exists a positive continuous function $\psi: M \to (0, \infty)$ such that

$$x \in C_i \Rightarrow \psi(x) \leqslant \delta_i \qquad \text{for every} \quad i \geqslant 2.$$

Let $R = \{Y^\alpha \mid \alpha \in \mathscr{A}\}$ be a family of C^1 vector fields on M indexed by $\mathscr{A}$ such that $\| Y^\alpha(x) - X^\alpha(x)\|_\omega < \psi(x)$ for every $x \in M$, for every $\alpha \in \mathscr{A}$. For $i \geqslant 2$ let $P_i = \{Y^\alpha \mid \alpha \in \mathscr{A}_i\} \subseteq R$. It is clear that for every $i \geqslant 2$ we have

$$\| Y^\alpha - X^\alpha \|_{C_i} < \delta_i, \qquad \alpha \in \mathscr{A}_i$$

(in fact this holds for every $\alpha \in \mathscr{A}$). We will show by induction on i that for every $i \geqslant 2$ and for every $(x, y) \in L_i \times L_i$ y is reachable from x via $\bigcup_{j=2}^{i} P_j$. Since $M = \bigcup_{i=1}^{\infty} L_i$, this will obviously imply that R is completely controllable.

For $i = 2$ the choice of C_2 and δ_2 and the inequalities $\| Y^\alpha - X^\alpha \|_{C_2} < \delta_2$, $\alpha \in \mathscr{A}_2$, imply that for every $(x, y) \in L_2 \times L_2$ y is reachable from x via P_2.

Let $i \geqslant 2$ and assume that for every $(x, y) \in L_i \times L_i$ y is reachable from x via $\bigcup_{j=2}^{i} P_j$. Let $(x, y) \in L_{i+1} \times L_{i+1}$. We must show that y is reachable from x via $\bigcup_{j=2}^{i+1} P_j$. It will be convenient to consider several cases.

Case 1. $x \in L_i$ and $y \in L_i$. Then y is reachable from x via $\bigcup_{j=2}^{i} P_j$ by the induction assumption, so obviously y is reachable from x via $\bigcup_{j=2}^{i+1} P_j$.

Case 2. $x \in L_{i+1} \backslash L_i$ and $y \in L_i$. Since $D_{i+1} \subseteq C_{i+1}$ and $\delta_{i+1} \leqslant \lambda_{i+1}$, we see that

$$\| Y^\alpha - X^\alpha \|_{C_{i+1}} < \delta_{i+1}, \qquad \alpha \in \mathscr{A}_{i+1},$$
$$\Rightarrow \| Y^\alpha - X^\alpha \|_{D_{i+1}} < \lambda_{i+1}, \qquad \alpha \in \mathscr{A}_{i+1}.$$

The choice of D_{i+1} and λ_{i+1} implies that there exists $z \in \operatorname{int} L_i \backslash L_{i-1}$ such that z is reachable from x via P_{i+1}. By the induction assumption y is reachable from z via $\bigcup_{j=2}^{i} P_j$. Hence y is reachable from x via $\bigcup_{j=2}^{i+1} P_j$.

Case 3. $x \in L_i$ and $y \in L_{i+1} \backslash L_i$. Since $E_{i+1} \subseteq C_{i+1}$ and $\delta_{i+1} \leqslant \rho_{i+1}$, we see that

$$\| Y^\alpha - X^\alpha \|_{C_{i+1}} < \delta_{i+1}, \qquad \alpha \in \mathscr{A}_{i+1},$$
$$\Rightarrow \| Y^\alpha - X^\alpha \|_{E_{i+1}} < \rho_{i+1}, \qquad \alpha \in \mathscr{A}_{i+1}.$$

The choice of E_{i+1} and ρ_{i+1} implies that there exists $w \in \operatorname{int} L_i \backslash L_{i-1}$ such that y is reachable from w via P_{i+1}. By the induction assumption w is reachable from x via $\bigcup_{j=2}^{i} P_j$. Hence y is reachable from x via $\bigcup_{j=2}^{i+1} P_j$.

Case 4. $x \in L_{i+1} \backslash L_i$ and $y \in L_{i+1} \backslash L_i$. Arguing as in Case 2, we obtain $z \in \operatorname{int} L_i \backslash L_{i-1}$ such that z is reachable from x via P_{i+1}. Arguing as in

Case 3, we obtain $w \in \operatorname{int} L_i \backslash L_{i-1}$ such that y is reachable from w via P_{i+1}. By the induction assumption w is reachable from z via $\bigcup_{j=2}^{i} P_j$. Hence y is reachable from x via $\bigcup_{j=2}^{i+1} P_j$. This completes the proof of the theorem. ∎

Remark 4.14. A result analogous to Theorem 4.13 holds for the accessibility property. One must make appropriate modifications to an argument given by Sussmann [6, Lemma 3.8], taking into account the results developed in the first part of this section. We omit the details.

We will conclude by briefly discussing a reinterpretation of Theorem 4.13 for control systems specified in a more traditional form.

DEFINITION 4.15 [1]. Let Ω be a Hausdorff topological space. A C^1 *control vector field* on M with *control space* Ω is a continuous mapping $\xi: M \times \Omega \to TM$ such that $(\pi \circ \xi)(x, w) = x$ for every $(x, w) \in M \times \Omega$ and the mapping $X^w: M \to TM$ given by $X^w(x) = \xi(x, w)$ is a C^1 vector field on M for every $w \in \Omega$. We let $\Gamma^1(M \times \Omega, TM)$ denote the set of all C^1 control vector fields on M with control space Ω.

In the case where $M = \mathbb{R}^n$, there is a natural identification of $T\mathbb{R}^n$ with $\mathbb{R}^n \times \mathbb{R}^n$, which in turn leads to an identification of a control vector field $\xi: \mathbb{R}^n \times \Omega \to T\mathbb{R}^n$ with a mapping $f: \mathbb{R}^n \times \Omega \to \mathbb{R}^n$ [4, Remark 2.5(ii)]. It is customary in this situation to refer to the control vector field ξ as the *control system* $\dot{x} = f(x, w)$.

Let $\xi: M \times \Omega \to TM$ be a C^1 control vector field. A *control* is a mapping $u: \mathbb{R} \to \Omega$ such that u is piecewise constant on every compact subinterval of $\mathbb{R}$. For $x \in M$ and u a control we let $J(x, u)$ denote the maximal subinterval of $\mathbb{R}$ containing 0 on which a solution of the initial-value problem

$$\dot{\sigma}(t) = \xi(\sigma(t), u(t)), \qquad \sigma(0) = x,$$

can be defined. We denote the solution defined on this maximal subinterval by $\mu_{(x,u)}: J(x, u) \to M$. The *attainable set* of ξ from x is defined by

$$A_\xi(x) = \{\mu_{(x,u)}(t) \mid u \text{ is a control, } t \in J(x, u), \text{ and } t \geqslant 0\}.$$

We say that ξ is *completely controllable* if $A_\xi(x) = M$ for every $x \in M$.

A C^1 control vector field $\xi: M \times \Omega \to TM$ induces a family of C^1 vector fields $S(\xi) = \{X^w \mid w \in \Omega\}$ on M, where $X^w(x) = \xi(x, w)$ for every $(x, w) \in M \times \Omega$. Since the controls have been stipulated to be piecewise-constant mappings of $\mathbb{R}$ into Ω, it follows easily that $A_\xi(x) = A_{S(\xi)}(x)$. In particular, ξ is completely controllable if and only if $S(\xi)$ is completely controllable.

For $\xi \in \Gamma^1(M \times \Omega, TM)$ and $\psi: M \times \Omega \to (0, \infty)$ continuous, we define

$$\mathcal{N}(\xi; \psi) = \{\eta \in \Gamma^1(M \times \Omega, TM) \mid \|\eta(x, w) - \xi(x, w)\|_\omega < \psi(x, w) \text{ for every } (x, w) \in M \times \Omega\}.$$

As ξ ranges over $\Gamma^1(M \times \Omega, TM)$ and ψ ranges over all continuous functions of $M \times \Omega$ into $(0, \infty)$, the resulting family of sets $\mathcal{N}(\xi; \psi)$ is easily seen to form an open basis for a topology on $\Gamma^1(M \times \Omega, TM)$, which we call the *fine C^0 topology*.

In this context Theorem 4.13 can be recast into the following form.

THEOREM 4.16. *Let $\xi: M \times \Omega \to TM$ be a completely controllable C^1 control vector field. Then there exists a continuous function $\psi: M \to (0, \infty)$ such that if $\eta: M \times \Omega \to TM$ is a C^1 control vector field satisfying*

$$\|\eta(x, w) - \xi(x, w)\|_\omega < \psi(x) \qquad \textit{for every} \quad (x, w) \in M \times \Omega,$$

then η is completely controllable. In particular, the set of completely controllable control vector fields in $\Gamma^1(M \times \Omega, TM)$ is an open subset of $\Gamma^1(M \times \Omega, TM)$ in the fine C^0 topology.

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