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Mathematics of Control, Signals, and Systems

Controllability on Classical Lie Groups*

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Abstract. We consider the problem of accessibility and controllability of certain bilinear systems. These evolve on Lie groups whose Lie algebras are the normal real forms of complex simple Lie algebras. Previous results by other authors were obtained under the assumption that the controlled vector field is strongly regular. Our paper is aimed at weakening this requirement, and involves relating the root structure of elements in a Lie algebra as above to the nodal connection graphs obtained from their standard matrix representations. This is in turn related to a standard irreducibility assumption on the uncontrolled vector field. The abstract results on generation of Lie algebras are of some independent interest.

Key words. Semisimple Lie algebras, Lie groups, Controllability.

1. Introduction

This paper is concerned with the study of global controllability of bilinear systems evolving on certain semisimple matrix Lie groups G. From the work of Levitt and Sussmann [LS] it follows that on each smooth connected paracompact manifold (in particular on G) there exists a globally controllable set of two smooth vector fields and, more recently, Aeyels [A] proved that there exists a large class of systems globally controllable by means of two vector fields. However, these results cannot be applied if we insist that the vector fields are left- (right)-invariant on G.

We consider control systems which are described by an equation in a connected matrix Lie group $G \subset \mathrm{GL}(n, \mathbb{R})$, of the form

$$\dot{x}(t) = (Au + Bv)x(t), \qquad u, v \in R, \quad x \in G, \tag{1}$$

or

$$\dot{x}(t) = (A + uB)x(t), \qquad u \in R, \quad x \in G, \tag{1'}$$

where A and B are elements of the corresponding matrix Lie algebra $\mathscr{L}$.

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In general, we may view system (1) or (1′) as evolving on an arbitrary connected Lie group with A and B left-invariant vector fields on G. It is well known that a necessary condition for global controllability of (1) or (1′) is that the Lie algebra generated by A and B be equal to $\mathscr{L}$. This condition is also sufficient in the case of system (1), and if G is compact or if the one-parameter subgroup generated by A is compact or quasi-periodic then it is also sufficient in the case of system (1′). The general problem of establishing conditions for global controllability has a considerable history and so we only mention some of the papers pertinent to our present work.

In 1981 Jurdjevic and Kupka [JK2] found conditions for a system $\dot{x} = (A + uB)x$, $u \in \mathbb{R}$, to be controllable in the case $G = \mathrm{SL}(n, \mathbb{R})$ and in [JK1] they generalized the previous work to systems defined on semisimple Lie groups with a finite center. Later, in 1982, and for $G = \mathrm{SL}(n, \mathbb{R})$ the Jurdjevic–Kupka criteria were improved by Gauthier and Bornard [GB] by proving that if B is a strongly regular (diagonal) matrix and $a_{1n} \cdot a_{n1} > 0$ for system (1′) then a necessary and sufficient condition for global controllability is that A be a permutation-irreducible matrix. More recently, Gauthier *et al.* [GKS] partially improved the results of [JK1] and generalized the results of [GB] for some classes of simple Lie groups.

The concept of a "strongly regular element" plays a very important role in all the papers [GB], [GKS], [JK1], [JK2]; indeed, it is a principal assumption in all of these papers that B is a strongly regular element. The purpose of this paper is to weaken this assumption for some classes of systems by replacing it by one depending on the other generator A.

The structure of this paper is as follows. In Section 2 we review some ideas and results used throughout the paper. In Section 3 we give sufficient conditions for the Lie algebra generated by A and B, denoted $\{A, B\}_{\mathrm{L.A.}}$, to equal $\mathscr{L}$ in the cases where $\mathscr{L}$ is a normal real form of one of the classical complex simple Lie algebras of type $\mathscr{A}_n$, $\mathscr{B}_n$, $\mathscr{C}_n$, or $\mathscr{D}_n$. In Section 4 we apply the results of the previous section and the main result in [GKS] to state a necessary and sufficient condition for a particular class of systems (1′) to be globally controllable when $\mathscr{L}$ is of type $\mathscr{A}_n$ or $\mathscr{D}_n$.

Besides trying to improve results in [GB], [GKS], [JK1], and [JK2] motivation for this paper comes from a search for generators of matrix Lie algebras which have a very simple structure, such as fewest possible nonzero entries. See [S] and [CS] for a problem in the generation of Lie groups which uses in its solution similar properties of Lie algebras. In an attempt to capture this idea, in Sections 3 and 4 we introduce the notion of minimal elements A, and show that they have an appealling interpretation in terms of the basic construction used in this paper, that of a root graph introduced in Section 3.

2. Basic Results and Definitions

We refer to Helgason [H] and Wan [W] for more details concerning the ideas of this section.

Let g be a finite-dimensional semisimple Lie algebra over $\mathbb{C}$, h a Cartan subalgebra of g, and φ the set of nonzero roots of g (with respect to h). For each $\alpha \in \varphi$, there exists a unique $H_\alpha \in h$ such that $\langle H, H_\alpha \rangle = \alpha(H)$, for all $H \in h$; $\langle \cdot, \cdot \rangle$ is the bilinear form on $g \times g$ defined by $\langle X, Y \rangle = \operatorname{trace}(\operatorname{ad} X \operatorname{ad} Y)$ and is called the killing form of g. Define $h_R = \sum_{\alpha \in \varphi} \mathbb{R} H_\alpha$; the killing form is strictly positive definite on $h_R \times h_R$ and $h = h_R \oplus i h_R$ where $i = \sqrt{-1}$ [H, p. 170]. It is convenient to identify φ with h_R^* and then define an ordering of φ induced by some vector space ordering of h_R^* (the dual space of h_R). A positive root is called fundamental if it cannot be written as a sum of two positive roots. Δ^+ will denote the set of fundamental roots.

For each $\alpha \in \varphi$, there also exists an element $E_\alpha \in g$ such that $\langle E_\alpha, E_{-\alpha} \rangle = 1$ and for all $\alpha, \beta \in \varphi$

$$\begin{aligned} [E_\alpha, E_{-\alpha}] &= H_\alpha, \\ [H, E_\alpha] &= \alpha(H) E_\alpha \qquad \text{for all} \quad H \in h, \\ [E_\alpha, E_\beta] &= \begin{cases} 0 & \text{if } \alpha + \beta \notin \varphi, \\ N_{\alpha,\beta} E_{\alpha+\beta} & \text{if } \alpha + \beta \in \varphi, \end{cases} \end{aligned} \tag{2}$$

where $N_{\alpha,\beta}$ are real constants satisfying $N_{\alpha,\beta} = -N_{-\alpha,-\beta}$. $\{H_\alpha, \alpha \in \Delta^+\} \cup \{E_\alpha, \alpha \in \varphi\}$ is called a *Weyl basis* of g (with respect to h).

Weyl bases are important for the study of normal real forms. In fact, the subspace

$$\mathscr{L} = \sum_{\alpha \in \varphi} \mathbb{R} H_\alpha \oplus \sum_{\alpha \in \varphi} \mathbb{R} E_\alpha \tag{3}$$

is a normal real form of g, unique up to an isomorphism [H, p. 426]. It is obvious that h_R is a maximal abelian subalgebra of the normal real form of g.

According to (3) every $A \in \mathscr{L}$ admits a unique decomposition of the form

$$A = A_0 + \sum_{\alpha \in \varphi_a} k_\alpha E_\alpha,$$

where $\varphi_a \subset \varphi$, $k_\alpha \in \mathbb{R} \backslash \{0\}$, and $A_0 \in h_R$. In all that follows, whenever we write A in the form above we always assume that $k_\alpha \neq 0$ for all $\alpha \in \varphi_a$ and $A_0 \in h_R$.

The next definition is fundamental for our objectives.

Definition 1. Given $A \in \mathscr{L}$ with $A \notin h_R$ we say that $B \in h_R$ is *A-strongly regular* if the elements $\alpha(B)$, $\alpha \in \varphi_a$, are nonzero and distinct.

In contrast, as in [GB], an element B is strongly regular if $B \in h_R$ and $\alpha(B)$, $\alpha \in \varphi$, are nonzero and distinct!

Obviously the set of A-strongly regular elements is open and dense in h_R, as is the set of strongly regular elements.

Another concept we need is that of an irreducible matrix, also used in [GB]; we include the definition for the sake of completeness.

Definition 2. We say that an $n \times n$ matrix A is P-irreducible if there exists no permutation matrix P such that

$$P^{-1} A P = \begin{bmatrix} A_1 & A_3 \\ 0 & A_2 \end{bmatrix},$$

where A_1 is a $k \times k$ matrix and $0 < k < n$.

Corresponding to every square matrix we can associate a graph as follows. If $A = [a_{ij}]$ is $n \times n$, then graph (A) consists of n nodes; an oriented edge joins node i to node j if and only if $a_{ij} \neq 0$. A graph is strongly connected if and only if any ordered pair of nodes can be joined by an oriented path. A classical result is the following:

Theorem 1 [V]. *A square matrix A is P-irreducible if and only if its graph is strongly connected.*

Another way of viewing P-irreducibility involves the notion of a strongly connected subgraph; namely, if $\Gamma \subset \text{graph}(A) = X$ is a subgraph then we say that Γ is strongly connected (to Σ = largest subgraph contained in $X \backslash \Gamma$) if there exists an oriented path (in X) joining Γ to Σ and an oriented path joining Σ to Γ.

A subgraph of X which is not strongly connected will be called strongly disconnected, and X and φ are considered to be strongly connected.

We then have the following simply proved result.

Lemma 1. *A square matrix A is P-irreducible if and only if its graph does not contain any strongly disconnected subgraph.*

3. Generators and Graphs

We consider systems (1) and (1′) in which A, $B \in \mathscr{L}$ and $B \in h_R$, where $\mathscr{L}$ is the normal form of a classical complex Lie algebra g of type $\mathscr{A}_n$, $\mathscr{B}_n$, $\mathscr{C}_n$, or $\mathscr{D}_n$. We may assume that B is a diagonal matrix.

As we have already stated, a necessary condition for controllability of systems (1) and (1′) is the controllability rank condition, $\{A, B\}_{\text{L.A.}} = \mathscr{L}$. A necessary condition for $\{A, B\}_{\text{L.A.}} = \mathscr{L}$ is that A be P-irreducible. In fact, since B is diagonal if A were reducible, A and B would generate a Lie subalgebra of $\mathscr{L}$ consisting of reducible matrices [GB].

We now refer to Chapter 3 of [H] or pp. 7–10 of [W] for explicit descriptions of the root structure and eigenvector structure of the Lie algebras g, and in particular for the explicit matrix representations of the elements E_α and H_α.

For each $\alpha \in \varphi$ we define an *elementary root graph* as simply the graph of E_α. Table 1 describes the elementary root-graphs for the classical simple Lie algebras g, and their corresponding root structures.

Definition 3. The *root-graph* of $A = A_0 + \sum_{\alpha \in \varphi_a} k_\alpha E_\alpha$ is the union of the elementary root-graphs of E_α, $\alpha \in \varphi_a$.

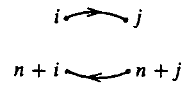
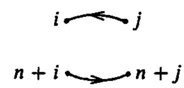
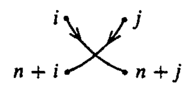
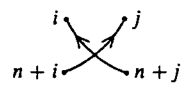
Remark 1. It is obvious that the root-graph of A is just the graph of $A - A_0$. However, since the component of A in the Cartan subalgebra does not interfere with the irreducibility of A we can say that A is P-irreducible if and only if its root-graph is strongly connected.

Table 1

$\mathfrak{g}$	Φ	Elementary root-graphs
$\mathscr{A}_n$	$\lambda_i - \lambda_j,\quad 1 \leqq i < j \leqq n+1$	$i \rightarrow j$
	$-\lambda_i + \lambda_j,\quad 1 \leqq i < j \leqq n+1$	$i \leftarrow j$
$\mathscr{B}_n$	$\lambda_i - \lambda_j,\quad 1 \leqq i < j \leqq n$	$i+1 \rightarrow j+1$ $n+i+1 \leftarrow n+j+1$
	$-\lambda_i + \lambda_j,\quad 1 \leqq i < j \leqq n$	$i+1 \leftarrow j+1$ $n+i+1 \rightarrow n+j+1$
	$\lambda_i + \lambda_j,\quad 1 \leqq i < j \leqq n$	$i+1 \rightarrow n+j+1$ $j+1 \rightarrow n+i+1$
	$-\lambda_i - \lambda_j,\quad 1 \leqq i < j \leqq n$	$n+j+1 \rightarrow i+1$ $n+i+1 \rightarrow j+1$
	$\lambda_i,\quad 1 \leqq i \leqq n$	$i+1 \rightarrow 1 \rightarrow n+i+1$
	$-\lambda_i,\quad 1 \leqq i \leqq n$	$n+i+1 \rightarrow 1 \rightarrow i+1$
$\mathscr{C}_n$	$\lambda_i - \lambda_j,\quad 1 \leqq i < j \leqq n$	$i \rightarrow j$ $n+i \leftarrow n+j$
	$-\lambda_i + \lambda_j,\quad 1 \leqq i < j \leqq n$	$i \leftarrow j$ $n+i \rightarrow n+j$
	$\lambda_i + \lambda_j,\quad 1 \leqq i < j \leqq n$	$i \rightarrow n+j$ $j \rightarrow n+i$
	$-\lambda_i - \lambda_j,\quad 1 \leqq i < j \leqq n$	$n+j \rightarrow i$ $n+i \rightarrow j$
	$2\lambda_i,\quad 1 \leqq i \leqq n$	$i \rightarrow n+i$
	$-2\lambda_i,\quad 1 \leqq i \leqq n$	$n+i \rightarrow i$

Continued

Table 1 *(continued)*

g	Φ	Elementary root-graphs
$\mathscr{D}_n$	$\lambda_i - \lambda_j, \quad 1 \leqq i < j \leqq n$	
	$-\lambda_i + \lambda_j, \quad 1 \leqq i < j \leqq n$	
	$\lambda_i + \lambda_j, \quad 1 \leqq i < j \leqq n$	
	$-\lambda_i - \lambda_j, \quad 1 \leqq i < j \leqq n$	

Remark 2. It is clear from Table 1 that we may assign to each ordered pair (i, j), $i \neq j$, of nodes a unique elementary root-graph except when $|i - j| = n$ if $\mathscr{L}$ is of type $\mathscr{D}_n$ or $|i - j| = n$ and $1 \notin \{i, j\}$ if $\mathscr{L}$ is of type $\mathscr{B}_n$, in which case we assign no elementary root graph. Equally, we may assign roots to pairs of nodes in the same way. For this reason and for convenience we sometimes use the notation (i, j) for the unique root assigned to the ordered pair (i, j). However, this correspondence is not one-to-one since the same root may be assigned to two different pairs of ordered nodes. The only cases where the correspondence root/ordered pair is one-to-one is when $\mathscr{L}$ is of type $\mathscr{A}_n$ or when $\mathscr{L}$ is of type $\mathscr{C}_n$ and the roots are of the form $\pm 2\lambda_i$, $i = 1, \ldots, n$. In this context we say that two ordered pairs are *equivalent* if they both correspond to the same root. The following pairs are equivalent:

$$\left.\begin{array}{r} (i, j) \approx (j + n, i + n) \\ (i, n + j) \approx (j, n + i) \\ (n + i, j) \approx (n + j, i) \end{array}\right\} \quad \begin{array}{l} 1 \leqq i, j \leqq n \text{ (if } \mathscr{L} \text{ is of type } \mathscr{C}_n \text{ or } \mathscr{D}_n) \\ \text{or} \\ 2 \leqq i, j \leqq n + 1 \text{ (if } \mathscr{L} \text{ is of type } \mathscr{B}_n), \end{array}$$

$$\left.\begin{array}{r} (i, 1) \approx (1, n + i) \\ (n + i, 1) \approx (1, i) \end{array}\right\} \quad 2 \leqq i \leqq n + 1 \text{ (if } \mathscr{L} \text{ is of type } \mathscr{B}_n).$$

Definition 4. Two roots $\alpha_1 = (i, j)$ and $\alpha_2 = (k, l)$ are consecutive if $j = k$.

Similarly, l roots $\alpha_1, \alpha_2, \ldots, \alpha_l$ are consecutive if, for all $i = 2, \ldots, l - 1$, α_i and α_{i+1} are consecutive.

Lemma 2. *Given l consecutive roots $\alpha_1 = (i_1, i_2)$, $\alpha_2 = (i_2, i_3)$, $\ldots$, $\alpha_l = (i_l, i_{l+1})$, then*

(1) *if $\mathscr{L}$ is of type $\mathscr{A}_n$ or $\mathscr{C}_n$ its sum is a root,*
(2) *if $\mathscr{L}$ is of type $\mathscr{D}_n$ its sum is a root if for all $j = 2, \ldots, l + 1$, $|i_1 - i_j| \neq n$,*

(3) *if $\mathscr{L}$ is of type $\mathscr{B}_n$ its sum is a root if, for all $j = 2, \ldots, l+1$ and $1 \notin \{i_1, i_j\}$, $|i_1 - i_j| \neq n$.*

Proof. For $l = 2$ the lemma is an immediate consequence of Table 1 and Remark 2. For $l > 2$ the result follows easily by induction on the number of roots. ■

Lemma 3. *Given l consecutive roots $\alpha_1 = (i_1, i_2), \alpha_2 = (i_2, i_3), \ldots, \alpha_l = (i, i_{l+1})$ such that the directed path joining i_1 to i_{l+1} does not pass through the same node twice, then*

(1) *if $\mathscr{L}$ is of type $\mathscr{D}_n$ its sum is a root if and only if $|i_1 - i_{l+1}| \neq n$,*
(2) *if $\mathscr{L}$ is of type $\mathscr{B}_n$ its sum is a root if and only if $|i_1 - i_{l+1}| \neq n$ when $1 \notin \{i_1, i_{l+1}\}$.*

Proof. By Lemma 2 it is enough to consider the situation where there exists an integer $j \in [2, l-1]$ such that $\alpha_1 + \cdots + \alpha_j$ is not a root. The conditions above imply that there can only be one such integer j. Assume that $j = l - 1$. That is every partial sum $\alpha_1 + \cdots + \alpha_i$, $i < l - 1$, is a root but $\alpha_1 + \cdots + \alpha_{l-1}$ is not a root. We will prove that the ordered sequence of roots $\alpha_l, \alpha_1, \alpha_2, \ldots, \alpha_{l-1}$ satisfy the conditions of Lemma 2 and so its sum is a root . To prove that $(i_l, i_{l+1}), (i_1, i_2), (i_2, i_3), \ldots, (i_{l-1}, i_l)$ are consecutive it is enough to show that (i_l, i_{l+1}) and (i_1, i_2) are consecutive. This follows from the fact that since $\alpha_1 + \cdots + \alpha_{l-1}$ is not a root then $|i_1 - i_l| = n$ (with $1 \notin \{i_1, i_l\}$ if $\mathscr{L}$ is of type $\mathscr{B}_n$). That is (i_l, i_{l+1}) is equivalent to (i_k, i_1) where $|i_k - i_{l+1}| = n$.

That every partial sum $\alpha_l + \sum_{i=1}^{s} \alpha_i$, $1 \leqq s < l - 1$, is a root, is a consequence of Lemma 2. In fact, we need to show that $|i_k - i_m| \neq n$ for all $m = 1, \ldots, l$. However, since $|i_k - i_{l+1}| = n$, $|i_k - i_m| = n$ if and only if $i_m = i_{l+1}$. But this is impossible since the nodes $i_1, \ldots, i_{l+1}$ are all distinct by assumption.

If $j \neq l - 1$ the same arguments can be used replacing α_1 by $\alpha_{j+1} + \cdots + \alpha_l$. This follows since $\alpha_{j+1} + \cdots + \alpha_l$ is a root. Otherwise the directed path joining i_1 to i_{l+1} would pass twice through the node i_1 which is impossible by assumption. So the results follow. ■

Corollary 1. *In the situation of Lemma 3 we can reorder roots in such a way that for the new ordering the roots $\alpha_{s_1}, \ldots, \alpha_{s_l}$ are consecutive and each partial sum*

$$\sum_{i=s_1}^{r} \alpha_i, \qquad r \leqq s_l,$$

is a root.

The result is straightforward from the proofs of Lemmas 2 and 3.

Theorem 2. *If $A = A_0 + \sum_{\alpha \in \varphi_\alpha} k_\alpha E_\alpha \in \mathscr{L}$ is P-irreducible then $\{E_\alpha, \alpha \in \varphi_\alpha\}_{\mathrm{L.A.}} = \mathscr{L}$.*

Proof. Since A is P-irreducible its root-graph is strongly connected (Remark 1). That is, every pair of ordered nodes (i, j), $i \neq j$, can be joined by an oriented path which does not pass twice through the same node. Let $i = i_1, i_2, \ldots, i_{l+1} = j$ be the consecutive nodes along that path. Clearly, to each pair of nodes (i_s, i_{s+1}), $s = 1, \ldots, l$, we can assign a unique $\alpha \in \varphi_a$ and to each $\beta \in \varphi$ we can assign an ordered pair of nodes (i_1, i_{l+1}) (not necessarily unique). Since β is a root the corresponding pair (i_1, i_{l+1}) always satisfies conditions (1) or (2) in Lemma 3 and so, as a consequence of Lemmas 2 and 3, every root $\beta \in \varphi$ can be written as a sum of roots belonging to φ_a. That is, for all $\beta \in \varphi$

$$\beta = \sum_{i=1}^{l} \alpha_i, \qquad \alpha_i \in \varphi_a \quad \text{for } i = 1, \ldots, l.$$

Hence, by Corollary 1, for some reordering of the roots we have

$$\beta = \sum_{j=1}^{l} \alpha_{i_j}, \qquad \{i_1, \ldots, i_l\} = \{1, \ldots, l\},$$

where for every $s = 1, \ldots, l$ $\sum_{j=1}^{s} \alpha_{i_j}$ is a root. Then, it follows from (2) (Section 2) that, for all $\beta \in \varphi$, $E_\beta = \lambda[\cdots[[E_{\alpha_{i_1}}, E_{\alpha_{i_2}}]E_{\alpha_{i_3}}]\cdots E_{\alpha_{i_l}}]$ for some $\lambda \in \mathbb{R}$ and since $[E_\beta, E_{-\beta}] = H_\beta$ we can conclude that $\{E_\alpha, \alpha \in \varphi_a\}_{\text{L.A.}} = \mathscr{L}$. ■

Theorem 3. *Let $A = A_0 + \sum_{\alpha \in \varphi_a} k_\alpha E_\alpha \in \mathscr{L}$ and $B \in h_R$ be A-strongly regular. Then $\{A, B\}_{\text{L.A.}} = \mathscr{L}$ if and only if A is P-irreducible.*

Proof. From the comment at the beginning of this section, it is enough to prove that under the above assumptions if A is P-irreducible then $\{A, B\}_{\text{L.A.}} = \mathscr{L}$. Let $\varphi_a = \{\alpha_1, \ldots, \alpha_l\}$. Using (2) of Section 2 to compute $\operatorname{ad}^i B \cdot A$, $i = 1, \ldots, l$, we obtain

$$\begin{bmatrix} \operatorname{ad} B \cdot A \\ \operatorname{ad}^2 B \cdot A \\ \vdots \\ \operatorname{ad}^l B \cdot A \end{bmatrix} = M \begin{bmatrix} E_{\alpha_1} \\ E_{\alpha_2} \\ \vdots \\ E_{\alpha_l} \end{bmatrix},$$

where

$$M = \begin{bmatrix} \alpha_1(B)k_{\alpha_1} & \alpha_2(B)k_{\alpha_2} & \cdots & \alpha_l(B)k_{\alpha_l} \\ \alpha_1^2(B)k_{\alpha_1} & \alpha_2^2(B)k_{\alpha_2} & \cdots & \alpha_l^2(B)k_{\alpha_l} \\ \vdots & \vdots & & \vdots \\ \alpha_1^l(B)k_{\alpha_1} & \alpha_2^l(B)k_{\alpha_2} & \cdots & \alpha_l^l(B)k_{\alpha_l} \end{bmatrix}$$

$$= \begin{bmatrix} \alpha_1(B) & \alpha_2(B) & \cdots & \alpha_l(B) \\ \alpha_1^2(B) & \alpha_2^2(B) & \cdots & \alpha_l^2(B) \\ \vdots & \vdots & & \vdots \\ \alpha_1^l(B) & \alpha_2^l(B) & \cdots & \alpha_l^l(B) \end{bmatrix} \begin{bmatrix} k_{\alpha_1} & & & \\ & k_{\alpha_2} & & 0 \\ & 0 & \ddots & \\ & & & k_{\alpha_l} \end{bmatrix}.$$

Since B is A-strongly regular and $k_\alpha \neq 0$ for all $\alpha \in \varphi_a$ we have

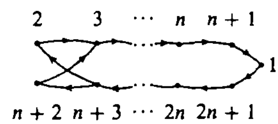
$$\det M = \prod_{1 \leqq j < i \leqq l} (\alpha_i(B) - \alpha_j(B)) \prod_{i=1}^{l} \alpha_i(B) \prod_{i=1}^{l} k_{\alpha_i}.$$

But M invertible implies $\{E_\alpha, \alpha \in \varphi_a\}_{\text{L.A.}} \subset \{A, B\}_{\text{L.A.}}$ and since A is P-irreducible, $\{E_\alpha, \alpha \in \varphi_a\}_{\text{L.A.}} = \mathscr{L}$ by Theorem 2. So the result follows. ■

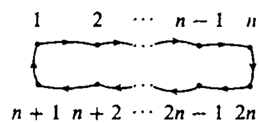
Table 2

($\mathscr{A}_n$) $$A = \begin{bmatrix} 0 & k_1 & 0 & \cdots & 0 \\ 0 & 0 & k_2 & \cdots & 0 \\ \vdots & & 0 & \ddots & \vdots \\ & & & \ddots & 0 \\ 0 & & & 0 & k_n \\ k_\beta & 0 & \cdots & & 0 \end{bmatrix}$$

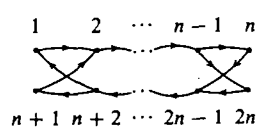
($\mathscr{B}_n$) $$A = \left[\begin{array}{c|cccc|cccc} 0 & 0 & & \cdots & 0 & 0 & \cdots & 0 & k_n \\ \hline 0 & 0 & k_1 & & & & & & \\ \vdots & & 0 & k_2 & 0 & & & & \\ & & & \ddots & k_{n-1} & & 0 & & \\ 0 & & 0 & & 0 & & & & \\ -k_n & & & & 0 & & & & \\ \hline 0 & 0 & -k_\beta & & & 0 & & & \\ & k_\beta & 0 & 0 & 0 & -k_1 & 0 & & 0 \\ \vdots & & 0 & \ddots & & & -k_2 & 0 & \\ & 0 & & \ddots & 0 & 0 & & \ddots & \ddots \\ 0 & & & 0 & 0 & & & -k_{n-1} & 0 \end{array}\right]$$



($\mathscr{C}_n$) $$A = \left[\begin{array}{cccc|cccc} 0 & k_1 & & & 0 & & & \\ & 0 & k_2 & 0 & & \ddots & & 0 \\ & & \ddots & \ddots & & & & \\ 0 & & 0 & k_{n-1} & & 0 & & 0 \\ & & & 0 & & & & k_n \\ \hline k_\beta & & & & 0 & & & \\ & 0 & & 0 & -k_1 & 0 & & 0 \\ & & & & & -k_2 & 0 & \\ & 0 & \ddots & & & 0 & \ddots & \ddots \\ & & & 0 & & & -k_{n-1} & 0 \end{array}\right]$$



($\mathscr{D}_n$) $$A = \left[\begin{array}{cccc|cccc} 0 & k_1 & & & 0 & 0 & & \\ & 0 & k_2 & 0 & 0 & \ddots & \ddots & 0 \\ & & \ddots & \ddots & & \ddots & & 0 \\ 0 & & 0 & k_{n-1} & 0 & 0 & 0 & k_n \\ & & & 0 & & & -k_n & 0 \\ \hline 0 & -k_\beta & & & 0 & & & \\ k_\beta & 0 & 0 & 0 & -k_1 & 0 & & 0 \\ & 0 & \ddots & & & -k_2 & 0 & \\ 0 & \ddots & & 0 & 0 & & \ddots & \ddots \\ & & 0 & 0 & & & -k_{n-1} & 0 \end{array}\right]$$



The converse of Theorem 2 can be easily proved using an argument similar to the one contained in the proof of Theorem 3. So, we state the following.

Corollary 2. *$A = A_0 + \sum_{\alpha \in \varphi_a} k_\alpha E_\alpha \in \mathscr{L}$ is P-irreducible if and only if $\{E_\alpha, \alpha \in \varphi_a\}_{\text{L.A.}} = \mathscr{L}$.*

We complete this section by constructing elements $A = A_0 + \sum_{\alpha \in \varphi_a} k_\alpha E_\alpha$ which are *minimal* in the sense that $\{E_\alpha, \alpha \in \varphi_a\}$ is a set of generators of $\mathscr{L}$ such that no subset of $\{E_\alpha, \alpha \in \varphi_a\}$ generates $\mathscr{L}$.

Let Δ^+ = the set of fundamental roots and β be the minimal root. So, for the complex Lie algebras of type $\mathscr{A}_n$, $\mathscr{B}_n$, $\mathscr{C}_n$, or $\mathscr{D}_n$ we have the following situations:

$$\Delta^+(\mathscr{A}_n) = \{\lambda_1 - \lambda_2, \lambda_2 - \lambda_3, \ldots, \lambda_n - \lambda_{n+1}\}, \qquad \beta(\mathscr{A}_n) = -\lambda_1 + \lambda_{n+1},$$

$$\Delta^+(\mathscr{B}_n) = \{\lambda_1 - \lambda_2, \lambda_2 - \lambda_3, \ldots, \lambda_{n-1} - \lambda_n, \lambda_n\}, \qquad \beta(\mathscr{B}_n) = -\lambda_1 - \lambda_2,$$

$$\Delta^+(\mathscr{C}_n) = \{\lambda_1 - \lambda_2, \lambda_2 - \lambda_3, \ldots, \lambda_{n-1} - \lambda_n, 2\lambda_n\} \qquad \beta(\mathscr{C}_n) = -2\lambda_1,$$

$$\Delta^+(\mathscr{D}_n) = \{\lambda_1 - \lambda_2, \lambda_2 - \lambda_3, \ldots, \lambda_{n-1} - \lambda_n, \lambda_{n-1} + \lambda_n\}, \qquad \beta(\mathscr{D}_n) = -\lambda_1 - \lambda_2.$$

Then we claim that

$$A = \sum_{\alpha \in \Delta^+} k_\alpha E_\alpha + k_\beta E_\beta \text{ is minimal.}$$

To demonstrate this we display these elements in Table 2 for the normal real forms of complex Lie algebras of type $\mathscr{A}_n$, $\mathscr{B}_n$, $\mathscr{C}_n$, and $\mathscr{D}_n$ and also their corresponding root-graphs ($k_i = k_{\alpha_i}$, $\Delta^+ = \{\alpha_1 \cdots \alpha_n\}$). Clearly, if we suppress one elementary root-graph from the root-graph of A, it is no longer strongly connected. Therefore, by Remark 1 and Corollary 2 no subset of $\{E_\beta, E_\alpha, \alpha \in \Delta^+\}$ generates $\mathscr{L}$, and so A is minimal.

4. Global Controllability

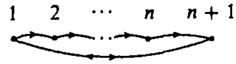
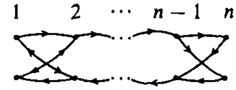
Theorem 4. *Let $\mathscr{L}$ be a normal real form of any complex Lie algebra of type $\mathscr{A}_n$, $\mathscr{B}_n$, $\mathscr{C}_n$, or $\mathscr{D}_n$, $A = A_0 + \sum_{\alpha \in \varphi_a} k_\alpha E_\alpha \in \mathscr{L}$, and $B \in h_R$ be A-strongly regular. Then system* (1) *is globally controllable if and only if A is P-irreducible.*

This result is clearly an immediate consequence of Theorem 3 and the fact that for system (1) the controllability rank condition is equivalent to controllability.

Theorem 5. *Let $\mathscr{L}$ be a normal real form of a complex simple Lie algebra of type $\mathscr{A}_n$ or $\mathscr{D}_n$. Let $A = A_0 + \sum_{\alpha \in \varphi_a} k_\alpha E_\alpha$ satisfy $\langle A(s), A(-s)\rangle < 0$ for $s = \sup\{\alpha: \alpha \in \varphi\}$ ($A(s) = k_s E_s$) and $B \in h_R$ be A-strongly regular. Then system* (1′) *is globally controllable if and only if A is P-irreducible.*

Proof. In [GKS] Theorem 1, Gauthier *et al.* proved that if $\mathscr{L}$ satisfies the above condition and if B is strongly regular and $\langle A(s), A(-s)\rangle < 0$, $s = \sup\{\alpha; \alpha \in \varphi\}$, then the controllability rank condition is equivalent to the global controllability of system (1′). However, it turns out that the proof of their theorem still works if we replace the strong regularity of B by the condition that B is A-strongly regular. So, our result is a consequence of Theorem 1 in [GKS] and Theorem 3. ∎

Table 3

$(\mathscr{A}_n)$	$A = \begin{bmatrix} 0 & k_1 & 0 & \cdots & k_{-\beta} \\ 0 & 0 & k_2 & \cdots & 0 \\ \vdots & & 0 & \ddots & \vdots \\ & & & \ddots & 0 \\ 0 & & & 0 & k_n \\ k_\beta & 0 & \cdots & & 0 \end{bmatrix}$	
$(\mathscr{D}_n)$	$A = \left[\begin{array}{cccc\|cccc} 0 & k_1 & & & 0 & k_{-\beta} & & \\ & 0 \; k_2 & 0 & & -k_{-\beta} & & 0 & 0 \\ & \ddots & \ddots & & & 0 & \ddots & 0 \\ 0 & & 0 & k_{n-1} & 0 & & 0 \quad 0 & k_n \\ & & & 0 & & & -k_n & 0 \\ \hline 0 & -k_\beta & & & 0 & & & \\ k_\beta & 0 & 0 & & -k_1 & 0 & & 0 \\ & 0 & \ddots & & & -k_2 & 0 & \\ 0 & & \ddots & 0 & 0 & & \ddots & \ddots \\ & & 0 & 0 & & & -k_{n-1} & 0 \end{array}\right]$	

It is clear that Theorems 4 and 5 improve on those in [GB] and improve on those in [GKS] in the particular cases studied here, by weakening the strong regularity condition on B.

We finally return to minimal elements A introduced in Section 3. By Corollary 2, P-irreducibility of A is characterized by the fact that $\{E_\alpha; \alpha \in \varphi_a\}_{\text{L.A.}} = \mathscr{L}$, and minimal elements A are those for which the latter condition is satisfied with φ_a having the minimum possible cardinality. Theorem 4 may now be applied directly to situations in which A is such a minimal element

$$A = \sum_{\alpha \in \Delta^+} k_\alpha E_\alpha + k_\beta E_\beta,$$

where $k_\beta, k_\alpha \neq 0$ and β is a minimal root. In this case we have $\{A, B\}_{\text{L.A.}} = \mathscr{L}$, and system (1) is globally controllable if B is A-strongly regular. In the case of Theorem 5 we must modify the definition of minimal element.

As in the previous section, let Δ^+ be the set of fundamental roots and β and $-\beta$ the minimal and the maximal roots, respectively. Clearly,

$$A = \sum_{\alpha \in \Delta^+} k_\alpha E_\alpha + k_\beta E_\beta + k_{-\beta} E_{-\beta},$$

where $k_\beta \cdot k_{-\beta} < 0$ is minimal in the sense that it satisfies the requirements of Theorem 5 but if we suppress one of its components it no longer satisfies the requirements. We display these minimal elements and their root graphs in Table 3.

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