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Invertibility of Quantum-Mechanical Control Systems[1]

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Abstract. This is the first of two papers concerned with the formulation of a continuous-time quantum-mechanical filter. Efforts focus on a quantum system with Hamiltonian of the form $H_0 + u(t)H_1$, where H_0 is the Hamiltonian of the undisturbed system, H_1 is a system observable which couples to an external classical field, and $u(t)$ represents the time-varying signal impressed by this field. An important problem is to determine when and how the signal $u(t)$ can be extracted from the time-development of the measured value of a suitable system observable C (invertibility problem). There exist certain quasiclassical observables such that the expected value and the measured value can be made to coincide. These are called quantum nondemolition observables. The invertibility problem is posed and solved for such observables. Since the physical quantum-mechanical system must be modelled as an *infinite*-dimensional bilinear system, the domain issue for the operators H_0, H_1, and C becomes nontrivial. This technical matter is dealt with by invoking the concept of an analytic domain. An additional complication is that the output observable C is in general time-dependent.

1. Introduction

A control system is said to be invertible if the corresponding input-output map is injective. The derivation of explicit conditions for invertibility is part of the larger problem of determining the input of the control system from a knowledge of its output. The latter problem may be rephrased as the problem of construction of an inverse system: when the output of the control system is fed into its inverse, the

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original input is generated. Invertibility is an important aspect not only of decoding problems [1, 2], but also of functional controllability [3] and filtering and prediction theory [4].

The theory of the existence and construction of inverses for time-invariant finite-dimensional linear systems (and certain nonlinear systems) is well developed [1–9]. The corresponding results are applicable to a variety of systems obeying classical dynamics. The next logical step is to consider quantum-mechanical systems coupled to weak classical fields. Indeed, the problem of extracting information on the time variation of a classical signal from measurements performed on an affected quantum system is the focus of a great deal of effort presently, especially in connection with the possibility of detecting gravitational waves from energetic cosmic sources [10]. In pursuing this problem one is led naturally to the general question of invertibility for quantum control systems. An intrinsic feature of most quantum-mechanical systems (and certainly the ones to be considered here) is the infinite dimensionality of the state space. In control parlance, a nonrelativistic quantum object (described in terms of a finite number of quantum-mechanical degrees of freedom) defines an infinite-dimensional bilinear system. The output of such a system is determined by an appropriate inner product in the infinite-dimensional state space. In general, this inner product only gives the expected value of the physical quantity being measured. However, in the forthcoming analysis we shall operate under the assumption that the chosen output in fact also yields the measurement result itself. The conditions for the latter assumption to hold, i.e., the conditions of *quantum nondemolition measurement* [11], are studied in a sequel to this paper.

To deal with the infinite-dimensional nature of the quantum problem, the notion of analytic domain [12] will be invoked. The analysis differs from that for the "classical", finite-dimensional case in that it is necessary to approximate states by analytic vectors. The treatment given here also differs from earlier work on invertibility in another respect: the output map is allowed to vary with time.

The quantum-mechanical concepts needed for an understanding of the present paper are minimal, being confined to the Schrödinger dynamics of the state vector ψ and the statistical interpretation of ψ. The reader may consult such standard texts as Messiah [13], Cohen-Tannoudji, Diu and Laloë [14] and Jauch [15] for a thorough discussion of these and other aspects of quantum theory.

2. Problem Formulation

We consider quantum systems described by a Hamiltonian

$$H = H_0 + u(t)H_1 \tag{1}$$

wherein neither H_0 nor H_1 depends explicitly on the time t, and $u(t)$ is a real analytic function of t. Such a Hamiltonian may be obtained (for instance) by quantizing a classical linear or bilinear system [16]. The time development of the state ψ of the system is governed by the Schrödinger equation

$$i\hbar\frac{d\psi}{dt} = [H_0 + u(t)H_1]\psi, \qquad \psi \in S_{\mathscr{H}}. \tag{2}$$

Here, H_0 and H_1 are self-adjoint operators in the underlying Hilbert space $\mathscr{H}$ associated with the quantum system, and consequently ψ remains on the unit sphere $S_{\mathscr{H}}$ in $\mathscr{H}$. It will be assumed that H_1 is H_0-bounded and $u(t)$ is suitably bounded to give a relative bound less than unity (Kato–Rellich condition [17]).

The operator H_0 can be interpreted as the Hamiltonian of the quantum system in isolation. The second term of (1) represents a coupling to an external field of strength $u(t)$ through some system observable H_1. Depending on the individual cases of interest, H_0 and H_1 take on special meaning. For example, in the case of an optical communication system, H_0 is the Hamiltonian of the unperturbed radiation field, H_1 is determined by the modulation scheme adopted (e.g., amplitude or phase modulation) and $u(t)$ is the impressed signal.

Without loss of generality, we may take $\hbar = 1$ and divide the Schrödinger equation (2) by i. Equation (2) now becomes

$$\frac{d}{dt}\psi(t) = [H_0 + u(t)H_1]\psi(t), \qquad \psi \in S_{\mathscr{H}}. \tag{3}$$

The new H_0 and H_1 are skew-adjoint operators and (3) is recognized as an infinite-dimensional bilinear control system.

Our model will not be complete without an output process. For this we suppose a precise measurement of an observable C is made. (More correctly, a measurement is made of a physical quantity $\mathscr{C}$ which is represented formally by a certain self-adjoint operator C acting in $\mathscr{H}$.) In the optical communication example, C determines the receiver structure. With the system in state ψ, the expected value of the measurement result is

$$y(t) = \langle \psi(t) | C(t)\psi(t) \rangle, \tag{4}$$

where $\langle \cdot | \cdot \rangle$ denotes the Hilbert-space inner product. The state $\psi(t)$ will evolve with time according to (3), and in addition we have assumed that C may have explicit dependence on time t. The quantum invertibility problem is again one of determining whether the output uniquely determines the input, and if so how to construct the inverse system. However, without certain restrictions on C, the irreversibility of the quantum measurement process [18] prevents us from making any further progress with the invertibility problem.

For the quantum invertibility problem to be of practical relevance it is of course essential that $y(t)$ of (4) be not merely an expected value but indeed the actually realized measurement result. For a *general* observable C this will not be the case: $\psi(t)$ will not evolve on an eigenmanifold of $C(t)$ and there would be some dispersion in the results of a set of measurements of C carried out at a given time on an ensemble of exact copies of the developing system [13–15]. Accordingly, we must restrict our attention to *special* "quasiclassical" observables C having the property that at any time after an initial measurement (or set of measurements) the aforesaid dispersion would be zero, $y(t)$ coinciding with the (unique) measurement result for C. Such observables are called *quantum nondemolition observables* (QNDO) and are the subject of much current research [11]. If a QNDO has the further property that the input $u(t)$ can be reconstructed from a knowledge of $y(t)$, it will be termed a *quantum nondemolition filter* (QNDF). In a sequel to the present article, we shall go on to establish necessary and sufficient conditions for an observable to qualify as a QNDF—based in part on the invertibility results obtained herein.

3. Analytic Domain for Quantum Control Systems

Bringing (3) and (4) together, the control system of interest is specified by

$$\frac{d}{dt}\psi(t) = [H_0 + u(t)H_1]\psi(t), \qquad \psi(t) \in S_{\mathscr{H}}, \qquad \psi(0) = \psi_0,$$
$$y(t) = \langle \psi(t)|C(t)\psi(t)\rangle. \tag{5}$$

In addition to our earlier delineations of the operators H_0, H_1 and C and the control $u(t)$, some further assumptions about the system are necessary. These assumptions will be stated first, and their relevance pointed out as we proceed with the development of results.

Assumptions. (a) Consider the Lie algebra $\mathscr{A} = \mathscr{L}(H_0, H_1)$ generated by the skew-adjoint operators H_0, H_1 under the bracket operation $[A, B] = AB - BA$, where A, B are elements of the Lie algebra. It is assumed that the tangent space $\mathscr{A}(\phi) \equiv \{X(\phi), X \in \mathscr{A}\}$ has constant, finite dimension for all ϕ belonging to a suitable domain to be prescribed below.

(b) The observable C is taken to have the structure

$$C(t) = \sum_{r=1}^{q<\infty} \gamma_r(t) iQ_r,$$

where the functions $\gamma_r(t)$ are real analytic in t and the Q_r are time-independent skew-adjoint operators. Typically, the Q_r's are elements of the Lie algebra $\mathscr{A}$.

(c) Assumption (a) raises the question of *domain* for the various operators which enter. Careful attention to this issue is imperative in a proper treatment of infinite-dimensional systems with unbounded operators. We shall invoke the existence of an *analytic domain* for the augmented Lie algebra $\mathscr{A}' = \mathscr{L}(H_0, H_1, Q_1, \ldots, Q_q)$ in the spirit of Nelson's work [12]. Note that if $Q_i \in \mathscr{A}$, $\forall i \in \{1, \ldots, q\}$, then $\mathscr{A} = \mathscr{A}'$.

(d) In order for the measurement result (4) to make sense we shall further assume that $\operatorname{dom} C \supset \operatorname{dom} H_0 \cap \operatorname{dom} H_1$.

Definition 1. System (5) is said to admit an analytic domain $\mathscr{D}_\omega$ if there exists a common, dense invariant domain $\mathscr{D}_\omega$ for the Lie algebra $\mathscr{A}'$ such that

(i) $\mathscr{D}_\omega$ is invariant under the corresponding unitary Lie group G as well as $\mathscr{A}'$,

(ii) on $\mathscr{D}_\omega$, an arbitrary element g of G can be written locally in a group parameter t as $g = \exp Xt$, where $X \in \mathscr{A}'$, and

(iii) this exponential expression can in fact be extended globally to all $t \in \mathbb{R}^+$.

Sufficient conditions for the existence of an analytic domain are given in [12] and [19]. The analytic domain proves useful in the analysis of many of the interesting examples of quantum-mechanical control systems, in particular the simple harmonic oscillator.

3.1. *Vector Fields, Integral Curves and Flows*

Geometrical terminology which relates the quantum control system (5) to finite-dimensional control systems such as those investigated by Sussmann and Jurdjevic [20, 21], Krener [22] and others will be introduced in this subsection. Since we are dealing with an infinite-dimensional system modelled on a Hilbert space, we appeal to [23] for the necessary definitions. In particular, there is a need to define concepts without the use of local coordinates $x_1, x_2, \ldots, x_m$ and their differentials $dx_1, dx_2, \ldots, dx_m$.

We shall begin by defining a time-dependent vector field.

Definition 2. Let M be a manifold and let TM be the tangent bundle of M. A C^p *time-dependent vector field* is a C^p map $X: \mathbb{R} \times M \to TM$ such that $X(t, \mu) \in T_\mu M$ for all $(t, \mu) \in \mathbb{R} \times M$.

Using this terminology, the skew-adjoint operators of $\mathscr{A}'$ are vector fields on $\mathscr{D}_\omega$. These vector fields are complete [12].

Definition 3. An *integral curve* for X starting at $\phi \in M$ at time t_0 is a C^r, $r \geq 1$, map $\alpha: I \times M \to M$ from $I \times M$ into M, where $I \subset \mathbb{R}$ is an open interval with $t_0 \in I$ and $\alpha(t_0, t_0; \phi) = \phi$, the map α being such that

$$\frac{d}{dt}\alpha(t, t_0; \phi) = X(t, \alpha(t, t_0; \phi))$$

for each $t \in I$.

Thus, the integral curves are the state trajectories if X is the Hamiltonian vector field.

Definition 4. A *time-dependent flow* or *evolution operator* $\alpha(t, t_0; \phi)$ of X is defined by the requirement that $t \mapsto \alpha(t, t_0; \phi)$ be the integral curve of X starting at ϕ at time t_0.

In terms of the flow α, we can define Lie derivatives without reference to local coordinates, as follows.

Definition 5. Let $f:\mathscr{H}\to\mathbb{R}$ be a real-valued functional defined on some domain $\mathscr{D}$ of the Hilbert space. The *Lie derivative* at time t of f with respect to the vector field X with flow α is prescribed by

$$\mathfrak{L}_X f(\zeta) = \lim_{s\to t}(s-t)^{-1}[f(\alpha(s,t;\zeta))-f(\alpha(t,t;\zeta))], \qquad \zeta\in\mathscr{D}. \tag{6}$$

The Lie derivative $\mathfrak{L}_X Y$ of a vector field Y with respect to X can be defined similarly.

It can be shown [23] based on the above definitions that

$$\mathfrak{L}_X f = XF+\frac{\partial f}{\partial t}, \qquad \mathfrak{L}_X Y = [X,Y]+\frac{\partial Y}{\partial t}.$$

In the forthcoming developments, we shall make use of the common notation $\mathrm{ad}_X Y=[X,Y]$ and generally $\mathrm{ad}_X^k Y=[X,\mathrm{ad}_X^{k-1}Y]$, $\mathrm{ad}_X^0 Y=Y$.

In the case where the functional f and the vector fields X and Y are time-independent, the above formulas simplify in obvious ways and a local flow becomes a one-parameter group action.

3.2. *Preliminary Results*

In preparation for the statement and proof of the main results we shall develop a series of lemmas concerning the system. It is presumed throughout that the quantum control system (5) admits an analytic domain $\mathscr{D}_\omega$. On such a domain, standard techniques, developed for the analysis of finite-dimensional bilinear systems, are applicable [24–26].

We are concerned with a manifold $M\subset S_{\mathscr{H}}$ on which the Schrödinger dynamical law (3) prevails. This manifold inherits the Hilbert space topology and therefore it is paracompact and connected. Referring to the assumptions set forth in Section 3 we note that, in general, M need not belong to $\mathscr{D}_\omega$. However, $\mathscr{D}_\omega\cap M$ is dense in M, and since $\mathscr{D}_\omega$ is invariant under $\mathscr{A}'(H_0,H_1,Q_1,\dots,Q_q)$ the submanifold $\mathscr{D}_\omega\cap M$ will actually be the main scene of action. Assumption (a) implies that the manifold M and hence $\mathscr{D}_\omega\cap M$ is finite-dimensional [24]. In addition, part (ii) of Definition 1 implies that $\mathscr{D}_\omega\cap M$ is an analytic manifold.

Suppose for the moment that the control $u(t)$ is a piecewise-constant function with a finite number of switching times, and recall that H_0 and H_1 are time-independent.

Lemma 1. *Considering the control system* (5), *assert that the Lie algebra* $\mathscr{A}=\mathscr{L}(H_0,H_1)$ *satisfies* $\dim\mathscr{A}(\phi)=m<\infty$, $\forall\phi\in\mathscr{D}_\omega\cap M$ (*cf. assumptions* (a), (c), *Section* 3). *Let* X_t *and* Y_t *denote* (*elements of*) *the one-parameter groups defined respectively by vector fields* X, $Y\in\mathscr{A}$. *Introduce the ideal in* $\mathscr{A}$ *generated by* H_1, *i.e.*, $\mathscr{I}=\mathscr{L}(\mathrm{ad}_{H_0}^j H_1,\ j=0,1\dots)$. *Then for all* $\phi\in\mathscr{D}_\omega\cap M$ *and* $\forall t\in\mathbb{R}$, *the sets* $X_t(I(\mathscr{I},\phi))$ *and* $Y_t(I(\mathscr{I},\phi))$ *are identical, where the argument* $I(\mathscr{I},\phi)$ *stands for the maximal integral manifold in* $\mathscr{D}_\omega\cap M$, *with tangent space characterized by* $\mathscr{I}$, *passing through ϕ. In particular, $H_{0_t}(I(\mathscr{I},\phi))$ is the unique maximal integral manifold in $\mathscr{D}_\omega \cap M$ for $\mathscr{I}$ through $H_{0_t}(\phi)$.*

Proof. This result corresponds to Lemma 3.6 of Sussmann and Jurdjevic [20] for finite-dimensional systems. Since $I(\mathscr{I},\phi) \subset \mathscr{D}_\omega \cap M$, while in turn $\mathscr{D}_\omega \cap M$ is an analytic connected manifold with paracompact topology (cf. Huang, Tarn and Clark [24]), the deliberations of [20] can be extended to confirm the above lemma. □

The concept of reachable or attainable sets is by now a familiar one in the literature [21, 22]. Here we shall denote the reachable set of $\phi \in \mathscr{D}_\omega \cap M$ at time t by $R^t(\phi)$. Thus, denoting the solution to (5) by $\psi(\phi, u, \cdot)$, $R^t(\phi)$ is the set of all ξ such that $\psi(\phi, u, t) = \xi$, for all u belonging to the admissible set. In the case that the control is restricted to the set of piecewise constant functions with a finite number of switchings, the reachable set of ϕ at time t will be designated by $R_p^t(\phi)$. We then have the following lemma.

Lemma 2. *For all $\phi \in \mathscr{D}_\omega \cap M$, and $\forall t > 0$, $R_p^t(\phi) \subset I(\mathscr{I}, H_{0_t}\phi)$.*

Proof. The assumption of an analytic domain implies the existence of a foliation of M into disconnected maximal integral manifolds. The invariance of $\mathscr{D}_\omega$ under the unitary group corresponding to $\mathscr{I}$ implies $R_p^t(\phi) \subset \mathscr{D}_\omega$. From that point on it is routine to adapt the argument of [21, 22] to complete the proof. □

The above two lemmas deal with piecewise constant controls with a finite number of switchings, whereas the permissible controls in the problem at hand comprise analytic controls. Nevertheless, we shall see in the next lemma that the situation remains the same in the sense that the state and control of (5) can be approximated arbitrarily closely by a state in $\mathscr{D}_\omega$ and a piecewise constant control (see later Remark 2).

Lemma 3. *Let $\psi_\omega(t) \in \mathscr{D}_\omega$ and $u_p(t) \in \{$piecewise constant controls$\}$ be a state and a control approximating, respectively, the true evolving state $\psi(t)$ and the analytic control $u(t)$ of (5), $\psi_\omega(t)$ and $u_p(t)$ being related by*

$$\frac{d}{dt}\psi_\omega(t) = \left[H_0 + u_p(t)H_1\right]\psi_\omega(t), \qquad \psi_\omega(0) = \psi(0) = \phi \in \mathscr{D}_\omega. \tag{7}$$

Then $\psi_\omega \to \psi$, i.e., $\|\psi(t) - \psi_\omega(t)\| \to 0$, $\forall t$, if $u_p(t) \to u(t)$, $\forall t$. The converse (namely, $u_p(t) \to u(t)$ $\forall t$, if $\psi_\omega \to \psi$) holds provided ψ does not belong to the null space $\mathscr{N}$ of H_1. Moreover, $\psi(t) \in \operatorname{cl} R_p^t(\phi)$.

Proof. First note that $\mathscr{D}_\omega$ is dense in M and the set of piecewise-constant controls is dense in the set of analytic controls.

Next consider the state evolution equations for $\psi(t)$ and $\psi_\omega(t)$; subtracting one from the other we may form

$$\frac{d}{dt}\psi_e = H_0\psi_e + uH_1\psi_e + (u - u_p)H_1\psi_\omega, \qquad \psi_e(0) = 0, \tag{8}$$

where $\psi_e(t) \equiv \psi(t) - \psi_\omega(t)$. The kernel $K(t,s)$ for the differential equation of (8) satisfies

$$\frac{d}{dt}K(t,s) = \left[H_0 + u(t)H_1\right]K(t,s), \qquad K(s,s) = I.$$

Since H_0, H_1 share a common dense invariant domain of analytic vectors, $H_0 + u(t)H_1$ is essentially self-adjoint. Therefore $K(t,s)$ exists and is unitary (Nelson's Theorem [12]). In terms of $K(t,s)$, ψ_e of (8) may be expressed as

$$\psi_e(t) = \int_0^t \left[u(s) - u_p(s)\right]K(t,s)\chi(s)\,ds,$$

where $\chi(s) \equiv H_1\psi_\omega(s)$ is of bounded norm since $\psi_\omega \in \mathscr{D}_\omega$. From the unitarity of $K(t,s)$, it follows that

$$\|\psi_e(t)\|^2 = \int_0^t \left[u(s) - u_p(s)\right]^2 \|\chi(s)\|^2\,ds < \infty.$$

Hence as $u_p \to u$, $\|\psi_e\| \to 0$. Accordingly, we may conclude that $\psi(t) \in \mathrm{cl}\, R_p^t(\phi)$. Thereupon it is clear that as $\psi_\omega \to \psi$, $\|\psi_e\| \to 0$ and correspondingly $u_p \to u$, $\forall \psi \notin \mathscr{N}(H_1)$. If $\psi \in \mathscr{N}(H_1)$, the presence or absence of the input makes no difference to the solution ψ_e, which is anyway assured to have zero norm. □

Lemma 4. *Under the hypotheses of Lemma* 1, *and taking* $u(t)$ *analytic*, $\mathrm{cl}\, H_{0_t}(I(\mathscr{I}, \phi))$ *is the unique maximal integral manifold for* $\mathscr{I}$ *through* $H_{0_t}(\phi)$ *in* M.

Proof. This result ensues directly from Lemmas 1 and 3.

We have indicated in this section some of the results for finite-dimensional systems that carry over for system (5) provided the analysis is carried out on an analytic domain. Further details of such an analysis are given in [24], where the question of the controllability of quantum systems is investigated. □

4. Analytic Invertibility

4.1. Definitions

It is again presumed throughout that $u(t)$ is real analytic and that the quantum control system (5) admits an analytic domain $\mathscr{D}_\omega$. If invertible on $\mathscr{D}_\omega$, the system will be called *analytically invertible*. Since system (5) is nonlinear, we follow the lead of Hirschorn [5] in formulating suitable definitions.

Definition 6.

(i) System (5) is *analytically invertible* at $\psi_0 = \psi(t=0) \in M \cap \mathscr{D}_\omega$ if distinct inputs $u_1(t), u_2(t)$ give rise to distinct outputs, i.e.,

$$y(t, u_1, \psi_0) \not\equiv y(t, u_2, \psi_0).$$

(ii) System (5) is *strongly analytically invertible at* ψ_0 if there exists an open neighborhood N of ψ_0 such that analytic invertibility holds at ξ for all $\xi \in N \cap \mathscr{D}_\omega$.

(iii) System (5) is *strongly analytically invertible* if there exists an open submanifold M_0 of M, dense in M, such that strong analytic invertibility holds at ζ, for all $\zeta \in M_0 \cap \mathscr{D}_\omega$.

Definition 7. Given system (5), we define a sequence of operators C_k, where k is a positive integral index, by the recursive relation

$$C_k(t) = [C_{k-1}(t), H_0] + \frac{d}{dt} C_{k-1}(t),$$

with $C_{k=0}(t) = C_0(t) = C(t)$. If the output observable C is independent of time, then

$$C_k = (-1)^k \mathrm{ad}_{H_0}^k C.$$

Noting that the output map is (in general) time-varying, we adopt the following definition for the relative order of the system.

Definition 8. The *relative order* μ of system (5) is the smallest positive integer k such that $[C_{k-1}(t), H_1] \neq 0$ for almost all t, where "for almost all t" means the Lebesgue measure of $\{t : [C_{k-1}(t), H_1] = 0\}$ is zero.

If a system is invertible, one can in principle recover the control $u(\cdot)$ by constructing an inverse system such that when driven by an appropriate derivative of y, the inverse system produces u as its output. As in finite-dimensional systems, the relative order in a sense gives the lowest order of differentiation necessary to recover u.

Since $C(t)$ is dependent on time, we could think of making observations on the time derivatives of C. However, as far as invertibility goes there is nothing to be gained from this, as we may see from the following remark.

Remark 1. If $d^\rho C/dt^\rho$, $\rho \geq 1$, is substituted for the observable C in the above, the relative order is greater than or equal to that found for C.

If $\mu < \infty$, it will be shown that system (5) is invertible and we can construct an inverse system. The pertinent dynamical manifold for the inverse system is a submanifold of M, specified as follows.

Definition 9. The *inverse submanifold* for the system (5) having relative order μ is defined as

$$M_\mu = \left\{\zeta \in M \cap \mathscr{D}_\omega : \left\langle \zeta | \left[C_{\mu-1}(t), H_1\right] \zeta \right\rangle \neq 0 \text{ for almost all t}\right\}.$$

Justification of the statement that M_μ is a submanifold of M runs as follows: On the analytic domain $\mathscr{D}_\omega$, $[C_{\mu-1}(t), H_1]\zeta$ is an analytic vector; hence $g(\zeta) = \left\langle \zeta | \left[C_{\mu-1}(t), H_1\right] \zeta \right\rangle$ is a nonzero real analytic function of ζ for almost all t.

The analyticity of $g(\zeta)$ means that it cannot vanish on any open subset of M. This fact together with the continuity of $g(\zeta)$ implies that M_μ is an open dense subset of M and therefore a submanifold of M.

4.2. *Main Results*

Besides the lemmas presented in Subsection 3.2, we shall need the following lemmas for the statement and proof of the main results.

Lemma 5. *Given operators H_0, H_1 and $C(t)$ such that assumptions* (a)–(d) *of Section* 3 *are valid, $y^{(n)}(t) = d^n y(t)/dt^n$ exists for $0 \le n \le \mu$.*

Proof. If t belongs to a time interval over which $\psi(t)$ obeys the controlled Schrödinger equation (3), we may reason as follows. The existence of $y(t) = \langle \psi(t) | C(t)\psi(t)\rangle$ itself is guaranteed by the assumption that the Kato-Rellich condition is satisfied and that $\operatorname{dom} C \supset \operatorname{dom} H_0 \cap \operatorname{dom} H_1$. By Lemma 3, $\psi(t) \in \operatorname{cl} R_p^t(\phi)$. Hence there exists a sequence of $\psi_\omega \in \mathscr{D}_\omega$, evolving according to (7), such that $\psi_\omega \to \psi$. Let $y_\omega(t) = \langle \psi_\omega(t) | C(t)\psi_\omega(t)\rangle$; then by the continuity of the inner-product map, $y_\omega \to y$ as $\psi_\omega \to \psi$. Using (7) and abbreviating dC/dt by $\dot{C}$, we obtain

$$\dot{y}_\omega = \left\langle \psi_\omega \middle| \left\{ [C, H_0] + u_p[C, H_1] + \dot{C} \right\} \psi_\omega \right\rangle.$$

which is well-defined under assumptions (c), (d) since $\psi_\omega \in \mathscr{D}_\omega$. Moreover, as $\psi_\omega \to \psi$, u_p will approximate u arbitrarily closely (Lemma 3) and thus, again invoking continuity of the inner product map, $\dot{y}_\omega \to \dot{y}$. (The broader message is that we can always work in the analytic domain if we are content with an arbitrarily good approximation.) If $[C, H_1] = 0$, the above argument may be repeated with $C_1 = [C, H_0] + \dot{C}$ replacing C. The process is continued until the μth order of t-differentiation.

If, on the other hand, a measurement is actually carried out at t, the idiosyncrasies of quantum observation [13–15] come to the fore: the measurement process, in general, suspends the Schrödinger equation (3) and produces an unpredictable jump in the state $\psi(t)$. In our treatment such discontinuous behavior is ruled out by the special circumstances of QND measurement [11], which we assume to apply. □

Remark 2. Within Lemma 5 and henceforth, when ψ is outside the analytic domain $\mathscr{D}_\omega$ but is inside the domain of an operator $X \in \mathscr{A}'$, the quantity $\langle \psi | X\psi \rangle$ is defined as $\langle \psi_\omega | X\psi_\omega \rangle$, where ψ_ω, as circumscribed above, belongs to $\mathscr{D}_\omega$ and is arbitrarily close to ψ. This is a suitable definition since $\mathscr{D}_\omega$ is dense and $\langle \zeta | X\zeta \rangle$, with $\zeta \in \mathscr{D}_\omega$, is a bounded, continous function of ζ.

Lemma 6. *Given operators H_0, H_1 and $C(t)$ such that assumptions* (a)–(d) *of Section* 3 *are valid, the Lie derivative of y with respect to H_0 is*

$$\mathfrak{L}_{H_0} y = \langle \psi | \{ [C, H_0] + \dot{C} \} \psi \rangle, \tag{9}$$

where $\psi = \psi(t)$ is the solution of the controlled Schrödinger equation. If in addition it is specified that the system (5) *has relative order $\mu > 1$, then*

$$\mathfrak{L}_{(\mathrm{ad}_{H_0}^{k-1} H_1)} y = \begin{cases} \langle \psi | \dot{C} \psi \rangle, & k = 1, \\ 0, & 1 < k < \mu, \\ (-1)^{\mu-1} \big\langle \psi | [C_{\mu-1}, H_1] \psi \big\rangle, & k = \mu. \end{cases} \tag{10}$$

Proof. Both components of this lemma rest on Definition 5. The first is demonstrated by the simple calculation

$$\begin{aligned} \mathfrak{L}_{H_0} y &= \frac{d}{ds} \langle \alpha(s, t; \psi) | C(t) \alpha(s, t; \psi) \rangle |_{s=t} \\ &= [\langle \dot{\alpha} | C\alpha \rangle + \langle \alpha | C\dot{\alpha} \rangle + \langle \alpha | \dot{C} \alpha \rangle] |_{s=t} \\ &= \big\langle \alpha | \{ [C, H_0] + \dot{C} \} \alpha \big\rangle \, |_{s=t} \\ &= \big\langle \psi(t) | \{ [C(t), H_0] + \dot{C}(t)] \} \psi(t) \big\rangle, \end{aligned} \tag{11}$$

wherein $\alpha(s, t; \psi)$ denotes the image of the relevant flow of H_0. To pass from line 2 of (11) to line 3 we recall that by definition $\dot{\alpha}$ (which here means $d\alpha/ds$) is just $H_0 \alpha$; the last step makes use of the boundary condition $\alpha(t, t; \psi) = \psi$. Proof of (10) requires the same reasoning together with a process of induction. Considerations of domain run as in Lemma 5, with attention to Remark 2. □

We are now ready to state and prove the main results of this paper.

Theorem 1. *Given that system* (5) *admits an analytic domain $\mathscr{D}_\omega$, the system is strongly analytically invertible if its relative order μ is finite. If indeed μ is finite and if the initial state ψ_0 belongs to the inverse submanifold M_μ, the system specified by*

$$\begin{aligned} \frac{d}{dt} \hat{\psi}(t) &= a(\hat{\psi}(t)) + \hat{u}(t) b(\hat{\psi}(t)), \qquad \hat{\psi}(0) = \psi_0, \\ \hat{y}(t) &= d(\hat{\psi}(t)) + \hat{u}(t) e(\hat{\psi}(t)), \end{aligned} \tag{12}$$

provides an acceptable inverse for the quantum control system (5) *with*

$$\begin{aligned} a(\hat{\psi}(t)) &= H_0 \hat{\psi} - \big\langle \hat{\psi} | [C_{\mu-1}, H_1] \hat{\psi} \big\rangle^{-1} \langle \hat{\psi} | C_\mu \hat{\psi} \rangle H_1 \hat{\psi}, \\ b(\hat{\psi}(t)) &= \big\langle \psi | [C_{\mu-1}, H_1] \hat{\psi} \big\rangle^{-1} H_1 \hat{\psi}, \\ d(\hat{\psi}(t)) &= - \big\langle \hat{\psi} | [C_{\mu-1}, H_1] \hat{\psi} \big\rangle^{-1} \langle \hat{\psi} | C_\mu \hat{\psi} \rangle, \\ e(\hat{\psi}(t)) &= \big\langle \hat{\psi} | [C_{\mu-1}, H_1] \hat{\psi} \big\rangle^{-1}. \end{aligned} \tag{13}$$

Proof. Since M_μ is open and dense in M, the system is strongly analytically invertible if it is analytically invertible for all $\psi_0 \in M_\mu \cap \mathscr{D}_\omega$. To demonstrate analytic invertibility at an arbitrary point $\psi_0 \in M_\mu \cap \mathscr{D}_\omega$, we examine the proposed system (12) and (13). Inserting $\hat{u}(t) = y_\omega^{(\mu)}(t)$ into the differential equation of (12) we have

$$\dot{\hat{\psi}} = a(\hat{\psi}(t)) + \left\langle \psi_\omega \middle| \left\{ \left[C_\mu(t) + u_p\left[C_{\mu-1}(t), H_1 \right] \right\} \psi_\omega \right\rangle b(\hat{\psi}(t)). \tag{14}$$

When $\psi_\omega(t)$ is substituted for $\hat{\psi}(t)$, the right-hand side of (14) collapses to $[H_0 + u_p(t)H_1]\psi_\omega(t)$, where by definition $\psi_\omega \in \mathscr{D}_\omega$. Thus $\hat{\psi}(t) = \psi_\omega(t)$ solves the state-evolution equation of (12) under the choice $\hat{u}(t) = y_\omega^{(\mu)}(t)$. But in turn $\psi_\omega(t)$ provides an approximate solution of the actual system (5). From Lemma 3, we know in fact $\psi_\omega(t) \to \psi(t)$ as $u_p(t) \to u(t)$ and conversely; i.e., there exists a sequence of analytic vectors that satisfy (5) arbitrarily well. (Note that $\psi(t) \in \mathscr{N}(H_1) \forall t$ implies $\mu = \infty$. Our definition of M_μ is designed to accomodate the case that $\psi(t) \in \mathscr{N}(H_1)$ at times belonging to a set of measure zero.) Finally, inserting the above $\hat{u}(t)$ into the output equation of (12), we may obtain $u(t)$ to arbitrary accuracy: $\hat{y}(t) \to u(t)$ as we proceed down the sequence of ψ_ω. In other words, the proposed inverse system, when steered by $y_\omega^{(\mu)}(t)$ as control, yields as output an arbitrarily accurate approximation to the signal $u(t)$ for $t \in [0, \eta)$, some $\eta > 0$. By analytic continuation, one may extend this result to all t.

The proposed construction (12), (13) may of course break down on a set of measure zero in t. However, this "flaw" is immaterial because $u(t)$ is specified to be analytic. □

Corollary 1.1. *Suppose* (5) *admits an analytic domain and also assume that* $d^\rho C/dt^\rho$ *vanishes identically in* t *for some positive integer* ρ. *Then* $\mu < \infty$ *provides a necessary as well as sufficient condition for the system to be strongly analytically invertible.*

Proof. Only the necessity of the stated condition remains to be shown.

Since $\mathscr{D}_\omega$ is dense in $\mathscr{H}$, we may without loss of generality take $\psi_0 = \phi \in \mathscr{D}_\omega \cap M$, where M is some submanifold of $\mathscr{H}$ (e.g., $S_{\mathscr{H}}$) on which the evolution of $\psi(t)$ takes place. Denote the set of integral-curve sections at time t of the Hamiltonian $H_0 + u(t)H_1$, starting at $\psi_0 = \phi$, by $R_a^t(\phi) = \{\psi(t; u, \phi),\ u(s)$ analytic on $s \in [0, t]\}$. Further let $\mathscr{I}$ and $I(\mathscr{I}, \phi)$ be as defined in Lemma 1. Under the assumption (a) that $\dim \mathscr{A}(\phi) = m < \infty$ we have $R_a^t(\phi) \subset \operatorname{cl} I(\mathscr{I}, H_0, \phi)$ by previous lemmas. This result implies that the system is not strongly analytically invertible if y turns out to be constant on the submanifold $I(\mathscr{I}, H_0, \phi)$. For if the latter situation holds, the outputs y_1 and y_2 for two different controls u_1 and u_2 will coincide on $[0, t)$, negating strong analytic invertibility.

Now assume that $\mu = \infty$. Then according to Lemma 6, the Lie derivative of y with respect to $\operatorname{ad}_{H_0}^{k-1} H_1$ vanishes for all integers $k > 1$, while $\mathfrak{L}_{H_1} y = \langle \psi | \dot{C} \psi \rangle$. Thus the only element of the set $\Sigma = \{\operatorname{ad}_{H_0}^j H_1,\ j = 0, 1, \ldots, \infty\}$ which can possibly distinguish between different controls is $\operatorname{ad}_{H_0}^0 H_1 = H_1$. If $\langle \psi | \dot{C} \psi \rangle$ is zero, then $\mathfrak{L}_{H_1}\ y = 0$ and this distinguishing element is rendered ineffectual. From Remark 1 we know that the relative order of the system, with C replaced by $\dot{C}$, is greater than or equal to μ. Therefore the considerations of the first paragraph can be repeated, with the output functional y replaced by $\langle \psi | \dot{C} \psi \rangle$; we conclude that $\mathfrak{L}_{H_1} y$ is zero provided $\langle \psi | \ddot{C} \psi \rangle$ is. And so the argument continues, until we reach an order ρ of t-differentiation such that $d^\rho C/dt^\rho$ vanishes, or, in the practical context, is negligible. Thereupon the Lie derivative of y with respect to $\mathrm{ad}^j_{H_0}H_1$ vanishes for *all* $j \geq 0$ and it follows that $s \to \langle [\mathrm{ad}^j_{H_0}H_1]_s\xi | C[\mathrm{ad}^j_{H_0}H_1]_s\xi\rangle$ is a constant map, where $\xi \in \mathscr{D}_\omega \cap M$ and $s \to [\mathrm{ad}^j_{H_0}H_1]_s\xi$ is an integral curve, $j = 0, 1, \ldots, \infty$. But since Σ as defined above is a set of generators for the Lie algebra $\mathscr{I}$, it follows from Chow's theorem [22] that each $\chi \in I(\mathscr{I}, \xi)$ is expressible as

$$\chi = X^1_{s_1} \circ X^2_{s_2} \circ \cdots \circ X^l_{s_l}\xi$$

where $X^i \in \Sigma$ and $s_i \in \mathbb{R}$, $i = 1, 2, \ldots, l$. Thus

$$\begin{aligned}\langle \chi | C\chi\rangle &= \langle X^1_{s_1} \ldots X^l_{s_l}\xi | X^1_{s_1} \ldots X^l_{s_l}\xi\rangle \\ &= \langle X^2_{s_2} \ldots X^l_{s_l}\xi | CX^2_{s_2} \ldots X^l_{s_l}\xi\rangle \\ &= \cdots = \langle \xi | C\xi\rangle.\end{aligned}$$

In other words, $\langle \psi | C\psi\rangle$ is a constant map on $I(\mathscr{I}, H_{0t}\phi)$, $\forall$ fixed t. This implies $\langle \psi | C\psi\rangle$ is constant on $R^t_a(\phi)$ since $\langle \psi | C\psi\rangle$ is continuous on $R^t_a(\phi) \subset \operatorname{dom} C$, and consequently the system is not strongly analytically invertible. □

Our findings simplify if C is independent of time.

Corollary 1.2. *Given that the system* (5) *admits an analytic domain and that C is time-independent, the condition $[C, \mathrm{ad}^{k-1}_{H_0}H_1] \neq 0$ for some positive integer $k < \infty$ is necessary and sufficient for strong analytic invertibility. If this condition is met and ν is the minimum such k, the inverse system is specified by* (12), (13) *with the replacements $\mu \to \nu$, $[C_{\mu-1}, H_1] \to [C, \mathrm{ad}^{\nu-1}_{H_0}H_1]$ and $C_\mu \to (-1)^{\nu-1}\mathrm{ad}^{\nu-1}_{H_0}C$ in* (12), (13) *and the definition of the inverse submanifold.*

Proof. The replacements follow trivially from the definition of C_k by setting the time derivatives of all C_k equal to zero and using Jacobi's identity. The condition is necessary from Corollary 1.1 since $\dot{C} \equiv 0$. □

5. Examples

In this section two elementary examples will be given to illustrate the results.

Example 1. (Electrooptic Amplitude Modulation)
Consider the Hamiltonian

$$H = \omega a^\dagger a + iu(t)(a^\dagger - a)$$

defined on the Schwartz space $\mathscr{S}(\mathbb{R})$, where a, $a^\dagger$ are respectively the annihilation and creation operators for the one-dimensional quantum system [14]. The set of finite linear combinations of the Hermite functions is a dense set of analytic vectors invariant under a, $a^\dagger$ and $a^\dagger a$ [19].

Suppose measurements are made on the time-varying observable

$$C = ae^{i\omega t} + a^{\dagger}e^{-i\omega t}.$$

We have

$$[C, H_1] = e^{i\omega t} + e^{-i\omega t} = 2\cos\omega t \neq 0$$

except at $t = l\pi/2$, $l = 1, 3, 5, \ldots$; therefore the invertibility condition is met. The input $u(t)$ is related to the output $y(t)$ via

$$u(t) = \left(\frac{dy}{dt}\right)(2\cos\omega t)^{-1}.$$

Thus by staying away from the zeroes of $\cos\omega t$, we can in effect reconstruct $u(t)$ from $y(t)$.

Example 2. As another example consider the Hamiltonian

$$H = p^2/2 + u(t)(px + xp),$$

where p, x are the momentum and position operators for the one-dimensional system. As output observable we choose $C = p^2$. The system again admits an analytic domain built from Hermite functions [19]. The invertibility condition is satisfied since

$$[C, H_1] = -4p^2 \neq 0.$$

Indeed, the input is given simply by

$$u(t) = -\frac{1}{4}\frac{1}{y(t)}\frac{dy}{dt}.$$

These examples will be developed more fully in the sequel to the present paper.

6. Conclusions

In this paper we have considered the invertibility of infinite-dimensional bilinear systems with states belonging to some Hilbert space. The output of the system is determined by an inner product depending on the state and on a self-adjoint operator C which is in general time-varying. Our model describes a quantum-mechanical system provided the output is the true measurement result and not merely the expected value. Based on the assumption that the system admits an analytic domain, the concept of analytic invertibility was defined and necessary and sufficient conditions for the system to be analytically invertible were determined. To our knowledge, results on invertibility of *time-varying* bilinear or nonlinear systems are not available in the literature, for either an infinite-dimensional or a finite-dimensional state space. The results obtained here simplify considerably if C is time-independent.

On an analytic domain, standard techniques, developed for the analysis of finite-dimensional bilinear and nonlinear systems, may be exploited to treat our model. However, the existing results on invertibility of such finite-dimensional systems are *not* directly applicable to the invertibility problem posed here, for the following reason. In the course of the analysis we are forced to approximate the analytic input signal by a piecewise-constant input, and there arises the possibility that the state of the system falls on the boundary of the closure of the set of analytic vectors. This distinguishes our problem from its finite-dimensional counterpart, in which no approximation is needed. We hasten to add that the analytic domain is by definition dense in the Hilbert space.

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