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CHOW'S CONNECTIVITY THEOREM

Notations

- The term *smooth* is always a synonym for C^∞;
- An m-dimensional *smooth manifold* is by definition a T_2 paracompact topological space $\mathscr{M}$ locally homeomorphic to $\mathbb{R}^m$ and with smooth chart-change maps (in particular $\mathscr{M}$ has a countable basis and its topology is metrizable);
- An a-dimensional *submanifold* of a smooth manifold $\mathscr{M}$ is a topological subspace $\mathscr{A} \subseteq \mathscr{M}$ such that for every point $p \in \mathscr{A}$ there exist an open neighborhood U of p in $\mathscr{M}$ and a diffeomorphism $U \xrightarrow{\cong} \mathbb{R}^m$ mapping $\mathscr{A} \cap U$ bijectively on $\mathbb{R}^a \subseteq \mathbb{R}^m$ ($\mathscr{A}$ is locally smoothly flattenable in the ambient $\mathscr{M}$);
- If $f : \mathscr{M} \to \mathscr{N}$ is a smooth map then its *differential* over the point $p \in \mathscr{M}$ is $\mathrm{d}f_{[p]} : T_p\mathscr{M} \to T_{f(p)}\mathscr{N}$

1. Sub-remannian manifolds and metrics

Usually, in Control Theory the condition of smoothness is too strong and must be replaced by the weaker *absolute continuity*; in particular we'll need to work with the family of *absolutely continuous path* $\mathrm{AC}\big([0,T],\mathscr{M}\big)$ on a manifold $\mathscr{M}$ that consists in path $\gamma : [0,T] \longrightarrow \mathscr{M}$ such that in every coordinate chart of $\mathscr{M}$ such path is AC in the standard sense (this works well since coordinate changes are smooth and thus preserve AC). In particular every AC-path is a.e. differentiable and in a coordinate chart it can be recovered by its derivative, althought this is meaningless in the whole manifold.

1.1. Definition (Sub-riemannian metric). *Let $\mathscr{M}$ be a smooth m-dimensional manifold and fix a k-uple $\mathbf{X}_1, \dots \mathbf{X}_k$ of smooth vector fields on $\mathscr{M}$ with $k \le m$; then the Sub-riemannian metric $g_-(-)$ associated to such vector fields is defined as:*

$$\forall x \in \mathscr{M}, \forall v \in T_x\mathscr{M} \qquad g_x(v) := inf\left\{ u_1^2 + \dots u_k^2 \,\middle|\, \begin{array}{l} u_1, \dots, u_k \in \mathbb{R} \\ v = u_1\mathbf{X}_1(x) + \dots + u_k\mathbf{X}_k(x) \end{array} \right\} \in \mathbb{R}^{\ge 0} \cup \{+\infty\}$$

The magnitude of the tangent vector v is defined as $\|v\|_x := \sqrt{g_x(v)} \in \mathbb{R}^{\ge 0} \cup \{+\infty\}$ and it is finite if only if $v \in span_{\mathbb{R}}\big\{\mathbf{X}_1(x), \dots, \mathbf{X}_k(x)\big\}$.

Given a path $\gamma \in \mathrm{AC}([a,b],\mathscr{M})$ one can define its lenght as $len(\gamma) := \int_{[a,b]} \|\dot\gamma(t)\|_{\gamma(t)} \mathrm{d}t$ (it's a good definition since it can be shown that the function $t \mapsto \|\dot\gamma(t)\|_{\gamma(t)}$ is Lebesgue-measurable, althought non almost everywhere finite in general)

1.2. Definition (Sub-riemannian distance). *Given two point $p, q \in \mathscr{M}$, their Sub-riemannian distance (relative to the fixed vector fields) is defined as:*

$$\mathrm{d}(p,q) := inf\left\{ len(\gamma) \,\middle|\, \gamma \in \mathrm{AC}([a,b],\mathscr{M}),\ \begin{array}{l} \gamma(a) = p \\ \gamma(b) = q \end{array} \right\} \in \mathbb{R}^{\ge 0} \cup \{+\infty\}$$

A *necessary condition* for an AC-path γ to have finite lenght is that the map $t \mapsto \|\dot\gamma(t)\|_{\gamma(t)}$ is finite almost everywhere; it's easy to see that this is equivalent to require that there exist k measurable functions $u_1, \dots u_k$ such that:

$$\dot\gamma(t) = \sum_{i=1}^{k} u_i(t)\mathbf{X}_i(\gamma(t)) \quad \text{for a.e. } t \in [0,T]$$

since this means that for a.e. t the tangent $\dot\gamma(t)$ of γ lives in the corresponding subspace of $T_{\gamma(t)}$ spanned by the tangent vectors $\mathbf{X}_1(\gamma(t)), \dots \mathbf{X}_k(\gamma(t))$.

In general there is no hypothesis about the punctual linear independence of the vector fields and hence the functions $u_1,\dots,u_k$ are not unique; a *path has finite lenght* iff there exist at least functions $u_1,\dots,u_k$ that belong to $\mathrm{L}^1\big([a,b],\mathbb{R}\big)$ such that γ satisfies the latter equation.

2. Accessibility and the theorem Sussmann

Let $u=(u_1,\dots,u_k)\in \mathrm{L}^1\big([0,T],\mathbb{R}^k\big)$ be a fixed control function and consider the following differential system:

$$\begin{cases} \gamma\in \mathrm{AC}\big([0,T],\mathscr{M}\big) \\ \dot\gamma(t)=\sum_{i=1}^k u_i(t)\mathbf{X}_i(\gamma(t)) \quad \text{for a.e. } t\in[0,T] \\ \gamma(0)=p \end{cases}$$

Standard theorems in Control Theory assert that the latter problem admits an unique absolutely continuous solution γ_u whenever the control u belong to a certain open neighborhood $\mathscr{U}_{p,T}$ of the trivial control 0 in $\mathrm{L}^1\big([0,T],\mathbb{R}^k\big)$; moreover the following definition-theorem is well known:

2.1. Definition (End-point map). *The end-point map is defined solving the system with small control u and taking the end point of the path γ_u; formally:*

$$E_{p,T}:\mathscr{U}_{p,T}\longrightarrow\mathscr{M} \qquad u\mapsto\gamma_u(T)$$

Moreover, the end-point map is Fréchet differentiable (the set $\mathscr{U}_{p,T}$ inherits a smooth structure as open set of the Banach manifold $\mathrm{L}^1\big([0,T],\mathbb{R}^k\big)$).

We are working with path defined only in the segment $[0,T]$ with T fixed, but this is not a serious limitation: in fact if $\gamma_u:[0,T']\longrightarrow\mathscr{M}$ is an AC-path related to the control u then the reparamentrization of γ_u to $[0,T]$ solves the problem for the control $\frac{T'}{T}u$. This leads to the following:

2.2. Definition (Accessible set). *The image of the map $E_{p,T}$ in $\mathscr{M}$ is called the accessible set $\mathscr{A}_p$ of p; it consists in the set of point of $\mathscr{M}$ that are reachable from p with a controlled path in finite time.*

2.3. Definition (Normal and abnormal control). *Let r be the maximal rank of the end-point map $E_{p,T}:\mathscr{U}_{p,T}\longrightarrow\mathscr{M}$; a control function $u\in\mathscr{U}_{p,T}$ and the corresponding path γ_u are called normal iff the rank of $E_{p,T}$ at u is exactly the maximal rank r. Otherwise we speak of abnormality.*

If $u\in\mathrm{L}^1\big([0,T],\mathbb{R}^m\big)$ and $v\in\mathrm{L}^1\big([0,T'],\mathbb{R}^m\big)$ their *concatenation* $u*v$ is defined as the new control function in $\mathrm{L}^1\big([T+T'],\mathbb{R}^m\big)$:

$$\big(u*v\big)(t):=\begin{cases} u(t) & \text{if } t\in[0,T[\\ v(t-T) & \text{if } t\in[T,T'] \end{cases}$$

and define the inverse control $\widehat{u}$ of u as (if γ_u maps p in q then $\gamma_{\widehat{u}}$ maps q in p):

$$\widehat{u}(t):=-u(T-t)$$

Now we use such notions to prove the following useful lemma:

2.4. Lemma (Normal accessibility). *Every point accessible from p is normally accessible from p.*

Proof. First we prove the following claim: if a control u is normal then its concatenation $u*v$ with any other control v is normal too. To show this we use the variational language and pick an infinitesimal variation δu in L^1; we'll work with special variations of $u*v$ of the form $(u+\delta u)*v$. Let $q=\gamma_u$ and $r=\gamma_{u*v}$. Since u is normal, when δu describes L^1, δu describes a subspace of $T_q\mathscr{M}$ of rank equal to the maximal rank of the end point map $\mathscr{E}_{p,T}$; notice that keeping fix the control v and integrating with starting point nearby q we obtain a local diffeomorphism Φ_v and consequently $\mathrm{D}\Phi_v(\delta q)$ describes a subspaces of $T_r\mathscr{M}$ with the same maximal dimension and this is enough.

Finishing the proof of the lemma is easy: suppose that the path γ_u associated to the control u connect p and q; it is enough to take a normal control v taking p somewhere (which exists by the definition of normal control) and defining the return control $v*\widehat{v}*u$ which is normal by the previous claim and connects the original points. $\square$

2.5. Definition (Immersed submanifold). *A subset $\mathscr{A}$ of a smooth manifold $\mathscr{M}$ is called an immersed submanifold iff:*

- *$\mathscr{A}$ is endowed with a smooth manifold structure;*

- *the inclusion* $i : \mathscr{A} \hookrightarrow \mathscr{M}$ *is a smooth immersion (i.e. smooth with differential injective);*
- *for every manifold* $\mathscr{N}$ *and continuous map* $f : \mathscr{N} \longrightarrow \mathscr{M}$ *such that* $f(\mathscr{N}) \subseteq \mathscr{A}$*, the restriction* $f| : \mathscr{N} \to \mathscr{A}$ *is continuous for the topology of* $\mathscr{A}$*.*

If the latter condition holds, the same is true even for smooth maps $\mathscr{N} \longrightarrow \mathscr{M}$ *and hence the manifold structure of* $\mathscr{A}$ *is unique.*

In general $\mathscr{A}$ *has a topology different from the topology inherited as subspace of* $\mathscr{M}$ *and it is not a submanifold (i.e. locally smoothly flattenable).*

In an equivalent fashion, an immersed submanifold can be thought as the image of a manifold via a smooth injective immersive map such that such image is well behaved in some sense...

2.6. Theorem (Sussmann). *The accessibility set* $\mathscr{A}_p$ *is an immersed submanifold of* $\mathscr{M}$ *for each* $p \in \mathscr{M}$*.*

Proof. The *normal controls* form an open subset $N_{p,T}$ of $U_{p,T}$ and by Lemma[..] the accessibile $\mathscr{A}_p$ is image of $N_{p,T}$ via a constant rank map, say $rank \mathscr{E}_{p,T} = r$; we are going to use the *Rank Theorem* for manifold locally diffeomorphic to Banach spaces (in this case the domain is an open set in a Banach space and the target is a manifold which is locally diffeomorphic to euclidean space).

The map $\mathscr{E}_{p,T}$ is Fréchet-differntiable and its rank (i.e. the dimension of the image of the differential $\mathrm{d}\mathscr{E}_{p,T}$ which is well defined since the target space $\mathscr{M}$ is finite dimensional) is constant by the Rank Theorem it's locally flat; this means that for every normal control $u \in N_{p,T}$ with terminal point $x := \mathscr{E}_{p,T}(u) = \gamma_u(T) \in \mathscr{A}_p$ there exist a triple $(\mathscr{V}, U, \psi)$:

- a smooth r-dimensional submanifold $\mathscr{V}$ of $\mathscr{M}$ with $x \in \mathscr{V}$;
- an open neighborhood U of u in $N_{p,T}$;
- a diffeomorphism $\psi : U \xrightarrow{\cong} \mathscr{V} \times \mathscr{B}$ where $\mathscr{B}$ is an open set in a Banach space and the diagram commutes:

$$\begin{tikzcd} U \arrow[rr, "\psi", "\cong"'] \arrow[dr, "\mathscr{E}_{p,T}"'] & & \mathscr{V} \times \mathscr{B} \arrow[dl, "\pi_1"] \\ & \mathscr{V} & \end{tikzcd}$$

This means that we have found new "coordinates" for U near u and for $\mathscr{M}$ near x such that the map is linearized....

Using this fact one can topologize the accessible set $\mathscr{A}_p$ by saying that the incusion $\mathscr{V} \subseteq \mathscr{A}_p$ is open when $\mathscr{V}$ is part of a triple $(\mathscr{V}, U, \psi)$; a simple check shows that this is indeed the coarsest topology that makes the map $\mathscr{E}_{p,T}$ open. Next step is to define a smooth structure on $\mathscr{A}_p$ in a similar manner saying that the inclusions $\mathscr{V} \subseteq \mathscr{A}_p$ are immersions (i.e. the differential is injective in any point); this time we need to check some compatibility condition, that if $\mathscr{V}_1$ and $\mathscr{V}_2$ are subsets as in the latter, the intersection $\mathscr{V}_1 \cap \mathscr{V}_2$ is a smooth submanifold of both sets. Suppose that $x = \mathscr{E}_{p,T}(u) = \mathscr{E}_{p,T}(u')$ and since $\mathscr{E}_{p,3T}$ has rank r in the point $u'' := u * \widetilde{u} * \widetilde{u}'$ the image of a small neighborhood of u'' is a submanifold of $\mathscr{M}$ containing x and whose germ at x contains the germ of $\mathscr{V}'$ and $\mathscr{V}''$. But both have the same dimension these germs are the same and this is enough.

Now $\mathscr{A}_p$ is a smooth manifold and the inclusion map in $\mathscr{M}$ is immersive since any point has a neighborhood $\mathscr{V}$-like.

Next we need to show that $\mathscr{A}_p$ is indeed an immersed submanifold and we begin with some preliminaries; suppose that $q_0 \in \mathscr{A}_p$ and let $u_0 \in N_{p,T}$ a normal control such that $\gamma_{u_0}(p_0) = q_0$. The end-point $E_{p,T}$ has constant rank r near u_0 there exists a submanifold $\mathscr{W}_0 \ni 0$ of L^1 and a submanifold $\mathscr{V}_0$ of $\mathscr{M}$ such that $E_{p,T}$ maps $u_0 + \mathscr{W}_0$ diffeomorphically onto $\mathscr{V}_0$; this implies also that $E_{q_0,2T}$ maps $\widehat{u}_0 * (u_0 + \mathscr{W}_0)$ diffeomorphically onto $\mathscr{V}_0$. Actually, for any q near q_0 the map $E_{q,2T} : \widehat{u}_0 * (u_0 + \mathscr{W}_0) \to \mathscr{M}$ is a diffeomorphism onto a submanifold $\mathscr{V}(q)$ of $\mathscr{M}$; moreover, if q is in $\mathscr{A}_p$ then $\mathscr{V}(q)$ is fully contained as open set in $\mathscr{A}_p$.

Chose a submanifold $\mathscr{Q}$ transverse locally to $\mathscr{A}_p$ at q_0 and note that the map:

$$\begin{aligned} \phi : \mathscr{Q} \times \mathscr{W}_0 &\longrightarrow \mathscr{M} \\ (q, w) &\mapsto E_{q,2T}(\widehat{u} * (u_0 + w)) \end{aligned}$$

is a diffeomorphism onto an open neighborhood of q_0; consequently, $\phi(\{q\} \times \mathscr{W}_0) = \mathscr{V}(q)$ and for a given q only two possibilities are allowed: either $\phi(\{q\} \times \mathscr{W}_0) \subseteq \mathscr{A}_p$ or $\phi(\{q\} \times \mathscr{W}_0) \cap \mathscr{A}_p$.

Now, the original topology of $\mathscr{A}_p$ has countable basis since L^1 is separable (see the beginning of the proof that describes such topology) and this implies that only countably many of the sets $\phi(\{q\} \times \mathscr{W}_0), q \in \mathscr{Q}$ are contained in $\mathscr{A}_p$ since they are mutually disjoint.

Let η a continuous path in $\mathscr{A}_p$ for the topology of $\mathscr{M}$ such that $\eta(t_0) = q_0$; note that $\eta(t) = q_0$ for t near t_0, otherwise $\eta(t)$ would meet uncountably many slices $\phi(\{q\} \times \mathscr{W}_0)$. This implies that $\eta(t)$ is contained in $\mathscr{V}_0$ and in a neighborhood of t_0 is continuous for the topology of $\mathscr{A}_p$; since continuity is a local property, the path η is continuous for the original topology of $\mathscr{A}_p$ and the same argument works for any continuous map.

□

3. Local flows and Lie brackets

3.1. Definition (Lie bracket). *Let $\mathbf{X}$ and $\mathbf{Y}$ smooth vector fields on an m-dimensional manifold $\mathscr{M}$ and let Φ the local flow generated by $\mathbf{X}$; then for every point $p \in \mathscr{M}$ the following assignment originates a vector field:*

$$[\mathbf{X}, \mathbf{Y}](p) := \lim_{\varepsilon \to 0} \frac{\mathrm{d}\Phi_{-\varepsilon[\Phi_\varepsilon(p)]}(\mathbf{Y}(\Phi_\varepsilon(p))) - \mathbf{Y}(p)}{\varepsilon} \in T_p\mathscr{M}$$

An equivalent definition is obtained iterating the vector field action on C^∞ maps (i.e. $(\mathbf{X}f)(p) := \mathrm{d}f_{[p]}(\mathbf{X}(p))$):

$$\forall f \in \mathrm{C}^\infty(\mathscr{M}) \quad [\mathbf{X}, \mathbf{Y}]f := \mathbf{X}(\mathbf{Y}f) - \mathbf{Y}(\mathbf{X}f)$$

which locally in a coordinate patch with coordinates $(x_1, \ldots, x_m)$ and constant coordinate fields $(e_1, \ldots e_m)$ is also equivalent to the following:

$$\begin{array}{l} \mathbf{X}(p) = a_1(p)e_1 + \ldots a_m(p)e_m \\ \mathbf{Y}(p) = b_1(p)b_1 + \ldots b_m(p)e_m \end{array} \qquad [\mathbf{X}, \mathbf{Y}] = \sum_i (\sum_j a_j \frac{\partial b_i}{\partial x_j} - b_j \frac{\partial a_i}{\partial x_j}) e_i$$

3.2. Remark. It's important to note that the Lie bracket is a differential topology matter: no metric is used at all...!!!

It can be shown that the bracket $[-,-]$ is $\mathbb{R}$-bilinear, anticommutative, without unity and in general non associative (however the loss of associativity is described by the Jacobi relation: $[\mathbf{X}, [\mathbf{Y}, \mathbf{Z}]] = [[\mathbf{X}, \mathbf{Y}], \mathbf{Z}] + [,]$); this new operation transforms the $\mathbb{R}$-vector space of smooth vector fields $\Xi^\infty(\mathscr{M})$ in an algebra (a Lie algebra).

The following theorems describes the importance of bracket operation in differential topology:

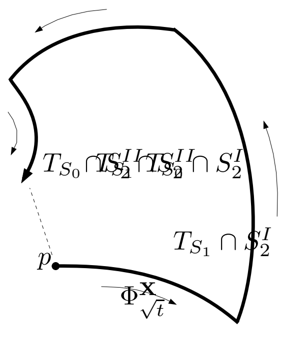

Figure 1. Geometric interpretation of the bracket.

3.3. Theorem (Flows commutation). *Suppose that $\mathbf{X}$ and $\mathbf{Y}$ are smooth vector fields on a manifold $\mathscr{M}$ and denote with $\Phi^{\mathbf{X}}$ and $\Phi^{\mathbf{Y}}$ are respectively the associated local flows; then for every point $p \in \mathscr{M}$ the following map is well defined admits the derivative in $0 \in \mathbb{R}$:*

$$\gamma(t) := \Phi^{\mathbf{Y}}_{-\sqrt{t}} \circ \Phi^{\mathbf{X}}_{-\sqrt{t}} \circ \Phi^{\mathbf{Y}}_{\sqrt{t}} \circ \Phi^{\mathbf{X}}_{\sqrt{t}}(p)$$

Thus γ defines a derivation and the following equality holds (as derivations):

$$[\mathbf{X}, \mathbf{Y}](p) = \dot\gamma(0)$$

Moreover, the local flows $\Phi^{\mathbf{X}}$ and $\Phi^{\mathbf{Y}}$ commute nearby p iff $[\mathbf{X}, \mathbf{Y}] = 0$ locally in p.

3.4. Theorem (Flattening vector fields). *Suppose given smooth vector fields* $\mathbf{X}_1, \dots, \mathbf{X}_k$ *on an* m*-dimensional smooth manifold and fix a point* $p \in \mathscr{M}$*; then the following are equivalent:*

- *the tangent vectors* $\mathbf{X}_1(p), \dots, \mathbf{X}_k(p)$ *are linearly independent in* $T_p\mathscr{M}$ *and they commute, i.e.* $[\mathbf{X}_i, \mathbf{Y}_j] = 0$ *for all* $i, j = 1, \dots, k$.
- *there are coordinates* $x_1, \dots x_m$ *around* p *such that the vector fields ... becomes the first* k *coordinate fields* $e_1, \dots, e_k$.

4. Connectivity and the theorem of Chow

4.1. Theorem (Chow). *Let* $\mathscr{M}$ *a smooth connected manifold, fix* k *vector fields* $\mathbf{X}_1, \dots \mathbf{X}_k$ *and a point* $p \in \mathscr{M}$*; then if the following relation holds:*

- *the vector fields* $\mathbf{X}_1, \dots \mathbf{X}_k$ *and all their iterated brackets* $[\mathbf{X}_a, \mathbf{X}_b]$, $[\mathbf{X}_a, [\mathbf{X}_b, \mathbf{X}_c]]$*,... generate in every point* $x \in \mathscr{M}$ *the full tangent space* $T_p\mathscr{M}$*;*

the accessible component $\mathscr{A}_p$ *of* p *is the whole* $\mathscr{M}$.

Proof. By construction, the flows $\Phi^{\mathbf{X}_1}, \dots \Phi^{\mathbf{X}_k}$ associated to the vector fields map $\mathscr{A}_p$ into itself; thus locally in $\mathscr{A}_p$ these fields are tangent. Due to the continuoty property of an immersed submanifold the vector fields $\mathbf{X}_1, \dots \mathbf{X}_k$ are indeed smooth vector fields in $\mathscr{A}_p$ for its manifold structure; looking at the definition of bracket [..] it's easy to see that the various Lie bracket of our fields are tangent to $\mathscr{A}_p$ too (the bracket operation can be seen as the derivative of a smooth curve in $\mathscr{A}_p$ and consequently it lives to $\mathscr{A}_p$) and hence can be transported in $\mathscr{A}_p$.

If Chow's condition holds we have that $T_y\mathscr{A}_p = T_y\mathscr{M}$ ($T_y\mathscr{A}_p$ is the image via differential of the inclusion of the tangent space of $\mathscr{A}_p$ at y for its manifold structure) of for every $y \in \mathscr{A}_p$ and hence $\mathscr{A}_p$ is open in $\mathscr{M}$ since is the image of a local diffeomorphism. The same reasoning can be done for the accessible component of every point not in $\mathscr{A}_p$, showing that the complement of $\mathscr{A}_p$ in $\mathscr{M}$ is open too. Since $\mathscr{M}$ is connected $\mathscr{A}_p$ is the whole manifold. □

5. Choice of controls

The set of points accessible from a given point in $\mathscr{M}$ by means of control functions belonging to any reasonable class of control functions, ranging from piecewise constant to L^1, is independent of the class of controls used; more precisely we have the following theorem:

5.1. Theorem (Changing controls). *Suppose* $\mathscr{C}$ *is a* L^1*-dense linear subspace in* $\mathrm{L}^1\big([0,T], \mathbb{R}^k\big)$*; then if two points* p *and* q *are accessible then they are accessbile using some control in* $\mathscr{C}$.

Proof. Using Lemma [] we can assume that p and q are connected by a path γ_u associated to a *normal control* u; this time for semplicity we call the end-point map $\mathrm{E}_{p,T}$ as E. By normality assumption, the continuous linear map $\mathrm{d}E[u] : \mathrm{L}^1 \to T_q\mathscr{A}_p$ is surjective.

The key trick is the following: the metric space L^1 is separable and thus with countable basis; consequently the subspace $\mathscr{C}$ is metric and with a countable basis, hence separable. Use a countable dense subset of $\mathscr{C}$ (which is automatically dense in L^1) to constuct a sequence of *finite dimensional* linear subspaces of $\mathscr{C}$ with striclty increasing dimensions and such that their union is dense L^1:

$$\mathscr{H}_1 \subsetneq \mathscr{H}_2 \ldots \subsetneq \mathscr{H}_k \subsetneq \ldots \qquad \mathrm{L}^1 = \overline{\cup_j \mathscr{H}_j}$$

Next use the continuity and surjectivity of differential $\mathrm{d}E[u]$ to pick an integer k_0 such that $\mathrm{d}E[u](\mathscr{H}_{k_0}) = T_q\mathscr{A}_p$ (just choose a basis for $T_q\mathscr{A}_p$ with elements belonging to the image of $\cup_j \mathscr{H}_j$ that is dense by continuity and surjectivity of the differential; then take any counterimage..). Chose a linear subspace $\mathscr{V}$ of $\mathscr{H}_{k_0}$ such that the linear map $\mathrm{d}E[u] : \mathscr{V} \to T_q\mathscr{A}_p$ is an isomorphism; then there exists $\varepsilon > 0$ such that:

- E is defined on $B^{\mathscr{V}}(0, 2\varepsilon)$;
- the mapping $\phi : h \mapsto E(u+h)$ is a diffeomorphism of the compact ball $\overline{B} := \overline{B}^{\mathscr{V}}(0,\varepsilon)$ onto $\phi(\overline{B}) = E(u + \overline{B})$.

Now let $u_k \in \mathscr{H}_k$ for $k = 1, 2, \dots$ a sequence of controls converging to u; if k is large enough then the map:

$$\phi_k : h \mapsto E(u_k + h)$$

is defined on all $\overline{B}$ and converges C^1-uniformly to ϕ. Now we need the following simple lemma:

Let $\overline{B}$ the compact ball in $\mathbb{R}^n$ of center q and radius ε; suppose that $\{f_k : \overline{B} \to \mathbb{R}^n\}_{k \geq 1}$ is a sequence of smooth mappings converging C^1-uniformly to the identity map of $\overline{B}$. Then for k large enough, the point q is contained in the image of f_k.

It suffices to prove that, for any smooth map $g : \overline{B} \to \mathbb{R}^n$ verifying $\|g(x) - x\| \leq \varepsilon/2$ and $\|\mathrm{d}g[x] - id\| \leq 1/2$ for all $x \in \overline{B}$ then the image of g contains q. Fot that purpose, consider the sequence in $\overline{B}$ defined by $x_0 := q$ and $x_{i+1} := q + x_i - g(x_i)$. It is clear by induction that this is well defined. Indeed, we have $\|x_i - q\| \leq \|x_1 - q\| + \|x_2 - x_1\| + \ldots + \|x_i - x_{i-1}\| \leq \varepsilon/2 + \varepsilon/4 + \ldots + \varepsilon/2^i < \varepsilon$. The same computation proves that the series $q + (x_1 - q) + (x_2 - x_1) + \ldots + (x_i - x_{i-1}) + \ldots$ converges. In other words, x_i converges to same x_∞ and clearly we have $\|x_\infty - q\| < \varepsilon$ and thus $g(x_\infty) = q$.

Applying this lemma to the sequence $f_k = \phi^{-1} \circ \phi_k$ it is easy to see that q belongs to the image of ϕ_k. Since $\phi_k(\overline{B}) = E(u_k + \overline{B})$ consists of images by E of elemens $\cup \mathscr{H}_k$ it results that q is accessible by means of a control function in the class $\mathscr{C}$. □

The simpler notiion of accessibility is obtained by using only concatenations of integral curves of the various vector fields; this is equivalent to use concatenations of controls of the type $(0, \ldots, 0, \pm 1, 0, \ldots, 0)$ which is called a Bang-Bang control. Controls of such type don't form a linear subspace of L^1 but the previous theorem holds with minor modification in the proof.

6. Examples

6.1. Example (Grusin's plane). Consider on the 2-manifold $\mathbb{R}^2$ the following two smooth vector fields:

$$\mathbf{X}(x,y) := \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \mathbf{Y}(x,y) := \begin{bmatrix} 0 \\ x \end{bmatrix}$$

And since the associated flow to $\mathbf{X}$ is just the horizontal translation:

$$[\mathbf{X}, \mathbf{Y}](x,y) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

This implies that at every pont $(x,y) \in \mathbb{R}^2$ the tangent vectors $\{\mathbf{X}(x,y), \mathbf{Y}(x,y), [\mathbf{X}, \mathbf{Y}](x,y)\}$ generates the tangent plane and thus *Chow's condition is satisfied everywhere.* This example is too easy since the only trouble come along the line $x = 0$. Note however that since the bracket is always differento from zero, the two flows

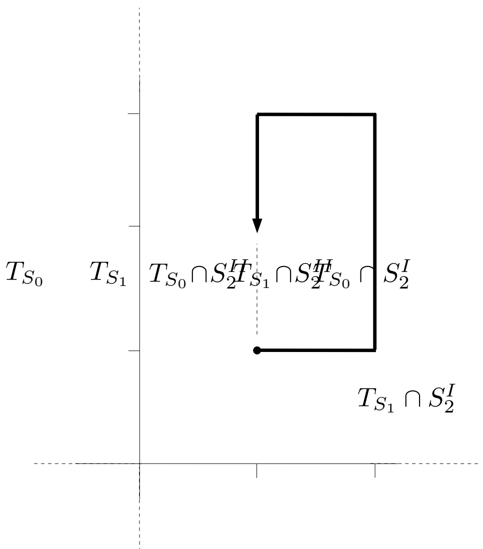

Figure 2. The flows don't commute.

never commutes.

6.2. Example (Combed plane). Take $\mathbb{R}^2$ with the standard euclidean coordinates (x, y) and consider the following smooth vector fields:

$$\mathbf{X}(x,y) := \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \mathbf{Y}(x,y) := \begin{bmatrix} 0 \\ f(x) \end{bmatrix}$$

where $f\colon \mathbb{R} \longrightarrow \mathbb{R}$ is a smooth map that is strictly positive in $]0, +\infty[$ and zero elsewhere (for example take the smooth extension by zero of $t \mapsto exp(-1/t^2)$). A simple computation gives:

$$[\mathbf{X}, \mathbf{Y}](x,y) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{ if } x \leq 0 \qquad [\mathbf{X}, \mathbf{Y}](x,y) = \begin{bmatrix} 0 \\ f'(x) \end{bmatrix} \text{ if } x > 0$$

Thus *Chow's condition is not satisfied* on the semiplane $x \leq 0$, however given any point the accessibe component is the whole plane; it's enough to use horizontal paths in the left semiplane to enter in the right part where every path is ammissible. Then clearly every pair of point can be connected.

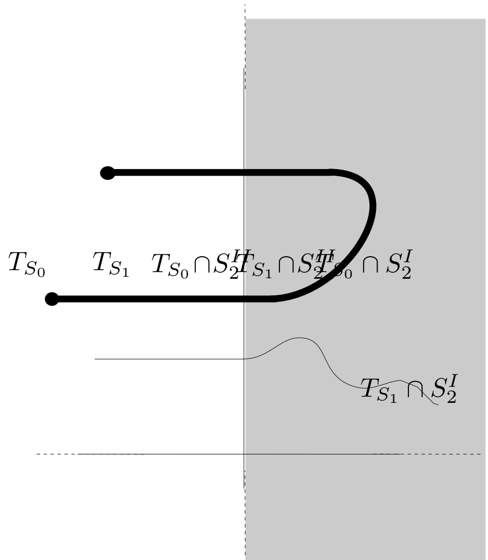

Figure 3. The appearance of a comb.

6.3. Example (Heisenberg's space). In the this example the role of non holonomy appears clearly and it serves as a paradigm for the whole theory; consider on the 3-manifold $\mathbb{R}^3$ with euclidean coordinates (x, y, z) the following smooth vector fields:

$$\mathbf{X}(x,y,z) := \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad \mathbf{Y}(x,y,z) := \begin{bmatrix} 0 \\ 1 \\ x \end{bmatrix}$$

Since the local flow associated to $\mathbf{X}$ is just the right translation along the x-axis, it is clear that:

$$[\mathbf{X}, \mathbf{Y}](x,y,z) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

and hence Chow's condition is satisfied. This implies that *every pair of points can be connected with a controlled path*; the following figure shows what is happening in this case; note that this time it is not so obvious why every pair of points can be connected since we work in a three dimensional space with only two vector fields. However, the "gap" originate by the non commutativity of the flows allows to walk even towards forbidden directions.

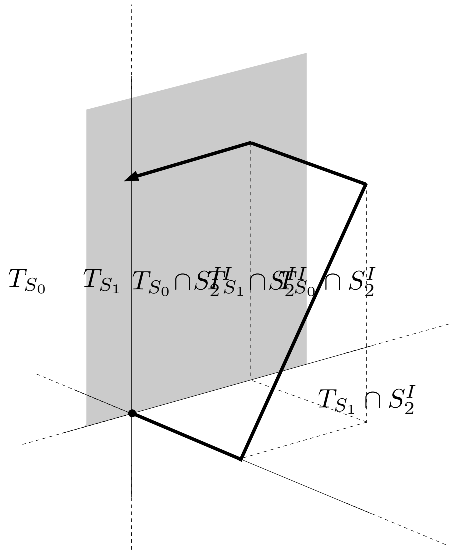

FIGURE 4. Why Chow's condition works.