ON THE ALGEBRAIC PROBLEM CONCERNING THE NORMAL FORMS OF LINEAR DYNAMICAL SYSTEMS.

By John Williamson.

Introduction. Let m be the number of degrees of freedom of a linear conservative dynamical system and let the point $(q_1, q_2, \cdots, q_m, p_1, p_2, \cdots, p_m)$ of the phase space be denoted by $x = (x_1, x_2, \cdots, x_{2m})$. A system of $2m$ ordinary differential equations of the first order, which are homogeneous, linear and do not contain t explicitly, is a canonical system if, and only if, there exists a symmetric matrix A, such that the differential equations may be written in the form

(i) $$Bdx/dt = Ax,$$

where B is the skew symmetric matrix $\begin{pmatrix} 0 & -E \\ E & 0 \end{pmatrix}$, and E the unit matrix of order m. In fact, apart from a factor 2, A is simply the matrix of the $2m$-ary quadratic form, which represents the Hamiltonian function. A non-singular matrix P is said to be a *Hamiltonian* matrix, if the transformation $x = Py$ sends every differential system of the form (i) into a differential system of the same form.

It has been pointed out by Wintner[1] that, if the system (i) is transformed into the system

(ii) $$Bdy/dt = Cy$$

by the transformation $x = Py$, then P is a Hamiltonian matrix if and only if

(iii) $$P'BP = sB \quad \text{and} \quad P'AP = sC.$$

In the following pages we use this result to determine a normal form for the system of equations (i). Equations (iii) imply,

$$P'(A - \lambda B)P = s(C - \lambda B),$$

or

$$P'_1(A - \lambda B)P_1 = \pm (C - \lambda B),$$

where $P_1 = (1/\sqrt{|s|})P$. Since $\begin{pmatrix} 0 & E \\ E & 0 \end{pmatrix} B \begin{pmatrix} 0 & E \\ E & 0 \end{pmatrix} = -B$, we have either

[1] A. Wintner, "On the linear conservative dynamical systems," *Annali di matematica pura ed applicata*, ser. 4, tomo 13 (1934-1935).

$$P'_1(A - \lambda B)P_1 = C - \lambda B \quad \text{or} \quad P'_2(A - \lambda B)P_2 = C_2 - \lambda B,$$

where $P_2 = P_1 \begin{pmatrix} 0 & E \\ E & 0 \end{pmatrix}$ and $C_2 = \begin{pmatrix} 0 & E \\ E & 0 \end{pmatrix} C \begin{pmatrix} 0 & E \\ E & 0 \end{pmatrix}$. In fact, if C is written as a matrix of matrices of order m, so that $C = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix}$, then $C_2 = \begin{pmatrix} C_{22} & C_{21} \\ C_{12} & C_{11} \end{pmatrix}$. It is therefore apparent that a normal form for the system (i) can be obtained from a suitable canonical form of the matrix pencil $A - \lambda B$.[2]

Accordingly we first consider the purely algebraic problem of determining a canonical form for a pencil $A - \lambda B$ where A is symmetric and B is skew symmetric and non-singular. For the sake of generality we assume that the elements of the matrices A and B lie in a commutative field K and that the transformation matrices are restricted to have elements in the same field. Later we particularize B to be the matrix $\begin{pmatrix} 0 & -E \\ E & 0 \end{pmatrix}$ and so obtain results applicable to the original dynamical problem. In the final section the simplifications, which arise when K is the field of all real numbers, are considered and a list of the possible normal forms of the matrix A in (i) is given for the case of two degrees of freedom ($m = 2$). These normal forms depend on the elementary divisors of $A - \lambda B$ which may be *real, complex or pure imaginary.* It is interesting to note that, if any of the elementary divisors of $A - \lambda B$ are pure imaginary, the elementary divisors alone are not sufficient to determine the normal form.

I. Let A and B be two square matrices of order n with elements in a commutative field K of characteristic zero. Further, let A be symmetric and B be skew-symmetric and non-singular, so that $A = A'$, $B = -B'$ and $|B| \neq 0$. If M is any matrix with elements in K, which is similar to AB^{-1}, there exists a non-singular matrix P, with elements in K, such that $P^{-1}AB^{-1}P = M$. Hence, if λ is any indeterminate, $P^{-1}(A - \lambda B)B^{-1}P = M - \lambda E$. Accordingly,

$$P'(B^{-1})'(A - \lambda B)B^{-1}P = P'(B^{-1})'P(M - \lambda E) = R(M - \lambda E),$$

where $R = P'(B^{-1})'P$. If $C = B^{-1}P$, we may write this last equation in the form

$$C'(A - \lambda B)C = R(M - \lambda E). \tag{1}$$

As a consequence of (1),

$$R = C'BC \quad \text{and} \quad RM = C'AC,$$

[2] Cf. C. Lanczos, "Eine neue Transformationstheorie linearer kanonischer gleichungen," *Annalen der Physik*, 5 Folge; ser. 653, Band 20 (1934).

so that R is skew symmetric and RM is symmetric. Therefore

$$RM = (RM)' = M'R' = -M'R. \tag{2}$$

The pencil $A - \lambda B$ is *equivalent* under a non-singular *congruent transformation* with elements in K to the pencil $RM - \lambda R$ and we may, without any risk of confusion, simply say that the two pencils are equivalent. We shall proceed to determine a canonical form for the pencil $A - \lambda B$ by choosing a suitable form for the matrix M and by reducing the matrix R. We first notice that, if W is a non-singular matrix with elements in K and if

$$W'R(M - \lambda E)W = S(M - \lambda E), \tag{3}$$

then

$$S = W'RW \tag{4}$$

and

$$W'RMW = SM = W'RWM \text{ by } (4),$$

so that, since $W'R$ is non-singular,

$$MW = WM. \tag{5}$$

Hence, in the reduction of the matrix R, we are only at liberty to use transformations, whose matrices are commutative with M.

Further, if Q is a non-singular matrix satisfying the equation

$$QM = -M'Q, \tag{6}$$

it follows easily from (2) that

$$R = QG, \tag{7}$$

where

$$GM = MG. \tag{8}$$

If M is a *diagonal block* matrix,

$$M = [M_1, M_2, \cdots, M_t], \tag{9}$$

where M_i is a square matrix of order e_i, we may write G as a matrix of matrices,

$$G = (G_{ij}) \qquad (i, j = 1, 2, \cdots, t),$$

where G_{ij} is a matrix of e_i rows and e_j columns.

If Q_i is a non-singular matrix of order e_i, such that

$$Q_i M_i = -M'_i Q_i \qquad (i = 1, 2, \cdots, t),$$

then the diagonal block matrix

$$(10) \qquad Q = [Q_1, Q_2, \cdots, Q_t]$$

is non-singular and satisfies (6).

We now prove,

LEMMA 1. *If $Q'_i = \rho_i Q_i$ $(i = 1, 2, \cdots, t)$, where $\rho_i = \pm 1$, and G_{11} is non-singular, there exists a non-singular matrix W commutative with M, such that $W'QGW = QH$, where $H = (H_{ij})$ $(i, j = 1, 2, \cdots, t)$, and $H_{11} = G_{11}$, $H_{1j} = H_{j1} = 0$ $(j \neq 1)$.*

Proof. Let

$$W = \begin{pmatrix} E_1 & -G_{11}^{-1}G_{12} & -G_{11}^{-1}G_{13} & \cdots & -G_{11}^{-1}G_{1t} \\ 0 & E_2 & 0 & \cdots & 0 \\ 0 & 0 & E_3 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & \cdots & E_t \end{pmatrix},$$

where E_i is the unit matrix of order e_i.

Since the matrix G satisfies (8), $G_{ij}M_j = M_iG_{ij}$ and consequently, $-G_{11}^{-1}G_{1j}M_j = -G_{11}^{-1}M_1G_{1j} = -M_1G_{11}^{-1}G_{1j}$. Hence the matrix W is commutative with M.

Since R is skew symmetric, it follows from (7) that

$$(11) \qquad Q_iG_{ij} = -G'_{ji}Q'_j = -\rho_jG'_{ji}Q_j.$$

The element in the j-th row, $j \neq 1$, of the first column of $W'Q$ is

$$-G'_{1j}(G_{11}^{-1})'Q_1.$$

But

$$-G'_{1j}(G_{11}^{-1})'Q_1 = -\rho_1G'_{1j}Q_1G_{11}^{-1} = -\rho_1^2Q_jG_{j1}G_{11}^{-1} = -Q_jG_{j1}G_{11}^{-1}$$

by (11) and the definition of ρ_1. Therefore $W'Q = QL$, where L is the matrix obtained from W' by substituting $-G_{j1}G_{11}^{-1}$ for $-G'_{1j}(G_{11}^{-1})'$ as the element in the j-th place of the first column, $j \neq 1$. Since the element in the first row and the first column of the product LGW is G_{11}, while all other elements in the first row or column are zero, the lemma is proved.

If the diagonal block matrix M in (9) is such that every matrix G commutative with M is also a diagonal block matrix $G = [G_1, G_2, \cdots, G_t]$, where G_i is a square matrix of order e_i, then

$$(12) \qquad G_iM_i = M_iG_i \qquad (i = 1, 2, \cdots, t).$$

Further, as a consequence of (7), R is a diagonal block matrix $[R_1, R_2, \cdots, R_t]$, where

$$R_i = Q_i G_i \qquad (i = 1, 2, \cdots, t), \tag{13}$$

and since W is commutative with M, $W = [W_1, W_2, \cdots, W_t]$ and the matrix S defined by (4) is a diagonal block matrix $[S_1, S_2, \cdots, S_t]$, where

$$S_i = W'_i R_i W_i \qquad (i = 1, 2, \cdots, t). \tag{14}$$

But, apart from the suffixes i, equations (12), (13) and (14) are the same as (8), (7) and (4) respectively. *Therefore, in reducing R to S, we need only consider the reduction of each block R_i separately by matrices commutative with M_i.*

2. *Form of the matrix M.* Since the elements of the matrices A and B lie in the field K, the invariant factors of the pencil $A - \lambda B$ are polynomials in λ with coefficients in K. We shall call the powers of the distinct irreducible factors of the invariant factors, *the elementary factors* (with respect to K) of the pencil. Since A is symmetric and B is skew symmetric, the invariant factors are unaltered, except perhaps in sign, by the interchange of λ and $-\lambda$. Hence each invariant factor is the product of an even polynomial in λ by a power of λ. Accordingly the elementary factors of the pencil $A - \lambda B$ are of three types:

Type a. $[p(\lambda)]^k$ together with $[p(-\lambda)]^k$, where $p(\lambda)$ is an irreducible polynomial but is not an even polynomial in λ and $p(\lambda) \neq \lambda$.

Type b. $[h(\lambda)]^k$ where $h(\lambda) = p(\lambda^2)$ is an even irreducible polynomial in λ.

Type c. λ^k.

We now proceed to determine matrices with elementary factors of types (a), (b) and (c) respectively.[3]

Type (a). Let $p(\lambda)$ be of degree m and let p be any matrix of order m with elements in K, whose characteristic equation is $p(\lambda)$ [4] $= 0$, and let e be the unit matrix of order m. Then, if

$$\pi = \begin{pmatrix} p & 0 \\ 0 & -p \end{pmatrix}, \qquad \phi = \begin{pmatrix} e & 0 \\ 0 & -e \end{pmatrix}, \qquad \epsilon = \begin{pmatrix} e & 0 \\ 0 & e \end{pmatrix}, \tag{15}$$

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[3] If D is a matrix with elements in K we shall mean by the elementary factors of D, the elementary factors of the pencil $D - \lambda E$, where E is the unit matrix.

[4] We may take as the matrix p the companion matrix of $p(\lambda)$. Cf. J. Williamson, "The equivalence of non-singular pencils of hermitian matrices in an arbitrary field," *American Journal of Mathematics*, vol. 57 (1935), p. 475.

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the matrix

$$N = \begin{pmatrix} \pi & \phi & 0 & \cdot & 0 \\ 0 & \pi & \phi & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & \phi \\ 0 & 0 & 0 & \cdot & \pi \end{pmatrix} \tag{16}$$

of order k, considered as a matrix of matrices of order $2m$, has the elementa[r] factors $[p(\lambda)]^k$, $[p(-\lambda)]^k$. For $[p(N)]^k[p(-N)]^k = 0$ and N satisfi[es] no equation of lower degree. We now write (16) in the more convenient for[m]

$$N = \pi E + \phi U, \tag{17}$$

where

$$E = \begin{pmatrix} \epsilon & 0 & \cdot & 0 \\ 0 & \epsilon & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & \epsilon \end{pmatrix} \quad \text{and} \quad U = \begin{pmatrix} 0 & \epsilon & 0 & \cdot & 0 \\ 0 & 0 & \epsilon & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & \epsilon \\ 0 & 0 & 0 & \cdot & 0 \end{pmatrix},$$

and proceed to determine a non-singular matrix V satisfying

$$VN = -N'V. \tag{18}$$

If

$$T = \begin{pmatrix} 0 & 0 & \cdot & 0 & \epsilon \\ 0 & 0 & \cdot & \epsilon & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \epsilon & \cdot & 0 & 0 \\ \epsilon & 0 & \cdot & 0 & 0 \end{pmatrix}, \tag{19}$$

we see immediately that $T^2 = E$ and that

$$TU = U'T. \tag{20}$$

Further, we can determine a non-singular symmetric matrix q such that [5]

$$qp = p'q. \tag{21}$$

Since the matrix τ, defined by

$$\tau = \begin{pmatrix} 0 & -e \\ e & 0 \end{pmatrix}, \tag{22}$$

satisfies the equation

$$\tau\phi = -\phi\tau = -\phi'\tau, \tag{23}$$

it follows that the matrix

$$V = q\tau T \tag{24}$$

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[5] J. Williamson, *loc. cit.*, p. 490.

is a non-singular skew symmetric matrix satisfying (18). In fact,

$$\begin{aligned} VN &= q\tau T(\pi E + \phi U) \text{ by } (17) \\ &= q\tau(\pi E + \phi U')T \text{ by } (20) \\ &= -(\pi' E + \phi' U')q\tau T \text{ by } (23) \\ &= -N'V. \end{aligned}$$

Type b. The characteristic equation of the matrix

$$\pi = \begin{pmatrix} 0 & e \\ p & 0 \end{pmatrix} \tag{25}$$

is $p(\lambda^2) = 0$, so that π has the single elementary divisor $p(\lambda^2)$ and the matrix

$$N = \pi E + U \tag{26}$$

has the single elementary factor $[p(\lambda^2)]^k$. If

$$X = \begin{bmatrix} 0 & 0 & \cdot & 0 & -\epsilon \\ 0 & 0 & \cdot & \epsilon & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & (-\epsilon)^{k-1} & \cdot & 0 & 0 \\ (-\epsilon)^k & 0 & \cdot & 0 & 0 \end{bmatrix}, \tag{27}$$

it is easily shown that

$$XU = -U'X$$

and that, if

$$V = q\tau X, \tag{28}$$

V is non-singular and satisfies (18). It should be noted that, since τ is skew symmetric, V is symmetric, if k is even, and skew symmetric if k is odd.

Type c. If, in (26), $\pi = 0$ and $\epsilon = 1$, so that U is the auxiliary unit matrix of order k,

$$N = U, \tag{29}$$

is a matrix with the single elementary factor λ^k. Moreover the matrix V defined by (28) where q, τ and ϵ all have the value unity, satisfies (18).

If $N_1, N_2, \cdots, N_r$ are r matrices, where N_i is the matrix N with $k = k_i$, the diagonal block matrix

$$M = [N_1, N_2, \cdots, N_r] \tag{30}$$

has the elementary factors $[p(\lambda)]^{k_i}$, $[p(-\lambda)]^{k_i}$, if each N_i is defined by (17); the elementary factors $[p(\lambda^2)]^{k_i}$, if each N_i is defined by (26); and the elementary factors λ^{k_i}, if each N_i is defined by (29) $(i=1,2,\cdots,r)$. Equation (24) or (28) with $k=k_i$ determines a non-singular matrix V_i such that $V_iN_i=-N'V_i$ and accordingly the matrix

$$(31) \qquad Q=[V_1,V_2,\cdots,V_r]$$

is non-singular and satisfies (6), when M has the value given by (30).

Let the elementary factors of $A-\lambda B$ be $[p_i(\lambda)]^{k_{ij}}$, $[p_i(-\lambda)]^{k_{ij}}$ $(i=1,2,\cdots,s)$, of type a, $[p_i(\lambda^2)]^{k_{ij}}$ $(i=s+1,\cdots,t-1)$, of type b, $\lambda^{k_{tj}}$ of type c,

$$j=1,2,\cdots,r_i;\ k_{i1}\geqq k_{i2}\geqq\cdots\geqq k_{ir_i},$$

where $p_i(\lambda)\neq p_j(\lambda)$, $p_i(\lambda^2)\neq p_j(\lambda^2)$, if $i\neq j$. Then the matrix

$$(32) \qquad M=[M_1,M_2,\cdots,M_t],$$

where M_i is the matrix corresponding to the matrix on the left of (30), when p is replaced by p_i and r by r_i, has the same elementary factors and therefore the same invariant factors as $A-\lambda B$. Hence the matrix M is similar to the matrix AB^{-1}. Moreover, if Q_i is obtained from (31) in the same way as M_i is obtained from (30), the diagonal block matrix,

$$(33) \qquad Q=[Q_1,Q_2,\cdots,Q_t],$$

satisfies the equation $QM=-M'Q$. We may accordingly take the matrices M and Q defined by (32) and (33) as the matrices M and Q of section 1. Since any matrix G, commutative with M in (32) is a diagonal block matrix [6] $[G_1,G_2,\cdots,G_t]$ by the remark at the end of section I we see that we may treat each block R_i separately.

3. *Reduction of R.* We consider the matrix $R_i=Q_iG_i$, where R_i is skew-symmetric and non-singular, G_i is commutative with M_i and $Q_iM_i=-M'_iQ_i$. We first treat the case, in which $i\leqq t-1$, so that the elementary factors of M_i are not of the form λ^k. For simplicity of writing we temporarily drop all suffixes i and write R, Q, G, M, etc. for R_i, Q_i, G_i, M_i respectively so that M and Q are the matrices defined by equations (30) and (31) respectively.

If,

$$G=(G_{ij})$$

is a *partition* of G *similar* to that of M in (30), i. e. if G_{ij} is a matrix with

[6] R. C. Trott, *Bulletin of the American Mathematical Society*, January 1935, Abstract No. 95.

the same number of rows k_i as N_i and the same number of columns k_j as N_j, it is known that, when $k_i \geqq k_j$,

$$G_{ij} = \begin{pmatrix} F_{ij} \\ 0 \end{pmatrix}, \qquad G_{ji} = (0 \;\; F_{ji}), \tag{34}$$

where F_{ij} and F_{ji} are square matrices of order $k = k_j$.[7] Moreover,

$$F_{ij} = f_{ij0}E + f_{ij1}U + \cdots + f_{ij,k-1}U^{k-1}, \tag{35}$$

where $f_{ijs} = f_{ijs}(\pi)$ is a polynomial in the matrix π with coefficients in K. Since $R = QG$ is skew-symmetric, it follows that

$$V_iG_{ij} = -G'_{ji}V'_j = \rho_jG'_{ji}V_j, \tag{36}$$

where $\rho_j = +1$, if V_j is skew symmetric and $\rho_j = -1$ if V_j is symmetric. In particular, if $k_i = k_j$, since $G_{ij} = F_{ij}$ and $V_i = V_j$, (36) becomes

$$V_iF_{ij} = \rho_jF'_{ji}V_j = \rho_iF'_{ji}V_i. \tag{37}$$

As a consequence of the definition of V_i,

$$V_i\pi U^a = -\pi'U'^aV_i.$$

Hence,

$$\begin{aligned} V_iF_{ij} &= V_i \sum_{s=0}^{k-1} f_{ijs}(\pi)\,U^s, \\ &= \sum_{s=0}^{k-1} f_{ijs}(-\pi')\,U'^sV_i, \\ &= \rho_i \sum_{s=0}^{k-1} f_{jis}(\pi')\,U'^sV_i \quad \text{by} \quad (37). \end{aligned}$$

Therefore, if $k_i = k_j$,

$$f_{ijs}(-\pi) = \rho_if_{jis}(\pi) \qquad (s = 1, 2, \cdots, k-1). \tag{38}$$

In particular,

$$f_{iis}(-\pi) = \rho_if_{iis}(\pi).$$

Hence $f_{iis}(\pi)$ is an even polynomial in π if $\rho_i = 1$ and an odd polynomial, if $\rho_i = -1$. In either case $f_{iis}(\pi)$ is singular, if and only if it is zero. Consequently we have the result; *G_{ii} is singular, if and only if its first element f_{ii0} is zero.*

Let $k_1 = k_2 = \cdots = k_c > k_{c+1} \geqq k_{c+j}$. Then, if G_{11} is singular, but for some value of j, $1 < j \leqq c$, G_{jj} is non-singular, we may interchange the first and j-th rows, and the first and j-th columns, thus bringing G_{jj} into the place

[7] Trott, *loc. cit.*, cf. Cullis, *Matrices and Determinoids*, vol. 3, chap. XXVII.

of G_{11} without disturbing M or Q. We therefore suppose that G_{jj} is singular for all values of j, $1 \leqq j \leqq c$. Since the first element of G_{i1} is zero, when $i > c$, (equation (34)) and G is non-singular, the first element f_{j10} of G_{j1} is different from zero for at least one value of j, $1 < j \leqq c$. Accordingly without any loss of generality we may suppose that $f_{210} \neq 0$.

Let

$$W = \left[\begin{pmatrix} E_1 & 0 \\ w(\pi) E_1 & E_1 \end{pmatrix}, E_2 \right],$$

where E_1 is the unit matrix of the same order as N_1 and E_2 the unit matrix of the same order as $[N_3, N_4, \cdots, N_r]$. The matrix W is commutative with M and

$$W'Q = \left[\begin{pmatrix} E_1 & w(\pi') E_1 \\ 0 & E_1 \end{pmatrix}, E_2 \right] Q,$$

$$= Q \left[\begin{pmatrix} E_1 & w(-\pi) E_1 \\ 0 & E_1 \end{pmatrix}, E_2 \right].$$

If $W'QGW = QH$, where $H = (H_{ij})$ is a partition of H similar to that of G,

$$H_{11} = G_{11} + w(-\pi) G_{21} + w(\pi) G_{12} + w(-\pi) w(\pi) G_{22}.$$

The first element h of H_{11} accordingly satisfies the equation

$$h = f_{110} + w(-\pi) f_{210} + w(\pi) f_{120} + w(-\pi) w(\pi) f_{220}$$

or

$$h = w(-\pi) f_{210} + w(\pi) f_{120},$$

since by hypothesis $f_{110} = f_{220} = 0$.

If $w(\pi)$ is the identity matrix h has the value $f_{210} + f_{120}$ and if $w(\pi) = \pi$, the value $\pi(f_{120} - f_{210})$. Since π is non-singular both these values of h cannot be zero, as otherwise f_{210} would be zero, contrary to our assumption. Thus we find a non-singular matrix W, such that $W'QG = QH$ where the first element of H_{11} is non-zero so that H_{11} is non-singular. We may therefore suppose that such a transformation has already been applied to G and accordingly may assume that G_{11} is non-singular.

The matrices Q and G now satisfy the hypothesis of Lemma 1, so that G may be reduced to a form, in which $G_{j1} = G_{1j} = 0$, $j \neq 1$. By $r-1$ repetitions of the above process we finally reduce G to the diagonal block matrix,

$$G = [G_1, G_2, \cdots, G_r], \tag{39}$$

where $G_j = \sum_{s=0}^{k_j - 1} \gamma_{js} U_j{}^s$ and $\gamma_{js} = \gamma_{js}(\pi)$ is a polynomial in the matrix π while γ_{j0} is non-singular.

We now proceed to reduce the matrix G_j to the form $\gamma_{j0} E_j$. Let

$$\gamma_{j1} = \gamma_{j2} = \cdot\cdot\cdot = \gamma_{jc} = 0, \quad \gamma_{j,c+1} \neq 0, \quad c \leqq k_{j-1}.$$

Then, if $W_j = E_j - \frac{1}{2}\gamma_{j0}{}^{-1}\gamma_{j\,c+1}U_j{}^{c+1}$,

$$W_j{}^2 G_j = H_j = \gamma_{j0}E_j + \sum_{s=c+2}^{k_j-1} h_{js}(\pi)\,U_j{}^s.$$

Moreover

$$\begin{aligned} W'_jV_j &= \{E_j - \tfrac{1}{2}(\gamma_{j0}{}^{-1}E_j)'(\gamma_{j\,c+1}U_j{}^{c+1})'\}V_j \\ &= \rho_j{}^2V_j(E_j - \tfrac{1}{2}\gamma_{j0}{}^{-1}E_j\gamma_{j\,c+1}U_j{}^{c+1}), \text{ by } (36), \\ &= V_jW_j. \end{aligned}$$

Therefore $$W'_jV_jG_jW_j = V_jW_j{}^2G_j = V_jH_j.$$

But H_j is of the same type as G_j, except that it contains no term in $U_j{}^{c+1}$. Accordingly, in at most $k_j - 1$ such steps, we may reduce G_j to the form

$$G_j = \gamma_jE_j. \tag{40}$$

The matrix γ_j in (40) is a polynomial in π, which is even or odd according as V_j is skew symmetric or symmetric.

It is now necessary to distinguish between the two cases;

case a. $$\pi = \begin{pmatrix} p & 0 \\ 0 & -p \end{pmatrix},$$

case b. $$\pi = \begin{pmatrix} 0 & e \\ p & 0 \end{pmatrix}.$$

In case a for all values of j, V_j is skew symmetric, so that γ_j in (40) is an even polynomial

$$\gamma_j = g_j(\pi^2) = \begin{pmatrix} g_j(p^2) & 0 \\ 0 & g_j(p^2) \end{pmatrix}.$$

Let $$r(\pi) = \begin{pmatrix} [g_j(p^2)]^{-1} & 0 \\ 0 & e \end{pmatrix}.$$

Then

$$\begin{aligned} r(\pi)'q\tau g_j(\pi^2)r(\pi) &= q\tau\begin{pmatrix} e & 0 \\ 0 & [g_j(p^2)]^{-1} \end{pmatrix}\begin{pmatrix} g_j(p^2) & 0 \\ 0 & g_j(p^2) \end{pmatrix}\begin{pmatrix} [g_j(p^2)]^{-1} & 0 \\ 0 & e \end{pmatrix}, \\ &= q\tau\begin{pmatrix} e & 0 \\ 0 & e \end{pmatrix} \text{ by (21) and (22).} \end{aligned}$$

Therefore, if $W_j = r(\pi)E_j$,

$$W'_jV_jG_jW_j = V_j\begin{pmatrix} e & 0 \\ 0 & e \end{pmatrix}E_j = V_j = q\tau T_j.$$

Hence in case a we may reduce R and RM by transformations permutable with M, to the forms

$$(41)\qquad [q\tau T_1, q\tau T_2, \cdots, q\tau T_r], \qquad [q\tau T_1N_1, q\tau T_2N_2, \cdots, q\tau T_rN_r]$$

respectively. The matrices (41) are uniquely determined by the matrices p and q and by the exponents k_i to which $p(\lambda)$ and $p(-\lambda)$ occur among the elementary factors of $A - \lambda B$.

We condense the above results in the following statement: *Corresponding to each pair of elementary factors $[p(\lambda)]^k$, $[p(-\lambda)]^k$ of type a in the canonical form of $A - \lambda B$, there is a block*

$$(42)\qquad VN - \lambda V,$$

where N is defined by (17) *and V by* (24).

case b. In this case no such great simplification is in general possible. Let

$$G = [\gamma_1 E_1, \gamma_2 E_2, \cdots, \gamma_r E_r], \qquad H = [\sigma_1 E_1, \sigma_2 E_2, \cdots, \sigma_r E_r],$$

where σ_i and γ_i are all non-singular polynomials in π. Further let W be a non-singular matrix commutative with M such that,

$$(43)\qquad W'QGW = QH.$$

If, as in previous cases, we write $W = (W_{ij})$ as a matrix of matrices, it is a consequence of (43) that

$$(44)\qquad \sum_{a=1}^{r} W'_{ai} V_a \gamma_a E_a W_{aj} = \delta_{ij} V_i \sigma_i E_i \qquad (i, j = 1, 2, \cdots, r),$$

δ_{ij} the Kronecker δ.

Since W is commutative with M

$$W_{ij} = \begin{pmatrix} F_{ij} \\ 0 \end{pmatrix}, \quad W_{ji} = (0\ F_{ji}), \quad i \leqq j \quad \text{i. e.} \quad k_i \geqq k_j,$$

where F_{ij} is defined by (35). Moreover,

$$\begin{aligned} F'_{ij} V_j &= \{f_{ij0}(\pi') E_j + \sum_{s=1}^{k_j - 1} f_{ijs}(\pi') U'^{s}_{j}\} V_j \\ &= V_j \{f_{ij0}(-\pi) E_j + \sum_{s=1}^{k_j - 1} (-1)^s f_{ijs}(-\pi) U_j^s\} \\ &= V_j \tilde{F}_{ij}. \end{aligned}$$

Accordingly if $i \leqq j$,

$$W'_{ij} V_i = (F'_{ij}\ 0) V_i = (0\ F'_{ij} V_j) = (0\ V_j \tilde{F}_{ij}) = V_j (0\ \tilde{F}_{ij})$$

and

$$W'_{ji}V_j = \begin{pmatrix} 0 \\ F'_{ji} \end{pmatrix} V_j = \begin{pmatrix} 0 \\ F'_{ji}V_j \end{pmatrix} = \begin{pmatrix} 0 \\ V_j\tilde{F}_{ji} \end{pmatrix} = V_i \begin{pmatrix} \tilde{F}_{ji} \\ 0 \end{pmatrix}.$$

Hence,

(45) $$W'_{ij}V_i = V_j\tilde{W}_{ij} \quad \text{and} \quad W'_{ji}V_j = V_i\tilde{W}_{ji}$$

where

$$\tilde{W}_{ij} = (0\ \tilde{F}_{ij}) \quad \text{and} \quad \tilde{W}_{ji} = \begin{pmatrix} \tilde{F}_{ji} \\ 0 \end{pmatrix} \quad k_i \geqq k_j.$$

Therefore, if $w_{ij}(\pi)$ and $\tilde{w}_{ij}(\pi)$ are the first elements of the matrices W_{ij} and $\tilde{W}_{ij}$ respectively we have the results

(46) $$\tilde{w}_{ij}(\pi) = w_{ij}(-\pi), k_i = k_j;\ \tilde{w}_{ij}(\pi) = 0, k_i > k_j;\ w_{ij}(\pi) = 0, k_i < k_j.$$

It follows from (44) and (45) that

$$V_i \sum_{\alpha=1}^{r} \tilde{W}_{\alpha i}\gamma_\alpha E_\alpha W_{\alpha j} = \delta_{ij} V_i \sigma_i E_i,$$

or since V_i is non-singular that

(47) $$\sum_{\alpha=1}^{r} \tilde{W}_{\alpha i}\gamma_\alpha E_\alpha W_{\alpha j} = \delta_{ij}\sigma_i E_i \qquad (i, j = 1, 2, \cdots, r).$$

As a consequence of the nature of the matrices $\tilde{W}_{\alpha i}$ and $W_{\alpha j}$, (47) remains true when each matrix is replaced by its first element, so that

$$\sum_{\alpha=1}^{r} \tilde{w}_{\alpha i}(\pi)\gamma_\alpha w_{\alpha j}(\pi) = \delta_{ij}\sigma_i \qquad (i, j = 1, 2, \cdots, r).$$

If $k_{c-1} > k_c = k_{c+1} = \cdots = k_d > k_{d+1}$, it follows from (46) and the last equation that

(48) $$\sum_{\alpha=c}^{d} w_{\alpha i}(-\pi)\gamma_\alpha w_{\alpha j}(\pi) = \sigma_i\delta_{ij}, \qquad c \leqq i \leqq d, \quad c \leqq j \leqq d.$$

The matrices γ_i and σ_i $(i = c, c+1, \cdots, d)$, are either all even polynomials in π or else all odd polynomials in π. We may therefore write

(49) $$\gamma_i = g_i(\pi^2)\pi^a, \qquad \sigma_i = h_i(\pi^2)\pi^a \qquad (a = 0 \text{ or } 1),$$

so that (48) becomes,

(50) $$\sum_{\alpha=c}^{d} w_{\alpha i}(-\pi) g_\alpha(\pi^2) w_{\alpha j}(\pi) = h_i(\pi^2)\delta_{ij}, \qquad c \leqq i \leqq d, \quad c \leqq j \leqq d.$$

If θ^2 is a zero of $p(x)$, the field $K_1 = K(\theta^2)$ is simply isomorphic to the field of all polynomials in π^2 with coefficients in K and the field $K_2 = K(\theta)$ is simply isomorphic to the field of all polynomials in π with coefficients in K. Accordingly (50) implies

$$\sum_{a=c}^{d} w_{ai}(-\theta) g_a(\theta^2) w_{aj}(\theta) = h_i(\theta^2)\delta_{ij}, \tag{51}$$

and conversely (51) implies (50). The field K_2 is quadratic over K_1 and, if $w(\theta)$ is an element of K_2, $w(-\theta) = \bar{w}$, is its conjugate. Hence, if

$$C = (c_{ij}) \qquad (i, j = 1, 2, \cdots, d + 1 - c),$$

where

$$c_{ij} = w_{c+i-1,c+j-1}(\theta),$$

(51) is equivalent to

$$C^*[g_c(\theta^2), g_{c+1}(\theta^2), \cdots, g_d(\theta^2)]\, C = [h_c(\theta^2), \cdots, h_d(\theta^2)], \tag{52}$$

where C^* is the conjugate transposed of C.

By a suitable interchange of rows and columns it can be shown that $|\, w_{ij}(\pi) \,|$, $c \leqq i$, $j \leqq d$ is a factor of $|\, W \,|$. Hence, since W is non-singular, $|\, w_{ij}(\pi) \,| \neq 0$ and therefore $|\, C \,| \neq 0$, so that C is non-singular. Hence, *the two matrices* $[g_c(\theta^2), g_{c+1}(\theta^2), \cdots, g_d(\theta^2)]$ *and* $[h_c(\theta^2), h_{c+1}(\theta^2), \cdots, h_d(\theta^2)]$ *with elements in* K_2 *are equivalent under a non-singular conjunctive transformation with elements in* K_1.

Conversely, if in (52) θ is replaced by π, we have

$$\tilde{C}(\pi)[g_c(\pi^2), \cdots, g_d(\pi^2)] C(\pi) = [h_c(\pi^2), \cdots, h_d(\pi^2)]$$

and, if W_c is the direct product of $C(\pi)$ and the unit matrix E_c,

$$W_c[g_c(\pi^2)E_c, \cdots, g_d(\pi^2)E_d]\, W_c = [h_c(\pi^2)E_c, \cdots, h_d(\pi^2)E_d].$$

Multiplying this last equation by π^a and using (49) we get,

$$\tilde{W}_c[\gamma_c E_c, \cdots, \gamma_d E_d]\, W_c = [\sigma_c E_c, \cdots, \sigma_d E_d]. \tag{53}$$

Equation (43) implies a set of equations (53), one for each distinct value of the exponents k_i of $p(\lambda^2)$. If $W = [W_1, W_2, \cdots, W_f]$ and $W_1, W_2, \cdots, W_f$ are the matrices W_c of (52) corresponding to the distinct equations of the set (53),

$$\tilde{W}GW = H \text{ or } Q\tilde{W}GW = QH. \text{ Since } Q\tilde{W} = W'Q,$$

it follows that $W'QGW = QH$ and, since each matrix W_c is non-singular, that W is non-singular. Hence in case b we may reduce R and RM by transformations permutable with M to the forms,

$$(54) \qquad [q\tau\gamma_1 X_1, \cdots, q\tau\gamma_r X_r] \quad \text{and} \quad [q\tau\gamma_1 X_1 N_1, \cdots, q\tau\gamma_r X_r N_r]$$

respectively where $\gamma_i = g_i(\pi^2)$, if k_i is odd and $\gamma_i = \pi g_i(\pi^2)$, if k_i is even. The matrices (54) are not uniquely determined by the matrices p, q and the exponents k_i of $p(\lambda^2)$. We may express the results as follows: *If* $[p(\lambda^2)]^k$ *occurs exactly* a *times among the elementary factors of the pencil* $A - \lambda B$, *corresponding to* $[p(\lambda^2]^k$ *in the canonical form there is a block*

$$(55) \qquad [V\gamma_1 N - \lambda V\gamma_1, \cdots, V\gamma_a N - \lambda V\gamma_a]$$

where N *is defined by* (26) *and* V *by* (28). *With this block, and so with* $[p(\lambda^2)]^k$, *is associated a diagonal matrix of order* a *with elements in the field* $K(\theta^2)$, *where* θ^2 *is a zero of* $p(x) = 0$. *This associated matrix is not uniquely determined but is determined apart from a non-singular conjunctive transformation in the field* $K(\theta)$.

If K is the field of all real numbers the only irreducible even polynomials are of the type $p(\lambda^2) = \lambda^2 + b^2$. Hence $K(\theta^2) = K$ and $K(\theta)$ is the field of all complex numbers. Since any real quadratic form is equivalent in the real field to a sum of a certain number of positive and negative squares, the matrix associated with an elementary factor $(\lambda^2 + b^2)^k$ may be reduced to the simple form $[\rho_1, \rho_2, \cdots, \rho_a]$ where $\rho_i = +1 \;\; i \leqq d$, $\rho_i = -1 \;\; i > d$, and d is uniquely determined. In fact d is the index of the quadratic form.

Case c. The reduction in this case is similar in many respects to that of the previous cases. Equations (34) and (35) are true, where f_{ijs} is now an ordinary number and no longer a matrix. We assume that

$$k_1 = k_2 = \cdots = k_c > k_{c+j}.$$

If k_1 is even V_1 is skew symmetric and, by a proof exactly similar to that in case a or b, we can reduce the matrix (G_{ij}) $(i, j = 1, 2, \cdots, c)$, to a diagonal matrix $[g_1 E_1, g_2 E_2, \cdots, g_c E_c]$ where, as in case b, the diagonal matrix $[g_1, g_2, \cdots, g_c]$ is only determined to within a non-singular congruent transformation with elements in K. Therefore corresponding to an elementary factor λ^{2k} there is in the canonical form of $A - \lambda B$ a block

$$gX_{2k}U_{2k} - \lambda g X_{2k},$$

where $g \neq 0$ and K_{2k} is defined by (27) with $\epsilon = 1$, while U_{2k} is the auxiliary unit matrix of order $2k$. On rearranging the rows and columns of X_{2k} and

U_{2k} in the order $1, 3, \cdots, 2k-1, 2, 4, \cdots, 2k$ we find that X_{2k} and U_{2k} are equivalent respectively to

$$\begin{pmatrix} 0 & -q \\ q & 0 \end{pmatrix} = q\tau \quad \text{and} \quad \pi = \begin{pmatrix} 0 & e \\ p & 0 \end{pmatrix}$$

where e is the unit matrix of order k and p is the auxiliary unit matrix of order k while q, which satisfies (21), is the matrix T of (19) when $\epsilon = 1$. Hence *an elementary factor* λ^{2k} *may be considered to be of type b where* $p(\lambda) = \lambda^{2k}$ *and* $p = U$.

If k_1 is odd, V_1 is symmetric and $f_{110} = 0$, so that G_{11} is singular. As in previous cases we may suppose that $f_{210} \neq 0$ and since $f_{120} = -f_{210}$, it is easily shown that $\begin{vmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{vmatrix} \neq 0$. By repeated applications of Lemma 1 it therefore follows that c must be even and that G may be reduced to the [8] diagonal block form

$$[H_1, H_2, \cdots, H_{c/2}],$$

where H_j is a square matrix of $2k_1$ rows of the same type as $H_1 = \begin{pmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{pmatrix}$. It is not possible to reduce the matrix H_1, for example, to diagonal form. Accordingly we proceed as follows and consider the pencil

$$(56) \qquad \begin{pmatrix} V_1 & 0 \\ 0 & V_1 \end{pmatrix} H_1 \left\{ \begin{pmatrix} U_1 & 0 \\ 0 & U_1 \end{pmatrix} - \lambda \begin{pmatrix} E_1 & 0 \\ 0 & E_1 \end{pmatrix} \right\} .$$

The elementary factors of this pencil are $\lambda^{k_1}, \lambda^{k_1}$. But with the notation of (15) the elementary factors of $\phi U_1 - \lambda\epsilon E_1$ are also $\lambda^{k_1}, \lambda^{k_1}$, if $e = 1$. Hence the pencil (56) is equivalent to

$$VG(N - \lambda E),$$

where $N = \phi U_1$ and V is defined by (24) with $e = 1$ and $q = 1$. The matrix G is permutable with N and, if G is considered as a matrix of two rowed matrices, every element to the left of the leading diagonal is zero so that,

$$G = G_0 + G_1 + G_2 + \cdots + G_{k-1},$$

where G_i is the matrix formed by the elements of G in the i-th diagonal to the right of the leading one. Moreover, since G is non-singular G_0 is non-singular. Further $G_iN = NG_i$ and $VG_i = -G'_iV' = G'_iV$. If $G_1 = G_2 = G_{f-1} = 0$ and $G_f \neq 0$, the matrix $W = E - 1/2G_0^{-1}G_f$ is permutable with N and satisfies the equation

$$W'VGW = V(G_0 + H_{f+1} + \cdots H_{k-1})$$

[8] This is a well known result. See Turnbull and Aitken, *Canonical Matrices*, p. 137.

where H_j is of the same type as G_j. Hence we may assume that $G_1 = G_2 = G_{k-1} = 0$. Since G_0 is commutative with N and VG_0 is skew symmetric,

$$G_0 = g \epsilon E_1,$$

where g is an element of K. A reduction similar to that in case *b* shows that g may be taken as $+1$.

Hence, *if* λ^k *occurs among the elementary factors of* $A - \lambda B$ *and* k *is odd it must occur an even number,* $2a$, *of times. In the canonical form of* $A - \lambda B$ *occur* a *blocks of the nature*

$$\tau T(\phi U - \lambda \epsilon E).$$

It should be noted that the two elementary factors λ^k, λ^k are accordingly of type (a) where $p(\lambda) = \lambda$ and π is the zero matrix. We have accordingly proved the theorem

THEOREM I. *A canonical form for the pencil* $A - \lambda B$, *where* A *is symmetric and* B *is skew symmetric, under non-singular congruent transformation in* K, *is a diagonal block matrix, whose component blocks are given by equation* (42) *or equation* (55).

COROLLARY. *Necessary and sufficient conditions that two such pencils* $A - \lambda B$ *and* $C - \lambda D$ *be equivalent in* K *are that,*

(a) *the elementary factors of* $A - \lambda B$ *be the same as those of* $C - \lambda D$.

(b) *the matrix associated with each elementary factor of the type* $[p(\lambda^2)]^k$ *in a normal form of* $A - \lambda B$ *be equivalent under a conjunctive transformation to the corresponding matrix in a normal form of* $C - \lambda D$.

4. *Reduction of B.* Since B is non-singular and skew symmetric there exists a non-singular matrix P with elements in K such that

$$P'BP = \begin{pmatrix} 0 & -E \\ E & 0 \end{pmatrix}, \tag{57}$$

where E is the unit matrix of order one half the order of B. We now proceed to find a canonical form for the pencil $A - \lambda B$, in which B is equivalent to the simple matrix on the right of (57). We start with the canonical form deduced in the previous sections and have in all to consider three cases.

Case a. Corresponding to the elementary factors $[p(\lambda)]^k$, $[p(-\lambda)]^k$ in the canonical form is the block $q\tau TN - \lambda q\tau T$ (equation (42)), where

$$\tau T = \begin{bmatrix} 0 & 0 & 0 & 0 & \cdot & 0 & -e \\ 0 & 0 & 0 & 0 & \cdot & e & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & -e & \cdot & 0 & 0 \\ 0 & 0 & e & 0 & \cdot & 0 & 0 \\ 0 & -e & 0 & 0 & \cdot & 0 & 0 \\ e & 0 & 0 & 0 & \cdot & 0 & 0 \end{bmatrix}.$$

By rearranging the rows and columns of τT in the order $1, 3, 5, \cdots, 2k-1, 2, 4, \cdots, 2k$, we see that $\tau T \approx \begin{pmatrix} 0 & -T_e \\ T_e & 0 \end{pmatrix}$ [9] where T_e is the matrix

$$\begin{bmatrix} 0 & 0 & \cdot & e \\ \cdot & \cdot & \cdot & \cdot \\ 0 & e & \cdot & 0 \\ e & 0 & \cdot & 0 \end{bmatrix}.$$

The same transformation applied to N shows that

$$N \approx \begin{pmatrix} L & 0 \\ 0 & -L \end{pmatrix},$$

where

$$L = \begin{bmatrix} p & e & \cdot & 0 & 0 \\ 0 & p & \cdot & 0 & 0 \\ \cdot & \cdot & \cdot & p & e \\ 0 & 0 & \cdot & 0 & p \end{bmatrix}. \tag{58}$$

If $W = \begin{pmatrix} q^{-1}T_e & 0 \\ 0 & E \end{pmatrix}$,

$$W'q\begin{pmatrix} 0 & -T_e \\ T_e & 0 \end{pmatrix} W = \begin{pmatrix} q^{-1}T_e & 0 \\ 0 & E \end{pmatrix}\begin{pmatrix} 0 & -qT_e \\ qT_e & 0 \end{pmatrix}\begin{pmatrix} q^{-1}T_e & 0 \\ 0 & E \end{pmatrix} = \begin{pmatrix} 0 & -E \\ E & 0 \end{pmatrix}.$$

Further

$$W'q\begin{pmatrix} 0 & -T_e \\ T_e & 0 \end{pmatrix}\begin{pmatrix} L & 0 \\ 0 & -L \end{pmatrix} W = \begin{pmatrix} 0 & L \\ qT_eLq^{-1}T_e & 0 \end{pmatrix} = \begin{pmatrix} 0 & L \\ L' & 0 \end{pmatrix},$$

since $qT_eL = L'qT_e$. Hence

$$q\tau TN - \lambda q\tau T \approx \begin{pmatrix} 0 & L \\ L' & 0 \end{pmatrix} - \lambda\begin{pmatrix} 0 & -E \\ E & 0 \end{pmatrix}.$$

It accordingly follows that, if $R_aM_a - \lambda R_a$ is the part of the canonical form of $A - \lambda B$ depending on elementary factors of type a, including those of type λ^k where k is odd,

$$R_aM_a - \lambda R_a \approx \begin{pmatrix} 0 & F \\ F' & 0 \end{pmatrix} - \lambda\begin{pmatrix} 0 & -E \\ E & 0 \end{pmatrix}. \tag{59}$$

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[9] We use $\approx$ to denote "is equivalent to."

where

$$F = [L_1, L_2, \cdots, L_w] \tag{60}$$

is a diagonal block matrix, which is the direct sum of all matrices L defined by (58), one for each pair of elementary factors $[p(\lambda)]^k$, $[p(-\lambda)]^k$ of type a.

case b. Corresponding to the elementary factor $[p(\lambda^2)]^k$ in the canonical form is the block $q\tau\gamma XN - \lambda q\tau\gamma X$ (equation 55). It is necessary to consider the cases in which k is even and in which k is odd separately. If $k = 2f$ is even, the matrix $V = q\tau X$ is skew symmetric, so that $\gamma = g(\pi^2)\pi$ is an odd polynomial in π. Hence

$$q\tau\gamma = -\gamma' q\tau. \tag{61}$$

By rearranging the rows and columns in the order $1, 3, \cdots, 2f-1, 2, 4, \cdots, 2f$ we find that

$$X \approx \begin{pmatrix} 0 & -T \\ T & 0 \end{pmatrix} \quad \text{and} \quad N \approx \begin{pmatrix} \pi E & E \\ U & \pi E \end{pmatrix}$$

where T is defined by (19) and E and U by (17).

If $\psi = q\tau\gamma$,

$$\psi' = \gamma' q' \tau' = -\gamma' q\tau = q\tau\gamma \text{ by (61)},$$

so that ψ is symmetric. Accordingly, if $W = \begin{pmatrix} E & 0 \\ 0 & \psi^{-1}T \end{pmatrix}$,

$$W' q\tau\gamma XW \approx \begin{pmatrix} E & 0 \\ 0 & T\psi^{-1} \end{pmatrix} \begin{pmatrix} 0 & -\psi T \\ \psi T & 0 \end{pmatrix} \begin{pmatrix} E & 0 \\ 0 & \psi^{-1}T \end{pmatrix} = \begin{pmatrix} 0 & -E \\ E & 0 \end{pmatrix}.$$

Similarly,

$$W' q\tau\gamma XNW \approx \begin{pmatrix} E & 0 \\ 0 & \psi^{-1}T \end{pmatrix} \begin{pmatrix} 0 & -\psi T \\ \psi T & 0 \end{pmatrix} \begin{pmatrix} \pi E & E \\ U & \pi E \end{pmatrix} \begin{pmatrix} E & 0 \\ 0 & \psi^{-1}T \end{pmatrix},$$
$$\approx \begin{pmatrix} -\psi TU & -\psi T\pi\psi^{-1}T \\ \pi E & \psi^{-1}T \end{pmatrix}.$$

Hence

$$q\tau\gamma XN \approx Z = \begin{pmatrix} -\psi TU & \pi' E \\ \pi E & \psi^{-1}T \end{pmatrix}. \tag{62}$$

For example if $k = 6$,

$$Z = \begin{bmatrix} 0 & 0 & 0 & \pi' & 0 & 0 \\ 0 & 0 & -\psi & 0 & \pi' & 0 \\ 0 & -\psi & 0 & 0 & 0 & \pi' \\ \pi & 0 & 0 & 0 & 0 & \psi^{-1} \\ 0 & \pi & 0 & 0 & \psi^{-1} & 0 \\ 0 & 0 & \pi & \psi^{-1} & 0 & 0 \end{bmatrix},$$

where $\pi = \begin{pmatrix} 0 & e \\ p & 0 \end{pmatrix}$ and $\psi = \begin{pmatrix} -qpg & 0 \\ 0 & qg \end{pmatrix}$ and $g = g(p)$ is a polynomial in p.

We may write the matrix Z of (62) in the form $Z = \begin{pmatrix} Z_{11} & Z_{12} \\ Z'_{12} & Z_{22} \end{pmatrix}$, where $Z_{11} = -\psi TU$ etc. Then if $R_{b_1}M_{b_1} - \lambda R_{b_1}$ is that part of the canonical form of $A - \lambda B$ depending on elementary factors of type b, i. e. on $[p(\lambda^2)]^k$, where k is even,

$$(63) \qquad R_{b_1}M_{b_1} - \lambda R_{b_1} \approx \begin{pmatrix} Y_{11} & Y_{12} \\ Y'_{12} & Y_{22} \end{pmatrix} - \lambda \begin{pmatrix} 0 & -E \\ E & 0 \end{pmatrix},$$

where

$$(64) \qquad Y_{ij} = [Z_{ij,1}, Z_{i,j,2}, \cdots, Z_{ij,w}] \qquad (i, j = 1, 2),$$

is a diagonal block matrix, which is the direct sum of all matrices $Z_{ij,r}$ one for each elementary factor $[p(\lambda^2)]^k$, k even.

If however k is odd, the matrix V in (55) is symmetric so that $\gamma = g(\pi^2)$ is an even polynomial in π. In fact $q\tau\gamma = \begin{pmatrix} 0 & -q \\ q & 0 \end{pmatrix}\begin{pmatrix} g & 0 \\ 0 & g \end{pmatrix}$, where $g = g(p)$ is a polynomial in p and

$$q\tau\gamma X = \begin{pmatrix} 0 & 0 & \cdot & 0 & 0 & 0 & qg \\ 0 & 0 & \cdot & 0 & 0 & -qg & 0 \\ 0 & 0 & \cdot & 0 & -qg & 0 & 0 \\ 0 & 0 & \cdot & qg & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & qg & \cdot & 0 & 0 & 0 & 0 \\ -qg & 0 & \cdot & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Rearranging the rows and columns of this matrix in the order 1, 3, 5, etc., we find

$$q\tau\gamma X \approx \begin{pmatrix} 0 & -qgX_e \\ qgX_e & 0 \end{pmatrix},$$

where X_e is symmetric, and is defined by (27) with ϵ replaced by e. The same transformation shows that $N \approx \begin{pmatrix} U & E \\ pE & U \end{pmatrix}$. If $W = \begin{pmatrix} (qg)^{-1}X_e & 0 \\ 0 & E \end{pmatrix}$,

$$W'q\tau\gamma XW \approx \begin{pmatrix} X_e(qg)^{-1} & 0 \\ 0 & E \end{pmatrix}\begin{pmatrix} 0 & -qgX_e \\ qgX_e & 0 \end{pmatrix}\begin{pmatrix} X_e(qg)^{-1} & 0 \\ 0 & E \end{pmatrix} = \begin{pmatrix} 0 & -E \\ E & 0 \end{pmatrix},$$

while

$$(65) \qquad W'q\tau\gamma XNW \approx \begin{pmatrix} -p(qg)^{-1}X_e & -U \\ -U' & qgX_e \end{pmatrix} = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix} = C.$$

For example, if $k = 3$,

$$C = \begin{pmatrix} 0 & 0 & p(qg)^{-1} & 0 & -e & 0 \\ 0 & -p(qg)^{-1} & 0 & 0 & 0 & -e \\ p(qg)^{-1} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -qg \\ -e & 0 & 0 & 0 & qg & 0 \\ 0 & -e & 0 & -qg & 0 & 0 \end{pmatrix}$$

Accordingly, if $R_{b_2}M_{b_2} - \lambda R_{b_2}$ is that part of the canonical form of $A - \lambda B$ depending on elementary factors of type b, i. e. on $[p(\lambda^2)]^k$, where k is odd,

$$(66) \qquad R_{b_2}M_{b_2} - \lambda R_{b_2} \approx \begin{pmatrix} D_{11} & D_{12} \\ D_{12} & D_{22} \end{pmatrix} - \lambda \begin{pmatrix} 0 & -E \\ E & 0 \end{pmatrix},$$

where

$$(67) \qquad D_{ij} = [C_{ij,1}, C_{ij,2}, \cdots, C_{ij,w}] \qquad (i, j = 1, 2),$$

is a diagonal block matrix which is the direct sum of all matrices C_{ij} one for each elementary factor $[p(\lambda^2)]^k$, k odd (including the case $\lambda^{2k} = p(\lambda^2)$). It is an immediate consequence of equations (59), (63) and (64) that

THEOREM 2. *The pencil $A - \lambda B$ is equivalent in K to the pencil*

$$A_1 - \lambda \begin{pmatrix} 0 & -E \\ E & 0 \end{pmatrix}$$

where

$$A_1 = \begin{pmatrix} 0 & 0 & 0 & F & 0 & 0 \\ 0 & Y_{11} & 0 & 0 & Y_{12} & 0 \\ 0 & 0 & D_{11} & 0 & 0 & D_{12} \\ F' & 0 & 0 & 0 & 0 & 0 \\ 0 & Y'_{12} & 0 & 0 & Y_{22} & 0 \\ 0 & 0 & D'_{12} & 0 & 0 & D_{22} \end{pmatrix},$$

and F, Y_{ij}, D_{ij} are defined by (60), (64) and (67) respectively.

COROLLARY. *The symmetric matrix A is equivalent to the matrix A_1 under a non-singular congruent transformation in K, which leaves the skew symmetric matrix $\begin{pmatrix} 0 & -E \\ E & 0 \end{pmatrix}$ invariant.*

This corollary follows immediately by substituting the matrix $\begin{pmatrix} 0 & -E \\ E & 0 \end{pmatrix}$ for B in the pencil $A - \lambda B$.

5. *K the field of all real numbers.* In the canonical forms of the previous sections occur the matrices p and q, where p is any matrix with elements in K

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whose characteristic equation is $p(\lambda) = 0$, and q is a non-singular symmetric matrix satisfying the equation $qp = p'q$. If p is chosen as the companion matrix of $p(\lambda) = 0$, a comparatively simple matrix q can be determined.[10] If however K is the field of all real numbers, there are only three possible types for the irreducible equation $p(\lambda)$, and the corresponding values of p and q are even more simple. These are

$$(1)\ p(\lambda) = \lambda - a;\ (2)\ p(\lambda) = \lambda^2 - 2a\lambda + a^2 + b^2;\ (3)\ p(\lambda^2) = \lambda^2 + a^2.$$

In case (1) $p = a,\ q = 1$;

In case (2) $p = \begin{pmatrix} a & b \\ -b & a \end{pmatrix},\ q = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$;

In case (3) $p = -a^2,\ q = 1,\ \pi = \begin{pmatrix} 0 & 1 \\ -a^2 & 0 \end{pmatrix}$.

Moreover each matrix g occurring in Y_{ij} or D_{ij} (equations (64) and (67)) now has the value ± 1.

The matrix $p = -a^2$, in case (3), is obtained by particularizing the general formula but for some purposes it is preferable to take $\pi = \begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix}$ instead of $\begin{pmatrix} 0 & 1 \\ -a^2 & 0 \end{pmatrix}$. If this is done, it is easily seen, that the matrix Z in (62) is unaltered, except that $\psi = \begin{pmatrix} ga & 0 \\ 0 & ga \end{pmatrix}$ where g is a real number. Since, in ψ, g may be replaced by any real number with the same sign, we may take $\psi = \begin{pmatrix} \rho & 0 \\ 0 & \rho \end{pmatrix}$ where $\rho = \pm 1$. Similarly the matrix C in (65) is only altered to the extent that

$$C_{11} = \rho a X_2; \qquad C_{22} = \rho a X_2.$$

In conclusion we exhibit the possible canonical forms, to one of which a real symmetric matrix A of order 4 can be reduced by a real non-singular congruent transformation, which leaves invariant the skew symmetric matrix $\begin{pmatrix} 0 & -E \\ E & 0 \end{pmatrix}$. If A is non-singular, the possible elementary divisors of $A - \lambda \begin{pmatrix} 0 & -E \\ E & 0 \end{pmatrix}$ are

(α) $(\lambda \pm a), (\lambda \pm b)$; (β) $(\lambda \pm a \pm ib)$; (γ) $(\lambda \pm ia), (\lambda \pm ib)$;
(δ) $(\lambda \pm a), (\lambda \pm ib)$; (ϵ) $(\lambda \pm a)^2$; (ζ) $(\lambda \pm ia)^2$.

The corresponding canonical forms for A are

[10] J. Williamson, *loc. cit.*, p. 490.

$$(\alpha)\quad \begin{bmatrix} 0 & 0 & a & 0 \\ 0 & 0 & 0 & b \\ a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \end{bmatrix}; \qquad (\beta)\quad \begin{bmatrix} 0 & 0 & a & -b \\ 0 & 0 & b & a \\ a & b & 0 & 0 \\ -b & a & 0 & 0 \end{bmatrix};$$

$$(\gamma)\quad \begin{bmatrix} \rho a & 0 & 0 & 0 \\ 0 & \sigma b & 0 & 0 \\ 0 & 0 & \rho a & 0 \\ 0 & 0 & 0 & \sigma b \end{bmatrix} \rho, \sigma = \pm 1; \quad (\delta)\quad \begin{bmatrix} 0 & 0 & a & 0 \\ 0 & \rho b & 0 & 0 \\ a & 0 & 0 & 0 \\ 0 & 0 & 0 & \rho b \end{bmatrix}, \ \rho = \pm 1;$$

$$(\epsilon)\quad \begin{bmatrix} 0 & 0 & a & 1 \\ 0 & 0 & 0 & a \\ a & 0 & 0 & 0 \\ 1 & a & 0 & 0 \end{bmatrix}; \qquad (\zeta)\quad \begin{bmatrix} 0 & 0 & 0 & a \\ 0 & 0 & -a & 0 \\ 0 & -a & \rho & 0 \\ a & 0 & 0 & \rho \end{bmatrix}, \ \rho = \pm 1.$$

The matrices in cases (α), (β), (ϵ) depend solely on the elementary divisors of the pencil; (γ) yields 4 or 3 non-equivalent matrices according as a is not or is the same as b while (δ) and (ζ) both yield two non-equivalent matrices.

If A is singular and the pencil has the pair of elementary divisors λ, λ the canonical form is obtained from (α) or (δ) by putting $a = 0$ and from (α) by putting $a = b = 0$ if the pencil contains the 4 elementary divisors $\lambda, \lambda, \lambda, \lambda$. If λ^2 occurs among the elementary divisors the canonical form is obtained from that corresponding to $(\lambda \pm ia)$ by replacing the first a by unity and the other by zero. If λ^4 is an elementary divisor the canonical form is

$$\begin{bmatrix} \rho & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\rho \\ 0 & 0 & -\rho & 0 \end{bmatrix}, \ \rho = \pm 1.$$

Thus we have determined a complete list of the possible canonical forms for the case $n = 4$.

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