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Controllability of a Class of Multi-input Bilinear Systems ⋆

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Abstract

This paper mainly studies the controllability of a class of multi-input time-invariant discrete-time bilinear systems. The necessary and sufficient conditions for controllability are derived when dimension of the bilinear system is two. At the same time, our results extend the existing controllability counterexample. Some sufficient conditions for uncontrollability are given for this class of bilinear systems with higher dimension.

Keywords: Bilinear System; Controllability; Commutative Matrix

1 Introduction and Statement of the Problem

Theory of controllability derives from Kalman's important work [10]. Roughly speaking, controllability means that it is possible to steer dynamical system from an arbitrary initial state to an arbitrary final state using the set of admissible controls. At present this theory of deterministic linear control systems is completely mature [3]. A legitimate conjecture may be that the controllability to nonlinear control systems can be more meaningful. Controllability research on general nonlinear systems is very difficult [9]. There has a kind of important nonlinear systems–bilinear systems.

Many systems have natural models which are bilinear systems, for example, microscopic processes and quantum ensembles [12], neural network [14] and population growth [15]. Such systems have been successfully used to model a variety of physical phenomena for which linear model representation has proved inadequate [13]. In addition, bilinear systems can well approximate a lot of nonlinear systems [16]. Furthermore, it is in general more accurate to use a bilinear model to represent the dynamics of a nonlinear system than to use a linear model. It is necessary to clarify fundamental properties of bilinear systems, say, controllability. Many literature focused on the study of controllability of bilinear systems, such as [4, 5, 7, 8, 11, 17, 18]. We often need computer simulation of a discrete-time approximation to a continuous bilinear system, and sometimes use

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Euler's method. A natural question is whether the discretization changes controllability of the original system. Elliott [2] gives a counterexample which is a class of single-input two-dimensional continuous bilinear systems:

$$\dot{x}(t) = u(t)Bx(t), \quad x(0) = \xi \tag{1}$$

which is uncontrollable. But the corresponding discrete system of (1) is controllable by the Euler discretization.

It reminds us that the study to continuous bilinear system by numerical methods must be more careful. In addition, it inspires us to find the essence of this category of bilinear systems resembling the controllability counterexample, and to consider whether multi-input bilinear systems have the same phenomena as (1). Note that in the existing literature [4, 5, 7, 17, 18], the controllability of homogeneous discrete time bilinear systems with single input was discussed. Now we consider controllability of the following system

$$\dot{x}(t) = \sum_{i=1}^{n} u_i(t)B_i x(t) \tag{2}$$

and corresponding discrete system

$$\begin{aligned} x(k+1) &= \left(I + \sum_{i=1}^{n} v_i(k)B_i\right) x(k) \\ v_i(k) &= \tau u_i(k), \quad k = 0, 1, \cdots \end{aligned} \tag{3}$$

by the Euler discretization, where $x(t) \in \mathbb{R}^K$ is the state variable, $u_i(t) \in \mathbb{R}$, and $B_i \in \mathbb{R}^{K\times K}$, $i = 1, 2, \cdots, n$.

The goal of this paper is to get controllability relations between (2) and (3), which can not be obtained by the existing results [7, 8], and others. The rest of this paper is organized as follows. In section 2, we introduce concepts of controllability, and give several useful lemmas used for the proof of the latter theorems. In section 3, we will propose the necessary and sufficient conditions of controllability of (2) and (3) for $n = 2$ and $K = 2$. In section 4, we will give some sufficient conditions of uncontrollability for (2) and (3) when $n \geq 2$ and $K \geq 2$.

2 Basic cConcepts and Fundamental Lemmas

Definition 2.1 *The bilinear systems (2) is said to be controllable if for any pair of vectors* $\xi, \eta \in \mathbb{R}^K_*(= \mathbb{R}^K \setminus \{0\})$, *there exists a real number* $T \geq 0$ *and a control* $u(\cdot) = (u_1(\cdot), \cdots, u_n(\cdot)) \in \mathbb{R}^n$ *such that the trajectory of the systems staring at* $x(0) = \xi$ *and finishing at* $x(T) = \eta$.

Definition 2.2 *The bilinear systems (3) is said to be controllable if for any initial state* $\xi \in \mathbb{R}^K_*$ *and any terminal state* $\eta \in \mathbb{R}^K_*$, *there exists a positive integer* L *and a finite control sequence* $v(i) = (v_1(i), \cdots, v_n(i))$ $(i = 0, 1, \cdots, L)$ *such that* ξ *can be transferred to* η.

Lemma 2.1 *([2]) The system* $x(k+1) = f(x(k), u(k))$ *is controllable on a connected submanifold* $S \subset \mathbb{R}^K$ *if and only if there exist controls such that for every initial state* ξ, *a neighborhood* $N(\xi) \subset S$ *of* ξ *is attainable.*

Lemma 2.2 *Given a matrix $P \in \mathbb{R}^{K \times K}$ is invertible, the following system defined by a differential equation*

$$\dot{x}(t) = \sum_{i=1}^{n} u_i(t) P^{-1} B_i P x(t) \tag{4}$$

is controllable if and only if the system (2) is controllable.

Proof Suppose the system (2) is controllable. For any $\xi, \eta \in \mathbb{R}_*^K$, we have $P\xi, P\eta \in \mathbb{R}_*^K$. There exists $T \geq 0$ and $u_1(\cdot), \cdots, u_n(\cdot) \in \mathbb{R}$ such that the trajectory $x(t)$ of (2) satisfies $x(0) = P\xi$, and $x(T) = P\eta$. Let $y(t) = P^{-1}x(t)$. Then $\dot{y}(t) = \sum_{i=1}^{n} u_i(t) P^{-1} B_i P y(t)$, and $y(0) = \xi$ and $y(T) = \eta$. It is shown that the systems (4) is controllable. Let $P = I$ is the identity matrix. It is clear that (2) is controllable if (4) is controllable.

Lemma 2.3 *If a matrix $P \in \mathbb{R}^{K \times K}$ is invertible, the following system defined by a difference equation*

$$x(k+1) = \left(I + \sum_{i=1}^{n} v_i(k) P^{-1} B_i P \right) x(k) \tag{5}$$

and the system (3) are both either controllable or uncontrollable.

Proof The process of its proof is very similar to the Lemma 2.2, and we do not repeat it.

3 Controllability of Bi-input Systems

In this section, we will discuss controllability of a class of second order bi-input systems, namely systems (2) and (3) with $n = 2$ and $K = 2$. Especially, the counterexample in [2] is a special case of the results.

Theorem 3.1 *Let $n = 2$ and $K = 2$. The system (2) becomes*

$$\dot{x}(t) = u_1(t) B_1 x(t) + u_2(t) B_2 x(t) \tag{6}$$

If matrix multiplication for matrices B_1 and B_2 is commutative, i.e. $B_1 B_2 = B_2 B_1$, and both B_1 and B_2 only have real eigenvalues, then the system (6) is always uncontrollable.

Proof For commutative B_1 and B_2, the trajectory $x(t)$ can be abtained from (6) that $x(t) = \exp(B_1 w_1(t)) \exp(B_2 w_2(t)) x(0)$ where $w_i(t) = \int_0^t u_i(s) ds$, $i = 1, 2$. Since both B_1 and B_2 only have real eigenvalues, and $B_1 B_2 = B_2 B_1$, there exists a nonsingular matrix P such that $P^{-1} B_1 P$ and $P^{-1} B_1 P$ are upper triangular matrices, i.e. $P^{-1} B_1 P = \begin{bmatrix} \lambda_1 & * \\ 0 & \lambda_2 \end{bmatrix}$, $P^{-1} B_2 P = \begin{bmatrix} \mu_1 & * \\ 0 & \mu_2 \end{bmatrix}$.
According to the Lemma 2.2, it is enough to study the controllability of the system

$$\dot{x}(t) = u_1(t) P^{-1} B_1 P x(t) + u_2(t) P^{-1} B_2 P x(t). \tag{7}$$

By simple calculation, we have

$$x(t)=\begin{bmatrix} e^{\lambda_1 w_1(t)} & * \\ 0 & e^{\lambda_2 w_1(t)} \end{bmatrix}\cdot\begin{bmatrix} e^{\mu_1 w_2(t)} & * \\ 0 & e^{\mu_2 w_2(t)} \end{bmatrix}\cdot x(0),\ w_i(t)=\int_0^t u_i(s)ds,\ i=1,2.$$

This demonstrates that (7) is not controllable and completes the proof.

Theorem 3.2 *Let $n=2$, $K=2$, and $B_1B_2=B_2B_1$. Suppose B_1 has a non-real pair of complex conjugate eigenvalues, which implies that there exists a nonsingular real matrix P such that*

$$P^{-1}B_1P=J_1=\begin{bmatrix} a & b \\ -b & a \end{bmatrix}\ (a,b\in\mathbb{R},\ b\neq 0), P^{-1}B_1P=J_2=\begin{bmatrix} c & d \\ -d & c \end{bmatrix}\ (c,d\in\mathbb{R}),$$

owing to B_1 and B_2 being commutative. Then the system (6) is controllable if and only if

$$ad-bc\neq 0 \tag{8}$$

Proof According to the Lemma 2.2, we only need to consider the controllability of the system

$$\dot{x}(t)=u_1(t)J_1x(t)+u_2(t)J_2x(t) \tag{9}$$

where its trajectory is

$$x(t)=\exp(aw_1(t)+cw_2(t))\begin{bmatrix} \cos(bw_1(t)+dw_2(t)) & \sin(bw_1(t)+dw_2(t)) \\ -\sin(bw_1(t)+dw_2(t)) & \cos(bw_1(t)+dw_2(t)) \end{bmatrix}x(0)$$

where $w_i(t)=\int_0^t u_i(s)ds$, $i=1,2$. It is not difficult to know that (9) is controllable if and only if for any $(\eta_1,\eta_2)\in\mathbb{R}\times[0,2\pi]$, there exist $w_1(t)$ and $w_2(t)$ such that $aw_1(t)+cw_2(t)=\eta_1$ and $bw_1(t)+dw_2(t)=\eta_2$, the equivalent condition of which is that (8) holds. It shows that (6) is controllable if and only if (8) holds.

Corollary 3.1 *Let $K=2$. The system*

$$\dot{x}(t)=u(t)Bx(t)$$

is always uncontrollable.

Proof The conclusion is directly derived from Theorem 3.1 and Theorem 3.2 by setting that B_2 is the zero matrix.

Theorem 3.3 *Let $n=2$ and $K=2$. The system (3) becomes*

$$x(k+1)=(I+v_1(k)B_1+v_2(k)B_2)\,x(k) \tag{10}$$

If $B_1B_2=B_2B_1$, then the system (10) is uncontrollable if and only if one of the following conditions is satisfied:
(i) both B_1 and B_2 have a real pair of eigenvalues;
(ii) both B_1 and B_2 have a pair of pure imaginary eigenvalues;
(iii) one of B_1 and B_2 has a pair of pure imaginary eigenvalues, another is the zero matrix.

Proof The proof is processed in four cases.

Case 1: If all of the eigenvalues of B_1 and B_2 are real numbers, noting that $B_1B_2 = B_2B_1$, there exists a nonsingular real matrix P such that $P^{-1}B_1P$ and $P^{-1}B_1P$ are upper triangular matrices, i.e.

$$P^{-1}B_1P = \begin{bmatrix} \lambda_1 & * \\ 0 & \lambda_2 \end{bmatrix} \quad \text{and} \quad P^{-1}B_2P = \begin{bmatrix} \mu_1 & * \\ 0 & \mu_2 \end{bmatrix}$$

According to the Lemma 2.3, it has been known that the below system

$$x(k+1) = \big(I + v_1(k)P^{-1}B_1P + v_2(k)P^{-1}B_2P\big)\, x(k) \tag{11}$$

has the same controllability as the systems (10). It has been shown that (11) is uncontrollable by considering the initial state of the form $\xi = [\xi_1, 0]^T$. So (10) is not controllable.

Case 2: If B_1 or B_2 has a non-real pair of complex conjugate eigenvalues except pure imaginary eigenvalues, then (10) is controllable by setting $v_2(k) = 0$ or $v_1(k) = 0$ according to [18].

Case 3: If both B_1 and B_2 only have a pair of pure imaginary eigenvalues, then there exists a nonsingular real matrix P such that

$$P^{-1}B_1P = \begin{bmatrix} 0 & b \\ -b & 0 \end{bmatrix} (b \neq 0).$$

Let $J_1 = P^{-1}B_1P$ and $J_2 = P^{-1}B_2P$. Then J_1 and J_2 are commutative. By simple calculation, J_2 has the below form:

$$J_2 = \begin{bmatrix} 0 & c \\ -c & 0 \end{bmatrix} (c \neq 0)$$

From Lemma 2.3, it is equivalent to consider the controllability of the following system

$$x(k+1) = (I + v_1(k)J_1 + v_2(k)J_2)\, x(k) \tag{12}$$

Note that

$$I + v_1(k)J_1 + v_2(k)J_2 = \sqrt{(bv_1(k) + cv_2(k))^2 + 1} \begin{bmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{bmatrix},$$

where $\tan\varphi = bv_1(k) + cv_2(k)$. It implies that any state η with its Euclidean norm less than that of the initial ξ can not be reached from ξ. Therefore, (12) is uncontrollable, and then (10) is uncontrollable.

Case 4: If one of B_1 and B_2 (for example B_1) has a pair of pure imaginary eigenvalues, another has a real pair of eigenvalues, we do some similar discussions with the case 3. There exists a nonsingular real matrix P such that $P^{-1}B_1P = \begin{bmatrix} 0 & b \\ -b & 0 \end{bmatrix}$ and $P^{-1}B_2P = \begin{bmatrix} c & 0 \\ 0 & c \end{bmatrix}$. When $c = 0$, it is easy to validate the uncontrollability of (12). It also means that B_2 is the zero matrix. When $c \neq 0$, let $J_1 = P^{-1}B_1P$, $J_2 = P^{-1}B_2P$. It is obvious that the system

$$x(k+1) = (I + v_1(k)J_1 + v_2(k)J_2)\, x(k) \tag{13}$$

has the same controllability as (10). Let $\tilde{J}_1 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ and $\tilde{J}_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Then the system

$$x(k+1) = \left(I + v_1(k)\tilde{J}_1 + v_2(k)\tilde{J}_2\right) x(k) \tag{14}$$

has the same controllability as (13). Now we only consider whether (14) is controllable.

Firstly, we will testify that every initial $\xi \in \mathbb{R}^2_*$ can be transferred to itself by a finite control sequence. We claim that there exist real numbers α_1, α_2, β_1 and β_2 satisfying

$$(I + \alpha_1\tilde{J}_1 + \alpha_2\tilde{J}_2)(I + \beta_1\tilde{J}_1 + \beta_2\tilde{J}_2) = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}, \tag{15}$$

where $\theta = \frac{2m}{r}\pi \in [0, \pi)$, m, $r \in \mathbb{Z}^+$. Then by using $v(2i-2) = (\alpha_1, \alpha_2)$, $v(2i-1) = (\beta_1, \beta_2)$, $i = 1, 2, \cdots, r$, where $v(i) = (v_1(i), v_2(i))$ for $i = 0, 1, \cdots, 2r-1$, the initial ξ can be transferred to itself, i.e. $\left[\prod_{i=0}^{2r-1}\left(I + v_1(i)\tilde{J}_1 + v_2(i)\tilde{J}_2\right)\right]\xi = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}^r \xi = \xi$. In fact, it follows from $(I + \alpha_1\tilde{J}_1 + \alpha_2\tilde{J}_2)(I + \beta_1\tilde{J}_1 + \beta_2\tilde{J}_2) = \begin{bmatrix} 1+\alpha_2 & \alpha_1 \\ -\alpha_1 & 1+\alpha_2 \end{bmatrix} \begin{bmatrix} 1+\beta_2 & \beta_1 \\ -\beta_1 & 1+\beta_2 \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ that

$$(\alpha_1^2 + (1+\alpha_2)^2)(\beta_1^2 + (1+\beta_2)^2) = 1,\ \theta = \theta_1 + \theta_2 \ \text{ or } \ \theta = \theta_1 + \theta_2 - 2\pi$$

with θ_1, θ_2 satisfying $\tan\theta_1 = \frac{\beta_1}{1+\beta_2}$, $\tan\theta_2 = \frac{\alpha_1}{1+\alpha_2}$, where $\theta_i \in [0, 2\pi) \setminus \{0, \frac{\pi}{2}, \frac{3\pi}{2}\}, i = 1, 2$. Here, we suppose $\alpha_1, \beta_1 \neq 0$ and $1+\alpha_2, 1+\beta_2 \neq 0$. Let

$$\beta_1^2 + (1+\beta_2)^2 = \gamma, \quad \alpha_1^2 + (1+\alpha_2)^2 = \frac{1}{\gamma}, \tag{16}$$

where $\gamma \in (0, 1)$. We can choose

$$\beta_1 = \sqrt{\frac{1}{2}\gamma} > 0,\ \alpha_1 = \sqrt{\gamma} > 0. \tag{17}$$

By substituting (17) into (16), we choose

$$\beta_2 = -1 - \sqrt{\frac{1}{2}\gamma}\,, \quad \alpha_2 = -1 - \sqrt{\frac{1}{\gamma} - \gamma}. \tag{18}$$

Since

$$\tan(\theta_1 + \theta_2) = \frac{\beta_1 + \alpha_2\beta_1 + \alpha_1 + \alpha_1\beta_2}{1 + \alpha_2 + \beta_2 + \alpha_2\beta_2 - \alpha_1\beta_1} \tag{19}$$

by substituting (17) and (18) into (19), it follows that $f(\gamma) := \tan(\theta) = \tan(\theta_1 + \theta_2) = \frac{-\sqrt{1-\gamma^2}-\gamma}{\sqrt{1-\gamma^2}-\gamma}$. It is obvious that $f(\gamma)$ is not a constant and has at most two discontinuous points on $(0, 1)$. This indicates there exists an interval $(d_1, d_2) \subset (0, 1)$ such that $f(\gamma)$ is continuous on (d_1, d_2). Consequently, a point $\bar{\gamma} \in (d_1, d_2)$ can be chosen such that $f(\bar{\gamma}) = \tan\theta$, $\theta = \frac{2m}{r}\pi \in [0, 2\pi) \setminus \left\{0, \frac{\pi}{2}, \frac{3\pi}{2}\right\}$, $m, r \in \mathbb{Z}^+$. Then $\alpha_1, \alpha_2, \beta_1$ and β_2 satisfying (15) are gained by substituting $\bar{\gamma}$ into (17) and (18).

Secondly, we will construct an attainable neighborhood of ξ for itself. Define a function $G(z_1, z_2, y_1, y_2)$ by $G(z_1, z_2, y_1, y_2) = (I + z_1\tilde{J}_1 + z_2\tilde{J}_2)\xi - (I + \alpha_1\tilde{J}_1 + \alpha_2\tilde{J}_2)(\xi + y)$, where $\xi = (\xi_1, \xi_2)^T, y = (y_1, y_2)^T \in \mathbb{R}^2_*, z_1, z_2 \in \mathbb{R}$. It is well known that $G = 0$ at the point $(\alpha_1, \alpha_2, 0, 0)$. Since $\left|\frac{\partial G(z_1,z_2,y_1,y_2)}{\partial(z_1,z_2)}\right| = \left|\tilde{J}_1\xi, \tilde{J}_2\xi\right| = \xi_1^2 + \xi_2^2 \neq 0$, it has been shown that, at the pint $(\alpha_1, \alpha_2, 0, 0)$, the above Jacobian determinant is not equal to zero for any $\xi \in \mathbb{R}^2_*$. According to the *Inverse Function Theorem* (see [1]), there exists an open neighborhood $B(\mathbf{0}, \delta)$ of $\mathbf{0} = (0, 0)^T$ such that $z_1 = z_1(y), z_2 = z_2(y)$, where $z_1 = z_1(y), z_2 = z_2(y)$ are both continuous and differentiable on $B(\mathbf{0}, \delta)$. For every fixed $\xi \in \mathbb{R}^2_*$, and any $y \in B(\mathbf{0}, \delta)$, we have

$$\left[\prod_{i=1}^{2r-1}\left(I + v_1(i)\tilde{J}_1 + v_2(i)\tilde{J}_2\right)\right](I + z_1(y)\tilde{J}_1 + z_2(y)\tilde{J}_2)\xi = \xi + y,$$

$v(0) = (z_1(y), z_2(y)), v(2i) = (\alpha_1, \alpha_2), v(2i-1) = (\beta_1, \beta_2),\ i = 1, 2, \cdots, r$. On the basis of the above facts, ξ can be transferred from an initial to any point in an open neighborhood of itself. In the end, we conclude that (14) is controllable according to Lemma 2.1.

Synthesizing the above four cases, we complete the proof.

Corollary 3.2 *Let $n = 2$ and $K = 2$. If $B_1B_2 = B_2B_1$, then the system (10) is controllable if and only if one of B_1 and B_2 has a non-real pair of complex conjugate eigenvalues except pure imaginary eigenvalues, or one of B_1 and B_2 has a pair of pure imaginary eigenvalues and another has nonzero real eigenvalues.*

Example 3.1 Let $B_1 = \begin{bmatrix} -\frac{1}{2} & -\frac{1}{2\sqrt{3}} \\ \frac{1}{2\sqrt{3}} & -\frac{1}{2} \end{bmatrix}$ and $B_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. Then the system (10) becomes the controllability counterexample in [2]. It is easy to gain two eigenvalues of $B_1 : \lambda_{B_1} = -\frac{1}{2} \pm \frac{1}{2\sqrt{3}}i$ $(i = \sqrt{-1})$. According to Corollary 3.2 and Corollary 3.1, we have the same result as [2] that (10) is controllable and (6) is not controllable.

4 Further Discussion

In this section, we plan to consider more general cases, i.e. how is the controllability when $n > 2$ or $K > 2$ in the bilinear (3)? We will give two sufficient conditions for uncontrollability.

Theorem 4.1 *Let $n > 2$, $K \geq 2$. If $B_iB_j = B_jB_i$ $(i, j = 1, 2, \cdots, n)$, and eigenvalues of each $B_i (i = 1, 2, \cdots, n)$ are all real, the system (3) is not controllable.*

Proof Since $B_iB_j = B_jB_i$, there exist a nonsingular real matrix P (see [6]) such that

$$P^{-1}B_iP = \begin{bmatrix} \lambda_{i_1} & & & * \\ & \lambda_{i_2} & & \\ & & \ddots & \\ 0 & & & \lambda_{i_K} \end{bmatrix} \quad (i = 1, 2, \cdots, n).$$

Obviously, $I + v_1(k)P^{-1}B_1P + \cdots + v_n(k)P^{-1}B_nP$ is an upper triangular matrix.Consider the case with the initial state $\xi = (\xi_1, \xi_2, \cdots, \xi_{K-1}, 0)^T$. Then the terminal state η always has this form $\eta = (*, *, \cdots, *, 0)^T$. It has been shown that (3) has no controllability.

5 Conclusions

In this paper, we present sufficient and necessary conditions for the controllability of the bilinear system (2) and (3) with $n = 2$, $K = 2$ and commutative matrices in systems, and conclude that uncontrollability of (3) implies uncontrollability of (2). Therefore, the controllability in this case is completely solved. The controllability counterexample in [2] is only a special case of our results. When the dimension of (3) is greater than two or the number of input in (3) is greater than two, some sufficient conditions has been given in some special circumstances.

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