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A Condition Equivalent to Global Controllability in Systems of Vector Fields

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1. INTRODUCTION

Let M be a connected differentiable manifold (of class at least C^2), let S be a family of C^1 vector fields on M, and for $x \in M$ let $A(x, S)$ denote the reachable set of S from x (precise definitions are given in Section 2). The family S is said to be *globally controllable* if $A(x, S) = M$ for every $x \in M$. Following Sussmann [6], we say that S has *property* (P) if $x \in \operatorname{int} A(x, S)$ for every $x \in M$. One could perhaps also refer to property (P) as the *local controllability* of S from every point of M. However, the term "local" is somewhat misleading, since it could happen that to go from a point x to a nearby point y one might have to follow a trajectory of S that wanders quite far from x before it eventually arrives at y.

When the state manifold M is compact, a theorem of Kupka and Sallet [4] states that S is globally controllable if and only if S has property (P). Thus property (P), which at first glance appears to be *local* in nature, is actually equivalent to *global* controllability. The main result of this paper will show that this equivalence continues to hold when the state manifold M is noncompact. As pointed out in [4], this result is obvious when the family S is *symmetric* (cf. Section 2), but for nonsymmetric S it is not quite so transparent.

We are obliged to point out that the work of Kupka and Sallet is carried out in the context of pseudosemigroups of local diffeomorphisms of M. Consequently, their results apply to discrete-time systems as well as to continuous-time systems on a compact manifold. Our method of proof, which differs considerably from that of Kupka and Sallet, enables us to treat families of vector fields (i.e., continuous-time systems) on a noncompact manifold, but it does not appear to generalize to the case of pseudosemigroups of local diffeomorphisms.

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2. Preliminary Definitions and Results

Let M denote a connected, finite-dimensional, second-countable, Hausdorff differentiable manifold of class C^k with $k \geqslant 2$ and set $n = \dim M$. For a C^1 vector field X on M and a point $x \in M$, we denote by $t \to X_t(x)$ the maximal integral curve of X passing through x at time $t = 0$. The mapping $(t, x) \to X_t(x)$ is defined and of class C^1 on an open subset of $\mathbb{R} \times M$ and is called the *global flow* of X.

Let S be a family of C^1 vector fields on M. We say that a point $y \in M$ is *reachable* from a point $x \in M$ via S if for some $q \in \mathbb{N}$ there exist a q-tuple $(X^1, ..., X^q)$ of elements of S and a q-tuple $(s_1, ..., s_q)$ of nonnegative real numbers such that the expression $(X^q_{s_q} \circ \cdots \circ X^1_{s_1})(x)$ is defined and equals y. The notation $A(x, S)$ stands for the set of all points in M that are reachable from x via S and $A(x, S)$ is called the *reachable set* of S from x.

The binary relation "y is reachable from x" is reflexive and transitive, although it is generally not symmetric. A sufficient condition for the symmetry of this relation is that

$$-S = \{-X | X \in S\} = S;$$

in this case we call the family S *symmetric*. If S is symmetric, then it follows that the collection of reachable sets $\{A(x, S) | x \in M\}$ forms a partition of M. If S is not symmetric, then this may no longer be true as easy examples show.

In the following theorem, we state some elementary and well-known properties of the reachable set. For a proof the reader can consult [1].

Theorem 2.1. *Let S be an arbitrary family of C^1 vector fields on M. Then the following properties hold*:

(i) $y \in A(x, S) \Leftrightarrow x \in A(y, -S)$;

(ii) $y \in M \backslash A(x, S) \Rightarrow A(y, -S) \subseteq M \backslash A(x, S)$;

(iii) $y \in \operatorname{int} A(x, S) \Rightarrow A(y, S) \subseteq \operatorname{int} A(x, S)$;

(iv) $A(x, S)$ *is open in* $M \Leftrightarrow x \in \operatorname{int} A(x, S)$;

(v) $y \in \overline{A(x, S)} \Rightarrow A(y, S) \subseteq \overline{A(x, S)}$.

Using Theorem 2.1, we can give a short proof of a weakened form of the main result, which appears in the next section.

Proposition 2.2. *Let S be an arbitrary family of C^1 vector fields on M. If $x \in \operatorname{int} A(x, S)$ for every $x \in M$, then $A(x, S)$ is an open dense subset of M for every $x \in M$.*

Proof. By Theorem 2.1(iv) we have that $A(x, S)$ is open for every

$x \in M$, so it suffices to show that $M \backslash \overline{A(x, S)}$ is empty for every $x \in M$. If this is not the case, then for some $x \in M$ the set $M \backslash \overline{A(x, S)}$ is nonempty and open. Since M is connected, the set $M \backslash \overline{A(x, S)}$ cannot also be closed, so there exists $z \in \overline{A(x, S)}$ such that z is a cluster point of $M \backslash \overline{A(x, S)}$. By assumption $A(z, S)$ contains an open neighborhood of z, so we infer that $A(z, S) \cap (M \backslash \overline{A(x, S)})$ is nonempty because z is a cluster point of $M \backslash \overline{A(x, S)}$. However, by Theorem 2.1 (v) $z \in \overline{A(x, S)}$ implies that $A(z, S) \subseteq \overline{A(x, S)}$. Thus we obtain a contradiction and it must be the case that $M \backslash \overline{A(x, S)}$ is empty. ∎

The next definition is due to Sussmann [6] and refines the notion of reachability.

DEFINITION 2.3. Let S be a family of C^1 vector fields on M and let k be an integer satisfying $0 \leqslant k \leqslant n = \dim M$. We say that $y \in M$ is *normally k-reachable* from $x \in M$ via S if for some $q \in \mathbb{N}$ there exist a q-tuple $(X^1, ..., X^q)$ of elements of S and a q-tuple $(s_1, ..., s_q)$ of positive real numbers such that the expression $(X^q_{s_q} \circ \cdots \circ X^1_{s_1})(x)$ is defined and equals y and the mapping

$$(t_1, ..., t_q) \to (X^q_{t_q} \circ \cdots \circ X^1_{t_1})(x),$$

which is defined and of class C^1 on an open neighborhood of $(s_1, ..., s_q)$ in $\mathbb{R}^q$, has rank k at $(s_1, ..., s_q)$.

PROPOSITION 2.4. *If $y \in M$ is normally n-reachable from $x \in M$ via S, then $y \in \operatorname{int} A(x, S)$.*

Proof. This is a direct consequence of the surjective-mapping theorem [2, p. 378]. ∎

The converse of Proposition 2.4 does not hold in general, even if the vector fields in S are C^∞ (cf. Example 3.5). One can show that the converse does hold if the vector fields in S are real analytic, although we omit the proof as it is not of essential importance here.

3. THE MAIN THEOREM

In this section we state and prove the main result of this paper on the equivalence of global controllability and property (P) for families of vector fields. The following definition and proposition are essential in the proof of this result.

DEFINITION 3.1. Let $N \subseteq M$ be a C^1 immersed submanifold of M and let $i: N \to M$ denote the inclusion mapping. A C^1 vector field X on M is said to be *tangent* to N if for every $x \in N$ we have $X(x) \in \text{image } di_x$ (di denotes the differential of i).

PROPOSITION 3.2. *Let X be a C^1 vector field on M that is tangent to a C^1 immersed submanifold $N \subseteq M$. Then for every $x \in N$ there exists an $\varepsilon > 0$ such that $|t| < \varepsilon$ implies $X_t(x) \in N$.*

Proof. See [3, Proposition 3.2]. ∎

THEOREM 3.3. *Let S be a family of C^1 vector fields on M. If $x \in \text{int } A(x, S)$ for every $x \in M$, then S is globally controllable.*

Proof. We argue by contradiction and assume that the theorem is false. Then for some $x \in M$ the set $M \backslash A(x, S)$ is nonempty. By Proposition 2.2, $A(x, S)$ is an open dense subset of M, so $M \backslash A(x, S)$ is closed and nowhere dense. Furthermore, by Theorem 2.1 (ii), if $z \in M \backslash A(x, S)$, then $A(z, -S) \subseteq M \backslash A(x, S)$. We next establish two claims which will readily yield the desired contradiction.

Claim 1. There exist $p \in M \backslash A(x, S)$ and $Y \in S$ such that $Y_t(p) \in A(x, S)$ for every $t > 0$ for which the expression $Y_t(p)$ is defined.

Proof of Claim 1. Fix $p_0 \in M \backslash A(x, S)$. Since $M \backslash A(x, S)$ is closed and nowhere dense and $A(p_0, S)$ is open, there exists $p_1 \in A(p_0, S)$ such that $p_1 \in A(x, S)$. Therefore we can find a q-tuple $(X^1, \ldots, X^q)$ of elements of S and a q-tuple $(s_1, \ldots, s_q)$ of nonnegative real numbers such that $(X^q_{s_q} \circ \cdots \circ X^1_{s_1})(p_0) = p_1$. Let $r_0 = 0$, let $r_i = \sum_{j=1}^{i} s_j$ for $1 \leqslant i \leqslant q$, and define a curve $\varphi\colon [0, r_q] \to M$ by

$$\varphi(t) = \begin{cases} X^1_t(p_0), & 0 \leqslant t \leqslant r_1, \\ X^i_{t-r_{i-1}}((X^{i-1}_{s_{i-1}} \circ \cdots \circ X^1_{s_1})(p_0)), & r_{i-1} \leqslant t \leqslant r_i,\ 2 \leqslant i \leqslant q. \end{cases}$$

Then φ is continuous, $\varphi(0) = p_0$, $\varphi(r_q) = p_1$, and for $1 \leqslant i \leqslant q$ $\varphi | [r_{i-1}, r_i]$ is an integral curve of X^i (φ is sometimes called an S-trajectory).

Let $T = \{t \in [0, r_q] \mid \varphi(t) \in M \backslash A(x, S)\}$. Since φ is continuous and $M \backslash A(x, S)$ is closed, T is a closed subset of $[0, r_q]$. Observe that $0 \in T$ and $r_q \notin T$. Thus, if we set $t^* = \sup T$, then $t^* \in T$ and $t^* < r_q$. Choose $i \in \{1, \ldots, q\}$ so that $r_{i-1} \leqslant t^* < r_i$. Then $p = \varphi(t^*) \in M \backslash A(x, S)$ and $X^i_t(p) \in A(x, S)$ at least for $0 < t \leqslant r_i - t^*$. However, Theorem 2.1 (iii) implies that we must actually have $X^i_t(p) \in A(x, S)$ for every $t > 0$ for which the expression $X^i_t(p)$ is defined. Hence we can take $Y = X^i$ and the proof of Claim 1 is complete.

Let $p \in M \backslash A(x, S)$ be as in Claim 1. For $y \in A(p, -S)$ we define

$$r(y) = \max\,\{k \in \{0, 1, \ldots, n\} \mid y \text{ is normally } k\text{-reachable from } p \text{ via } -S\}$$

and we set

$$l = \max\{r(y) \mid y \in A(p, -S) \cap A(p, S)\}.$$

Claim 2. Let $y \in A(p, -S) \cap A(p, S)$ be such that $r(y) = l$. Then there exists a C^1 embedded l-dimensional submanifold N of M such that $y \in N \subseteq A(p, -S) \cap A(p, S)$ and every vector field of $-S$ is tangent to N.

Proof of Claim 2. By assumption there exist a q-tuple $(Z^1, ..., Z^q)$ of vector fields of $-S$ and a q-tuple $(s_1, ..., s_q)$ of positive real numbers such that $(Z^q_{s_q} \circ \cdots \circ Z^1_{s_1})(p) = y$ and the mapping

$$f(t_1, ..., t_q) = (Z^q_{t_q} \circ \cdots \circ Z^1_{t_1})(p),$$

which is defined and of class C^1 on an open neighborhood of $(s_1, ..., s_q)$ in $\mathbb{R}^q$, has rank l at $(s_1, ..., s_q)$. Let W_1 be an open neighborhood of $(s_1, ..., s_q)$ in $\mathbb{R}^q$ such that $W_1 \subseteq \text{domain } f$ and $(t_1, ..., t_q) \in W_1$ implies $t_i > 0$ for $1 \leqslant i \leqslant q$. The existence of W_1 follows from the fact that the domain of f is open and the real numbers s_i are positive for $1 \leqslant i \leqslant q$. Observe that $f(W_1) \subseteq A(p, -S)$. Since f is continuous, $f(s_1, ..., s_q) = y \in A(p, S)$, and $A(p, S)$ is open in M, there exists an open neighborhood W_2 of $(s_1, ..., s_q)$ in $\mathbb{R}^q$ such that $W_2 \subseteq W_1$ and $f(W_2) \subseteq A(p, S)$. We can further shrink W_2 to an open neighborhood W_3 of $(s_1, ..., s_q)$ in $\mathbb{R}^q$ such that f has rank l at each point of W_3. The existence of W_3 follows from the maximality property of l and the fact that the rank of a C^1 mapping is locally nondecreasing. Finally, the rank theorem [5, p. 18] yields an open neighborhood W_4 of $(s_1, ..., s_q)$ in $\mathbb{R}^q$ such that $W_4 \subseteq W_3$ and $f(W_4)$ is a C^1 embedded l-dimensional submanifold of M. If we set $N = f(W_4)$, then by construction $y \in N$ and $N \subseteq A(p, -S) \cap A(p, S)$. The maximality property of l implies by an easy argument (cf. the proof of Theorem 3.12, Claim 1, in [3]) that every vector field in $-S$ is tangent to N. This proves Claim 2.

We can now complete the proof of the theorem. Let $y \in A(p, -S) \cap A(p, S)$ be such that $r(y) = l$. Then y is normally l-reachable from p via $-S$ and, since $y \in A(p, S)$, we see that p is reachable from y via $-S$. It follows that p is normally l-reachable from itself via $-S$, so that $r(p) = l$. Applying Claim 2 to the point p in place of the point y, we obtain a C^1 embedded l-dimensional submanifold N of M such that $p \in N \subseteq A(p, -S) \cap A(p, S)$ and every vector field of $-S$ is tangent to N. It is clear that every vector field in S is also tangent to N. In particular the vector field $Y \in S$ given by Claim 1 is tangent to N. Proposition 3.2 yields an $\varepsilon > 0$ such that $|t| < \varepsilon$ implies $Y_t(p) \in N$. The contradiction is now apparent. On one hand, by Claim 1, $0 < t < \varepsilon$ implies $Y_t(p) \in A(x, S)$. On the other hand, $p \in M \backslash A(x, S)$ implies $A(p, -S) \subseteq M \backslash A(x, S)$, as was observed just prior to Claim 1, so for $0 < t < \varepsilon$ we infer that

$$Y_t(p) \in N \subseteq A(p, -S) \cap A(p, S) \subseteq M \backslash A(x, S).$$

This is the desired contradiction and the proof of the theorem is complete. ▮

COROLLARY 3.4. *Let S be a family of C^1 vector fields on M. Then the following statements are equivalent.*

(i) *S is globally controllable.*

(ii) *$x \in \operatorname{int} A(x, S)$ for every $x \in M$.*

(iii) *$A(x, S)$ is open for every $x \in M$.*

(iv) *x is normally n-reachable from x for every $x \in M$.*

(v) *x is normally n-reachable from y for every $(x, y) \in M \times M$.*

Proof. The implication (ii) ⇒ (i) is given by Theorem 3.3 and the implication (i) ⇒ (ii) is obvious. The equivalence (ii) ⇔ (iii) follows from Theorem 2.1 (iv). Finally, the equivalence of (i), (iv), and (v) is due to Sussmann [6, Theorem 4.3]. ▮

We conclude this paper by giving an example which shows that the conclusion of Theorem 3.3 can fail if the hypothesis $x \in \operatorname{int} A(x, S)$ is violated at precisely one point of M.

EXAMPLE 3.5. Let $M = \mathbb{R}^2$ and denote the coordinates on $\mathbb{R}^2$ by (x, y). Choose a C^∞ function $\varphi\colon \mathbb{R} \to \mathbb{R}$ such that $\varphi(x) = 0$ if $x \leqslant 0$ and $\varphi(x) > 0$ if $x > 0$. Let

$$C = \{(x, 0) \mid x \geqslant 0\}, \qquad D = \{(x, 0) \mid x \leqslant 0\}, \qquad \Omega = \mathbb{R}^2 \backslash D,$$

and let $\psi\colon \mathbb{R}^2 \to \mathbb{R}$ be a C^∞ function such that $\psi(p) \geqslant 0$ for every $p \in \mathbb{R}^2$ and $\psi^{-1}(0) = C$. We consider the family of C^∞ vector fields on M defined by

$$S = \{\partial/\partial x,\ -\psi(x, y)\, \partial/\partial x,\ \pm\varphi(x)\, \partial/\partial y\}.$$

A routine verification shows that

$$A(p, S) = \begin{cases} \mathbb{R}^2, & p = (x, 0),\ x < 0, \\ \Omega \cup \{(0, 0)\}, & p = (0, 0), \\ \Omega, & p \in \Omega. \end{cases}$$

Hence $p \in \operatorname{int} A(p, S)$ for every $p \in \mathbb{R}^2$ except the origin, but S is not globally controllable. We also observe that if $p = (x, 0)$, $x < 0$, then $p \in \operatorname{int} A(p, S) = \mathbb{R}^2$, but p is not normally 2-reachable from itself via S (see the remarks following Proposition 2.4).

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