Notes on the symplectic control of Gaussian states

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INTRODUCTION

It is known that for qudit systems full unitary controllability and pure state reachability does not coincide if the local dimension d is even []. One does not need the full $\mathfrak{su}(d)$ dynamical algebra to be able to connect any to pure states, only the full unitary symplectic algebra $\mathfrak{usp}(d)$,. Similarly, it was recently shown that for fermionic systems the pure Gaussian state reachability one does not need the full quadratic control algebra $\mathfrak{so}(2n)$, only the subalgebra $\mathfrak{so}(2n+1)$.

We will address the analogous question for bosonic systems. In the case controllability and reachability questions become more complex, because the underlying Hilbert space is infinite dimensional and the group of symplectic transformations is non-compact. This means, for example, that there are symplectic transformations that cannot be obtained by the application of any single quadratic Hamiltonian. To obtain these, one needs the application of at least two quadratic Hamiltonians. This also implies that to connect two isospectral bosonic Gaussian states one cannot generally only apply one single quadratic Hamiltonian. However, also in this case the pure state reachability turns out to be an exception, here it always exists a single Hamiltonian with the application of which one can obtain a particular pure Gaussian state from another one.

BASICS, NOTATIONS

We consider pairs of canonically conjugated operators $\{\hat{x}_j, \hat{p}_j\}$ acting on the bosonic Fock space. Using the notation $\hat{R} = (\hat{x}_1, \hat{p}_1, \ldots, \hat{x}_n, \hat{p}_n)$, The canonical commutation relations for the $\hat{R}_i$ can be expressed in terms of the symplectic form Ω

$$[\hat{R}_j, \hat{R}_k] = 2i\Omega_{jk} \ , \tag{1}$$

$$\text{where} \quad \Omega = \bigoplus_{j=1}^{n} \omega \ , \quad \omega = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} .$$

In these notes, we'll focus on *Gaussian states.* A Gaussian state is completely characterised by the first and second moments of the canonical operators. For simplicity, first we consider states with null first moments, completely determined by the symmetric covariance matrix $\boldsymbol{\sigma}$ with entries $\boldsymbol{\sigma}_{jk} = \langle(\hat{R}_j\hat{R}_k + \hat{R}_k\hat{R}_j)\rangle$.

Let us also recapitulate some useful facts about symplectic operations and Gaussian states. An n–mode Gaussian state can always be written as

$$\boldsymbol{\sigma} = S^T \nu S \ , \tag{2}$$

with $S \in Sp_{(2n,\mathbb{R})}$ and

$$\nu = \operatorname{diag}(\nu_1, \nu_1, \ldots, \nu_n, \nu_n) \,. \tag{3}$$

The quantities $\{\nu_j\}$ are the so-called *symplectic eigenvalues* of $\boldsymbol{\sigma}$, and the transformation S is said to perform a *symplectic diagonalisation* of $\boldsymbol{\sigma}$. The Gaussian state is pure if and only if the symplectic eigenvalues are all 1.

We will also use the following general decomposition of a symplectic transformation S:

$$S = O_1 Z O_2, \tag{4}$$

where $O_1, O_2 \in Sp(2n, \mathbb{R}) \cap SO(2n) \cong U(n)$ are orthogonal symplectic transformations, while

$$Z = \oplus_{j=1}^{n} \begin{pmatrix} z_j & 0 \\ 0 & \frac{1}{z_j} \end{pmatrix} \ ,$$

with $z_j \geq 1 \,\forall\, j$. The set of such Z's forms a non-compact subgroup of $Sp_{2n,\mathbb{R}}$ comprised of single-mode squeezing operations.

THE POLAR DECOMPOSITION OF $Sp(2n, \mathbb{R})$

Since $\mathfrak{sp}(2n, \mathbb{R})$ is not a compact Lie algebra, and $Sp(2n, \mathbb{R})$ is non-compact Lie group, it is not any more true that the exponential map $\mathfrak{exp} : \mathfrak{sp}(2n, \mathbb{R}) \to Sp(2n, \mathbb{R})$ is surjective. However, it turns out that one can reach $Sp(2n, \mathbb{R})$ by multiplying two exponentials of $\mathfrak{sp}(2n, \mathbb{R})$ elements, i.e. as $e^X e^Y$ where $X, Y \in$

CHARACTERIZATION OF MIXED AND PURE GAUSSIAN STATES