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ANALYTIC VECTORS

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1. Introduction

If A is an operator on a Banach space $\mathfrak{X}$, we will call an element x of $\mathfrak{X}$ an *analytic vector* for A in case the series expansion of $e^{As}x$ has a positive radius of absolute convergence; that is, in case

$$\sum_{n=0}^{\infty} \frac{\| A^n x \|}{n!} s^n < \infty$$

for some $s > 0$. For example, if $\mathfrak{X} = C([0, 1])$ is the Banach space of continuous functions on the closed unit interval, in the supremum norm, and if A is differentiation, an analytic vector for A is simply an analytic function on $[0, 1]$. If A is a bounded operator then every vector in $\mathfrak{X}$ is an analytic vector for A, so that only unbounded operators will be of interest.

A number of problems in analysis may be regarded as the study of analytic vectors for an operator. It is a classical result that if A is an elliptic partial differential operator with analytic coefficients and if $Au=0$, then u is an analytic function. In the last section we show more generally that if u is an analytic vector for A then u is an analytic function. For example, if A is self-adjoint then any element of a spectral subspace corresponding to a bounded interval is an analytic function.

Most of our results concern representations of Lie groups and Lie algebras. In § 7–8 we show that any (strongly continuous) representation of a Lie group on a Banach space has a dense set of analytic vectors (well-behaved vectors) and in § 9 a simple criterion is found in order that a Lie algebra of unbounded skew-symmetric operators on a Hilbert space give rise to a unitary representation of a Lie group. Detailed descriptions of these problems will be found in the opening paragraphs of the mentioned sections. Here we shall discuss briefly the general method.

Suppose that A and X are two operators on a Banach space $\mathfrak{X}$. The main theorem in § 3 gives a sufficient condition in order that every analytic vector for A be an analytic vector for X. This involves estimating $\| X^n x \|$ in terms of $\| x \|, \| Ax \|, \cdots, \| A^n x \|$. Suppose that $\| Xx \| \leqq \| Ax \|$ for all x (leaving aside the question of domains of opera-

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tors). If X and A commute, then $\| X^n x \| \leqq \| A^n x \|$ for all n. In the general case this is not so. However, it is true that $\| X^2 x \| \leqq \| AXx \| = \| XAx - (XA - AX)x \| \leqq \| A^2 x \| + \| (XA - AX)x \|$, so that we need some estimate for $XA - AX = (\operatorname{ad} X)A$, and in general for the higher order commutators $(\operatorname{ad} X)^n A$. It turns out to be sufficient to require (for all x and n) $\| (\operatorname{ad} X)^n Ax \| \leqq c_n \| Ax \|$, where the power series with coefficients $c_n/n!$ has a positive radius of convergence.

In all of the applications, A will be an elliptic operator (either in the ordinary sense as a partial differential operator or in an abstract sense as an element of the enveloping algebra of a Lie algebra) and X will be a first order operator. Then $(\operatorname{ad} X)^n A$ is of order less than or equal to the order of A, and the ellipticity of A enables us to estimate $(\operatorname{ad} X)^n A$ in terms of A.

The results on elliptic partial differential operators in the last section depend only on § 2-4. Section 8, in which it is shown that a representation of a Lie group on a Banach space has a dense set of analytic vectors, is based on somewhat different considerations, and is practically independent of the rest of the paper. The treatment of unitary representations in § 7 is simpler.

2. The calculus of absolute values

It will be convenient to develop some simple rules for calculating with relations of the form : $\| Cx \| \leqq \| Ax \| + \| Bx \|$ for all x. We write this as $| C | \leqq | A | + | B |$ and call the symbol $| A |$ the *absolute value* of A. A formal definition is given below. We abbreviate

$$\sum_{n=0}^{\infty} \frac{\| A^n x \|}{n!} s^n \text{ by } \| e^{|A|s} x \| .$$

Let $\mathfrak{X}$ be a Banach space. By an *operator* A on $\mathfrak{X}$ is meant a linear transformation defined on a linear subspace $\mathfrak{D}(A)$, called the *domain* of A, and taking values in $\mathfrak{X}$. By the *absolute value* $| A |$ of A we shall mean the set consisting of A alone (for the sake of definiteness). We denote the set of all operators on $\mathfrak{X}$ by $\mathcal{O}(X)$. Let $| \mathcal{O}(\mathfrak{X}) |$ be the free abelian semigroup with the set of all $| A |$, with A in $\mathcal{O}(\mathfrak{X})$, as generators. That is, an element α of $| \mathcal{O}(\mathfrak{X}) |$ is a finite formal sum

(2.1) $$\alpha = | A_1 | + \cdots + | A_l | .$$

If

(2.2) $$\beta = | B_1 | + \cdots + | B_m |$$

then α and β are equal elements of $| \mathcal{O}(\mathfrak{X}) |$ if they are identical except possibly for the order of the summands. If a is a positive number we shall identify a with $|aI|$, where I is the identity operator on $\mathfrak{X}$. We define the product of α and β, given by (2.1) and (2.2), by

$$\alpha\beta = \sum_{i=1}^{l}\sum_{j=1}^{m} |A_iB_j| . \tag{2.3}$$

We recall that the product AB of two operators on $\mathfrak{X}$ is the operator whose domain $\mathfrak{D}(AB)$ consists of all vectors x in $\mathfrak{D}(B)$ such that Bx is in $\mathfrak{D}(A)$, in which case $(AB)x = A(Bx)$. The sum $A + B$ has as domain $\mathfrak{D}(A) \cap \mathfrak{D}(B)$. With the definition (2.3), $|\mathcal{O}(\mathfrak{X})|$ is a semiring and, because of the idetification of a with $|aI|$, a semialgebra.

We shall adopt the convention of writing $\|Ax\| = \infty$ to mean that the vector x is not in the domain of the operator A. With α in $|\mathcal{O}(\mathfrak{X})|$ given by (2.1), we define $\|\alpha x\|$, for all x in $\mathfrak{X}$, by

$$\|\alpha x\| = \|A_1x\| + \cdots + \|A_lx\| \tag{2.4}$$

We define a relation $\leqq$ on $|\mathcal{O}(\mathfrak{X})|$ by putting $\alpha \leqq \beta$ in case $\|\alpha x\| \leqq \|\beta x\|$ for all x in $\mathfrak{X}$.

Notice that the following are true, for all operators A, B, and C on $\mathfrak{X}$:

$$|A + B| \leqq |A| + |B| \tag{2.5}$$

$$\text{if } |A| \leqq |B|, \text{ then } |AC| \leqq |BC| . \tag{2.6}$$

In fact, for all x in $\mathfrak{D}(A) \cap \mathfrak{D}(B)$, $\|(A + B)x\| \leqq \|Ax\| + \|Bx\|$. If x is not in $\mathfrak{D}(A) \cap \mathfrak{D}(B)$, $\|Ax\| + \|Bx\| = \infty$, so that (2.5) is true. If $|A| \leqq |B|$ then in particular for a vector of the form Cx, $\|ACx\| \leqq \|BCx\|$, proving (2.6).

We shall have occasion to use elements $\varphi = \sum_{n=0}^{\infty} \alpha_n s^n$ of the semialgebra of all power series in a variable s with coefficients in $|\mathcal{O}(\mathfrak{X})|$. If also $\psi = \sum_{n=0}^{\infty} \beta_n s^n$, we define $\varphi \leqq \psi$ in case $\alpha_n \leqq \beta_n$ for each n, and we define $\|\varphi x\|$, for x in $\mathfrak{X}$, by $\|\varphi x\| = \sum_{n=0}^{\infty} \|\alpha_n x\| s^n$. We define $(d/ds)\varphi(s)$ and $\int_0^s \varphi(t)dt$ by formal differentiation and integration.

A vector x in $\mathfrak{X}$ is called an *analytic vector for* α in $|\mathcal{O}(\mathfrak{X})|$ in case $\|e^{\alpha s}x\| < \infty$ for some $s > 0$.

If α is given by (2.1), then

$$\|e^{\alpha s}x\| = \sum_{n=0}^{\infty} \frac{1}{n!} \|(|A_1| + \cdots + |A_l|)^n x\| s^n$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{1 \leqq i_1, \cdots, i_n \leqq l} \|A_{i_1} \cdots A_{i_n} x\| s^n .$$

Thus $\|e^{\alpha s}x\| < \infty$ if and only if the series expansion of $e^{(A_1+\cdots+A_l)s}\, x$ is absolutely convergent. If $s_1 \leqq s, \cdots, s_l \leqq s$ then $e^{A_1 s_1 + \cdots + A_l s_n} x$ is *a fortiori* absolutely convergent. Thus x is an analytic vector for α if and only if the series expansion of $e^{A_1 s_1 + \cdots + A_l s_l} x$ converges absolutely for $(s_1, \cdots, s_l)$ sufficiently small.

Let ξ be an element of $|\mathcal{O}(\mathfrak{X})|$, $\xi = |X_1| + \cdots + |X_d|$. We define ad ξ on $|\mathcal{O}(\mathfrak{X})|$ by

$$(\mathrm{ad}\ \xi)\alpha = \sum_{i=1}^{d} \sum_{j=1}^{l} |X_i A_j - A_j X_i| \tag{2.7}$$

where α is given by (2.1). If X is in $\mathcal{O}(\mathfrak{X})$, we define ad$X$ on $\mathcal{O}(\mathfrak{X})$ by $(\mathrm{ad}X)A = XA - AX$. The following combinatorial lemmas will be needed in § 3–4. We recall that $A \supset B$ mean that $\mathfrak{D}(A) \supset \mathfrak{D}(B)$ and A agrees with B on $\mathfrak{D}(B)$. The domain $\mathfrak{D}(A)$ of an operator A is called *invariant* in case A maps $\mathfrak{D}(A)$ into $\mathfrak{D}(A)$.

LEMMA. 2.1. *If $X_1, \cdots, X_n$, A are operators on $\mathfrak{X}$ then*

$$X_n \cdots X_1 A \supset \sum_{k=0}^{n} \sum_{\sigma \in (n,k)} (\mathrm{ad}\ X_{\sigma(k)} \cdots \mathrm{ad}\ X_{\sigma(1)} A) X_{\sigma(n)} \cdots X_{\sigma(k+1)} \tag{2.8}$$

where (n, k) denotes the set of all $\binom{n}{k}$ permutations σ of $1, \cdots, n$ such that $\sigma(n) > \sigma(n-1) > \cdots > \sigma(k+1)$ and $\sigma(k) > \sigma(k-1) > \cdots > \sigma(1)$. If $X_1, \cdots, X_n$, A have a common invariant domain, then equality holds in (2.8).

PROOF. For $n = 0$, (2.8) states that $A \supset A$ and for $n = 1$ it states that $X_1 A \supset AX_1 + (\mathrm{ad}\ X_1)A$, which is true. Suppose that (2.8) holds for n and let X_{n+1} be an operator on $\mathfrak{X}$. Then

$$\begin{aligned} X_{n+1} X_n \cdots X_1 A &\supset \sum_{k=0}^{n} \sum_{\sigma \in (n,k)} X_{n+1} (\mathrm{ad}\ X_{\sigma(k)} \cdots \mathrm{ad}\ X_{\sigma(1)} A) X_{\sigma(n)} \cdots X_{\sigma(k+1)} \\ &\supset \sum_{k=0}^{n} \sum_{\sigma \in (n,k)} \{(\mathrm{ad}\ X_{\sigma(k)} \cdots \mathrm{ad}\ X_{\sigma(1)} A) X_{n+1} X_{\sigma(n)} \cdots X_{\sigma(k+1)} \\ &\qquad + (\mathrm{ad}\ X_{n+1}\ \mathrm{ad}\ X_{\sigma(k)} \cdots \mathrm{ad}\ X_{\sigma(1)} A) X_{\sigma(n)} \cdots X_{\sigma(k+1)}\} . \end{aligned}$$

Let τ be a permutation in $(n + 1, k)$. Then.

$$(\mathrm{ad}\ X_{\tau(k)} \cdots \mathrm{ad}\ X_{\tau(1)} A) X_{\tau(n+1)} \cdots X_{\tau(k+1)}$$

occurs as a term before the + sign in the braces (corresponding to a σ in (n, k)) if $\tau(n + 1) = n+1$, and as a term after the + sign (corresponding to a σ in $(n, k - 1)$) otherwise, for either $\tau(n + 1)$ or $\tau(k)$ must be equal to $n + 1$, by the definition of (n, k). Since $\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$, the correspondence is one-to-one, and (2.8) holds for $n+1$. The last statement of the lemma is obvious, for then the two sides of (2.8) have the same domain.

LEMMA 2.2. *If $X_1, \cdots, X_n$, are operators on $\mathfrak{X}$ then*

$$AX_n \cdots X_1 \supset X_n \cdots X_1 A - f_n \tag{2.9}$$

where

$$f_n = \sum_{k=1}^{n} \sum_{\sigma \in (n,k)} (\mathrm{ad}\, X_{\sigma(k)} \cdots \mathrm{ad}\, X_{\sigma(1)} A) X_{\sigma(n)} \cdots X_{\sigma(k+1)} . \tag{2.10}$$

If $X_1, \cdots, X_n$ have a common invariant domain, then equality holds in (2.9).

PROOF. The last statement is an immediate consequence of the preceding lemma. To prove (2.9), notice that it is true for $n = 0$ and $n=1$, assume it for n, and let X_{n+1} be an operator on $\mathfrak{X}$. Then (2.9) holds with $X_n, \cdots, X_1$ replaced by $X_{n+1}, \cdots, X_2$ and f_n modified accordingly; calling f'_n the expression which replaces f_n, we have

$$\begin{aligned} AX_{n+1} \cdots X_2 X_1 &\supset X_{n+1} \cdots X_2 A X_1 - f'_n X_1 \\ &\supset X_{n+1} \cdots X_2 X_1 A - X_{n+1} \cdots X_2 (\mathrm{ad}\, X_1) A - f'_n X_1 . \end{aligned}$$

Applying Lemma 2.1 to $X_{n+1} \cdots X_2 (\mathrm{ad}\, X_1) A$ (with $(\mathrm{ad}\, X_1) A$ playing the rôle of A in that lemma), we see that $X_{n+1} \cdots X_2 (\mathrm{ad}\, X_1) A + f'_n X_1 \supset f_{n+1}$, each permutation τ in $(n+1, k)$ with $\tau(1) = 1$ corresponding to a term in $X_{n+1} \cdots X_2 (\mathrm{ad}\, X_1) A$ and each τ with $\tau(1) \neq 1$ corresponding to a term in f_{n+1}.

LEMMA 2.3. *Let ξ and α be in $|\mathcal{O}(\mathfrak{X})|$.* Then

$$\alpha \xi^n \leqq \xi^n \alpha + \sum_{k=1}^{n} \binom{n}{k} ((\mathrm{ad}\, \xi)^k \alpha) \xi^{n-k} \tag{2.11}$$

PROOF. Let $\xi = |X_1| + \cdots + |X_d|$, $\alpha = |A_1| + \cdots + |A_l|$, and let $\sum^*$ denote the sum over all $1 \leqq l$, $1 \leqq g_1 \leqq d, \cdots, 1 \leqq g_n \leqq d$. Then, using Lemma 2.2,

$$\begin{aligned} \alpha \xi^n &= \sum{}^* |A X_{g_n} \cdots X_{g_1}| \leqq \sum{}^* |X_{g_n} \cdots X_{g_1} A_\iota| \\ &\qquad + \sum{}^* \Big| \sum_{k=1}^{n} \sum_{\sigma \in (n,k)} (\mathrm{ad}\, X_{\sigma(g_k)} \cdots \mathrm{ad}\, X_{\sigma(g_1)} A_\iota) X_{\sigma(g_n)} \cdots X_{\sigma(g_{k+1})} \Big| \\ &= \xi^n \alpha + \sum{}^* \Big| \sum_{k=1}^{n} \binom{n}{k} (\mathrm{ad}\, X_{g_k} \cdots \mathrm{ad}\, X_{g_1} A_\iota) X_{g_n} \cdots X_{g_{k+1}} \Big| \end{aligned}$$

since there are $\binom{n}{k}$ permutations in (n, k), so that each term $(\mathrm{ad}\, X_{g_k} \cdots \mathrm{ad}\, X_{g_1} A_\iota) X_{g_n} \cdots X_{g_{k+1}}$ occurs $\binom{n}{k}$ times. But this is $\leqq$ the right hand side of (2.11), concluding the proof.

3. The main theorem

THEOREM 1. *Let $\mathfrak{X}$ be a Banach space, ξ and α in $|\mathcal{O}(\mathfrak{X})|$. Let $\xi \leqq c\alpha$, $(\text{ad}\,\xi)^n\alpha \leqq c_n\alpha$, and*

$$\nu(s) = \sum_{n=1}^{\infty} \frac{c_n}{n!} s^n$$

$$\kappa(s) = \int_0^s \frac{dt}{1 - \nu(t)}$$

Then $e^{\xi s} \leqq e^{c\alpha\kappa(s)}$.

If c and c_n are such that $c < \infty$ and $\nu(s)$ has a positive radius of convergence, $\nu(s) < \infty$ for some $s > 0$, we shall say that α *analytically dominates* ξ. If ν has a positive radius of convergence, so does κ. Therefore the following corollary follows at once from the theorem.

COROLLARY 3.1. *Let $\mathfrak{X}$ be a Banach space, ξ and α in $|\mathcal{O}(\mathfrak{X})|$. If α analytically dominates ξ then every analytic vector for α is an analytic vector for ξ.*

We state the next corollary, which is the consequence of Theorem 1 which will be used most often, without using the terminology of § 2.

COROLLARY 3.2. *Let $X_1, \cdots, X_d, A$ be operators on a Banach space $\mathfrak{X}$. Let k and k_n, $n = 1, 2, \cdots$, be such that for all x in the domain of A*

$$\| X_i x \| \leqq k (\| Ax \| + \| x \|) , \qquad 1 \leqq i \leqq d$$

$$\| ad\, X_{i_1} \cdots \text{ad}\, X_{i_n} Ax \| \leqq k_n(\| Ax \| + \| x \|) , \quad 1 \leqq i_1, \cdots, i_n \leqq d.$$

Suppose that $k < \infty$ and that $\sum_{n=1}^{\infty} (k_n/n!)\, s^n < \infty$ for some $s < 0$. If there is an $s > 0$ such that

$$\sum_{n=0}^{\infty} \frac{\| A^n x \|}{n!} s^n < \infty$$

then for $(s_1, \cdots, s_d)$ sufficiently close to $(0, \cdots, 0)$

$$\sum_{n=0}^{\infty} \frac{1}{n!} \sum_{1 \leqq i_1, \cdots, i_n \leqq d} \| X_{i_1} \cdots X_{i_n} x \| s_{i_1} \cdots s_{i_n} < \infty .$$

PROOF. Set $\xi = |X_1| + \cdots + |X_d|$ and $\alpha = |A| + |I|$. Notice that $(\text{ad}\,\xi)^n\alpha = (\text{ad}\,\xi)^n |A|$. Therefore α analytically dominates ξ with c given by dk and c_n given by $d^n k_n$. Corollary 3.2 now follows from Corollary 3.1.

We turn now to the proof of Theorem 1. Define by recursion

$$\pi_0 = |I| \tag{3.1}$$
$$\pi_{n+1} = c\pi_n\alpha + \sum_{k=1}^n \binom{n}{k} c_k \pi_{n+1-k} .$$

Thus $\pi_1 = c\alpha$, $\pi_2 = c^2\alpha^2 + c_1 c\alpha$, and each π_n is a polynomial in α. We will show that for all n

$$\xi^n \leqq \pi_n . \tag{3.2}$$

In fact, we will prove that for all $n \geqq 1$

$$\alpha\xi^{n-1} \leqq \frac{1}{c}\pi_n \tag{3.3}$$

Since $\xi \leqq c\alpha$, $\xi^n \leqq c\alpha\,\xi^{n-1}$, so that (3.3) implies (3.2). For $n = 1$, (3.3) says that $\alpha \leqq (1/c)c\alpha$. Suppose that (3.3) holds for all $k \leqq n$. Then, by Lemma 2.3,

$$\begin{aligned} \alpha\xi^n &\leqq \xi^n\alpha + \sum_{k=1}^n \binom{n}{k} ((\operatorname{ad}\xi)^k\alpha)\xi^{n-k} \\ &\leqq \xi^n\alpha + \sum_{k=1}^n \binom{n}{k} c_k \alpha\xi^{n-k} \\ &\leqq \pi_n\alpha + \sum_{k=1}^n \binom{n}{k} c_k \frac{1}{c}\pi_{n+1-k} \leqq \frac{1}{c}\pi_{n+1} \end{aligned}$$

proving (3.3).

Let $\pi(s)$ be the power series

$$\pi(s) = \sum_{n=0}^\infty \frac{\pi_n}{n!} s^n .$$

By (3.1) and the relation $\binom{n}{k} = n!/k!\,(n-k)!$ we have

$$(n+1)\frac{\pi_{n+1}}{(n+1)!} = c\frac{\pi_n}{n!}\alpha + \sum_{k=1}^n \frac{c_n}{k!}(n+1-k)\frac{\pi_{n+1-k}}{(n+1-k)!} .$$

By definition of $\nu(s)$, this says that

$$\frac{d}{ds}\pi(s) = c\pi(s)\alpha + \nu(s)\frac{d}{ds}\pi(s) . \tag{3.4}$$

That is, $(d/ds)\pi(s) = c\alpha\pi(s)/(1-\nu(s))$. Using the definition of $\kappa(s)$, this implies that $\pi(s) = e^{c\alpha\kappa(s)}$, as may be verified by differentiating both sides, giving (3.4), and observing that both sides have the same constant term $|I|$. By (3.2), $e^{\xi s} \leqq \pi(s) = e^{c\alpha\kappa(s)}$, concluding the proof.

We have stated the theorem in terms of Banach spaces, but we have nowhere used the fact that $\mathfrak{X}$ is complete. The results and proofs of §2–3 remain valid if $\mathfrak{X}$ is a vector space over the real or complex numbers with a "norm" $\|\ \|$ satisfying $0 \leqq \| x \| \leqq \infty$, $\| x + y \| \leqq \| x \| + \| y \|$, $\| ax \| = | a |\, \| x \|$ for all x, y in $\mathfrak{X}$ and scalars a (with the convention that $0 \cdot \infty = 0$). Also, we have nowhere used the fact that $A_1, \cdots, A_l$ are linear. However, only the theorem as stated will be used in the sequel.

4. Non-commuting vector fields

The theorem proved in this section will not be needed until § 12, but it serves to clarify § 7 and to illustrate Theorem 1.

Let U be an open neighborhood of the origin in R^d (d-dimensional euclidean space). A function u defined in U and taking values in a Banach space $\mathfrak{X}$ is said to be of class C^∞ in case all mixed partial derivatives of u exist in the strong topology; e. g.,

$$\frac{1}{h}\,(u(x_1 + h, x_2, \cdots, x_n) - u(x_1, x_2, \cdots, x_n))$$

converges in $\mathfrak{X}$ as $h \to 0$, for all $(x_1, \cdots, x_n)$ in U. Let K be a compact subset of U and let

$$\| u \|_K = \sup\nolimits_{x \in K} \| u(x) \| \;. \tag{4.1}$$

The function u is said to be analytic in U in case for each point x in U, there is an $\varepsilon > 0$ such that if K is the closed sphere with center x and radius ε, then u is an analytic vector for $| \partial/\partial x_1 | + \cdots + | \partial/\partial x_d |$ in the norm $\|\ \|_K$; i. e., in case

$$\sum\nolimits_{n=0}^{\infty} \frac{1}{n!} \sum\nolimits_{1 \leqq i_1, \cdots, i_n \leqq d} \left\| \frac{\partial}{\partial x_{i_1}} \cdots \frac{\partial}{\partial x_{i_n}} u \right\|_K s_{i_1} \cdots s_{i_n}$$

is absolutely convergent for $s_1, \cdots, s_d$ sufficiently small. The property of being analytic is, of course, invariant under analytic changes of coordinates. In some contexts, notably the study of Lie groups, it is more convenient to use a family of d linearly independent analytic vector fields $X_1, \cdots, X_d$ which do not commute, rather than the vector fields $(\partial/\partial x_1), \cdots, (\partial/\partial x_d)$ of a coordinate system. (We recall that a vector field $X = \sum_{i=1}^d a_i(x)(\partial/\partial x_i)$ is called analytic in case the a_i are analytic. If X and Y are two vector fields, so is $XY - YX$, but it is not 0 in general.) The next theorem asserts that the notion of analyticity with respect to $X_1, \cdots, X_d$ is the same as the usual notion.

Let $\mathfrak{X}_K$ be the Banach space of all continuous functions from K to $\mathfrak{X}$ in the norm (4.1), and let X be a vector field with continuous coefficients,

$X = \sum a_i(x)(\partial/\partial x_i)$. If K is the closure of its interior, then we define an operator, also denoted X, with domain $\mathfrak{D}(X)$ consisting of all u in $\mathfrak{X}_K$ which are in $C^\infty(K)$; i. e., have extensions $\tilde{u}$ to U of class C^∞, with Xu being defined as the restriction of $\sum a_i(x)(\partial/\partial x_i)\,\tilde{u}$ to K. If $u = 0$ on K then $Xu = 0$, since K is the closure of its interior. (If K were not the closure of its interior, we could take $\mathfrak{X}_K$ to be the space of functions defined on U of class C^∞ with the "norm" (4.1). Then differentiation of X might take a function of "norm" 0 into a function with positive "norm". By the remarks at the end of § 3, the proof of the following theorem would apply to that case as well.)

THEOREM 2. *Let U be an open set in R^d, K a compact subset of U which is the closure of its interior. Let $Y_1, \cdots, Y_d$ be analytic vector fields on U which are linearly independent at each point of U, and let $X_1, \cdots, X_l$ be analytic vector fields on U. Then any analytic vector for*

$$\eta = |\,Y_1\,| + \cdots + |\,Y_d\,|$$

is an analytic vector for $\xi = |\,X_1\,| + \cdots + |\,X_l\,|$. In fact, η analytically dominates ξ.

PROOF. Let $X_i = \sum_{j=1}^d a_{ij}(x)(\partial/\partial x_j)$, $Y_i = \sum_{j=1}^d b_{ij}(x)(\partial/\partial x_j)$, and $\delta = |\,\partial/\partial x_1\,| + \cdots + |\,\partial/\partial x_d\,|$.

Let a_n be the maximum absolute value on K of all mixed partial derivatives of order n of the a_{ij} and b_{ij}. Notice that since the a_{ij}, b_{ij} are a finite set of analytic functions and K is compact,

$$a(s) = \sum_{n=0}^\infty \frac{a_n}{n!} s^n < \infty \tag{4.2}$$

for some $s > 0$. Now $|\,X_i\,| \leqq a_0\delta$, so that $\xi \leqq la_0\delta$. Let b be the maximum absolute value on K of elements of the matrix inverse to $b_{ij}(x)$, so that $\delta \leqq db\eta$. Since $Y_1, \cdots, Y_d$ are everywhere linearly independent and K is compact, $b < \infty$. Letting $c = la_0\,db$, we have $\xi \leqq c\eta$ with $c < \infty$.

Define by recursion

$$\begin{aligned} k_0(s) &= a(s) \\ k_{n+1}(s) &= d\cdot a(s)Dk_n(s) + d\cdot k_n(s)Da(s) \end{aligned} \tag{4.3}$$

where $D = d/ds$ (and d is the dimension of R^d). We need to estimate $(\mathrm{ad}\xi)^n\eta$. Now $(\mathrm{ad}\,\xi)^n\eta = \sum |\,\mathrm{ad}\,X_{i_n} \cdots \mathrm{ad}\,X_{i_1}\,Y_j\,|$, where the summation is over all $1 \leqq i_n, \cdots, i_1 \leqq l$ and $1 \leqq j \leqq d$, so that there are $l^n d$ terms in $(\mathrm{ad}\,\xi)^n\eta$. Let

$$\mathrm{ad}\,X_{i_n} \cdots \mathrm{ad}\,X_{i_1} Y_j = \sum_{h=1}^d p_{hi_n \cdots i_1 j}(x)\frac{\partial}{\partial x_h}\,. \tag{4.4}$$

If p is a function on K and k is a power series with positive coefficients in one variable s, we say that p is *majorized* by k in case we always have $|(\partial/\partial x_{i_m}) \cdots (\partial/\partial x_{i_1})\, p(x)| \leqq D^m k(0)$ for all x in K and $m = 0, 1, 2, \cdots$. It is clear that

(i) if p is majorized by k then $(\partial/\partial x_g)p$ is majorized by Dk, and

(ii) if $p, \hat{p}$ are majorized by $k, \hat{k}$ respectively then $p\hat{p}$ is majorized by $k\hat{k}$ and $p \pm \hat{p}$ is majorized by $k + \hat{k}$.

Now we shall prove by induction that each $p_{hi_n \cdots i_1 j}$ is majorized by k_n, for all n. For $n = 0$ this is clearly true, by definition of $a(s)$. Suppose that it is true for n. Now

$$\operatorname{ad} X_{i_{n+1}} \operatorname{ad} X_{i_n} \cdots \operatorname{ad} X_{i_1} Y_j = \sum_{g,h=1}^{d} a_{i_{n+1}g}\left(\frac{\partial}{\partial x_g} p_{hi_n \cdots i_1 j}\right)\frac{\partial}{\partial x_h} - \sum_{g,h=1}^{d} p_{hi_n \cdots i_1 j}\left(\frac{\partial}{\partial x_h} a_{i_{n+1}g}\right)\frac{\partial}{\partial x_g}$$

so that

$$p_{hi_{n+1}i_n \cdots i_1 j} = \sum_{g=1}^{d} a_{i_{n+1}g}\left(\frac{\partial}{\partial x_g} p_{hi_n \cdots i_1 j}\right) - \sum_{g=1}^{d} p_{gi_n \cdots i_1 j}\left(\frac{\partial}{\partial x_g} a_{i_{n+1}h}\right). \tag{4.5}$$

The first term on the right hand side of (4.5) is majorized by the first term on the right hand side of (4.3), by (i) and (ii), and similarly for the second terms. This completes the induction. Consequently,

$$(\operatorname{ad} \xi)^n \eta \leqq l^n d^2 k_n(0)\delta \leqq l^n d^3 k_n(0) b\eta .$$

It remains only to show that if $c_n = l^n d^3 k_n(0) b$ then $\sum_{n=0}^{\infty}(c_n/n!)\, s^n < \infty$ for some $s > 0$, which is equivalent to showing that $\sum_{n=0}^{\infty}(k_n(0)/n!)t^n < \infty$ for some $t > 0$.

We shall in fact show that $\sum_{n=0}^{\infty}(k_n(s)/n!)t^n$ is absolutely convergent for s and t sufficiently small. By (4.3), $k_n(s) = (D \cdot a(s))^n a(s)$ where the operator $D \cdot a(s)$ is defined by $(D \cdot a(s))f(s) = D(a(s)f(s))$. But $(D \cdot a(s))^n a(s) = a(s)^{-1}(a(s)D)^n a(s)^2$ and by the analytic change of variables given by $r = \int_0^s \frac{dw}{a(w)}$ this is equal to $c(r)^{-1}\left(\frac{d^n}{dr^n}\right)c(r)^2$, where $c(r)$ is the analytic function $c(r) = a(s(r))$. Since $c(r)^2$ is an analytic function of r, the series $\sum_{n=0}^{\infty}(1/n!)c(r)^{-1}(d^n/dr^n)c(r)^2 t^n$ is absolutely convergent for r and t sufficiently small. That is, $\sum_{n=0}^{\infty}(k_n(s)/n!)t^n$ is absolutely convergent for s and t sufficiently small, concluding the proof. Notice that this argument shows that $\sum_{n=0}^{\infty}(k_n(s)/n!)t^n$ is absolutely convergent for all complex values of s and t sufficiently close to 0.

COROLLARY 4.1. *Let ξ and η be as in Theorem 2 and let m be a positive integer. Then* $(\mathrm{ad}\,\xi)^n\eta$ *is the sum of $l^n d$ terms of the form* $\left|\sum_{h=1}^{d} p_h(x)(\partial/\partial x_h)\right|$ *such that if*

$$c_n = \sup_{0\leqq j\leqq m,\, x\in K}\left|\frac{\partial}{\partial x_{i_1}}\cdots\frac{\partial}{\partial x_{i_j}}\, p_h(x)\right| \tag{4.6}$$

then the power series with coefficients $c_n/n!$ has a positive radius of convergence.

PROOF. We have seen that $(\mathrm{ad}\,\xi)^n\eta$ is the sum of $l^n d$ terms each of which is the absolute value of a vector field of the form (4.4), where p_h is majorized by k_n. Since p_h is majorized by k_n, the right hand side of (4.6) is bounded by $k_n(0)+Dk_n(0)+\cdots+D^m k_n(0)$. Since $\sum_{n=0}^{\infty}(k_n(s)/n!)\,t^n$ is absolutely convergent for complex values of s and t sufficiently close to 0, $\sum_{n=0}^{\infty}(D^j k_n(s)/n!)t^n$ is absolutely convergent for each j, if s and t are sufficiently small. This proves the corollary.

5. Extensions of operators

At various places in the following sections we shall show that the hypotheses of Corollary 3.2 are satisfied by operators $X_1, \cdots, X_d$, A on a certain domain and we shall also know that the closure $\bar{A}$ of A has a dense set of analytic vectors. (We recall that the closure $\bar{A}$ of an operator A, if it exists, is defined on the domain $\mathfrak{D}(\bar{A})$ consisting of all x such that there is a sequence x_j of vectors in $\mathfrak{D}(A)$ with $x_j \to x$ and $Ax_j \to y$ for some y in $\mathfrak{X}$, in which case $\bar{A}x = y$. In order for this to be well defined, it is necessary and sufficient that whenever x_j are in $\mathfrak{D}(A)$ with $x_j \to 0$ and $Ax_j \to y$, then $y = 0$.) We shall want to conclude that $|\bar{X}_1| + \cdots + |\bar{X}_d|$ has a dense set of analytic vectors. I do not know whether this is true in general. In fact, let A and X be two operators having closures and such that $|A|$ analytically dominates $|X|$. Then it is not even clear that if x is an analytic vector for $\bar{A}$ then x is in the domain of $\bar{X}^2$. However, there are two ways of extending an operator A. Besides the closure, or strong extension, we may consider weak extensions. This theory is simplest for the case of a symmetric operator A on Hilbert space, in which case we mean A^*. This case will suffice for the applications. In Lemma 5.2 we show that in case $\bar{A}=A^*$, so that the two extensions coincide, the desired inequalities do hold for the closures of the operators involved.

See [30] for the theory of symmetric operators. Here we recall the fundamental definitions. An operator A is called *densely defined* in case $\mathfrak{D}(A)$ is dense. An operator A on a Hilbert space $\mathfrak{H}$ is called *symmetric* in case A is densely defined and $(Ax, y) = (x, Ay)$ for all x and y in $\mathfrak{D}(A)$, *skew-symmetric* in case it is densely defined and $(Ax, y) = -(x, Ay)$ for all x and y in $\mathfrak{D}(A)$. The *adjoint* A^* of a densely defined operator A is defined on the domain $\mathfrak{D}(A^*)$ of all y in $\mathfrak{H}$ such that (Ax, y) (defined for x in $\mathfrak{D}(A)$) is a continuous linear functional of x. Since $\mathfrak{D}(A)$ is dense, there is a unique vector A^*y such that $(Ax, y) = (x, A^*y)$ for all x in $\mathfrak{D}(A)$. An operator A is *self-adjoint* in case $A = A^*$, *skew-adjoint* in case $A = -A^*$, *essentialy self-adjoint* in case $\bar{A} = A^*$, and *essentially skew-adjoint* in case $\bar{A} = -A^*$.

The following lemma, and the last statements in Lemma 5.2, will not be used before § 9.

LEMMA 5.1. *Let X be a closed symmetric operator on a Hilbert space. Then X is self-adjoint if and only if it has a dense set of analytic vectors.*

The lemma remains true if " symmetric " is replaced by " skew-symmetric " and " self-adjoint " by " skew-adjoint ", by considering iX (which has the same analytic vectors as X).

PROOF. The necessity of the condition is a trivial consequence of the spectral theorem. In fact, if X is self-adjoint and φ is a Baire function of a real variable such that $|\varphi(\lambda)| \leqq ke^{-c|\lambda|}$ for some $c > 0$, $k < \infty$, then any vector in the range of $\varphi(X)$ is an analytic vector for X. (Conversely, any analytic vector x for X is in the range of $\varphi_c(A)$ for c sufficiently small, where $\varphi_c(\lambda) = e^{-c|\lambda|}$, since we may write $x = \varphi_c(A)\varphi_{-c}(A)x$ for c sufficiently small.) For example, if X has resolution of the identity $E(\cdot)$, and Φ is a bounded Borel set, then any vector in the range of the spectral projection $E(\Phi)$ is an analytic vector for X. Such vectors are dense, by the spectral theorem.

We must prove that the condition is sufficient. For each $s > 0$ let $\mathfrak{H}_s$ be the set of vectors x in $\mathfrak{H}$ such that $\| e^{|X|s}x \| < \infty$. Then $\mathfrak{H}_s$ is a linear sub-space of $\mathfrak{H}$. Let $\mathfrak{H}_0 = \bigcup_{s>0}\mathfrak{H}_s$. Our assumption is that $\mathfrak{H}_0$ is dense in $\mathfrak{H}$.

If x is in $\mathfrak{H}_s$ and $|t| \leqq s$ define

$$U(t)x = \sum_{n=0}^{\infty} \frac{(iX)^n}{n!} xt^n . \tag{5.1}$$

This converges absolutely in $\mathfrak{H}$. We will prove that if x is in $\mathfrak{H}_s$ and $|t_1| + |t_2| \leqq s$ then $U(t_1)x$ is in $\mathfrak{H}_{t_2}$ and

$$U(t_2)U(t_1)x = U(t_2 + t_1)x . \tag{5.2}$$

Let $y = U(t_1)x$, $y_j = \sum_{n=0}^{j}\{(iX)^n/n!\}xt_1^n$. Then y_j is in $\mathfrak{D}(X)$ and $X(y_j - y_k) \to 0$ as $j, k \to \infty$ since x is in $\mathfrak{H}_{t_1}$. Since X is a closed operator, this mean that y is in $\mathfrak{D}(X)$ and $Xy_j \to Xy$. Now $Xy_j \to U(t_1)Xx$, so that $XU(t_1)x = U(t_1)Xx$. Consequently

$$\sum_{k=0}^{l}\frac{(iX)^k}{k!}\Big(\sum_{n=0}^{\infty}\frac{(iX)^n}{n!}xt_1^n\Big)t_2^k = \sum_{n=0}^{\infty}\sum_{k=0}^{l}\frac{(iX)^{k+n}}{k!n!}xt_1^n t_2^k$$

converges absolutely as $l \to \infty$ to

$$\sum_{m=0}^{\infty}\frac{(iX)^m}{m!}x(t_1 + t_2)^m = U(t_1 + t_2)x$$

proving (5.2).

Now if x is in $\mathfrak{H}_s$ and $|t| < s$ then

$$\frac{d}{dt}(U(t)x,\ U(t)x) = 2\,\mathrm{Re}\,\lim_{h\to 0}\Big(\frac{U(t+h) - U(t)}{h}x, U(t)x\Big)$$

$$= 2\,\mathrm{Re}\,\lim_{h\to 0}\Big(\frac{U(h) - 1}{h}U(t)x,\ U(t)x\Big) = 2\,\mathrm{Re}\,\big(iXU(t)x,\ U(t)x\big) = 0$$

since X is symmetric. Therefore for all x in $\mathfrak{H}_s$,

$$\|U(t)x\| = \|x\| \tag{5.3}$$

if $|t| < s$. Consequently

$$\sum_{n=0}^{\infty}\frac{\|X^nU(t)x\|}{n!}s^n = \sum_{n=0}^{\infty}\frac{\|U(t)X^nx\|}{n!}s^n = \sum_{n=0}^{\infty}\frac{\|X^nx\|}{n!}s^n < \infty$$

if x is in $\mathfrak{H}_s$ and $|t| < s$. That is, if $|t| < s$ then $U(t)\mathfrak{H}_s \subset \mathfrak{H}_s$. Now for an arbitrary real number t let the integer m be such that $|t| < ms$ and define $U(t)x$, for x in $\mathfrak{H}_s$, by $U(t)x = \{U(t/m)\}^m x$. Then $U(t)$ is well defined for all real t on $\mathfrak{H}_s$, $U(t)\mathfrak{H}_s \subset \mathfrak{H}_s$, and (5.2) and (5.3) are satisfied for all x in $\mathfrak{H}_s$. This may be done for each $s > 0$, and the resulting definitions of $U(t)x$ are consistent if x is in $\mathfrak{H}_s$ and $\mathfrak{H}_{s'}$, so that $U(t)$ is defined on $\mathfrak{H}_0$, $U(t)\mathfrak{H}_0 \subset \mathfrak{H}_0$, and (5.2), (5.3) are satisfied. Since $\mathfrak{H}_0$ is dense in $\mathfrak{H}$ and $\|U(t)\| = 1$ we may extend $U(t)$ to all of $\mathfrak{H}$, preserving (5.2) and (5.3). In short, there is a one parameter unitary group $U(t)$ on $\mathfrak{H}$ such that (5.1) holds if x is in $\mathfrak{H}_s$ and $|t| \leqq s$.

By construction of $U(t)$, for x in $\mathfrak{H}_0$, $U(t)x$ is a continuous function of t. Since $\mathfrak{H}_0$ is dense and $U(t)$ is uniformly bounded ($\|U(t)\| = 1$), $U(t)x$ is a continuous function of t for all x in $\mathfrak{H}$. That is, the one parameter unitary group $U(t)$ is strongly continuous. By Stone's theorem [29], there is a unique self-adjoint operator Y such that $U(t) = e^{itY}$. We wish to show that $X = Y$. Let X_0 be the restriction of X to $\mathfrak{H}_0$. Then $X_0 \subset Y$ since for each x in $\mathfrak{H}_0$,

$$iYx = \lim_{h\to 0} \frac{U(h) - 1}{h} = iXx \; .$$

Now let Z be an arbitrary self-adjoint extension of X_0 and let $V(t) = e^{itZ}$. Then for all x in $\mathfrak{H}_s$ and $|t| \leqq s$ we have by (5.1) and the fact that Z and X agree on $\mathfrak{H}_s$,

$$U(t)x = \sum_{n=0}^{\infty} \frac{(iZ)^n}{n!} xt^n \; .$$

Since the series is absolutely convergent it must be $V(t)x$, by the spectral theorem. Thus for $|t| \leqq s$, $U(t)$ and $V(t)$ agree on $\mathfrak{H}_s$. Repeating the argument used in the definition of $U(t)$ on $\mathfrak{H}_0$ for all real t, we see that $U(t)$ and $V(t)$ agree on $\mathfrak{H}_0$ and consequently on $\mathfrak{H}$. Therefore $Z = Y$, and X_0 has a unique self-adjoint extension. The means [30, Chap. IX] that the closure of X_0 is self-adjoint. Since X is a closed symmetric extension of X_0, X is self-adjoint (in fact, $X = \bar{X}_0 = Y = Z$), which concludes the proof.

LEMMA 5.2. *Let $X_1, \cdots, X_d$, A be symmetric operators on a Hilbert space $\mathfrak{H}$ with a common invariant domain $\mathfrak{D}$, and suppose that A is essentially self-adjoint.*

Let $\xi = |X_1| + \cdots + |X_d|$, $\alpha = |A| + |I|$, $\xi \leqq c\alpha$ and $(\operatorname{ad} \xi)^n \alpha \leqq c_n \alpha$ with $c < \infty$ and $c_n < \infty$, for all $n \geqq 1$. For all finite sequences $i_1 \cdots, i_n$

$$\mathfrak{D}(\bar{A}^n) \subset \mathfrak{D}(\bar{X}_{i_1} \cdots \bar{X}_{i_n}) \tag{5.4}$$

Let $\tilde{\mathfrak{D}} = \bigcap_{n=1}^{\infty} \mathfrak{D}(\bar{A}^n)$ and let $\tilde{X}_1, \cdots, \tilde{X}_d$, $\tilde{A}$ be the restrictions of $\bar{X}_1, \cdots, \bar{X}_d, \bar{A}$, respectively, to $\tilde{\mathfrak{D}}$. Let $\tilde{\xi} = |\tilde{X}_1| + \cdots + |\tilde{X}_d|$, $\tilde{\alpha} = |\tilde{A}| + |I|$. Then $\tilde{\xi} \leqq c\tilde{\alpha}$, $(\operatorname{ad} \tilde{\xi})^n \tilde{\alpha} \leqq c_n \tilde{\alpha}$ for all $n \geqq 1$.

If α analytically dominates ξ then there is an $s > 0$ such that the set of x in $\tilde{\mathfrak{D}}$ for which $\| e^{\tilde{\xi} s} x \| < \infty$ is dense in $\mathfrak{H}$, and each X_i is essentially self-adjoint.

In the applications the X_i will be skew-symmetric, but this does not alter the validity of the lemma (if "self-adjoint" is replaced by "skew-adjoint" in the last statement), by considering $iX_1, \cdots, iX_d$ instead.

The reason for restricting the closures to $\tilde{\mathfrak{D}}$ is that $\mathfrak{D}(\bar{A})$ need not be included in $\mathfrak{D}(\bar{X}_1 \bar{A} - \bar{A} \bar{X}_1)$, for example. By restricting to $\tilde{\mathfrak{D}}$ it suffices to consider $\overline{X_1 A - A X_1}$ instead. However, any analytic vector for $\bar{A}$ is an analytic vector for $\tilde{A}$, by definition of $\tilde{\mathfrak{D}}$, so there is no loss in considering

$\tilde{A}$.

PROOF. First we will prove (5.4), beginning with the case $n = 1$. If x is in $\mathfrak{D}(\bar{A})$ there is a sequence x_j in $\mathfrak{D}$ with $x_j \to x$ and $Ax_j \to \bar{A}x$. For all all $i = 1, \cdots, d$, $\| X_i(x_j - x_k) \| \leqq c(\| A(x_j - x_k) \| + \| x_j - x_k \|) \to 0$ as $j, k \to \infty$ since $c < \infty$. Therefore x is in $\mathfrak{D}(\bar{X}_i)$. The same argument shows that if $C = \operatorname{ad} X_{i_k} \cdots \operatorname{ad} X_{i_1} A$ then x is in $\mathfrak{D}(\bar{C})$, since $c_k < \infty$.

Instead of proving (5.4) directly we shall prove

$$\mathfrak{D}(\bar{A}^n) \subset \mathfrak{D}(\bar{A}\bar{X}_{i_1} \cdots \bar{X}_{i_{n-1}}) \tag{5.5}$$

This implies (5.4) by the case $n = 1$ above. Suppose that (5.5) holds for n, and let x be in $\mathfrak{D}(\bar{A}^{n+1})$. Since $\bar{A} = A^*$, we need to show that $\bar{X}_{i_1} \cdots \bar{X}_{i_n} x$ is in $\mathfrak{D}(A^*)$; i. e., that $(X_{i_n} \cdots X_{i_1} Ay, x)$ is a continuous linear functional of y in $\mathfrak{D}$. Now $X_{i_n} \cdots X_{i_1} A = AX_{i_n} \cdots X_{i_1} + S$, where $AX_{i_n} \cdots X_{i_1}$ is the term $k = 0$ in (2.8) and S is the remainder. By the induction hypothesis and Lemma 2.1, (Sy, x) is a continuous linear functional of y, and since x is in $\mathfrak{D}(\bar{A}^{n+1})$, A^*x is in $\mathfrak{D}(\bar{A}^n)$ and $(AX_{i_n} \cdots X_{i_1} y, x) = (X_{i_n} \cdots X_{i_1} y, A^*x)$ is by the induction hypothesis a continuous linear functional of y. This proves (5.5).

As a consequence of (5.4), the operators $\tilde{X}_1, \cdots, \tilde{X}_d$ (and $\tilde{A}$) leave $\tilde{\mathfrak{D}}$ invariant. Let x be in $\tilde{\mathfrak{D}}$. Then x is in $\mathfrak{D}(\bar{A})$, and there is a sequence x_j in $\mathfrak{D}$ with $x_j \to x$, $Ax_j \to \bar{A}x$. Then

$$\begin{aligned} \textstyle\sum_{i=1}^d \| \tilde{X}_i x \| &= \textstyle\sum_{i=1}^d \lim_{j \to \infty} \| X_i x_j \| \\ &\leqq \lim_{j \to \infty} c(\| Ax_j \| + \| x_j \|) = c(\| \tilde{A}x \| + \| x \|) \end{aligned}$$

so that $\tilde{\xi} \leqq c\tilde{\alpha}$, and similarly $(\operatorname{ad} \tilde{\xi})^n \tilde{\alpha} \leqq c_n \tilde{\alpha}$.

Now suppose that α analytically dominates ξ, so that the power series $\nu(s)$ and $\kappa(s)$ of Theorem 1 have a positive radius of convergence. Let $E(\cdot)$ be the resolution of the identity for the self-adjoint operator $\bar{A}$, and let $\mathfrak{B}$ be the set of all vectors x such that for some bounded Borel set Φ, $E(\Phi)x = x$. Then $\mathfrak{B} \subset \tilde{\mathfrak{D}}$, $\mathfrak{B}$ is dense in $\mathfrak{H}$, and $\| e^{\tilde{\alpha}t} x \| < \infty$ for all x in $\mathfrak{B}$ and $0 \leqq t < \infty$, by the spectral theorem. If s is small enough so that $\kappa(s) < \infty$, then by Theorem 1, $\| e^{\tilde{\xi}s} x \| \leqq \| e^{c\tilde{\alpha}\kappa(s)} x \| < \infty$ for all x in $\mathfrak{B}$. Any analytic vector for $\tilde{\xi}$ is *a fortiori* an analytic vector for each $\tilde{X}_i$, so that by Lemma 5.1 each $\tilde{X}_i$ is essentially self-adjoint. Since $X_i \subset \tilde{X}_i \subset \bar{X}_i$, each X_i is essentially self-adjoint, concluding the proof.

6. Domination in the enveloping algebra of a Lie algebra

Lemma 6.2 below is the basic estimate used in § 7 and § 9. Throughout this section we shall assume that $\mathfrak{H}$ is a Hilbert space, $\mathfrak{D}$ a linear subspace of $\mathfrak{H}$, $\mathfrak{g}$ a Lie algebra of skew-symmetric operators having $\mathfrak{D}$ as common invariant domain. Notice that the Lie product $[X, Y] = XY - YX$ of two skew-symmetric operators is skew-symmetric. Let $\mathcal{A}$ be the associative algebra, over the field of real numbers, of operators on $\mathfrak{D}$ generated by $\mathfrak{g}$. In other words, $\mathcal{A}$ (or strictly speaking, $\mathcal{A}$ with the identity operator on $\mathfrak{D}$ adjoined if necessary) is the enveloping algebra of $\mathfrak{g}$. It is a homomorphic image of the universal enveloping algebra of $\mathfrak{g}$. An element of $\mathcal{A}$ is said to be of order $\leqq n$ in case it is a real linear combination of operators of the form $Y_1 \cdots Y_k$ with $k \leqq n$ and each Y_i in $\mathfrak{g}$. The set of elements of $\mathcal{A}$ of order $\leqq n$ will be denoted $\mathcal{A}_n$. Let $X_1, \cdots, X_d$ be a basis for $\mathfrak{g}$. Then it is clear that

$$X_i, \quad H_{ij} = X_iX_j + X_jX_i\,; \qquad 1 \leqq i, j \leqq d \tag{6.1}$$

constitutes a set of linear generators for $\mathcal{A}_2$. Let

$$\Delta = X_1^2 + \cdots + X_d^2\,. \tag{6.2}$$

LEMMA 6.1 *If the operator B is in $\mathcal{A}_2$ then for some $k < \infty$, $|B| \leqq k|\Delta - I|$.*

PROOF. It is clearly sufficient to prove this for the operators (6.1). For the X_i we have

$$\sum_{i=1}^d \| X_ix \|^2 = \sum_{i=1}^d (X_ix, X_ix) = (-\Delta x, x)$$
$$\leqq \left(\left(\frac{1}{2}\Delta^2 - \Delta + \frac{1}{2}\right)x, x\right) = \left(\frac{1}{2}(\Delta - I)^2x, x\right) = \frac{1}{2}\|(\Delta - I)x\|^2\,.$$

By the Schwarz inequality for finite sequences,

$$\sum_{i=1}^d \| X_ix \| \leqq \sqrt{\frac{d}{2}}\|(\Delta - I)x\|\,. \tag{6.3}$$

It remains to consider the H_{ij}. We shall use the notation B^+ for the restriction of B^* to $\mathfrak{D}$, if B is in $\mathcal{A}$. Thus $(X_{i_1}\cdots X_{i_n})^+ = (-1)^n X_{i_n}\cdots X_{i_1}$. Let $\mathcal{P}$ be the set of elements in $\mathcal{A}$ of the form $\sum_r Q_r^+ Q_r$ (using finite sums only). Note that $-\Delta$ is in $\mathcal{P}$. Now

$$\begin{aligned} -\Delta + H_{ij} &= (X_i - X_j)^+(X_i - X_j) + \textstyle\sum_{k\neq i,j} X_k^+X_k\,; \qquad i \neq j \\ -2\Delta + H_{ii} &= 2\textstyle\sum_{k\neq i} X_k^+X_k \\ -\Delta - H_{ij} &= (X_i + X_j)^+(X_i + X_j) + \textstyle\sum_{k\neq i,j} X_k^+X_k\,. \end{aligned} \tag{6.4}$$

In all cases, therefore, if the a_{ij} are real and $B = \sum a_{ij}H_{ij}$ then there is an $a \geqq 0$ such that

$$-a\Delta + B \in \mathcal{P}\,. \tag{6.5}$$

Consider now $4\Delta^2 - H_{ij}^2 = (2\Delta - H_{ij})(2\Delta + H_{ij}) + B_1$ where B_1 is in $\mathcal{A}_3$. Now $-(2\Delta - H_{ij}) = \sum_k Y_k^+ Y_k$ with the Y_k in $\mathfrak{g}$ by (6.4), and similarly $-(2\Delta + H_{ij}) = \sum_l Z_l^+ Z_l$ with the Z_l in $\mathfrak{g}$. Therefore

$$4\Delta^2 - H_{ij}^2 = \sum_{k,l} (Y_k Z_l)^+ (Y_k Z_l) + B_2 \tag{6.6}$$

where B_2 is in $\mathcal{A}_3$ (using the fact that if we permute the factors in the terms of highest order of an element of $\mathcal{A}$ the error committed is of lower order). Now $\mathcal{A}_3$ is spanned by the operators (6.1) and those of the form $H_{ijk} = X_i X_j X_k + X_i X_k X_j + X_j X_i X_k + X_j X_k X_i + X_k X_i X_j + X_k X_j X_i$. Therefore we may write $B_2 = \sum a_{ij} H_{ij} + S$, where S is a real linear combination of the X_i and the H_{ijk}. Since the other terms in (6.6) are symmetric, S must be symmetric. But the X_i and H_{ijk} are skew-symmetric, so that S is also skew-symmetric, and consequently $S = 0$. Therefore, by (6.5), there is an $a \geqq 0$ such that $-a\Delta + B_2$ is in $\mathcal{P}$. Hence $4\Delta^2 - H_{ij}^2 - a\Delta$ is in $\mathcal{P}$. Completing the square, we have for all x in $\mathfrak{D}$, $\| H_{ij} x \|^2 \leqq ((4\Delta^2 - a\Delta)x, x) \leqq \| (2\Delta - (a/4))x \|^2$. Letting k be the maximum of $\sqrt{2}$ and $\sqrt{a}/2$, we have $\| H_{ij} x \| \leqq k \| (\Delta - I)x \|$, which concludes the proof.

LEMMA 6.2. *Let $\xi = | X_1 | + \cdots + | X_d |$ and let $\alpha = | \Delta - I |$. Then α analytically dominates ξ. In fact, $\xi \leqq \sqrt{d/2}\ \alpha$ and there is a $c < \infty$ such that for all $n \geqq 1$, $(\mathrm{ad}\ \xi)^n \alpha \leqq c^n \alpha$. Also, $| \Delta | + | I |$ analytically dominates ξ.*

PROOF. The fact that $\xi \leqq \sqrt{d/2}\ \alpha$ is precisely (6.3). Now $\mathcal{A}_2$ is a finite dimensional vector space, of dimension at most $d + d(d+1)/2$, since it is spanned by (6.1). If B is in $\mathcal{A}_2$ we define $||| B |||$ to be the least number k such that $| B | \leqq k\alpha$. By Lemma 6.1 this is always finite, and if $||| B ||| = 0$ then $B = 0$, so that $\mathcal{A}_2$ with this norm is a finite dimensional Banach space. For each $i = 1, \cdots, d$, ad X_i is a linear transformation taking $\mathcal{A}_2$ into itself, so there is a $c_i < \infty$ such that $||| (\mathrm{ad}\ X_i) B ||| \leqq c_i ||| B |||$. Let $c = d \max_i c_i$. Now $(\mathrm{ad}\ \xi)^n \alpha$ is the sum of d^n terms of the form $| \mathrm{ad}\ X_{i_n} \cdots \mathrm{ad}\ X_{i_1} \Delta |$, which is $\leqq c_{i_n} \cdots c_{i_1} \alpha$. Therefore $(\mathrm{ad}\ \xi)^n \alpha \leqq c^n \alpha$. The last statement holds *a fortiori*, concluding the proof.

In § 11 we will need an extension of these lemmas.

LEMMA 6.3. *For all positive integers m, if B is in $\mathcal{A}_{2m}$ then for some $k < \infty$,*

$$| B | \leqq k\alpha^m \tag{6.7}$$

where $\alpha^m = | (\Delta - I)^m |$. If $\eta = | Y_1 | + \cdots + | Y_l |$ where Y_j is in $\mathcal{A}_{2m}$ and ad Y_j *maps $\mathcal{A}_{2m}$ into itself, for $j = 1, \cdots, l$, then α^m analytically dominates η. For some $c < \infty$, $(\mathrm{ad}\ \eta)^n \alpha^m \leqq c^n \alpha^m$ for all $n \leqq 1$.*

PROOF. It suffices to prove (6.7) for operators of the form $B = X_{i_1} \cdots X_{i_s}$ with $s \leqq 2m$, since by definition of $\mathcal{A}_{2m}$ they span $\mathcal{A}_{2m}$. For $m = 1$, (6.7) is true by Lemma 6.1. Suppose it to be true for $m - 1$. Making the convention that if $s = 0$ then $X_{i_1} \cdots X_{i_s} = I$, (6.7) is true for $s = 0$. Suppose it to be true for $s-1$. Then

$$| X_{i_1} \cdots X_{i_s} | \leqq k \, | (\Delta - 1) X_{i_3} \cdots X_{i_s} | \leqq k \, | X_{i_3} \cdots X_{i_s} (\Delta - I) | + k \, | B' |$$

where $B' = (\Delta - I) X_{i_3} \cdots X_{i_s} - X_{i_3} \cdots X_{i_s} (\Delta - I)$ is in A_{s-1}, so that $|B'| \leqq k' \alpha^m$ (with $k, k' < \infty$). Now by the induction hypothesis on m, $| X_{i_3} \cdots X_{i_s} | \leqq \alpha^{m-1}$, so that $| X_{i_1} \cdots X_{i_s} | \leqq k\alpha^m + kk'\alpha^m$, completing the induction on s and therefore on m. This proves (6.7). The last statements are proved just as in the proof of Lemma 6.2, since $\mathcal{A}_{2m}$ is finite dimensional.

It would be interesting to find a better method of studying order properties of the enveloping algebra of a Lie algebra. For example, it was proved in [23] that in a unitary representation of a Lie group an elliptic element A of the enveloping algebra is, if symmetric, essentially self-adjoint on the Gårding space. One would expect that if A has real coefficients it would be semi-bounded, but the authors were unable to prove this. Even in a polynomial algebra (which is the universal enveloping algebra of an abelian Lie algebra) Hilbert showed that a positive element need not be a sum of squares, so that the crude computational method used above will not work except in special cases.

7. Analytic vectors for unitary representations of Lie groups

Let G be a Lie group and $\mathfrak{X}$ a Banach space. By a *representation* T of G is meant a mapping of G into the set of bounded operators with domain $\mathfrak{X}$ such that $T(e) = I$, where e is the identity element of G, $T(\sigma\tau) = T(\sigma)T(\tau)$ for all σ and τ in G, and such that for all x in $\mathfrak{X}$, $\sigma \to T(\sigma)x$ is a continuous mapping of G into $\mathfrak{X}$ (where $\mathfrak{X}$ has the strong, i. e., norm, topology). The representation is called *unitary* in case $\mathfrak{X}$ is a Hilbert space and each $T(\sigma)$ is a unitary operator. An element x of $\mathfrak{X}$ is said to be an *infinitely differentiable vector for* T in case the mapping $\sigma \to T(\sigma)x$ of G into $\mathfrak{X}$ is of class C^∞. The set of all infinitely differentiable vectors will be denoted $\mathfrak{C}$. It is dense in $\mathfrak{X}$. In fact, Gårding showed [12] that if the function φ on G is of class C^∞ and has compact support and if the operator $T(\varphi)$ on $\mathfrak{X}$ is defined by

$$T(\varphi)x = \int_G T(\sigma)x\varphi(\sigma)d\sigma \tag{7.1}$$

where $d\sigma$ is left-invariant Haar measure on G, then for all x in $\mathfrak{X}$, $T(\varphi)x$ is in $\mathfrak{C}$ and the set of such vectors is dense in $\mathfrak{X}$. The set of all finite linear combinations of vectors of the form (7.1) is called the *Gårding space*. Let $\mathfrak{g}$ be the Lie algebra of G and let exp denote the exponential mapping (see [5]). If X is in $\mathfrak{g}$, then $T(X)x$ is defined for all x in $\mathfrak{C}$ by

$$(7.2) \qquad T(X)x = \lim_{h\to 0} \frac{T(\exp hX)x - x}{h} \, .$$

Then $T(X)$ is an operator with $\mathfrak{G}$ as invariant domain. If T is unitary it follows from (7.2) that $T(X)$ is skew-symmetric. Since $\mathfrak{G}$ is invariant under each $T(X)$ for X in $\mathfrak{g}$, T extends to a homomorphism of the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ of $\mathfrak{g}$ onto an algebra of operators with domain $\mathfrak{G}$.

The study of a representation on the enveloping algebra is a powerful tool in the study of Lie group representations. Harish-Chandra pointed out in [15] that for many purposes $\mathfrak{G}$ is the wrong domain for the operators $T(X)$. For example, a subspace of $\mathfrak{G}$ may be invariant under all the $T(X)$ without its closure being invariant under the $T(\sigma)$. For this reason Harish-Chandra introduced the notion of *well-behaved* vectors. Following Cartier and Dixmier [4], we will call them analytic vectors. A vector x in $\mathfrak{X}$ is an *analytic vector for* T in case the mapping $\sigma \to T(\sigma)x$ of G into $\mathfrak{X}$ is analytic.

For this to be a useful notion it is necessary to show that the set of analytic vectors for T is dense in $\mathfrak{X}$. Harish-Chandra [15] showed this to be the case for certain representations of semi-simple Lie groups. Using similar methods, Cartier and Dixmier [4] showed that if T is either bounded or scalar-valued on a certain discrete central subgroup Z of G then the set of analytic vectors for T is dense. In particular, their result includes all unitary representations. This method involves proving the theorem first for certain Lie groups which as analytic manifolds are isomorphic to euclidean space and then using structure theory of Lie groups to extend the result.

In this section we shall give a new proof of the denseness of the analytic vectors for a unitary representation. The proof makes no use of structure theory. Since the proof is entirely infinitesimal, it may be applied to representations of Lie algebras (see § 9). Besides showing that the analytic vectors are dense, Theorem 3 describes a useful set of vectors which is a dense set of analytic vectors. An application of this additional information is made in §11. In §8 it is shown that an arbitrary representation has a dense set of analytic vectors. This proof is by a different, though related, method.

First we need to establish the connection between analytic vectors for a group representation and analytic vectors in the sense of § 2.

LEMMA 7.1. *Let T be a representation of the Lie group G on the Banach space X. Let $X_1, \cdots, X_d$ be a basis for the Lie algebra $\mathfrak{g}$ of G, and let $\xi = | T(X_1) | + \cdots + | T(X_d) |$. Then x in $\mathfrak{X}$ is an analytic vector for T if* *and only if x is an analytic vector for ξ.*

PROOF. Let $x_1, \cdots, x_d$ be analytic coordinates in a neighborhood of e. Since translations by elements of G are analytic isomorphisms, $\sigma \to T(\sigma)x$ is analytic on all of G if and only if it is analytic in a neighborhood of e (see [15, p. 209]). Let $K = \{\sigma \in G : |x_1(\sigma)| \leqq \varepsilon, \cdots, |x_d(\sigma)| \leqq \varepsilon\}$. Then x is an analytic vector if and only if for some $\varepsilon > 0$, $\sigma \to T(\sigma)x$ is an analytic vector in the norm (4.1) for $|\partial/\partial x_1| + \cdots + |\partial/\partial x_d|$.

We shall identify $\mathfrak{g}$ with the space of all vector fields on G which commute with all right translations by elements of G. If X is in $\mathfrak{g}$ and x is in $\mathfrak{E}$ then, by (7.2)

$$T(X)x = XT(\sigma)x|_{\sigma=e} .$$

Therefore x is an analytic vector for $\xi = |T(X_1)| + \cdots + |T(X_d)|$ if and only if for some $\varepsilon > 0$, $\sigma \to T(\sigma)x$ is an analytic vector in the norm (4.1) for $|X_1| + \cdots + |X_d|$. By Theorem 2, the proof is complete.

The sufficiency of the condition, which is all that we shall need, may also be proved without appealing to Theorem 2. Since the exponential mapping is an analytical isomorphism of a neighborhood of 0 in $\mathfrak{g}$ with a neighborhood of e in G, it is sufficient to prove that if x is an analytic vector for ξ then $X \to T(\exp X)x$ is analytic in some neighborhood of 0 in $\mathfrak{g}$. Let $X = X_1t_1 + \cdots + X_dt_d$. Then this occurs if and only if

$$\sum_{n=0}^{\infty} \frac{1}{n!} \sum_{a_1+\cdots+a_d=n} \| \psi_{a_1 \cdots a_d} \| t_1^{a_1} \cdots t_d^{a_d} < \infty \tag{7.3}$$

for $t_1, \cdots, t_d$ sufficiently small, where $\psi_{a_1\cdots a_d}$ is the coefficient of $t_1^{a_1} \cdots t_d^{a_d}$ in the expansion of $e^{X_1t_1+\cdots+X_dt_d}x$. Thus $\| \psi_{a_1\cdots a_d} \|$ is the norm of the sum of several terms whose norms occur in the expansion of $\| e^{\xi s}x \|$, if we take $t_1 = \cdots = t_d = s$. Therefore the left hand side of (7.3) is $\leqq \| e^{\xi s}x \|$, which is $< \infty$ if x is an analytic vector for ξ.

The first statement in the conclusion of the following theorem is due to Cartier and Dixmier [4].

THEOREM 3. *Let U be a unitary representation of the Lie group G on the Hilbert space $\mathfrak{H}$. Then the set of analytic vectors for U is dense in $\mathfrak{H}$.*

Let $X_1, \cdots, X_d$ be a basis for the Lie algebra $\mathfrak{g}$ of G, and let $\Delta = X_1^2 + \cdots + X_d^2$ in the universal enveloping algebra of $\mathfrak{g}$. Then any analytic vector for $\overline{U(\Delta)}$ is an analytic vector for U, and the set of such vectors is dense in $\mathfrak{H}$.

PROOF. Let $\xi = |U(X_1)| + \cdots + |U(X_d)|$, $\alpha = |U(\Delta) - I|$. By Lemma 6.2, α analytically dominates ξ. (This lemma is applicable since the $U(X)$, with X in $\mathfrak{g}$, are skew-symmetric and since $U(\Delta) = U(X_1)^2 + \cdots + U(X_d)^2$).

By [23], $U(\Delta)$ is essentially self-adjoint. By (5.4) in Lemma 5.2, every vector in $\tilde{\mathfrak{E}} = \bigcap_{n=1}^{\infty} \mathfrak{D}(\overline{U(\Delta)}^n)$ is in $\mathfrak{D}(\overline{U(X_{i_1})} \cdots \overline{U(X_{i_n})})$ for all finite sequences $i_1, \cdots, i_n$. Now if X is in $\mathfrak{g}$ then $\mathfrak{D}(\overline{U(X)})$ is simply the set of all vectors x for which the limit in (7.2) (with T replaced by U) exists (this follows from Stone's theorem [29] and Segal's theorem [27] (or [23]) that $U(X)$ is essentially skew-adjoint, for example). Therefore if x is in $\tilde{\mathfrak{E}}$, $U(\sigma)x$ has all partial derivatives at $\sigma = e$. Now $\overline{U(X)}U(\sigma)x = U(\sigma)\overline{U(Y)}x$ where $Y = (\mathrm{Ad}\,\sigma^{-1})x$, so that $U(\sigma)x$ has all partial derivatives at all σ, and x is in $\mathfrak{E}$. That is, $\tilde{\mathfrak{E}} \subset \mathfrak{E}$. Since the reverse inclusion is obvious, $\tilde{\mathfrak{E}} = \mathfrak{E}$, and any analytic vector for $\overline{U(\Delta)}$ is an analytic vector for $U(\Delta)$. Now $\overline{U(\Delta)}$ is self-adjoint and so has a dense set of analytic vectors, by the spectral theorem (see Lemma 5.1). By Corollary 3.1 these are all analytic vectors for ξ, and by Lemma 7.1 they are analytic vectors for U. This concludes the proof.

We sketch here a brief proof, due to W.F. Stinespring and myself, that $U(\Delta)$ is essentially self-adjoint. Since it is semi-bounded from above by 0, it is enough to show that the range $\mathfrak{R}$ of $U(\Delta) - I$ is dense. Suppose that x is orthogonal to $\mathfrak{R}$. This implies (using (7.1)) that the numerical function $(U(\sigma)x, x)$ is a weak solution of the equation $(\Delta - 1)(U(\sigma)x, x) = 0$. By the regularity theorem for elliptic equations (e. g., [24]), it is a solution in the ordinary sense. But $(U(\sigma)x, x)$ is a positive definite function, and so has a maximum at e. But this contradicts the maximum principle unless $x = 0$.

8. The heat equation on Lie groups

Gårding showed [12] that a representation of a Lie group on a Banach space has a dense set of infinitely differentiable vectors by means of the integral $\int_G T(\sigma)x\varphi(\sigma)d\sigma$ where φ is in $C_0^\infty(G)$ (i. e., of class C^∞ on G and having compact support). If φ is an analytic function on G such that it and all its mixed partial derivatives decrease rapidly enough at infinity, the same integral will give analytic vectors (see [15] and [4]). This argument was used by Gelfand [14] in 1939 to show that a one-parameter group of operators has a dense set of analytic vectors. Gelfand considered only bounded representations (as he was investigating spectral properties which depend on boundedness) but his argument holds in general if we choose a suitable kernel. Let

$$P^t x = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} T(\sigma) x e^{-(\sigma^2/4t)} d\sigma \ . \tag{8.1}$$

We must have $\| T(\sigma) \| \leqq k_1 e^{c_1|\sigma|}$ for some $k_1, c_1 < \infty$, so that the integral in (8.1) is absolutely convergent, and $P^t x \to x$ as $t \to 0$. Furthermore, the integral remains absolutely convergent when the kernel is extended to the complex domain, and from this it follows that $P^t x$ is an analytic vector for T. We will extend this argument to an arbitrary Lie group. The kernel in (8.1) is the fundamental solution of the heat equation $\partial u/\partial t = \partial^2 u/\partial \sigma^2$ on the line. We shall use the fundamental solution of the heat equation $\partial u/\partial t = \Delta u$ on a Lie group.

Let G be a connected Lie group, $X_1, \cdots, X_d$ a basis for its Lie algebra, U the left regular representation of G (given by $U(\sigma)f(\tau) = f(\sigma^{-1}\tau)$ for all f in $\mathfrak{L}^2(G)$, where $\mathfrak{L}^2(G)$ is formed with respect to left invariant Haar measure $d\tau$ on G), $\Delta = X_1^2 + \cdots + X_d^2$, and let A be the closure of $U(\Delta)$. Then (by [23]) A is a self-adjoint negative operator. Let P^t be the operator e^{tA}, for $0 < t < \infty$. Any function in the range of P^t is in $\mathfrak{D}(A^n)$ for all n and, since A is elliptic, is equal a. e. to a continuous function (see [24]). By [1] or [20], this implies that P^t is an integral operator with a kernel of Carleman type.

That is, there is a function $p^t(\sigma, \tau)$ such that for all σ, $p^t(\sigma, \cdot)$ is in $\mathfrak{L}^2(G)$ and

$$P^t f(\sigma) = \int_G p^t(\sigma, \tau) f(\tau) d\tau$$

for all f in $\mathfrak{L}^2(G)$ and σ in G. Since the operator $P^t = e^{tA}$ arises from the left regular representation of G, it commutes with all right translations. That is, for all υ in G

$$P^t f(\sigma\upsilon) = \int_G p^t(\sigma, \tau) f(\tau\upsilon) d\tau = \int_G p^t(\sigma, \tau\upsilon^{-1}) f(\tau)\theta(\upsilon^{-1}) d\tau$$

where θ is the modular function. Since we also have

$$P^t f(\sigma\upsilon) = \int_G p^t(\sigma\upsilon, \tau) f(\tau) d\tau$$

for all f in $\mathfrak{L}^2(G)$, $p^t(\sigma, \tau\upsilon^{-1})\theta(\upsilon^{-1}) = p^t(\sigma\upsilon, \tau)$. Therefore if we denote the function of a single variable $p^t(\sigma, e)$ by $p^t(\sigma)$, we have that $p^t(\sigma, \tau) = p^t(\sigma\tau^{-1})\theta(\tau^{-1})$. This gives us

$$P^t f(\sigma) = \int_G p^t(\sigma\tau) f(\tau^{-1}) d\tau = p^t * f(\sigma) \ .$$

Since A is real and self-adjoint, $p^t(\sigma, \tau) = p^t(\tau, \sigma)$, so that letting $\sigma = e$, $p^t(\tau^{-1})\theta(\tau^{-1}) = p^t(\tau)$ and p^t is in $\mathfrak{L}^2(G)$. Since $P^t P^s = P^{t+s}$, $p^t * p^s = p^{t+s}$ so that p^t is in $C(G)$ (continuous on G and vanishing at infinity).

We call p^t the *fundamental solution of the heat equation on* G. It depends on a choice of left versus right and a choice of basis for the Lie algebra. The function p^t has the property that $p^t(\sigma) \geqq 0$ for all σ in G and $\int_G p^t(\sigma)d\sigma = 1$. This follows (e.g., by [16, p. 354, E_9]) from the fact that for all $\lambda > 0$ the resolvent kernel $r_\lambda(\sigma) = \int_0^\infty e^{-\lambda t}p^t(\sigma)dt$ satisfies $r_\lambda(\sigma) \geqq 0$ for all σ in G and $\lambda \int_G r_\lambda(\sigma)d\sigma = 1$, as follows easily by the maximum principle since convolution by r_λ inverts $\lambda - \Delta$. (Or we may observe that G.A. Hunt has shown [17] that there is a unique positivity preserving semigroup generated by some extension of $U(\Delta)$ and since $U(\Delta)$ on $\mathfrak{L}^2(G)$ is essentially self-adjoint, P^t must be that semigroup.)

Let ρ be any left invariant metric giving the topology of G and let $N_r = \{\sigma \in G : \rho(\sigma, e) \leqq r\}$. For any Borel set B in G, we will write $p^t(B)$ for $\int_B p^t(\sigma)d\sigma$.

LEMMA 8.1. *For all $t > 0$ and $c < \infty$ there exists a $k < \infty$ such that*

$$p^s(G - N_r) \leqq ke^{-cr} \tag{8.2}$$

for all $r \geqq 0$ and $s \leqq t$.

The proof we give is based on the theory of Markoff processes. As this subject is not widely known, we include a large amout of explanatory material after the proof. A purely analytical proof of the lemma could presumably be constructed, perhaps showing first that the resolvent kernel r_λ decreases faster than any given exponential if λ is sufficiently large, but the present proof is conceptually very simple.*

PROOF. We consider the diffusion process starting at e with sub-stochastic transition density function $e^{-\lambda t}p^t(\sigma^{-1}\tau)$ (so that λdt is the probability of the particle being killed in the time interval dt, where λ is a positive constant). With probability one the sample paths are continuous until the particle is killed, since Δ is a local operator. Now $e^{-\lambda t}p^t(G - N_r)$ is the probability that the particle is in $G - N_r$ at time t. Let $q(\lambda, t, r)$ be the probability that the particle is in $G - N_r$ at some time s with $0 \leqq s \leqq t$, so that for all $s \leqq t$ and $r \geqq 0$,

$$e^{-\lambda s}p^s(G - N_r) \leqq q(\lambda, t, r) . \tag{8.3}$$

Since the particle travels continuously before being killed, to go distance $r + r'$ it must first travel distance r and then travel distance at least r' from that point, by the triangle inequality. Since both the metric and the transition probabilities are invariant under left translations, $p^t((\upsilon\sigma)^{-1}\upsilon\tau) =$

* *Added in proof.* Gårding has found a simpler proof of the theorem of this section, to appear in Kungl. Fysiografiska Sällskapets i Lund Förhandlingar, using the fact that Lemma 8.1 is essentially a special case of results proved by M. P. Gaffney in *The conservation property of the heat equation on Riemannian manifolds*, Comm. Pure Appl. Math. 12 (1959), 1-11.

$p^t(\sigma^{-1}\tau)$, this implies (using the strong Markoff property) that

$$q(\lambda, t, r + r') \leqq q(\lambda, t, r)q(\lambda, t, r') . \tag{8.4}$$

In particular, for all $a \geqq 0$ and positive integers n, $q(\lambda, t, na) \leqq q(\lambda, t, a)^n$, and so for all $r \geqq 0$,

$$q(\lambda, t, r) \leqq q\left(\lambda, t, \left[\frac{r}{a}\right]a\right) \leqq q(\lambda, t, a)^{(r/a)-1} \tag{8.5}$$

Choose $a > 0$ so that $p^t(G - N_a) > 0$ (actually, this is true for any $a > 0$ such that $G - N_a$ is non-empty, since G is assumed to be connected). Then by (8.3), $q(\lambda, t, a) > 0$ for all $\lambda < \infty$. Now

$$q(\lambda, t, a) \to 0 \text{ as } \lambda \to \infty \tag{8.6}$$

so that for any $c < \infty$, if λ is sufficiently large, $q(\lambda, t, a)^{1/a} \leqq e^{-c}$. By (8.5), $q(\lambda, t, r) \leqq q(\lambda, t, a)^{-1}e^{-cr}$ and by (8.3) if we let $k = e^{\lambda t}q(\lambda, t, a)^{-1}$ then (8.2) holds, as was to be proved.

In greater detail, let $\bar{G}$ be the one point compactification of G by the point ∞, let

$$\Omega = \prod\nolimits_{0 \leqq t < \infty} \bar{G}$$

so that Ω is a compact Hausdorff space in the product topology, and let $\xi_t(\omega) = \omega(t)$ for all ω in Ω and $0 \leqq t < \infty$. Then there is a unique regular Borel probability measure Pr on Ω (see [22]) such that the ξ_t are a Markoff process (in the sense of Doob [7, p. 80]) with stochastic transition function [7, p. 256]

$$\begin{aligned} p_\lambda^t(\sigma, E) &= e^{-\lambda t}p^t(\sigma^{-1}(E \cap G)) + (1 - e^{-\lambda t})\chi_E(\infty) , \qquad \sigma \neq \infty \\ p_\lambda^t(\infty, E) &= \chi_E(\infty) \end{aligned}$$

(where E is a Borel set in $\bar{G}$ and χ_E is its characteristic function) and with initial probability distribution Pr $(\{\omega : \xi_0(\omega) = e\}) = 1$. Let Δ_∞ be the set of all ω in Ω such that $\xi_t(\omega)$ is a continuous function of t for all $0 \leqq t < \infty$ except possibly for one value $t_\infty(\omega)$ of t, and such that $\xi_t(\omega) \in G$ for $t < t_\infty(\omega)$ and $\xi_t(\omega) = \infty$ for $t > t_\infty(\omega)$. Then $\Pr(\Delta_\infty) = 1$, as the proof of Theorem 3 in [21] shows, or by [25]. Let

$$\Gamma(t, r) = \bigcup\nolimits_{0 \leqq s \leqq t} \{\omega : \xi_s(\omega) \in G - N_r\} .$$

Since $\Gamma(t, r)$ is a union (albeit an uncountable one) of open sets, $\Gamma(t, r)$ is open, and $q(\lambda, t, r) = \Pr(\Gamma(t, r))$ is well defined. By definition,

$$e^{-\lambda t}p^t(G - N_r) = \Pr(\{\omega : \xi_t(\omega) \in G - N_r\}) ,$$

and since $\{\omega : \xi_t(\omega) \in G - N_r\} \subset \Gamma(t, r)$, (8.3) is clear.

Let $\tau_r(\omega) = \inf\{t : \xi^t(\omega) \in G - N_r\}$ if $\{t : \xi_t(\omega) \in G - N_r\}$ is non-empty, $\tau_r(\omega) = \infty$ otherwise. Then the random variable τ_r, $0 \leqq \tau_r \leqq \infty$, is a stopping time in the sense of R. M. Blumenthal [3]. Since our process satisfies the regularity conditions in the hypothesis of Theorem 1.1 of [3], τ_r is a Markoff time, which means that $\xi'_t(\omega) = \xi_{t+\tau_r(\omega)}(\omega)$ is also a Markoff process with the same transition probabilities as the original process, and independent of events before the time τ_r in the sense defined in [3]. Now by the triangle inequality,

$$\Delta_\infty \cap \Gamma(t, r + r') \subset \Gamma(t, r) \cap \Delta_\infty \cap \Gamma$$

where

$$\Gamma = \bigcup_{0 \leqq s \leqq t} \{\omega : \xi_{\tau_r(\omega)}(\omega) \in G, \xi_{s+\tau_r(\omega)}(\omega) \in G, \rho(\xi_{s+\tau_r(\omega)}(\omega), \xi_{\tau_r(\omega)}(\omega)) > r'\}.$$

By the strong Markoff property (Theorem 1.1 of [3]), $q(\lambda, t, r + r') \leqq q(\lambda, t, r)\Pr(\Gamma)/\Pr(\{\omega : \tau_r(\omega) < \infty\})$ since $\Pr(\Gamma)/\Pr(\{\omega : \tau_r(\omega) < \infty\})$ is the conditional probability, given that $\tau_r(\omega) < \infty$, of Γ. (Notice that $\Gamma \subset \{\omega : \tau_r(\omega) < \infty\}$ since $\xi_{\tau_r(\omega)}(\omega) \in G$ for all ω in Γ.) By the invariance under left translations of the metric ρ and the stochastic transition function defining Pr, $\Pr(\Gamma) = \Pr(\Gamma(t, r')) \cdot \Pr(\{\omega : \tau_r(\omega)) < \infty\})$, so that (8.4) holds.

The remaining point which may need clarification is (8.6). First, if $s \to 0$ then $q(0, s, a) \to 0$ (in fact, it is shown in [21] that $q(0, s, a) = o(s)$. Thus for any $\xi > 0$ and $t > 0$, there is an s, with $0 < s \leqq t$, such that $q(0, s, a) \leqq \varepsilon$, and since $q(\lambda, s, a)$ is a decreasing function of λ, $q(\lambda, s, a) \leqq \varepsilon$ for all $0 \leqq \lambda < \infty$. Now if ω is in Δ_∞ and $\Gamma(t, a)$ but not $\Gamma(s, a)$, then $\xi_s(\omega) \in G$. That is, $q(\lambda, t, a) \leqq q(\lambda, s, a) + e^{-\lambda s}p^s(G) \leqq \varepsilon + e^{-\lambda s}$. Therefore $\lim_{\lambda \to \infty} q(\lambda, t, a) \leqq \varepsilon$ for all $\varepsilon > 0$, and (8.6) holds.

The function $p^t(\tau)$ satisfies a parabolic partial differential equation with analytic coefficients, namely $(\Delta - \partial/\partial t)p^t(\tau) = 0$. By results of Eidelman [9], [10, pp. 86–89], or Avner Friedman [11], $p^t(\tau)$ is an analytic function of τ, for each $t > 0$. (Also, for any $t > 0$, $p^{2t} = p^t * p^t$ is in the range of P^t and since A is a negative operator, is an analytic vector for A by the spectral theorem. By Theorem 3, p^{2t} is analytic vector for the left regular representation U. Therefore

$$\begin{aligned} p^{4t}(\tau) = p^{2t} * p^{2t}(\tau) &= \int_G p^{2t}(\tau\sigma^{-1})\theta(\sigma^{-1})p^{2t}(\sigma)d\sigma \\ &= \int_G p^{2t}(\tau\sigma^{-1})\theta(\tau\sigma^{-1})\theta(\tau^{-1})p^{2t}(\sigma)d\sigma \\ &= \int_G p^{2t}(\tau\sigma)\theta(\tau\sigma)\theta(\tau^{-1})p^{2t}(\sigma^{-1})\theta(\sigma^{-1})d\sigma \\ &= \theta(\tau^{-1})(U(\tau^{-1})p^{2t}, p^{2t}) = \theta(\tau^{-1})(U(\tau)p^{2t}, p^{2t}) . \end{aligned}$$

Since p^{2t} is an analytic vector for U, p^{2t} is in $\mathfrak{L}^2(G)$, and $\theta(\tau^{-1})$ is an analytic function of τ, $p^{4t}(\tau)$ is an analytic function of τ for all $t>0$, i.e., $p^t(\tau)$ is an analytic function of τ for all $t>0$. This fact also follows from Theorem 8 in § 12. Let $x_1, \cdots, x_d$ be analytic coordinates at e, for $\varepsilon > 0$ let V_ε be the set of points in G whose coordinates satisfy $|x_1| \leqq \varepsilon, \cdots, |x_d| \leqq \varepsilon$, and let $\tilde{V}_\varepsilon$ be the set of n-tuples of complex numbers $z_k = x_k + iy_k$ such that $|z_1| \leqq \varepsilon, \cdots, |z_d| \leqq \varepsilon$. We identify V_ε with the subset of $\tilde{V}_\varepsilon$ consisting of all $(z_1, \cdots, z_d)$ in $\tilde{V}_\varepsilon$ with $y_1 = \cdots = y_d = 0$. Let $t_0 > 0$. The estimates used in showing that $p^t(\tau)$ is analytic are uniform in $t \geqq t_0$ (for example, in the above deduction of this fact from Theorem 3), so that there is an $\varepsilon > 0$ and an $M < \infty$ such that for each $t \geqq t_0$ there exists a complex analytic function defined on $\tilde{V}_\varepsilon$, bounded in absolute value by M, and agreeing with p^t on V_ε. We shall denote this function by the same symbol p^t, so that $p_t(\tilde{\tau})$ is its value for $\tilde{\tau}$ in $\tilde{V}_\varepsilon$. Also, if ε is sufficiently small then the mapping on $V_\varepsilon \times V_\varepsilon$ given by $(\tau, \sigma) \to \tau\sigma$ extends to a complex analytic mapping on $\tilde{V}_\varepsilon \times \tilde{V}_\varepsilon$ taking values in $\tilde{V}_\delta$ for some $\delta > \varepsilon$. We shall again denote this mapping by juxtaposition: $(\tilde{\tau}, \tilde{\sigma}) \to \tilde{\tau}\tilde{\sigma}$, and shall assume ε sufficiently small so that it is defined.

We have now derived a quantitative estimate on the decrease of $p^t(\tau)$ for τ near infinity and the qualitative fact that it is an analytic function. We need to combine these two and have quantitative information about the analyticity of $p^t(\tau)$ — the behavior of all its spatial derivatives at infinity. I am grateful to Professor Gårding for showing me how to do this.

Let I be an open interval on the real axis, and let u be a function defined on $I \times V_\varepsilon$ and satisfying the heat equation

$$\left(\Delta_\tau - \frac{\partial}{\partial t}\right) u(t, \tau) = 0 \tag{8.7}$$

there. (A subscript on Δ indicates the variable on which it operates.) We wish to obtain a complex analytic extension of u and obtain a bound for it. The fact that we have a fundamental solution of the heat equation which is analytic in τ enables us to do this, as follows.

Let us define $p^s(\sigma) = 0$ if $s \leqq 0$. Then on $R \times G$, $(\Delta_\sigma - \partial/\partial s)p^s(\sigma) = \delta$ in the sense of distributions, where δ is the distribution $\varphi \to \varphi(0, e)$. Therefore convolution on $R \times G$ by $p^s(\sigma)$ inverts $\Delta_\sigma - \partial/\partial s$. Convolution by $p^s(\sigma)$ is integration with respect to the kernel $p^{s-t}(\sigma\tau^{-1})\theta(\tau^{-1})$. Now the adjoint operator to $\Delta_\sigma - \partial/\partial s$ is $\Delta_\sigma + \partial/\partial s$, since Δ_σ is symmetric. Therefore $\Delta_\sigma + \partial/\partial s$ is inverted by integration with the kernel $p^{t-s}(\tau\sigma^{-1})\theta(\sigma^{-1})$. That is, if φ is in $C_0^\infty(R \times G)$ then

$$(8.8)\qquad \Big(\Delta_\sigma + \frac{\partial}{\partial s}\Big)\int_G\int_R p^{t-s}(\tau\sigma^{-1})\theta(\sigma^{-1})\varphi(t, \tau)dtd\tau = \varphi(\sigma, s)$$

Let I' be an open subinterval of I whose closure is contained in I, let $0 < \varepsilon' < \varepsilon$, and let ψ be in $C_0^\infty(I \times V_\varepsilon)$ and be identically 1 on $I' \times V_{\varepsilon'}$. Then for all (t, τ) in $I' \times V_{\varepsilon'}$, if u satisfies (8.7),

$$(8.9)\quad u(t, \tau) = \int_I\int_{V_\varepsilon} u(s, \sigma)\Big(\Delta_\sigma + \frac{\partial}{\partial s}\Big)\{p^{t-s}(\tau\sigma^{-1})\theta(\sigma^{-1})(1 - \psi(s, \sigma))\}d\sigma ds\,.$$

This is an instance of Gårding's formula (3) in [13]. It is proved by computing the inner product of both sides of (8.9) with an arbitrary function φ in $C_0^\infty(I' \times V_{\varepsilon'})$, using (8.8). Now the integrand of (8.9) is 0 except on $I \times V_\varepsilon - I' \times V_{\varepsilon'}$ because $1 - \psi(s, \sigma) = 0$ on $I' \times V_{\varepsilon'}$. (Gårding's formula is a "smeared-out" version of the Poisson formula, expressing a solution of the partial differential equation in terms of its values in the neighborbood of the boundary.) Let I'' consists of all t in I' whose distance from the complement of I' is at least t_0. Then if t is in I'' only values of s with $t - s \geqq t_0$ contribute to (8.9). Hence there is a constant M' such that for all t in I'' and $\tilde\tau$ in $\tilde V_\varepsilon$,

$$\Big|\Big(\Delta_\sigma + \frac{\partial}{\partial s}\Big)\{p^{t-s}(\tilde\tau\sigma^{-1})\theta(\sigma^{-1})(1 - \psi(s, \sigma))\}\Big| \leqq M'.$$

If u is a solution of (8.7) then (8.9) with τ replaced by $\tilde\tau$ defines a complex analytic extension of $u(t, \cdot)$ to $\tilde V_{\varepsilon'}$ for all t in I'', such that

$$|u(t, \tilde\tau)| \leqq M'\int_I\int_{V_\varepsilon} |u(s, \sigma)|\, d\sigma\, ds\,.$$

Any right translate $p^t(\tau\sigma)$ of $p^t(\tau)$ satifies (8.7) since Δ_τ commutes with all right translations. Therefore for any fixed $t > 0$ if $0 < t_1 < t$ there is an $M' < \infty$ and an $\varepsilon > 0$ such that for all σ in G, $p^t(\tau\sigma)$ has a complex analytic extension $p^t(\tilde\tau\sigma)$ defined for $\tilde\tau$ in $\tilde V_\varepsilon$ and satisfying

$$(8.10)\qquad |\, p^t(\tilde\tau\sigma)\,| \leqq M'\int_{t_1}^t\int_{V_\varepsilon} p^s(\sigma_1\sigma)d\sigma_1\, ds\,.$$

(Notice that we have not defined the expression $\tilde\tau\sigma$ for arbitrary σ in G, but merely use the notation $p^t(\tilde\tau\sigma)$ for the extension of $p^t(\tau\sigma)$ to $\tilde\tau$ in $\tilde V_\varepsilon$.)

LEMMA 8.2. *Let* $t > 0$ and $c < \infty$. *Then there exists an* $\varepsilon > 0$ *and a* $\tilde k < \infty$ *such that for all* σ *in* G, $p^t(\tilde\tau\sigma)$ *is a complex analytic function of* $\tilde\tau$ *in* $\tilde V_\varepsilon$ *agreeing with* $p^t(\tau\sigma)$ for $\tilde\tau$ in V_ε *and satisfying*

$$\int_{G-N_r} |\, p^t(\tilde\tau\sigma)\,|\, d\sigma \leqq \tilde k e^{-cr}$$

for all $\tilde{\tau}$ in $\tilde{V}_\varepsilon$ and $r \geqq 0$.

PROOF. Let ε be sufficiently small so that (8.10) holds, $V_\varepsilon^{-1} \subset N_1$, and the volume of V_ε is less than 1. Then $\sigma_2 = \sigma_1\sigma$ is in $G - N_{r-1}$ if σ_1 is in V_ε and σ is in $G-N_r$, since $r<\rho(e, \sigma)\leqq\rho(e, \sigma_1^{-1})+\rho(\sigma_1^{-1}, \sigma)\leqq 1+\rho(e, \sigma_1 \sigma)$. We have

$$\int_{G-N_r} | p^t(\tilde{\tau}\sigma) | \, d\sigma \leqq M' \int_{G-N_r} \int_{t_1}^{t} \int_{V_\varepsilon} p^s(\sigma_1\sigma)d\sigma_1 \, ds \, d\sigma$$

$$\leqq M' \int_{t_1}^{t} \int_{G-N_{r-1}} p^s(\sigma_2)d\sigma_2 \leqq M'(t - t_1)ke^c e^{-cr}$$

by Lemma 8.1, which proves the lemma.

We shall now assume that the metric ρ is the geodesic metric of some left-invariant Riemannian metric on G. Such a metric has the property that N_1 is compact and if σ is in N_r then there exist $\sigma_1, \sigma_2, \cdots, \sigma_{[r+1]}$ in N_1 with $\sigma_1\sigma_2 \cdots \sigma_{[r+1]} = \sigma$.

LEMMA 8.3. *Let T be a representation of a Lie group G on a Banach space $\mathfrak{X}$. Then there exist k_1, $c_1 < \infty$ such that for all $r>0$ and σ in N_r,*

$$|| T(\sigma) || \leqq k_1 e^{c_1 r} \tag{8.11}$$

PROOF. We have $|| T(\sigma) || \leqq || T(\sigma_1) T(\sigma_2) \cdots T(\sigma_{[r+1]}) || \leqq a^{[r+1]}$ where $a = \sup_{\sigma \in N_1} || T(\sigma) ||$. Since T is strongly continuous and N_1 is compact, $\sup_{\sigma \in N_1} || T(\sigma)x || < \infty$ for all x in $\mathfrak{X}$. The principle of uniform boundedness [16, p. 26] states that $a<\infty$. Letting $k_1=a$, $c_1=\log a$ we have (8.11).

THEOREM 4. *Let T be a representation of a Lie group G on a Banach space $\mathfrak{X}$. Then T has a dense set of analytic vectors in $\mathfrak{X}$.*

Let $p^t(\sigma)$ be the fundamental solution of the heat equation on the connected component of the identity G_0, $p^t(\sigma) = 0$ for σ not in G_0. For all x in $\mathfrak{X}$ and $t > 0$

$$P^t x = \int_G p^t(\sigma)T(\sigma)x d\sigma \tag{8.12}$$

exists and is an analytic vector for T. As $t \to 0$, $P^t x \to x$.

PROOF. To show that the integral in (8.12) exists, observe that by Lemmas 8.1 and 8.3,

$$\int_G p^t(\sigma) \, || T(\sigma)x || \, d\sigma \leqq \sum_{n=0}^{\infty} \int_{N_{n+1}-N_n} p^t(\sigma) \, || T(\sigma) || \, \, || x || \, d\sigma$$

$$\leqq \sum_{n=0}^{\infty} ke^{-c(n+1)} k_1 e^{c_1 n} \, || x ||$$

which is finite if we choose $c > c_1$. Since this also holds for all $s \leqq t$, by Lemma 8.1, we have by the Lebesgue dominated convergence theorem that for all $r > 0$, $\lim_{s\to 0}\int_{G-N_r} p^s(\sigma)\,\|\,T(\sigma)x\,\|\,d\sigma = 0$, so that $P^s\,x \to x$ as $s \to 0$. In the same way, if $t>0$ then by Lemmas 8.2 and 8.3, for all $\tilde{\tau}$ in a neighborhood $\widetilde{V}_\varepsilon$, and for all x in $\mathfrak{X}$, the integral

$$\int_G p^t(\tilde{\tau}\sigma)T(\sigma)x d\sigma \tag{8.13}$$

is absolutely convergent. Since $p^t(\tilde{\tau}\sigma)$ is complex analytic in $\tilde{\tau}$, (8.13) is complex analytic in τ. But for $\tilde{\tau} = \tau$ in G, (8.13) is equal to $T(\tau^{-1})P^t x$, so that $\tau \to T(\tau^{-1})P^t x$ is analytic for τ in a neigborhood V_ε of e. Since $\tau \to \tau^{-1}$ is analytic, $\tau \to T(\tau)P^t x$ is analytic in a neighborhood of e, and so $P^t x$ is an analytic vector for T, concluding the proof.

Our method of proving that a group of operators has a dense set of analytic vectors has used as an intermediary a one-parameter semigroup of operators, the one associated with the heat equation. E. Hille and R.S. Phillips raise the question in [16, p. 310, (p. 229 in the first edition)] as to whether every strongly continuous one-parameter semigroup of operators has a dense set of analytic vectors. The following example answers this question in the negative (cf., Lemma 5.1).

Define T^t, for $0 \leqq t \leqq \infty$, on $\mathfrak{L}^2(0, \infty)$ by

$$\begin{aligned} T^t u(x) &= u(x-t)\,, & x &\geqq t \\ T^t u(x) &= 0\,, & 0 &\leqq x < t. \end{aligned}$$

Thus T^t is simply translation to the right by t, and is a strongly continuous semigroup. Suppose that u is an analytic vector for T^t and let v be in $\mathfrak{L}^2(0, \infty)$. Then $(T^t u, v)$ is an analytic function of t for $0 < t < \infty$. If v has compact support then $(T^t u, v) = 0$ for t sufficiently large, and so $(T^t u, v) = 0$ for all t, $(u, v) = 0$, and so u must be 0. That is, the only analytic vector for this semigroup is 0.

9. Lie algebras of operators

Suppose we have a representation of a Lie algebra $\mathfrak{g}$ by skew-symmetric operators defined on a common invariant domain in a Hilbert space $\mathfrak{H}$, and let G be the simply connected Lie group with Lie algebra $\mathfrak{g}$. In this section we shall give an answer to the question : When does the representation of $\mathfrak{g}$ come from a unitary representation of G ?

This question first arose in the special case of skew-symmetric operators P, Q satisfying the commutation relation $PQ - QP \subset i$ (or for a finite number n of degrees of freedom $P_j Q_k - G_k P_j \subset \delta_{jk}\, i$, for $1 \leqq j, k \leqq n$). Both the matrices of Heisenberg and the Schroedinger operators $P = d/dx$ and $Q =$ multiplication by $i\ x$ on $\mathfrak{L}^2(R)$ satisfy this relation, and

Dirac showed their equivalence. The example of d/dx and ix on $\mathfrak{L}^2(0, 1)$ shows that some condition on the operators is required. Weyl [31] expressed the commutation relation in a bounded form equivalent to postulating a unitary representation of the Lie group G whose Lie algebra has a basis X, Y, Z satisfying $[X, Y] = Z, [X, Z] = 0, [Y, Z] = 0$, and von Neumann showed that there is only one irreducible unitary representation (up to unitary equivalence) of G in which Z goes into i. The problem of finding purely infinitesimal conditions of P and Q was solved by Rellich [26] who showed that if the energy operator $P^2 + Q^2$ is essentially self-adjoint then P and Q are what they should be. Theorem 5 is in a way an extension of Rellich's result to arbitrary Lie algebras, but the method of proof is quite different. Also, Rellich made the stronger assumption that the range of every $E(\Phi)$, where $E(\cdot)$ is the solution of the identity for $\overline{P^2+Q^2}$ and Φ is a bounded interval, is contained in the (invariant) domain of P and Q.*

First we describe the connection with analytic vectors.

LEMMA 9.1. *Let* $\mathfrak{g}$ *be a Lie algebra of skew-symmetric operators on a Hilbert space* $\mathfrak{H}$ *having a common invariant domain* $\mathfrak{D}$. *Let* $X_1, \cdots, X_d$ *be a basis for* $\mathfrak{g}$, $\xi = |X_1| + \cdots + |X_d|$. *If for some* $s > 0$ *the set of vectors* x *in* $\mathfrak{D}$ *such that* $\| e^{\xi s}x \| < \infty$ *is dense in* $\mathfrak{H}$, *then there is on* $\mathfrak{H}$ *a unique unitary representation* U *of the simply connected Lie group* G *having* $\mathfrak{g}$ *as its Lie algebra such that for all* X *in* $\mathfrak{g}$, $\overline{U(X)} = \bar{X}$.

PROOF. By Lemma 5.1, if X is in $\mathfrak{g}$ then $i\bar{X}$ is self-adjoint, since any analytic vector for ξ is an analytic vector for X. Let exp be the exponential mapping in the sense of the theory of Lie groups [5] and let N be a neighborhood of e in G such that exp is one-to-one from a neighborhood of 0 in $\mathfrak{g}$ to N. For $\sigma = \exp X$ in N, define $U(\sigma)$ to be the unitary operator $e^{\bar{X}}$.

Let X, Y, and Z be in $\mathfrak{g}$ and suppose that $(\exp X)(\exp Y) = \exp Z$ in G. Then [5, p. 121] the two power series $\sum_{n=0}^{\infty} (1/n!)Z^n$ and $\sum_{k,l=0}^{\infty} (1/k!l!)X^k Y^l$ are formally equal. Consequently if x is a vector such that $\|e^{|X|+|Y|}x\| < \infty$, $\| e^{|Z|}x \| < \infty$ then $U(\exp X)\, U(\exp Y)x = U(\exp Z)x$. Now for X, Y, and Z sufficiently close to 0 in $\mathfrak{g}$ (such that the absolute values of their coordinates with respect to the basis $X_1, \cdots, X_d$ are less than $(1/2)s$) there is by hypothesis a dense set of vectors x such that $\| e^{|X|+|Y|} x \| < \infty$, $\| e^{|Z|}x \| < \infty$. Therefore, if σ and τ are sufficiently close to e in G, $U(\sigma)U(\tau) = U(\sigma\tau)$. That is, U in a neighborhood of e defines a unitary representation of the local group. (Strong continuity follows from the

* *Added in proof.* This restriction was removed by J. Dixmier, *Sur la relation* $i(PQ - QP) = 1$, Comp. Math. 13 (1958), 263–269.

fact that if $\| e^{\xi s} x \| < \infty$ then $\| e^{\bar{X}} x - x \| \to 0$ as $X \to 0$ in $\mathfrak{g}$, and it is sufficient to verify strong continuity on a dense set, since $U(\sigma)$ is uniformly bounded, $\| U(\sigma) \| = 1$.) Since G is simply connected, there is a unique extension of U to be a unitary representation of G on $\mathfrak{H}$, concluding the proof.

This lemma is not a strict analogue of Lemma 5.1, since in this lemma we assume that for some fixed $s > 0$, $\mathfrak{H}_s = \{x : \| e^{\xi s} x \| < \infty\}$ is dense. In Lemma 5.1 we did not need to assume this because each $\mathfrak{H}_s = \{x : \| e^{|X| s} x \| < \infty\}$ was invariant under $U(t)$, using the fact that the various powers of X all commute. I do not know whether Lemma 9.1 remains true if we merely assume the existence of a dense set of analytic vectors for ξ, but we shall not need such a result.

THEOREM 5. *Let $\mathfrak{g}$ be a Lie algebra of skew-symmetric operators on a Hilbert space $\mathfrak{H}$ having a common invariant domain $\mathfrak{D}$. Let $X_1, \cdots, X_d$ be a basis for $\mathfrak{g}$, $\Delta = X_1^2 + \cdots + X_d^2$. If Δ is essentially self-adjoint then there is on $\mathfrak{H}$ a unique unitary representation U of the simply connected Lie group G having $\mathfrak{g}$ as its Lie algebra such that for all X in $\mathfrak{g}$, $\overline{U(X)} = \bar{X}$.*

PROOF. Let $\xi = |X_1| + \cdots + |X_d|$. By Lemma 6.2, $|\Delta| + |I|$ analytically dominates ξ. The theorem follows by Lemma 5.2 and Lemma 9.1.

In Theorem 5 we assumed that $\mathfrak{D}$ was invariant under $\mathfrak{g}$ and constructed a group representation by means of analytic vectors. Roughly speaking, the theorem concerns the passage from C^∞ to C^ω. The following refinement of the theorem makes only C^2 assumptions, so to speak.

COROLLARY 9.1. *Let $\mathfrak{g}$ be a real Lie algebra, $\mathfrak{H}$ a Hilbert space. For each X in $\mathfrak{g}$ let $\rho(X)$ be a skew-symmetric operator on $\mathfrak{H}$. Let $\mathfrak{Q}$ be a dense linear subspace of $\mathfrak{H}$ such that for all X, Y in $\mathfrak{g}$, $\mathfrak{Q}$ is contained in the domain of $\rho(X)\rho(Y)$. Suppose that for all X, Y in $\mathfrak{g}$, x in $\mathfrak{Q}$, and real numbers a and b,*

$$\rho(aX + bY)x = a\rho(X)x + b\rho(Y)x$$
$$\rho([X, Y])x = (\rho(X)\rho(Y) - \rho(Y)\rho(X))x$$

Let $X_1, \cdots, X_d$ be a basis for $\mathfrak{g}$. If the restriction A of $\rho(X_1)^2 + \cdots + \rho(X_d)^2$ to $\mathfrak{Q}$ is essentially self-adjoint, then there is on $\mathfrak{H}$ a unique unitary representation U of the simply connected Lie group G having $\mathfrak{g}$ as its Lie algebra such that for all X in $\mathfrak{g}$, $\overline{U(X)} = \overline{\rho(X)}$.

PROOF. First, for each n, $\mathfrak{D}(\bar{A}^n) \subset \mathfrak{D}(\overline{\rho(X_{i_1})} \cdots \overline{\rho(X_{i_n})})$. The proof of this exactly parallels the proof of (5.4) in Lemma 5.2 except that instead of considering (ad $\rho(X))A$, which might have only 0 in its domain, we consider $\rho((\text{ad } X)\Delta)$, where $\Delta = X_1^2 + \cdots + X_d^2$ in the universal enveloping algebra of $\mathfrak{g}$. This is well-defined on $\mathfrak{Q}$, since $(\mathrm{ad}\, X)\Delta$ is a linear combination of the $X_i X_j$. Define $\tilde{\mathfrak{D}} = \bigcap_{n=1}^{\infty} \mathfrak{D}(\bar{A}^n)$, $\tilde{A}$ the restriction of $\bar{A}$ to $\tilde{\mathfrak{D}}$, and X_i the restriction of $\overline{\rho(X_i)}$ to $\tilde{\mathfrak{D}}$. Then we are in the situation of Theorem 5, and the corollary follows.

Two unbounded self-adjoint operators are said to *commute* (or be permutable [30]) in case their spectral resolutions commute. An operator $C = A + iB$ with A and B self-adjoint is called *normal* in case A and B commute. The next corollary is a criterion for normality.

COROLLARY 9.2. *Let A and B be symmetric operators on a Hilbert space $\mathfrak{H}$ and let $\mathfrak{Q}$ be a dense linear subspace of $\mathfrak{H}$ such that $\mathfrak{Q}$ is contained in the domain of A, B, A^2, AB, BA, and B^2, and such that $ABx = BAx$ for all x in $\mathfrak{Q}$. If the restricton of $A^2 + B^2$ to $\mathfrak{Q}$ is essentially self-adjoint then A and B are essentially self-adjoint and $\bar{A}$ and $\bar{B}$ commute.*

PROOF. This follows from Corollary 9.1 with $\mathfrak{g}$ the two-dimensional abelian Lie algebra with basis X, Y and $\rho(aX + bY) = i(aA + bB)$.

That some such condition is required is shown by the counter-example in the following section. Also, Theorem 2 of [23] is a converse of Theorem 5, except that it is necessary to assume that the operators have a domain invariant under the group representation. In fact, the following corollary gives necessary and sufficient conditions for a representation of a Lie algebra by skew-symmetric operators to be the infinitesimal representation associated with a group representation.

COROLLARY 9.3. *Let $\mathfrak{g}$ be a real Lie algebra with a basis $X_1, \cdots, X_d$, G the simply connected Lie group with Lie algebra $\mathfrak{g}$, $\mathfrak{H}$ a Hilbert space, $\mathfrak{E}$ a dense linear subspace of $\mathfrak{H}$. Let ρ be a representation of $\mathfrak{g}$ by skew-symmetric operators with domain $\mathfrak{E}$. Then there is a unitary representation U of G such that $\mathfrak{E}$ is the space of infinitely differentiable vectors for U and $U(X) = \rho(X)$ for all X in $\mathfrak{g}$ if and only if*

$$A = \rho(X_1)^2 + \cdots + \rho(X_d)^2$$

is essentially self-adjoint and $\mathfrak{E} = \bigcap_{n=1}^{\infty} \mathfrak{D}(\bar{A}^n)$.

PROOF. By Theorem 5 and [23], we need only show that if U is a unitary representation of G and $A = U(X_1)^2 + \cdots + U(X_d)^2$ then $\mathfrak{E} = \bigcap_{n=1}^{\infty} \mathfrak{D}(\bar{A}^n)$. The inclusion $\subset$ is obvious, since $\mathfrak{E}$ is invariant under A, and the reverse inclusion follows by (5.4) of Lemma 5.2.

10. A counter-example

Let $\mathfrak{g}$ be the non-abelian three dimensional nilpotent Lie algebra, with basis X, Y, Z such that $[X, Y] = Z$. Let $\mathfrak{H} = \mathfrak{L}^2(0, 1)$. Then a representation of $\mathfrak{g}$ by skew-symmetric operators is given by

$$
\begin{aligned}
X &\longrightarrow \frac{d}{dx} \\
Y &\longrightarrow ix \qquad (10.1)\\
Z &\longrightarrow i
\end{aligned}
$$

all of these oserators having $C_0^\infty(0, 1)$ as domain. (By ix we mean the operator $u(x) \to ixu(x)$.) It is clear that this representation does not give rise to a group representation. One shortcoming of this as a counter example is that d/dx is not essentially skew-adjoint on $C_0^\infty(0, 1)$. Let us now consider a different representation of $\mathfrak{g}$, also given by (10.1) but with domain for all of the operators being now the set of C^∞ functions u on $[0, 1]$ with $u(0) = u(1)$. Then d/dx on this domain is essentially skew adjoint but, of course, the representation still does not give rise to a group representation. The shortcoming of this as a counter-example is that the domain is not invariant. In fact, the set of functions u for which the commutation relation $(d/dx)(ixu) - ix(d/dx)u = iu$ has meaning is so small that d/dx is no longer essentially skew-adjoint on it. Other examples of representations of Lie algebras which do not generate group representations are given in [28].

In this section we give a counter-example having both the features of essential skew-adjointness and invariance of the domain. It appears to be the first example of this nature. The Lie algebra in question is the two dimensional abelian Lie algebra, so it is also an example of two essentially self-adjoint operators defined on a common invariant domain which commute on that domain but whose special resolutions do not commute.

Suppose we have a C^∞ vector field X on a C^∞ manifold M. Then X gives rise to a flow $x \to T_t x$ on M obtained by integrating X, except that each T_t may be only partially defined, since the trajectories obtained by integrating X may lead outside of M. For each t, let E_t denote the closure of the set where T_t is not defined.

LEMMA 10.1. *Let M be a C^∞ manifold, X a C^∞ vector field on M. Let $\mathfrak{L}^2(M)$ be formed with respect to a non-vanishing C^∞ exterior d-fòrm (where d is the dimension of M) and suppose that X is skew-symmetric on the domain $C_0^\infty(M)$. Let E_t be defined as above. If for some $t > 0$ the sets E_t and E_{-t} have measure 0 then X is essentially skew-adjoint.*

The differentiability assumptions could, of course, be relaxed. Also, it may be shown that the converse of the lemma is true (if X is essentially skew-adjoint, then for all real t, E_t has measure 0).

PROOF. For all real t, define $U(t)$ by

$$U(t)u(x) = u(T_t(x)) , \qquad x \notin E_t$$
$$U(t)u(x) = 0 , \qquad x \in E_t .$$

Since X is a skew-symmetric vector field, T_t is a measure-preserving map of $M - E_t$ into M, and so for all t

$$\| U(t)u \| \leqq \| u \| , \qquad u \in \mathfrak{L}^2(M) . \tag{10.2}$$

Any function in $\mathfrak{L}^2(M)$ which is orthogonal to $C_0^\infty(M - E_t)$ must vanish a. e. on $M - E_t$, so that if E_t has measure 0 then $C_0^\infty(M - E_t)$ is dense in $\mathfrak{L}^2(M)$.

To show that X is essentially skew-adjoint it suffices to show that if $X^*u = u$ or $X^*u = -u$ then $u=0$ [30, Chap. IX]. Suppose that $X^*u = u$ and let $t > 0$ be such that E_t has measure 0. Notice that for $0 \leqq s \leqq t$, $U(s)$ maps $C_0^\infty(M - E_t)$ into $C_0^\infty(M)$, since X is a C^∞ vector field. Now if $X^*u = u$, φ is in $C_0^\infty(M - E_t)$, and $0 < s < t$ then

$$\frac{d}{ds}(U(s)\varphi, u) = (XU(s)\varphi, u) = (U(s)\varphi, X^*u) = (U(s)\varphi, u)$$

Therefore $(U(s)\varphi, u) = (\varphi, u)e^s$ for $0 \leqq s \leqq t$. Since $C_0^\infty(M - E_t)$ is dense in $\mathfrak{L}^2(M)$, $\| U(s)^*u \| = e^s \| u \|$, and $u = 0$ by (10.2). The analogous proof holds in case $X^*u = -u$, by choosing $t < 0$. This concludes the proof.

Let M_1 be the set of all points in the xy-plane such that $0 \leqq x \leqq 3$ and $0 \leqq y \leqq 3$ but not both $1 < x < 2$ and $1 < y < 2$. That is, M_1 is the closed band between two concentric squares. Let M_2 be the set M_1 minus the points with integral coordinates at least one of which is 1 or 2. That is, we remove the four vertices of the inside square and the eight points on the sides of the outside square facing them. Let C be the set of points in both M_2 and the boundary of M_2. Each point z in C except (0, 0), (0, 3), (3, 0), (3, 3) has a unique *opposite point* z' in C, namely the point z' in C such that the straight line segment joining z and z' is either horizontal or vertical and lies entirely in M_2. Note that $(z')' = z$. Let M be the C^∞ manifold obtained by identifying opposite points of C in M_2 and by identifying (0, 0), (0, 3), (3, 0), (3, 3). (It may be verified that M is the closed surface of genus 4 with one point removed, which we have given a flat Riemannian metric.)

Now let $\mathfrak{g}$ consist of all operators $a(\partial/\partial x) + b(\partial/\partial y)$, where a and b are real constants, with domain $C_0^\infty(M)$. It is easily seen that each X in $\mathfrak{g}$ is a C^∞ vector field on M. Let $\mathfrak{L}^2(M)$ be formed with respect to Lebesgue measure. Then $\mathfrak{g}$ is a Lie algebra (abelian) of skew-symmetric operators having $C_0^\infty(M)$ as common invariant domain. For each $X=a(\partial/\partial x)+b(\partial/\partial y)$ in $\mathfrak{g}$ and real t, E_t consists of a finite number of line segments (in fact seven, if t is sufficiently small, of slope b/a and leading into points of $M_1 - M_2$). Therefore E_t has measure 0 and by Lemma 10.1 each X in $\mathfrak{g}$ is essentially skew-adjoint. However, $\mathfrak{g}$ clearly does not give rise to a group representation, for let X be the closure of $\partial/\partial x$, Y the closure of $\partial/\partial y$. If u has support in the lower left hand square $0 \leqq x \leqq 1, 0 \leqq y \leqq 1$ then $e^X e^Y u$ has support in the square $0 \leqq x \leqq 1, 1 \leqq y \leqq 2$ above it whereas $e^Y e^X u$ has support in the square $1 \leqq x \leqq 2, 0 \leqq y \leqq 1$ to its right, so that $e^X e^Y \neq e^Y e^X$.

We have proved the following result:

There exist two symmetric operators A and B on a Hilbert space $\mathfrak{H}$ having a common invariant domain $\mathfrak{D}$ such that for all real a and b, $aA + bB$ is essentially self-adjoint and such that for all x in $\mathfrak{D}$, $ABx = BAx$, but such that the spectral resolutions of $\bar{A}$ and $\bar{B}$ do not commute. The set of all $i(aA + bB)$ is a Lie algebra of skew-symmetric operators having a common invariant domain, such that each operator is essentially skew-adjoint but not giving rise to a unitary group representation.

11. An application to a paper of Dixmier

The purpose of this section is to show that our method yields some useful information other than the mere existence of a dense set of analytic vectors for a unitary representation. This will be done by giving a more direct proof of a lemma of Dixmier [6]. The application is based on the connection between analytic vectors and the operator Δ.

THEOREM 6. *Let G be a simply connected Lie group with Lie algebra $\mathfrak{g}$. Let $\mathfrak{g}'$ be an ideal in $\mathfrak{g}$, G' the connected subgroup of G with Lie algebra $\mathfrak{g}'$. Let U' be a unitary representation of G' on a Hilbert space $\mathfrak{H}$. Let r be a homomorphism of the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ of $\mathfrak{g}$ onto the image under U' of the universal enveloping algebra $\mathcal{U}(\mathfrak{g}')$ of $\mathfrak{g}'$, such that $r(B') = U'(B')$ for all B' in $\mathcal{U}(\mathfrak{g}')$ (in the natural identification of $\mathcal{U}(\mathfrak{g}')$ as a subalgebra of $\mathcal{U}(\mathfrak{g})$) and such that $r(X)$ is skew-symmetric for each X in $\mathfrak{g}$. Then there is on $\mathfrak{H}$ a unique unitary representation U of G such that $U(\sigma') = U'(\sigma')$ for all σ' in G'.*

PROOF. Let $X_1, \cdots, X_d$ be a basis for $\mathfrak{g}$ such that $X_1, \cdots, X_{d'}$ is a basis for $\mathfrak{g}'$. Let $\Delta' = X_1^2 + \cdots + X_{d'}^2$ in $\mathcal{U}(\mathfrak{g}')$. Let $\mathfrak{E}'$ be the space of infinitely differentiable vectors for the representation U'. Let $\eta = |r(X_1)| + \cdots + |r(X_d)|$ and $\alpha = |U'(\Delta') - I|$. Notice that $r(X_j) = U'(X_j)$ for $1 \leqq j \leqq d'$. Let m be a positive integer such that $r(X_j)$ is of order $\leqq 2m$ for $1 \leqq j \leqq d$.

Since $\mathfrak{g}'$ is an ideal in $\mathfrak{g}$, $\mathcal{U}(\mathfrak{g}')$ is invariant under ad X for all X in $\mathfrak{g}$. Therefore for all X in $\mathfrak{g}$ and B' in $\mathcal{U}(\mathfrak{g}')$—in particular for $B'=(\Delta'-I)^m$—we have $r(X)U'(B') - U'(B')r(X) = r(XB' - B'X) = U'(XB' - B'X)$, and this in an element of order not greater than the order of B'. Therefore Lemma 6.3 applies, and α^m analytically dominates η. By [23], $(U'(\Delta')-I)^m$ is essentially self-adjoint and by Lemma 5.2 and Lemma 9.1 there is a unique unitary representation U of G such that $\overline{U(X)} = \overline{r(X)}$, the representation U agreeing with U' on G'. This concludes the proof. Since $\mathfrak{E}' = \bigcap_{n=1}^{\infty}\mathfrak{D}((\Delta' - I)^{mn})$, by Lemma 5.2, this shows also that $\mathfrak{E}'$ is the set of infinitely differentiable vectors for U.

Precisely the situation described in Theorem 6 arises in a paper [6] (especially pp. 347–352) of Dixmier. In this case G is a simply connected nilpotent Lie group, $\mathfrak{g}'$ is an ideal of codimension 1 in $\mathfrak{g}$, and, letting X be in $\mathfrak{g}$ but not $\mathfrak{g}'$, there is a non-zero element A_1 in the center $\mathcal{Z}(\mathfrak{g}')$ of $\mathcal{U}(\mathfrak{g}')$ and an element A_2 in $\mathcal{U}(\mathfrak{g}')$ such that each of A_1, A_2, $XA_1 + A_2$ is either symmetric or skew-symmetric and XA_1+A_2 is in the center $\mathcal{Z}(\mathfrak{g})$ of $U(\mathfrak{g})$. By a *Hermitian character* of $\mathcal{Z}(\mathfrak{g})$ is meant a homomorphism of $\mathcal{Z}(\mathfrak{g})$ into the complex numbers sending symmetric elements into real numbers and skew-symmetric elements into purely imaginary numbers. If U is an irreducible unitary representation then $A \to U(A)$, for A in $\mathcal{Z}(\mathfrak{g})$, is a Hermitian character. By Λ is meant the set of Hermitian characters of $\mathcal{Z}(\mathfrak{g}')$ corresponding to a unique (up to unitary equivalence) irreducible unitary representation of G'. It is required to prove that every Hermitian character of $\mathcal{Z}(\mathfrak{g})$ whose restriction to $\mathcal{Z}(\mathfrak{g}')$ is in Λ and which is non-zero on A_1 corresponds to a unique (up to unitary equivalence) irreducible representation of G, and that its restriction to G' is also irreducible (see [6, Lemma 21]). To see this, let χ be the character, χ' its restriction to $\mathcal{Z}(\mathfrak{g}')$ (which is contained in $\mathcal{Z}(\mathfrak{g})$ in the case under consideration), and U' an irreducible unitary representation of G' with character χ'. This exists since χ' is in Λ. Define a homomorphism r of $\mathcal{U}(\mathfrak{g})$ by letting it be U' on $U(\mathfrak{g}')$ and setting

$$r(X) = \frac{\chi(XA_1 + A_2) - U'(A_2)}{\chi'(A_1)}.$$

By Theorem 6 there is a unique unitary representation U of G extending U'. The mapping r is so defined that U has the character χ (since the field of fractions of the ring generated by XA_1+A_2 and $\mathcal{Z}(\mathfrak{g}')$ is the same as the field of fractions of $\mathcal{Z}(\mathfrak{g})$; see [6]). Since U' is irreducible, U is *a fortiori* irreducible. Now let U be any irreducible unitary representation of G, U' its restriction to G'. If U' is not irreducible, let $\mathfrak{H}_1$ and $\mathfrak{H}_2$ be two closed invariant subspaces which are orthogonal complements to each other, and consider the representations U_1' and U_2' of G' on $\mathfrak{H}_1$ and $\mathfrak{H}_2$. Again by Theorem 6, each of these extends to a representation U_1 and U_2 of G, with character χ. Thus U and the direct sum of U_1 and U_2 are both extensions of U', and by uniqueness they must be equal. Therefore the irreducibility of U implies the irreducibility of U'.

12. Elliptic partial differential operators

Suppose that A is an elliptic partial differential operator with analytic coefficients, and let u be a solution of the equation $Au = 0$. Then it is a classical result that u is analytic (see F. John's book [18] and the references there). The most general result of this type is due to Morrey and Nirenberg [19, § 5], who show that a solution of a general elliptic system (in the sense of Douglis and Nirenberg [8]) with analytic coefficients is analytic. Here we shall show more generally that if u is an analytic vector for A then u is analytic. For simplicity we consider only single operators rather than systems. This result is then applied to self-adjoint A to show that if f decreases exponentially fast then the operator $f(A)$ has an analytic kernel.

Let U be an open set in R^d, K a compact subset of U which is the closure of its interior. Let A be a partial differential operator on U. We write A as $A = \sum_\rho a_\rho(x) D^\rho$, where ρ ranges over finite sequences $\rho_1, \cdots, \rho_k$ of integers between 1 and d, $D^\rho = \partial/\partial x_{\rho_1} \cdots \partial/\partial x_{\rho_k}$, and the coefficients a_ρ are invariant under all permutations of the indices in ρ. Letting $|\rho| = k$ be the length of ρ, the operator A is of order $\leqq m$ in case $a_\rho = 0$ whenever $|\rho| > m$. The characteristic polynomial $a(x, \xi)$ of A, if A has order m, is defined for each x in U by $a(x, \xi) = \sum_{|\rho| = m} a_\rho(x) \xi^\rho$, where $\xi = (\xi_1, \cdots, \xi_d)$ is a d-tuple of real numbers and $\xi^\rho = \xi_{\rho_1} \cdots \xi_{\rho_k}$. The operator A is called *elliptic* in case $a(x, \xi) \neq 0$ for $\xi \neq 0$.

We use $\| \ \|$ for the norm in $\mathfrak{L}^2(U)$, $\| u \|^2 = \int_U | u(x) |^2 \, dx$, and $\| \ \|_K$ for the norm in $\mathfrak{L}^2(K)$, $\| u \|_K^2 = \int_K | u(x) |^2 dx$. Let A be an elliptic operator of order m with C^∞ coefficients on U. As the domain of A we take all functions u in $\mathfrak{L}^2(U)$ such that the distribution Au is also in $\mathfrak{L}^2(U)$. In other words, if A^+ is the operator with domain $C_0^\infty(U)$ given by the formal adjoint $\sum_{|\rho| \leqq m} (-1)^{|\rho|} D^\rho (\bar{a}_\rho(x) \cdot)$ of A, then $A = (A^+)^*$. It is known (see [24, Theorem 1 of § 4]) that if u is in the domain of A then $\| D^\rho u \|_K < \infty$ for all $|\rho| \leqq m$. By the closed graph theorem, therefore, there is a $k < \infty$ such that for all u in the domain of A,

$$\sum_{|\rho|\leq m} \| D^\rho u \|_K \leqq k(\| Au \| + \| u \|) . \tag{12.1}$$

(Actually, the reference to the closed graph theorem is not necessary, since the value of k may be derived from the proof that $\| D^\rho u \|_K < \infty$ for $|\rho| \leqq m$.) If B is an operator of order $\leqq m$, $\sum_{|\rho|\leq m} b_\rho(x) D^\rho$, let

$$||| B ||| = \sup_{x\in K,\rho} | b_\rho(x) | . \tag{12.2}$$

Then it follows at once from (12.1) that

$$\| Bu \|_K \leqq k\, ||| B |||\, (\| Au \| + \| u \|) . \tag{12.3}$$

Let $\mathfrak{D}_m$ be the Banach space of all partial differential operators of order $\leqq m$ with coefficients in $C(K)$, with the norm (12.2). We say that a function is in $C^\infty(K)$ if it has a C^∞ extension to a neighborhood of K, and similarly that it is in $C^\omega(K)$ in case it has an analytic extension to a neighborhood of K. If X is a vector field with coefficients in $C^\infty(K)$ we define the operator ad$X$ on the Banach space $\mathfrak{D}_m$ to have as domain all elements B of $\mathfrak{D}_m$ which have coefficients in $C^\infty(K)$, with $(\operatorname{ad} X)B = XB - BX$. Notice that $\operatorname{ad} X \operatorname{ad} Y - \operatorname{ad} Y \operatorname{ad} X = \operatorname{ad}(XY - YX)$ (the Jacobi identity).

LEMMA 12.1. *Let $X_1, \cdots, X_l$ be vector fields with coefficients in $C^\omega(K)$, let $\hat{\xi} = |\operatorname{ad} X_1| + \cdots + |\operatorname{ad} X_l|$, and let $\hat{\delta} = |\operatorname{ad}(\partial/\partial x_1)| + \cdots + |\operatorname{ad}(\partial/\partial x_d)|$. Then $\hat{\delta} + |I|$ analytically dominates $\hat{\xi}$. If A has coefficients in $C^\omega(K)$ then A is an analytic vector for $\hat{\xi}$.*

PROOF. The operator $(\operatorname{ad}(\partial/\partial x_i))\, A$ is merely the operator obtained from A by applying $\partial/\partial x_i$ to its coefficients. (This is not true for the general vector field X, since derivatives of the coefficients of X occur in $(\operatorname{ad} X)A$.) Since K is compact, to say that A has coefficients in $C^\omega(K)$ is equivalent to saying that A is an analytic vector for $\hat{\delta} + |I|$. Therefore the last statement of the lemma follows from the preceding one by Theorem 1, and we need only show that $\hat{\delta} + |I|$ analytically dominates $\hat{\xi}$.

Let $c_n = \sum_{i=1}^{l} ||| \hat{\delta}^n X_i |||$, so that c_n is the sum from $i = 1$ to $i = l$ of the largest absolute value on K of derivatives of order n of the coefficients of X_i. Since the X_i have coefficients in $C^\omega(K)$, the power series with coefficients $c_n/n!$ has a positive radius of convergence.

For a certain integer r (depending only on m and d) we have for all vector fields X with coefficients in $C^\infty(K)$,

$$|\operatorname{ad} X| \leqq r \sum_{j=0}^{m} ||| \hat{\delta}^j X |||\, (\hat{\delta} + |I|) \tag{12.4}$$

since the coefficients of $(\operatorname{ad} X)B$ (for any B in $\mathfrak{D}_m$ with coefficients in $C^\infty(K)$) involve only the coefficients of X times first order derivatives of the coefficients of B and the coefficients of B times derivatives of order

$\leqq m$ (since B has order m) of the coefficients of X. By (12.4),

$$\hat{\xi} \leqq r(c_0 + \cdots + c_m)(\hat{\delta} + |I|) . \tag{12.5}$$

By Corollary 4.1, letting $\eta = \delta$, $(\text{ad}\ \xi)^n\delta$ is the sum of $l^n d$ terms of the form $|X|$ with $\sum_{j=0}^{m} ||| \delta^j X ||| \leqq k_n$, where the power series with coefficients $k_n/n!$ has a positive radius of convergence. By (12.4)(and the Jacobi identity) this gives

$$(\text{ad}\ \xi)^n\, \delta \leqq l^n d r k_n(\delta + |I|) . \tag{12.6}$$

By (12.5) and (12.6), $\delta + |I|$ analytically dominates ξ, concluding the proof.

THEOREM 7. *Let U be an open set in R^d, A an elliptic partial differential operator with analytic coefficients in U. Let u be a function of class C^∞ on U such that*

$$\sum_{n=0}^{\infty} \frac{\| A^n u \|}{n!} s^n < \infty \tag{12.7}$$

for some $s > 0$, where $\| v \|^2 = \int_U | v(x) |^2\, dx$. Then u is an analytic function in U.

For each $s > 0$ there is an open set $\tilde{U}_s$ in the complex space C^d containing U and such that all functions u of class C^∞ on U satisfying (12.7) *have a complex analytic extension to $\tilde{U}_s$.*

We may remark that the assumption that u is C^∞ is unnecessary, since by the regularity theorem any function in the domain of A^n for all n is automatically a C^∞ function.

PROOF. Let x_0 be a point in U. By an affine change of coordinates we may assume that $x_0 = 0$ and that the sphere of radius 2 and center 0 is contained in U. We choose the closed unit sphere for the compact set K contained in U. Let m be the order of A and let

$$\varphi(x) = \begin{cases} (1 - (x_1^2 + \cdots + x_d^2))^{m+1} , & x \in K \\ 0 , & x \notin K . \end{cases} \tag{12.8}$$

Although there is no such thing as a non-zero analytic function with compact support, the function φ will behave like one for all our purposes. Notice that the restriction of φ to K is in $C^\omega(K)$, so that if we define $X_1 = \varphi(x)(\partial/\partial x_1), \cdots, X_d = \varphi(x)(\partial/\partial x_d)$ then Lemma 12.1 applies to them.

Let $\xi = |X_1| + \cdots + |X_d|$. We shall show that any u satisfying (12.7) is an analytic vector for ξ. To do this it is enough, by Theorem 1, to show that $\alpha = |A| + |I|$ analytically dominates ξ. By (12.3) there is a

$c < \infty$ ($c = k\sum_{i=1}^{l} ||| X_i |||$) such that

$$\xi \leqq c\alpha \, . \tag{12.9}$$

By Lemma 12.1 there is a sequence c_n, such that the power series with coefficients $c_n/n!$ has a positive radius of convergence and such that $||| \hat{\xi}^n A ||| \leqq c_n$.

Let $\mathcal{V}$ be the set of all partial differential operators $B = \sum_{|\rho| \leqq m} b_\rho(x) D^\rho$ of order $\leqq m$ defined on U such that the coefficients b_ρ are continuous, vanish outside K, and such that their restrictions to K are in $C^\infty(K)$. Notice that

$$(\text{ad } X_i)\, A = \sum_{|\rho| \leqq m} \left(\varphi \frac{\partial}{\partial x_i} a_\rho D^\rho - a_\rho (D^\rho \cdot \varphi) \frac{\partial}{\partial x_i} \right)$$

is in $\mathcal{V}$ since φ vanishes to the order $m + 1$ on the boundary of K. For the same reason, if B is in $\mathcal{V}$ then $(\text{ad } X_i)B$ is in $\mathcal{V}$, $i = 1, \cdots, d$. By induction, therefore, each $\text{ad } X_{i_n} \cdots \text{ad } X_{i_1} A$ is in $\mathcal{V}$. Consequently $\| \text{ad } X_{i_n} \cdots \text{ad } X_{i_1} Au \|_K = \| \text{ad } X_{i_n} \cdots \text{ad } X_{i_1} Au \|$. By (12.3), therefore

$$\| \text{ad } X_{i_n} \cdots \text{ad } X_{i_1} Au \| \leqq k \, ||| \text{ad } X_{i_n} \cdots \text{ad } X_{i_1} A ||| (\| Au \| + \| u \|)$$

so that

$$(\text{ad } \xi)^n \alpha \leqq k c_n \alpha \, . \tag{12.10}$$

By (12.9) and (12.10), α analytically dominates ξ, and any u satisfying (12.7) is an analytic vector for ξ.

Let K_0 be the closed sphere of radius 1/2 and center 0. Since K_0 is contained in the interior of K it is obvious (and also a special case of Theorem 2) that any analytic vector for ξ is an analytic vector for

$$\delta_0 = \left| \frac{\partial}{\partial x_1} \right| + \cdots + \left| \frac{\partial}{\partial x_d} \right|$$

with respect to $\mathfrak{L}^2(K_0)$. That is, letting $\| v \|_{K_0}^2 = \int_{K_0} | v(x) |^2 \, dx$,

$$\sum_{|\rho|=0}^{\infty} \frac{\| D^\rho u \|_{K_0}}{|\rho|!} s_0^n < \infty \tag{12.11}$$

for some $s_0 > 0$. By a theorem of Sobolev (see [24, p. 655]) there is a constant $k_0 < \infty$ such that

$$\sup_{x \in K_0} | v(x) | \leqq k_0 \sum_{|\rho| \leqq [d/2]+1} \| D^\rho v \|_{K_0} \tag{12.12}$$

Applying (12.12) to $v = D^\rho u$ in (12.11), we have that

$$\sum_{|\rho|=0}^{\infty} \frac{\sup_{x \in K_0} | D^\rho u(x) |}{| \rho |!} s_0^n < \infty \tag{12.13}$$

for some $s_0 > 0$, so that u is indeed an analytic function. The operator A being fixed, the value of s_0 in (12.13) depends only on the s in (12.7). However, any u satisfying (12.13) has a complex analytic extension to the set of all z in C^d such that the distance from z to K_0 is less than s_0, proving the last statement of the theorem. This concludes the proof.

THEOREM 8. *Let M be a paracompact real analytic manifold, μ a measure on M which in each analytic local coordinate system has a non-vanishing analytic density. Let A be a self-adjoint elliptic partial differential operator on $\mathfrak{L}^2(M, \mu)$ with analytic coefficients. Let f be a Baire function of a real variable such that for some $k < \infty$ and $c>0$, $|f(\lambda)| \leqq ke^{-c|\lambda|}$ for all λ in the spectrum of A. Then there is a unique analytic function F on $M\times M$ such that for all x in M, $F(x, \cdot)$ is in $\mathfrak{L}^2(M, \mu)$ and for all u in $\mathfrak{L}^2(M, \mu)$*

$$(12.14) \qquad f(A)u(x) = \int_M F(x, y)u(y)d\mu(y)$$

Any element of $\mathfrak{L}^2(M, \mu)$ in the range of $f(A)$ is equal a. e. to an analytic function. The mapping $x \rightarrow F(x, \cdot)$ is analytic from M into $\mathfrak{L}^2(M, \mu)$. There is a neighborhood $\tilde{M}$ of M in a complex analytic manifold containing M and an extension $F(\tilde{x}, \tilde{y})$ of F to $\tilde{M} \times \tilde{M}$ which is complex analytic in $\tilde{x}$ and $\tilde{y}$, and such that the mapping $\tilde{x} \rightarrow F(\tilde{x}, \cdot)$ is complex analytic from $\tilde{M}$ into $\mathfrak{L}^2(M, \mu)$.

PROOF. The existence and uniqueness of a continuous function F having these properties is proved, for example, in [20], using the regularity theorem for elliptic operators (see [24]). We must prove that F is analytic.

By the spectral theorem, any element of $\mathfrak{L}^2(M, \mu)$ in the range of $f(A)$ is an analytic vector for A. By the regularity theorem (see [24]) any element of $\mathfrak{L}^2(M, \mu)$ in the domain of A^n for all n (in particular, any element in the range of $f(A)$) is of class C^∞. (We identify an element of $\mathfrak{L}^2(M, \mu)$, which is an equivalence class of functions equal a.e., with a continuous representative of it, which must be unique if it exists.)

Let U be an analytic coordinate neighborhood in M whose closure is contained in an analytic coordinate neighborhood. Then $d\mu$ on U is given by $d\mu = \psi(x)dx_1 \cdots dx_n$ where ψ is bounded away from 0. Therefore $\mathfrak{L}^2(U, \mu)$ and $\mathfrak{L}^2(U)$(formed with respect to Lebesgue measure) are the same spaces, with equivalent norms. Consequently, any u which is an analytic vector for A in $\mathfrak{L}^2(M, \mu)$ is an analytic vector for A in $\mathfrak{L}^2(U)$. By Theorem 7, therefore, such a u is an analytic function. In particular, any element of $\mathfrak{L}^2(M, \mu)$ in the range of $f(A)$ is an analytic function.

By the last statement of Theorem 7, if $\tilde{M}$ is a sufficiently small neighborhood of M in a complex analytic manifold containing M (which exists by [32, p. 133]) then any element of $\mathfrak{L}^2(M, \mu)$ in the range of $f(A)$ has a complex analytic extension to $\tilde{M}$. If $\tilde{x}$ is in $\tilde{M}$ then $u \to f(A)u(\tilde{x})$ is an everywhere defined linear functional on $\mathfrak{L}^2(M, \mu)$, and it is easily seen to be continuous (either by the closed graph theorem as in [1] or by the fact that all our estimates depend only on $\| u \|_{\mathfrak{L}^2}$). Therefore there is a unique element $F(\tilde{x}, \cdot)$ of $\mathfrak{L}^2(M, \mu)$ such that

$$f(A)u(\tilde{x}) = \int_M F(\tilde{x}, y)u(y)d\mu(y) . \tag{12.15}$$

Comparing (12.14) with (12.15) we see that $F(\tilde{x}, y)$ is (as the notation suggests) an extension of $F(x, y)$. Since each $f(A)u(\tilde{x})$ is a complex analytic function, the map $\tilde{x} \to F(\tilde{x}, \cdot)$ from $\tilde{M}$ to $\mathfrak{L}^2(M, \mu)$ is complex analytic (by [16, p. 96]). Since $\overline{F(y, x)}$ is the kernel of $\bar{f}(A)$, the same argument shows that there is a complex analytic mapping $\tilde{y} \to F(\cdot, \tilde{y})$ of $\tilde{M}$ into $\mathfrak{L}^2(M, \mu)$ extending $y \to F(\cdot, y)$.

Let $g(\lambda) = f(\lambda)^2$ and let G be the kernel of $g(A)$, so that

$$G(x, y) = \int_M F(x, z)F(z, y)d\mu(z) = (F(x, \cdot), \overline{F(\cdot, y)})$$

Let us define $G(\tilde{x}, \tilde{y}) = (F(\tilde{x}, \cdot), \overline{F(\cdot, \tilde{y})})$. Then $G(\tilde{x}, \tilde{y})$ is an extension of $G(x, y)$ to $\tilde{M} \times \tilde{M}$ which is complex analytic in $\tilde{x}$ and $\tilde{y}$ separately, and hence [2, p. 33] jointly. Consequently $G(x, y)$ is a real analytic function of x and y. But the function f is itself the square of a Baire function which is exponentially decreasing on the spectrum of A, so that the same argument applies to $F(x, y)$, so that $F(x, y)$ is a real analytic function of x and y. This concludes the proof.

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