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Semigroups of Simple Lie Groups and Controllability

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Abstract In this paper, we consider a subsemigroup S of a real connected simple Lie group G generated by $\{\exp tX : X \in \Gamma, t \geq 0\}$ for some subset Γ of L, the Lie algebra of G. It is proved that for an open class $\Gamma = \{A, \pm B\}$ and a generic pair (A, B) in $L \times L$, if S contains a subgroup isomorphic to $SL(2, \mathbb{R})$, associated to an arbitrary root, then S is the whole G. In a series of previous papers, analogous results have been obtained for the maximal root only. Recently, a similar result for complex connected simple Lie groups was proved. The proof uses special root properties that characterize some particular subalgebras of L. In control theory, this case $\Gamma = \{A, \pm B\}$ is specially important since the control system, $\dot{g} = (A + uB)g$, where $u \in \mathbb{R}$, is controllable on G if and only if $S = G$.

Keywords Simple Lie groups · Invariant vector fields · Root systems · Controllability

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1 Introduction

We deal with controllability on a real finite-dimensional connected simple Lie group G, of right invariant control systems of the form $\dot{g} = (A + uB)g$, where $g \in G$, $u \in \mathbb{R}$ and $A, B \in L$, the Lie algebra of G. By invariance of the vector fields, controllability on G of those systems is equivalent to the fact that the semigroup S of G generated by $\{\exp tX : X \in \Gamma, t \geq 0\}$, where $\Gamma = \{A, \pm B\}$, is exactly G. It is known that the rank condition (i.e., the Lie subalgebra of L generated by Γ equal to L) is a necessary condition for controllability, and if the group G is compact, it is also sufficient. It is also known [10] that the semigroup generated by a symmetric set of the form $\{\pm A, \pm B\}$ for a generic pair (A, B) in $L \times L$ generates G as a group. Various results on controllability of systems

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on Lie groups can be found in [1, 7, and 12]. In [9], sufficient conditions for controllability of $\Gamma = \{A, \pm B\}$, where the pair (A, B) is generic and B is strongly regular, were obtained for noncompact semi-simple Lie groups with finite center. Those conditions were improved for some particular cases in [3, 5, and 13] and for the general case in [4]. In those papers, it has been proved that if the union of one-parameter groups $\{\exp tE_{\mu},\ t \in \mathbb{R}\}$ and $\{\exp tE_{-\mu},\ t \in \mathbb{R}\}$ is included in S, where $\mathbb{R}E_{\mu} = L_{\mu}$ denotes the root space for the maximal root μ, then the semigroup S generates G. Our paper generalizes those previous results, obtained for the maximal root μ only, to an arbitrary root. Moreover, here, B is not necessarily a strongly regular element, B is only A-strongly regular (see Definition 1). Recently, a similar result was shown in [11] for complex simple Lie groups. The authors used some nontrivial results on the geometry of flag manifolds.

In this work, we prove the result for a real connected simple Lie group G, whose Lie algebra is the split real form of any complex simple Lie algebra and for an arbitrary long root α. For a short root α, the result remains valid in the case where the Lie algebra of G is the split real form of a complex simple Lie algebra of type G_2. In the other cases of Dynkin diagrams with double edges F_4, $B_n, n \geq 2$, and $C_n, n \geq 3$, we add an additional condition to obtain the result. Moreover, in Theorem 3, we give weaker conditions for controllability in the exceptional case, where G is of type G_2, as follows: if the semigroup S contains *only* one-parameter group $\{\exp tE_{\alpha},\ t \in \mathbb{R}\}$ for an arbitrary root α and the components of A in the root spaces L_{β} are nonzero, for some roots β, then $S = G$. The proof of our results is based on algebraic facts concerning special properties of long and short roots that characterize some particular subalgebras of L.

The paper is organized as follows. Section 2 describes some basic results on root systems of semi-simple Lie algebras, reviews some facts about controllability of right invariant systems, and specifies the notations used in this paper. In Section 3, we give the statements of the main results. We also give definitions and properties that are essential for the proofs. In Section 4, we formulate and prove properties concerning long and short roots used in the proofs of the main results. In Section 5, we give the proofs of the main results.

2 Basic Notions and Notations

In this section, we give some known facts about semi-simple Lie algebras and known results about controllability which we shall use in the proofs of our results.

- Let L be a real semi-simple finite dimensional Lie algebra and $L_{\mathbb{C}} = L \oplus_{\mathbb{R}} \mathbb{C}$, its complexification Lie algebra. Consider $\mathfrak{h}$ a Cartan subalgebra of $L_{\mathbb{C}}$ and $\mathcal{R}$ a root system associated to $(L_{\mathbb{C}}, \mathfrak{h})$. We refer to [2, 6, and 14] for more details. The root space decomposition of semi-simple Lie algebras is $L = \mathfrak{h}_{\mathbb{R}} \oplus \sum_{\alpha \in \mathcal{R}} L_{\alpha}$, where $L_{\alpha} = \{X \in L_{\mathbb{C}} : \forall\ H \in \mathfrak{h},\ [H, X] = \alpha(H)X\}$ is the root space of a root α. Clearly, $[\mathfrak{h}, L_{\alpha}] \subset L_{\alpha}$. It is known that $dim_{\mathbb{C}} L_{\alpha} = 1$ and for any $\alpha, \beta \in \mathcal{R}$, $[L_{\alpha}, L_{\beta}] = L_{\alpha+\beta}$ (resp. $\{0\}$) if $\alpha + \beta \in \mathcal{R}$ (resp. $\alpha + \beta \notin \mathcal{R}$ and $\alpha + \beta \neq 0$), and $[L_{\alpha}, L_{-\alpha}] \subset \mathfrak{h}$. For every $\alpha \in \mathcal{R} \subset \mathfrak{h}^*$, the dual space of $\mathfrak{h}$, there exists a unique element $h_{\alpha} \in \mathfrak{h}$ such that for any $H \in \mathfrak{h}$, $\alpha(H) = (h_{\alpha}; H)$ and $\alpha(h_{\alpha}) \neq 0$, where $(;)$ is the Cartan–Killing symmetric and nondegenerate bilinear form on $L_{\mathbb{C}}$. Moreover, for any $\alpha \in \mathcal{R}$, there exist unique elements $H_{\alpha} \in \mathfrak{h}$ and $E_{\alpha} \in L_{\alpha}$ such that $\alpha(H_{\alpha}) = 2$, $(E_{\alpha}; E_{-\alpha}) = 1$ and $[E_{\alpha}, E_{-\alpha}] = H_{\alpha}$. Let $<, >$ denote the nondegenerate symmetric bilinear form on $\mathfrak{h}^*$, defined by $<\alpha, \beta>=(h_\alpha; h_\beta)$ for $\alpha, \beta \in \mathfrak{h}^*$. For any root α, the reflection r_α on $\mathfrak{h}^*$, defined by $r_\alpha(\lambda) = \lambda - 2\frac{<\alpha,\lambda>}{<\alpha,\alpha>}\alpha$, for $\lambda \in \mathfrak{h}^*$, satisfies that for any $\alpha, \beta \in \mathcal{R}$, $r_\alpha(\beta) = \beta - \beta(H_\alpha)\alpha \in \mathcal{R}$. The constants $\beta(H_\alpha) = 2\frac{<\alpha,\beta>}{<\alpha,\alpha>}$ are integers, called the Cartan integers. We know that $-3 \leq \beta(H_\alpha) \leq 3$, for any $\alpha, \beta \in \mathcal{R}$. For nonproportional roots α and β, there exist two unique positive integers $p(\alpha, \beta)$ and $q(\alpha, \beta)$, such that for any integer m, $-p(\alpha, \beta) \leq m \leq q(\alpha, \beta)$, $m\alpha + \beta$ is a root and $\beta(H_\alpha) = p(\alpha, \beta) - q(\alpha, \beta)$.

For the Dynkin diagrams that have only simple edges A_n, D_n, E_6, E_7, and E_8, the action of the Weyl group $\mathcal{W}$, generated by the reflections r_α, $\alpha \in \mathcal{R}$, on the set of roots $\mathcal{R}$ is transitive. For the other diagrams B_n, C_n, F_4, and G_2, there are two orbits which are given by the long and short roots, respectively.

There exists one split real form of $L_{\mathbb{C}}$ isomorphic to the following semi-simple real Lie algebra:

$$\mathcal{L} = \mathfrak{h}_{\mathbb{R}} \oplus \sum_{\alpha \in \mathcal{R}} \mathcal{L}_\alpha, \text{ where } \mathcal{L}_\alpha = \mathbb{R}E_\alpha \text{ and } \mathfrak{h}_{\mathbb{R}} = \sum_{\alpha \in \mathcal{R}} \mathbb{R}H_\alpha.$$

$\mathcal{R}$ is a root system associated to $(\mathcal{L}, \mathfrak{h}_{\mathbb{R}})$. Write the root space decomposition of $X \in \mathcal{L}$:

$$X = X_0 + \sum_{\alpha \in \mathcal{R}} X_\alpha \ , \text{ where } X_0 \in \mathfrak{h}_{\mathbb{R}} \text{ and } X_\alpha \in \mathcal{L}_\alpha.$$

Recall that an element $B \in L$ (L a real semi-simple Lie algebra of finite dimensional) is regular if and only if B is semi-simple and Ker(ad$B$) is of minimal dimension. $B \in L$ is strongly regular if and only if B is regular and all the eigenspaces corresponding to the nonzero eigenvalues of ad$B$ are one-dimensional in $L_{\mathbb{C}}$. $B \in L$ is real strongly regular if and only if B is strongly regular and all the eigenvalues of ad$B$ are real. It is known that the set of strongly regular elements is open and dense in L. The set of real strongly regular elements is open and nonempty in L.

- Now, we state some results about controllability (see [8] and [9]). For $\Gamma \subset L$, let $LS(\Gamma)$ denote the Lie-saturated cone of Γ, i.e., the set of all elements X of the Lie algebra generated by Γ such that $\{\exp tX,\ t \geq 0\} \subset clS$, where cl denotes closure. We know that $LS(\Gamma) = L$ is a necessary and sufficient condition for controllability of Γ on G, i.e., the semigroup generated by $\{\exp tX : X \in \Gamma,\ t \geq 0\}$ is equal to G. We will give some properties of $LS(\Gamma)$ in Lemma 5.

3 Main Results

Hereafter, L is assumed to be the split real form of a simple Lie algebra $L_{\mathbb{C}}$. Write $L = \mathfrak{h}_{\mathbb{R}} \oplus \sum_{\alpha \in \mathcal{R}} L_\alpha$, with $L_\alpha = \mathbb{R}E_\alpha$, the root space of a root α and represent any $X \in L$ as $X = X_0 + \sum_{\alpha \in \mathcal{R}} X_\alpha$, where $X_0 \in \mathfrak{h}_{\mathbb{R}}$ and $X_\alpha \in L_\alpha$. To simplify the notation, the subalgebra $\mathfrak{h}_{\mathbb{R}} = \sum_{\alpha \in \mathcal{R}} \mathbb{R}H_\alpha$ will be denoted by $\mathfrak{h}$.

Definition 1 Let $A \in L$, $A \notin \mathfrak{h}$. An element $B \in \mathfrak{h}$ is called A-strongly regular if for any $\alpha, \beta \in \mathcal{R}$, $\alpha \neq \beta$, such that $A_\alpha \neq 0$ and $A_\beta \neq 0$, the eigenvalues $\alpha(B)$ and $\beta(B)$ of ad$B$ are nonzero and distinct.

Clearly, every strongly regular element is in particular A-strongly regular. It follows that for any $A \in L$ ($A \notin \mathfrak{h}$), the set of A-strongly regular elements is dense in L. Consider $\Gamma = \{A, \pm B\}$, where $(A, B) \in L \times \mathfrak{h}$ and B is A-strongly regular. Write the root space decomposition of A, $A = A_0 + \sum_{\alpha \in \mathcal{R}} A_\alpha$, where $A_0 \in \mathfrak{h}$ and $A_\alpha \in L_\alpha$. Note that saying that the pair (A, B) generates the Lie algebra L means that the set $\{A_\alpha,\ \alpha \in \mathcal{R}\}$ generates L (since $B, A_0 \in \mathfrak{h}$ and $[\mathfrak{h}, L_\alpha] \subset L_\alpha$). For the main results, we use the fact that for any nonproportional roots α and β, there exist two unique positive integers $p(\alpha, \beta)$ and $q(\alpha, \beta)$ less than or equal to 3, such that for every integer m, $-p(\alpha, \beta) \leq m \leq q(\alpha, \beta)$, $m\alpha + \beta$ is a root. We consider the following sets of roots that characterize some particular subalgebras of L (see Corollary 2). Those sets have been considered in [3] and [4] for the maximal root which is a long root. Here, we will give some important properties of those sets for an arbitrary long or short root.

Definition 2 For a root α, consider the following:

$$\mathcal{R}'(\alpha) = \{\beta \in \mathcal{R} : p(\alpha, \beta) \geq 1 \text{ or } q(\alpha, \beta) \geq 1\}.$$
$$\mathcal{R}''(\alpha) = \{\beta \in \mathcal{R} : p(\alpha, \beta) = q(\alpha, \beta) = 0\}.$$

For a short root α, consider the following:

$$\mathcal{R}'_1(\alpha) = \{\beta \in \mathcal{R}'(\alpha) : p(\alpha, \beta) = 1 \text{ or } q(\alpha, \beta) = 1 \text{ and } p(\alpha, \beta)q(\alpha, \beta) \leq 1\}.$$
$$\mathcal{R}'_2(\alpha) = \{\beta \in \mathcal{R}'(\alpha) : p(\alpha, \beta) = 2 \text{ or } q(\alpha, \beta) = 2 \text{ and } p(\alpha, \beta) \leq 2,\ q(\alpha, \beta) \leq 2\}.$$
$$\mathcal{R}'_3(\alpha) = \{\beta \in \mathcal{R}'(\alpha) : p(\alpha, \beta) = 3 \text{ or } q(\alpha, \beta) = 3\}.$$

Remark 1 Note that the integers $p(\alpha, \beta)$ and $q(\alpha, \beta)$ are defined for two nonproportional roots α and β (α and β are proportional if and only if $\beta = \pm\alpha$). Therefore, if $\beta \in \mathcal{R}'(\alpha) \cup \mathcal{R}''(\alpha)$, then β is not proportional to α.

Theorem 1 *Let G be a real connected Lie group, whose Lie algebra L is the split real form of a complex simple Lie algebra $L_{\mathbb{C}}$. Then, for $A, B \in L$, the semigroup S generated by the union* $\{\exp tA,\ t \geq 0\} \cup \{\exp tB,\ t \in \mathbb{R}\}$ *generates G if*

1. *$B \in \mathfrak{h}$, B is A-strongly regular and the pair (A, B) generates the Lie algebra L.*
2. *S contains the one-parameter groups* $\{\exp tE_\alpha,\ t \in \mathbb{R}\}$ *and* $\{\exp tE_{-\alpha},\ t \in \mathbb{R}\}$*, where $\mathbb{R}E_\alpha$ is the root space of a **long** (or a **short** if G is of type $\mathbf{G}_2$) root α.*

In the other cases, namely B_n, $n \geq 2$, C_n, $n \geq 3$, and F_4, for a short root α, we add an additional condition on the sets $\mathcal{R}'_2(\alpha)$ and $\mathcal{R}''(\alpha)$.

Theorem 2 *Let G be a real connected Lie group, whose Lie algebra L is the split real form of a complex simple Lie algebra $L_{\mathbb{C}}$ of type $\mathbf{B_n}$, $n \geq 2$, $\mathbf{C_n}$, $n \geq 3$, or $\mathbf{F}_4$. Then for $A, B \in L$, the semigroup S generated by the union* $\{\exp tA,\ t \geq 0\} \cup \{\exp tB,\ t \in \mathbb{R}\}$ *generates G if*

1. *$B \in \mathfrak{h}$, B is A-strongly regular and the pair (A, B) generates the Lie algebra L.*
2. *There exists a **short** root α such that S contains the union of the one-parameter groups* $\{\exp tE_\alpha,\ t \in \mathbb{R}\} \cup \{\exp tE_{-\alpha},\ t \in \mathbb{R}\}$ *and for any $\beta \in \mathcal{R}'_2(\alpha)$, we have $A_\beta = 0$.*
3. *In the case of type $\mathbf{F}_4$ or $\mathbf{C_n}$, $n \geq 3$, for any **short** root $\beta \in \mathcal{R}'(\alpha)$ and any $\gamma \in \mathcal{R}''(\alpha)$ such that $q(\beta, \gamma) = 2$, if $\alpha + \beta$ and $-\alpha + \beta$ are not long roots and $2\beta + \gamma \neq \pm\alpha$, then $A_\gamma = 0$.*

In the special case G_2, we give a weaker condition than condition 2 of Theorem 1 and Theorem 2.

Theorem 3 *Let G be a real connected Lie group, whose Lie algebra L is the split real form of a complex simple Lie algebra $L_{\mathbb{C}}$ of type $\mathbf{G}_2$. Then for $A, B \in L$, the semigroup S generated by $\{\exp tA,\ t \geq 0\} \cup \{\exp tB,\ t \in \mathbb{R}\}$ generates G if:*

1. *$B \in \mathfrak{h}$, B is A-strongly regular and the pair (A, B) generates the Lie algebra L.*
2. *There exists a **short** (resp. **long**) root α such that S contains the one-parameter group $\{\exp tE_\alpha,\ t \in \mathbb{R}\}$ and $A_\beta \neq 0$, for exactly two (resp. three) long roots $\beta \in \mathcal{R}$ such that $q(\alpha, \beta) = 3$ (resp. $q(\alpha, \beta) = 1$ or $\beta = -\alpha$).*

In the following remark, we give a condition on $A \in \Gamma$ (if B is A-strongly regular), such that the semigroup S contains the union of the one-parameter groups $\{\exp tE_\alpha,\ t \in \mathbb{R}\}$ and $\{\exp tE_{-\alpha},\ t \in \mathbb{R}\}$, for a root α.

Remark 2 Consider $\Gamma = \{A, \pm B\}$, where B is A-strongly regular. If there exists a root α such that $A_\beta = 0$, for any root β such that $|\beta(B)| > |\alpha(B)|$, then (see the proof of Lemma 5–4) A_α and $A_{-\alpha}$ belong to $LS(\Gamma)$. If, moreover, we suppose $(A_\alpha; A_{-\alpha}) < 0$, then $A_\alpha + A_{-\alpha}$ is compact as soon as the group G has a finite center and therefore $\mathbb{R}E_\alpha \cup \mathbb{R}E_{-\alpha} \subset LS(\Gamma)$, which means that S contains $\{\exp tE_\alpha, t \in \mathbb{R}\}$ and $\{\exp tE_{-\alpha}, t \in \mathbb{R}\}$ (see [9] for more details).

Using Remark 2, we obtain immediately the following corollary.

Corollary 1 *Let G be a real connected Lie group with finite center and whose Lie algebra L is the split real form of a complex simple Lie algebra $L_{\mathbb{C}}$. For $A, B \in L$, the semigroup S generated by $\{\exp tA,\ t \geq 0\} \cup \{\exp tB,\ t \in \mathbb{R}\}$ generates G if*

1. *$B \in \mathfrak{h}$, B is A-strongly regular and the pair (A, B) generates the Lie algebra L.*
2. *There exists a root α such that $(A_\alpha, A_{-\alpha}) < 0$, and $A_\beta = 0$ for any root β such that $|\beta(B)| > |\alpha(B)|$ and in the case B_n, $n \geq 2$, C_n, $n \geq 3$ or F_4, if α is a **short** root, then $A_\beta = 0$ for any $\beta \in \mathcal{R}'_2(\alpha)$.*
3. *If α is a **short** root, in the case of type $\mathbf{F}_4$ or $\mathbf{C_n}$, $n \geq 3$, for any **short** root $\beta \in \mathcal{R}'(\alpha)$ and any $\gamma \in \mathcal{R}''(\alpha)$ such that $q(\beta, \gamma) = 2$, if $\alpha + \beta$ and $-\alpha + \beta$ are not long roots and $2\beta + \gamma \neq \pm\alpha$, then $A_\gamma = 0$.*

Remark 3 In the case of type $B_n, n \geq 2$, or F_4 (resp. $C_n, n \geq 3$), we will prove in Lemma 4 that for every **short** root α, the set $\mathcal{R}'_2(\alpha) \cup \mathcal{R}''(\alpha)$ consists exactly of all long roots (resp. $\mathcal{R}'_2(\alpha)$ contains exactly two positive long roots and their opposites). Moreover, the root system $\mathcal{R}$ is generated by the roots of $\mathcal{R}'_1(\alpha) \cup \{\pm\alpha\}$ which is the set of short roots (resp. contains short and long roots).

4 Root Properties

In this section, we will formulate and prove properties of roots, on which our proofs are based and which are of independent interest. We start by giving some properties of the sets $\mathcal{R}''(\alpha)$, $\mathcal{R}'(\alpha)$ and $\mathcal{R}'_i(\alpha)$, $1 \leq i \leq 3$, for a root α. Clearly, for nonproportional roots α and β, we have $\beta \in \mathcal{R}'(\alpha)$ signifies that $\alpha+\beta$ or $-\alpha+\beta$ is a root. Moreover, a root $\beta \in \mathcal{R}''(\alpha)$

signifies that β is not proportional to α and $\alpha+\beta$, $\alpha-\beta$ are both not roots. We thus have the following:

$$\begin{aligned}\mathcal{R} &= \mathcal{R}'(\alpha)\cup\mathcal{R}''(\alpha)\cup\{\alpha,-\alpha\} \text{ and } \mathcal{R}'(\alpha)\cap\mathcal{R}''(\alpha)=\emptyset\\ \mathcal{R}'(\alpha) &= \mathcal{R}'_1(\alpha)\cup\mathcal{R}'_2(\alpha)\cup\mathcal{R}'_3(\alpha) \text{ and } \mathcal{R}'_i(\alpha)\cap\mathcal{R}'_j(\alpha)=\emptyset,\ \text{ for } 1\le i\ne\ j\le 3.\end{aligned}$$

Since $\mathcal{R}=-\mathcal{R}$, we easily verify that

$$\begin{aligned}\mathcal{R}'(\alpha) &= -\mathcal{R}'(\alpha)=\mathcal{R}'(-\alpha),\\ \mathcal{R}''(\alpha) &= -\mathcal{R}''(\alpha)=\mathcal{R}''(-\alpha),\\ \mathcal{R}'_i(\alpha) &= -\mathcal{R}'_i(\alpha)=\mathcal{R}'_i(-\alpha) \text{ for } 1\le i\le 3.\end{aligned}$$

Lemma 1 *For every root α, the following equivalent implications hold:*

1. *For any root $\beta,\gamma\in\mathcal{R}''(\alpha)$, if $\beta+\gamma$ is a root, then $\beta+\gamma\in\mathcal{R}''(\alpha)$.*
2. *For any root $\beta\in\mathcal{R}'(\alpha)$ and $\gamma\in\mathcal{R}''(\alpha)$, if $\beta+\gamma$ is a root, then $\beta+\gamma\in\mathcal{R}'(\alpha)$.*

Proof Let $\alpha\in\mathcal{R}$ and $\beta,\gamma\in\mathcal{R}''(\alpha)$. If $\beta+\gamma\in\mathcal{R}$, then $\beta+\gamma$ is not proportional to α, because if $\beta+\gamma=\alpha$ (resp. $\beta+\gamma=-\alpha$), then $\gamma=\alpha-\beta$ (resp. $\gamma=-\alpha-\beta$) and hence, by definition of $\mathcal{R}'(\alpha)$, we get $\beta\in\mathcal{R}'(\alpha)$. This contradicts the fact that $\mathcal{R}'(\alpha)\cap\mathcal{R}''(\alpha)=\emptyset$. By definition of $\mathcal{R}''(\alpha)$, both $\alpha+\beta$ and $\alpha+\gamma$ are not roots and thus $[E_\alpha,E_\gamma]=[E_\beta,E_\alpha]=0$. By the Jacobi identity, $[E_\alpha,[E_\beta,E_\gamma]]=0$, implying that $\alpha+(\beta+\gamma)$ is not a root. Similarly, we show that $-\alpha+(\beta+\gamma)$ is not a root either. Therefore, $\beta+\gamma\in\mathcal{R}''(\alpha)$ which proves 1). Now, we will show that 1 implies 2. Let $\beta\in\mathcal{R}'(\alpha)$ and $\gamma\in\mathcal{R}''(\alpha)$ such that $\beta+\gamma$ is a root. Assume that $\beta+\gamma\in\mathcal{R}''(\alpha)$ since $\beta=(\beta+\gamma)-\gamma$ and $-\gamma\in\mathcal{R}''(\alpha)$ by applying 1, we obtain that $\beta\in\mathcal{R}''(\alpha)$. This is a contradiction since $\mathcal{R}'(\alpha)\cap\mathcal{R}''(\alpha)=\emptyset$. As above, we prove that $\beta+\gamma$ is not proportional to α. Therefore, since $\mathcal{R}=\mathcal{R}'(\alpha)\cup\mathcal{R}''(\alpha)\cup\{\alpha,-\alpha\}$, we obtain $\beta+\gamma\in\mathcal{R}'(\alpha)$. The other implication can be proved similarly. □

Recall that $\beta\in\mathcal{R}$ if and only if $w\beta\in\mathcal{R}$, for every element w of the Weyl group $\mathcal{W}$ (i.e., $\mathcal{W}\mathcal{R}=\mathcal{R}$). Then, by Definition 2, we prove immediately the following lemma.

Lemma 2 *For every root α and $w\in\mathcal{W}$, we have the following:*
$\mathcal{R}'(w\alpha)=w\mathcal{R}'(\alpha)$, $\mathcal{R}''(w\alpha)=w\mathcal{R}''(\alpha)$, and $\mathcal{R}'_i(w\alpha)=w\mathcal{R}'_i(\alpha)$, for $1\le i\le 3$.

We give in Lemma 3 and Lemma 4 some properties of the sets $\mathcal{R}'(\alpha)$ and $\mathcal{R}''(\alpha)$ for a long and short root α, respectively. We know that the action of the Weyl group $\mathcal{W}$ is transitive on the set of roots for the Dynkin diagrams that have only simple edges $A_n, n\ge 1$, D_n, $n\ge 4$, E_6, E_7, and, E_8, and thus, every root α is long. For the other diagrams with multiple edges B_n, $n\ge 2$, C_n, $n\ge 3$, F_4, and G_2, there are two orbits which are given by the long and short roots, respectively.

Lemma 3 *For every **long** root α and $\beta\in\mathcal{R}$, we have the following:*

1. *$\beta\in\mathcal{R}'(\alpha)$ if and only if $<\alpha,\beta>=\pm\frac{1}{2}<\alpha,\alpha>$.*
2. *$\beta\in\mathcal{R}''(\alpha)$ if and only if $<\alpha,\beta>=0$ (i.e., $\mathcal{R}''(\alpha)$ is the orthogonal to α).*
3. *$2\alpha+\beta$ is a root if and only if $\beta=-\alpha$ and then $2\alpha+\beta=\alpha$.*
4. *$\mathcal{R}'_2(\alpha)=\mathcal{R}'_3(\alpha)=\emptyset$.*

Proof For nonproportional roots α and β, the Cartan integers satisfy (see Section 2) $\beta(H_\alpha) = p(\alpha,\beta) - q(\alpha,\beta) = 2\frac{<\alpha,\beta>}{<\alpha,\alpha>}$. Also, for a long root α and a root $\beta \neq \pm\alpha$, we have $2\frac{<\alpha,\beta>}{<\alpha,\alpha>} = 0$ or ± 1. That is clear for $\alpha = \mu$, the maximal root (μ is positive and $\mu + \beta \notin \mathcal{R}$ for every positive root β). Therefore, since there exists $w \in \mathcal{W}$ such that $w\alpha = \mu$, $\mathcal{W}\mathcal{R} = \mathcal{R}$, $<,>$ is invariant by the Weyl group and by Lemma 2, we obtain the result. □

The following corollary is an immediate consequence of Lemma 1 and Lemma 3. It characterizes some particular subalgebras of L depending on the sets $\mathcal{R}''(\alpha)$ and $\mathcal{R}'(\alpha)$.

Corollary 2 *Let $\mathcal{R}$ be a root system of a simple Lie algebra L.*

1. *For any **long** (resp. **short**) root α, the set $\mathcal{R}''(\alpha)$ is equal to (resp. contained in) the orthogonal to α. It defines a subalgebra of L generated by the root spaces L_β , $\beta \in \mathcal{R}''(\alpha)$ which commutes with the subalgebra isomorphic to $sl(2,\mathbb{R})$ generated by $L_{\pm\alpha}$.*
2. *For any **long** root α, the subalgebra of L generated by L_α and L_β, $\beta \in \mathcal{R}'(\alpha)$, such that $< \alpha, \beta >= \frac{1}{2} < \alpha, \alpha >$, is the Heisenberg Lie algebra.*

Proof For 1, it suffices to apply Lemma 1-1 and Lemma 3-2. Clearly, by definition, for any $\beta \in \mathcal{R}''(\alpha)$, we have $[L_{\pm\alpha}, L_\beta] = \{0\}$, and $[H_\alpha, L_\beta] = \{0\}$ since $\gamma(H_\alpha) = 2\frac{<\alpha,\beta>}{<\alpha,\alpha>} = 0$. For 2), the Heisenberg Lie algebra is given by $[L_\beta, L_{\alpha-\beta}] = L_\alpha$ with $\beta \in \mathcal{R}'(\alpha)$ such that $< \alpha, \beta >= \frac{1}{2} < \alpha, \alpha >$ and all other brackets vanish. Because for $\beta_i \in \mathcal{R}'(\alpha)$, $i = 1, 2$, such that $< \alpha, \beta_i >= \frac{1}{2} < \alpha, \alpha >$, if $\beta_1 + \beta_2$ is a root, then by Lemma 3-1, $< \alpha, \beta_1 + \beta_2 >=< \alpha, \alpha >$. Therefore, $\beta_1 + \beta_2 \notin \mathcal{R}'(\alpha) \cup \mathcal{R}''(\alpha)$, then $\beta_1 + \beta_2 = \alpha$ and, hence, $\beta_2 = \alpha - \beta_1$. □

In the exceptional case G_2, we know that for a short root α, $3\alpha + \beta$ can be a root for a root β. In the cases, $B_n, n \geq 2$, $C_n, n \geq 3$, and F_4, $m\alpha + \beta$ can be a root for some integers m, $-2 \leq m \leq 2$, and then we have $\mathcal{R}'_3(\alpha) = \emptyset$, for every short root α.

Lemma 4 *Let α be a **short** root.*

1. *In the case G_2, if $\beta \in \mathcal{R}'(\alpha)$ is short and if $q(\alpha,\beta) = 2$, then $2\alpha + \beta \in \mathcal{R}'_3(\alpha)$ and for every $\gamma \in \mathcal{R}''(\alpha)$ such that $q(\beta,\gamma) = 2$, we have $2\beta + \gamma = \pm\alpha$.*
2. *In the cases $\mathbf{B_n}$, $n \geq 2$, $\mathbf{C_n}$, $n \geq 3$, and $\mathbf{F}_4$ if $\beta \in \mathcal{R}'_2(\alpha)$, then β is long. The root system $\mathcal{R}$ is generated by $\mathcal{R}'_1(\alpha) \cup \{\pm\alpha\}$.*
3. *In the case $\mathbf{B_n}$, $n \geq 2$, for any **short** root β nonproportional to α, we have that both $\alpha + \beta$ and $-\alpha + \beta$ are long roots*

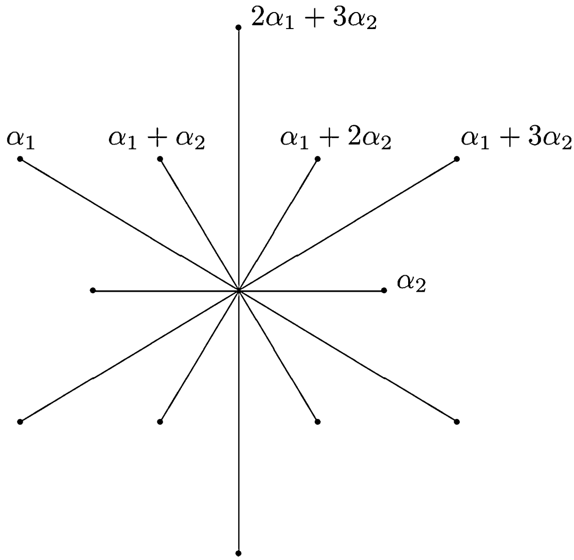
Remark 4 In the cases F_4 and $\mathbf{C_n}$, $n \geq 3$, we will show that the sum or the difference of two nonproportional short roots can be a long or short root.

Proof To prove Lemma 4, we give in each case of Dynkin diagrams with multiple edges the sets of roots $\mathcal{R}''(\alpha)$, $\mathcal{R}'(\alpha)$, and $\mathcal{R}'_i(\alpha)$, $1 \leq i \leq 3$, for a short root α. We can choose α by Lemma 2. Recall that $\mathcal{R} = -\mathcal{R}$ and for any $\alpha \in \mathcal{R}$, $2\alpha \notin \mathcal{R}$. A positive root is primitive if it cannot be written as a sum of two other positive roots. Any positive root α can be written as a sum of primitive roots, $\alpha = n_1\alpha_1 + ... + n_n\alpha_n$, where $n = dim\mathfrak{h}$, α_i is primitive and

n_i are unique positive integers for $1 \leq i \leq n$. Denote $\mathcal{R}^+$ the set of positive roots, and put $\mathcal{R}''^+(\alpha) = \mathcal{R}''(\alpha) \cap \mathcal{R}^+$ and $\mathcal{R}_i'^+(\alpha) = \mathcal{R}_i'(\alpha) \cap \mathcal{R}^+$, for $1 \leq i \leq 3$.

1. Consider the Dynkin diagram of G_2, where α_1 and α_2 are primitive roots:

Long and short roots are given by the following picture:



The set of positive short roots is $\{\alpha_2,\ \alpha_1 + \alpha_2,\ \alpha_1 + 2\alpha_2\}$. By Lemma 2, we can assume that $\alpha = \alpha_2$, and we verify easily that

$\mathcal{R}_1'(\alpha_2) = \emptyset,\ \mathcal{R}_2'^+(\alpha_2) = \{\alpha_1 + \alpha_2,\ \alpha_1 + 2\alpha_2\},\ \mathcal{R}_3'^+(\alpha_2) = \{\alpha_1,\ \alpha_1 + 3\alpha_2\}$, and $\mathcal{R}''^+(\alpha_2) = \{2\alpha_1 + 3\alpha_2\}$.

Note that $\mathcal{R}_2'(\alpha_2) \cup \{\pm\alpha_2\}$ and $\mathcal{R}_3'(\alpha_2) \cup \mathcal{R}''(\alpha_2)$ are exactly the sets of short and long roots, respectively. If $\beta \in \mathcal{R}_2'(\alpha_2)$, β is short and we can verify easily that if $2\alpha_2 + \beta \in \mathcal{R}$, then $-3\alpha_2 + (2\alpha_2 + \beta) = -\alpha_2 + \beta$ is a root and hence $2\alpha_2 + \beta \in \mathcal{R}_3'(\alpha_2)$. Further, for $\gamma \in \mathcal{R}''(\alpha_2)$, we have that $2\beta + \gamma$ is a root only if $\beta \in \{\alpha_1 + \alpha_2, -\alpha_1 - \alpha_2\}$ and then $2\beta + \gamma = \pm\alpha_2$. This proves Lemma 4-1.

2. Consider the Dynkin diagram of B_n, $n \geq 2$:

Primitive roots are $\{\alpha_1, \alpha_2, ..., \alpha_n\}$. The set of positive roots is as follows:

$$\begin{aligned}\mathcal{R}^+ &= \{\alpha_i, 1 \leq i \leq n\} \cup \{\alpha_i + ... + \alpha_j, 1 \leq i < j \leq n\} \\ &\quad \cup \{\alpha_i + ... + \alpha_{j-1} + 2(\alpha_j + ... + \alpha_n), 1 \leq i < j \leq n\}.\end{aligned}$$

The set of positive short roots is as follows:

$$\{\alpha_n\ ,\ \alpha_j + ... + \alpha_n\ ,\ 1 \leq j \leq n-1\}.$$

By Lemma 2, we can assume that $\alpha = \alpha_n$. For $n = 2$, we obtain $\mathcal{R}''(\alpha_n) = \emptyset$, and for $n \geq 3$, we obtain the following:

$$\mathcal{R}''^{+}(\alpha_n) = \{\beta \in \mathcal{R}^+, \beta = n_1\alpha_1 + ... + n_n\alpha_n \ / \ n_{n-1} = n_n = 0 \text{ or } n_{n-1} = n_n = 2\}.$$

For $n \geq 2$, we have the following:

$$\mathcal{R}'^{+}(\alpha_n) = \{\beta \in \mathcal{R}^+, \beta = n_1\alpha_1 + ... + n_n\alpha_n \ / \ n_{n-1} = 1\}.$$

Now, let β be a positive short root different from α, we have $\beta = \alpha_j + ... + \alpha_n$ with $1 \leq j \leq n-1$. Clearly, $\alpha_n + \beta = \alpha_j + ... + \alpha_{n-1} + 2\alpha_n$ and $-\alpha_n + \beta = \alpha_j + ... + \alpha_{n-1}$ are long roots, in particular $\beta \in \mathcal{R}'(\alpha_n)$.
We obtain easily that

$$\mathcal{R}_2'^{+}(\alpha_n) = \{\alpha_i + ... + \alpha_{n-1} \ , \ \alpha_i + ... + \alpha_{n-1} + 2\alpha_n, 1 \leq i \leq n-1\}.$$

Obviously, for any $\beta \in \mathcal{R}''(\alpha_n) \cup \mathcal{R}_2'(\alpha_n)$, β is long. Also, note that $\mathcal{R}$ can be generated by $\mathcal{R}_1'(\alpha_n) \cup \{\pm\alpha_n\}$, since every $\beta_j = \alpha_j + ... + \alpha_n \ , \ 1 \leq j \leq n-1$ belongs to $\mathcal{R}'(\alpha)$ and $\beta_j - \beta_{j-1} = \alpha_j$. Note that we can use the standard realization of B_n as $sl(2n+1, \mathbb{R})$. This is the Lie algebra of real matrices:

$$\begin{pmatrix} 0 & v & w \\ -w^T & A & B \\ -v^T & C & -A^T \end{pmatrix} \quad B + B^T = C + C^T = 0.$$

The Cartan subalgebra of real diagonal matrices

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & \Lambda & 0 \\ 0 & 0 & -\Lambda \end{pmatrix}$$

where $\Lambda = \text{diag}\{a_1, a_2, ..., a_n\}$, with a_i, $1 \leq i \leq n$ are nonzero and distinct. Short roots are $\pm a_i$ and long roots are $\pm(a_i \pm a_j)$ for $i \neq j$. Then, we can deduce that any sum (or difference) of short is a long root. One can identify $a_j = \alpha_j + ... + \alpha_n$ for $1 \leq j \leq n-1$ and $a_n = \alpha_n$. This proves 2 and 3 of Lemma 4.

3. Consider the Dynkin diagram of C_n, $n \geq 3$:

The set of primitive roots is $\{\alpha_1, \alpha_2, ..., \alpha_n\}$, and every primitive root α_i , $1 \leq i \leq n-1$ is short. We have the following:

$$\begin{aligned} \mathcal{R}^+ &= \{\alpha_i \ , 1 \leq i \leq n\} \cup \{\alpha_i + ... + \alpha_j \ , 1 \leq i < j \leq n\} \\ &\cup \{\alpha_i + ... + \alpha_{j-1} + 2(\alpha_j + ... + \alpha_{n-1}) + \alpha_n \ , 1 \leq i < j \leq n-1\} \\ &\cup \{2(\alpha_i + ... + \alpha_{n-1}) + \alpha_n \ , 1 \leq i \leq n-2\}. \end{aligned}$$

The set of positive *long* roots is

$$\{\alpha_n \ , 2(\alpha_i + ... + \alpha_{n-1}) + \alpha_n \ , 1 \leq i \leq n-1\}.$$

By Lemma 2, we can assume that $\alpha = \alpha_1$. We obtain easily that

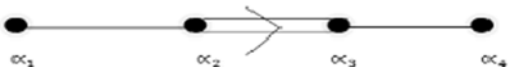
$$\mathcal{R}''(\alpha_1) = \{\beta \in \mathcal{R} \ , \beta = n_1\alpha_1 + ... + n_n\alpha_n \ /n_1 = n_2 = 0\}.$$

Note that if $\beta \in \mathcal{R}''(\alpha_1)$, then β can be short or long. Also, we can verify that if $\beta \in \mathcal{R}$ satisfies $2\alpha_1+\beta \in \mathcal{R}$, then $\beta = 2(\alpha_2+...+\alpha_{n-1})+\alpha_n$ or $\beta = -2(\alpha_1+...+\alpha_{n-1})-\alpha_n$. So, β is long and consequently

$$\mathcal{R}_2'^{+}(\alpha_1) = \{2(\alpha_2 + ... + \alpha_{n-1}) + \alpha_n \ , \ 2(\alpha_1 + ... + \alpha_{n-1}) + \alpha_n\}.$$

Observe that every root $\beta_j = \alpha_1 + ... + \alpha_j$, $2 \leq j \leq n$ is short and belongs to $\mathcal{R}_1'(\alpha_1)$. Since $\beta_j - \beta_{j-1} = \alpha_j$, we deduce that $\mathcal{R}$ can be generated by $\mathcal{R}_1'(\alpha_1) \cup \{\pm\alpha_1\}$. This shows Lemma 4-2. For Remark 4, observe that $\alpha_1 + \beta$ is long (resp. short) if $\beta = \alpha_1 + 2(\alpha_2 + ... + \alpha_{n-1}) + \alpha_n$ (reps. $\beta = \alpha_2 + ... + \alpha_{n-1} + \alpha_n$).

4. Consider the Dynkin diagram of F_4:



The set of primitive roots is $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$. A positive root $\alpha = n_1\alpha_1 + n_2\alpha_2 + n_3\alpha_3 + n_4\alpha_4$, with positive integers n_i, $1 \leq i \leq 4$, will be denoted by $n_1n_2n_3n_4$. The set of positive short roots is

$$\{0010;\ 0001;\ 0011;\ 0110;\ 0111;\ 0121;\ 1110;\ 1111;\ 1121;\ 1221;\ 1231;\ 1232\}.$$

The set of positive long roots is

$$\{1000;\ 0100;\ 1100;\ 1120;\ 1220;\ 0120;\ 0122;\ 1222;\ 1122;\ 1242;\ 1342;\ 2342\}.$$

By Lemma 2, we can assume that $\alpha = \alpha_3$. We obtain

$$\begin{aligned} \mathcal{R}''^{+}(\alpha_3) &= \{1000;\ 0122;\ 1220;\ 1122;\ 1342;\ 2342\}, \\ \mathcal{R}_2'^{+}(\alpha_3) &= \{0100;\ 0120;\ 1100;\ 1120;\ 1242;\ 1222\}. \end{aligned}$$

Observe that the union $\mathcal{R}''(\alpha_3) \cup \mathcal{R}_2'(\alpha_3)$ is exactly the set of all long roots, and hence, the set of short roots is $\mathcal{R}_1'(\alpha_3) \cup \{\pm\alpha_3\}$. Clearly, the root system $\mathcal{R}$ can be generated by $\mathcal{R}_1'(\alpha_3)$. One can check that for every short root β, nonproportional to α_3, we have $\alpha_3 + \beta$ or $-\alpha_3 + \beta$ is a root, and it can be long or short ($0010 + 0110 = 0120$ is long and $0010 + 0001 = 0011$ is short). This shows Remark 4. Also, for the benefit of the reader, the set of positive short roots β satisfying $\beta + \alpha_3$ or $\beta - \alpha_3$ is a long root, and there exists $\gamma \in \mathcal{R}''(\alpha)$ such that $2\beta + \gamma$ is a root, is $\{0110\ ;\ 1110\ ;\ 1232\}$. This ends the proof of Lemma 4. □

5 Proofs of the Main Results

In this section, we will prove Theorem 1, Theorem 2, Theorem 3, and the following proposition on which their proofs are based.

Proposition 1 *Let $\Gamma = \{A, \pm B\}$ and $L_\alpha \cup L_{-\alpha} \subset LS(\Gamma)$ for a root α, satisfying the conditions of Theorem 1 or Theorem 2. Then*

1. *For any $\beta \in \mathcal{R}'(\alpha)$ such that $A_\beta \neq 0$, we have $L_\beta \subset LS(\Gamma)$.*
2. *If $L_\beta \subset LS(\Gamma)$, for $\beta \in \mathcal{R}'(\alpha)$, then for any $\gamma \in \mathcal{R}''(\alpha)$ such that $A_\gamma \neq 0$, we have $[L_\beta, L_\gamma] \subset LS(\Gamma)$.*

In the proof of Proposition 1, we use some known ([8] and [9]) properties of a Lie-saturated cone $LS(\Gamma)$ for a subset $\Gamma \subset L$, that we recall in Lemma 5, and we show the property 4 of this lemma. The notation $\pm X \in LS(\Gamma)$ means that X and $-X$ belongs to $LS(\Gamma)$.

Lemma 5

1. *$LS(\Gamma)$ is a closed convex cone in L.*
2. *If $\pm X \in LS(\Gamma)$ and $\pm Y \in LS(\Gamma)$, then $\pm[X, Y]LS \in (\Gamma)$.*
3. *If $\pm X \in LS(\Gamma)$ and $Y \in LS(\Gamma)$, then for any $v \in \mathbb{R}$, $e^{v\mathrm{ad}X}(Y) \in LS(\Gamma)$.*
4. *Let $X \in L$, $X = X_0 + \sum_{\alpha\in\mathcal{R}} X_\alpha$, where $X_\alpha \in \mathbb{R}E_\alpha$. If $\pm X \in LS(\Gamma)$ and for any $X_\alpha \neq 0$, the eigenvalue of* ad$B$, *$\alpha(B)$ is nonzero and $\alpha(B) \neq \beta(B)$, for every root $\beta \neq \alpha$ such that $X_\beta \neq 0$, then $\pm X_\alpha \in LS(\Gamma)$.*

Proof For 4, the same proof given in [9] in the case where B is real strongly regular still holds: By 3, for any $v \in \mathbb{R}$, $e^{v\mathrm{ad}B}(X) \in LS(\Gamma)$. Write $e^{v\mathrm{ad}B}(X) = X_0 + \sum_{\alpha\in\mathcal{R}} e^{v\alpha(B)}(X_\alpha)$. Let $\alpha(B)$ be the maximal eigenvalue (nonzero) of ad$B$ such that $X_\alpha \neq 0$. Then, $e^{-v\alpha(B)}e^{v\mathrm{ad}B}(X)$ belongs to $LS(\Gamma)$ and $\lim_{v\to+\infty} e^{-v\alpha(B)}e^{v\mathrm{ad}B}(X) = X_\alpha \in LS(\Gamma)$. Since $-X \in LS(\Gamma)$, we deduce similarly that $-X_\alpha \in LS(\Gamma)$ and therefore $\pm(X - X_\alpha) \in LS(\Gamma)$. It suffices to apply the same argument to $X - X_\alpha$, and we obtain the result. $\square$

To prove Proposition 1, we show the following lemma:

Lemma 6

1. *For any root α and for any integer k, $2 \leq k \leq 3$, if $\pm\mathrm{ad}^k E_\alpha(A) \in LS(\Gamma)$, then for every root β, we have $\pm\mathrm{ad}^k E_\alpha(A_\beta) \in LS(\Gamma)$.*
2. *For any root α if $L_\alpha \subset LS(\Gamma)$, $\pm A_{-\alpha} \in LS(\Gamma)$ and $\pm\mathrm{ad}E_\alpha(A) \in LS(\Gamma)$, then $\pm\mathrm{ad}E_\alpha(A_\beta) \in LS(\Gamma)$ for every root β.*
3. *Let α be a long root or a short in the case of type G_2. Then*
   (a) *If $L_\alpha \cup L_{-\alpha} \subset LS(\Gamma)$, then $\pm\mathrm{ad}^2 E_\alpha(A) \in LS(\Gamma)$.*
   (b) *If $L_\beta \subset LS(\Gamma)$ for $\beta \in \mathcal{R}'(\alpha)$, then $\pm\mathrm{ad}^2 E_\beta(A_\gamma) \in LS(\Gamma)$, for every $\gamma \in \mathcal{R}''(\alpha)$.*

Proof Let $1 \leq k \leq 3$, and set $X = \mathrm{ad}^k E_\alpha(A) = \sum_{\beta\in\mathcal{R}(\alpha)} \mathrm{ad}^k E_\alpha(A_\beta)$. Clearly, $\mathrm{ad}^k E_\alpha(A_\beta) \neq 0$ if and only if $A_\beta \neq 0$ and $k\alpha+\beta$ is a root, and $(k\alpha+\beta)(B) = (k\alpha+\delta)(B)$ is equivalent to $\beta(B) = \delta(B)$. Since B is A-strongly regular, by applying Lemma 5-4, we obtain that if $(k\alpha + \beta)(B) \neq 0$, then $\pm X_{k\alpha+\beta} = \pm\mathrm{ad}^k E_\alpha(A_\beta) \in LS(\Gamma)$. Note that if there exists a root β such that $X_{k\alpha+\beta} \neq 0$ and $(k\alpha + \beta)(B) = 0$, then β is unique (because as above, if $(k\alpha + \beta)(B) = (k\alpha + \delta)(B) = 0$, then $\beta(B) = \delta(B)$ for $A_\beta \neq 0$ and $A_\delta \neq 0$, we obtain that $\beta = \delta$, since B is A-strongly regular). Denote this root by β_0, and write

$$X_{k\alpha+\beta_0} = X - X_0 - \sum_{\beta\in\mathcal{R},\ \beta\neq\beta_0} X_{k\alpha+\beta}.$$

Observe that for $2 \leq k \leq 3$, we have $X_0 = 0$ and then $\pm X_0 \in LS(\Gamma)$. For $k = 1$, we have $X_0 = A_{-\alpha}$, in this case $\pm A_{-\alpha} \in LS(\Gamma)$ and $\pm[E_\alpha, A_0] \in L_\alpha \subset LS(\Gamma)$. Since $\pm X \in LS(\Gamma)$ and $\pm\mathrm{ad}^k E_\alpha(A_\beta) \in LS(\Gamma)$ for $\beta \neq \beta_0$, we deduce that $\pm X_{k\alpha+\beta_0} \in LS(\Gamma)$. This proves 1 and 2. For 3,

- if α is a long root, then by Lemma 3-3, we have that $2\alpha+\beta \in \mathcal{R}$ if and only if $\beta = -\alpha$ and hence $2\alpha+\beta = \alpha$. Therefore, $\pm\mathrm{ad}^2 E_\alpha(A) \in L_\alpha \subset LS(\Gamma)$. Let $L_\beta \subset LS(\Gamma)$, for a root $\beta \in \mathcal{R}'(\alpha)$ and $\gamma \in \mathcal{R}''(\alpha)$. If β is long, then as above, we have $2\beta+\gamma$ is a root if and only if $\gamma = -\beta$. Since $\mathcal{R}'(\alpha) \cap \mathcal{R}''(\alpha) = \emptyset$, we deduce that $2\beta+\gamma \notin \mathcal{R}$ for every $\gamma \in \mathcal{R}''(\alpha)$, and then in this case, $\mathrm{ad}^2 E_\beta(A_\gamma) = 0$. If β is short and $2\beta+\gamma \in \mathcal{R}$, for a root $\gamma \in \mathcal{R}''(\alpha)$, then by Lemma 3, we deduce that $2\beta+\gamma = \pm\alpha$ (because $<\alpha, 2\beta+\gamma> = 2<\alpha,\beta> = \pm <\alpha,\alpha>$ implies that $2\beta+\gamma \notin \mathcal{R}''(\alpha)\cup\mathcal{R}'(\alpha)$). Then, in this case $\mathrm{ad}^2 E_\beta(A_\gamma) \in L_{\pm\alpha} \subset LS(\Gamma)$. Therefore, $\pm\mathrm{ad}^2 E_\beta(A_\gamma) \in LS(\Gamma)$.
- If α is a short root in the case G_2, we have for any root β, $4\alpha+\beta \notin \mathcal{R}$ and then $\mathrm{ad}^4 E_\alpha(A) = 0$. Using Lemma 5, we obtain that for every $v \in \mathbb{R}$,

  $$\lim_{v\to\pm\infty} \frac{1}{|v|^3} e^{v\mathrm{ad}E_\alpha}(A) = \pm\mathrm{ad}^3 E_\alpha(A) \in LS(\Gamma).$$

  Applying 1 to $X = \mathrm{ad}^3 E_\alpha(A) = \sum_{\beta\in\mathcal{R}} \mathrm{ad}^3 E_\alpha(A_\beta)$, we deduce that $\pm X_{3\alpha+\beta} = \pm\mathrm{ad}^3 E_\alpha(A_\beta) \in LS(\Gamma)$. Therefore, if there exists a root $\beta \in \mathcal{R}'_3(\alpha)$ such that $X_{3\alpha+\beta} \neq 0$, then $L_{3\alpha+\beta} \subset LS(\Gamma)$. Since $L_{-\alpha} \subset LS(\Gamma)$, it follows that $L_{2\alpha+\beta} \subset LS(\Gamma)$ and then $\pm\mathrm{ad}^2 E_\alpha(A) \in LS(\Gamma)$. Now, if $X = 0$, we have the following:

  $$Y = \lim_{v\to+\infty} \frac{1}{|v|^2} e^{v\mathrm{ad}E_\alpha}(A) = +\mathrm{ad}^2 E_\alpha(A) \in LS(\Gamma).$$

  Clearly, if $Y = 0$, we obtain the result. If $Y \neq 0$ since $\pm E_{-\alpha}$ and Y belong to $LS(\Gamma)$, we have by Lemma 5 that for every $v \in \mathbb{R}$,

  $$\lim_{v\to\pm\infty} \frac{1}{|v|^3} e^{v\mathrm{ad}E_{-\alpha}}(Y) = \pm\mathrm{ad}^3 E_{-\alpha}(Y) \in LS(\Gamma).$$

  Set $Z = \mathrm{ad}^3 E_{-\alpha}(Y) = \mathrm{ad}^3 E_{-\alpha}(\mathrm{ad}^2 E_\alpha(A))$. If $Y_{2\alpha+\beta} \neq 0$ (i.e., $A_\beta \neq 0$ and $2\alpha+\beta \in \mathcal{R}$), by Lemma 4–1), we have $2\alpha+\beta \in \mathcal{R}'_3(\alpha)$ and $-3\alpha+(2\alpha+\beta) = -\alpha+\beta$ is a root, then we obtain that $Z_{-\alpha+\beta} \neq 0$ and $\pm Z_{-\alpha+\beta} \in LS(\Gamma)$. Therefore, $L_{-\alpha+\beta} \subset LS(\Gamma)$. We deduce that $L_\beta \subset [L_\alpha, L_{-\alpha+\beta}] \subset LS(\Gamma)$. Then, $\pm A_\beta \in LS(\Gamma)$ and hence $\pm\mathrm{ad}^2 E_\alpha(A) \in LS(\Gamma)$. For 3-b, let $L_\beta \subset LS(\Gamma)$, for $\beta \in \mathcal{R}'(\alpha)$ and $\gamma \in \mathcal{R}''(\alpha)$ such that $A_\gamma \neq 0$. If β is long, we have seen that $2\beta+\gamma \notin \mathcal{R}$ and hence $\mathrm{ad}^2 E_\alpha(A_\beta) = 0$. If β is short and $2\beta+\gamma$ is a root, then by Lemma 4-1, $2\beta+\gamma = \pm\alpha$ and hence $\pm\mathrm{ad}^2 E_\alpha(A_\gamma) \in LS(\Gamma)$. This ends the proof of Lemma 6. □

5.1 Proof of Proposition 1

Let α be a root such that $L_\alpha \cup L_{-\alpha} \subset LS(\Gamma)$. If α is long or α is short in the case of type G_2, then by Lemma 6-3-a, we have $\pm\mathrm{ad}^2 E_\alpha(A) \in LS(\Gamma)$. If α is a short root in the case of type B_n, $n \geq 2$, C_n, $n \geq 3$, or F_4, we have by the condition 2 of Theorem 2 that $\mathrm{ad}^2 E_\alpha(A) = \sum_{\beta\in\mathcal{R}} \mathrm{ad}^2 E_\alpha(A_\beta) = 0$ and in this case $\pm\mathrm{ad}^2 E_\alpha(A) \in LS(\Gamma)$. Then by Lemma 6-1 for $k = 2$, we obtain that $\pm\mathrm{ad}^2 E_\alpha(A_\beta) \in LS(\Gamma)$. Therefore, if $A_\beta \neq 0$ and $2\alpha+\beta \in \mathcal{R}$, then $L_{2\alpha+\beta} \subset LS(\Gamma)$. Since $L_{-\alpha} \subset LS(\Gamma)$, by Lemma 5-2, we have $L_\beta = [L_{-\alpha}, [L_{-\alpha}, L_{2\alpha+\beta}]] \subset LS(\Gamma)$. This proves 1.

For 2, let $L_\beta \subset LS(\Gamma)$, for $\beta \in \mathcal{R}'(\alpha)$. It suffices to prove that $\pm\mathrm{ad}^2 E_\beta(A) \in LS(\Gamma)$, and then we deduce that $\pm[E_\beta, A] \in LS(\Gamma)$ as follows: If $\pm\mathrm{ad}^2 E_\beta(A) \in LS(\Gamma)$, then by

Lemma 5-2, $\pm\mathrm{ad}^k E_\beta(A) \in LS(\Gamma)$, for every $k \geq 3$. Therefore, for any $v \in \mathbb{R}$, $e^{v\mathrm{ad}E_\alpha}(A) - \sum_{k\geq 2} \frac{v^k}{k!}\mathrm{ad}^k E_\alpha(A) = E_\beta + v[E_\beta, A] \in LS(\Gamma)$ and hence

$$\lim_{v\to\pm\infty} \frac{1}{|v|}(A + v[E_\beta, A]) = \pm[E_\beta, A] \in LS(\Gamma).$$

Then by Lemma 6-2, we obtain that $\pm[E_\beta, A_\gamma] \in LS(\Gamma)$, for every root γ. In particular for $\gamma \in \mathcal{R}''(\alpha)$. Therefore, if $A_\gamma \neq 0$ and $\beta + \gamma \in \mathcal{R}$, then $[L_\beta, L_\gamma] \subset LS(\Gamma)$.

Now, we prove that $\pm\mathrm{ad}^2 E_\beta(A) \in LS(\Gamma)$. Since $\mathcal{R} = \mathcal{R}'(\alpha) \cup \mathcal{R}''(\alpha) \cup \{\alpha, -\alpha\}$, write

$$A = A_0 + A_\alpha + A_{-\alpha} + \sum_{\beta\in\mathcal{R}'(\alpha)} A_\beta + \sum_{\gamma\in\mathcal{R}''(\alpha)} A_\gamma.$$

Obviously, $\pm A_{\pm\alpha} \in L_{\pm\alpha} \subset LS(\Gamma)$ and by 1, we have $\pm A_\beta \in LS(\Gamma)$, for every $\beta \in \mathcal{R}'(\alpha)$.

- If α be a long root or a short in the case of type G_2, Lemma 6-3-b shows that $\pm\mathrm{ad}^2 E_\beta(A_\gamma) \in LS(\Gamma)$, for every $\gamma \in \mathcal{R}''(\alpha)$. Therefore, $\pm\mathrm{ad}^2 E_\beta(A) \in LS(\Gamma)$. Assume that α is a short root in the case of type B_n, $n \geq 2$, C_n, $n \geq 3$, or F_4. If β is long, we know that $2\beta + \gamma \notin \mathcal{R}$ (by Lemma 3-3) and since $\mathcal{R}'(\alpha) \cap \mathcal{R}''(\alpha) = \emptyset$) and then $\mathrm{ad}^2 E_\beta(A_\gamma) = 0$.
- If β is short and $2\beta + \gamma$ is a root, in the case B_n, $n \geq 2$, by Lemma 4-3, we have $\alpha + \beta$ and $-\alpha + \beta$ are long roots. If $2\beta + \gamma$ is a root, then $\beta + \gamma$ is also a root and by Lemma 1-2, we have $\beta + \gamma \in \mathcal{R}'(\alpha)$ and hence $\alpha + \beta + \gamma$ or $-\alpha + \beta + \gamma$ is also a root. Consequently, in the cases B_n, $n \geq 2$, if $\alpha + \beta$ is a long root, we have $L_{\alpha+\beta} = [L_\alpha, L_\beta] \subset LS(\Gamma)$ and then by Lemma 3-3, $L_{2(\alpha+\beta)+\gamma} \subset L_{\alpha+\beta} \subset LS(\Gamma)$, we deduce that if $\alpha+\beta+\gamma$ is a root, then $[L_{\alpha+\beta}, L_\gamma] = L_{\alpha+\beta+\gamma} \subset LS(\Gamma)$. Therefore, $L_{\beta+\gamma} \subset [L_{-\alpha}, L_{\alpha+\beta+\gamma}] \subset LS(\Gamma)$. Similarly, if $-\alpha + \beta$ is long, we obtain the result. In the case F_4, if $\alpha + \beta$ and $-\alpha + \beta$ are not long roots, the condition 2 of Theorem 2 gives that $A_\gamma = 0$ and thus $\mathrm{ad}^2 E_\beta(A_\gamma) = 0$.

In the case of type C_n, $n \geq 3$, or F_4, as above, if $\alpha + \beta$ or $-\alpha + \beta$ is a long root, we obtain the result if $\alpha + \beta$ and $-\alpha + \beta$ are not long roots and if $2\beta + \gamma$ is a root not proportional to $\pm\alpha$, the condition 2 of Theorem 2 gives that $A_\gamma = 0$ and thus $\mathrm{ad}^2 E_\beta(A_\gamma) = 0$. Therefore, $\pm\mathrm{ad}^2 E_\beta(A) \in LS(\Gamma)$. This ends the proof of Proposition 1.

5.2 Proof of Theorem 1 and Theorem 2

Recall that $LS(\Gamma) = L$ is equivalent to the fact that the semigroup generated by $\{\exp tX : X \in \Gamma, t \geq 0\}$ is equal to G. We consider the Lie subalgebra of L, denoted by $\mathcal{I}$, generated by $L_{\pm\alpha}$ and $L_\beta \subset LS(\Gamma)$ for $\beta \in \mathcal{R}'(\alpha)$. Clearly, by Lemma 5-2, it follows $\mathcal{I} \subset LS(\Gamma)$. By Proposition 1-1, if $A_\beta \neq 0$, $\beta \in \mathcal{R}'(\alpha)$, then $L_\beta \subset \mathcal{I}$. It suffices to show that $\mathcal{I}$ is an ideal (nontrivial) of L and therefore (since L is simple), we obtain $\mathcal{I} = L$. To prove this, since L is generated by $\{A_\delta \neq 0,\ \delta \in \mathcal{R}\}$ and $\mathcal{R} = \mathcal{R}'(\alpha) \cup \mathcal{R}''(\alpha) \cup \{\alpha, -\alpha\}$, it is enough to check that $[\mathcal{I}, L_\gamma] \subset LS(\Gamma)$ for every $\gamma \in \mathcal{R}''(\alpha)$ such that $A_\gamma \neq 0$. We have $[L_{\pm\alpha}, L_\gamma] = \{0\}$ and by Proposition 1-2, if $L_\beta \subset LS(\Gamma)$ for $\beta \in \mathcal{R}'(\alpha)$, then $[L_\beta, L_\gamma] \subset LS(\Gamma)$. Also, by Lemma 2-2, if $\beta + \gamma$ is a root, then it belongs to $\mathcal{R}'(\alpha)$. Therefore, we deduce that $[L_\beta, L_\gamma] = L_{\beta+\gamma} \subset \mathcal{I}$ and then $L \subset LS(\Gamma)$.

5.3 Proofs of Theorem 3

Now, we prove Theorem 3 concerning the case of type G_2. Assume that there exists a root α such that $L_\alpha \subset LS(\Gamma)$.

- If α is short, we have seen that $\pm \mathrm{ad}^3 E_\alpha(A) \in LS(\Gamma)$ and $\pm \mathrm{ad}^3 E_\alpha(A_\beta) \in LS(\Gamma)$. Also, $\mathrm{ad}^3 E_\alpha(A_\beta) \neq 0$, for any $\beta \in \mathcal{R}$ such that $3\alpha + \beta \in \mathcal{R}$ and $A_\beta \neq 0$. By Lemma 2, we can assume that $\alpha = \alpha_2$, then there exist exactly two roots β such that $3\alpha_2 + \beta \in \mathcal{R}$, namely, $\beta \in \{\alpha_1, -3\alpha_2 - \alpha_1\}$. Hence, if $A_{\alpha_1} \neq 0$ and $A_{-3\alpha_2-\alpha_1} \neq 0$, then $L_{3\alpha_2+\alpha_1} \subset LS(\Gamma)$ and $L_{-\alpha_1} \subset LS(\Gamma)$. But $3\alpha_2 + \alpha_1$ is a long root and since $A_{\alpha_1} \neq 0$, we obtain $L_{3\alpha_2+2\alpha_1} \subset LS(\Gamma)$. Also, $3\alpha_2 + 2\alpha_1$ is a long root and $A_{-3\alpha_2-\alpha_1} \neq 0$, hence $L_{\alpha_1} \subset LS(\Gamma)$. Therefore, $L_{\alpha_1} \cup L_{-\alpha_1} \subset LS(\Gamma)$, by Theorem 1, we obtain the result.
- if α is long, by Lemma 2, we can assume that $\alpha = \alpha_1$. For a long root β, if $\alpha_1 + \beta \in \mathcal{R}$, then $\beta \in \{\alpha_1 + 3\alpha_2, -2\alpha_1 - 3\alpha_2\}$. As above, we deduce that if $A_\beta \neq 0$, then $L_{3\alpha_2+2\alpha_1} \cup L_{-3\alpha_2-\alpha_1} \subset LS(\Gamma)$. Note that the root $-3\alpha_2 - \alpha_1$ is long and hence if $A_{-\alpha_1} \neq 0$, then $L_{-3\alpha_2-2\alpha_1} \subset LS(\Gamma)$. Therefore by Theorem 1, we obtain the result.

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