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Positive Orthant Scalar Controllability of Bilinear Systems

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ABSTRACT. For the bilinear control system $\dot{x} = (A + uB)x$, $x \in \mathbb{R}^n$, $u \in \mathbb{R}$, where A is an $n \times n$ essentially nonnegative matrix, and B is a diagonal matrix, the following controllability problem is investigated: can any two points with positive coordinates be joined by a trajectory of the system? For $n > 2$, the answer is negative in the generic case: hypersurfaces in $\mathbb{R}^n$ are constructed that are intersected by all the trajectories of the system in one direction.

§1. Introduction

Consider the following control system:

$$\dot{x} = (A + uB)x, \tag{1}$$

where $x \in \mathbb{R}^n$, A and B are constant real $n \times n$ matrices, and $u: \mathbb{R} \to \mathbb{R}$ is a piecewise constant unbounded control.

For any point $x \in \mathbb{R}^n$, we denote by $R(x)$ the set of all points in $\mathbb{R}^n$ which is accessible from x in arbitrary nonnegative time, i.e.,

$$R(x) = \{\gamma(T) : \gamma(\cdot) \ \text{ is the trajectory of system (1)}, \ \gamma(0) = x, \ T \geq 0\}.$$

The set $\mathbb{R}^n_+$, called the *positive orthant*, and the set $\mathring{\mathbb{R}}^n_+$, called the *open positive orthant*, are defined as

$$\mathbb{R}^n_+ = \{x \in \mathbb{R}^n : x \geq 0\}, \qquad \mathring{\mathbb{R}}^n_+ = \{x \in \mathbb{R}^n : x > 0\}.$$

The following definition was proposed by W. M. Boothby in [1]:

Definition 1. The system (1) is *positive orthant controllable* if for any $x \in \mathbb{R}^n \setminus \{0\}$ we have $R(x) \supset \mathring{\mathbb{R}}^n_+$.

The property of positive orthant controllability is meaningful only if any solution which lies in $\mathbb{R}^n \setminus \{0\}$ at $t = 0$ will remain in this set for all $t \geq 0$, i.e., when

$$\begin{aligned} &\bullet \text{ all off-diagonal elements } A = (a_{ij}) \text{ are nonegative;} \\ &\bullet\ B \text{ is diagonal} : B = \operatorname{diag}(b_1, \dots, b_n). \end{aligned} \tag{2}$$

This paper is concerned with positive orthant controllability of system (1) for $n > 2$, provided that conditions (2) are satisfied. In the generic case the answer to this question is shown to be negative. This follows from the existence of hypersurfaces which divide $\mathring{\mathbb{R}}^n_+$ into two parts and are intersected by all the trajectories of system (1) in one direction only. The result obtained is a generalization of the author's result for the three-dimensional case (see [2]).

For $n = 2$, this problem was fully investigated by A. Bacciotti in [3]. In the two-dimensional case the set of pairs of matrices (A, B) satisfying conditions (2) (for which system (1) is positive orthant controllable) has a nonempty interior and is not everywhere dense.

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§2. Auxiliary lemmas

First, we study the generic case, in which the largest and the smallest eigenvalues of the matrix B are distinct from all other eigenvalues:

$$B = \operatorname{diag}(b_1, \dots, b_n), \qquad b_1 < b_2 \le \dots \le b_{n-1} < b_n. \tag{3}$$

Consider the following function defined on $\overset{\circ}{\mathbb{R}}{}^n_+$:

$$H(x) = \sum_{i=2}^{n-1} x_1^{c_i} x_i x_n^{d_i},$$

where for $i = 1, \dots, n$ the coefficients c_i and d_i are defined by

$$c_i = \frac{b_i - b_n}{b_n - b_1}, \qquad d_1 = \frac{b_1 - b_i}{b_n - b_1}.$$

Note the following properties of these coefficients:

$$\begin{gathered} c_1 = d_n = -1, \qquad c_n = d_1 = 0, \qquad -1 < c_i < 0, \quad -1 < d_i < 0, \qquad i = 2, \dots, n-1; \\ c_i + d_i = -1, \quad b_1 c_i + b_i + b_n d_i = 0, \qquad i = 1, \dots, n. \end{gathered} \tag{4}$$

Lemma 2.1. *Let conditions* (3) *be satisfied. Then the function $H(x)$ is a first integral of the equation $\dot{x} = Bx$ in $\overset{\circ}{\mathbb{R}}{}^n_+$.*

Proof. Let us calculate the derivative of $H(x)$ using the equation given above. In view of (4), we have

$$\frac{dH}{dt} = \sum_{i=2}^{n-1} (c_i b_1 + b_i + d_i b_n) x_1^{c_i} x_i x_n^{d_i} \equiv 0.$$

The proof of the lemma is complete. $\square$

Thus, the vector field Bx touches any hypersurface $\{H(x) = C\}$ at all its points in $\overset{\circ}{\mathbb{R}}{}^n_+$. The direction of the intersection of the hypersurface $\{H(x) = C\}$ by the field Ax is determined by the sign of the function

$$f(x) = \langle \operatorname{grad} H(x), Ax \rangle.$$

Lemma 2.2. *Let $n > 2$ and $A = (a_{ij})$ be an $n \times n$ matrix such that*

$$a_{1n} > 0, \quad a_{n1} > 0, \quad a_{1j} \ge 0, \quad a_{nj} \ge 0, \quad a_{1j} + a_{nj} > 0, \qquad j = 2, \dots, n-1.$$

Let condition (3) *be satisfied. Then for sufficiently large C the following inequality holds*:

$$\left(f(x) \big|_{H(x)=C} \right) < 0.$$

Proof. We introduce homogeneous coordinates $u = (u_1, \dots, u_n)$: $u_i = x_i / x_1$, $i = 2, \dots, n$, and, moreover, we set $u_1 \equiv 1$. In the new coordinates, we have

$$f(u) = \sum_{i=2}^{n-1} \sum_{j=1}^{n} u_j u_n^{d_i} \left(c_i a_{1j} u_i + a_{ij} + d_i a_{nj} \frac{u_i}{u_n} \right), \qquad H(u) = \sum_{i=2}^{n-1} u_i u_n^{d_i}.$$

We want to show that

$$\left(f(x) \big|_{H(x)=C} \right) < 0 \quad \text{for sufficiently large } C. \tag{5}$$

It is easily seen that

$$f(u) = P_1(u) + N_1(u) + P_2(u) + N_2(u) + P_3(u) + N_3(u),$$

where $N_i(u)$ are nonnegative terms and $P_i(u)$ are terms with indeterminate sign:

$$N_1(u) = a_{1n} \sum_{i=2}^{n-1} c_i u_i u_n^{d_i+1}, \qquad N_2(u) = a_{n1} \sum_{i=2}^{n-1} d_i u_i u_n^{d_i-1},$$

$$N_3(u) = \sum_{i,j=2}^{n-1} (c_i a_{1j} u_i u_j u_n^{d_i} + d_i a_{nj} u_i u_j u_n^{d_i-1}),$$

$$P_1(u) = a_{11} \sum_{i=2}^{n-1} c_i u_i u_n^{d_i}, \qquad P_2(u) = \sum_{i=2}^{n-1} \sum_{j=1}^{n} a_{ij} u_j u_n^{d_i}, \qquad P_3(u) = a_{nn} \sum_{i=2}^{n-1} d_i u_i u_n^{d_i}.$$

Let us prove the assertion (5) by dividing the variation region of u into three parts and by making appropriate estimates in each part.

(A) First we prove that

$$\exists \varepsilon > 0 \, \forall C > 0 \quad \left(f(u)\big|_{H(u)=C, u_n<\varepsilon}\right) < 0.$$

Indeed, for small u_n, the sign of $\left(f(u)\big|_{H(u)=C}\right)$ is determined by the term $N_2(u)$: it is easily seen that

$$P_k(u) = o\big(N_2(u)\big), \qquad u_n \to 0, \quad k = 1, 2, 3.$$

(B) Further, we derive similar estimates for large u_n, i.e., we show that

$$\exists K > 0 \, \forall C > 0 \quad \left(f(u)\big|_{H(u)=C, u_n>K}\right) < 0,$$

using the fact that, for large u_n, the sign of $\left(f(u)\big|_{H(u)=C}\right)$ is defined by the term $N_1(u)$.

(C) Next we prove that

$$\exists \varepsilon > 0 \;\; \forall K > 0 \;\; \text{for sufficiently large } C \left(f(u)\big|_{H(u)=C, \varepsilon \le u_n \le K}\right) < 0,$$

The leading term is now $N_3(u)$: for $\varepsilon \le u_n \le K$ and sufficiently large C all the terms $P_k(u)$, $k = 1, 2, 3$, are suppressed by the negative term $N_3(u)$.

Equation (5) now follows, obviously, from (A), (B), and (C). □

§3. Conditions for uncontrollability

In this section two conditions are given for system (1) to be positive orthant uncontrollable. The first result embraces the generic case.

Theorem 3.1. *Let $n > 2$ and conditions* (2) *be satisfied. Suppose also that the following inequalities hold*:

$$a_{1n} > 0, \quad a_{n1} > 0, \quad a_{1j} + a_{nj} > 0, \qquad j = 2, \ldots, n-1; \qquad b_1 < b_2 \le \cdots \le b_{n-1} < b_n.$$

Then system (1) *is not positive orthant controllable.*

Proof. For sufficiently large C the hypersurface $\{H(x) = C\}$ is intersected by all trajectories of system (1) in one direction only, namely that of decreasing values of $H(x)$ (see Lemmas 2.1 and 2.2). □

Let us now prove another sufficient condition for uncontrollability; it relates to certain degenerate cases not included in the conditions of Theorem 3.1.

Theorem 3.2. *Suppose that $n > 2$ and conditions (2) are satisfied. Suppose that for some i, j such that $1 \leq i < j \leq n$ we have*

$$b_i = b_j, \qquad (\forall\, k \neq i \quad a_{ik} > 0) \quad \text{or} \quad (\forall\, k \neq j \quad a_{jk} > 0).$$

Then system (1) is not positive orthant controllable.

Proof. Suppose that the first condition ($\forall k \neq i$, $a_{ik} > 0$) is satisfied; if the second one is satisfied, we replace j by i.

Let us consider the function $G(x) = x_j/x_i$ and show that for a sufficiently large C the hyperplane $\{G(x) = C\}$ is intersected by all trajectories of system (1) in $\mathring{\mathbb{R}}^n_+$ in one direction only.

Indeed, the field Bx touches the hyperplane $\{G(x) = C\}$ by virtue of the relation $b_i = b_j$. The direction of intersection of this hyperplane by the field Ax is determined by the sign of the function

$$p(x) = \langle \operatorname{grad} G(x), Ax \rangle.$$

We introduce homogeneous coordinates: $v = (v_1, \dots, v_{i-1}, v_{i+1}, \dots, v_n)$, where $v_k = x_k/x_i$ for $k \neq i$. In these coordinates we have

$$G(v) = v_j, \qquad p(v) = \sum_{k \neq i,j} (a_{jk} - G(v)a_{ik})v_k + a_{ji} + a_{jj}G(v) - G(v)a_{ii} - G^2(v)a_{ij}.$$

By the method used in Lemma 2.2, it is readily shown that for sufficiently large C we have

$$\left(p(v)\big|_{G(v)=C}\right) < 0.$$

Therefore, for sufficiently large C the hyperplane $\{G(x) = C\}$ in $\mathring{\mathbb{R}}^n_+$ is intersected by all trajectories of system (1) in one direction only, namely that of decreasing values of $G(x)$. □

Although for $n > 2$ almost all systems (1) satisfying conditions (2) are not positive orthant controllable, for $n = 3$ we can give examples of such positive orthant controllable systems and generally furnish some sufficient conditions for controllability (related to degenerate cases only).

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