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Invariant Convex Cones and Causality in Semisimple Lie Algebras and Groups

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The convex cones in a simple Lie algebra $\mathfrak{G}$ invariant under the adjoint group G of $\mathfrak{G}$ are studied. Using a earlier abstract classification of such cones, we find explicit algebraic presentations of such cones in all the classical hermitian symmetric Lie algebras. (Nontrivial such cones exist only in these cases.) The G-orbits in such cones are listed. The notion of a *temporal action* of a Lie group with an invariant causal orientation upon a causally oriented manifold is defined. The canonical actions of such classical groups G as above on the Šhilov boundaries of the associated (tube-type) hermitian symmetric spaces are shown to be temporal actions. Corollaries are (1) the existence of nontrivial (Lie) semigroups S in the infinite-sheeted coverings $\tilde{G}$ of G, which are invariant under conjugation by $\tilde{G}$ and satisfy $S \cap S^{-1} = \{e\}$, and (2) the *global causality* (i.e., no "closed time-like curves") of such covering groups $\tilde{G}$.

Contents.

INTRODUCTION

Let G be any Lie group. A *causal orientation* (i.e., a specification of a convex cone of "future directions" in the tangent space at each point) of G,

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which is invariant under all left and right translations, is clearly completely determined by a convex cone in the Lie algebra $\mathfrak{G}$ of G, which is invariant under the adjoint group $\mathrm{Ad}(G)$ of $\mathfrak{G}$. We call an invariant convex cone C in $\mathfrak{G}$ a *causal cone* if C is nontrivial, closed, and satisfies $C \cap -C = \{0\}$.

Such causal cones do not always exist; in the case of G semisimple, they exist precisely when $\mathrm{Ad}(G)$ has a noncompact simple factor which is associated to an hermitian symmetric space, or equivalently when the Lie algebra of a maximal compact subgroup of $\mathrm{Ad}(G)$ has a nontrivial center. This fact is a corollary of results of Kostant, appearing in [16] (and presented in Section 2), concerning the existence of invariant convex cones in the space of a finite-dimensional representation of a Lie group. An additional necessary and sufficient condition in this general context has also been given recently by Vinberg [17].

The above work, however, leaves the questions of possible uniqueness and/or classification of such cones quite open, even in the especially interesting case (connected with causality in groups as above) of the adjoint representation. In [12], the causal cones in the classical simple Lie algebras were classified by a rather awkward case-by-case analysis, which, however, had sufficient repetition to suggest a general method [14], applicable also to the two exceptional algebras.

The general (abstract) picture obtained is that causal cones C are usually quite nonunique; in special cases, though, certain such C are distinguished by the possession of simple algebraic characterizations. The suggested infinite-dimensional analogues have led to new stability criteria, extending in part the theory of Krein and his school [10], for differential equations in Hilbert space [13], e.g., the hyperbolic P.D.E.studied in quantum field theory [15], chiefly by virtue of the circumstance that the interiors of such finite-dimensional C have interiors consisting of elliptic (in the group-theoretical sense) elements.

In this paper we recall and use the abstract theory in [14], identify in the classical algebras these distinguished cones (e.g., the unique, up to sign, maximal causal cones in $\mathfrak{G}$), and apply the results toward showing the *global causality* of the infinite-dimensional coverings of the adjoint groups. The orbits on the boundaries ∂C of the causal cones and some of their properties are also determined, using the classification due to Burgoyne and Cushman [1], relevant parts of which are summarized here. This classification is probably considerably more than what might be ultimately needed, as the nilpotent parts of the $X \in \partial C$ (for $\mathfrak{G}$ classical, at least) all turn out to have square zero, suggesting a possible algebraic characterization. This orbit classification is not needed to identify the maximal (or minimal) causal cones, or to prove the global causality results. Finally, we examine aspects of the cases of groups of ranks two and three in some detail in Chapter VII.

One geometrical feature seems worth pointing out explicitly. Given a real noncompact simple Lie algebra $\mathfrak{G}$, it seems well known (to representation theorists, at least [20]), that $\mathfrak{G}$ has exactly two (exactly one) minimal nilpotent orbits (resp., orbit) if and only if $\mathfrak{G}$ is (resp., is not) hermitian symmetric. Our "explanation" for the circumstance of two such orbits $\pm\mathcal{O}_+$ in the hermitian symmetric case, is that the convex hulls (no closure) of $\mathcal{O}_+$ and $-\mathcal{O}_+$, with $0 \in \mathfrak{G}$ adjoined to each, are in fact the two unique minimal causal cones in $\mathfrak{G}$ (cf. Section 5). In other words, one minimal nilpotent orbit is in the "past," and the other is in the "future."

Additional points of contact between causal cones and representation theory (holomorphic (relatively) discrete series) have emerged in the work [11] of Olshansky in the Soviet Union (private communications). Propositions 19.1 and 19.2 here are applicable to the "ladder" representations of $SU(2,2)$, which are analytic continuations of discrete series representations. However, because of the already observable applications of this theory in functional analytic contexts referred to above, we have here rigorously avoided the techniques of semisimple Lie theory without infinite dimensional analogues, and tried to be as algebraic as possible. For example, root systems are not used at all.

I. Preliminaries

1. *Preliminaries on Convex Cones*

We will need some standard facts about convex cones as, for example, in [3], to which we refer for proofs. In this section all vector spaces are finite-dimensional.

Definition. Let E be a real vector space. $C \subseteq E$ is *convex* if $\lambda x + (1-\lambda)x \in C$ whenever $x, y \in C$ and $\lambda \in [0,1]$. $C \subseteq E$ is a *cone* if $x \in C$ implies $\lambda x \in C$ for all $\lambda > 0$.

Elementary properties of convex cones are, for example: the closure or interior of a convex cone is also a convex cone. Less intuitive is

Lemma 1.1. *A convex cone which is dense in E is equal to E.*

There is a "separating hyperplane theorem" for convex cones.

Lemma 1.2. *If C is a closed convex cone in E and $x \notin C$, then there exists a linear functional f on E such that $f(x) > 0$ and $f(y) \leqslant 0$ for all $y \in C$.*

Corollary 1.3. *A convex cone, not all of E, is in some closed half-space.*

It is convenient to introduce a real positive-definite scalar product $\langle \cdot, \cdot \rangle$ in E to state the duality theorem for cones.

DEFINITION. Given any cone C in E, let $C^* = \{y: \langle y, x\rangle \geqslant 0$ for all $x \in C\}$. C^* is a closed convex cone, called the *dual cone* of C.

COROLLARY 1.4. *For any cone C in E, $(C^*)^*$ is the closed convex hull of C. In particular, if C is closed and convex, $(C^*)^* = C$.*

Proof. It follows from the definitions and Lemma 1.2.

Furthermore, an open (closed) convex cone is regularly open (resp., regularly closed) with respect to the space it spans. In fact, if C is any convex cone which spans E, $\overline{C^{\text{int}}} = \bar{C}$, $(\bar{C})^{\text{int}} = C^{\text{int}}$, so $C^{\text{int}} + \bar{C} = C^{\text{int}}$, where C^{int}, $\bar{C}$ denote the interior and closure of C.

2. *Existence of the Minimal Cone*

In this section we recall the general criterion of Kostant [16, p. 29] for the existence of invariant convex cones in semisimple Lie algebras.

DEFINITION. A *causal cone* in a real Lie algebra $\mathfrak{G}$ is a nonzero closed convex cone C in $\mathfrak{G}$ invariant under the adjoint group of $\mathfrak{G}$ and satisfying $C \cap -C = \{0\}$.

LEMMA 2.1. *Any nontrivial closed invariant convex cone C in a real simple Lie algebra satisfies $C \cap -C = \{0\}$.*

Proof. $C \cap -C$ is an ideal in $\mathfrak{G}$, hence must be $\{0\}$ by the invariance.

THEOREM 2.2 (Kostant). *Let G be a connected semisimple Lie group acting in a real finite-dimensional space V. Assume that K is a maximal compact subgroup of G. Then there exists a closed G-invariant convex cone C in V satisfying $C \cap -C = \{0\}$ if and only if V has a nonzero K-invariant vector.*

Proof. Let C be a cone with the given properties. By Lemma 1.2, there exists a linear functional f such that $f(x) \geqslant 0$ for all $x \in C$ and $f(z) > 0$ for some $z \in C$. Then

$$w = \int_K k(z)\, dk$$

is K-invariant and $f(w) > 0$ so $w \neq 0$.

Conversely, let $w \neq 0$ be K-invariant. Let $\mathfrak{k}$ be the Lie algebra of K and $\mathfrak{G} = \mathfrak{k} + \mathfrak{p}$ the Cartan decomposition. As G is a matrix group, its complexification G_c acts on $V_c = V + iV$, and the connected subgroup G_u corresponding to $\mathfrak{k} + i\mathfrak{p}$ is compact. We have $K \subset G_u$, and G_u leaves invariant some complex Hilbert structure $\langle \cdot, \cdot \rangle$. All $X \in \mathfrak{k} + i\mathfrak{p}$ are then skew-hermitian on V_c, so all $X \in \mathfrak{p}$ are hermitian and $g \in \exp \mathfrak{p}$ positive-definite hermitian.

Any $g \in G$ can be written uniquely as $(\exp X) k$ for $X \in \mathfrak{p}$, $k \in K$. Thus $\langle gw, w \rangle = \langle (\exp X) w, w \rangle > 0$. Now $\langle gu, v \rangle = \langle u, \Theta(g^{-1}) v \rangle$ for all $g \in G$, $u, v \in V$, where $\Theta: G \to G: (\exp X) k \to (\exp -X) k$ is the Cartan involution corresponding to $\theta: X + Y \to X - Y$ for $X \in \mathfrak{k}$, $Y \in \mathfrak{p}$. Letting C_0 denote the convex cone generated by the gw for $g \in G$, it is clear that C_0 is a G-invariant convex cone such that $\langle u, v \rangle > 0$ for all $u, v \in C_0$, whose closure $\overline{C_0}$ has the desired properties. Q.E.D.

COROLLARY 2.3. *Let $\mathfrak{G}$ be a semisimple Lie algebra and G the adjoint group of $\mathfrak{G}$. Let K be any maximal compact subgroup of G, with Lie algebra $\mathfrak{k}$. Then there is a causal cone in $\mathfrak{G}$ if and only if $\mathfrak{k}$ has a nontrivial center.*

Proof. Take $V = \mathfrak{G}$ in Theorem 2.2. As K acts irreducibly on the simple components of the orthogonal complement of $\mathfrak{k}$ in $\mathfrak{G}$, any nonzero K-fixed vector must be in $\mathfrak{k}$. Q.E.D.

It is clear that any invariant convex cone in a semisimple $\mathfrak{G}$ is contained in the direct sum of invariant cones in the simple summands, and we restrict to the simple case from now on. It is well known that if $\mathfrak{G}$ is a real simple Lie algebra the dimension of the center $Z(\mathfrak{k})$ is either 0 or 1. Thus by Lemma 2.1 a simple $\mathfrak{G}$ admits a nontrivial invariant convex cone if and only if $\dim Z(\mathfrak{k}) = 1$. It is a theorem of Cartan that such is the case if and only if the associated symmetric space G/K is hermitian symmetric.

The integration argument above shows that in this case there are unique minimal causal cones $\pm\overline{C_0}$. By the simplicity of $\mathfrak{G}$, the positive-definite K-invariant form $\langle \cdot, \cdot \rangle$ used in the proof of Theorem 2.2 was on $\mathfrak{G}$ equal to $B_\theta(\cdot, \cdot) = -B(\cdot, \theta \cdot)$ up to a scalar, where B is the Killing form of $\mathfrak{G}$. The proof shows that $B_\theta(X, Y) \geqslant 0$ for all $X, Y \in \overline{C_0}$, but this is also a consequence of the following two observations: (1) If C is a causal cone in $\mathfrak{G}$ then its dual C^* with respect to the K-invariant form B_θ, is also a causal cone, i.e., is G-invariant, and (2) $\overline{C_0}$, being minimal, must contain its dual cone $(\overline{C_0})^*$. Minimality of $\overline{C_0}$ also implies that $(\overline{C_0})^*$ is maximal, i.e., is contained in no larger causal cone.

We see that no compact or complex simple Lie algebra admits a causal cone. There are four classical families of hermitian symmetric algebras,

$$sp(n, \mathbb{R}) \qquad (n \geqslant 1),$$

$$su(p, q) \qquad (p \geqslant q \geqslant 1),$$

$$so^*(2n) \qquad (n \geqslant 3),$$

$$so(2, n) \qquad (n \geqslant 3),$$

in the notation of [6], as well as two exceptional algebras. With the above restrictions on the indices the coincidental isomorphisms are

$$sp(1, \mathbb{R}) \approx su(1, 1), \qquad sp(2, \mathbb{R}) \approx so(2, 3),$$

$$su(2, 2) \approx so(2, 4), \qquad su(3, 1) \approx so^*(6), \qquad so(2, 6) \approx so^*(8).$$

A glance at the list on p. 516 of [6] indicates:

COROLLARY. *A complex simple Lie algebra $\mathfrak{G}$ admits a noncompact real form having a causal cone if and only if the adjoint group corresponding to a compact real form of $\mathfrak{G}$ is not simply connected.*

3. *Certain Linear Groups*

We would like to be able to describe in some fashion the orbits on the boundaries of the causal cones (the interiors being generic and elliptic, cf. III, Section 5). The language we adopt is the orbit classification for linear groups of Burgoyne and Cushman [1]. This classification is not needed in order to identify the minimal and maximal causal cones, and is used only to treat the boundaries.

We first give their uniform algebraic description of the nonexceptional semisimple Lie groups. Let V be a finite-dimensional complex vector space, always nonzero, and $G = GL(V, \mathbb{C})$. As in [1], we let $G(V, \sigma, \tau)$ (resp., $L(V, \sigma, \tau)$) be a generic symbol for the groups (resp., Lie algebras) described below. In this notation, σ is a conjugate linear operator in V with square $\pm I$, and τ is either a nondegenerate symmetric or antisymmetric complex-bilinear form on V, *or* a nondegenerate hermitian form on V. In this latter case τ will be written τ_*. Then $G(V, \sigma, \tau)$ $(L(V, \sigma, \tau))$ denotes the $g \in G$ (resp., $X \in gl(V, \mathbb{C})$) commuting with σ and preserving (resp., skew with respect to) τ. Of course for certain groups either σ or τ or both may not actually occur in the definition of the group. We agree that σ is absent when τ denotes τ_*, and that when τ and σ are both present they satisfy $\tau(\sigma v, \sigma w) = \overline{\tau(v, w)}$ for all $v, w \in V$. Therefore in this picture there are three families of complex Lie groups and seven families of real Lie groups.

If $\sigma^2 = I$, $V_\sigma^+ = \{v \in V: \sigma v = v\}$ is a real form of V, and $g \in G$ commutes with σ if and only if g leaves V_σ^+ invariant.

If $\sigma^2 = -I$, V must have even (complex) dimension, and V receives the structure of a quaternionic vector space as follows. Letting 1, i, j, k be a basis for the quaternions $\mathbb{Q}$ as usual, define $(\alpha+\beta j)v = \alpha v + \beta\sigma(v)$ for $\alpha, \beta \in \mathbb{C}$, $v \in V$. If τ is a complex-bilinear form on V,

$$\tau_-(u, v) = \tau(u, v) + \tau(u, \sigma v) j, \qquad \text{for } u, v \in V,$$

defines a nondegenerate $\mathbb{Q}$-valued form on V. It satisfies $\tau_-(\lambda u, \mu v) = \lambda\tau_-(u, v)\mu^q$, $\lambda, \mu \in \mathbb{Q}$, where $\lambda \in \mathbb{Q} \to \lambda^q \in \mathbb{Q}$ is the unique antiautomorphism of $\mathbb{Q}$ fixing 1, i, k and sending j to $-j$. If $\tau(u, v) = (-1)^m \tau(v, u)$, then $\tau_-(u, v) = (-1)^m \tau_-(v, u)^q$.

Real forms of $O(n, \mathbb{C})$. If τ is symmetric and $\sigma^2 = I$, $\tau|_{V_\sigma}$ has a certain real signature (p, q), and $G(V, \sigma, \tau)$ is isomorphic to some $O(p, q)$. If $\sigma^2 = -I$, $G(V, \sigma, \tau)$ is always isomorphic to $SO^*(2n)$ ($\dim V = 2n$), a connected group. Note that $SO^*(2n)$ transforms those *quaternionic bases* $\{a_i\}$ of V such that $\tau_-(a_i, a_j) = \delta_{ij}$.

Real forms of $Sp(n, \mathbb{C})$. If τ is antisymmetric dim V is necessarily even. If $\sigma^2 = I$, $G(V, \sigma, \tau)$ is always isomorphic to some $Sp(n, \mathbb{R})$, a connected group. If $\sigma^2 = -I$, $G(V, \sigma, \tau)$ is isomorphic to some $Sp(p, q)$, the pair (p, q) being determined by the quaternionic signature of τ_-. In this case $\tau_-(u, v) = -\tau_-(v, u)^q$, and $\mathbb{Q}$-bases $\{a_i\}$ exist for V such that $\tau_-(a_r, a_s) = \pm\delta_{rs} j$ [2].

4. *Classification of Orbits* (*Burgoyne and Cushman*)

We define the notion of a *type* Δ, an equivalence class of pairs (A, V), $A \in L(V, \sigma, \tau)$, under the obvious notion of equivalence, so that if $A, B \in L(V, \sigma, \tau)$, there exists $g \in G(V, \sigma, \tau)$ such that $g^{-1}Ag = B$ if and only if (A, V) and (B, V) belong to the same type. There is the obvious notion of the sum $\Delta_1 + \Delta_2$ of two types Δ_1, Δ_2 belonging to the same family. A type Δ is *indecomposable* if it cannot be written as the sum of two other types.

Given any $A \in L(V, \sigma, \tau)$, one can write it uniquely as $A = S + N$, where $S, N \in L(V, \sigma, \tau)$, S is semisimple, N is nilpotent, and $SN = NS$.

DEFINITIONS. (1) Let $m \geqslant 0$ be the unique integer such that $N^m \neq 0$ and $N^{m+1} = 0$ in the above. m is called the *height* of (A, V), and the notation $ht\,\Delta$ for any type Δ is well defined. Clearly $\operatorname{Ker} N^m \supseteq NV$; if equality holds, we say that the type of (A, V) is *uniform*.

(2) If $\operatorname{ht}\Delta = 0$ call Δ a *semisimple type*. A semisimple type is uniform.

There is a natural mapping of uniform types to semisimple types, defined as follows. Let Δ be uniform and $m = \operatorname{ht}\Delta$. If $(A, V) \in \Delta$ put $\bar{V} = V/NV$, and for $v \in V$ put $\bar{v} = v + NV$. Define $\bar{A}$, $\bar{\sigma}$, and $\bar{\tau}$ on $\bar{V}$ by $\bar{A}\bar{v} = \overline{Av}$, $\bar{\sigma}\bar{v} = \overline{\sigma v}$, and $\bar{\tau}(\bar{u}, \bar{v}) = \tau(u, N^m V)$. Since τ is nondegenerate on V and (A, V) is uniform, $\bar{\tau}$ is nondegenerate on $\bar{V}$. $G(\bar{V}, \bar{\sigma}, \bar{\tau})$ and $L(\bar{V}, \bar{\sigma}, \bar{\tau})$ are well defined, and

$A \in L(\bar{V}, \bar{\sigma}, \bar{\tau})$. Let $\bar{\Delta}$ denote the type containing $(\bar{A}, \bar{V})$; $\bar{\Delta}$ is semisimple. Note $\bar{\Delta}$ has nothing to do with complex conjugation in $\mathbb{C}$.

Remark. $G(\bar{V}, \bar{\sigma}, \bar{\tau})$ may be a group in a different class than $G(V, \sigma, \tau)$: if τ is complex bilinear and $\tau(u, v) = \lambda\tau(v, u)$, $\lambda = \pm 1$, then $\bar{\tau}$ satisfies $\bar{\tau}(\bar{u}, \bar{v}) = \lambda(-1)^m \bar{\tau}(\bar{v}, \bar{u})$. If τ denotes τ_* one can assume that $\bar{\tau}$ is again an Hermitian form by replacing $\bar{\tau}$ by $i\bar{\tau}$ if necessary.

THEOREM 4.1. *The decomposition of a type $\Delta = \Delta_1 + \cdots + \Delta_s$ into indecomposable types (which clearly exists) is unique.*

PROPOSITION 4.2. *If Δ is indecomposable then Δ is uniform and $\bar{\Delta}$ is indecomposable.*

PROPOSITION 4.3. *If Δ is uniform it is uniquely determined by* $\operatorname{ht} \Delta$ *and* $\bar{\Delta}$.

The proofs of these results [1] involve only linear algebra and one quoted result, namely, Sylvester's theorem on the signature.

We next give the simple construction recovering a uniform type Δ from $\operatorname{ht} \Delta$ and the semisimple type $\bar{\Delta}$, as in Proposition 4.3. Let $(S, E) \in \bar{\Delta}$ and $S \in L(E, \sigma, \bar{\tau})$. Let V be the direct sum of $m+1$ copies of E, written conveniently as $E + NE + \cdots + N^m E$. Extend σ and S to V, acting componentwise on this decomposition, and let N act on V in the obvious way. Then $N^m \neq 0$, $N^{m+1} = 0$, and $\ker N^m = NV$. To define τ on V, let $\tau(N^r E, N^s E) = 0$ unless $r + s = m$, and in that case let $\tau(N^r e_1, N^s e_2) = (-1)^r \bar{\tau}(e_1, e_2)$ for all $e_1, e_2 \in E$. Then clearly $G(V, \sigma, \tau)$ is one of the above groups, $A = S + N \in L(V, \sigma, \tau)$, and $(A, V) \in \Delta$.

By the above, the problem of the classification of the orbits under the adjoint group of any of these algebras reduces to the determination of the semisimple indecomposable types, at least in the case where G is connected. This description is straightforward for the complex groups and the $U(p, q)$. The results are as follows. Let Δ be a semisimple indecomposable type for either a complex G or $U(p, q)$ and $(S, W) \in \Delta$.

$G = GL(W)$. W is one-dimensional and Δ is determined by a single eigenvalue $\zeta \in \mathbb{C}$. Denote this type by $\Delta(\zeta)$.

$G = O(W, \tau)$. If $S = 0$, W is one-dimensional. There is a basis element e such that $\tau(e, e) = 1$. Denote this type by $\Delta(0)$.

If $S \neq 0$ the set of eigenvalues for S is $\{\zeta, -\zeta\}$ for some $0 \neq \zeta \in \mathbb{C}$. W is two-dimensional and there is a basis $\{e, f\}$ such that $Se = \zeta e$, $Sf = -\zeta f$, $\tau(e, e) = \tau(f, f) = 0$ and $\tau(e, f) = 1$. Denote this type by $\Delta(\zeta, -\zeta)$.

$G = Sp(W, \tau)$. Whether $S \neq 0$ or not, $\dim W = 2$, and there is a basis $\{e, f\}$ and $\zeta \in \mathbb{C}$ such that $Se = \zeta e$, $Sf = -\zeta f$, and $\tau(e, f) = 1$. Denote this type also by $\Delta(\zeta, -\zeta)$.

$G = U(p, q)$. $\dim W$ is either 2 or 1. When $\dim W = 2$ there is a basis $\{e, f\}$ and $\zeta \in \mathbb{C}$ such that $\zeta \neq -\bar{\zeta}$, where $\tau_*(e, e) = \tau_*(f, f) = 0$, $\tau_*(e, f) = 1$, and $Se = \zeta e$, $Sf = -\bar{\zeta} f$. Denote this type by $\Delta(\zeta, -\bar{\zeta})$.

When $\dim W = 1$ there is $\zeta \in \mathbb{C}$ such that $\bar{\zeta} = -\zeta$ and a basis element e such that $Se = \zeta e$ and $\tau_*(e, e) = \pm 1$. The two signs give different types. Denote these by $\Delta^{\pm}(\zeta)$.

For the real groups $G(V, \sigma, \tau)$ one proceeds as follows. An indecomposable semisimple type Δ for $G(V, \sigma, \tau)$ clearly gives rise to a type Δ^c for the corresponding complex group $G(V, \tau)$: just omit σ. Let $(S, V) \in \Delta$. Δ^c is a sum of semisimple indecomposable types for $G(V, \tau)$, so let Δ_1^c be an indecomposable component of Δ^c. Let $(S, W) \in \Delta_1^c$, where $W \subseteq V$. Clearly $(S, \sigma W)$ is also a type for $G(V, \tau)$, and we have either $\sigma W = W$ or or $\sigma W \cap W = \{0\}$. Since $\sigma^2 = \pm I$ and Δ is indecomposable we have three possible decompositions for Δ^c:

(a) $\Delta^c = \Delta_1^c + \sigma\Delta_1^c$ and $\Delta_1^c \neq \sigma\Delta_1^c$;

(b) $\Delta^c = \Delta_1^c + \sigma\Delta_1^c$ and $\Delta_1^c = \sigma\Delta_1^c$;

(c) $\Delta^c = \Delta_1^c$ and $\Delta_1^c = \sigma\Delta_1^c$.

Let $\operatorname{eig} \Delta_1^c$ denote the set of eigenvalues of S on W. They prove the following [1].

LEMMA A.1. *Decomposition* (a) *occurs iff* $\overline{\operatorname{eig} \Delta_1^c} \neq \operatorname{eig} \Delta_1^c$.

LEMMA A.2. *Suppose* $S \neq 0$. *Then* (b) *occurs iff* $\sigma^2 = -I$ *and all elements of* $\operatorname{eig} \Delta_1^c$ *are real.*

We consider next each of the six remaining classes of real groups, and indicate the classification of and notation for the semisimple indecomposable types, based mostly on Lemmas A.1 and A.2.

$G = GL(V)$

$\sigma^2 = I$. Let $\Delta_1^c = \Delta(\zeta)$.

(1) $\zeta \neq \bar{\zeta}$; (a) by A.1; type $\Delta(\zeta, \bar{\zeta})$.

(2) $\zeta = \bar{\zeta} \neq 0$; (c) by A.1 and A.2; type $\Delta(\zeta)$.

(3) $\zeta = 0$; (c) because if (b) and $W = \mathbb{C}e$, $\mathbb{C}(e + \sigma e)$ is σ-invariant, contrary to indecomposability; type $\Delta(0)$.

$\sigma^2=-I$. Let $\Delta_1^c=\Delta(\zeta)$.

(1) $\zeta\neq\bar\zeta$; (a) by A.1; type $\Delta(\zeta,\bar\zeta)$.

(2) $\zeta=\bar\zeta\neq 0$; (b) by A.2; type $\Delta(\zeta,\zeta)$.

(3) $\zeta=0$; (b) as dim V must be even; type $\Delta(0,0)$.

$G=O(V,\tau)$

$\sigma^2=I$. Let $\Delta_1^c=\Delta(0)$.

(1) Clearly (c); sig $\tau_+=(1,0)$ or $(0,1)$; two types $\Delta^\pm(0)$.

Let $\Delta_1^c=\Delta(\zeta,-\zeta)$, $\zeta\neq 0$.

(2) $\zeta\neq\pm\bar\zeta$; (a) by A.1; sig $\tau_+=(2,2)$; type $\Delta(\zeta,-\zeta,\bar\zeta,-\bar\zeta)$.

(3) $\zeta=\bar\zeta$; (c) by A.1, A.2; sig $\tau_+=(1,1)$; type $\Delta(\zeta,-\zeta)$.

(4) $\zeta=-\bar\zeta$; again (c); sig $\tau_+=(2,0)$ or $(0,2)$; two types $\Delta^\pm(\zeta,-\zeta)$.

$\sigma^2=-I$. Let $\Delta_1^c=\Delta(0)$.

(1) Decomposition (b) as V must be even-dimensional; type $\Delta(0,0)$.

Let $\Delta_1^c=\Delta(\zeta,-\zeta)$, $\zeta\neq 0$.

(2) $\zeta\neq\pm\bar\zeta$; (a) by A.1; type $\Delta(\zeta,-\zeta,\bar\zeta,-\bar\zeta)$.

(3) $\zeta=\bar\zeta$; (b) by A.2; type $\Delta(\zeta,-\zeta,\bar\zeta,-\bar\zeta)$.

(4) $\zeta=-\bar\zeta$; (c) by A.1, A.2; two types $\Delta^\pm(\zeta,-\zeta)$ depending on the signature of $\tau_-(\cdot,S\cdot)$ (V is one-dimensional over $\mathbb{Q}$.)

$G=Sp(V,\tau)$

$\sigma^2=I$. Let $\Delta_1^c=\Delta(\zeta,-\zeta)$.

(1) $\zeta\neq\pm\bar\zeta$; (a) by A.1; type $\Delta(\zeta,-\zeta,\bar\zeta,-\bar\zeta)$.

(2) $\zeta=\bar\zeta\neq 0$; (c) by A.1, A.2; type $\Delta(\zeta,-\zeta)$.

(3) $\zeta=-\bar\zeta\neq 0$; (c) by A.1, A.2; $\tau(S\cdot,\cdot)$ either positive or negative definite on V_σ^+; two types $\Delta^\pm(\zeta,-\zeta)$.

(4) $\zeta=0$; clearly (c); type $\Delta(0,0)$.

$\sigma^2=-I$. Let $\Delta_1^c=\Delta(\zeta,-\zeta)$.

(1) $\zeta\neq\pm\bar\zeta$; (a) by A.1; type $\Delta(\zeta,-\zeta,\bar\zeta,-\bar\zeta)$.

(2) $\zeta=\bar\zeta\neq 0$; (b) A.2; type $\Delta(\zeta,-\zeta,\zeta,-\zeta)$.

(3) $\zeta=-\bar\zeta\neq 0$; (c) by A.1, A.2; two types $\Delta^\pm(\zeta,-\zeta)$ depending on sig $\tau_-(\cdot,\cdot)$.

(4) $\zeta=0$; clearly (c); again two types $\Delta^\pm(0,0)$.

Write $\sigma=\sigma_\pm$, $\tau=\tau_{\pm 1}$ according to whether $\sigma^2=\pm I$, $\tau(u,v)=(\pm 1)\,\tau(v,u)$.

Using the construction outlined before we can now list the indecomposable types in the hermitian symmetric case. If $\bar{\Delta}(\cdots)$ $(\bar{\Delta}^{\epsilon}(\cdots))$ is a semisimple type for $G(V, \sigma_{\pm}, \tau_j)$ $(j=0$ or $1)$ and $m \geqslant 0$, let $\Delta_m(\cdots)$ (resp. $\Delta^{\epsilon}_m(\cdots)$) denote the corresponding indecomposable type for $G(V + \cdots + N^m V,\ \sigma_{\pm},\ \tau_{j(-1)^m})$ (see the *Remark* in this section) of height m. Set $\delta = (-1)^{m/2}\varepsilon$ for m even, $\varepsilon = \pm 1$.

Indecomposable Types for Hermitian Symmetric Lie Algebras

Group	Type	Conditions	Signature
$GL(V, \tau_*)$ $\lvert \supset SU(p, q) \rvert$	$\Delta_m(\zeta, -\bar{\zeta})$	$\zeta \neq -\bar{\zeta}$	$(m+1, m+1)$
	$\Delta^{\epsilon}_m(\zeta)$	$\zeta = -\bar{\zeta}$	$\begin{cases} m \text{ even} \left(\frac{1}{2}(m+1+\delta), \frac{1}{2}(m+1-\delta)\right) \\ m \text{ odd} \left(\frac{m+1}{2}, \frac{m+1}{2}\right) \end{cases}$
$O(V, \sigma_+, \tau)$ $\lvert = O(p, q) \rvert$	$\Delta_m(\zeta, -\zeta, \bar{\zeta}, -\bar{\zeta})$	$\zeta \neq \pm\bar{\zeta}$	$(2(m+1), 2(m+1))$
	$\Delta_m(\zeta, -\zeta)$	$\zeta = \bar{\zeta} \neq 0$	$(m+1, m+1)$
	$\Delta^{\epsilon}_m(\zeta, -\zeta)$	$\zeta = -\bar{\zeta} \neq 0$	$\begin{cases} m \text{ even} (m+1+\delta, m+1-\delta) \\ m \text{ odd} (m+1, m+1) \end{cases}$
	$\Delta^{\epsilon}_m(0)$	m even	$\left(\frac{1}{2}(m+1+\delta), \frac{1}{2}(m+1-\delta)\right)$
	$\Delta_m(0, 0)$	m odd	$(m+1, m+1)$
$O(V, \sigma_-, \tau)$ $\lvert = SO^*(2n) \rvert$	$\Delta_m(\zeta, -\zeta, \bar{\zeta}, -\bar{\zeta})$	$\zeta \neq -\bar{\zeta}$	
	$\Delta^{\epsilon}_m(\zeta, -\zeta)$	$\zeta = -\bar{\zeta} \neq 0$	
	$\Delta_m(0, 0)$	m even	
	$\Delta^{\epsilon}_m(0, 0)$	m odd	
$Sp(V, \sigma_+, \tau)$ $\lvert = Sp(n, \mathbb{R}) \rvert$	$\Delta_m(\zeta, -\zeta, \bar{\zeta}, -\bar{\zeta})$	$\zeta \neq \pm\bar{\zeta}$	
	$\Delta_m(\zeta, -\zeta)$	$\zeta = \bar{\zeta} \neq 0$	
	$\Delta^{\epsilon}_m(\zeta, -\zeta)$	$\zeta = -\bar{\zeta} \neq 0$	
	$\Delta_m(0, 0)$	m even	
	$\Delta^{\epsilon}_m(0)$	m odd	

The indecomposable types which can contribute to a type for $o(2, n)$ are quite limited, and listed below.

$\Delta_m(\zeta, -\zeta, \bar{\zeta}, -\bar{\zeta})$	$m = 0$
$\Delta_m(\zeta, -\zeta)$	$m = 0, 1$
$\Delta^{\epsilon}_m(\zeta, -\zeta)$	$m = 0, \varepsilon = \pm 1$; $m = 2$ with $\varepsilon = 1$;
	$m = 1, \varepsilon = \pm 1$
$\Delta^{\epsilon}_m(0)$	$m = 0, \varepsilon = \pm 1$; $m = 2, \varepsilon = \pm 1$; $m = 4$ with $\varepsilon = -1$
$\Delta_m(0, 0)$	$m = 1$

5. *Results of the Abstract Classification of Cones*

In this section we present the main results of [14] in general, yet precise, terms. What is obtained there is a uniform abstract classification of causal cones in simple Lie algebras (including exceptional cases) in terms of cones invariant under finite groups. In the following four chapters we consider each of the classical families of hermitian symmetric Lie algebras separately, obtaining more detailed information in all cases. This section attempts to bridge the rather wide gap between these two approaches, by describing the classification results for a general $\mathfrak{G}$ independently of the specialized semisimple theory, e.g., root systems. (However, of course such latter notions are essential for the proofs of the abstract classification.) With such results in hand, the main work in Chapters II to V is the identification of those cones predicted by the general theory.

Let $\mathfrak{G}$ be hermitian symmetric, $\mathfrak{k}$ a maximal compact subalgebra, and let $B_\theta(\cdot, \cdot)$ be the positive-definite form on $\mathfrak{G}$ as in Section 2. Let G be the adjoint group of $\mathfrak{G}$ and K the subgroup corresponding to $\mathfrak{k}$. Let $\mathfrak{h}$ be a maximal abelian subalgebra of $\mathfrak{k}$, and let $z \in \mathfrak{k}$ span $z(\mathfrak{k})$, the center of $\mathfrak{k}$. We have $Z \in \mathfrak{h}$ by maximal commutativity of $\mathfrak{h}$. One finds that there is a closed polyhedral cone $c_{\min} \subset \mathfrak{h}$ (that generated by the finitely many noncompact positive root vectors) which contains Z in its interior, and is in turn contained in its dual cone $c_{\max} = (c_{\min})^* \supseteq c_{\min}$. (Thus $c_{\min}, c_{\max}$ possess no full lines.) There is also a finite group W_K of isometries of $\mathfrak{h}$, coming from the adjoint action of $\mathfrak{k}$, which preserves $c_{\min}, c_{\max}$, and fixes Z.

The main results of [14] are I–VI below. We note that in a recent paper [17] Vinberg showed independently I and VI, proved the existence of unique minimal and maximal cones, and established the ellipticity of their interiors. (Cf. also Section 14 for a description of his results in [17] on invariant causal structures in the corresponding groups.)

(I) Each open invariant convex cone (the interiors of the causal cones) contains Z or $-Z$.

(II) The causal (invariant open convex) cones in $\mathfrak{G}$ containing Z are in 1–1 correspondence, via intersection with $\mathfrak{h}$, with those W_K-invariant convex cones in $\mathfrak{h}$ containing $c_{\min}$ (resp. $(c_{\min})^{\mathrm{int}}$) and contained in $c_{\max}$ (resp., $(c_{\max})^{\mathrm{int}}$).

(III) If C is a causal cone, C is equal to the closure of the union of orbits $\mathrm{Ad}(G)X$, where $X \in C \cap \mathfrak{h}$. If C is an open invariant convex cone, each $X \in C$ is in the G-orbit of a $Y \in C \cap \mathfrak{h}$.

(IV) If C is a causal cone, $(C \cap \mathfrak{h})^* = C^* \cap \mathfrak{h}$ (answering a question listed in [17]), the dual on the l.h.s. being taken in $\mathfrak{h}$.

(V) (Noncompact convexity theorem.) Let $\Gamma: \mathfrak{G} \to \mathfrak{h}$ be the $B_\theta(\cdot, \cdot)$-orthogonal projection onto $\mathfrak{h}$. If $X \in c_{\max}$, then

$$\Gamma\{\mathrm{Ad}(g)\, X: g \in G\} = \{Y + W: Y \in c_{\min},\, W \text{ in the convex hull of the } W_K\text{-orbit of } X\}. \tag{5.1}$$

The similar equality obtained from (5.1) by changing $g \in G$ to $g \in K$, and omitting the $Y \in c_{\min}$ term, is the Horn–Kostant [7, 9] convexity theorem (cf. also [5]).

(VI) It is well known that in the hermitian symmetric case there are exactly two G-orbits $\mathcal{O}_\pm$ in $\mathfrak{G}$ which are nilpotent and of minimal dimension (discussed in [20]). In fact the convex hulls of $\mathcal{O}_\pm$ (no closure) are precisely the minimal causal cones minus the origin.

It seems likely that $\mathcal{O}_\pm$ are the unique orbits in the minimal causal cones with the above property. This is easily seen in the classical cases, to which we now turn.

II. $sp(n, \mathbb{R})$

6. *Uniqueness of Causal Cones*

Let $n \geqslant 1$ and define

$$\mathfrak{G} = \left\{ \begin{pmatrix} A & B \\ C & -A^t \end{pmatrix} : \begin{array}{r} A, B, C \text{ real } n \times n \text{ matrices,} \\ B, C \text{ symmetric, } A \text{ arbitrary} \end{array} \right\},$$

$$\mathfrak{k} = \left\{ \begin{pmatrix} A & -B \\ B & A \end{pmatrix} \in \mathfrak{G}: A \text{ skew, } B \text{ symmetric} \right\}.$$

$$\mathfrak{h} = \left\{ \begin{pmatrix} 0 & -D \\ D & 0 \end{pmatrix} \in \mathfrak{G}: D \text{ diagonal} \right\}.$$

Given $X \in \mathbb{R}^n$, let $\mathfrak{h}(X) = \left(\begin{smallmatrix} 0 & -D \\ D & 0 \end{smallmatrix}\right) \in \mathfrak{h}$ be that element where $\operatorname{diag} D = X$. Identifying $\mathfrak{h}$ with $\mathbb{R}^n$ in this way, the finite group W_K is given by all permutations of the matrix entries of the $X \in \mathbb{R}^n$. The center of $\mathfrak{k}$ is spanned by $\mathfrak{h}(1, \dots, 1)$. The noncompact root vectors generate the convex cone $c_{\min} = \{\mathfrak{h}(X): X_j \geqslant 0,\ 1 \leqslant j \leqslant n\}$, which is self-dual (a positive orthant), so $c_{\max} = c_{\min}$. The positive-definite form B_θ on $\mathfrak{G}$, invariant under the group $K \approx U(n)$ generated by $\mathfrak{k}$, is $B_\theta(X, Y) = \operatorname{tr}(XY^t)$.

Define the symplectic form $\mathcal{Q}(\cdot, \cdot)$ on $\mathbb{R}^{2n}$ by

$$\mathcal{Q}\left(\begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} u \\ v \end{pmatrix} \right) = y \cdot u - x \cdot v$$

for $x, y, u, v \in \mathbb{R}^n$. It is invariant under the group $G = Sp(n, \mathbb{R})$ corresponding to $\mathfrak{G}$.

By the general theory, $c_{\min} = c_{\max}$ implies that there are unique causal cones $\pm C$ in $\mathfrak{G}$. As

$$C = \{X \in \mathfrak{G} : \mathcal{O}\!\!\!\mathcal{l}(Xv, v) \geqslant 0 \text{ for all } v \in \mathbb{R}^n\} \tag{6.1}$$

is nontrivial and invariant, it must be one of them.

The minimal orbit $\mathcal{O}_+$ in C is the image of the 2-to-1 covering and $Sp(n, \mathbb{R})$-equivariant map $v \in \mathbb{R}^{2n} - \{0\} \to Y_v \in \mathfrak{G}$, where $\mathcal{O}\!\!\!\mathcal{l}(Xv, v) = -\operatorname{tr}(XY_v)$ for all $X \in \mathfrak{G}$. One element of $\mathcal{O}_+$ is $\left(\begin{smallmatrix} 0 & D \\ 0 & 0 \end{smallmatrix}\right)$, where D is diagonal, diag $D = (-1, 0, \ldots, 0)$.

The B_θ-orthogonal projection $\Gamma \colon \mathfrak{G} \to \mathfrak{h}$ is clearly

$$\Gamma \begin{pmatrix} A & B \\ C & -A^t \end{pmatrix} = \mathfrak{h}\left(\frac{-B_{11} + C_{11}}{2}, \ldots, \frac{-B_{nn} + C_{nn}}{2}\right)$$

for $\left(\begin{smallmatrix} A & B \\ C & -A^t \end{smallmatrix}\right) \in \mathfrak{G}$.

Let $d = (d_1, \ldots, d_n) \in \mathbb{R}^n$, where all $d_j \geqslant 0$. The Horn–Kostant convexity theorem, applied to this situation, says

$$\begin{aligned} &\Gamma(\{k\mathfrak{h}(d)\,k^{-1} : k \in K\}) \\ &\quad = \left\{ Y : Y = \sum_{\sigma \in W_K} \lambda_\sigma \sigma(d), 0 \leqslant \lambda_\sigma \leqslant 1, 1 = \sum_{\sigma \in W_K} \lambda_\sigma \right\} \equiv \chi(d). \end{aligned}$$

The noncompact convexity theorem (Section 5) says

$$\Gamma(\{g\mathfrak{h}(d)\,g^{-1} : g \in G\}) = \{X + Y : X \in c_{\min}, Y \in \chi(d)\}.$$

7. *Boundary Orbits and a Geometrical Duality*

From the form (6.1) of the cone C, it is clear that Δ is a type having a representative in C if and only if the indecomposable summands of Δ have representatives contained in the (possibly) lower dimensional causal cones.

LEMMA 7.1. *Only the types $\Delta_1^+(0)$, $\Delta_0(0,0)$, and $\Delta_0^+(\zeta, -\zeta)$ $(\zeta = -\bar{\zeta} \neq 0)$ can contribute to an indecomposable type in a causal cone.*

Proof. As the eigenvalues of the $X \in C^{\text{int}}$ are purely imaginary, it suffices to examine only $\Delta_m^\pm(\zeta, -\zeta)$ $(\zeta = -\bar{\zeta} \neq 0)$, $\Delta_m(0,0)$ (m even), and $\Delta_m^\pm(0)$ (m odd). Let $A = S + N$ be a representative of one of these types, of height m. The representation space is a direct sum $E + NE + \cdots + N^m E$, where $\mathcal{O}\!\!\!\mathcal{l}(\cdot, N^m \cdot)$ is nondegenerate on E. Take $v, w \in E$, and assume first $m \geqslant 2$. Then

$$\mathcal{O}\!\!\!\mathcal{l}(A(N^{m-1}v + w), N^{m-1}v + w) = (-1)^{m-1}\, \mathcal{O}\!\!\!\mathcal{l}(AN^{2m-2}v, v) + 2\mathcal{O}\!\!\!\mathcal{l}(N^m v, w),$$

and this expression is clearly of indefinite sign. If $m = 1$, consider

$$\mathcal{O}(A(Nv + w), Nv + w) = -2\mathcal{O}(NSw, v) + \mathcal{O}(Nw, w).$$

This can be of definite sign only if $S = 0$.

This leaves only $\Delta_0(0, 0)$, $\Delta_1^+(0)$, $\Delta_0^+(\zeta, -\zeta)$, which all are orbits in $sp(1, \mathbb{R}) \approx sl(2, \mathbb{R})$. $\Delta_0(0, 0)$ is just $\left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right) \cdot \left(\begin{smallmatrix} a & b \\ c & -a \end{smallmatrix}\right)$ represents $\Delta_0^+(\zeta, -\zeta)$ $(\zeta \neq 0)$ iff $-a^2 - bc > 0$, $b < 0$, and $c > 0$, and represents $\Delta_1^+(0)$ iff $-a^2 - bc = 0$, $b \leqslant 0$, $c \geqslant 0$, and $b^2 + c^2 > 0$, e.g., $\left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)$. Q.E.D.

In the remainder of this section we show how to classify *all* invariant convex cones in $sp(n, \mathbb{R})$. The first result is, there are exactly $2^{n+1} - 2$ invariant convex cones not containing 0 between C^{int} and C (inclusively), in a natural 1–1 correspondence with the collection of nonempty and proper subsets of $\{0, \ldots, n\}$ (Proposition 7.3). Furthermore, we prove that the geometrical duality mapping

$$C_1 \to \hat{C}_1 = \{Y \in \mathfrak{G} : B_\theta(Y, X) > 0 \text{ for all } X \in C_1\}$$

corresponds simply to complementation among these subsets (Proposition 7.4).

By Lemma 7.1 each $X \in C$ is conjugate under G to some $\left(\begin{smallmatrix} 0 & -D \\ E & 0 \end{smallmatrix}\right)$, where D, E are nonnegative diagonal matrices. It is clear that then $d_i = \mathcal{O}(Ya_i, a_i)$, $e_i = \mathcal{O}(Yb_i, b_i)$ for some *symplectic basis* $\varphi = (a_i, b_i)_{i=1}^n$, where the collection of all symplectic bases, on which G acts, is

$$\Phi = \{(a_i, b_i) : \mathcal{O}(a_i, a_j) = \mathcal{O}(b_i, b_j) = 0, \mathcal{O}(a_i, b_j) = \delta_{ij}\}.$$

If $\varphi = (a_i, b_j) \in \Phi$ and $X \in C$, let

$$M_\varphi(X) = (\mathcal{O}(Xa_1, a_1), \ldots, \mathcal{O}(Xa_n, a_n); \mathcal{O}(Xb_1, b_1), \ldots, \mathcal{O}(Xb_n, b_n)) \in \mathscr{P}^n,$$

where $\mathscr{P}^n = \{v = (x_1, \ldots, x_n; y_1, \ldots, y_n) : x_i, y_j \geqslant 0\}$. If such a $v \in \mathscr{P}^n$ as above has exactly Z pairs $(x_i, y_i) = (0, 0)$, and exactly N pairs $(x_i, x_j) = (0, \lambda)$ or $(\lambda, 0)$, $\lambda > 0$, write $\eta(v) = (N, Z)$.

The following is a "boundary" version of the noncompact convexity theorem.

LEMMA 7.2. *Let $X \in C$, and let the type of X involve P summands of the form $\Delta_0^+(\zeta, -\zeta)$ $(\zeta = -\bar{\zeta} \neq 0)$, N of the form $\Delta_1^+(0)$, and Z of the form $\Delta_0(0, 0)$, so that there exists $\varphi \in \Phi$ such that $\eta(M_\varphi(X)) = (N, Z)$.*

If $\psi \in \Phi$ is arbitrary, and $\eta(M_\psi(X)) = (N', Z')$, then $Z' \leqslant Z$ and $N' + Z' \leqslant N + Z$.

Proof. Let $\varphi = (a_i, b_j)$, $\psi = (c_i, d_j)$. We may assume that if $x_i = \mathcal{O}(Xa_i, a_i)$, $y_j = \mathcal{O}(Xb_j, b_j)$, that $x_i, y_i > 0$ for $i = 1, \ldots, P$; $x_i > 0$, $y_i = 0$ for

$i = P+1,\ldots, P+N$; $x_i = y_i = 0$ for $i = P+N+1,\ldots, n$; and the same with c_i, d_i, P', N', Z' replacing a_i, b_i, P, N, Z, respectively.

Now each c_j, d_j is a linear combination of the a_i, b_i. If $j > P' + N'$, c_j and d_j can involve no a_i or b_i with $i \leqslant P$, or a_i such that $P+1 \leqslant i \leqslant P+N$. The $2Z' \times 2Z'$ matrix

$$\begin{pmatrix} \mathfrak{A}(c_k, c_l) & \mathfrak{A}(c_k, d_l) \\ \mathfrak{A}(d_k, c_l) & \mathfrak{A}(d_k, d_l) \end{pmatrix}_{k,l=P'+N'+1,\ldots,n}$$

(depending only on the combinations of the a_i, b_i for $P+N+1 \leqslant i \leqslant n$) is of rank $2Z'$ as $\psi \in \Phi$, but is equal to $A^t\left(\begin{smallmatrix} 0 & I \\ -I & 0 \end{smallmatrix}\right)A$, where A is of order $2Z \times 2Z'$, so $Z' \leqslant Z$.

Likewise express d_j for $j = P'+1,\ldots, n$. The above expressions for d_j can involve no a_i or b_i for $i = 1,\ldots, P$. Now the maximal dimension of an isotropic subspace in a $2(N+Z)$ dimensional symplectic space is $N+Z$, so $N' + Z' \leqslant N + Z$. Q.E.D.

DEFINITIONS. Let $\mathscr{C}$ be the collection of invariant convex cones D in $\mathfrak{G}$ such that $C^{\text{int}} \subseteq D \subseteq C$ and $0 \notin D$.

Define the set of lattice points

$$\mathscr{L} = \{(n_1, n_2) \in \mathbb{Z} \times \mathbb{Z} : 0 \leqslant n_1, n_2;\ n_1 + n_2 \leqslant n;\ n_2 < n\},$$

and the collection of subsets of $\mathscr{L}$

$$\tilde{\mathscr{C}} = \{\mathscr{S} \subseteq \mathscr{L} : (s_1, s_2) \in \mathscr{S} \text{ implies } (s_3, s_4) \in \mathscr{S} \text{ whenever } (s_3, s_4) \in \mathscr{L},\ s_4 \leqslant s_2, \text{ and } s_3 + s_4 \leqslant s_1 + s_2\}.$$

PROPOSITION 7.3. *If $D \in \mathscr{C}$, then*

$$\tilde{D} \equiv \{\eta(M_\varphi(X)) : X \in D, \varphi \in \Phi\} \in \tilde{\mathscr{C}},$$

and $D \in \mathscr{C} \to \tilde{D} \in \tilde{\mathscr{C}}$ is a 1–1 correspondence between the above collections $\mathscr{C}$, $\tilde{\mathscr{C}}$ of invariant convex cones and sets of lattice points, respectively.

Proof. Let $D \in \mathscr{C}$, $X \in D$, and choose $\varphi = (a_i, b_i) \in \Phi$ which "diagonalizes" X as in Lemma 7.2, so that if $\eta(M_\varphi(X)) = (N, Z)$, the decomposition of the type of X involves N summands of the form $\Delta_1^+(0)$ and Z of the form $\Delta_0(0, 0)$. Let $(N', Z') \in \mathscr{L}$ such that $Z' \leqslant Z$ and $N' + Z' \leqslant N + Z$.

Now to show $\tilde{D} \in \tilde{\mathscr{C}}$, by Lemma 7.2 it suffices to show there exists $Y \in D$ such that $\eta(M_\varphi(Y)) = (N', Z')$. It is easy to see that if $N \neq 0$, there exists Y in the convex hull of the transforms of X by those elements of G generated by the permutations $\{(a_i, b_i) \to (a_j, b_j), (a_j, b_j) \to (a_i, b_i)\}$ for some $i \neq j$, and $\{a_i \to b_i, b_i \to -a_i\}$, such that $\eta(M_\varphi(Y)) = (N', Z')$.

The case $N = 0$ is more difficult. Since $Z < n$, part of X involves

$Y = \binom{0\ -1}{1\ \ 0} \oplus \binom{0\ 0}{0\ 0} \in sp(2, R)$, and to complete the proof of $\tilde{D} \in \tilde{\mathscr{C}}$, we must show how to obtain $\binom{0\ -1}{1\ \ 1} \oplus \binom{0\ -1}{0\ \ 0}$ from an average of Y and gYg^{-1} for some $g \in Sp(2, \mathbb{R})$. Suppose then $Ya_1 = -b_1$, $Yb_1 = a_1$, $Ya_2 = Yb_2 = 0$. It is easy to see that $(c_1, c_2; d_1, d_2) \in \Phi$, where $c_1 = a_1$, $c_2 = a_2 + ka_1$, $d_1 = b_1 - kb_2$, $d_2 = b_2$, if and only if $(a_1, a_2; b_1, b_2) \in \Phi$. It follows that $Yc_1 = -d_1 - kd_2$, $Yc_2 = -kd_1 - k^2 d_2$, $Yd_1 = c_1$, and $Yd_2 = 0$; the average of this for $k = \pm 1$ gives $Yc_1 = -d_1$, $Yd_1 = c_1$, $Yc_2 = -d_2$, $Yd_2 = 0$. Therefore $\tilde{D} \in \tilde{\mathscr{C}}$.

If $C_1, C_2 \in \mathscr{C}$, and $\tilde{C}_1 = \tilde{C}_2$, the above argument shows that C_1 and C_2 contain the same orbits, so $C_1 = C_2$. Finally, if $D \in \tilde{\mathscr{C}}$, let D_1 be the convex cone generated by the representatives of the types determined by D. By Lemma 7.2, one sees that $\eta(M_\varphi(X)) \in D$ for all $X \in D_1$, $\varphi \in \Phi$, and that $\tilde{D}_1 = D$. Q.E.D.

It is easy to see that any $\mathscr{S} \in \tilde{\mathscr{C}}$ has the form

$$\{(0, 0), \ldots, (N_0, 0), (0, 1), \ldots, (N_1, 1), \ldots, (0, m), \ldots, (N_m, m)\}$$

for a unique m, where $0 \leqslant m \leqslant n - 1$, and unique sequence $\{N_j\} \subset \{0, \ldots, n\}$, where $0 \leqslant N_m < \cdots < N_1 < N_0 \leqslant n$. Thus by Proposition 7.3 the elements of $\mathscr{C}$ or $\tilde{\mathscr{C}}$ are in 1–1 correspondence with the nonempty and proper subsets of $\{0, \ldots, n\}$, so there are exactly $2^{n+1} - 2$ invariant convex cones contained in C not containing 0.

PROPOSITION 7.4. *Let the invariant convex cone $C_1 \in \mathscr{C}$ correspond to $\Omega = \{N_m, \ldots, N_0\} \subset \{0, \ldots, n\}$ in the above manner. Then the "dual cone"*

$$D_1 \equiv \{X \in \mathfrak{G} : B_\theta(X, Y) > 0 \text{ for all } Y \in C_1\} \in \mathscr{C},$$

and D_1 similarly corresponds to the complement $\{0, \ldots, n\} - \Omega$.

LEMMA 7.5. *If $v \in \mathscr{S}^n - \{0\}$, $\eta(v) = (N, Z)$, then there exists $w \in \mathscr{S}^n - \{0\}$ such that $v \cdot w = 0$ (orthogonality in $\mathbb{R}^{2n}$) and $\eta(w) = (N', Z')$ if and only if*

$$n \leqslant N + Z + Z' \tag{7.1}$$

and

$$n \leqslant N' + Z + Z'. \tag{7.2}$$

Proof. Define P, P' such that $P + N + Z = n$, $P' + N' + Z' = n$. Clearly such a w exists if and only if $P' \leqslant Z$ and $Z' \geqslant P$, which are equivalent to the cited conditions. Q.E.D.

Proof of Proposition 7.4. Clearly $D_1 \in \mathscr{C}$. Let D_1 correspond to $\Omega_1 \subset \{0,..., n\}$. If $N_0 < n$, then $n \leqslant N_0 + 0 + Z'$ fails for $0 \leqslant Z' \leqslant n - N_0 - 1$, which is (7.1) for $(N, Z) = (N_0, 0)$. Therefore (7.1) fails whenever $(N_0, 0)$ is replaced by any $(N, Z) \in \tilde{C}_1$. Thus $\{N_0 + 1,..., n-1, n\} \subseteq \Omega_1$ by Lemma 7.5.

On the other hand, suppose $\{n-j,..., n-1, n\} \subseteq \Omega$, but $n-j-1 \notin \Omega$. Then $\{n-j,..., n\} \cap \Omega_1 = \varnothing$ since (7.1) and (7.2) are both satisfied if $(N, Z) = (n-j, j)$, $(N', Z') = (n-k, 0)$ for $0 \leqslant k \leqslant j$.

Therefore $n \in \Omega$ if and only if $n \notin \Omega_1$. To finish the proof we appeal to induction on n. Note $n \in \Omega$ iff $(n, 0) \in \tilde{C}_1$ (notation as in the statement of Proposition 7.3) iff the cone in $\mathscr{S}^n$ corresponding to $C_1 \subset \mathfrak{G}$ (via the M_φ, $\varphi \in \Phi$, as before), call it c_1, contains a vector containing a pair (x_i, y_i) such that not both x_i and y_i are positive. Similiarly $n \notin \Omega_1$ iff the cone in $\mathscr{S}^n$, say d_1, corresponding to $D_1 \subset \mathfrak{G}$ has the property that each vector in d_1 contains a pair (x_i, y_i) with both $x_i, y_i > 0$.

Suppose first that $n \in \Omega$. By the above property of d_1, $v \in d_1$ iff v has a positive inner product with each $w \in C_1$ such that w has at least one pair $(x_i, y_i) = (0, 0)$. (This is true, because if a $w \in c_1$ had no such pair, any $v \in d_1$ would automatically satisfy $v \cdot w > 0$.) The collection of such w is determined precisely by the collection $\{N_1,..., N_m\}$ (recall N_j = the maximal number of $\Delta_1^+(0)$-types among the orbits in C_1 which have exactly j $\Delta_0(0, 0)$-types). As each such w has at least one $(0, 0)$ pair, and each $v \in d_1$ has at least one pair with $x_i, y_i > 0$, the problem of determining Ω_1 from c_1 has the value of n one less, and by induction Ω_1 is the complement of $\{N_1,..., N_m\}$ in $\{0, 1,..., n-1\}$, as desired.

On the other hand, if $n \notin \Omega$ and $n \in \Omega_1$, every vector in c_1 has at least one $x_i, y_i > 0$ pair, and the issue remaining in determining Ω_1 concerns those vectors in d_1 which have at least one $(0, 0)$ pair. The reasoning is similar to the previous case, leading to an appeal to induction on n as before. Q.E.D.

III. $su(p, q)$

8. *Causal Cones and Hermitian–Symplectic Forms*

We take $p \geqslant q \geqslant 1$, $n = p + q$, and for convenience $n > 2$. Let

$$\mathfrak{G} = \left\{ \begin{pmatrix} B & A \\ A^* & C \end{pmatrix} : \begin{array}{l} A, B, C \text{ complex matrices; } B \text{ and } C \text{ skew-} \\ \text{hermitian of order } p \text{ and } q, \text{ resp.,} \\ A \text{ arbitrary; } \operatorname{Tr} B + \operatorname{Tr} C = 0 \end{array} \right\},$$

$$\mathfrak{k} = \left\{ \begin{pmatrix} B & 0 \\ 0 & C \end{pmatrix} \in \mathfrak{G} \right\},$$

and

$$\mathfrak{h} = \left\{ \begin{pmatrix} B & 0 \\ 0 & C \end{pmatrix} \in \mathfrak{k} \colon B \text{ and } C \text{ diagonal} \right\}.$$

The positive-definite form $B_\theta(\cdot,\cdot)$ on $\mathfrak{G}$, invariant under the group $K = S(U(p) \times U(q))$ generated by $\mathfrak{k}$, is defined by $B_\theta(X, Y) = \operatorname{tr}(X\bar{Y}^t)$.

Let

$$E = \{(\lambda_1, \dots, \lambda_p, \sigma_1, \dots, \sigma_q) \in \mathbb{R}^{p+q} \colon \textstyle\sum \lambda_i = \sum \sigma_j\},$$

$$E_1 = \{(\lambda, \sigma) \in E \colon \lambda_i + \sigma_j \geqslant 0 \text{ for all } i, j\},$$

$$E_0 = \{(\lambda, \sigma) \in E \colon \text{all } \lambda_i, \sigma_j \geqslant 0\},$$

and for $X = (\lambda, \sigma) \in E$, let $\mathfrak{h}(X) = (\begin{smallmatrix} B & 0 \\ 0 & C \end{smallmatrix}) \in \mathfrak{h}$ be that element where $\operatorname{diag} B = (-i\lambda_1, \dots, -i\lambda_p)$, $\operatorname{diag} C = (i\sigma_1, \dots, i\sigma_q)$. The finite group W_K acting on $\mathfrak{h}$ is then all permutations of the λ_i, together with all permutations of the σ_j (separately). The center of $\mathfrak{k}$ is spanned by $\mathfrak{h}$ $((1/p, \dots, 1/p, 1/q, \dots, 1/q))$. The noncompact root vectors generate $c_{\min} = \mathfrak{h}(E_0)$, whose dual cone is $c_{\max} = c^*_{\min} = \mathfrak{h}(E_1)$, as is easily seen by induction on n.

Define the hermitian form $H(\cdot,\cdot)$ on $\mathbb{C}^n$ by

$$H\left(\begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} u \\ v \end{pmatrix} \right) = \sum_{i=1}^{p} x_i \overline{u_i} - \sum_{j=1}^{q} y_j \overline{v_j},$$

for $x, u \in \mathbb{C}^p$, $y, v \in \mathbb{C}^q$, and the symplectic form $\mathcal{O}(u, v) = -\operatorname{Im} H(u, v)$, $u, v \in \mathbb{C}^n$. H and $\mathcal{O}$ are $G = SU(p, q)$-invariant. Note that $iH(Xv, v) = \mathcal{O}(Xv, v)$ is real for all $X \in \mathfrak{G}$, $v \in \mathbb{C}^n$.

By the general theory there are unique causal cones in $\mathfrak{G}$ whose intersections with $\mathfrak{h}$ are $\mathfrak{h}(E_0)$ and $\mathfrak{h}(E_1)$, these being minimal and maximal, respectively. It is fortuitous that these can be identified so simply. Let

$$C_0 = \{X \in \mathfrak{G} \colon \mathcal{O}(Xv, v) \geqslant 0 \text{ for all } v \in \mathbb{C}^n\},$$

and

$$C_1 = \{X \in \mathfrak{G} \colon \mathcal{O}(Xv, v) \geqslant 0 \text{ for all } v \text{ such that } H(v, v) = 0\}. \tag{8.1}$$

which are clearly causal cones, if nonempty.

LEMMA 8.1. *Let $X = (\lambda, \sigma) \in E$. Then $\mathfrak{h}(X) \in C_1$ if and only if $X \in E_1$, and $\mathfrak{h}(X) \in C_0$ if and only if $X \in E_0$.*

Proof. $X \in E_1$ if and only if there exists c such that $\lambda_i \geqslant c \geqslant -\sigma_j$ for all i, j, and $X \in E_0$ furthermore if and only if $c = 0$ may be chosen.

If

$$y = \begin{pmatrix} u \\ v \end{pmatrix} \in \mathbb{C}^n,$$

$$iH(h(X)y, y) = \sum_{i=1}^{p} \lambda_i |u_i|^2 + \sum_{j=1}^{q} \sigma_j |v_j|^2;$$

if $0 = H(y, y)$, $X \in E_1$, this is $\geqslant cH(y, y) = 0$. Q.E.D.

By III, Section 5, each $X \in C_1^{\text{int}} = \{X \in \mathfrak{G}: \mathcal{O}(Xv, v) > 0$ for all $v \neq 0$ such that $H(v, v) = 0\}$ is G-conjugate to some $Y \in \mathfrak{h}(E_1^{\text{int}})$.

The minimal nilpotent orbit $\mathcal{O}_+$ in C_0 is the image of the $U(p, q)$-equivariant map $v \in \mathbb{C}^n - 0 \to Y_v \in \mathfrak{G}$, where $\mathcal{O}(Xv, v) = -\text{tr}(XY_v)$ for all $X \in \mathfrak{G}$, whose fibers are circles. One element of $\mathcal{O}_+$ is

$$\begin{pmatrix} D & 0 & D \\ 0 & 0 & 0 \\ -D & 0 & -D \end{pmatrix},$$

where D is $q \times q$ $(p \geqslant q)$, diagonal, and diag $D = (-i, 0, \ldots, 0)$.

Let Γ: $\mathfrak{G} \to \mathfrak{h}$ be the B_θ-orthogonal projection; clearly $\Gamma\left(\begin{smallmatrix} B & A \\ A^* & C \end{smallmatrix}\right) = \mathfrak{h}(iB_{11}, \ldots, iB_{pp}, -iC_{11}, \ldots, -iC_{qq})$. Let $d = (\lambda, \sigma) \in E_1$; the noncompact convexity theorem says

$$\Gamma(\{ g\mathfrak{h}(d) g^{-1}: g \in G\}) = \{X + Y: X \in c_{\min}, Y \text{ a convex combination of the } \sigma(\mathfrak{h}(d)), \sigma \in W_K\}.$$

9. *Boundary Orbits in* $su(p, q)$

As for $sp(n, \mathbb{R})$, only indecomposable types with purely imaginary eigenvalues can contribute to types represented in the maximal causal cone C_1.

LEMMA 9.1. *Among the indecomposable types for* $u(p, q)$, *only* $\Delta_0^{\pm}(\zeta)$, $\Delta_1^{\pm}(\zeta)$ $(\zeta = -\bar{\zeta})$ *can contribute to* $\pm C_1$.

Proof. We need only consider the types $\Delta_m^{\pm}(\zeta)$, $\zeta \in i\mathbb{R}$. The cases $m \geqslant 3$ are ruled out exactly as in the proof of Lemma 7.1: the vectors $v + N^{m+1}w \in E + NE + \cdots + N^m E$ appearing there are $H(\cdot, \cdot)$-isotropic.

If $m = 2$, set $y = N^2 v + Nw + u$, $u, v, w \in E$, and note $H(y, y) = 2 \operatorname{Re} H(u, N^2 v) - H(w, N^2 w)$, and $H((S + N)y, y) = \zeta(H(y, y)) + 2i \operatorname{Im} H(N^2 w, u)$. Take u, v, w all nonzero such that $H(y, y) = 0$, and note that as w is rotated by a phase $iH((S + N)y, y)$ takes both positive and negative real values. Q.E.D.

Let $\lambda \in \mathbb{R}$. Recall that $\Delta_0^{\pm}(i\lambda)$ represents multiplication by $i\lambda$ in a one-dimensional space $\mathbb{C}e$ where $H(e, e) = \pm 1$. It is easily seen that $\Delta_1^{\pm}(-i\lambda)$ is represented by

$$\begin{pmatrix} -i\lambda & 0 \\ 0 & -i\lambda \end{pmatrix} \pm \begin{pmatrix} -i & i \\ -i & i \end{pmatrix}$$

acting in $\mathbb{C}^2$, where $H(\binom{x}{y}), \binom{u}{v}) = x\bar{u} - y\bar{v}$ as usual, $x, y, u, v \in \mathbb{C}$. Note that $\begin{pmatrix} -i & i \\ -i & i \end{pmatrix}$ is the limit of $SU(1, 1)$-transforms of elements of the form

$$\begin{pmatrix} -i\lambda & 0 \\ 0 & i\lambda \end{pmatrix}, \qquad \lambda > 0.$$

By this analysis and Lemma 8.1, it is a simple matter to list the indecomposable types represented in $C_1 \subset su(p, q)$; matrix representatives are easily found.

PROPOSITION 9.2. *The types Δ in C_1^{int} are of the form*

$$\Delta = \Delta^+(-i\lambda_1) + \cdots + \Delta^+(-i\lambda_p) + \Delta^-(i\sigma_1) + \cdots + \Delta^-(i\sigma_q), \qquad (9.1)$$

where all $\lambda_i + \sigma_j > 0$ and $\sum_i \lambda_i = \sum_j \sigma_j$. Δ is furthermore in C_0^{int} iff all $\lambda_i, \sigma_j > 0$.

The types Δ in C_0 are of either of two forms:

(1) *Types of the form* (9.1) *with all $\lambda_i, \sigma_j \geqslant 0$ (and such a Δ may also be in C_1^{int}), or*

$$(2) \qquad \Delta = \Delta^+(-i\lambda_1) + \cdots + \Delta^+(-i\lambda_{p-l}) + \underbrace{\Delta_1^+(-i\lambda) + \cdots + \Delta_1^+(-i\lambda)}_{l \text{ times}}$$

$$+ \Delta^-(i\sigma_1) + \cdots + \Delta^-(i\sigma_{q-l}) \qquad (9.2)$$

for $\lambda = 0$, some l with $1 \leqslant l \leqslant q$, all $\lambda_i, \sigma_j \geqslant 0$ and $\sum_i \lambda_i = \sum_j \sigma_j$. Such a type is on the boundaries of C_0 and C_1.

The remaining types in C_1 are those either of the form (9.1) *with all $\lambda_i + \sigma_j \geqslant 0$, $\sum_i \lambda_i = \sum_j \sigma_j$, and some $\lambda_i = -\sigma_j \neq 0$, or* (9.2) *for $\lambda \neq 0$, some l with $q \geqslant l \geqslant 1$, λ_i and σ_j satisfying $\lambda_i \geqslant \lambda \geqslant -\sigma_j$ for all i, j, and $2l\lambda + \sum_{i=1}^{p-l} \lambda_i = \sum_{j=1}^{q-l} \sigma_j$.*

It follows from an observation in Section 1, that any invariant convex cone C_1 in a general $\mathfrak{G}$ is "sandwiched" between its interior and closure: $C_1^{\text{int}} \subseteq C_1 \subseteq \overline{C_1}$. It seems likely, but not immediately clear, that there are only finitely many such C_1, for a given closed $\overline{C_1}$.

The following lemma may be useful in classifying such C_1 for a given closed $\overline{C_1}$ in $su(p, q)$. It represents the analogue of the $Sp(2, \mathbb{R})$-symmetry used in the proof of Proposition 7.3, and is applied in Section 17 to find all invariant convex cones in $su(2, 1)$.

LEMMA 9.3. *If C is an invariant convex cone in $su(p, q)$ which contains a type Δ having the decomposition $\Delta_1 + \Delta_2$, where $\Delta_1 = \Delta^+(-id) + \Delta^+(i\lambda) + \Delta^-(i\lambda)$, and $d + \lambda > 0$, then C also contains a representative of $\tilde{\Delta}_1 + \Delta_2$, where $\tilde{\Delta}_1 = \Delta^+(-id) + \Delta_1^+(i\lambda)$.*

Proof. We will conjugate

$$D = \begin{pmatrix} i\lambda & 0 & 0 \\ 0 & -id & 0 \\ 0 & 0 & i\lambda \end{pmatrix}$$

by $\exp tN$, where

$$N = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

is a representative of the principal nilpotent orbit ($N^3 = 0$) in $su(2, 1)$, and average for $t = \pm\text{const.}$

One computes that the terms in $e^{tN}De^{-tN}$ involving even powers of t are

$$D - t^2NDN + (t^2/2)(DN^2 + N^2D) + (t^4/4)\, N^2DN^2.$$

The last term is 0, and the second and third contribute

$$t^2 \begin{pmatrix} -id & 0 & id \\ 0 & 0 & 0 \\ -id & 0 & id \end{pmatrix} \quad \text{and} \quad t^2 \begin{pmatrix} -i\lambda & 0 & i\lambda \\ 0 & 0 & 0 \\ -i\lambda & 0 & i\lambda \end{pmatrix},$$

respectively, which changes the type $\Delta^+(i\lambda) + \Delta^-(i\lambda)$ to $\Delta_1^+(i\lambda)$, as desired.

Q.E.D.

IV. $so^*(2n)$

10. *Causal Cones and Quaternionic–Symplectic Forms*

Take $n \geqslant 3$, and

$$\mathfrak{G} = \left\{ \begin{pmatrix} A & -B \\ \bar{B} & \bar{A} \end{pmatrix} : \begin{matrix} A, B \text{ complex } n \times n \text{ matrices;} \\ A \text{ skew, } B \text{ Hermitian} \end{matrix} \right\},$$

$$\begin{aligned} \mathfrak{k} &= \{X \in \mathfrak{G} : X \text{ real}\} \\ &= \left\{ \begin{pmatrix} A & -B \\ B & A \end{pmatrix} \in \mathfrak{G} : A \text{ skew, } B \text{ symmetric} \right\}, \end{aligned}$$

and

$$\mathfrak{h} = \left\{ \begin{pmatrix} 0 & -D \\ D & 0 \end{pmatrix} \in \mathfrak{k} : D \text{ diagonal} \right\}.$$

As with $sp(n, \mathbb{R})$, given $X \in \mathbb{R}^n$ let $\mathfrak{h}(X) = \left(\begin{smallmatrix} 0 & -D \\ D & 0 \end{smallmatrix}\right) \in \mathfrak{h}$ be that element where $\operatorname{diag} D = X$. Again, the finite group W_K acting on $\mathbb{R}^n \approx \mathfrak{h}$ comes from just the permutations of the coordinates of $\mathbb{R}^n$. The center of $\mathfrak{k}$ is spanned by $\mathfrak{h}(1,\dots,1)$, and the positive-definite form $B_\theta(\cdot,\cdot)$ on $\mathfrak{G}$, invariant under the group $K \approx U(n)$ generated by $\mathfrak{k}$, is given by $B_\theta(X, Y) = \operatorname{tr}(X\bar{Y}^t)$ for $X, Y \in \mathfrak{G}$.

Thus far the situation as regards $\mathfrak{k}$ and $\mathfrak{h}$ here is the same as for $sp(n, \mathbb{R})$ (cf. Section 6). However, the first essential difference is that $c_{\min}$ is generated as a convex cone by the permutations of $\mathfrak{h}(1, 1, 0, 0,\dots, 0)$, and is not the entire positive orthant. One sees easily that $c_{\min} = h(E_0)$, where

$$E_0 = \left\{ (\lambda_1,\dots,\lambda_n) \in \mathbb{R}^n : 0 \leqslant \lambda_i \leqslant \sum_{\substack{j=1 \\ j\neq i}}^{n} \lambda_j,\ i = 1,\dots, n \right\}.$$

The dual cone is $c_{\max} = c^*_{\min} = \mathfrak{h}(E_2)$, where

$$E_2 = \{(\lambda_1,\dots,\lambda_n) \in \mathbb{R}^n : \lambda_i + \lambda_j \geqslant 0 \text{ for } i \neq j\}.$$

Also let $E_1 = \{\lambda \in \mathbb{R}^n : \lambda_i \geqslant 0, \text{ all } i\}$.

Define the complex symmetric form $\tau(\cdot,\cdot)$ on $\mathbb{C}^{2n}$ by

$$\tau(x, y) = \sum_{j=1}^{2n} x_i y_i, \qquad x, y \in \mathbb{C}^{2n},$$

$J = \left(\begin{smallmatrix} 0 & -I_n \\ I_n & 0 \end{smallmatrix}\right)$, $\mathscr{C}: \mathbb{C}^{2n} \to \mathbb{C}^{2n} : X \to \bar{X}$ (complex conjugation), and $\sigma = \mathscr{C}J = J\mathscr{C}$, so $\sigma^2 = -I$. Also define $H(x, y) = -i\tau(x, \sigma y)$, an hermitian form of signature (n, n), and $\mathcal{O}(\cdot,\cdot) = \operatorname{Re} \tau(\cdot, \sigma\cdot) = -\operatorname{Im} H(\cdot,\cdot)$, a (real) symplectic form. $\mathfrak{G}$ can be characterized (cf. Section 3) as those complex linear transformations of $\mathbb{C}^{2n}$ skew with respect to τ and commuting with σ, or equivalently as those real linear infinitesimally $\mathcal{O}(\cdot,\cdot)$-symplectic transformations which commute with i and σ. (i, σ are infinitesimally $\mathcal{O}(\cdot,\cdot)$-symplectic, are both in a distinguished $sp(\mathcal{O})$-orbit, and can be intrinsically characterized there.)

Let $G = SO^*(2n)$ be the group generated by $\mathfrak{G}$. In Section 3 we defined the quaternionic form

$$\tau_-(u, v) = \tau(u, v) + \tau(u, \sigma v)j, \qquad u, v \in \mathbb{C}^{2n}$$

(cf. also [2].) We define a collection of bases of $\mathbb{C}^{2n}$ over $\mathbb{C}$ which G transforms:

$$\Phi = \{(e_i, f_i)_{i=1}^n : e_i, f_i \in \mathbb{C}^{2n},\ \tau(e_i, e_j) = \tau(f_i, f_j) = 0,$$
$$\tau(e_j, f_k) = \delta_{jk},\ \sigma e_j = if_j, \text{ so } \sigma f_j = -ie_j\}.$$

An example is $e_j=(\tilde{e}_j+i\tilde{e}_{j+n})/\sqrt{2}$, $f_j=(\tilde{e}_j-i\tilde{e}_{j+n})/\sqrt{2}$, where $\tilde{e}_1,\ldots,\tilde{e}_{2n}$ are the standard unit basis vectors for $\mathbb{C}^{2n}$. In terms of τ_-, $(e_i,f_j)\in\Phi$ is equivalent to $\tau_-(e_i,f_j)=\delta_{ij}$, $\tau_-(e_i,e_j)=\delta_{ij}k$, $\tau_-(f_i,f_j)=-\delta_{ij}k$.

The e_i,f_i are related simply to the quaternionic bases $\{a_i\}$ of Section 3: if $a_i=(e_i+f_i)/\sqrt{2}$, $\tau_-(a_i,a_j)=\delta_{ij}$, and the a_i span $\mathbb{C}^{2n}$ over $\mathbb{Q}$. Note that in the example of $(e,f)\in\Phi$ above, the corresponding $a_i=\tilde{e}_i$.

Note that if $(e_i,f_j)\in\Phi$, $a_i=(e_i+f_j)/\sqrt{2}$, and $X\in\mathfrak{G}$, then

$$\mathcal{O}(Xe_i,e_i)=\mathcal{O}(Xf_i,f_i)=\mathcal{O}(Xa_i,a_i).$$

If $e=\tilde{e}_1+i\tilde{e}_2$, or more generally $e=a_1\pm ia_2$, $\{a_i\}$ a τ_--orthogonal $\mathbb{Q}$-basis, then $\tau_-(e,e)=0$. With $e=\tilde{e}_1\pm i\tilde{e}_2$, $X=\begin{pmatrix} A & -B \\ B & A\end{pmatrix}\in\mathfrak{G}$, we have

$$\mathcal{O}(Xe,e)=B_{11}+B_{22}\pm 2\operatorname{Im}B_{12}. \tag{10.1}$$

In analogy with C_1 for $su(p,q)$ (i.e., formula (8.1)), let

$$C_2=\{X\in\mathfrak{G}:\mathcal{O}(Xe,e)\geqslant 0 \text{ whenever } \tau_-(e,e)=0\}.$$

As in the two previous chapters, these τ_--isotropic vectors e give rise to a minimal nilpotent orbit $\mathcal{O}_+$ in $\mathfrak{G}$. For a matrix representation of an element of $\mathcal{O}_+$, see the remarks following Lemma 11.1. The natural mapping of the set of nonzero τ_--isotropic vectors onto $\mathcal{O}_+$ defined exactly as in Sections 6 and 8) has fibres homeomorphic to $SU(2)$.

LEMMA 10.1. *$\mathfrak{h}(X)\in C_2$ if and only if $X\in E_2$. Therefore C_2 is the unique (up to sign) maximal causal cone in $\mathfrak{G}$.*

Proof. Let $e=\begin{pmatrix} c+id \\ f+ig\end{pmatrix}\neq 0$ satisfy $\tau(e,e)=\tau(e,\sigma e)=0$, where $c,d,f,g\in\mathbb{R}^n$. The conditions are equivalent to $c\cdot g=f\cdot d$, $c\cdot c+f\cdot f=d\cdot d+g\cdot g$, and $c\cdot d+f\cdot g=0$. In terms of vectors in $\mathbb{C}^n$, they become $\|c+if\|^2=\|d+ig\|^2$, $\langle c+if,d+ig\rangle=0$, where $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$ are the standard sesquilinear form and norm on $\mathbb{C}^n$. Recalling now the isomorphism $\mathbb{R}^{2n}\approx\mathbb{C}^n$ which implements $K\approx U(n)$, and the fact that $U(n)$ acts transitively on pairs of orthonormal vectors in $\mathbb{C}^n$, there clearly exists $k\in K$ and $s>0$ such that

$$k\left(s\begin{pmatrix} c \\ f\end{pmatrix}+is\begin{pmatrix} d \\ g\end{pmatrix}\right)=\tilde{e}_1+i\tilde{e}_2=\tilde{e}.$$

Then $\mathcal{O}(\mathfrak{h}(X)e,e)=s^{-2}\mathcal{O}((k\mathfrak{h}(X)k^{-1})e,e)$. Now $h\mathfrak{h}(X)k^{-1}\in\mathfrak{k}$ is real, so $\mathcal{O}(\mathfrak{h}(X)e,e)\geqslant 0$ follows from (10.1), $X\in E_2$, and the Horn–Kostant convexity theorem (cf. V, Section 5). The necessity of $X\in E_2$, e.g., $X_1+X_2\geqslant 0$, follows from taking $e=e_1+ie_2$, etc. Q.E.D.

COROLLARY 10.2. *If* $X \in C_2^{\text{int}} = \{X \in \mathfrak{G}: \mathcal{Q}(Xe, e) > 0$ *for all nonzero* e *such that* $\tau_-(e, e) = 0\}$, *then* X *is* G*-conjugate to an element of* $c_{\max}^{\text{int}} \subset \mathfrak{h}$.

Let $\Gamma: \mathfrak{G} \to \mathbb{R}^n: \left(\begin{smallmatrix} A & -B \\ B & A \end{smallmatrix}\right) \to \operatorname{diag} B$, which via the isomorphism $\mathbb{R}^n \approx \mathfrak{h}$ earlier is just the B_θ-orthogonal projection onto $\mathfrak{h}$. The noncompact convexity theorem then says that if $X \in E_2$,

$$\begin{aligned}&\Gamma(\{gh(X)g^{-1}: g \in G\})\\ &\quad = \{Y + W: Y \in E_0,\ W \text{ in the convex hull of the } \sigma(X),\ \sigma \in W_K\}.\end{aligned}$$

It follows that if we define

$$C_1 = \{X \in \mathfrak{G}: \mathcal{Q}(Xv, v) \geqslant 0 \text{ for all } v \in \mathbb{C}^{2n}\},$$

then

$$C_1 = \{X \in \mathfrak{G}: \Gamma(gXg^{-1}) \in E_1 \text{ for all } g \in G\},$$

and

$$\begin{aligned}C_0 &= \{X \in \mathfrak{G}: \Gamma(gXg^{-1}) \in E_0 \text{ for all } g \in G\}\\ &= \left\{X \in C_1: \mathcal{Q}(Xe_1, e_1) \leqslant \sum_{j=2}^{n} \mathcal{Q}(Xe_j, e_j) \text{ for all } (e_i, f_j) \in \Phi\right\}\end{aligned}$$

is a minimal causal cone.

11. *Boundary Orbits in* $so^*(2n)$

We shall see that if Δ is a type represented in C_2, and $\Delta = \Delta_1 + \cdots + \Delta_m$ is its decomposition into indecomposable types, then each Δ_j is represented in $so^*(4)$ *or* $so^*(2) = so(2)$. Now $so^*(4) \approx sl(2, \mathbb{R}) \oplus su(2)$; the $sl(2, \mathbb{R})$-component is spanned by

$$\begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix},$$

and

$$\begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}$$

and the (commuting) $su(2)$ component is spanned by

$$\begin{pmatrix} 0 & 1 & & \\ -1 & 0 & & \\ & & 0 & 1 \\ & & -1 & 0 \end{pmatrix}, \quad \begin{pmatrix} & & 1 & 0 \\ & & 0 & -1 \\ -1 & 0 & & \\ 0 & 1 & & \end{pmatrix},$$

and

$$\begin{pmatrix} & & 0 & 1 \\ & & 1 & 0 \\ 0 & -1 & & \\ -1 & 0 & & \end{pmatrix}.$$

LEMMA 11.1. *Only the indecomposables $\Delta_0(0,0)$, $\Delta_1^{\pm}(0,0)$, $\Delta_0^{\pm}(\zeta,-\zeta)$, and $\Delta_1^{\pm}(\zeta,-\zeta)$ $(\zeta=-\bar{\zeta}\neq 0)$ can contribute to a type in $\pm C_2$.*

Proof. As before, it suffices to check $\Delta_m^{\pm}(\zeta,-\zeta)$ $(m\geqslant 0,\ \zeta\neq 0)$, $\Delta_m(0,0)$ (m even), and $\Delta_m^{\pm}(0,0)$ (m odd). Assume first $m\geqslant 3$. In the notation of the earlier Lemmas 7.1 and 9.1, we considered $\mathcal{O}\!\ell(Aa,a)$ for $a\in N^{m-1}E+E$, and found it to be of indeterminant sign. τ vanishes on the σ-invariant $N^{m-1}E+E$, so clearly no such type can contribute to $\pm C_2$.

It remains to rule out $\Delta_2(0,0)$, $\Delta_2^{\pm}(\zeta,-\zeta)$ $(\zeta\neq 0)$. We could argue as in Lemma 9.1 for the case $m=2$, but here will take a simpler approach. As $\Delta_2(0,0)$ has representatives in the closure of the rays generated by representatives of $\Delta_2^{\pm}(\zeta,-\zeta)$ $(\zeta\neq 0)$, it suffices to rule out $\Delta_2(0,0)$ from $\pm C_2$.

Let N represent $\Delta_2(0,0)$, acting on $E+NE+N^2E$, where $\dim E=2$. Define $T=+I$ on $E+N^2E$ and $T=-I$ on NE; clearly $T\in SO^*(6)$ and $TNT=-N$, so that the nilpotent N cannot contribute to any orbit in a causal cone C (as always $C\cap -C=\{0\}$). Q.E.D.

It is not hard to see that $\Delta_0(0,0)$ corresponds to $\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right)$, $\Delta^{\pm}(\zeta,-\zeta)$ to $\left(\begin{smallmatrix}0&-i\zeta\\i\zeta&0\end{smallmatrix}\right)$ where $\pm i\zeta>0$, and $\Delta_1^{\pm}(\zeta,-\zeta)$ $(\zeta=-\bar{\zeta})$ to $N+V$, where

$$N=\pm\begin{pmatrix} & & -1 & i \\ & & -i & -1 \\ 1 & i & & \\ -i & 1 & & \end{pmatrix}$$

and V is an element in the $su(2)$ component of $so^*(4)$, nonzero if and only if $\zeta\neq 0$. We note that $N(\tilde{e}_1+i\tilde{e}_2)=0$. The direct sum of N with the necessary number of zeroes gives an element of $\mathfrak{g}$ in the minimal nilpotent orbit $\mathcal{O}_+$.

Finally, we list the types in $C_2\subset so^*(2n)$, $n\geqslant 3$.

PROPOSITION 11.2. *Types in C_1^{int} are those of the form*

$$\Delta = \Delta^+(\zeta_1, -\zeta_1) + \cdots + \Delta^+(\zeta_n, -\zeta_n). \tag{11.1}$$

Δ is furthermore in C_0^{int} if $(|\zeta_1|,..., |\zeta_n|) \in E_0^{\text{int}}$. Other types in C_2^{int} have either the form

$$\Delta = \Delta(0, 0) + \Delta^+(\zeta_2, -\zeta_2) + \cdots + \Delta^+(\zeta_n - \zeta_n) \tag{11.2}$$

or

$$\Delta = \Delta^-(\zeta_1, -\zeta_1) + \Delta^+(\zeta_2, -\zeta_2) + \cdots + \Delta^+(\zeta_n, -\zeta_n) \tag{11.3}$$

for $(-|\zeta_1|, |\zeta_2|,..., |\zeta_n|) \in E_2^{\text{int}}$. (We recall that the notation $\Delta^\pm(\zeta, -\zeta)$ $(= \Delta_0^\pm(\zeta, -\zeta))$ <u>implies</u> $\zeta \neq 0$.)

Types in C_0 have either the form (11.1) *with $(|\zeta_1|,..., |\zeta_n|) \in E_0$ or*

$$\Delta = \underbrace{\Delta_1^+(0, 0) + \cdots + \Delta_1^+(0, 0)}_{l \text{ times}} + \underbrace{\Delta(0, 0) + \cdots + \Delta(0, 0)}_{m \text{ times}}$$

$$+ \Delta^+(\zeta_{2l+m+1}, -\zeta_{2l+m+1}) + \cdots + \Delta^+(\zeta_n - \zeta_n), \tag{11.4}$$

with $(0,..., 0, |\zeta_{2l+m+1}|,..., |\zeta_n|) \in E_0$ and $2l + m \geqslant 1$.

Types in ∂C_1 have the form (11.4) *with $2l + m \geqslant 1$.*

Finally, types in ∂C_2 have either the form (11.3) *with $(-|\zeta_1|, |\zeta_2|,..., |\zeta_n|) \in \partial E_2$,* (11.4) *with $l \geqslant 1$ and/or $m \geqslant 2$, or*

$$\Delta = \Delta_1^+(\zeta_1, -\zeta_1) + \Delta^+(\zeta_3, -\zeta_3) + \cdots + \Delta^+(\zeta_n, -\zeta_n),$$

where $(|\zeta_1|, -|\zeta_1|, |\zeta_3|,..., |\zeta_n|) \in \partial E_2$ and $\zeta_1 \neq 0$.

V. $so(2, n)$

12. *Identification of Causal Cones in $so(2, n)$*

Let $n \geqslant 3$, $l = [n/2]$,

$$\mathfrak{G} = \left\{ \begin{pmatrix} A & B \\ B^t & C \end{pmatrix} : A, B, C \text{ real}; B\ 2 \times n; A, C \text{ skew} \right\},$$

and

$$\mathfrak{k} = \left\{ \begin{pmatrix} A & 0 \\ 0 & C \end{pmatrix} : A, C \text{ skew} \right\}.$$

If $d = (d_0, d_1, \ldots, d_l) \in \mathbb{R}^{1+l}$, let

$$\mathfrak{h}(d) = \begin{pmatrix} \begin{pmatrix} 0 & -d_0 \\ d_0 & 0 \end{pmatrix} & & 0 \\ & \begin{pmatrix} 0 & -d_1 \\ d_1 & 0 \end{pmatrix} & \\ 0 & & \ddots \end{pmatrix},$$

and set $\mathfrak{h} = \{\mathfrak{h}(d) : d \in \mathbb{R}^{1+l}\}$. The group W_K acting on $\mathfrak{h}$ is all permutations and sign changes of the $d_1, \ldots, d_l$ for n odd, and all permutations and even numbers of sign changes of the $d_1, \ldots, d_l$ for n even. The center of $\mathfrak{k}$ is spanned by $Z = \mathfrak{h}(1, 0, \ldots, 0)$. We also define $B_\theta(X, Y) = \operatorname{tr}(XY^t)$ for $X, Y \in \mathfrak{G}$, which is invariant under the group K generated by $\mathfrak{k}$.

Let

$$E_1 = \{(d_0, \ldots, d_l) : d_0 \geqslant |d_j|, j = 1, \ldots, l\}$$

and

$$E_0 = \left\{(d_0, \ldots, d_l) : d_0 \geqslant \sum_{j=1}^{l} |d_j|\right\}.$$

Then $c_{\min} = \mathfrak{h}(E_0)$, $c_{\max} = \mathfrak{h}(E_1)$, and $c_{\min} = c_{\max}$ if and only if $l = 1$; i.e., $n = 3$. For convenience in the following *we will always assume* $n \geqslant 4$.

Define the symmetric form τ by

$$\tau\left(\begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} z \\ w \end{pmatrix}\right) = x_1 z_1 + x_2 z_2 - \sum_{j=1}^{n} y_j w_j,$$

where $x, z \in \mathbb{R}^2$, $y, w \in \mathbb{R}^n$. The identity component $G = SO_0(2, n)$ is characterized as follows. Let $e_1 = (1, 0, \ldots, 0)$, $e_2 = (0, 1, 0, \ldots, 0) \in \mathbb{R}^{2+n}$, and let $p: \mathbb{R}^{2+n} \to \mathbb{R}^2$ be the orthogonal projection. If $g \in SO(2, n)$, then $g \in G$ if and only if the linearly independent vectors $p(g(e_1))$ and $p(g(e_2))$ have the same orientation as e_1 and e_2.

Let Ω be the set of all pairs (e, f) of linearly independent vectors in $\mathbb{R}^{2+n}$ satisfying $\tau(e, e) = \tau(f, f) = \tau(e, f) = 0$, such that the (automatically linearly independent) $\mathbb{R}^2$ components of e and f are oriented the same as e_1 and e_2. (Also considered in [4, p. 175].) It is clear that G acts on Ω, and that given any $(e, f) \in \Omega$ there exists $r > 0$ and $k \in K$ such that $g(re) = (1, 0, 1, 0, \ldots, 0)$, $g(rf) = (a, b, a, b, 0, \ldots, 0)$ for $a, b \in \mathbb{R}$ with $b > 0$.

Let Φ be the collection of all ordered bases of $\mathbb{R}^{2+n}$ which are transforms by a $g \in G$ of the standard basis $e_1, e_2, \ldots, e_{2+n}$. Also define

$$C_1 = \{X \in \mathfrak{G}: \tau(Xe, f) \geqslant 0 \text{ for all } (e, f) \in \Omega\},$$

a causal cone in $\mathfrak{G}$ (if nonempty).

LEMMA 12.1. *Let $d \in \mathbb{R}^{1+l}$. Then $\mathfrak{h}(d) \in C_1$ if and only if $d \in E_1$. Therefore C_1 is the maximal causal cone containing Z.*

Proof. The necessity of $d \in E_1$ is immediate, e.g., $d_0 - |d_1| \geqslant 0$ follows by taking

$$(e_1 + e_3, e_2 + e_4), (e_1 + e_4, e_2 + e_3) \in \Omega.$$

Conversely, by the remark following the definition of Ω, it suffices to compute $\tau(Y\tilde{e}, \tilde{f})$, where $\tilde{e} = (1, 0, 1, 0, \ldots, 0)$, $\tilde{f} = (a, b, a, b, 0, \ldots, 0) \in \mathbb{R}^{2+n}$ with $b > 0$, and Y is conjugate to $\mathfrak{h}(d)$ under K. By the Kostant–Horn theorem (cf. Section 5), we may assume $Y = \mathfrak{h}(d_0, h, \ldots)$, where $|h| \leqslant \max_{j>0} |d_j| < d_0$. One computes

$$\tau(Y\tilde{e}, \tilde{f}) = b(d_0 - h) \geqslant 0. \qquad \text{Q.E.D.}$$

Let $\Gamma: \mathfrak{G} \to \mathfrak{h}$ be the B_θ-orthogonal projection. The noncompact convexity theorem says that for all $d \in E_1$,

$$\begin{aligned} &\Gamma(\{g\mathfrak{h}(d)g^{-1}: g \in G\}) \\ &\qquad = \{X + Y: X \in c_{\min}, Y \text{ in the convex hull of } \sigma(d), \sigma \in W_K\}. \end{aligned}$$

Note that as $n \geqslant 4$, $(v_1, \ldots, v_{2+n}) \in \Phi$ implies $(v_1 \pm v_3, v_2 \pm v_4) \in \Omega$, so $X \in C_1$ implies $\tau(Xv_1, v_2) \geqslant |\tau(Xv_3, v_4)|$. Similarly, it follows that the minimal causal cone containing Z is

$$C_0 = \left\{X \in C_1: \tau(Xv_1, v_2) \geqslant \sum_{j=1}^{l} |\tau(Xv_{2j+1}, v_{2j+2})| \right.$$
$$\left. \text{for all } (v_1, \ldots, v_{2+n}) \in \Phi \right\}$$

As before each X in

$$C_1^{\text{int}} = \{X \in \mathfrak{G}: \tau(Xe, f) > 0 \text{ for all } (e, f) \in \Omega\}$$

is conjugate under G to some $Y \in \mathfrak{h}(E_1^{\text{int}})$.

13. *Boundary Orbits in $so(2, n)$*

Just as the types in the maximal cones of $so^*(2n)$ $(n \geqslant 3)$ were composed of indecomposable types coming from $so^*(4) \approx sl(2, \mathbb{R}) \oplus su(2)$, we will see that corresponding orbits for $so(2, n)$ come from orbits of $o(2, 2) \approx sl(2, \mathbb{R}) \oplus sl(2, \mathbb{R})$. For these algebras we know of no better procedure to determine them than to examine the list at the very end of Section 4.

We first give the decomposition of $o(2, 2)$. One $sl(2, \mathbb{R})$ is spanned by

$$J_1 = \begin{pmatrix} \begin{matrix} 0 & -1 \\ 1 & 0 \end{matrix} & 0 \\ 0 & \begin{matrix} 0 & -1 \\ 1 & 0 \end{matrix} \end{pmatrix} \quad \text{and all} \quad \begin{pmatrix} 0 & \begin{matrix} x & y \\ y & -x \end{matrix} \\ \begin{matrix} x & y \\ y & -x \end{matrix} & 0 \end{pmatrix};$$

the other is spanned by

$$J_2 = \begin{pmatrix} \begin{matrix} 0 & -1 \\ 1 & 0 \end{matrix} & 0 \\ 0 & \begin{matrix} 0 & 1 \\ -1 & 0 \end{matrix} \end{pmatrix} \quad \text{and all} \quad \begin{pmatrix} 0 & \begin{matrix} x & y \\ -y & x \end{matrix} \\ \begin{matrix} x & -y \\ y & x \end{matrix} & 0 \end{pmatrix}.$$

Of the outer automorphisms from $O(2, 2)$, conjugation by

$$\begin{pmatrix} I & 0 \\ 0 & \begin{matrix} -1 & 0 \\ 0 & 1 \end{matrix} \end{pmatrix}$$

exchanges the two subalgebras (and exchanges J_1 and J_2), whereas conjugation by

$$\begin{pmatrix} \begin{matrix} -1 & 0 \\ 0 & 1 \end{matrix} & 0 \\ 0 & \begin{matrix} -1 & 0 \\ 0 & 1 \end{matrix} \end{pmatrix}$$

"inverts" each $sl(2, \mathbb{R})$ separately, exchanging the positive and negative cones. In $\mathrm{Ad}(G)$ the latter does not exist, but the former does in the sense that $o(2, 2) \subset o(2, n)$ and $n \geqslant 4$. Note $J = \frac{1}{2}(J_1 + J_2)$ with this inclusion.

Of all the indecomposable types for the $o(2, n)$ (i.e., orbits under the disconnected groups $O(2, n)$), all but $\Delta_4^{-1}(0)$ (signature $(2, 3)$) and $\Delta_2^1(\zeta, -\zeta)$ $(\zeta = -\bar{\zeta} \neq 0$, signature $(2, 4))$ lie in an $o(2, 2)$. One sees by writing out the matrices [12] that $\Delta_4^{-1}(0)$ and $\Delta_2^1(\zeta, -\zeta)$ cannot contribute to types in $\pm C_1$.

For the rest we give the signature of τ, a representative matrix, and a description of the image in $o(2,2)$ in terms of the $sl(2,\mathbb{R})$'s.

$\Delta^{\pm}(0)$. Sig$(1,0)$ or $(0,1)$; a 1×1 zero (0).

$\Delta_2^{+}(0)$. Sig $(1,2)$;

$$\left(\begin{array}{c|cc} 0 & 0 & 1 \\ \hline 0 & 0 & -1 \\ 1 & 1 & 0 \end{array}\right).$$

In $o(2,2)$ it is the sum of two nonzero nilpotents in the $sl(2,\mathbb{R})$'s, one with positive J_i component, the other with negative such. It cannot contribute to a type in C_1.

$\Delta_2^{-}(0)$. Sig $(2,1)$;

$$\left(\begin{array}{cc|c} 0 & -1 & 0 \\ 1 & 0 & 1 \\ \hline 0 & 1 & 0 \end{array}\right).$$

This is the other sum of nonzero nilpotents in the $sl(2,\mathbb{R})$'s; it can contribute to types in C_1.

$\Delta_1(0,0)$. Sig $(2,2)$;

$$\left(\begin{array}{cc|cc} 0 & -1 & 0 & -1 \\ 1 & 0 & 1 & 0 \\ \hline 0 & 1 & 0 & 1 \\ -1 & 0 & -1 & 0 \end{array}\right).$$

It is a nonzero nilpotent in one $sl(2,\mathbb{R})$ and 0 in the other; it can contribute to C_1.

$\Delta^{+}(\zeta,-\zeta)$. Sig $(2,0)$ or $(0,2)$;

$$\begin{pmatrix} 0 & -d \\ d & 0 \end{pmatrix}, \qquad d\neq 0.$$

$\Delta_1^{\pm}(\zeta,-\zeta)$. $(\zeta=-\bar{\zeta}\neq 0)$ Sig $(2,2)$. The $\pm$ cases are the two possibilities for an elliptic element in one $sl(2,\mathbb{R})$ and a nonzero nilpotent in the other. One can contribute to $\pm C$, the other cannot. Let us say $\Delta_1^{+}(\zeta,-\zeta)$ can contribute. A typical representative is

$$X=\begin{pmatrix} 0 & -\lambda & & \\ \lambda & 0 & \multicolumn{2}{c}{0} \\ \multicolumn{2}{c}{0} & 0 & -\lambda \\ & & \lambda & 0 \end{pmatrix}+\begin{pmatrix} 0 & -1 & 0 & -1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ -1 & 0 & -1 & 0 \end{pmatrix}, \qquad \lambda>0. \quad (13.1)$$

The remaining cases $\Delta_{0,1}(\zeta, -\zeta)$ ($0 \neq \zeta$ real) and $\Delta_0(\zeta, -\zeta, \bar{\zeta}, -\bar{\zeta})$ ($\zeta \neq \pm\bar{\zeta}$) are in $o(2, 2)$ and involve hyperbolic components in the $sl(2, \mathbb{R})$'s, so cannot contribute to a causal cone.

It is clear from this list that types in C_1 are composed of $\Delta_0^{\pm}(0)$ (1×1 zero), $\Delta_2^-(0)$ (nilpotent $\oplus$ nilpotent), $\Delta_1(0, 0)$ (nilpotent $\oplus$ zero), $\Delta_1^+(\zeta, -\zeta)$ (nilpotent $\oplus$ elliptic), and $\Delta_0^{\pm}(\zeta, -\zeta)$ (the $\begin{pmatrix} 0 & -d \\ d & 0 \end{pmatrix}$'s). Furthermore, representatives of $\Delta_2^-(0)$ and $\Delta_1(0, 0)$ are conjugate to arbitrarily small multiples of themselves, so can appear in a type in C_1 if and only if they are followed by all $\Delta_0^-(0)$'s. These types are on the boundaries of C_1 and C_0. Finally, as we see from (13.1) above, representatives of $\Delta_1^+(\zeta, -\zeta)$ are sums of certain representatives of the elliptic types $\Delta_0^{\pm}(\zeta, -\zeta)$ and those of $\Delta_1(0, 0)$. Using these observations a list of types in the causal cones, like Propositions 9.2 and 11.2, could easily be written out, but is omitted.

VI. Global Causality of the Covering Groups

14. *Definition of Causal and Temporal Actions*

A *causal structure* [16, Chap. 2] on a manifold M is a smooth assignment to each point $p \in M$ of a nontrivial closed convex cone in the tangent space $T_p(M)$. A causal manifold M is called *globally causal* if there exists no closed nontrivial piecewise C^1-curve in M, the differential of which lies in the causal cone at each point. A causal structure on a Lie group is said to be *invariant* if it is invariant under both left and right translations. Clearly such a structure is completely determined by a causal cone in its Lie algebra.

The purpose of his chapter is to show that the simply connected Lie groups $\tilde{G}$ associated to the classical Lie algebras studied in Chapters II to V are globally causal with respect to certain of the available causal cones. A corollary is the existence of invariant semigroups S in $\tilde{G}$ having the property that $S \cap S^{-1} = \{e\}$. The finite coverings of the adjoint groups, not being globally causal, have no such (Lie) semigroups.

In the (classical) tube-type cases we show that the universal covering is globally causal with respect to all such causal structures. In fact the argument given applies (and the same is true) for precisely those coverings of the adjoint groups which have infinite centers. These are exactly those coverings which act on infinite-sheeted coverings of the Bergman–Šhilov boundaries of the associated hermitian symmetric spaces. (In the case of $su(2, 2)$ there are two such groups, the universal covering $\tilde{G}$ of $SU(2, 2)$ and the quotient of $\tilde{G}$ modulo its unique $\mathbb{Z}_2$-central subgroup.) In [17] Vinberg shows abstractly that a covering of the adjoint group of an *arbitrary* hermitian symmetric Lie algebra in globally causal with respect to the *minimal* causal cone if and only if the covering has infinite center, but does not consider such groups' nonlinear realizations. (As remarked there, such groups have no faithful (finite-dimensional) linear representations.)

If a Lie group G acts transitively on a manifold M, then it is well known that the universal covering group $\tilde{G}$ acts on the universal cover $\tilde{M}$. In fact, if $M = G/H$ for some subgroup H, and if $0 \to D \to \tilde{G} \to^{\pi} G \to 0$ is the exact sequence of groups, then there is the sequence of covering maps

$$\tilde{G}/\tilde{H}' \to \tilde{G}/\tilde{H} \cong G/H_0 \to G/H,$$

where H_0 is the identity component of H, $\tilde{H} = \pi^{-1}(H_0)$, and $\tilde{H}'$ is the identity component of $\tilde{H}$. $\tilde{G}/\tilde{H}'$ is simply connected, so $\tilde{M} \cong \tilde{G}/\tilde{H}'$. (See, for example, [16, p. 33].) Furthermore, the composition $\tilde{H}' \to \pi^{-1}(H_0) \to^{\pi} H_0$ is also a covering map.

Recall that G is said to act *effectvely* on M if the identity eement $e \in G$ is the only element of G acting trivially on M. It is easily seen that if $M = G/H$, then the subgroup of G acting trivially on M is precisely the largest normal subgroup of G contained in H.

The proof of the following is not difficult, and we will not use it here.

PROPOSITION 14.1. *With the above notation, if G acts effectively on M, then $\tilde{G}$ acts effectively on $\tilde{M}$ if and only if $\pi: \tilde{H}' \to H_0$ is an isomorphism. In any case, the subgroup of $\tilde{G}$ acting trivially on $\tilde{M}$ is $D \cap \tilde{H}'$.*

DEFINITION. Let G be a Lie group acting smoothly on a manifold M with causal structure $\{C_p\}_{p \in M}$. Say that G *acts causally* on M if the group actions preserve this structure. If in addition G has an invariant causal cone C, say that G acts *temporally* [12] if G acts causally and

$$\frac{d}{dt} \exp tX \cdot y|_{t=0} \in C_y \tag{14.1}$$

for all $X \in C$, $y \in M$. If $\Phi: G \times M \to M$ is the group action, these conditions are equivalent to requiring that the differential $d\Phi_{(g,p)}$ map the direct sum of cones $C_g \times C_p$ into $C_{g(p)}$ for all $g \in G$, $p \in M$, where of course $C_g = dL_g C = dR_g C$, L_g, R_g being left and right translations by g.

Remark. In [16] the cone C was defined implicity by projecting the inverse images $(d\Phi_{(e,p)})^{-1}(C_p)$ onto $Te(G)$, and letting C be the intersection over all $p \in M$ of these projections. (That is, C is all those infinitesimal group transformations which move all points $p \in M$ into the "infinitesimal future" C_p at p. Given any causal action of a Lie group G on a causally oriented manifold M, this procedure defines a closed invariant convex cone in the Lie algebra of G, which of course is quite possibly $\{0\}$.

15. *Alternate Presentations of* $sp(n, \mathbb{R})$, $so^*(2n)$

In the remainder of this chapter we specialize to the cases of the simple groups treated in Chapters II to V. Our first step toward proving the global causality results mentioned above will be to embed copies of $sp(n, \mathbb{R})$ and $so^*(2n)$ into $su(n, n)$ by means of the mapping

$$\mathscr{C}: \mathbb{C}^{2n} \to \mathbb{C}^{2n}: X \to \frac{1}{\sqrt{2}} \begin{pmatrix} i & 1 \\ -i & 1 \end{pmatrix} X,$$

the matrix on the r.h.s. being more than superficially related to the Cayley transform for symmetric spaces [18].

Now $sp(n, \mathbb{R})$, as presented in Chapter II, acts on $\mathbb{R}^{2n}$; we extend its action to $\mathbb{C}^{2n}$ by complex-linearity, and extend the symplectic form $\mathscr{A}(\cdot, \cdot)$ (defined originally on $\mathbb{R}^{2n}$) to a real symplectic form, also denoted $\mathscr{A}(\cdot, \cdot)$, on $\mathbb{C}^{2n}$, by requiring that $\mathscr{A}(ix, y) = 0$ and $\mathscr{A}(ix, iy) = \mathscr{A}(x, y)$ for all $x, y \in \mathbb{R}^{2n}$. One checks easily that

$$\mathscr{A}(v, w) = \operatorname{Re}\{iH(\mathscr{C}v, \mathscr{C}w)\} \tag{15.1}$$

for all $v, w \in \mathbb{C}^{2n}$, where $H(\cdot, \cdot)$ is the hermitian form defined in Chapter III for $su(n, n)$:

$$H\left(\begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} u \\ v \end{pmatrix}\right) = \sum_{i=1}^{n} x_i \overline{u_i} - \sum_{j=1}^{n} y_j \overline{v_j}, \qquad x, y, u, v \in \mathbb{C}^n.$$

(Note that the symplectic form $-\operatorname{Im} H(\cdot, \cdot)$ defined for $su(p, q)$ in Chapter III is *not* the one employed here.) Thus

$$\mathscr{C}\, sp(n, \mathbb{R})\, \mathscr{C}^{-1} \equiv sp(n, \mathbb{R})_1 \subseteq su(n, n),$$

with equality only if $n = 1$.

For $so^*(2n)$ $(n \geqslant 3)$, it was defined as acting in $\mathbb{C}^{2n}$, and $so^*(2n) \cap sp(n, \mathbb{R})$ is just the $\mathfrak{k} \approx u(n)$ defined for either algebra. Also, the symplectic form $\mathscr{A}(\cdot, \cdot)$ on $\mathbb{C}^{2n}$ defined in Chapter III is the *same* as the one above. (However, the hermitian form $H(\cdot, \cdot)$ defined in Chapter IV for $so^*(2n)$ is not the hermitian form employed here.) Therefore by (15.1)

$$\mathscr{C}\, so^*(2n)\, \mathscr{C}^{-1} \equiv so^*(2n)_1 \subsetneq su(n, n)$$

for all n considered here. Also by (15.1) we can compare directly the matrix elements $\mathscr{A}(Xv, v)$, $v \in \mathbb{C}^{2n}$, $X \in sp(n, \mathbb{R})$ or $so^*(2n)$, with corresponding $\operatorname{Re}\{iH(Yu, u)\}$, where $u = \mathscr{C}v$, and $Y \in sp(n, \mathbb{R})_1$ or $so^*(2n)_1$.

It is useful to have the explicit forms of these new algebras and their correspnding connected groups in $SU(n, n)$. We have

$$sp(n, \mathbb{R})_1 = \left\{ \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix} : \begin{matrix} \alpha, \beta \; n \times n \text{ complex matrices,} \\ \alpha \text{ skew Hermitian}, \beta \text{ symmetric} \end{matrix} \right\}$$

and

$$\mathscr{C} \begin{pmatrix} A & B \\ C & -A^t \end{pmatrix} \mathscr{C}^{-1} = \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix},$$

where

$$\alpha = \tfrac{1}{2}(A - A^t + i(B - C))$$

and

$$\beta = \tfrac{1}{2}(-A - A^t + i(B + C)).$$

Also,

$$Sp(n, \mathbb{R})_1 = \left\{ \begin{pmatrix} F & G \\ \bar{G} & \bar{F} \end{pmatrix} : G^*F \text{ symmetric}, F^*F - G^t\bar{G} = I \right\},$$

the two conditions being equivalent to FG^t symmetric and $FF^* - GG^* = I$.

Analogously,

$$so^*(2n)_1 = \left\{ \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix} : \begin{matrix} \alpha, \beta \; n \times n \text{ complex matrices,} \\ \alpha \text{ skew Hermitian}, \beta \text{ skew} \end{matrix} \right\}$$

and

$$\mathscr{C} \begin{pmatrix} A & B \\ -\bar{B} & \bar{A} \end{pmatrix} \mathscr{C}^{-1} = \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix},$$

where $\alpha = \frac{1}{2}(A + \bar{A} + i(B + \bar{B}))$ and $\beta = \frac{1}{2}(-A + \bar{A} + i(B - \bar{B}))$. Finally,

$$SO^*(2n)_1 = \left\{ \begin{pmatrix} F & G \\ -\bar{G} & \bar{F} \end{pmatrix} : \begin{matrix} G^*F \text{ skew,} \\ F^*F - G^t\bar{G} = I \end{matrix} \right\},$$

the two conditions equivalent to FG^t skew and $FF^* - GG^* = I$.

From (15.1) it is clear that the causal cones in $sp(n, \mathbb{R})_1$ are contained in the minimal causal cones of $su(n, n)$. Secondly, recall the self-dual cone C_1 in $so^*(2n)$ defined in Chapter IV; define $\tilde{C}_1 = \mathscr{C} C_1 \mathscr{C}^{-1}$. Again by (15.1) we see that $\tilde{C}_1$ is contained in a minimal causal cone in $su(n, n)$. However, it is easily seen that the maximal causal cones in $so^*(2n)_1$ extend outside the maximal causal cones in $su(n, n)$.

16. *Temporal Actions on Šhilov Boundaries*

We recall the standard action of $SU(n, n)$ on the group of unitaries $U(n)$ by fractional linear transformations [16, p. 35]. Given $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in SU(n, n)$ and $U \in U(n)$, define

$$(\rho(g))\, U = (AU + B)(CU + D)^{-1} \in U(n).$$

ρ defines a left group action. Giving $U(n)$ the standard $U(n)$-group translation-invariant causal structure determined by nonnegative hermitian matrices, the action ρ is causal, as shown in [16, p. 35].

Let

$$U_s(n) = \{U \in U(n)\colon U \text{ symmetric}\}$$

and (only for n even)

$$U_k(n) = \{U \in U(n)\colon U \text{ skew}\}.$$

One checks that $Sp(n, \mathbb{R})_1$ and $SO^*(2n)_1$ transform $U_s(n)$, $U_k(n)$, respectively. Restricting the causal structure of $U(n)$ to these closed submanifolds, one finds that the resulting cone fields on $U_s(n)$ and $U_k(2n)$ are nontrivial for all $n \geqslant 1$. Thus $Sp(n, \mathbb{R})_1$ and $SO^*(4n)_1$ act causally on $U_s(n)$ and $U_k(2n)$, respectively. (These two actions and the action of $SU(n, n)$ on $U(n)$ are individually isomorphic to the actions of each group G on the Šhilov boundary of its associated hermitian symmetric space G/K.)

THEOREM 16.1. *$SU(n, n)$ acts temporally on $U(n)$, the causal structure on $SU(n, n)$ coming from the maximal causal cone C_1.*

Proof. As $SU(n, n)$ acts causally on $U(n)$ and its causal structure is invariant, it suffices to check the condition (14.1) at $U \in U(n)$; we compute the differential of ρ at this point. For $X = \begin{pmatrix} A & B \\ B^* & C \end{pmatrix}$ in the Lie algebra, $(U + \varepsilon AU + \varepsilon B)(\varepsilon B^* U + I + \varepsilon C)^{-1} = U(I + \varepsilon L) + O(\varepsilon^2)$, where $L = U^{-1}AU - C - B^*U + U^{-1}B$. We check that iL is nonnegative if $X \in C_1$.

Given $y \in \mathbb{C}^n$, set $x = Uy$, and note

$$\langle iLy, y\rangle = i\langle Ax, x\rangle - i\langle Cy, y\rangle - i\langle B^*x, y\rangle + i\langle By, x\rangle.$$

In fact this expression is just $iH(Xw, w)$ for $w = \begin{pmatrix} x \\ y \end{pmatrix}$ and $H(w, w) = 0$, so $\langle iLy, y\rangle \geqslant 0$ for all $y \in \mathbb{C}^n$ by the definition of C_1. Q.E.D.

In particular, $SU(n, n)$ acts temporally with the causal structure obtained from the minimal cone C_0, and by the last paragraph in Section 15 we have:

COROLLARY 16.2. *$Sp(n, \mathbb{R})_1$, given its unique invariant causal structure,* *acts temporally on the Šhilov boundary $U_s(n)$. $SO^*(2n)_1$, given the invariant causal structure from the self-dual cone $\tilde{C}_1$ (cf. end of Section* 15), *acts temporally on $U_k(n)$ for $n \geqslant 4$ even, and on $U(n)$ for all $n \geqslant 3$.*

The case of $SO^*(4n)$ can be strengthened.

THEOREM 16.3. *$SO^*(4n)_1$ ($n \geqslant 2$), given the invariant causal structure from a maximal causal cone in its Lie algebra, acts temporally on the Šhilov boundary $U_k(2n)$.*

Proof. Using the computations in the proof of Theorem 16.1 and (15.1), it suffices to show that if $U \in U_k(2n)$ and $v \in \mathbb{C}^{2n}$,

$$\mathscr{C}^{-1}\begin{pmatrix} Uv \\ v \end{pmatrix} = \begin{pmatrix} iv - iUv \\ v + Uv \end{pmatrix}$$

is isotropic with respect to the quaternionic form τ_- appearing in the definition of the maximal causal cone C_2 (cf. Section 10), and this is a short computation. Q.E.D.

Finally, we turn to $so(2, n)$, $n \geqslant 3$. $SO_0(2, n)$ acts on the projective quadric

$$\mathscr{Q} = \{[x \in \mathbb{R}^{2+n} : \tau(x, x) = 0]\},$$

$[x]$ meaning the line determined by x. (See [16, Section II.4.]) The tangent space to $[x] \in \mathscr{Q}$ can be identified with the projective space of vectors $y \in \mathbb{R}^{2+n}$ such that $\tau(y, x) = 0$, modulo $\mathbb{R}x$. τ factors to a conformal structure τ_1 on $\mathscr{Q}$ with signature $(1, n)$. As $S^1 \times S^n$ is a double cover of $\mathscr{Q}$ a system of forward cones can easily be chosen. Clearly G leaves τ_1 invariant, so G acts causally on $\mathscr{Q}$.

THEOREM 16.4. *$SO_0(2, n)$, given the invariant causal structure from a maximal causal cone, acts tempotally on $\mathscr{Q}$.*

Proof. Recall the maximal causal cone C_1 defined in Chapter V: $X \in C_1$ if and only if $\tau(Xe, f) \geqslant 0$ for all $e, f \in \mathbb{R}^{2+n}$, $e \neq 0 \neq f$ such that $\tau(e, e) = \tau(f, f) = \tau(e, f) = 0$, and such that the $\mathbb{R}^2$ components of e and f are oriented the same as $(1, 0)$ and $(0, 1)$.

Given any $[e] \in \mathscr{Q}$, $X \in \mathfrak{G}$, we have $[Xe] \in T_{[e]}(\mathscr{Q})$, the tangent space at $[e]$, as $\tau(Xe, e) = 0$ always. Now the space of all $f \in \mathbb{R}^{2+n}$ such that $\tau(f, f) = \tau(f, e) = 0$ and satisfying the above orientation condition, modulo $\mathbb{R}e$, and projectivized by positive scalars, is one of the cones $\tilde{C}$ (not convex) in $T_{[e]}(\mathscr{Q})$ of τ_1-isotropic vectors. It follows that $X \in C_1$ is equivalent to $\tau_1([Xe], w) \geqslant 0$ for all $w \in \tilde{C}$, which implies (as τ_1 has signature $(1, n)$) that

$[Xe]$ is a tangent vector in the convex cone generated by $\tilde{C}$. Thus $[Xe]$ is a forward-pointing tangent vector at $[e]$ for all $[e] \in \mathscr{Q}$. Q.E.D.

THEOREM 16.5. *The universal covers* $\widetilde{Sp(n, \mathbb{R})}$, $\widetilde{SU(n, n)}$, $\widetilde{SO^*(4n)}$, $\widetilde{SO_0(2, n+2)}$ $(n \geqslant 1)$ *are globally causal with respect to causal structures from any causal cone in their Lie algebras.* $SU(p, q)$ $(p > q \geqslant 1)$ *is globally causal with respect to the causal structure from a minimal causal cone in* $su(p, q)$, *and* $so^*(4n+2)$ $(n \geqslant 1)$ *is globally causal with respect to the causal structure from the self-dual cone* $C_1 \subset so^*(4n+2)$.

Proof. The first four series of groups act temporally on $\widetilde{U_s(n)}$, $\widetilde{U(n)}$, $\widetilde{U_k(2n)}$, and $\mathbb{R} \times S^n$, respectively, by the initial remarks in Section 14 and Theorems 16.1–4. $\widetilde{U(n)}$ and $\mathbb{R} \times S^n$ are globally causal by Corollary 2.3.1 and Scholium 2.11 in [16].

To identify $\widetilde{U_s(n)}$ and $\widetilde{U_k(2n)}$, we set

$$M_s(n) = \{U \in U_s(n) \colon U \in SU(n)\} = \{UU^t \colon U \in SU(n)\}$$

and

$$M_k(2n) = \{UDU^t \colon U \in SU(2n)\},$$

$$D = \underbrace{\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \oplus \cdots \oplus \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}}_{n \text{ times}}.$$

They are coset spaces, isomorphic to $SU(n)/SO(n)$ and $SU(2n)/Sp(n)$, respectively, and are simply connected because $SU(n)$ is simply connected and the factor spaces are connected. Thus

$$\widetilde{U_s(n)} \cong M_s(n) \times \mathbb{R} \qquad \text{and} \qquad \widetilde{U_k(2n)} \cong M_k(2n) \times \mathbb{R},$$

and the spaces on the r.h.s. are contained component-wise in $SU(n) \times \mathbb{R} \cong \widetilde{U(n)}$. Clearly then $\widetilde{U_s(n)}$ and $\widetilde{U_k(2n)}$ are also globally causal.

We can assume that any closed time-like curve $g(t)$ $(0 \leqslant t \leqslant 1)$ in any of the covering groups $\tilde{G}$ starts and ends at $e \in \tilde{G}$. As the action of $\tilde{G}$ on the pertinent manifold $\tilde{S}$ is temporal, $g(t)p$ (for all $p \in \tilde{S}$) is a closed time-like curve in $\tilde{S}$, contradicting the global causality unless $g(t)p = p$ for all t, so $g(t) = e$ for all t.

For the last two statements, note that the connected subgroups of $\widetilde{SU(n+1, n+1)}$ $(n > 1)$ corresponding to the embedded subalgebras $su(n+1, q) \hookrightarrow su(n+1, n+1)$ $(q \leqslant n)$ and $so^*(2(n+1)) \hookrightarrow su(n+1, n+1)$ (as in Section 15) also act temporally on $\widetilde{U(n+1)}$, hence are globally causal by the preceeding paragraphs. This implies immediately that their universal covers are also globally causal. Q.E.D.

COROLLARY 16.6. *If $\tilde{G}$ is any of the above (Theorem 16.5) groups with the causal cone C indicated, then there exists a semigroup S in $\tilde{G}$ such that $S \cap S^{-1} = \{e\}$ and $gSg^{-1} = S$ for all $g \in \tilde{G}$, generated by $\{\exp X : X \in C\}$.*

Proof. If $\exp X_1 \cdots \exp X_n = e$, $X_j \in C$, the expression gives a piecewise C^1 time-like curve in $\tilde{G}$, so all X_j must be 0. This suffices to get $S \cap S^{-1} = \{e\}$, and $gSg^{-1} = S$ is clear. Q.E.D.

VII. LOW-DIMENSIONAL EXAMPLES

17. *Classification of Cones in $su(2, 1)$*

In this section we classify all the invariant convex cones in $su(2, 1)$. This algebra and $sp(2, \mathbb{R})$ are the only hermitian symmetric Lie algebras of rank 2.

Let C_1 be the maximal causal cone in $\mathfrak{G} = su(2, 1)$, defined in Section 8, and C_0 the minimal causal cone contained in C_1. For brevity we let $\langle \cdot, \cdot \rangle$ denote the inner product $B_\theta(\cdot, \cdot)$ on $\mathfrak{G}$. One obtains the following picture of the compact Cartan subalgebra $\mathfrak{h}$ (Fig. 1).

The Weyl group W_K consists of reflections about the line $D'D$; the minimal (maximal) closed cone $c_{\min}$ (resp., $c_{\max}$) in $\mathfrak{h}$ is generated by $0B$ (resp., $0A$) and its Weyl reflection. The angle between $0B$ and $0E$ is 60°. Clearly there is a one-dimensional continuum C_θ $(0 \leqslant \theta \leqslant 1)$ of causal cones in $\mathfrak{G}$, so that $(C_\theta)^* = C_{1-\theta}$ (cf. IV, Section 5). Let $c_\theta = C_\theta \cap \mathfrak{h}$, so that $c_0 = c_{\min}$, $c_1 = c_{\max}$.

We recall from Section 9 our notation for the orbits in C_1. The elliptic types in $\mathfrak{G}$ have the form

$$\Delta = \Delta^+(-i\lambda_1) + \Delta^+(-i\lambda_2) + \Delta^-(i(\lambda_1 + \lambda_2)),$$

i.e., a 3×3 diagonal matrix X with $\operatorname{diag} X = (-i\lambda_1, -i\lambda_2, i(\lambda_1 + \lambda_2))$. Such Δ

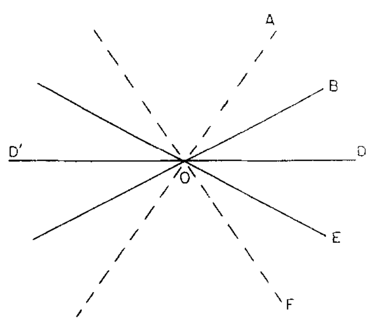

FIGURE 1

is in C_0 when $\lambda_1, \lambda_2 \geqslant 0$, and, for example, on the boundary of C_1 when $\lambda_2 = -2\lambda_1$, etc. In addition, there is the minimal nilpotent type

$$\mathcal{O}_+ = \Delta^+(0) + \Delta_1^+(0)$$

and

$$\mathcal{O}_2 \equiv \Delta^+(-2i\lambda) + \Delta_1^+(i\lambda) \qquad (\lambda > 0),$$

which is represented by a sum of representatives from $\mathcal{O}_+$ and $\Delta^+(-2i\lambda) + \Delta^+(i\lambda) + \Delta^-(i\lambda)$. For convenience let ε_θ be some nonzero elliptic element in G on the boundary of C_θ.

It is useful to observe that

$$\langle X, Y\rangle > 0 \qquad \text{for all } X \in C_0^{\text{int}}, \quad Y \in C_1. \tag{17.1}$$

This is seen from the classification of orbits in C_1 above, and an application of the noncompact convexity theorem (cf. Section 5).

PROPOSITION 17.1. *Let C_0^e be the invariant convex cone in $\mathfrak{G}$ generated by ε_0. Then*

$$C_0^{\text{int}} \subsetneq C_0^e \subsetneq C_0,$$

and C_0^e is the unique invariant convex cone in $\mathfrak{G}$ with this property.

Proof. $c_{\min}$ is included in the convex hull of the orbit of ε_0, and since $\mathcal{O}_+$ is the only other orbit in C_0, it suffices to show $\mathcal{O}_+ \not\subseteq C_0^e$. This follows from the fact that $\langle X, Y\rangle > 0$ for all $X \in C_0^e$, $Y \in C_0$ (proof follows the same as for (17.1) above), whereas this is not true for $X \in \mathcal{O}_+$. Q.E.D.

PROPOSITION 17.2. *Let $0 < \theta < 1$, and let C_θ^e (resp., C_θ^n) be the invariant convex cone generated by ε_θ (resp., C_θ^{int} and $\mathcal{O}_+$). Then $C_\theta^e \neq C_\theta^n$,*

$$\begin{array}{ccccc} & & C_\theta^n & & \\ & \subsetneq & & \supsetneq & \\ C_\theta^{\text{int}} & & & & C_\theta, \\ & \supsetneq & & \subsetneq & \\ & & C_\theta^e & & \end{array}$$

and there are no other invariant convex cones between C_θ^{int} and C_θ.

Proof. Note that $\mathcal{O}_+$ and the orbits of $r\varepsilon_\theta$ $(r > 0)$ are the only orbits in C_θ which are not in C_θ^{int}, as $\theta < 1$. Therefore it suffices to show that C_θ^n and C_θ^e are distinct from C_θ. For this it suffices to observe that $\langle X, Y\rangle > 0$ for all

$X \in C^e_\theta$, $Y \in C^n_{1-\theta}$, and that strict positivity fails for arbitrary $X \in C_\theta$, or $Y \in C_{1-\theta}$. Q.E.D.

The case $\theta = 1$ is rather more complicated. Up to multiplication by positive scalars, there are three orbits in C_1 not contained in C_1^{int}: $\mathcal{O}_+$, $\mathcal{O}_2$, and the elliptic orbit of ε_1. Therefore there are at most $2^3 - 2 = 6$ invariant convex cones strictly between C_1^{int} and C_1; however, the fact that any invariant convex cone containing ε_1 also contains $\mathcal{O}_2$, is the content of Lemma 9.3, so from this there are at most four such cones. We will see that in fact these four invariant convex cones do occur, i.e., are distinct.

Let C_1^2, C_1^e, C_1^n be the invariant convex cones generated by C_1^{int} and $\mathcal{O}_2$, ε_1, C_1^{int} and $\mathcal{O}_+$, respectively. Let $C_1^{2,n}$ be the invariant convex cone generated by C_1^{int}, $\mathcal{O}_2$, and $\mathcal{O}_+$. Thus $C_1^2 \subseteq C_1^e$ by Lemma 9.3.

PROPOSITION 17.3. *We have the diagram*

$$\begin{array}{ccccc} & \subsetneq & C_1^n \subsetneq C_1^{2,n} & \supsetneq & \\ C_1^{\text{int}} & & & & C_1. \\ & \supsetneq & C_1^2 \subsetneq C_1^e & \subsetneq & \end{array}$$

All six cones are distinct, and are the only invariant convex cones between C_1^{int} and C_1.

Proof. Recall that $\langle X, Y\rangle > 0$ for all $X \in \mathcal{O}_+$, $Y \in C^e_0$ (noted in the proof of Proposition 17.1). Therefore we have $\langle X, Y\rangle > 0$ for all $X \in C_1^{2,n}$, $Y \in C^e_0$, since such clearly holds for $X \in C_1^{\text{int}}$, and elements of $\mathcal{O}_2$ are sums of elements in $\mathcal{O}_+$ and C_1. Therefore $C_1^2 \subsetneq C_1^e$ and $C_1^e \subsetneq C_1^{2,n}$, whence also $C_1^{2,n} \subsetneq C_1$.

To see that $C_1^n \subsetneq C_1^{2,n}$, it suffices to note that $C_1^n = \bigcup_{\theta<1} C_\theta$, and $\mathcal{O}_2 \nsubseteq C_\theta$ for all $\theta < 1$ by the noncompact convexity theorem. To show $C_1^e \subsetneq C_1$, note that no nilpotent element, such as an $X \in \mathcal{O}_+$, can be a convex combination of elliptic elements in an invariant convex cone, as no elliptic orbit can approach 0. Also from these arguments it follows that all six cones are distinct. Q.E.D.

18. *Two Self-Dual Causal Cones in* $su(3, 1)$

There are three hermitian symmetric Lie algebras of rank 3: $su(3, 1) \approx so^*(6)$, $su(2, 2) \approx so(2, 4)$, and $so(2, 5)$. We recall from Section 10 that all the $so^*(2n)$ possess self-dual causal cones, whose intersection with the compact Cartan subalgebra $\mathfrak{h}$ is isometric to a positive orthant. However, as noted in [14], all the $su(n, 1)$ algebras (and $so^*(8)$) also possess self-dual causal cones, whose intersection with $\mathfrak{h}$ is a "light-cone", i.e., the cone of rays making angles $\leqslant 45°$ with the ray $\mathbb{R}^+Z$ spanning the center of the maximal compact. (We note here also that the $su(n, 1)$ algebras are unique in that (notation cf. Section 5) always $c_{\min} - \{0\}$ is contained in the interior (relative to $\mathfrak{h}$) of $c_{\max}$. Thus should not be taken to imply that the minimal causal cones in $\mathfrak{G}$ minus $\{0\}$ are then contained in the interiors of the maximal causal cones; in fact, for a general $\mathfrak{G}$ the minimal nilpotent orbits are always on the boundaries of all causal cones.)

When the rank of $\mathfrak{G}$ is three, one captures most of the essential information about the mutual relations of the causal cones, by taking a slice in $\mathfrak{h}$ through Z perpendicular to $\mathbb{R}Z$. We do this in the next section for $su(2, 2)$, and here for $so^*(6)$.

Recall the $\mathfrak{h}$ for $so^*(6)$ is isometric to $\mathbb{R}^3$, such that $c_{\min}$ is generated by

$$(1, 1, 0), \qquad (1, 0, 1), \qquad (0, 1, 1),$$

and $c_{\max}$ is generated by

$$(-1, 1, 1), \qquad (1, -1, 1), \qquad (1, 1, -1).$$

Also

$$v_1 = \frac{1}{\sqrt{3}}(1, 1, 1), \qquad v_2 = \frac{1}{\sqrt{2}}(1, -1, 0), \qquad v_3 = \frac{1}{\sqrt{6}}(1, 1, -2)$$

is a convenient orthonormal basis of $\mathfrak{h}$. We express

$$\begin{aligned}
(1, 0, 1) &= \frac{2}{\sqrt{3}} v_1 + \frac{1}{\sqrt{2}} v_2 - \frac{1}{\sqrt{6}} v_3,\\
(1, 1, 0) &= \frac{2}{\sqrt{3}} v_1 \qquad\quad + \frac{2}{\sqrt{6}} v_3,\\
(0, 1, 1) &= \frac{2}{\sqrt{3}} v_1 - \frac{1}{\sqrt{2}} v_2 - \frac{1}{\sqrt{6}} v_3,\\
(-1, 1, 1) &= \frac{1}{\sqrt{3}} v_1 - \frac{2}{\sqrt{2}} v_2 - \frac{2}{\sqrt{6}} v_3,\\
(1, -1, 1) &= \frac{1}{\sqrt{3}} v_1 + \frac{2}{\sqrt{2}} v_2 - \frac{2}{\sqrt{6}} v_3,\\
(1, 1, -1) &= \frac{1}{\sqrt{3}} v_1 \qquad\quad + \frac{4}{\sqrt{6}} v_3.
\end{aligned}$$

We take a slice through $(2/\sqrt{3})\, v_1$, and plot $\sqrt{6}$ times the coefficients of v_2, v_3 for the six points above in Fig. 2. The triangles ABC, IJK are the

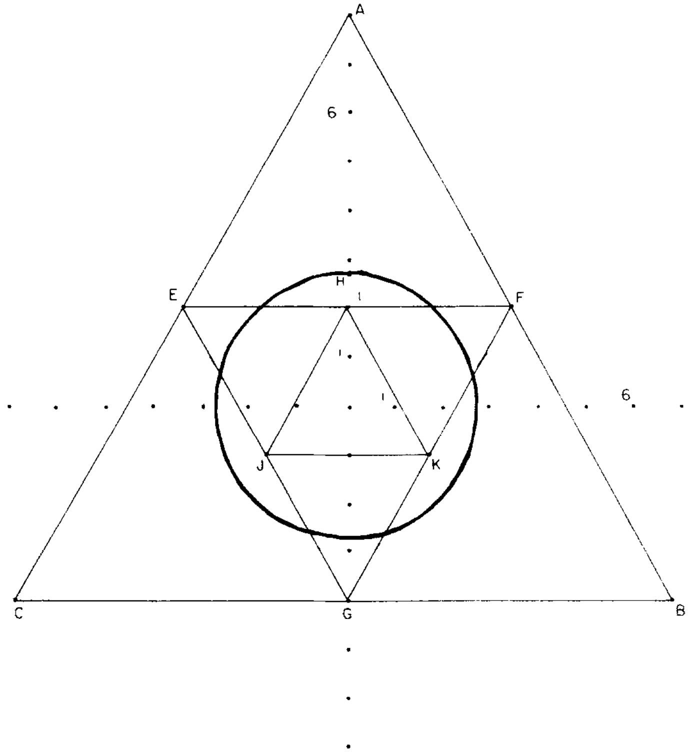

FIGURE 2

intersections of $c_{\max}$, $c_{\min}$ with the slice. The self-dual cones in $so^*(6)$ mentioned above similiarly give rise to EFG and the circle through $H = (0, 2\sqrt{2})$.

19. $su(2, 2)$: *A Characterization of Its Unique Self-Dual Causal Cone*

As in Section 8, we associate

$$\begin{pmatrix} -i\lambda_1 & & & 0 \\ & -i\lambda_2 & & \\ & & i\sigma_1 & \\ 0 & & & i\sigma_2 \end{pmatrix} \in su(2, 2)$$

with

$$(\lambda_1, \lambda_2, \sigma_1, \sigma_2) \in E = \{(\lambda_i, \sigma_j) \in \mathbb{R}^4 : \lambda_1 + \lambda_2 = \sigma_1 + \sigma_2\}.$$

Then

$$c_{\min} = \{(\lambda, \sigma) \in E: \lambda_i, \sigma_j \geqslant 0\},$$
$$c_{\max} = \{(\lambda, \sigma) \in E: \lambda_i + \sigma_j \geqslant 0, i, j = 1, 2\},$$

so $c_{\min}$ is generated by

$$(1, 0, 1, 0), \qquad (1, 0, 0, 1), \qquad (0, 1, 1, 0), \qquad (0, 1, 0, 1),$$

and $c_{\max}$ is generated by

$$\tfrac{1}{2}(3, -1, 1, 1), \qquad \tfrac{1}{2}(-1, 3, 1, 1), \qquad \tfrac{1}{2}(1, 1, 3, -1),$$
$$\tfrac{1}{2}(1, 1, -1, 3).$$

We consider the slice (as in the previous section) where $\lambda_1 + \lambda_2 = \sigma_1 + \sigma_2 = 1$, and there define the coordinates $x = \lambda_1 - \lambda_2$, $y = \sigma_1 - \sigma_2$. Then the above sets of generators for $c_{\min}$, $c_{\max}$ correspond in the x–y plane to

$$\{(1, 1), (1, -1), (-1, 1), (-1, -1)\},$$
$$\{(0, 2), (0, -2), (2, 0), (-2, 0)\},$$

respectively. We have

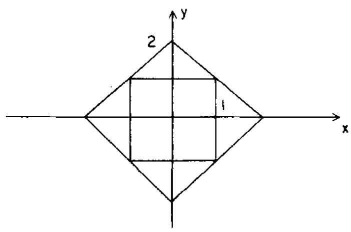

FIGURE 3

The inner square corresponds to $c_{\min}$, the outer to $c_{\max}$. The Weyl group action consists of reflections about the x and y axes. The outer automorphism $X \to \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} X \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$ of $su(2, 2)$ (termed "parity" because it is induced by a parity transformation on the Šhilov boundary on which the group acts, cf. Section 16) takes $\mathfrak{h}$ to itself, and acts in the x–y plane as a reflection about the line through $(1, -1)$ and $(-1, 1)$.

We remark that $\mathfrak{h}$, Z, $c_{\min}$, and $c_{\max}$ are (up to linear isometry) all the same for $so(2, 5)$, and that the Weyl group W_K for $so(2, 5)$ is generated by the above Weyl group for $su(2, 2)$ and the above reflection about the line through $(1, -1)$ and $(-1, 1)$.

Our final result, suggested by I. E. Segal, concerns (roughly speaking) a relation between so-called "finite propagation velocity" (or "real mass") representations of $su(2, 2)$, and this algebra's unique (up to sign) self-dual causal cone $C_{1/2}$, represented in Fig. 3 by the circle about 0 between the two squares.

For convenience we first label a basis of $\mathfrak{k}$. Let σ_1, σ_2, σ_3 be the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

also set $\sigma_0 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, and define $b_j = i\sigma_j$ for $j = 0, \ldots, 3$. Set

$$X_0 = \tfrac{1}{2} \begin{pmatrix} -b_0 & 0 \\ 0 & b_0 \end{pmatrix} \qquad \text{and} \qquad Y_j = \begin{pmatrix} b_j & 0 \\ 0 & 0 \end{pmatrix},$$

$$X_j = \begin{pmatrix} 0 & 0 \\ 0 & -b_j \end{pmatrix} \qquad \text{for } j = 1, 2, 3.$$

The irreducible representations of $\mathfrak{k}$ are labeled by triples (μ, j_1, j_2), where $\mu \in \mathbb{R}$, and $j_1, j_2 \geqslant 0$ are half-integral: in the representation, X_0 goes to $i\mu$, and the representation of the $su(2) \oplus su(2)$ spanned by the X_j, Y_j is a tensor product of two irreducible representations of $su(2)$ of spins j_1, j_2, i.e. so that

$$-\sum_{j=1}^{3} X_j^2, \quad -\sum_{j=1}^{3} Y_j^2 \qquad \text{go to} \qquad 4j_1(j_1+1), \quad 4j_2(j_2+1),$$

respectively. In this case each $\frac{1}{2}X_j$, $\frac{1}{2}Y_j$ $(j > 0)$ has eigenvalues $-ij_1$, $-i(j_1+1), \ldots, ij_1$.

Definitions. An irreducible representation (μ, j_1, j_2) of $\mathfrak{k}$ has *real mass* (more precisely, real K-invariant mass, and in particular, positive frequency) if $\mu \geqslant 0$ and

$$-X_0^2 + \tfrac{1}{2}\left(\sum_{j=1}^{3} (X_j^2 + Y_j^2)\right) = \mu^2 - 2(j_1^2 + j_1 + j_2^2 + j_2) \qquad (19.1)$$

is nonnegative.

Let ρ be a finite-dimensional linear representation of $\mathfrak{k}$ by skew-adjoint operators. We say ρ has real mass if each of its irreducible constituents has real mass. We define the *cone of positivity* of ρ to be

$$C_\rho = \{X \in \mathfrak{k} \colon -i\rho(X) \geqslant 0\}, \qquad (19.2)$$

which is clearly determined by its intersection with $\mathfrak{h}$.

Let $\tilde{C}_{1/2} = C_{1/2} \cap \mathfrak{k}$ and $c_{1/2} = C_{1/2} \cap \mathfrak{h}$.

PROPOSITION 19.1. *Let $\rho: \mathfrak{k} \to \mathscr{L}(\mathscr{H})$ be a linear finite-dimensional representation by skew-adjoint operators. If ρ has real mass, then the cone of positivity of ρ contains $\tilde{C}_{1/2}$.*

Conversely, if $X \notin \tilde{C}_{1/2}$, then there exists a real mass representation ρ of $\mathfrak{k}$ such that $X \notin C_\rho$.

Proof. It is easy to see that $X_0 + rX_3 + sY_3 \in c_{1/2}$ if and only if $r^2 + s^2 \leqslant 1/2$. It suffices to take $\rho = (\mu, j_1, j_2)$ irreducible, and conjugate the $X \in \tilde{C}_{1/2}$ in question to $\mathfrak{h}$. The maximal eigenvalue (absolute value) of $rX_3 + sY_3$ is then

$$\begin{aligned} 2(|r| j_1 + |s| j_2) &\leqslant 2(r^2 + s^2)^{1/2} (j_1^2 + j_2^2)^{1/2} \\ &\leqslant 2 \frac{1}{\sqrt{2}} \left(\frac{\mu^2}{2}\right)^{1/2} = \mu, \end{aligned}$$

so $\tilde{C}_{1/2} \subseteq C_\rho$.

For the converse, take $r, s \geqslant 0$ such that $X = X_0 + rX_3 + sY_3 \notin c_{1/2}$, so $r^2 + s^2 > \frac{1}{2}$. We need to find $j_1, j_2 \geqslant 0$ half-integral such that

$$2(j_1^2 + j_2^2 + j_1 + j_2) \leqslant \mu^2 < 4(rj_1 + sj_2)^2,$$

or simply

$$1 < \frac{1}{\sqrt{j_1^2 + j_2^2 + j_1 + j_2}} (uj_1 + vj_2)$$

when $u^2 + v^2 > 1$, $u, v \geqslant 0$. This is clear for particular j_1, j_2 sufficiently large.

Q.E.D.

COROLLARY 19.2. *Let ρ be a continuous unitary representation of the universal covering of $SU(2, 2)$ which has positive energy, i.e., $-id\rho(X_0)$ is nonnegative* [8]. *Then the cone of positivity*

$$C_\rho = \{X \in \mathfrak{G}: -id\rho(X) \geqslant 0\}$$

is a causal cone in $\mathfrak{G}$. If furthermore each K-type in $d\rho$ has real mass, then C_ρ contains the self-dual causal cone $C_{1/2}$.

Proof. It is clear that C_ρ is closed, by approximation by K-finite vectors. By Prop. 19.1, C_ρ contains $c_{1/2} \subset \mathfrak{h}$, so by the general theory (Section 5) it must also contain $C_{1/2}$. Q.E.D.

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