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Controllability properties of a class of systems modeling swimming microscopic organisms

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Abstract

We consider a finite-dimensional model for the motion of microscopic organisms whose propulsion exploits the action of a layer of *cilia* covering its surface. The model couples Newton's laws driving the organism, cosidered as a rigid body, with Stokes equations governing the surrounding fluid. The action of the *cilia* is described by a set of controlled velocity fields on the surface of the organism. The first contribution of the paper is the proof that such a system is generically controllable when the space of controlled velocity fields is at least three-dimensional. We also provide a complete characterization of controllable systems in the case in which the organism has a spherical shape. Finally, we offer a complete picture of controllable and non-controllable systems under the additional hypothesis that the organism and the fluid have densities of the same order of magnitude.

1 Introduction

The mathematical description of the motion of swimming organisms and of the mechanisms of their propulsion is a challenging and wide field of research. This paper aims at giving a contribution to such a domain in a comparatively narrow direction.

The object of our study are microscopic organisms. The most prominent aspect of the models describing their motion is the very high level of viscosity ([3, 15, 19]). The mechanism of their propulsion depends heavily on the different species and admits a great variety of models (see, for instance, [2, 7, 9, 14] and references therein).

Among microscopic organisms we will further restrict our attention to the class of *ciliata*, whose propulsion is determined by *cilia*, hair-like organelles covering their surface and whose size is very small compared with that of the organism (see [4, 5, 9]). Recently San Martín, Takahashi and Tucsnak [16] proposed a control theoretic approach to the description of ciliata. Following the former literature (and in particular [9, 13]) they assume that the organism is a rigid body and that the motion of the cilia is described by a set of controlled velocity fields on a surface enclosing the layer of cilia. They are therefore faced to an infinite-dimensional control system coupling Navier–Stokes equations, which describe the motion of the fluid, and the equations for the rigid body arising form Newton's laws. The hypothesis of high viscosity (ie, of low Raynolds number) mentioned above is exploited to reduce such a control system to a finite-dimensional version of it. The state-space is given by the coordinates of the center of mass of the organism, its orientation and its angular and linear velocities.

Summarizing, San Martín, Takahashi and Tucsnak single out a 12-dimensional nonlinear control system which is affine in the controls. The control system depends on a number of parameters which are explicitly derived as functions of the shape of the organism, of its mass distribution, and of the surface velocity fields describing the motion of the cilia. The authors study the control properties of the system proving that it is generically controllable when the number of controlled vector fields

is at least six (see Section 5 for details). They also provide some example of controllable systems in the case in which the organism has a spherical shape. In this special case they also study the controllability properties of the linearization of the system at its rest position.

Our scope is to enhance the results of [16] basically in three directions: firstly we want to give sharper results on the generic controllability of the system; secondly we want to obtain a complete characterization of controllable systems in the case in which the organism is spherical; thirdly we want to provide a complete picture of controllable and non-controllable systems in the simplified situation which –following again [16]– comes into effect in the case in which the organism and the fluid have densities of the same order of magnitude. The overall aim is to provide a better understanding of the dependence of the controllability properties of the model proposed in [16] on the physical quantities characterizing it. The hard aspect of the model is indeed the difficulty to obtain reliable expressions for the surface velocity fields describing the motions of the cilia. In this regard, it is important to provide as precise as possible parametric analysis of the controllability properties of the system.

The paper is organized as follows. Section 2 introduces the control theoretical language adopted throughout the paper. Section 3 presents the control system obtained in [16] and recalls the relations between the physical quantities characterizing the organism and the parameters appearing in the model. Section 4 is technical and contains an abstract controllability result used in the later sections. Section 5 is devoted to the analysis of the genericity of the controllability of the system introduced in Section 3. Genericity is initially formulated in terms of the parameters appearing in the finite-dimensional system and then expressed in terms of the physical objects they depend on (Section 5.3). The main result is that the system is generically controllable if the number of controlled vector fields is at least three and that one controlled vector field suffices for its generic accessibility (Theorem 5.1). Section 6 analyzes the case in which the organism is spherical and provides a complete characterization of the sets of parameters that make the system controllable (Theorem 6.1). Finally, Section 7 establishes a characterization of controllable systems when the organism can be assumed to have the same density as the fluid (Theorem 7.2). The result follows from a further reduction of the system that trasforms it into a six-dimensional one. The reduction procedure, suggested in [16], is carried out in full details.

2   Notations and definitions

Let us first introduce some notations: in the following Id denotes the 3×3 identity matrix, $0_{m \times n}$ is the zero $m \times n$ matrix, $0_m = 0_{m \times 1}$ is the zero vector in $\mathbf{R}^m$. We denote by $\mathscr{M}_{m \times n}$ the set of $m \times n$ real matrices. The interior of a set W is denoted by $\mathrm{Int}(W)$ and its closure by $\mathrm{Clos}(W)$. For simplicity of notation, when no confusion is possible, row and column vectors are identified. The canonical orthonormal basis of $\mathbf{R}^3$ is denoted by $\{e_1, e_2, e_3\}$.

In what follows we deal with control systems of the type

$$\dot{q} = f(q, u), \quad q \in M, \ u \in \mathbf{R}^m, \tag{1}$$

where M is a smooth ($\mathcal{C}^\infty$) manifold and $f : M \times \mathbf{R}^m \to TM$ is smooth with respect to both variables. Moreover, for every $T \geq 0$ the *endpoint mapping* E_T which associates to an initial condition $q_0 \in M$ and a control function $u \in L^\infty([0, T], \mathbf{R}^m)$ the final point of the corresponding trajectory of (1) is well defined and continuous on $M \times L^\infty([0, T], \mathbf{R}^m)$, where the L^1 norm is chosen on $L^\infty([0, T], \mathbf{R}^m)$. (For these and finer properties of the endpoint mapping see, for instance, [1].)

Let $\mathscr{F} = \{f(\cdot, u) \mid u \in \mathbf{R}^m\}$ be the family of vector fields characterizing (1). For every $X = f(\cdot, u) \in \mathscr{F}$ we denote by e^{tX} the one-parameter group generated by X, that is, $e^{tX}(q) = E_t(q, u1_{[0,t]})$. The *attainable set from* $q \in M$ is the set

$$\mathcal{A}(q, \mathscr{F}) = \cup_{T \geq 0} E_T(q, L^\infty([0, T], \mathbf{R}^m)).$$

When no confusion is possible we write $\mathcal{A}_q = \mathcal{A}(q, \mathscr{F})$.

We say that (1) is *controllable* (respectively, *approximatively controllable*) if for every $q \in M$ the set $\mathcal{A}_q$ is equal to (respectively, dense in) M.

Give two vector fields X and Y on M, the *Lie bracket* between X and Y is defined, in a local system of coordinates, by the relation

$$[X, Y] = (DY) X - (DX) Y$$

where DX denotes the derivative of X. Denote by $\mathscr{L}$ the Lie algebra generated by $\mathscr{F}$. System (1) is *Lie-bracket generating* if for every $q \in M$ the set $\mathscr{L}(q) = \{V(q) \mid V \in \mathscr{L}\}$ is equal to $T_q M$.

3 The dynamics of swimming microscopic organisms

In [16] the authors describe the swimming of a microscopic organism immersed in an infinite volume of fluid by coupling the dynamics of a rigid body (representing the organism) with the Navier–Stokes equations describing the behavior of the fluid outside the body. Exploiting the hypothesis of very low Reynolds number, they derive a simplified version of the coupled system. The system obtained in this way is finite-dimensional and nonlinear; more precisely, it has the following expression:

$$\dot{z} = Az + E(z) + Bu, \tag{2}$$
$$\dot{\zeta} = R\xi, \tag{3}$$
$$\dot{R} = RS(\omega), \tag{4}$$

where $z = (\xi, \omega) \in \mathbf{R}^3 \times \mathbf{R}^3$, $\zeta \in \mathbf{R}^3$, $R \in SO(3)$, and

$$E(z) = \begin{pmatrix} \xi \times \omega \\ J^{-1}((J\omega) \times \omega) \end{pmatrix}.$$

Here $J \in \mathscr{M}_{3 \times 3}$ denotes the inertia matrix of the rigid body representing the organism. Recall that J is symmetric and positive definite. The matrix $A \in \mathscr{M}_{6 \times 6}$ is a function of the rigid body and the viscosity of the fluid and its expression is given at the end of the section.

The control function u takes values in $\mathbf{R}^m$, $m \geq 1$, and B is a 6-by-m matrix, which we write as

$$B = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}$$

where both B_1 and B_2 belong to $\mathscr{M}_{3 \times m}$.

An important feature of the system above is that (2) is a well-defined control system on $\mathbf{R}^6$, since it does not depend on ζ nor R.

Denote by X_0 the vector field on $\mathbf{R}^6 \times \mathbf{R}^3 \times \mathrm{SO}(3)$ which is the drift of the control system (2)–(4), that is,

$$X_0(z, \zeta, R) = \begin{pmatrix} Az + E(z) \\ R\xi \\ RS(\omega) \end{pmatrix}.$$

Let moreover $X_1, \ldots, X_m$ be the controlled vector fields, so that system (2)–(4) can be written as

$$\dot{q} = X_0(q) + \sum_{i=1}^{m} u_i X_i(q), \qquad q \in \mathbf{R}^9 \times SO(3).$$

Define $\mathscr{L}$ as the Lie algebra generated by $X_0, X_1, \ldots, X_m$.

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We say that a vector field X on $\mathbf{R}^9 \times SO(3)$ is a *constant vector field* if there exist $v_1, v_2 \in \mathbf{R}^3$ such that $X : (\xi, \omega, \zeta, R) \mapsto (v_1, v_2, 0_6)$ and we write $X = (v_1, v_2, 0_6)$. Clearly, $X_1, \ldots, X_m$ are constant vector fields. Notice that if $V = (v_1, v_2, 0_6)$ and $W = (w_1, w_2, 0_6)$ are constant vector fields, then

$$[[X_0, V], W] = \begin{pmatrix} v_2 \times w_1 + w_2 \times v_1 \\ J^{-1}\left(Jw_2 \times v_2 + Jv_2 \times w_2\right) \\ 0_6 \end{pmatrix} \tag{5}$$

is a constant vector field as well.

Remark 3.1. *For every constant vector field $V = (v_1, v_2, 0_6)$, the bracket $[X_0, V]$ is of the form*

$$[X_0, V](z, \zeta, R) = (G(z, V), -Rv_1, -RS(v_2))$$

for some smooth map G with values in $\mathbf{R}^6$. Therefore, if the constant vector fields contained in $\mathscr{L}$ span $\mathbf{R}^6 \times \{0_6\}$, then system (2)–(4) is Lie-bracket generating.

We conclude the section by recalling the analytic characterization of the coefficients appearing in A, B and J. Let $S \subset \mathbf{R}^3$ be an open bounded connected set whose boundary is of class $\mathcal{C}^2$, representing the (rigid) shape of the organism. Then the coefficients J_{ij} of J are given by

$$J_{ij} = \int_S \rho(e_i \times \bar{\xi}) \cdot (e_j \times \bar{\xi}) dx, \tag{6}$$

where $\bar{\xi}$ denotes the mass center of S and $\rho : S \to (0, \infty)$ is the (possibly non-constant) density function.

Denote by F the set $\mathbf{R}^3 \setminus \text{Clos}(S)$ and, for $l \in \mathbf{N}$, $1 \leq q \leq \infty$, let

$$D^{l,q}(F) = \{\varphi \in L^1_{\text{loc}} \mid \partial^\alpha \varphi \in L^q(F) \text{ for all } \alpha \in \mathbf{N}^3, |\alpha| = l\}.$$

According to [16, Lemma 3.1], for every $i \in \{1, 2, 3\}$ there exists an unique solution $(h^{(i)}, p^{(i)})$ of

$$\begin{cases} -\Delta h^{(i)} + \nabla p^{(i)} &=& 0 & \text{on } F \\ \text{div} h^{(i)} &=& 0 & \text{on } F \\ h^{(i)} &=& e_i & \text{on } \partial S \\ \lim_{|y| \to \infty} h^{(i)}(y) &=& 0 \end{cases}$$

and an unique solution $(H^{(i)}, P^{(i)})$ of

$$\begin{cases} -\Delta H^{(i)} + \nabla P^{(i)} &=& 0 & \text{on } F \\ \text{div} H^{(i)} &=& 0 & \text{on } F \\ H^{(i)}(y) &=& e_i \times y & \text{on } \partial S \\ \lim_{|y| \to \infty} H^{(i)}(y) &=& 0 \end{cases}$$

such that $h^{(i)}, H^{(i)} \in L^s(F) \cap D^{1,r}(F) \cap D^{2,\theta}(F) \cap \mathcal{C}^\infty(F)$ and $p^{(i)}, P^{(i)} \in L^r(F) \cap D^{1,\theta}(F) \cap \mathcal{C}^\infty(F)$ for $s \in (3, \infty]$, $r \in (3/2, \infty]$ and $\theta \in (1, \infty)$.

The *Cauchy stress* is the tensor σ defined by the relation

$$\sigma(v, p) = -p\text{Id} + \mu\left(\frac{\partial v_k}{\partial y_l} + \frac{\partial v_l}{\partial y_k}\right)_{1 \leq l, k \leq 3} \tag{7}$$

where μ is the Reynolds number of the fluid. For $i = 1, 2, 3$, define

$$g^{(i)} = \sigma(h^{(i)}, p^{(i)})n|_{\partial S}, \quad G^{(i)} = \sigma(H^{(i)}, P^{(i)})n|_{\partial S},$$

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where n is the unit inner normal to ∂S.

Denote by $\Theta^1, \Theta^2, \Upsilon^1, \Upsilon^2$ the matrices in $\mathscr{M}_{3\times 3}$ defined by

$$\Theta^1_{ij} = -\int_{\partial S} g^{(i)}_j\, ds, \quad \Theta^2_{ij} = -\int_{\partial S} \left(x \times g^{(i)}\right)_j ds,$$

$$\Upsilon^1_{ij} = -\int_{\partial S} G^{(i)}_j\, ds, \quad \Upsilon^2_{ij} = -\int_{\partial S} \left(x \times G^{(i)}\right)_j ds,$$

where ds is the surface element of ∂S. It turns out (see [10]) that $\Upsilon^1 = \left(\Theta^2\right)^{\mathrm{T}}$ and that

$$A = \begin{pmatrix} \bar{m}^{-1}\Theta^1 & \bar{m}^{-1}\Theta^2 \\ J^{-1}\Upsilon^1 & J^{-1}\Upsilon^2 \end{pmatrix}, \tag{8}$$

where $\bar{m}$ denotes the mass of the organism. Moreover, with respect to the inner product in $\mathbf{R}^6$ defined by

$$\langle a, b \rangle_J = \bar{m} \sum_{l=1}^{3} a_l b_l + \sum_{l,k=1}^{3} J_{lk} a_{3+l} b_{3+k}, \tag{9}$$

A is self-adjoint and negative-definite.

The matrix B is defined by

$$B = \begin{pmatrix} \bar{m}^{-1}\kappa \\ J^{-1}\mathcal{K} \end{pmatrix} \tag{10}$$

where the entries of κ and $\mathcal{K}$ are

$$\kappa_{ij} = -\int_{\partial S} g^{(i)} \cdot \psi_j\, ds, \quad \mathcal{K}_{ij} = -\int_{\partial S} G^{(i)} \cdot \psi_j\, ds,$$

and $\psi_1, \ldots, \psi_m$ are fixed functions in $\mathcal{C}^2(\partial S, \mathbf{R}^3)$, each one associated to a component of the control u, that describe the admissible propulsive actions of the organism.

4 A controllability result in a more general setting

The following proposition provides sufficient (and necessary) conditions for the controllability of systems that generalize those introduced in the previous section, in the sense that a more general structure for the dynamics in the coordinates z is allowed.

Proposition 4.1. *Consider the following control system*

$$\dot{z} = f(z, u), \quad z \in \mathbf{R}^6, \quad u \in \mathbf{R}^m. \tag{11}$$

Then (3),(4),(11) is controllable if and only if it is Lie-bracket generating at $(z, \zeta, R) = (0_9, \mathrm{Id})$ and system (11) is controllable.

Proof. One direction of the equivalence being obvious, let us assume that system (11) is controllable and that (3),(4),(11) is Lie-bracket generating at $(z, \zeta, R) = (0_9, \mathrm{Id})$.

Notice that

$$(z', \zeta', R') \in \mathcal{A}_{(z,\zeta,R)} \quad \Longrightarrow \quad (z', \bar{\zeta} + \bar{R}\zeta', \bar{R}R') \in \mathcal{A}_{(z, \bar{\zeta}+\bar{R}\zeta, \bar{R}R)} \tag{12}$$

for every $(\bar{\zeta}, \bar{R}) \in \mathbf{R}^3 \times SO(3)$, where the letter $\mathcal{A}$ is used here to denote attainable sets for system (3),(4),(11). Indeed, if $t \mapsto u(t)$ is an admissible control steering (z, ζ, R) to (z', ζ', R') and $t \mapsto$

$(z(t), \zeta(t), R(t))$ is the corresponding trajectory, then $t \mapsto (z(t), \bar{\zeta} + \bar{R}\zeta(t), \bar{R}R(t))$ is an admissible trajectory corresponding to the same control u.

Fix $(z_0, \zeta_0, R_0), (z_1, \zeta_1, R_1) \in \mathbf{R}^9 \times SO(3)$. Since (11) is controllable, then there exist $\eta_0, \eta_1 \in \mathbf{R}^3$ and $Q_0, Q_1 \in SO(3)$ such that $(0_6, \eta_0, Q_0) \in \mathcal{A}_{(z_0, \zeta_0, R_0)}$ and $(z_1, \zeta_1, R_1) \in \mathcal{A}_{(0_6, \eta_1, Q_1)}$. We are left to prove that $\mathcal{A}_{(0_6, \eta_0, Q_0)}$ contains $\{0_6\} \times \mathbf{R}^3 \times SO(3)$.

Since (11) is controllable and (3),(4),(11) is Lie-bracket generating at $(z, \zeta, R) = (0_9, \mathrm{Id})$, then there exists an open nonempty subset V of $\mathbf{R}^3 \times SO(3)$ such that $\{0_6\} \times V$ is contained in $\mathcal{A}_{(0_9, \mathrm{Id})}$.

Let
$$G = \mathrm{Int}(\mathcal{A}_{(0_9, \mathrm{Id})})$$
and notice the following consequence of (12): if $(0_6, \bar{\zeta}, \bar{R}) \in \mathcal{A}_{(0_6, \eta_0, Q_0)}$ and $(0_6, \zeta', R') \in G$, then

$$(0_6, \bar{\zeta} + \bar{R}\zeta', \bar{R}R') \in \mathcal{A}_{(0_6, \eta_0, Q_0)}. \tag{13}$$

Notice that for every nonempty open subset O of $SO(3)$

$$W = \{P_1 P_2 \cdots P_k \mid k \in \mathbf{N}, P_1, \ldots, P_k \in O\} \tag{14}$$

is equal to $SO(3)$. Indeed, since O contains at least one element P_0 of finite order, ie, an axial rotation of angle commensurable with π, then Id belongs to the interior of W. The completeness of $SO(3)$ guarantees that $W = SO(3)$.

Take as O the projection of V on $SO(3)$. Then for every $R \in SO(3)$ there exists $\zeta \in \mathbf{R}^3$ such that $(0_6, \zeta, R)$ belongs to G; this can be seen by noticing that there exist k elements $(0_6, \zeta_1, P_1), \ldots, (0_6, \zeta_k, P_k)$ in V such that $P_i \in O$ and $P_1 P_2 \cdots P_k = R$ and by applying repeatedly (13). In particular, without loss of generality, $Q_0 = \mathrm{Id}$ and $Q_1 = \mathrm{Id}$. Indeed, since there exists $\zeta_0 \in \mathbf{R}^3$ such that $(0_6, \zeta_0, Q_0^{-1}) \in G$, equation (12) with $(\bar{\zeta}, \bar{R}) = (\eta_0, Q_0)$ shows that $(0_6, \eta_0 + Q_0\zeta_0, \mathrm{Id}) \in \mathcal{A}_{(0_6, \eta_0, Q_0)}$; similarly, the existence of $\zeta_1 \in \mathbf{R}^3$ such that $(0_6, \zeta_1, Q_1) \in G$ implies that $(0_6, \eta_1, Q_1) \in \mathcal{A}_{(0_6, \eta_1 - \zeta_1, \mathrm{Id})}$. Moreover, for every element p of $\mathbf{RP}^2$, the Grassmannian of one-dimensional subspaces of $\mathbf{R}^3$, there exists $\zeta_p \in \mathbf{R}^3$ such that $(0_6, \zeta_p, R_p)$ lies in G, where R_p denotes the rotation of angle π around the axis p. Applying (13) to $(\bar{\zeta}, \bar{R}) = (\zeta', R') = (\zeta_p, R_p)$, we obtain that $(0_6, \zeta_p + R_p\zeta_p, \mathrm{Id})$ lies in G. Notice that $\zeta_p + R_p\zeta_p = 2\langle \zeta_p, p \rangle p$ belongs to the axis p. Therefore, G intersects $\{0_6\} \times p \times \{\mathrm{Id}\}$ for every $p \in \mathbf{RP}^2$.

Of special interest will be for us the expression of (13) when $\bar{R} = R' = \mathrm{Id}$, namely

$$(0_6, z', \mathrm{Id}) \in G \implies (0_6, \bar{\zeta} + \zeta', \mathrm{Id}) \in \mathcal{A}_{(0_6, \bar{\zeta}, \mathrm{Id})}. \tag{15}$$

Let
$$\Omega_\pm = \{v \in \mathcal{S}^2 \mid \{0_6\} \times \mathbf{R}_{\geq 0}(\pm v) \times \{\mathrm{Id}\} \cap \mathrm{Int}\mathcal{A}_{(0_9, \mathrm{Id})} \neq \emptyset\}$$
where $\mathcal{S}^2$ denotes the unit sphere in $\mathbf{R}^3$. As we remarked above, Ω_+ and Ω_- cover $\mathcal{S}^2$ and, by construction, they are open. Moreover, both are nonempty, since $v \in \Omega_\pm$ implies that $-v \in \Omega_\mp$. Since $\mathcal{S}^2$ is connected we have $\Omega_+ \cap \Omega_- \neq \emptyset$, ie, there exist $v \in \mathbf{R}^3$ and two non-negative constants λ_1, λ_2 such that both $(0_6, \lambda_1 v, \mathrm{Id})$ and $(0_6, -\lambda_2 v, \mathrm{Id})$ lie in G; moreover, due to the openness of G, these constants can be assumed to be positive and commensurable and so there exist two positive integers n_1 and n_2 such that $n_1\lambda_1 - n_2\lambda_2 = 0$. Applying repeatedly (15), we obtain that G contains $(0_9, \mathrm{Id})$ and, therefore, $\{0_6\} \times \mathbf{R}^3 \times \{\mathrm{Id}\}$.

Applying (15) again with $\bar{\zeta} = \eta_0$ we deduce that $(0_6, \eta_1, \mathrm{Id}) = (0_6, \eta_1, Q_1)$ is attainable from $(0_6, \eta_0, \mathrm{Id}) = (0_6, \eta_0, Q_0)$. $\qquad \square$

5 Generic properties

The expression *generic* is commonly used to denote a property of a system that is, in a suitable sense, stable under small perturbations and that, even when it fails to apply, can be forced to hold by applying to the system an arbitrary small perturbation.

In order to define precisely what a generic property means in the present context, define

$$\Xi_0^m = \{(A, B, J) \in \mathscr{M}_{6\times 6} \times \mathscr{M}_{6\times m} \times \mathscr{M}_{3\times 3} \mid J = J^{\mathrm{T}} > 0, \ A \in \mathrm{Sym}_J^-\},$$

where

$$\mathrm{Sym}_J^- = \{A \in \mathscr{M}_{6\times 6} \mid A \text{ is symmetric and negative definite with respect to } \langle \cdot, \cdot \rangle_J\}$$

and $\langle \cdot, \cdot \rangle_J$ is defined as in (9).

We say that a property is *generic for system (2)–(4) when $m = m_0$* if there exists an open and dense set O in $\Xi_0^{m_0}$ such that the property holds for every system defined by a triple in O.

Such a definition of genericity, given in terms of the entries of the matrices A, J and B, is adapted to the finite-dimensional formulation (2)–(4) but can appear unsatisfactory from the point of view of the physical model. Section 5.3 discusses how to define the genericity directly in the space of configurations of the organism and provides a physically justified counterpart of the following result.

Theorem 5.1. *(a) Generically when $m \geq 1$ the rank of $\mathscr{L}$ is maximal at every point. (b) Generically when $m \geq 3$ system (2)–(4) is controllable.*

The generic conditions ensuring the Lie-bracket generating condition and the controllability of (2)–(4) are given explicitly in the Propositions 5.2 and 5.3. Notice that the techniques applied in this section are independent of the structure imposed on A by the physical motivations of the model, namely, the fact that $A \in \mathrm{Sym}_J^-$. Indeed, the conditions thatare obtained in the following sections define an open and dense set in

$$\Xi_1^m = \{(A, B, J) \in \mathscr{M}_{6\times 6} \times \mathscr{M}_{6\times m} \times \mathscr{M}_{3\times 3} \mid J = J^{\mathrm{T}} > 0\}$$

for $m \geq 1$ (for the Lie-bracket generating condition) or $m \geq 3$ (for the controllability) whose intersection with Ξ_0^m is open and dense in Ξ_0^m. In other words, Lie-bracket generating condition and controllability are generic properties (for $m = 1$ and $m = 3$, respectively) also in the class of systems of the form (2)–(4) for which the triple (A, B, J) is taken in Ξ_1^m.

5.1 Generic Lie-bracket generating condition for $m = 1$

When $m = 1$ the two matrices B_1 and B_2 are column vectors that we denote by b_1 and b_2 respectively. Then $X_1 = (b_1, b_2, 0_6)$ and $\mathscr{L}$ is the Lie algebra generated by X_0 and X_1.

Proposition 5.2. *Let $i = 0$ or $i = 1$. There exists an open and dense set of triples (A, B, J) in Ξ_i^1 such that system (2)–(4) is Lie-bracket generating. More precisely, if J, b_1, and b_2 satisfy the three following conditions*

$$\det(b_1, Jb_2, J^2 b_2) \ \neq \ 0, \tag{16}$$

$$\det(b_2, Jb_2, J^2 b_2) \ \neq \ 0, \tag{17}$$

$$\det\left(J - \frac{\|Jb_2\|^2}{\langle Jb_2, b_2 \rangle}\mathrm{Id}\right) \ \neq \ 0, \tag{18}$$

then $\mathscr{L}$ is maximal at every point.

Proof. Let us prove, first of all, that if $C \in \mathcal{M}(3 \times 3)$ is symmetric and invertible, and if $b \in \mathbf{R}^3$ is such that Cb and C^2b are linearly independent, then

$$C^{-1}(Cb \times b) = \det(C^{-1})(C^2b \times Cb). \tag{19}$$

Notice that $\langle C^{-1}(Cb \times b), Cb \rangle = 0 = \langle C^{-1}(Cb \times b), C^2b \rangle$, and therefore

$$C^{-1}(Cb \times b) = k(C^2b \times Cb), \tag{20}$$

for some $k \in \mathbf{R}$. For every $x \in \mathbf{R}^3$ we have $\det(C^{-1}x, Cb, b) = (\det C^{-1})\det(x, C^2b, Cb)$ and also

$$\det(C^{-1}x, Cb, b) = \langle C^{-1}x, Cb \times b \rangle = \langle x, C^{-1}(Cb \times b) \rangle = k \langle x, C^2b \times Cb \rangle = k \det(x, C^2b, Cb).$$

Therefore

$$(\det C^{-1})\det(x, C^2b, Cb) = k \det(x, C^2b, Cb)$$

for every $x \in \mathbf{R}^3$, proving that (21) holds true. In particular, taking $J = C$ and $b = b_2$, we have

$$J^{-1}(Jb_2 \times b_2) = \det(J^{-1})(J^2b_2 \times Jb_2). \tag{21}$$

Denote by V_1 the Lie bracket $[[X_0, X_1], X_1]$. According to (5) we have

$$V_1 = 2\left((b_2 \times b_1), J^{-1}(Jb_2 \times b_2), 0_6\right).$$

Similarly, the definition $V_2 = \dfrac{1}{4}[[X_0, V_1], V_1]$ leads to the expression

$$V_2 = \left(-\langle J^{-1}(Jb_2 \times b_2), b_2 \rangle\, b_1 + \langle J^{-1}(Jb_2 \times b_2), b_1 \rangle\, b_2, -\langle J^{-1}(Jb_2 \times b_2), b_2 \rangle\, b_2, 0_6\right).$$

Since $\langle J^{-1}(Jb_2 \times b_2), b_2 \rangle\, X_1 + V_2 = \left(\langle J^{-1}(Jb_2 \times b_2), b_1 \rangle\, b_2, 0_9\right)$ we have that

$$Z_1 = (b_2, 0_9)$$

belongs to $\mathcal{L}$, as it follows from (16) and from (21).

Let $Z_2 = \dfrac{1}{2}[[X_0, Z_1], V_1]$. Applying again (5), we have that

$$Z_2 = (-b_2 \times J^{-1}(Jb_2 \times b_2), 0_9).$$

Equation (21) implies that

$$Z_2 = (\det J^{-1})\, b_2 \times (Jb_2 \times J^2b_2).$$

Letting

$$b_3 = b_2 \times (Jb_2 \times J^2b_2)$$

we can represent $Z_3 = \dfrac{1}{2}[[X_0, Z_2], X_1]$ as

$$Z_3 = (b_2 \times b_3, 0_9).$$

We have shown that the Lie algebra $\mathcal{L}$ contains the vector fields

$$Z_1 = (b_2, 0_9), \quad Z_2 = (b_3, 0_9), \quad Z_3 = (b_2 \times b_3, 0_9).$$

Notice that

$$b_3 = \langle b_2, J^2b_2 \rangle\, Jb_2 - \langle b_2, Jb_2 \rangle\, J^2b_2$$

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is not in span(b_2), since $\langle b_2, Jb_2 \rangle > 0$ and because of (17). As a consequence, the vectors b_2, b_3 and $b_2 \times b_3$ are linearly independent.

Therefore, $\mathscr{L}$ contains every constant vector field of the type $(v, 0_9)$, with v in $\mathbf{R}^3$. It follows that

$$W_1 = (0_3, b_2, 0_6)$$

is in $\mathscr{L}$. As a consequence of (5), $[[X_0, W_1], W_1] = 2(0_3, J^{-1}(Jb_2 \times b_2), 0_6)$ lies in $\mathscr{L}$ as well. Due to (21), we deduce that

$$W_2 = (0_3, J^2 b_2 \times Jb_2, 0_6)$$

belongs to $\mathscr{L}$. Let

$$b_4 = J^2 b_2 \times Jb_2$$

and let $W_3 = [[X_0, W_1], W_2]$; we have $W_3 = (0_3, b_5, 0_6)$ with $b_5 = J^{-1}(Jb_4 \times b_2 + Jb_2 \times b_4)$. We are left to prove that the vectors b_2, b_4 and b_5 are linearly independent. Let $D = \det(b_2, b_4, b_5)$ and notice that

$$
\begin{aligned}
D &= \det(b_2, b_4, J^{-1}(Jb_4 \times b_2 + Jb_2 \times b_4)) \\
&= (\det J^{-1}) \det(Jb_2, Jb_4, Jb_4 \times b_2 + Jb_2 \times b_4) \\
&= (\det J^{-1}) \langle Jb_2, Jb_4 \times (Jb_4 \times b_2) + Jb_4 \times (Jb_2 \times b_4) \rangle \\
&= (\det J^{-1}) \langle Jb_2, \langle Jb_4, b_4 \rangle Jb_2 - \langle Jb_4, Jb_2 \rangle b_4 + \langle Jb_4, b_2 \rangle Jb_4 - \|Jb_4\|^2 b_2 \rangle .
\end{aligned}
$$

Since $\langle Jb_4, Jb_2 \rangle = \langle b_4, J^2 b_2 \rangle = 0$ and $\langle Jb_4, b_2 \rangle = \langle b_4, Jb_2 \rangle = 0$ we have

$$
\begin{aligned}
D &= (\det J^{-1}) \langle Jb_2, \langle Jb_4, b_4 \rangle Jb_2 - \|Jb_4\|^2 b_2 \rangle \\
&= (\det J^{-1})(\|Jb_2\|^2 \langle Jb_4, b_4 \rangle - \|Jb_4\|^2 \langle Jb_2, b_2 \rangle).
\end{aligned}
$$

Now, taking $C = J^{-1}$ and $b = J^2 b_2$ in (19), we have $Jb_4 = (\det J)(Jb_2 \times b_2)$, from which we deduce

$$\langle Jb_4, b_4 \rangle = (\det J)(\langle Jb_2, b_2 \rangle \langle Jb_2, J^2 b_2 \rangle - \|Jb_2\|^4).$$

We also have

$$
\begin{aligned}
\|Jb_4\|^2 &= (\det J)^2 \|Jb_2 \times b_2\|^2 \\
&= (\det J)^2 (\|Jb_2\|^2 \|b_2\|^2 - \langle Jb_2, b_2 \rangle^2).
\end{aligned}
$$

Defining $\lambda = \|Jb_2\|^2 / \langle Jb_2, b_2 \rangle$ we get

$$
\begin{aligned}
D &= \|Jb_2\|^2 (\langle Jb_2, b_2 \rangle \langle J^2 b_2, Jb_2 \rangle - \|Jb_2\|^4) - (\det J) \langle Jb_2, b_2 \rangle (\|Jb_2\|^2 \|b_2\|^2 - \langle Jb_2, b_2 \rangle^2) \\
&= \langle Jb_2, b_2 \rangle^3 \left(\frac{\langle J^3 b_2, b_2 \rangle}{\langle Jb_2, b_2 \rangle} \lambda - \lambda^3 - (\det J) \left(\frac{\|b_2\|^2}{\langle Jb_2, b_2 \rangle} \lambda - 1 \right) \right).
\end{aligned}
$$

Cayley–Hamilton theorem implies that

$$J^3 = a_2 J^2 + a_1 J + (\det J)\mathrm{Id}, \qquad a_1, a_2 \in \mathbf{R},$$

and thus

$$
\begin{aligned}
\frac{\langle J^2 b_2, Jb_2 \rangle}{\langle Jb_2, b_2 \rangle} &= a_2 \frac{\langle J^2 b_2, b_2 \rangle}{\langle Jb_2, b_2 \rangle} + a_1 + (\det J) \frac{\|b_2\|^2}{\langle Jb_2, b_2 \rangle} \\
&= a_2 \lambda + a_1 + (\det J) \frac{\|b_2\|^2}{\langle Jb_2, b_2 \rangle}.
\end{aligned}
$$

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Therefore,

$$
\begin{aligned}
D &= \langle Jb_2, b_2\rangle^3 \left(\left(a_2\lambda + a_1 + (\det J)\frac{\|b_2\|^2}{\langle Jb_2, b_2\rangle} \right) \lambda - \lambda^3 - (\det J)\left(\frac{\|b_2\|^2}{\langle Jb_2, b_2\rangle}\lambda - 1 \right) \right) \\
&= \langle Jb_2, b_2\rangle^3 \left(-\lambda^3 + a_2\lambda^2 + a_1\lambda + \det J \right) \\
&= \langle Jb_2, b_2\rangle^3 \det(J - \lambda\mathrm{Id})
\end{aligned}
$$

which is different from zero because of (18). $\qquad\square$

5.2 Generic controllability in the case $m = 3$

The aim of this section is to prove the genericity of the controllability of (2) in the case $m = 3$. Let us introduce the notation

$$
A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}
$$

with each A_{ij} belonging to $\mathscr{M}_{3\times 3}$.

Proposition 5.3. *Let $i = 0$ or $i = 1$. There exists an open and dense set of triples (A, B, J) in Ξ_i^3 such that system (2) is controllable. More precisely, if B_2 and $\tilde{A}_{11} + \tilde{A}_{11}^{\mathrm{T}}$ are invertible with $\tilde{A}_{11} = A_{11} - B_1 B_2^{-1} A_{21}$ and if the sets of eigenvectors of $B_1 B_2^{-1}$ and J are not equal, then system (2) is controllable.*

Propositions 5.2 and 5.3 can be summed up in in the following statement, which follows from Proposition 4.1.

Corollary 5.4. *For $m \geq 3$ and $i = 0$ or $i = 1$, there exists an open and dense set of triples (A, B, J) in Ξ_i^3 such that system (2)–(4) is controllable.*

The proof of Proposition 5.3 will be split in several steps. The following lemma allows us to study the controllability of (2) by investigating the controllability of an equivalent system in $\mathbf{R}^3$ (instead of $\mathbf{R}^6$).

Lemma 5.5. *Assume that B_2 is invertible and define $\tilde{B} = B_1 B_2^{-1}$ and $\tilde{A}_{11} = A_{11} - \tilde{B}A_{21}$. Then, the controllability of (2) is equivalent to the controllability of the system*

$$
\dot{x} = \tilde{A}_{11}x + v \times x + (\tilde{A}_{11}\tilde{B} + A_{12} - \tilde{B}A_{22})v + v \times \tilde{B}v - \tilde{B}J^{-1}(Jv \times v), \quad x \in \mathbf{R}^3, \qquad (22)
$$

with the control v taking values in $\mathbf{R}^3$.

Proof. Suppose that system (2) is controllable. Let $x = \xi - \tilde{B}\omega$ and rewrite system (2) as

$$
\begin{cases}
\dot{x} = \tilde{A}_{11}x + \omega \times x + (\tilde{A}_{11}\tilde{B} + A_{12} - \tilde{B}A_{22})\omega + \omega \times \tilde{B}\omega - \tilde{B}J^{-1}(J\omega \times \omega), \\
\dot{\omega} = A_{12}x + (A_{12}\tilde{B} + A_{22})\omega + J^{-1}(J\omega \times \omega) + B_2\,u.
\end{cases}
$$

Given two points (x_0, ω_0) and (x_1, ω_1) in $\mathbf{R}^6$, there exists a control $u : [0, T] \to \mathbf{R}^3$ that steers (x_0, ω_0) to (x_1, ω_1). Thus, denoting by $(x(\cdot), \omega(\cdot))$ the corresponding trajectory, the control $v(t) = \omega(t)$ steers system (22) from x_0 to x_1.

Conversely, let system (22) be controllable and fix two pairs (ξ_0, ω_0) and (ξ_1, ω_1) in $\mathbf{R}^6$. Notice that (22) can be controlled by smooth controls $v : [0, T] \to \mathbf{R}^3$ satisfying

$$
v(0) = \omega_0, \qquad v(T) = \omega_1. \qquad (23)
$$

(The result follows from the controllability of (22) and the density of $\{v \in \mathcal{C}^\infty([0,T], \mathbf{R}^3) \mid v(0) = \omega_0, \ v(T) = \omega_1\}$ in $L^\infty(0,T)$ with respect to the L^1-norm. The proof can be easily adapted from [6] and [11, Theorem 4, p. 110].)

Therefore, if $v \in \mathcal{C}^\infty([0,T], \mathbf{R}^3)$ satisfies (23) and steers (22) from $\xi_0 - \tilde{B}\omega_0$ to $\xi_1 - \tilde{B}\omega_1$, then

$$u(t) = B_2^{-1} \left(\dot{v}(t) - A_{12}x(t) - (A_{12}\tilde{B} + A_{22})v(t) - J^{-1}(Jv(t) \times v(t)) \right)$$

steers (2) from (ξ_0, ω_0) to (ξ_1, ω_1). $\qquad \square$

The following lemma establishes the controllability of system (22) in a first case.

Lemma 5.6. *Assume that the matrix $\tilde{A}_{11} + \tilde{A}_{11}^{\mathrm{T}}$ has two nonzero eigenvalues with opposite signs. Then there exists an open and dense subset O of $\{(B,J) \in \mathcal{M}_{6\times 3} \times \mathcal{M}_{3\times 3} \mid J = J^{\mathrm{T}} > 0\}$ such that system (22) is controllable if $(B,J) \in O$. More precisely, if B_2 is invertible and if the sets of eigenvectors of $B_1 B_2^{-1}$ and J are not equal, then system (22) is controllable.*

For the proof of this lemma we shall use the following result, proven in [12], about the controllability of a family of affine vector fields in $\mathbf{R}^n$. An *affine vector field* X is a mapping from $\mathbf{R}^n$ to $\mathbf{R}^n$ of the form $X : x \mapsto Cx + c$ where $C \in \mathcal{M}_{n\times n}$ and c is a constant vector in $\mathbf{R}^n$; the *linear part* of X, denoted by $\Lambda(X)$, is the linear vector field $x \mapsto Cx$. If $\mathscr{F}$ is a family of affine vector fields, we denote by $\Lambda(\mathscr{F})$ the set of linear parts of the vector fields in $\mathscr{F}$ and we say that $\mathscr{F}$ has no fixed point if there does not exist a point $x_0 \in \mathbf{R}^n$ such that $X(x_0) = 0$ for every $X \in \mathscr{F}$.

Theorem 5.7 (Jurdjevic and Sallet). *Let $\mathscr{F}$ be a family of affine vector fields given on $\mathbf{R}^n$. Assume that $\mathscr{F}$ has no fixed point. If $\Lambda(\mathscr{F})$ is controllable on $\mathbf{R}^n \smallsetminus \{0\}$ then $\mathscr{F}$ is controllable on $\mathbf{R}^n$.*

Proof of Lemma 5.6. From Lemma 5.5, we know that it is sufficient to prove the controllability of the system defined by the family of affine vector fields $\mathscr{F} = \{ C(v)x + c(v) \mid v \in \mathbf{R}^3 \}$ where

$$C(v) = \tilde{A}_{11} + S(v), \qquad c(v) = (\tilde{A}_{11}\tilde{B} + A_{12} - \tilde{B}A_{22})v + v \times \tilde{B}v - \tilde{B}J^{-1}(Jv \times v).$$

Because of the unboundedness of the parameter v, in order to check the controllability of $\Lambda(\mathscr{F})$ on $\mathbf{R}^3 \smallsetminus \{0\}$ it is enough to prove it for the family of linear vector fields $\mathscr{G} = \{ \tilde{A}_{11} + S(v_0), S(v) \mid v \in \mathbf{R}^3 \}$, where v_0 is a fixed vector chosen in such a way that the matrix $\tilde{A}_{11} + S(v_0)$ is symmetric.

The trajectories of the vector field $S(v)$ are circles contained in planes orthogonal to v and whose centers are at the intersections of these planes with the line $\mathbf{R}v$. Let $x \in \mathbf{R}^3 \smallsetminus \{0\}$; thanks to the vector fields $S(v)$, the attainable set from x for $\mathscr{G}$ contains the sphere of center 0_3 passing through x. Now, as $\tilde{A}_{11} + S(v_0)$ has two eigenvalues with opposite sign, we can move (thanks to this vector field) along a direction towards the origin and along an half-line exiting the sphere and going to infinity. Therefore, using again the fact that the family $\{S(v) \mid v \in \mathbf{R}^3\}$ is transitive on every sphere, we proved that $\mathscr{G}$, and thus $\Lambda(\mathscr{F})$, is controllable on $\mathbf{R}^3 \smallsetminus \{0\}$.

Assume now that we can find a fixed point, denoted by x_0, which is common to all the vector fields in $\mathscr{F}$. For every vector $v \in \mathbf{R}^3$, we have

$$C(v)x_0 + c(v) = C(-v)x_0 + c(-v) = 0_3$$

from which we deduce

$$\tilde{A}_{11}x_0 + v \times \tilde{B}v - \tilde{B}J^{-1}(Jv \times v) = 0. \tag{24}$$

Taking $v = 0_3$ gives $\tilde{A}_{11}x_0 = 0_3$ and therefore (24) cannot be identically satisfied if the sets of eigenvectors of $\tilde{B}$ and J are not equal. $\qquad \square$

We are left to deal with the case where the eigenvalues of the symmetric part of $\tilde{A}_{11}$ are all positive or all negative. Although in this case the linear part of the family $\mathscr{F}$ is not controllable on $\mathbf{R}^3 \smallsetminus \{0\}$, we can nevertheless adapt the method introduced in [12] in order to prove the controllability of (22).

Lemma 5.8. *Assume that the matrix $\tilde{A}_{11} + \tilde{A}_{11}^{\mathrm{T}}$ is invertible and that its eigenvalues have all the same sign. Then there exists an open and dense subset O of $\{(B, J) \in \mathscr{M}_{6\times3} \times \mathscr{M}_{3\times3} \mid J = J^{\mathrm{T}} > 0\}$ such that system (22) is controllable if $(B, J) \in O$. More precisely, if B_2 is invertible and if the sets of eigenvectors of $\tilde{B}$ and J are not equal, then system (22) is controllable.*

Notice that, together with Lemma 5.6, Lemma 5.8 concludes the proof of Proposition 5.3. In order to show Lemma 5.8 we need some definitions and have to prove some intermediate results.

Let $\mathscr{F}$ and $\Lambda(\mathscr{F})$ be defined as above. When the eigenvalues of A^s are of the same sign, the set $\mathcal{A}(x, \Lambda(\mathscr{F}))$ is easy to describe:

$$\mathcal{A}(x, \Lambda(\mathscr{F})) = \begin{cases} \{x\} \cup \{y \in \mathbf{R}^3 \mid 0 < \|y\| < \|x\|\} & \text{if } A^s < 0, \\ \{x\} \cup \{y \in \mathbf{R}^3 \mid \|y\| > \|x\|\} & \text{if } A^s > 0. \end{cases}$$

In the sequel, we shall need the notion of normal accessibility, which is recalled below.

Definition 5.9. *Let $\mathscr{V}$ be a family of complete vector fields on a manifold M. The point y is said to be normally $\mathscr{V}$-accessible from x if there exist $X^1, \ldots, X^p$ in $\mathscr{V}$ and $t_1, \ldots, t_p > 0$ such that*

$$y = e^{t_1 X^1} \circ \cdots \circ e^{t_p X^p}(x)$$

and the mapping

$$(\tau_1, \ldots, \tau_p) \longmapsto e^{\tau_1 X^1} \circ \cdots \circ e^{\tau_p X^p}(x),$$

defined in a neighborhood of $(t_1, \ldots, t_p)$, is of rank equal to $\dim M$ at $(t_1, \ldots, t_p)$.

In [12] the authors exploit the fact that if a family of vector fields is controllable, then every point y is normally accessible from every point x. As the family $\Lambda(\mathscr{F})$ is not controllable on $\mathbf{R}^3 \smallsetminus \{0\}$, we are lead to prove directly the following normal accessibility property.

Lemma 5.10. *Assume that the eigenvalues of $\tilde{A}_{11} + \tilde{A}_{11}^{\mathrm{T}}$ have all the same sign. Then we can extract from $\mathscr{F}$ a finite family $\mathscr{F}^0$ such that there exists a sphere of center 0_3 and radius r of points normally $\Lambda(\mathscr{F}^0)$-accessible from every point of the unit sphere.*

Proof. Let $v_0 \in \mathbf{R}^3$ be such that the matrix $A^s = \tilde{A}_{11} + S(v_0)$ is proportional to $\tilde{A}_{11} + \tilde{A}_{11}^{\mathrm{T}}$. Let, moreover, w_1 be an eigenvector of A^s and take $w_2, w_3 \in \mathbf{R}^3$ such that the family (w_1, w_2, w_3) is an orthogonal basis of $\mathbf{R}^3$. Consider now the family

$$\Lambda(\mathscr{F}^0) = \{ A^s, A^s + k_2 S(w_2), A^s + k_3 S(w_3) \}$$

extracted from $\Lambda(\mathscr{F})$. The quantities k_2 and k_3 can be chosen in such a way that the two last matrices in $\Lambda(\mathscr{F}^0)$ have a pair of non-real eigenvalues. By projecting the vector fields $A^s + k_2 S(w_2)$ and $A^s + k_3 S(w_3)$ on the unit sphere, it is easy to see that the family $\Lambda(\mathscr{F}^0)$ is transitive on the directions, that is, from any half-line starting from the origin, one can reach any other half-line. Moreover, thanks to the eigendirection w_1 of the vector field A^s we can go as far as we want (case $A^s > 0$) or as close as we want to the origin (case $A^s < 0$). Hence, for every $x \in \mathbf{R}^3 \smallsetminus \{0_3\}$, $\mathcal{A}(x, \Lambda(\mathscr{F}^0))$ contains a set of the form $\mathbf{R}^3 \smallsetminus B(0, r)$ (case $A^s > 0$) or $B(0, r) \smallsetminus \{0_3\}$ (case $A^s < 0$). The interior of $\mathcal{A}(x, \Lambda(\mathscr{F}^0))$ being nonempty, there exists at least one point, denoted by y, that can be normally $\Lambda(\mathscr{F}^0)$-accessed from x (see [18]). Notice that since y is normally $\Lambda(\mathscr{F}^0)$-accessible from x, then y is also normally $\Lambda(\mathscr{F}^0)$-accessible from every point in a (sufficiently small) neighborhood of x. Moreover, all the points in $\mathcal{A}(y, \Lambda(\mathscr{F}^0))$ are normally $\Lambda(\mathscr{F}^0)$-accessible from x.

We conclude by using the compactness of the unit sphere: for every $x \in S^2$, there exists a neighborhood V_x of x and a set $\mathbf{R}^3 \smallsetminus B(0, r_x)$ (case $A^s > 0$) or $B(0, r_x) \smallsetminus \{0_3\}$ (case $A^s < 0$) whose points are normally $\Lambda(\mathscr{F}^0)$-accessible from every point of V_x. As we can include the unit sphere in a finite union of neighborhoods V_x, we can claim the existence of a sphere of radius r whose points are normally $\Lambda(\mathscr{F}^0)$-accessible from every point of the unit sphere. $\qquad\square$

The following lemma guarantees the unboundedness of the sets $\mathcal{A}(x, \mathscr{F})$ and $\mathcal{A}(x, -\mathscr{F})$.

Lemma 5.11. *If B_2 is invertible and if the sets of eigenvectors of $\tilde{B}$ and J are not equal, then for every $x \in \mathbf{R}^3$ both sets $\mathcal{A}(x, \mathscr{F})$ and $\mathcal{A}(x, -\mathscr{F})$ are unbounded.*

Proof. Take $v \in \mathbf{R}^3$ such that $q(v) = v \times \tilde{B}v - \tilde{B}J^{-1}(Jv \times v) \neq 0_3$; this is possible since J and $\tilde{B}$ have different sets of eigenvectors. For every $\alpha \in \mathbf{R}$ the vector field $x \mapsto C(\alpha v)x + c(\alpha v)$ is in $\mathscr{F}$ and

$$\lim_{\alpha \to \infty} \frac{C(\alpha v)x + c(\alpha v)}{\alpha^2} = q(v).$$

This proves that, given a time $T > 0$, the solution of $\dot{x} = C(\alpha v)x + c(\alpha v)$ on the interval $[-T\alpha^{-2}, T\alpha^{-2}]$ is as close as we want to the solution of $\dot{x} = q(v)$ on $[-T, T]$ (with the same initial condition) provided that α is large enough. Now, the solution of $\dot{x} = q(v)$, $x(0) = x_0$, leaves any fixed bounded set both in time T and in time $-T$, provided that T is large enough. $\qquad\square$

We consider now, as in [12], the family $\{h_{\lambda,w} \mid \lambda > 0, \; w \in \mathbf{R}^n\}$ of affine diffeomorphisms of $\mathbf{R}^n$ defined by $h_{\lambda,w}(x) = w + \lambda(x - w)$. An easy computation shows that, for every $X \in \mathscr{F}$,

$$(h_{\lambda,w_*}X)(x) = \Lambda(X)(x - w) + \lambda X(w)$$

and in particular

$$\lim_{\lambda \to 0}(h_{\lambda,w_*}X)(x) = \Lambda(X)(x - w). \tag{25}$$

Denote by $\mathscr{F}_{\lambda,w}$ the image of the family $\mathscr{F}$ by h_{λ,w_*}, that is, the transformation of $\mathscr{F}$ under the change of coordinates $h_{\lambda,w}$.

Proof of Lemma 5.8. Let $\Lambda(\mathscr{F}^0)$ and r be as in the statement of Lemma 5.10. Consider the finite family $\Lambda(\mathscr{F}^0_w) = \{\Lambda(X)(x - w) \mid X \in \mathscr{F}^0\}$ and notice that each point of the sphere of center w and radius r is normally $\Lambda(\mathscr{F}^0_w)$-accessible from every point of the sphere of center w and radius 1. Let $\mathscr{F}_{\lambda,w} = \{h_{\lambda,w_*}X \mid X \in \mathscr{F}\}$. Thanks to formula (25) we can assert that, if λ is chosen sufficiently small, the sphere of center w and radius r is contained in the reachable set for $\mathscr{F}_{\lambda,w}$ from every point of the sphere of center w and radius 1 (see [18, Lemma 3.2]). Fix such a $\lambda > 0$.

Given w in $\mathbf{R}^3$ we claim that $\mathcal{A}(w, \mathscr{F})$ contains a neighborhood of w. Indeed, since the sets $\mathcal{A}(x, \mathscr{F})$ and $\mathcal{A}(x, -\mathscr{F})$ are unbounded, the same is true for the sets $\mathcal{A}(x, \mathscr{F}_{\lambda,w})$ and $\mathcal{A}(x, -\mathscr{F}_{\lambda,w})$, for every $x \in \mathbf{R}^3$. This implies in particular that there exists $y \in \mathcal{A}(w, \mathscr{F}_{\lambda,w})$ such that $\|y - w\| = 1$. Let $\rho < r$ and fix $z \in B(w, \rho)$. Since $\mathcal{A}(z, -\mathscr{F}_{\lambda,w})$ is unbounded, there exists $z' \in \mathcal{A}(z, -\mathscr{F}_{\lambda,w})$ such that $\|z' - w\| = r$. Since from every point of the sphere of center w and radius 1, we can reach any point of the sphere of center w and radius r, the point z' is reachable from y by the family $\mathscr{F}_{\lambda,w}$. Finally, every point $z \in B(w, \rho)$ belongs to $\mathcal{A}(w, \mathscr{F}_{\lambda,w})$ (see Figure 1). Since $\mathscr{F}_{\lambda,w}$ is the transformation of $\mathscr{F}$ by a diffeomorphism preserving w, we deduce that $\mathcal{A}(w, \mathscr{F})$ contains a neighborhood of w. Therefore, every attainable set $\mathcal{A}(x, \mathscr{F})$ is open.

Since the control system defined by $-\mathscr{F}$ has the same form as that defined by $\mathscr{F}$ and verifies the hypotheses of both Lemmas 5.10 and 5.11, the reasoning above proves also that every attainable set $\mathcal{A}(x, -\mathscr{F})$ is open. Take now y in the closure of $\mathcal{A}(x, \mathscr{F})$. The set $\mathcal{A}(y, -\mathscr{F})$ being open, there exists $z \in \mathcal{A}(y, -\mathscr{F}) \cap \mathcal{A}(x, \mathscr{F})$ which proves that y is reachable from x by the family $\mathscr{F}$. Every attainable set $\mathcal{A}(x, \mathscr{F})$ is therefore both open and closed; thus it is equal to $\mathbf{R}^3$. $\qquad\square$
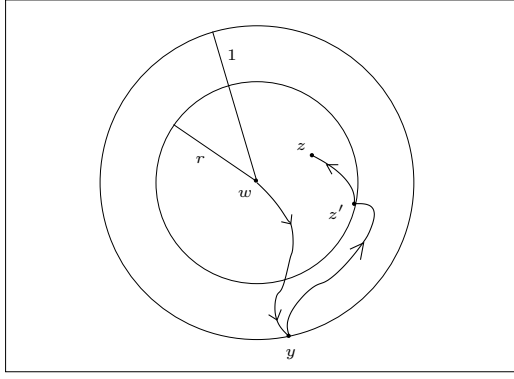
13

Figure 1: z belongs to $\mathcal{A}(w, \mathscr{F}_{\lambda,w})$ (the case $r < 1$).

5.3 Generic controllability: physical interpretation

The scope of this section is to provide a better physical insight of Theorem 5.1. More precisely, we want to check that for $m \geq 3$ there exists an open and dense set in the (suitably defined) space of microscopic organisms modeled here such that the corresponding control system (2)-(4) is controllable.

The space of organisms, denoted by $\Sigma^{(m)}$, will be identified with the set of pairs (S, Ψ), where $S \subset \mathbf{R}^3$ is open, connected, bounded, and of class $\mathcal{C}^2$, while Ψ is the m-uple of functions in $\mathcal{C}^2(\partial S, \mathbf{R}^3)$ determining the action of the controls (see Section 3). The topology on $\Sigma^{(m)}$ can be defined assigning the basis of open subsets defined by

$$\mathscr{N}_\varepsilon(S, \Psi) = \big\{ ((\mathrm{Id} + \upsilon)(S), \Phi) \mid \upsilon \in W^{3,\infty}(S, \mathbf{R}^3), \Phi \in (\mathcal{C}^2(\partial S, \mathbf{R}^3))^m,$$
$$\|\upsilon\|_{3,\infty} < \varepsilon, \|\Phi \circ (\mathrm{Id} + \upsilon) - \Psi\|_{\mathcal{C}^2(S)} < \varepsilon \big\}$$

for all $\varepsilon \in (0, 1/2)$ and all $(S, \Psi) \in \Sigma^{(m)}$ (see [17] or, for a different approach, [8]). Similarly, let $\overline{\Sigma}$ be the topological space of $\mathcal{C}^2$ domains.

Theorem 5.12. *Assume that $m \geq 3$. There exists an open and dense set $\mathcal{O}$ in $\Sigma^{(m)}$ such that (2)-(4) is controllable if (S, Ψ) belongs to $\mathcal{O}$. Moreover, there exists an open and dense set $\mathcal{P}$ in $\overline{\Sigma}$ such that $\mathcal{O} \cap (\{S\} \times (\mathcal{C}^2(\partial S, \mathbf{R}^3))^m)$ is dense in $\{S\} \times (\mathcal{C}^2(\partial S, \mathbf{R}^3))^m$ for every S in $\mathcal{P}$.*

Proof. Denote by $F(S, \Psi)$ the triple (A, B, J) associated to an element (S, Ψ) of Σ through (6), (8) and (10). We want to show that each condition appearing in the statement of Propositions 5.2 and 5.3 is satisfied by the elements of the image through F of an open and dense subset of Σ.

Notice that the map $\Psi \mapsto B$ is onto when considered from $(\mathcal{C}^2(\partial S, \mathbf{R}^3))^m$ to $\mathscr{M}_{6 \times m}$ (S fixed). Indeed, according to [16, Lemma 4.1], $g^{(1)}, g^{(2)}, g^{(3)}, G^{(1)}, G^{(2)}, G^{(3)}$ are linearly independent in $L^2(\partial S, \mathbf{R}^3)$. Therefore, since the orthogonal to a smooth function in $\mathcal{C}^2(\partial S, \mathbf{R}^3)$ with respect to the L^2-product has infinite codimension, the relation (10) defines a surjective map. In particular, the pre-image of an open and dense set in $\mathscr{M}_{6 \times m}$ is open and dense in $(\mathcal{C}^2(\partial S, \mathbf{R}^3))^m$.

Let us now take into account the dependence on S. It is easy to check by using axially symmetric perturbations that the map from the space of $\mathcal{C}^2$ bounded sets to $\mathbf{R}^3$ that associates to S the spectrum of J –with eigenvalues repeated according to their multiplicity– is a submersion (i.e., its differential is onto at every point). In particular, the eigenvalues of J are simple for an open and dense subset of the space of bounded $\mathcal{C}^2$ subsets of $\mathbf{R}^3$, independently of Ψ. If b_2 does not belong to any of the three planes generated by pairs of eigenvectors of J, then (17) is automatically verified.

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Similarly, one notices that all the assumptions appearing in the statements of Propositions 5.2 and 5.3 can be represented in the form $G(S, \Psi) \in O$ where O is an open and dense subset of a finite-dimensional vector space V, the map $G : \Sigma \to V$ is continuous and for S in an open and dense subset of $\overline{\Sigma}$ we have that $G(S, \Psi)$ belongs to $\mathcal{O}$ if B belongs to an open and dense subset of $\mathscr{M}_{6 \times m}$ (possibly depending on S). Therefore, $G^{-1}(\mathcal{O})$ is open and dense in Σ. $\qquad\square$

Remark 5.13. *As recalled above it follows from [16] that, once S is fixed, the linear map associating to the m-uple Ψ the matrix B through (10) is onto as a map from $(L^2(\partial S, \mathbf{R}^3))^m$ to $\mathscr{M}_{6 \times m}$. Thanks to this result, and to the remark that (2)-(4) is controllable if $m = 6$ and B is invertible, the authors prove that for $m \geq 6$, for an open and dense subset of m-uples in $\mathcal{C}^2(\partial S, \mathbf{R}^3)$ (S fixed) the corresponding system is controllable (see [16, Theorem 1.1]). In the language adopted here the result of [16] says that, for every fixed S and for $m \geq 6$, controllability is a generic property with respect to Ψ.*

6 Swimming spherical microscopic organisms

When the microscopic organism happens to be spherical, the equations presented in the previous sections have a very special form. As described in [16], general results in hydrodynamics (see, e.g., [10]) show that the matrix A corresponding to a spherical organism is diagonal and, more precisely, of the form

$$A = \begin{pmatrix} -\rho_1 \mathrm{Id} & 0 \\ 0 & -\rho_2 \mathrm{Id} \end{pmatrix},$$

with

$$\rho_2 > \rho_1 > 0. \tag{26}$$

Moreover, since the inertia matrix J is proportional to the identity (assuming that the mass distribution is homogenous inside S), the nonlinear term E appearing in (2) is given by

$$E(z) = \begin{pmatrix} \omega \times \xi \\ 0_3 \end{pmatrix}.$$

Notice that in the spherical case ω satisfies a well-defined control subsystem, ie,

$$\dot{\omega} = -\rho_2 \omega + B_2 u. \tag{27}$$

We are going to prove the following result.

Theorem 6.1. *Let S be a ball and assume that the mass distribution of S is homogenous. Then the control system (2)–(4) is controllable if and only if the rank of B_2 is equal to three and B_1 is different from $0_{3 \times m}$.*

The easier part of the proof is to show that the controllability of (2)–(4) implies that

$$\mathrm{rank}\, B_2 = 3, \qquad B_1 \neq 0_{3 \times m}. \tag{28}$$

Indeed, the linear control system (27) is, as it is well known, controllable if and only if rank $B_2 = 3$. Moreover, if $B_1 = 0_{3 \times m}$, then the space $\{0_3\} \times \mathbf{R}^3$ is invariant for the dynamics of the control system (2), which is therefore non-controllable.

The converse implication will be proven in several steps.

6.1 The case where B_1 and B_2 are linearly independent

In this section we study the case where the organism is spherical and

$$\operatorname{rank} B_2 = 3, \qquad B_1 \notin \operatorname{span}(B_2). \tag{29}$$

Assumption (29) guarantees that there exists an m-by-3 matrix Γ such that

$$B_1 \Gamma v = \sum_{i=1}^{3} v_i c_i, \qquad B_2 \Gamma v = \sum_{i=1}^{3} v_i d_i,$$

with $c_1 \times d_1 \neq 0_3$ and $\{d_1, d_2, d_3\}$ an orthonormal basis of $\mathbf{R}^3$.

Notice now that for every $Q \in SO(3)$, the change of coordinates $\xi' = Q\xi, \omega' = Q\omega, \zeta' = \zeta, R' = RQ^{-1}$ preserves the dynamics of (2)–(4). This is equivalent to say that, up to a change of coordinates, we can replace (B_1, B_2) by (QB_1, QB_2), for every choice of $Q \in SO(3)$. Choose Q such that $Qd_i = e_i$ for $i = 1, 2, 3$ (where we recall that $(e_i)_{i=1,2,3}$ denotes the canonical basis in $\mathbf{R}^3$). The above transformations show that without loss of generality $m = 3$, $B_2 = \operatorname{Id}$, and the first column of B_1 is not proportional to e_1.

Denote by b_1, b_2, b_3 the columns of B_1. Accordingly, the controlled vector fields are the three constant vector fields $X_i = (b_i, e_i, 0_6)$, $i = 1, 2, 3$.

Lemma 6.2. *If (29) holds then system (2)–(4) is Lie-bracket generating.*

Proof. Applying (5) in the case $J = \operatorname{Id}$ we obtain that for every $v_1, v_2 \in \mathbf{R}^3$ the constant vector field $V = (v_1, v_2, 0_6)$ satisfies

$$[[X_0, X_i], V] = (e_i \times v_1 - b_i \times v_2, 0_9). \tag{30}$$

In particular, taking $i = 1$ and $V = X_1$ in (30), we deduce that $Z = (e_1 \times b_1, 0_9)$ belongs to $\mathscr{L}$. Applying again (30) to $V = Z$ we get that, for $i = 1, 2, 3$, $Z_i = (e_i \times (e_1 \times b_1), 0_9)$ belongs to $\mathscr{L}$. Since Z, Z_1, Z_2, and Z_3 span $\mathbf{R}^3 \times \{0_9\}$, it follows that $\mathscr{L}$ contains

$$\operatorname{span}(X_1, X_2, X_3) + \mathbf{R}^3 \times \{0_9\} = \mathbf{R}^6 \times \{0_6\}.$$

As noticed in Remark 3.1 this proves the lemma. $\qquad\square$

The controllability of system (2) follows from Proposition 5.3: indeed, $\tilde{A}_{11} = \rho_1 \operatorname{Id}$ is symmetric and invertible and hypothesis (29) implies that $B_1 = B_1 B_2^{-1}$ has not the same set of eigenvectors as $J = \operatorname{Id}$. Theorem 6.1 is therefore proven, because of Proposition 4.1, under the stronger assumption that (29) holds true.

6.2 The proportional case

In order to complete the proof of Theorem 6.1, we have to prove that system (2)–(4) is controllable when

$$\operatorname{rank} B_2 = 3, \qquad B_1 = \lambda B_2, \qquad \lambda \neq 0. \tag{31}$$

Using the same normalization argument as in the previous section, we can assume that $m = 3$ and $B_2 = \operatorname{Id}$. Moreover, we can take $\lambda = 1$ by performing the change of coordinates $\xi' = \xi/\lambda, \omega' = \omega, \zeta' = \zeta/\lambda, R' = R$. Therefore, without loss of generality, $X_i = (e_i, e_i, 0_6)$ for $i = 1, 2, 3$.

Let us prove the Lie-bracket generating condition. We have

$$[X_0, X_i] = (-\rho_1 e_i, -\rho_2 e_i, Re_i, RS(e_i))$$

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and
$$[[X_0, X_i], [X_0, X_j]] = (0_6, 2R(e_j \times e_i), RS(e_j \times e_i)).$$

A further computation yields
$$[[[X_0, X_i], [X_0, X_j]], [X_0, X_k]] = (0_6, 3R(e_k \times (e_j \times e_i)), RS(e_k \times (e_j \times e_i))).$$

Therefore, the Lie algebra $\mathscr{L}$ contains the following vector fields

$$\begin{pmatrix} e_i \\ e_i \\ 0_3 \\ 0_3 \end{pmatrix}, \qquad \begin{pmatrix} -\rho_1 e_j \\ -\rho_2 e_j \\ Re_j \\ RS(e_j) \end{pmatrix}, \qquad \begin{pmatrix} 0_3 \\ 0_3 \\ 2Re_k \\ RS(e_k) \end{pmatrix}, \qquad \begin{pmatrix} 0_3 \\ 0_3 \\ 3Re_l \\ RS(e_l) \end{pmatrix},$$

for $i, j, k, l = 1, 2, 3$. As a consequence, $\mathscr{L}$ contains also

$$\begin{pmatrix} (\rho_2 - \rho_1)e_i \\ 0_3 \\ Re_i \\ RS(e_i) \end{pmatrix}, \qquad \begin{pmatrix} 0_3 \\ (\rho_1 - \rho_2)e_j \\ Re_j \\ RS(e_j) \end{pmatrix}, \qquad \begin{pmatrix} 0_3 \\ 0_3 \\ Re_k \\ 0_3 \end{pmatrix}, \qquad \begin{pmatrix} 0_3 \\ 0_3 \\ 0_3 \\ RS(e_l) \end{pmatrix},$$

for $i, j, k, l = 1, 2, 3$. These vector fields form a moving frame on the manifold $\mathbf{R}^9 \times \mathrm{SO}(3)$ (since $\rho_1 \neq \rho_2$).

In order to conclude the proof of Theorem 6.1 we need to show that (2) is controllable. Applying Lemma 5.5 this turns out to be equivalent to the controllability of the following system,

$$\dot{x} = -\rho_1 x + v \times x + (\rho_2 - \rho_1)v, \qquad x \in \mathbf{R}^3, \tag{32}$$

with the control v taking values in $\mathbf{R}^3$. Decomposing the control as $v = wx + W$ with $w \in \mathbf{R}$ and W orthogonal to x, equation (32) rewrites as

$$\dot{x} = (-\rho_1 + (\rho_2 - \rho_1)w)x + W \times x.$$

Clearly, choosing $W = 0_3$ allows to move as desired along $\mathrm{span}(x)$, while the choice $w = \rho_1/(\rho_2 - \rho_1)$ permits to attain the sphere of radius $\|x\|$ centered at the origin. The controllability of (2) is thus proven and the proof of Theorem 6.1 completed.

7 The case where the densities of the microscopic organism and of the fluid have the same order of magnitude

As pointed out in [16, Section 2], the assumption that the densities of the microscopic organism and of the fluid have the same order of magnitude leads to stronger simplifications than in the general case studied in the previous sections. In particular, according to the remarks of page 6 the control

system (2.18)–(2.27) presented in [16] reduces in this case to

$$-\Delta v + \nabla p = 0 \qquad \text{on } F \times (0, T), \tag{33}$$

$$\text{div} v = 0 \qquad \text{on } F \times (0, T), \tag{34}$$

$$v(y, t) = \xi(t) + \omega(t) \times y + \sum_{i=1}^{m} u_i(t) \psi_i(y) \qquad \text{for } (y, t) \in \partial S \times (0, T), \tag{35}$$

$$\lim_{|y| \to \infty} v(y, t) = 0 \qquad \text{for } t \in (0, T), \tag{36}$$

$$\int_{\partial S} \sigma(v, p) n \, ds = 0 \qquad \text{on } (0, T), \tag{37}$$

$$\int_{\partial S} y \times \sigma(v, p) n \, ds = 0 \qquad \text{on } (0, T), \tag{38}$$

$$\dot\zeta = R\xi \qquad \text{on } (0, T), \tag{39}$$

$$\dot R = RS(\omega) \qquad \text{on } (0, T). \tag{40}$$

Physically, v represents the field of velocities of the fluid in F (the coordinates are attached to the body), p is the pressure, ξ and ω are the linear and angular velocities of the organism and finally ζ and R give its position and orientation with respect to a fixed frame. Recall that $\psi_1, \ldots, \psi_m$ are the functions characterizing the control actions and that σ is the Cauchy stress defined in (7).

Given a time-interval $[0, T]$ and a control $u \in L^\infty([0, T], \mathbf{R}^m)$ there exists a unique solution $(v, p, \xi, \omega, \zeta, R)$ of (33)–(40) satisfying

$$v \in H^1([0, T], L^s(F) \cap D^{1,r}(F) \cap D^{2,\theta}(F) \cap \mathcal{C}^\infty(F)),$$

$$\sup_{y \in F}(1 + \|y\|)\|v(y, t)\| < \infty \quad \text{for almost every } t \in [0, T],$$

$$p \in H^1([0, T], L^r(F) \cap D^{1,\theta}(F) \cap \mathcal{C}^\infty(F)),$$

$$\xi \in H^1([0, T], \mathbf{R}^3), \ \omega \in H^1([0, T], \mathbf{R}^3), \ \zeta \in \mathcal{C}^1([0, T], \mathbf{R}^3), \ R \in \mathcal{C}^1([0, T], \mathrm{SO}(3)),$$

for $s \in (3, \infty]$, $r \in (3/2, \infty]$ and $\theta \in (1, \infty)$ (see [9, 16]).

The finite-dimensional reduction of system (33)–(40) can be obtained following the procedure proposed in [16]. To this extent, as in Section 3, define the fundamental solutions $(h^{(i)}, p^{(i)})$ and $(H^{(i)}, P^{(i)})$ of the Stokes system and associate to them the matrices $\Theta^k, \Upsilon^k, \kappa, \mathcal{K}$. Let L be the $n \times m$ matrix

$$L = - \begin{pmatrix} \Theta^1 & \Theta^2 \\ \Upsilon^1 & \Upsilon^2 \end{pmatrix}^{-1} \begin{pmatrix} \kappa \\ \mathcal{K} \end{pmatrix}.$$

Lemma 7.1. *For almost every $t \in (0, T)$ we have*

$$\begin{pmatrix} \xi(t) \\ \omega(t) \end{pmatrix} = Lu(t).$$

Proof. According to [16, Lemma 3.4] we have

$$\int_{\partial S} g^{(i)} \cdot v \, ds = \left[\int_{\partial S} \sigma(v, p) n \, ds \right]_i = 0$$

$$\int_{\partial S} G^{(i)} \cdot v \, ds = \left[\int_{\partial S} y \times \sigma(v, p) n \, ds \right]_i = 0$$

where the equalities in the right-hand sides follow from (37) and (38).

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On the other hand, for almost every $t \in (0, T)$,

$$\int_{\partial S} g^{(i)} \cdot v \, ds = \int_{\partial S} g^{(i)} \cdot (\xi + \omega \times y + \sum_{j=1}^{m} u_j \psi_j) \, ds = -(\Theta^1 \xi + \Theta^2 \omega + \kappa u)_i,$$

and, similarly

$$\int_{\partial S} G^{(i)} \cdot v \, ds = -(\Upsilon^1 \xi + \Upsilon^2 \omega + \mathcal{K} u)_i.$$

Therefore,

$$\begin{pmatrix} \kappa \\ \mathcal{K} \end{pmatrix} u = -\begin{pmatrix} \Theta^1 & \Theta^2 \\ \Upsilon^1 & \Upsilon^2 \end{pmatrix} \begin{pmatrix} \xi \\ \omega \end{pmatrix}$$

which proves the lemma. $\qquad\square$

System (33)–(40) therefore reduces to the control system

$$\begin{cases} \dot\zeta = RL^1 u, \\ \dot R = RS(L^2 u), \end{cases} \qquad u \in \mathbf{R}^m, \tag{41}$$

where $L^1, L^2 \in \mathscr{M}_{3 \times m}$ are such that

$$L = \begin{pmatrix} L^1 \\ L^2 \end{pmatrix}.$$

By homogeneity, since the set of admissible controls is the whole $\mathbf{R}^m$, the controllability in arbitrary small time of system (41) is equivalent to the controllability of the control system whose admissible velocities are given by the family of vector fields $\mathcal{F} = \{\pm X_i \mid 1 \le i \le m\}$ defined by

$$X_i(\zeta, R) = (R \, b_i, R \, S(c_i)) \tag{42}$$

where the b_i's denote the columns of L^1 and the c_i's those of L^2. Moreover, since $\mathcal{F}$ is symmetrical, the controllability of this family is equivalent to the fact that the Lie algebra generated by $\mathcal{F}$, denoted by $\mathscr{L}$, is of rank 6 at every point of $\mathbf{R}^3 \times \mathrm{SO}(3)$.

Theorem 7.2. *The control system* (41) *is controllable in arbitrary small time if and only if one of the following conditions is satisfied:*

1. $\mathrm{rank}(L^2) \ge 2$ *and* $(L^1)^{\mathrm{T}} L^2 + (L^2)^{\mathrm{T}} L^1 \ne 0_{m \times m}$;

2. $\mathrm{rank}(L^2) = 2$ *and* $\mathrm{rank}(L) \ge 3$.

Proof. First of all notice that the Lie bracket between two vector fields of the form $Z_i(\zeta, R) = (R \, v_i, R \, S(w_i))$, $i = 1, 2$, is given by

$$[Z_1, Z_2](\zeta, R) = (R(w_1 \times v_2 - w_2 \times v_1), R \, S(w_1 \times w_2)). \tag{43}$$

Let us prove the "if" part of the theorem. Suppose that we can find in $\mathscr{L}$ two vector fields Y_1, Y_2 of the form

$$Y_i(\zeta, R) = (R \, \beta_i, R \, S(\gamma_i)), \quad i = 1, 2,$$

such that the vectors γ_1 and γ_2 are linearly independent and that the inner product $\langle \beta_1, \gamma_1 \rangle$ is nonzero. We prove here below that, in this case, the Lie algebra $\mathscr{L}$ is of full rank at every point of $\mathbf{R}^3 \times \mathrm{SO}(3)$. Then we show that both 1. and 2. guarantee that, without loss of generality, such Y_1 and Y_2 can be found.

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Let $Y_3 = [Y_1, Y_2]$ whose expression can be obtained using (43). Define the vector field W_1 as the bracket between Y_3 and Y_1. Applying again (43) we obtain that

$$W_1(\zeta, R) = [Y_3, Y_1](\zeta, R) = \begin{pmatrix} R\left(2\langle\beta_1, \gamma_1\rangle\gamma_2 - (\langle\beta_1, \gamma_2\rangle + \langle\beta_2, \gamma_1\rangle)\gamma_1 - \langle\gamma_1, \gamma_2\rangle\beta_1 + \|\gamma_1\|^2\beta_2\right) \\ R\,S(-\langle\gamma_1, \gamma_2\rangle\gamma_1 + \|\gamma_1\|^2\gamma_2) \end{pmatrix}.$$

Let W_2 be the vector field obtained adding $\langle\gamma_1, \gamma_2\rangle Y_1 - \|\gamma_1\|^2 Y_2$ to W_1, ie,

$$W_2(\zeta, R) = \begin{pmatrix} R\left(2\langle\beta_1, \gamma_1\rangle\gamma_2 - (\langle\beta_1, \gamma_2\rangle + \langle\beta_2, \gamma_1\rangle)\gamma_1\right) \\ 0_3 \end{pmatrix}.$$

Since W_2 is in $\mathscr{L}$, then the vector field $Y_4 = [W_2, Y_1]$ belongs to $\mathscr{L}$ as well and we have that $Y_4(\zeta, R) = (R(2\langle\beta_1, \gamma_1\rangle\gamma_1 \times \gamma_2), 0_3)$. Consider now the vector fields $Y_5 = [Y_4, Y_1]$ and $Y_6 = [Y_4, Y_2]$, both belonging to $\mathscr{L}$, and whose expressions are $Y_5(\zeta, R) = (R(2\langle\beta_1, \gamma_1\rangle\gamma_1 \times (\gamma_1 \times \gamma_2)), 0_3)$ and $Y_6(\zeta, R) = (R(2\langle\beta_1, \gamma_1\rangle\gamma_2 \times (\gamma_1 \times \gamma_2)), 0_3)$. Since each matrix R in $SO(3)$ is invertible, and because $\langle\beta_1, \gamma_1\rangle \neq 0$, the vector fields $Y_1, \ldots, Y_6$ span $T_{(\zeta, R)}\mathbf{R}^3 \times SO(3)$ at every $(\zeta, R) \in \mathbf{R}^3 \times SO(3)$ if and only if the matrix

$$\Delta = \begin{pmatrix} \beta_1 & \beta_2 & \gamma_2 \times \beta_1 - \gamma_1 \times \beta_2 & \gamma_1 \times \gamma_2 & \gamma_1 \times (\gamma_1 \times \gamma_2) & \gamma_2 \times (\gamma_1 \times \gamma_2) \\ \gamma_1 & \gamma_2 & \gamma_2 \times c_1 & 0_3 & 0_3 & 0_3 \end{pmatrix}$$

is full-rank. The non-degeneracy of Δ follows easily from the assumption that the vectors γ_1 and γ_2 are linearly independent.

We shall prove now that the hypotheses 1. and 2. imply the existence of two vector fields Y_1 and Y_2 as above.

Let $r = \text{rank}(L^2)$. Notice that for every matrix $\Gamma \in GL(m)$ the reparamerization $u \to \Gamma u$ transforms the matrix (L^1, L^2) into $(L^1\Gamma, L^2\Gamma)$. The condition $(L^1)^\mathrm{T}L^2 + (L^2)^\mathrm{T}L^1 \neq 0_{m\times m}$ is preserved by this transformation, since, $(L^1\Gamma)^\mathrm{T}L^2\Gamma + (L^2\Gamma)^\mathrm{T}L^1\Gamma = \Gamma^T((L^1)^\mathrm{T}L^2 + (L^2)^\mathrm{T}L^1)\Gamma$. Thus, without loss of generality, we can assume that $L^2 = (L^{2,1} \,|\, 0_{3\times(m-r)})$ where $L^{2,1} \in \mathscr{M}_{3\times r}$ is of rank r. Let us write L^1 as $(L^{1,1} \,|\, L^{1,2})$ where $L^{1,1} \in \mathscr{M}_{3\times r}$ and $L^{1,2} \in \mathscr{M}_{3\times(m-r)}$.

Suppose that 1. holds. If $(L^{1,1})^\mathrm{T}L^{2,1} + (L^{2,1})^\mathrm{T}L^{1,1} \neq 0_{r\times r}$, then we can find $1 \leq j_1, j_2 \leq r$ such that $\langle b_{j_1}, c_{j_1}\rangle \neq 0$, or $\langle b_{j_2}, c_{j_2}\rangle \neq 0$, or $\langle b_{j_1}, c_{j_2}\rangle + \langle b_{j_2}, c_{j_1}\rangle \neq 0$. If $\langle b_{j_1}, c_{j_1}\rangle = \langle b_{j_2}, c_{j_2}\rangle = 0$, then the inner product $\langle b_{j_1} + b_{j_2}, c_{j_1} + c_{j_2}\rangle = \langle b_{j_1}, c_{j_2}\rangle + \langle b_{j_2}, c_{j_1}\rangle$ is nonzero, and thus the vector fields $Y_1 = X_{j_1} + X_{j_2}$ and $Y_2 = X_{j_2}$ fulfill the required conditions. Otherwise, $(L^{2,1})^\mathrm{T}L^{1,2} \neq 0_{r\times(m-r)}$ and we can fix $1 \leq j_1, j_2 \leq r$ and $j_3 > r$ such that $j_1 \neq j_2$, $\langle b_{j_1}, c_{j_1}\rangle = 0$ and $\langle b_{j_3}, c_{j_1}\rangle \neq 0$. Then $Y_1 = X_{j_1} + X_{j_3}$ and $Y_2 = X_{j_2}$ fulfill the required conditions.

Assume now that 2. holds and 1. does not. Then c_1 and c_2 are linearly independent and

$$\langle b_1, c_1\rangle = \langle b_2, c_2\rangle = \langle b_1, c_2\rangle + \langle b_2, c_1\rangle = 0.$$

Moreover, there exists $j > r = 2$ such that $b_j \neq 0_3$ and $\langle b_j, c_1\rangle = \langle b_j, c_2\rangle = 0$. Therefore, there exists a real number $\alpha \neq 0$ such that $b_3 = \alpha(c_1 \times c_2)$. Then, according to (43),

$$[X_1, X_j](\zeta, R) = (R(\alpha\, c_1 \times (c_1 \times c_2)), 0_3).$$

As $\langle c_2, c_1 \times (c_1 \times c_2)\rangle \neq 0$ the vector fields $Y_1 = X_1$ and $Y_2 = X_2 + [X_1, X_j]$ fulfill the required conditions.

Let us prove now the "only if" part of the statement. First notice that if $r \leq 1$ then there exists a vector $x_0 \in \mathbf{R}^3 \setminus \{0_3\}$ such that $L^2 u \times x_0 = 0$ for every control u. Multiplying equation (41) by x_0, we get that $\dot{R}x_0 = 0$ and so $R(t)x_0 = R(0)x_0$ for every time t. Therefore, system (41) is not controllable.

Assume now that $r = \text{rank}(L) = 2$ and $(L^1)^{\text{T}} L^2 + (L^2)^{\text{T}} L^1 = 0_{m \times m}$. Let $Y = [X_1, X_2]$, whose expression, according to (43), is given by

$$Y(\zeta, R) = (R(c_1 \times b_2 - c_2 \times b_1), R\, S(c_1 \times c_2)).$$

Since $\langle b_i, c_i \rangle = 0$ for $i = 1, 2$ and $\langle b_1, c_2 \rangle + \langle b_2, c_1 \rangle = 0$, we obtain that

$$
\begin{aligned}
[Y, X_1] &= -\langle c_1, c_2 \rangle X_1 + \|c_1\|^2 X_2 \\
[Y, X_2] &= -\|c_2\|^2 X_1 + \langle c_1, c_2 \rangle X_2.
\end{aligned}
$$

Therefore, the Lie algebra $\mathscr{L}$, which is generated by X_1 and X_2, is equal to the linear space of vector fields spanned by X_1, X_2 and Y and cannot be of full rank.

Let now $r = 3$ and $(L^1)^{\text{T}} L^2 + (L^2)^{\text{T}} L^1 = 0_{m \times m}$. The condition $(L^{2,1})^{\text{T}} L^{1,2} = 0_{3 \times (m-3)}$ implies that the columns of $L^{1,2}$ are orthogonal to all the elements of a basis of $\mathbf{R}^3$. Therefore, $L^{1,2} = 0_{3 \times (m-3)}$. As for the columns of $L^{1,1}$, we easily obtain from $(L^{1,1})^{\text{T}} L^{2,1} + (L^{2,1})^{\text{T}} L^{1,1} = 0_{3 \times 3}$ that

$$
\begin{aligned}
b_1 &= -\,\alpha_{12}\, c_2 - \alpha_{13}\, c_3, \\
b_2 &= \alpha_{12}\, c_1 - \alpha_{23}\, c_3, \\
b_3 &= \alpha_{13}\, c_1 + \alpha_{23}\, c_2,
\end{aligned}
$$

for some $\alpha_{12}, \alpha_{13}, \alpha_{23} \in \mathbf{R}$. Without loss of generality we can assume that (c_1, c_2, c_3) is a positively oriented orthonormal basis and an easy computation gives

$$
\begin{aligned}
[X_1, X_2] &= X_3, \\
[X_1, X_3] &= -\,X_2, \\
[X_2, X_3] &= X_1,
\end{aligned}
$$

which proves that $\mathscr{L}$ is equal to the linear space of vector fields spanned by X_1, X_2 and X_3 and cannot be of full rank. $\qquad\square$

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