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2006 J. Phys. A: Math. Gen. 39 13531

(http://iopscience.iop.org/0305-4470/39/43/010)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **39** (2006) 13531–13551 doi:10.1088/0305-4470/39/43/010

Quantum control by decomposition of $SU(1,1)$

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Received 16 April 2006, in final form 4 September 2006
Published 11 October 2006
Online at stacks.iop.org/JPhysA/39/13531

Abstract

Constructive algorithms are presented for controlling quantum systems evolving on the $SU(1,1)$ Lie group. These procedures are performed via structured decomposition of $SU(1,1)$, which achieve precise controls without any approximations or iterative computations, under the sufficient condition that examines the existence of such decomposition. The technique is applied to controlling transitions between $SU(1,1)$ coherent states. These results open up new perspectives on the control design of infinite-dimensional quantum systems involving discrete or continuous spectra.

PACS numbers: 32.80.Qk, 02.20.Bb, 07.05.Dz, 85.70.Rp

1. Introduction

Group theoretical techniques have been widely applied in quantum control systems whose propagators evolve on compact Lie groups [1–6]. An important method in the control design is to decompose the target system propagator into a relatively simple sequence of factors that can be directly implemented by piecewise constant or sinusoidal control pulses. This technique is quite useful for many two-level or multi-level quantum systems [7–9], e.g. the movement of spin-$\frac{1}{2}$ particles [4, 6], the manipulation of electronic states of rubidium and the Morse oscillator model of vibrational modes of hydrogen fluoride [5]. Among the existing studies, factorization algorithms for the simplest nontrivial compact Lie group $SU(2)$ have been investigated in both classical and quantum cases [2, 3] under certain constraint.

In a wider context, many fundamental quantum systems possess noncompact dynamical groups, e.g., $SU(1,1)$ for quantum systems with Poschl–Teller [10] or Morse [11] potential, $SO(4,2)$ for the hydrogen atoms [12–15]. Thus it is necessary to extend the study to noncompact cases. In the literature, much attention has been attracted on the analysis of dynamical properties [16–24] with known quantum Hamiltonians. However, to the authors' knowledge, no studies published to date have examined the inverse problem, i.e. the design of proper Hamiltonian to realize desired dynamics for such quantum systems.

In this paper, we will initiate the study of the control pulses design for quantum control systems with noncompact symmetry groups. Here we consider the simplest class of quantum systems whose Hamiltonians can be written as linear combination of the $su(1,1)$ Lie algebra generators. The involved quantum control systems obey the following Schrödinger equation (setting $\hbar = 1$):

$$\mathrm{i}\dot{U}(t) = H_0 U(t) + u(t) H_I U(t), \qquad U(0) = I, \tag{1}$$

where H_0 and H_I are the internal and interaction Hamiltonians, respectively, and they generate a $su(1,1)$ Lie algebra. The scalar control $u(t)$ represents some adjustable external field coupled to the system that is to be designed in order to achieve certain system evolutions. Related physical examples can be seen in many situations such as coherent states in quantum optics [25–28], spin wave in solid-state physics [29], the quantized vibrational motion of a trapped ion [30], laser–plasma scattering [31] and so on. Here we assume that the admissible control is a piecewise constant function of time, which is widely used in laboratory directly or after rotating wave approximations. The system (1) has a 'drift' term H_0 that is manipulated via switching on and off the only one 'perturbation' H_I. According to the group representation theory [32, 12], the unitary propagator $U(t)$ must act on an *infinite*-dimensional Hilbert space, hence carries an infinite-dimensional unitary irreducible representation (UIR) of $SU(1,1)$. Nevertheless, since all faithful representations of $SU(1,1)$ are algebraically isomorphic on which the design of control functions does not rely, one can always focus the study on the simplest two-dimensional non-unitary representation to be described in section 2.

We ascribe the control design of the above $SU(1,1)$-type quantum control systems to the following structured decomposition of the target system propagator $U_f = U(T_f)$:

$$U_f = \prod_{k=1}^{Q} \mathrm{e}^{-\mathrm{i}t_k(H_0 + u_k H_I)}, \tag{2}$$

where the $su(1,1)$ generators H_0 and H_I are assumed linearly independent, and u_k is a constant with respect to t_k. A physical realistic decomposition should satisfy that (i) positive time durations **O1**: $t_k > 0$, and (ii) bounded control pulses **O2**: $|u_k| \leqslant C$ for some prescribed constant C. Obviously, for a desired system propagator U_f, piecewise constant control laws can be naturally determined once a decomposition in the form of (2) is found to satisfy **O1** and **O2**. The idea here is parallel with those on the compact Lie group $SU(2)$ studied in [2], however, the structured decomposition of $SU(1,1)$ is far more complicated and limited as will be seen in the following sections.

The balance of this paper is organized as follows. Section 2 gives preliminaries of the Lie group $SU(1,1)$ and its Euler parametrizations. Section 3 constructs main factorization algorithms subjected to **O1** and **O2**, along with a sufficient condition to examine the existence of such decomposition. Section 4 applies the algorithms to the control of transitions between $SU(1,1)$ coherent states. Finally, conclusions are drawn in section 5.

2. Preliminaries on $SU(1,1)$

The Lie group $SU(1,1)$ consists of two-dimensional complex pseudo-unitary matrices parametrized by

$$\begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix}, \qquad |a|^2 - |b|^2 = 1, \tag{3}$$

where $\bar{a}$ denotes the complex conjugate of a. The corresponding Lie algebra $su(1, 1)$ has generators, say $\{K_1, K_2, K_3\}$, whose commutations read

$$[K_1, K_2] = -\mathrm{i}K_3, \qquad [K_2, K_3] = \mathrm{i}K_1, \qquad [K_3, K_1] = \mathrm{i}K_2. \tag{4}$$

Suppose the Hilbert space of quantum states should carry a positive discrete UIR $\mathscr{D}^+(k)$, where $k \in \mathbb{N}^+$ is the Bargmann index [12, 33]. Then one can choose an orthonormal basis $\{|m, k\rangle, m = 0, 1, 2, \ldots\}$ under which the Casimir operator $C = K_3^2 - K_1^2 - K_2^2$ and the compact generator K_3 are simultaneously diagonalized

$$C|m, k\rangle = k(k-1)|m, k\rangle, \qquad K_3|m, k\rangle = (m+k)|m, k\rangle.$$

Correspondingly, the operators $K_\pm = K_1 \pm \mathrm{i}K_2$ will act as raising and lowering operators, i.e.,

$$\begin{aligned} K_+|m, k\rangle &= [(m+1)(m+2k)]^{1/2}|m+1, k\rangle, \\ K_-|m, k\rangle &= [m(m+2k-1)]^{1/2}|m-1, k\rangle. \end{aligned} \tag{5}$$

Therefore, the $SU(1, 1)$ propagators of real quantum systems are represented by infinite-dimensional matrices under the above basis. However, as argued in section 1, one can adopt the simplest faithful non-unitary representation (3) for the purpose of control design, of which the algebra generators are identified as:

$$K_1 = \frac{\mathrm{i}}{2}\sigma_y, \qquad K_2 = -\frac{\mathrm{i}}{2}\sigma_x, \qquad K_3 = \frac{1}{2}\sigma_z, \tag{6}$$

where $\sigma_{x,y,z}$ are Pauli matrices. K_1 and K_2 generate the noncompact one-parameter $O(1, 1)$ subgroups of $SU(1, 1)$, respectively, as follows

$$\exp(-\mathrm{i}\alpha K_1) = \begin{pmatrix} \cosh\frac{\alpha}{2} & -\mathrm{i}\sinh\frac{\alpha}{2} \\ \mathrm{i}\sinh\frac{\alpha}{2} & \cosh\frac{\alpha}{2} \end{pmatrix},$$

$$\exp(-\mathrm{i}\beta K_2) = \begin{pmatrix} \cosh\frac{\beta}{2} & -\sinh\frac{\beta}{2} \\ -\sinh\frac{\beta}{2} & \cosh\frac{\beta}{2} \end{pmatrix},$$

while K_3 generates a compact $O(2)$ subgroup

$$\exp(-\mathrm{i}\gamma K_3) = \begin{pmatrix} \mathrm{e}^{-\mathrm{i}\frac{\gamma}{2}} & 0 \\ 0 & \mathrm{e}^{\mathrm{i}\frac{\gamma}{2}} \end{pmatrix}.$$

The $SU(1, 1)$ matrices can be decomposed as products of the above factors, which are called Euler-type decompositions [34]. In this paper, two different Euler-type decompositions will be used. The first type is as follows:

$$g(\xi, \eta, \zeta) = \mathrm{e}^{-\mathrm{i}\xi K_3}\,\mathrm{e}^{-\mathrm{i}\eta K_2}\,\mathrm{e}^{-\mathrm{i}\zeta K_3}, \tag{7}$$

where $-2\pi \leqslant \xi, \zeta \leqslant 2\pi$ and $0 \leqslant \eta < \infty$. Here the first and the third factors are $O(2)$ transformations generated by K_3. Every $SU(1, 1)$ element possesses this kind of decomposition.

The second kind of decomposition is achieved with its first and last factors generated by K_2, i.e.

$$h(\xi, \mu, \zeta) = \mathrm{e}^{-\mathrm{i}\xi K_2}\,\mathrm{e}^{-\mathrm{i}\mu K_3}\,\mathrm{e}^{-\mathrm{i}\zeta K_2}, \tag{8}$$

or

$$k_l(\xi, \nu, \zeta) = \mathrm{e}^{-\mathrm{i}\xi K_2}\,\mathrm{e}^{-\mathrm{i}\nu K_1}\,\mathrm{e}^{-\mathrm{i}l\pi K_3}\,\mathrm{e}^{-\mathrm{i}\zeta K_2}, \tag{9}$$

where $-\infty < \xi, \zeta, \nu < \infty$, $-2\pi \leqslant \mu \leqslant 2\pi$, $l = 0, 1, 2, 3$. Almost every $SU(1, 1)$ element has the second kind of decomposition except a set of exceptional elements with zero measure in $SU(1, 1)$ [34].

In addition, the following formula will be frequently used throughout this paper

$$\exp[-\mathrm{i}(xK_1 + yK_2 + zK_3)] = \begin{cases} \cos\frac{r}{2} I_2 - \mathrm{i}\frac{2}{r}\sin\frac{r}{2} \times (xK_1 + yK_2 + zK_3), & (\text{if } z^2 \geqslant x^2 + y^2), \\ \cosh\frac{r}{2} I_2 - \mathrm{i}\frac{2}{r}\sinh\frac{r}{2} \times (xK_1 + yK_2 + zK_3), & (\text{if } z^2 < x^2 + y^2), \end{cases} \tag{10}$$

where $r = \sqrt{|z^2 - x^2 - y^2|}$. The $SU(1,1)$ element $\mathrm{e}^{-\mathrm{i}(xK_1+yK_2+zK_3)}$ is called elliptical when $z^2 > x^2 + y^2$, hyperbolic when $z^2 < x^2 + y^2$, and parabolic when $z^2 = x^2 + y^2$.

3. Main results

This section contains the main algorithms to compute the structured decomposition of $SU(1,1)$ matrices for two representative cases: (1) $H_0 = K_3$, $H_I = K_2$ and (2) $H_0 = K_2$, $H_I = K_3$ in subsections 3.1 and 3.2, respectively. The influence of the amplitude bound of the control pulses on the time duration will be investigated. In subsection 3.3, we extend the algorithms to systems with more general Hamiltonians, which can also be taken as a sufficient condition that guarantees the existence of a desired decomposition.

3.1. The case $H_0 = K_3$, $H_I = K_2$

Given a target transformation parametrized in the Euler form (7), the factors like $\mathrm{e}^{-\mathrm{i}\theta K_3}$ can be realized as free evolutions of system (1). Thus, it suffices to decompose an arbitrary $SU(1,1)$ element by finding the following class of decomposition:

$$\mathrm{e}^{-\mathrm{i}\theta K_2} = \prod_{k=1}^{3} \mathrm{e}^{-\mathrm{i}t_k(K_3 + u_k K_2)} \tag{11}$$

for arbitrary nonzero θ, where $t_k \geqslant 0$, $|u_k| \leqslant C$, $k = 1, 2, 3$.

Proposition 1.

(1) If $C > 1$, then for any $-1 < u_1, u_3 < 1$ and $1 < |u_2| < C$, the decomposition (11) can be realized with time durations

$$\begin{cases} t_1 = \frac{2}{r_1}\left[\operatorname{arccot}\left(\frac{u_1 - u_2}{r_1}\coth\frac{\theta}{2} - \frac{1}{r_1}\sqrt{\triangle}\right) + m\pi\right], \\ t_2 = \frac{2}{r_2}\operatorname{arcoth}\left(\frac{1}{r_2}\sqrt{\triangle}\right), \\ t_3 = \frac{2}{r_3}\left[\operatorname{arccot}\left(\frac{u_3 - u_2}{r_3}\coth\frac{\theta}{2} - \frac{1}{r_3}\sqrt{\triangle}\right) + n\pi\right], \end{cases} \tag{12}$$

where $r_1 = \sqrt{1 - u_1^2}$, $r_2 = \sqrt{u_2^2 - 1}$, $r_3 = \sqrt{1 - u_3^2}$ and $\triangle = (u_2 - u_1)(u_2 - u_3)\left(\coth^2\frac{\theta}{2} - 1\right) + \left(u_2^2 - 1\right)$. The integers m and n are chosen so that t_1 and t_2 are positive, and their parities are identical when $(u_2 - u_1)\theta \geqslant 0$ and opposite when $(u_2 - u_1)\theta < 0$.

(2) If $C \leqslant 1$, then for any $-C < u_1, u_3 < C$ the decomposition (11) exists if

$$|\theta| \leqslant 2\max\left\{\operatorname{arcoth}\sqrt{\frac{1 + u_1u_3 - Cu_1 - Cu_3}{(C - u_1)(C - u_3)}},\ \operatorname{arcoth}\sqrt{\frac{1 + u_1u_3 + Cu_1 + Cu_3}{(C + u_1)(C + u_3)}}\right\}, \tag{13}$$

and u_2 satisfies

$$\begin{cases} (u_2-u_1)(u_2-u_3)>0, \\ \dfrac{1}{2\cosh^2\frac{\theta}{2}}\left|\sqrt{u_1^2+u_3^2-2u_1u_3\cosh\theta+\sinh^2\theta}+u_1+u_3\right| \leqslant |u_2| \leqslant C, \end{cases} \tag{14}$$

the corresponding time durations are given as

$$\begin{cases} t_1 = \dfrac{2}{r_1}\left[\operatorname{arccot}\left(\dfrac{u_1-u_2}{r_1}\coth\dfrac{\theta}{2} \pm \dfrac{1}{r_1}\sqrt{\triangle}\right)+m\pi\right], \\ t_2 = \dfrac{2}{r_2}\left[\operatorname{arccot}\left(\mp\dfrac{1}{r_2}\sqrt{\triangle}\right)+l\pi\right], \\ t_3 = \dfrac{2}{r_3}\left[\operatorname{arccot}\left(\dfrac{u_3-u_2}{r_3}\coth\dfrac{\theta}{2} \pm \dfrac{1}{r_3}\sqrt{\triangle}\right)+n\pi\right], \end{cases} \tag{15}$$

where $r_k=\sqrt{1-u_k^2}$, $(k=1,2,3)$*;* $\triangle=(u_2-u_1)(u_2-u_3)\left(\coth^2\frac{\theta}{2}-1\right)+\left(u_2^2-1\right)$. *The integers m, n and l are chosen so that t_1, t_2 and t_3 are positive, and the parities of m and n are identical when $(u_2-u_1)\theta \geqslant 0$ and opposite when $(u_2-u_1)\theta<0$.*

Proof. See appendix A. □

The above results provide a rather wide class of control laws for a special target evolution operator. For $C>1$, the target $\mathrm{e}^{-\mathrm{i}\theta K_2}$ can be constructed within three control pulses, of which the amplitudes of the first and the last terms are bounded by 1. Consequently, the two factors $\mathrm{e}^{-\mathrm{i}t_1(K_3+u_1K_2)}$ and $\mathrm{e}^{-\mathrm{i}t_3(K_3+u_3K_2)}$ are elliptical. They are periodic terms and thus can be used to keep t_1, t_2 and t_3 positive at the same time (see appendix A). In particular these two terms can be realized as pure free evolutions, i.e., $u_1=u_3=0$, resulting the time durations

$$\begin{cases} t_1 = 2\operatorname{arccot}\left(-u_2\coth\dfrac{\theta}{2}-\sqrt{u_2^2\coth^2\dfrac{\theta}{2}-1}\right)+2m\pi, \\ t_2 = \dfrac{2}{r_2}\operatorname{arcoth}\left(\dfrac{1}{r_2}\sqrt{u_2^2\coth^2\dfrac{\theta}{2}-1}\right), \\ t_3 = 2\operatorname{arccot}\left(-u_2\coth\dfrac{\theta}{2}-\sqrt{u_2^2\coth^2\dfrac{\theta}{2}-1}\right)+2n\pi. \end{cases} \tag{16}$$

Moreover, the formulae in (12) indicate that the control amplitude of the second pulse can take any value in $(1, C]$. The corresponding t_2 decreases as $|u_2|$ increases, implying a trade-off between the amplitude u_2 and the time duration t_2, while t_1 and t_3 increase (see figure 1 for an example). As a result, the total evolution time of the employed three pulses in our scheme is always nonzero even when the control pulses are unbounded because

$$\begin{aligned} \lim_{\substack{|u_2|\to\infty \\ |u_1|,|u_3|<1}} (t_1+t_2+t_3) &= \lim_{\substack{|u_2|\to\infty \\ |u_1|,|u_3|<1}} \frac{2}{r_1}\left[\operatorname{arccot}\left(\frac{u_1-u_2}{r_1}\coth\frac{\theta}{2}-\frac{1}{r_1}\sqrt{\triangle}\right)\right]+\frac{2}{r_1}m\pi \\ &\quad + \lim_{\substack{|u_2|\to\infty \\ |u_1|,|u_3|<1}} \frac{2}{r_3}\left[\operatorname{arccot}\left(\frac{u_3-u_2}{r_3}\coth\frac{\theta}{2}-\frac{1}{r_3}\sqrt{\triangle}\right)\right]+\frac{2}{r_3}n\pi \\ &= \frac{\pi}{4}\left\{\frac{2}{r_1}\left[4m+1-\operatorname{sign}((u_2-u_1)\theta)\right]+\frac{2}{r_3}[4n+1-\operatorname{sign}((u_2-u_3)\theta)]\right\} \\ &\geqslant \min\left[\left(\frac{3}{r_1}+\frac{1}{r_3}\right)\pi, \left(\frac{2}{r_1}+\frac{2}{r_3}\right)\pi, \left(\frac{1}{r_1}+\frac{3}{r_3}\right)\pi\right] > 0. \end{aligned}$$

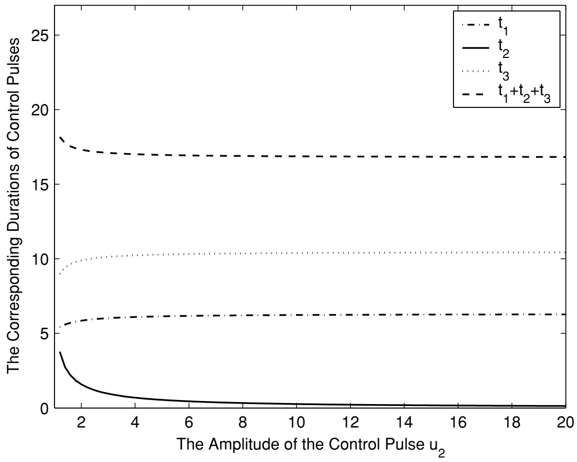

Figure 1. The durations of the control pulse for achieving $e^{-i2.5K_2}$ with respect to different u_2 according to the formulae in (12), where $u_1 = 0.1$, $u_3 = 0.8$, $m = n = 1$.

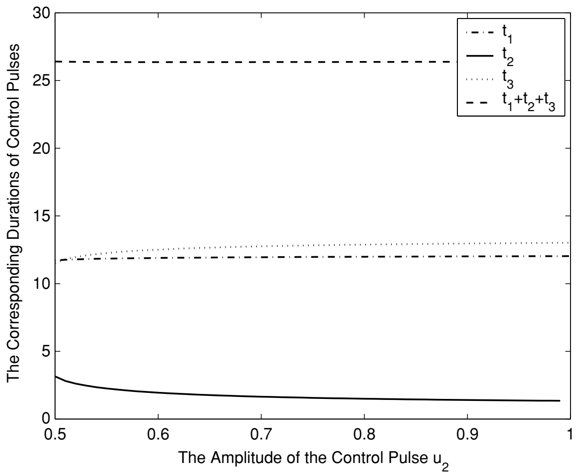

Figure 2. The durations of the control pulse for achieving $e^{-i0.2K_2}$ with respect to different u_2 according to formula (15), where $u_1 = 0$, $u_3 = 0.4$, $m = n = 1$, $l = 0$.

The case of $0 < C \leqslant 1$ is more complicated. However, one can still observe similar trade-off between the values of t_2 and u_2 when θ satisfies the conditions in proposition 1 (see figure 2 for an example). Particularly, $u_1 = u_3 = 0$ reduces equation (15) to

$$\begin{cases} t_1 = 2\,\mathrm{arccot}\left(-u_2\coth\dfrac{\theta}{2} \pm \sqrt{u_2^2\coth^2\dfrac{\theta}{2} - 1}\right) + 2m\pi, \\ t_2 = \dfrac{2}{r_2}\left[\mathrm{arccot}\left(\mp\dfrac{1}{r_2}\sqrt{u_2^2\coth^2\dfrac{\theta}{2} - 1}\right) + l\pi\right], \\ t_3 = 2\,\mathrm{arccot}\left(-u_2\coth\dfrac{\theta}{2} \pm \sqrt{u_2^2\coth^2\dfrac{\theta}{2} - 1}\right) + 2n\pi. \end{cases} \tag{17}$$

The range of the realizable operators within three pulses is largely limited when the bound $C < 1$. Nevertheless, one can find an appropriate integer N such that $\theta' = \frac{\theta}{N}$ satisfies (13) when θ is beyond the region defined in (13), and find a decomposition $\mathrm{e}^{-\mathrm{i}\frac{\theta}{N}K_2} = \prod_{k=1}^{3} \mathrm{e}^{-\mathrm{i}t_k(K_3+u_kK_2)}$ so that $\mathrm{e}^{-\mathrm{i}\theta K_2}$ can be realized with repeated sequences, i.e., $\mathrm{e}^{-\mathrm{i}\theta K_2} = \left[\prod_{k=1}^{3} \mathrm{e}^{-\mathrm{i}t_k(K_3+u_kK_2)}\right]^N$.

Explicit structured decomposition algorithms for an arbitrary target $g(\xi,\mu,\zeta)$ with a prescribed bound on the amplitude of the control pulses are summarized as follows:

Algorithm (a). The decomposing algorithm for $C > 1$

step 1. Select an appropriate control amplitude $|u_2| \in (1, C]$;
step 2. Determine the sign of u_2 by the constraint $u_2\mu \geqslant 0$. Set $u_1 = u_3 = 0$ and have $(m, n) = (1, 1)$. Use equation (16) to calculate t_1, t_2 and t_3;
step 3. Modulo the parameters $\xi + t_1$ and $\zeta + t_3$ by 2π so that they both fall in $[0, 2\pi)$;
step 4. The resulting factorization is $g(\xi,\mu,\zeta) = \mathrm{e}^{-\mathrm{i}(\xi+t_1)K_3}\,\mathrm{e}^{-\mathrm{i}t_2(K_3+u_2K_2)}\,\mathrm{e}^{-\mathrm{i}(\zeta+t_3)K_3}$.

Algorithm (b). The decomposing algorithm for $C \leqslant 1$

step 1. Let N be the minimum integer such that $N \geqslant \frac{|\eta|}{2\,\mathrm{artanh}\, C}$;
step 2. Select an appropriate u_2 such that $|u_2| \in \left[|\tanh\frac{\eta}{2N}|, C\right]$;
step 3. Determine the sign of u_2 and let $u_2\mu > 0$. Set $u_1 = u_3 = 0$ and have $(m, n) = (1, 1)$, then use equation (17) to calculate t_1, t_2 and t_3;
step 4. Modulo the parameters $\xi + t_1$, $\zeta + t_3$ and $t_1 + t_3$ by 2π so that they fall in $[0, 2\pi)$;
step 5. The resulting decomposition is $\mathrm{e}^{-\mathrm{i}(\xi+t_1)K_3}[\mathrm{e}^{-\mathrm{i}t_2(K_3+u_kK_2)}\,\mathrm{e}^{-\mathrm{i}(t_1+t_3)K_3}]^{N-1}\,\mathrm{e}^{-\mathrm{i}t_2(K_3+u_kK_2)} \times \mathrm{e}^{-\mathrm{i}(\zeta+t_3)K_3}$.

Example 1. The hyperbolic type transformation $\mathrm{e}^{-\mathrm{i}4K_2}$ can be realized with $H_0 = K_3$, $H_I = K_2$ and $C = 0.6$. One may choose $N = \left[\frac{4}{2\,\mathrm{artanh}(0.6)}\right] + 1 = 3$, $\theta = 4/N = 4/3$, $u_1 = u_3 = 0$, $u_2 = C = 0.6$, then compute from formula (17) that $t_1 = 4.4722$, $t_2 = 4.6696$, $t_3 = 4.4722$. Thus a possible decomposition for $\mathrm{e}^{-\mathrm{i}4K_2}$ is $\prod_{k=1}^{9} \mathrm{e}^{-\mathrm{i}t_k(K_3+u_kK_2)}$, with $t_1 = t_3 = t_4 = t_6 = t_7 = t_9 = 4.4722$, $t_2 = t_5 = t_8 = 4.6696$; $u_1 = u_3 = u_4 = u_6 = u_7 = u_9 = 0$, $u_2 = u_5 = u_8 = 0.6$.

3.2. The case $H_0 = K_2$, $H_I = K_3$

In this case, the factors that can be realized by the free evolution of the quantum system (1) are those generated by K_2. Based on the Euler parametrization (7), similarly, the key to construct a structured decomposition for an arbitrary $SU(1,1)$ element is to decompose $\mathrm{e}^{-\mathrm{i}\theta K_3}$ as

$$\mathrm{e}^{-\mathrm{i}\theta K_3} = \prod_{k=1}^{3} \mathrm{e}^{-\mathrm{i}t_k(K_2+u_kK_3)}, \tag{18}$$

where $t_k \geqslant 0$, $|u_k| \leqslant C$, $k = 1, 2, 3$.

According to [34], however, it is more convenient to use the Euler parametrization (8) and (9), where the generator K_2 is diagonalized, to compute the representation matrix (or propagator) elements under noncompact basis of the UIR's of $SU(1,1)$. Or in the language of quantum control theory, as the free Hamiltonian of the involved quantum control system here is the noncompact operator K_2, the Euler parametrization (8) and (9) will be more convenient to describe the population transition between different eigenstates of K_2. Thus algorithms are needed to find the structured decomposition

$$\mathrm{e}^{-\mathrm{i}\theta K_1} = \prod_{k=1}^{3} \mathrm{e}^{-\mathrm{i}t_k(K_2+u_kK_3)}, \tag{19}$$

where $t_k \geqslant 0$, $|u_k| \leqslant C$, $k = 1, 2, 3$.

Proposition 2. *For any $\theta \in (-\pi, \pi)$ and $C > 1$, the decomposition (18) can be realized with the time durations*

$$\begin{cases} t_1 = \dfrac{2}{r_1}\left[\operatorname{arcoth}\left(\dfrac{u_1-u_2}{r_1}\cot\dfrac{\theta}{2}+\dfrac{1}{r_1}\sqrt{\triangle}\right)\right], \\ t_2 = \dfrac{2}{r_2}\left[\operatorname{arccot}\left(-\dfrac{1}{r_2}\sqrt{\triangle}\right)+2l\pi\right], \\ t_3 = \dfrac{2}{r_3}\left[\operatorname{arcoth}\left(\dfrac{u_3-u_2}{r_3}\cot\dfrac{\theta}{2}+\dfrac{1}{r_3}\sqrt{\triangle}\right)\right], \end{cases} \tag{20}$$

where $(u_1, u_2, u_3) \in \Xi_+ \cap \Xi_0$,

$$\Xi_+ = \left\{(u_1, u_2, u_3)|\triangle \geqslant 0;\ \frac{u_k-u_2}{r_k}\cot\frac{\theta}{2}+\frac{1}{r_k}\sqrt{\triangle} \geqslant 1, k = 1, 3\right\},$$
$$\Xi_0 = \{(u_1, u_2, u_3)| -1 < u_1, u_3 < 1, 1 < |u_2| < C\},$$

$r_1 = \sqrt{1-u_1^2}$, $r_2 = \sqrt{u_2^2-1}$, $r_3 = \sqrt{1-u_3^2}$, $\triangle = (u_2-u_1)(u_2-u_3)\left(\cot^2\frac{\theta}{2}+1\right)+\left(1-u_2^2\right)$. *The integer l is chosen to keep t_2 positive.*

Proof. See appendix B. □

The result stated in proposition 2 also provides a rather wide class of control laws for achieving a given target. In such decomposition, only the second term $e^{-it_2(K_2+u_2K_3)}$ is elliptical, which can be used to adjust t_1, t_2 and t_3 to be positive. Similarly, setting the first and the second pulses u_1 and u_3 to zero will lead to the simplest expression of time durations

$$\begin{cases} t_1 = 2\operatorname{arcoth}\left(-u_2\cot\dfrac{\theta}{2}+\sqrt{1+u_2^2\cot^2\dfrac{\theta}{2}}\right), \\ t_2 = \dfrac{2}{r_2}\left[\operatorname{arccot}\left(-\dfrac{1}{r_2}\sqrt{1+u_2^2\cot^2\dfrac{\theta}{2}}\right)+2l\pi\right], \\ t_3 = 2\operatorname{arcoth}\left(-u_2\cot\dfrac{\theta}{2}+\sqrt{1+u_2^2\cot^2\dfrac{\theta}{2}}\right). \end{cases} \tag{21}$$

In comparison with the case when $H_0 = K_3$, $H_I = K_2$, the pulse durations t_1, t_2 and t_3 can be designed to be arbitrary short in the limit of unbounded controls (see figure 3 for an example), because one can verify from equation (20) that

$$\lim_{\substack{|u_2|\to\infty \\ |u_1|,|u_3|<1}} t_1 = \lim_{\substack{|u_2|\to\infty \\ |u_1|,|u_3|<1}} t_2 = \lim_{\substack{|u_2|\to\infty \\ |u_1|,|u_3|<1}} t_3 = 0. \tag{22}$$

It should be noted that all the three factors $e^{-it_1(K_2+u_1K_3)}$, $e^{-it_2(K_2+u_2K_3)}$ and $e^{-it_3(K_2+u_3K_3)}$ on the right-hand side of (18) can be designed to be simultaneously elliptical, i.e., $|u_1|, |u_2|, |u_3| \in (1, C]$, with the corresponding control durations being

$$\begin{cases} t_1 = \dfrac{2}{r_1}\left[\operatorname{arccot}\left(\dfrac{u_1-u_2}{r_1}\cot\dfrac{\theta}{2}\pm\dfrac{1}{r_1}\sqrt{\triangle}\right)+2m\pi\right], \\ t_2 = \dfrac{2}{r_2}\left[\operatorname{arccot}\left(\mp\dfrac{1}{r_2}\sqrt{\triangle}\right)+2l\pi\right], \\ t_3 = \dfrac{2}{r_3}\left[\operatorname{arccot}\left(\dfrac{u_3-u_2}{r_3}\cot\dfrac{\theta}{2}\pm\dfrac{1}{r_3}\sqrt{\triangle}\right)+2n\pi\right], \end{cases} \tag{23}$$

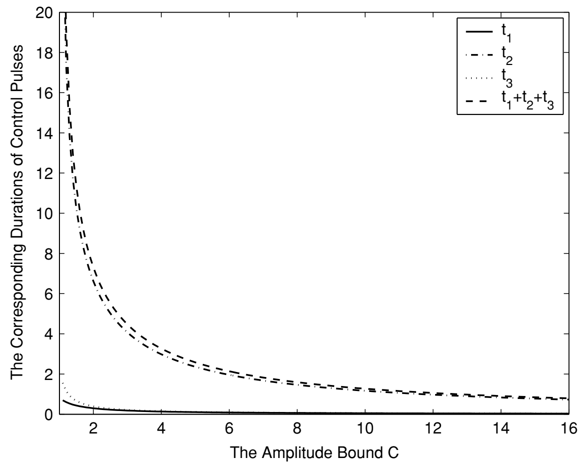

Figure 3. The durations of the control pulse for achieving $e^{i0.9K_3}$ with respect to different u_2 according to formula (20), where $u_1 = 0.1, u_3 = 0.9, l = 1$.

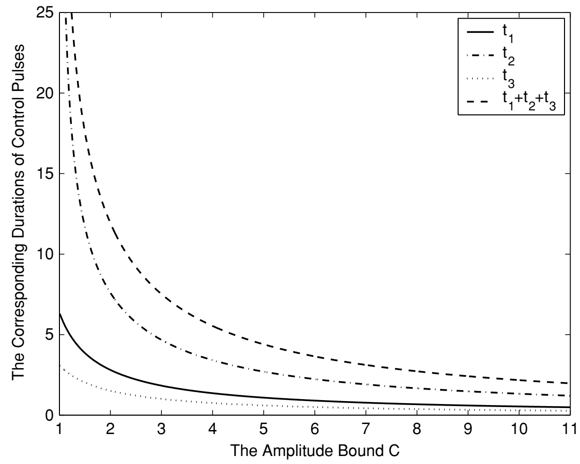

Figure 4. The durations of the control pulse for achieving $e^{-i1.5K_3}$ with respect to different u_2 according to formula (23), where $u_1 = 2u_2, u_3 = 5u_2, m = l = n = 1$.

where the integers m, n and l are introduced so that t_1, t_2 and t_3 are positive. The parities of m and n are identical when $(u_1 - u_2)\theta \geqslant 0$, and opposite when $(u_1 - u_2)\theta < 0$. From equation (23), we can obtain the following limitations for t_1, t_2 and t_3 as

$$\lim_{\substack{|u_2|\to\infty \\ |u_1|=O(|u_2|) \\ |u_3|=O(|u_2|)}} t_1 = \lim_{\substack{|u_2|\to\infty \\ |u_1|=O(|u_2|) \\ |u_3|=O(|u_2|)}} t_2 = \lim_{\substack{|u_2|\to\infty \\ |u_1|=O(|u_2|) \\ |u_3|=O(|u_2|)}} t_3 = 0. \tag{24}$$

The above equation indicates that the action time t_1, t_2 and t_3, can be designed arbitrary small as long as the upper bound of the control pulses is large enough (see figure 4 for an example).

Proposition 3. *For any $\theta \in \mathbb{R}$ and $C > 1$, the decomposition (19) can be realized with time durations*

$$
\begin{cases}
t_1 = \dfrac{1}{r_1}\left[\operatorname{arccot}\left(-\dfrac{\operatorname{sign}(\triangle_{13}/\triangle_{23})}{r_1}\sqrt{\dfrac{\triangle_{12}\triangle_{13}}{\triangle_{23}}}\right) + 2m\pi\right], \\
t_2 = \dfrac{1}{r_2}\operatorname{arcoth}\left[\dfrac{1}{r_2}\sqrt{\dfrac{\triangle_{12}\triangle_{23}}{\triangle_{13}}}\right], \\
t_3 = \dfrac{1}{r_3}\left\{\arctan\left[\dfrac{r_3(\triangle_{13}-\triangle_{23})\operatorname{sign}(\triangle_{23})}{(u_1-u_2)(u_3-\coth\theta)}\sqrt{\dfrac{\triangle_{12}}{\triangle_{13}\triangle_{23}}}\right] + 2n\pi\right\},
\end{cases}
\tag{25}
$$

where $r_1 = \sqrt{u_1^2-1}$, $r_2 = \sqrt{1-u_2^2}$, $r_3 = \sqrt{u_3^2-1}$, $\triangle_{12} = 1-u_1u_2+(u_1-u_2)\coth\theta$, $\triangle_{13} = 1-u_1u_3+(u_1-u_3)\coth\theta$, $\triangle_{23} = 1-u_2u_3+(u_2-u_3)\coth\theta$, $\Theta_0 = \{(u_1,u_2,u_3)|1 < |u_1|, |u_3| \leqslant C, |u_2| < 1\}$, $\Theta_+ = \left\{(u_1,u_2,u_3)\middle|\frac{\triangle_{12}\triangle_{13}}{\triangle_{23}} \geqslant 0, \frac{1}{r_2}\sqrt{\frac{\triangle_{12}\triangle_{23}}{\triangle_{13}}} \geqslant 1\right\}$, $(u_1,u_2,u_3) \in \Theta_0 \cap \Theta_+$, *and the integers m and n are chosen so that t_1 and t_3 are positive.*

Proof. The cumbersome proof is omitted since similar to that of the last two propositions. □

In the structured decomposition (19), the two terms $\mathrm{e}^{-\mathrm{i}t_1(K_2+u_1K_3)}$ and $\mathrm{e}^{-\mathrm{i}t_3(K_2+u_3K_3)}$ are elliptical, while the second hyperbolic term can be taken as a free evolution. Similarly, with the increase of the amplitudes of the control pulses, the corresponding durations will decrease and tend to zero.

The propositions 2 and 3 provide constructive algorithms for decomposing elements in $SU(1,1)$ in the forms of $g(\xi,\mu,\zeta)$, $h(\xi,\mu,\zeta)$ and $k_l(\xi,\nu,\zeta)(l=0,1,2,3)$ described in (7), (8) and (9), respectively, as follows.

Algorithm (c). The algorithm for $g(\xi,\eta,\zeta)$

step 1. Let $N = \left[\frac{|\xi|}{\pi}\right]+1$ and $M = \left[\frac{|\zeta|}{\pi}\right]+1$;

step 2. Let $|u_{21}| \in (1,C]$ and achieve the decomposition $\mathrm{e}^{-\mathrm{i}\frac{\xi}{N}K_3} = \mathrm{e}^{-\mathrm{i}t_{11}K_2}\,\mathrm{e}^{-\mathrm{i}t_{21}(K_2+u_{21}K_3)} \times \mathrm{e}^{-\mathrm{i}t_{31}K_2}$ according to (21);

step 3. Let $|u_{22}| \in (1,C]$, repeatedly, achieve the decomposition $\mathrm{e}^{-\mathrm{i}\frac{\zeta}{M}K_3} = \mathrm{e}^{-\mathrm{i}t_{12}K_2} \times \mathrm{e}^{-\mathrm{i}t_{22}(K_2+u_{22}K_3)}\,\mathrm{e}^{-\mathrm{i}t_{32}K_2}$;

step 4. The required decomposition is then

$$
g(\xi,\eta,\zeta) = [\mathrm{e}^{-\mathrm{i}t_{11}K_2}\,\mathrm{e}^{-\mathrm{i}t_{21}(K_2+u_{21}K_3)}\,\mathrm{e}^{-\mathrm{i}t_{31}K_2}]^N\,\mathrm{e}^{-\mathrm{i}\eta K_2}[\mathrm{e}^{-\mathrm{i}t_{12}K_2}\,\mathrm{e}^{-\mathrm{i}t_{22}(K_2+u_{22}K_3)}\,\mathrm{e}^{-\mathrm{i}t_{32}K_2}]^M. \tag{26}
$$

Algorithm (d). The algorithm for $h(\xi,\mu,\zeta)$

step 1. Let $p = \frac{1-\operatorname{sign}(\xi)}{2}$ and $q = \frac{1-\operatorname{sign}(\zeta)}{2}$;

step 2. Decompose $\mathrm{e}^{-\mathrm{i}\frac{\pi}{2}K_3}$ into $\mathrm{e}^{-\mathrm{i}t_{11}K_2}\,\mathrm{e}^{-\mathrm{i}t_{21}(K_2+u_{21}K_3)}\,\mathrm{e}^{-\mathrm{i}t_{31}K_2}$ by equation (21), where $|u_{21}| \in (1,C]$;

step 3. Repeatedly, obtain the decomposition of $\mathrm{e}^{-\mathrm{i}\mu K_3}$ as $[\mathrm{e}^{-\mathrm{i}t_{12}K_2}\,\mathrm{e}^{-\mathrm{i}t_{22}(K_2+u_{22}K_3)}\,\mathrm{e}^{-\mathrm{i}t_{32}K_2}]^N$ in the same way as that introduced in algorithm(c), where $|u_{23}| \in (1,C]$;

step 4. Note that $\mathrm{e}^{-\mathrm{i}3\pi K_3}\,\mathrm{e}^{-\mathrm{i}\xi K_2}\,\mathrm{e}^{-\mathrm{i}\pi K_3} = \mathrm{e}^{-\mathrm{i}(-\xi)K_2}$, thus we finally have

$$
\begin{aligned}
h(\xi,\mu,\zeta) = {} & [\mathrm{e}^{-\mathrm{i}t_{11}K_2}\,\mathrm{e}^{-\mathrm{i}t_{21}(K_2+u_{21}K_3)}\,\mathrm{e}^{-\mathrm{i}t_{31}K_2}]^{6p}\,\mathrm{e}^{-\mathrm{i}|\xi|K_2}[\mathrm{e}^{-\mathrm{i}t_{11}K_2}\,\mathrm{e}^{-\mathrm{i}t_{21}(K_2+u_{21}K_3)}\,\mathrm{e}^{-\mathrm{i}t_{31}K_2}]^{2p} \\
& \times [\mathrm{e}^{-\mathrm{i}t_{12}K_2}\,\mathrm{e}^{-\mathrm{i}t_{22}(K_2+u_{22}K_3)}\,\mathrm{e}^{-\mathrm{i}t_{32}K_2}]^N[\mathrm{e}^{-\mathrm{i}t_{11}K_2}\,\mathrm{e}^{-\mathrm{i}t_{21}(K_2+u_{21}K_3)}\,\mathrm{e}^{-\mathrm{i}t_{31}K_2}]^{6q} \\
& \times \mathrm{e}^{-\mathrm{i}|\zeta|K_2}[\mathrm{e}^{-\mathrm{i}t_{11}K_2}\,\mathrm{e}^{-\mathrm{i}t_{21}(K_2+u_{21}K_3)}\,\mathrm{e}^{-\mathrm{i}t_{31}K_2}]^{2q}.
\end{aligned}
$$

Algorithm (e). The algorithm for $k_l(\xi,\nu,\zeta)$

step 1. Calculate the decomposition of $\mathrm{e}^{-\mathrm{i}\frac{\pi}{2}K_3}$, $\mathrm{e}^{-\mathrm{i}\xi K_2}$ and $\mathrm{e}^{-\mathrm{i}\zeta K_2}$ according to algorithm (d);
step 2. Make use of equation (25) and decompose $\mathrm{e}^{-\mathrm{i}\nu K_1}$ into $\mathrm{e}^{-\mathrm{i}t_1(K_2+u_1K_3)}\,\mathrm{e}^{-\mathrm{i}t_2K_2}\,\mathrm{e}^{-\mathrm{i}t_3(K_2+u_3K_3)}$, where $|u_1|,|u_3|\in(1,C]$, $u_2=0$;
step 3. Obtain the decomposition of $k_l(\xi,\nu,\zeta)(l=0,1,2,3)$ as

$$\begin{aligned}k_l(\xi,\nu,\zeta)=&\left[\mathrm{e}^{-\mathrm{i}t_{11}K_2}\mathrm{e}^{-\mathrm{i}t_{21}(K_2+u_{21}K_3)}\mathrm{e}^{-\mathrm{i}t_{31}K_2}\right]^{6p}\mathrm{e}^{-\mathrm{i}|\xi|K_2}\left[\mathrm{e}^{-\mathrm{i}t_{11}K_2}\mathrm{e}^{-\mathrm{i}t_{21}(K_2+u_{21}K_3)}\mathrm{e}^{-\mathrm{i}t_{31}K_2}\right]^{2p}\\&\times\mathrm{e}^{-\mathrm{i}t_1(K_2+u_1K_3)}\mathrm{e}^{-\mathrm{i}t_2K_2}\mathrm{e}^{-\mathrm{i}t_3(K_2+u_3K_3)}\left[\mathrm{e}^{-\mathrm{i}t_{12}K_2}\mathrm{e}^{-\mathrm{i}t_{22}(K_2+u_{22}K_3)}\mathrm{e}^{-\mathrm{i}t_{32}K_2}\right]^{2l}\\&\times\left[\mathrm{e}^{-\mathrm{i}t_{11}K_2}\mathrm{e}^{-\mathrm{i}t_{21}(K_2+u_{21}K_3)}\mathrm{e}^{-\mathrm{i}t_{31}K_2}\right]^{6q}\mathrm{e}^{-\mathrm{i}|\zeta|K_2}\left[\mathrm{e}^{-\mathrm{i}t_{11}K_2}\mathrm{e}^{-\mathrm{i}t_{21}(K_2+u_{21}K_3)}\mathrm{e}^{-\mathrm{i}t_{31}K_2}\right]^{2q}.\end{aligned}$$

3.3. The general case

In this section, H_0 and H_I are allowed to be arbitrary linearly independent elements in the $su(1,1)$ Lie algebra, i.e., $H_0=a_1K_1+b_1K_2+c_1K_3$ and $H_I=a_2K_1+b_2K_2+c_2K_3$, where the two real vectors $[a_1,b_1,c_1]^T$ and $[a_2,b_2,c_2]^T$ are linearly independent.

Similarly, to accomplish the structured decomposition for $SU(1,1)$ elements, we only need to decompose $\mathrm{e}^{-\mathrm{i}\theta K_3}$ and $\mathrm{e}^{-\mathrm{i}\tau K_2}$. For simplicity, let

$$\begin{aligned}\Omega&=\left\{u\in\mathbb{R}\,\middle|\,\frac{(a_1+ua_2)^2+(b_1+ub_2)^2}{(c_1+uc_2)^2}<1,\ |u|\leqslant C\right\},\\\Theta(u_1,u_2)&=\frac{(a_1+u_2a_2)(b_1+u_1b_2)-(a_1+u_1a_2)(b_1+u_2b_2)}{(a_1+u_1a_2)(a_1+u_2a_2)+(b_1+u_1b_2)(b_1+u_2b_2)},\end{aligned}\tag{27}$$

and

$$\Upsilon=\{\sigma|\sigma=\arctan\Theta(u_1,u_2);\,u_1,u_2\in\Omega\}.$$

Obviously, there exists a closed interval $[\theta_{\min},\theta_{\max}]$ in Υ.

First, let us study the procedure for realizing the structured decomposition of $\mathrm{e}^{-\mathrm{i}\theta K_3}$. It is obvious that if $\theta\in[2\theta_{\min},2\theta_{\max}]$ there exist at least two different elements u_1 and u_2, which satisfy the equation

$$\Theta(u_1,u_2)=\tan\frac{\theta}{2},\tag{28}$$

in the defined set Ω above. Let u_{11} and u_{12} satisfy $\Theta(u_{11},u_{12})=\tan\frac{\theta}{2}$, and then with a few routine calculations we can obtain the following decomposition

$$\mathrm{e}^{-\mathrm{i}\theta K_3}=\prod_{k=1}^{2}\mathrm{e}^{-\mathrm{i}t_{1k}(H_0+u_{1k}H_I)},\tag{29}$$

where

$$\begin{cases}t_{11}=\dfrac{2}{r_1}\left[\arctan\dfrac{r_1\sin\frac{\theta}{2}\left(y_2\sin\frac{\theta}{2}-x_2\cos\frac{\theta}{2}\right)}{x_1z_2+y_2z_1\sin\frac{\theta}{2}\cos\frac{\theta}{2}-x_2z_1\cos^2\frac{\theta}{2}}+2m\pi\right],\\[2ex]t_{12}=\dfrac{2}{r_2}\left[\arctan\dfrac{x_1r_2\sin\frac{\theta}{2}}{(x_1z_2-x_2z_1)\cos\frac{\theta}{2}+y_2z_1\sin\frac{\theta}{2}}+2n\pi\right],\end{cases}\tag{30}$$

where $x_k=a_1+u_{1k}a_2$, $y_k=b_1+u_{1k}b_2$, $z_k=c_1+u_{1k}c_2$, $r_k=\sqrt{z_k^2-x_k^2-y_k^2}$, $k=1,2$. The integers m and n are chosen to keep t_{11} and t_{12} positive.

If $\theta\notin[2\theta_{\min},2\theta_{\max}]$ we can find two integers $N(>0)$ and M, which satisfy the inequality

$$2\theta_{\min}\leqslant\widetilde{\theta}=\frac{M\times2\pi+\theta}{N}\leqslant2\theta_{\max}.\tag{31}$$

Thus,

$$\exp(-\mathrm{i}\theta K_3) = \exp[-\mathrm{i}(M\times 2\pi + \theta)K_3] = \left[\exp\left(-\mathrm{i}\frac{M\times 2\pi + \theta}{N}K_3\right)\right]^N$$
$$= [\exp(-\mathrm{i}\tilde{\theta}K_3)]^N. \tag{32}$$

Consequently, it can be concluded that for arbitrary θ the factor $\mathrm{e}^{-\mathrm{i}\theta K_3}$ can be decomposed as

$$\mathrm{e}^{-\mathrm{i}\theta K_3} = \prod_{k=1}^{Q_1} \mathrm{e}^{-\mathrm{i}t_k^{(1)}(H_0 + u_k^{(1)}H_I)} \tag{33}$$

when the set Ω is nonempty, where $t_k^{(1)} \geqslant 0$, $\left|u_k^{(1)}\right| \leqslant C$, $k = 1, 2, \ldots, Q_1$.

Next, let us study the structured decomposition for $\mathrm{e}^{-\mathrm{i}\tau K_2}$. Let $u_{21}, u_{22}(\neq u_{21}), u_{23}(= u_{21}) \in \Omega$, $u'_{2k} = \frac{\sqrt{(a_1+u_{2k}a_2)^2+(b_1+u_{2k}b_2)^2}}{|c_1+u_{2k}c_2|} (k = 1, 2, 3)$ and make use of proposition 1, then we can decompose $\mathrm{e}^{-\mathrm{i}\tau K_2}$ for any $\tau \in [0, \infty)$ as

$$\exp(-\mathrm{i}\tau K_2) = \left\{\prod_{k=1}^{3} \exp[-\mathrm{i}t'_{2k}(K_3 + u'_{2k}K_2)]\right\}^{Q'}, \tag{34}$$

where $t'_{2k} \geqslant 0$, $k = 1, 2, 3$ and Q' is a positive integer. Therefore

$$\begin{aligned}
\exp(-\mathrm{i}\tau K_2) &= \left\{\prod_{k=1}^{3} \mathrm{e}^{-\mathrm{i}t'_{2k}(K_3+u'_{2k}K_2)}\right\}^{Q'} \\
&= \left\{\prod_{k=1}^{3} \mathrm{e}^{-\mathrm{i}\alpha_k K_3} \exp\left(-\mathrm{i}t'_{2k}\left[u'_{2k}\frac{(a_1+u_{2k}a_2)K_1 + (b_1+u_{2k}b_2)K_2}{\sqrt{(a_1+u_{2k}a_2)^2 + (b_1+u_{2k}b_2)^2}} + K_3\right]\right)\mathrm{e}^{-\mathrm{i}\gamma_k K_3}\right\}^{Q'} \\
&= \left\{\prod_{k=1}^{3} \mathrm{e}^{-\mathrm{i}(\alpha_k+\beta_k)K_3} \exp\left(-\mathrm{i}t'_{2k}\left[u'_{2k}\frac{(a_1+u_{2k}a_2)K_1 + (b_1+u_{2k}b_2)K_2}{\sqrt{(a_1+u_{2k}a_2)^2 + (b_1+u_{2k}b_2)^2}}\right.\right.\right. \\
&\quad \left.\left.\left. + \operatorname{sign}(c_1+u_{2k}c_2)K_3\right]\right)\mathrm{e}^{-\mathrm{i}(\gamma_k+\beta_k)K_3}\right\}^{Q'} \\
&= \left\{\prod_{k=1}^{3} \mathrm{e}^{-\mathrm{i}(\alpha_k+\beta_k)K_3} \exp\left(-\mathrm{i}\frac{t'_{2k}}{|c_1+u_{2k}c_2|}[(a_1K_1 + b_1K_2 + c_1K_3)\right.\right. \\
&\quad \left.\left. + u_{2k}(a_2K_1 + b_2K_2 + c_2K_3)]\right)\mathrm{e}^{-\mathrm{i}(\gamma_k+\beta_k)K_3}\right\}^{Q'} \\
&= \left\{\prod_{k=1}^{3} \mathrm{e}^{-\mathrm{i}(\alpha_k+\beta_k)K_3}\,\mathrm{e}^{-\mathrm{i}t_{2k}(H_0+u_{2k}H_I)}\,\mathrm{e}^{-\mathrm{i}(\gamma_k+\beta_k)K_3}\right\}^{Q'},
\end{aligned} \tag{35}$$

where

$$t_{2k} = \frac{t'_{2k}}{|c_1+u_{2k}c_2|} \geqslant 0, \tag{36}$$

$$\alpha_k = \arccos\frac{\operatorname{sign}(a_1+u_{2k}a_2)(b_1+u_{2k}b_2)}{\sqrt{(a_1+u_{2k}a_2)^2 + (b_1+u_{2k}b_2)^2}} + \frac{3}{2}[1 - \operatorname{sign}(a_1+u_{2k}a_2)]\pi, \tag{37}$$

$$\gamma_k = 4\pi - \alpha_k, \tag{38}$$

$$\beta_k = \begin{cases} 0 \\ (\text{if } c_1 + u_{2k}c_2 \geqslant 0), \\ \operatorname{sign}\left(w_k^2 - v_k^2\right) \arcsin \dfrac{-2w_k v_k}{w_k^2 + v_k^2} + \dfrac{1 - \operatorname{sign}\left(w_k^2 - v_k^2\right)}{2}\pi \\ (\text{if } c_1 + u_{2k}c_2 < 0), \end{cases} \tag{39}$$

$$w_k = \begin{cases} \cosh\left[t_{2k}\sqrt{(a_1 + u_{2k}a_2)^2 + (b_1 + u_{2k}b_2)^2 - (c_1 + u_{2k}c_2)^2}\right] \\ \quad (\text{if } (a_1 + u_{2k}a_2)^2 + (b_1 + u_{2k}b_2)^2 - (c_1 + u_{2k}c_2)^2 \geqslant 0), \\ \cos\left[t_{2k}\sqrt{(c_1 + u_{2k}c_2)^2 - (a_1 + u_{2k}a_2)^2 - (b_1 + u_{2k}b_2)^2}\right] \\ \quad (\text{if } (a_1 + u_{2k}a_2)^2 + (b_1 + u_{2k}b_2)^2 - (c_1 + u_{2k}c_2)^2 < 0), \end{cases} \tag{40}$$

$$v_k = \begin{cases} -\dfrac{c_1 + u_{2k}c_2}{\sqrt{(a_1 + u_{2k}a_2)^2 + (b_1 + u_{2k}b_2)^2 - (c_1 + u_{2k}c_2)^2}} \\ \quad \times \sinh\left(t_{2k}\sqrt{(a_1 + u_{2k}a_2)^2 + (b_1 + u_{2k}b_2)^2 - (c_1 + u_{2k}c_2)^2}\right) \\ \quad (\text{if } (a_1 + u_{2k}a_2)^2 + (b_1 + u_{2k}b_2)^2 - (c_1 + u_{2k}c_2)^2 \geqslant 0), \\ -\dfrac{c_1 + u_{2k}c_2}{\sqrt{(c_1 + u_{2k}c_2)^2 - (a_1 + u_{2k}a_2)^2 - (b_1 + u_{2k}b_2)^2}} \\ \quad \times \sin\left(t_{2k}\sqrt{(c_1 + u_{2k}c_2)^2 - (a_1 + u_{2k}a_2)^2 - (b_1 + u_{2k}b_2)^2}\right) \\ \quad (\text{if } (a_1 + u_{2k}a_2)^2 + (b_1 + u_{2k}b_2)^2 - (c_1 + u_{2k}c_2)^2 < 0). \end{cases} \tag{41}$$

From equations (33) and (35), we can draw the conclusion that for arbitrary τ the factor $\mathrm{e}^{-\mathrm{i}\tau K_2}$ can be decomposed as

$$\mathrm{e}^{-\mathrm{i}\tau K_2} = \prod_{k=1}^{Q_2} \mathrm{e}^{-\mathrm{i}t_k^{(2)}\left(H_0 + u_k^{(2)} H_I\right)} \tag{42}$$

when the set Ω is nonempty, where $t_k^{(2)} \geqslant 0$, $|u_k^{(2)}| \leqslant C$, $k = 1, 2, \ldots, Q_2$.

Finally, we have the following theorem:

Theorem 1. *For any given $g(\xi, \eta, \zeta) \in SU(1,1)$, the decomposition $g(\xi, \eta, \zeta) = \prod_{k=1}^{Q} \mathrm{e}^{-\mathrm{i}t_k(H_0 + u_k H_I)}$ with $t_k \geqslant 0$ and $|u_k| \leqslant C$ always exists if the set Ω in (27) is nonempty.*

In this theorem, the nonemptiness of set Ω assures that $H_0 + uH_I$ can be adjusted to be the generator of a compact one-parameter subgroup of $SU(1,1)$. It is evident that if the structured decomposition $g(\xi, \eta, \zeta) = \prod_{k=1}^{Q} \mathrm{e}^{-\mathrm{i}t_k(H_0 + u_k H_I)}$ is realizable for arbitrary (ξ, η, ζ), every element in $SU(1,1)$ will be attainable for the involved quantum system (1). The controls $u(t)$ that will send the system from the initial $U(0) = I$ to the terminal $U(t) = g(\xi, \eta, \zeta)$ are piecewise constant functions of time. Thus the nonemptiness of the set Ω is also a sufficient condition of system controllability.

The following is the summarized algorithm for the structured decomposition in the general case.

Algorithm (f). Decomposing algorithm for the general case

step 1. Check Ω for the given Hamiltonians H_0 and H_I and the prescribed control bound C. If the set is nonempty goto next step, otherwise the required structured decomposition is likely to be inexistent, thus stop the procedure;

step 2. Decompose $\mathrm{e}^{-\mathrm{i}\eta K_2}$ into (34) according to algorithm (a) or (b);

step 3. Based on the result obtained in step 2, make use of formulae (36)–(41) and obtain the decomposition for $\mathrm{e}^{-\mathrm{i}\eta K_2}$ provided in (35);

step 4. Prepare the compact terms $\mathrm{e}^{-\mathrm{i}\theta K_3}$ appeared in the above steps and in the parametrization formula $g(\xi, \eta, \zeta)$ as (33);

step 5. Unify the above steps and complete the structured decomposition for $g(\xi, \eta, \zeta)$.

4. Applications

In this section, we will show how the decomposition technique discussed above can be applied to control a realistic physical system evolving on $SU(1,1)$. Generally speaking, the relevant control objectives can be categorized as follows: (i) realize a transition between two different states in $\mathscr{D}^+(k)$, such as the basis states $|m,k\rangle$ or the superposition states $\sum_{m=0}^{\infty} c_m|m,k\rangle$ (where $\sum_{m=0}^{\infty}|c_m|^2 = 1$); (ii) maximize the expectation value $\langle\psi|\hat{F}|\psi\rangle$ of a selected observable $\hat{F}$, such as the expectation value of the free Hamiltonian H_0, which corresponds to the energy of the undergoing quantum system. In this section we will concentrate on the first case.

In order to realize a transition between two discrete basis $|m_1,k\rangle$ and $|m_2,k\rangle$, one may find a (nonunique) propagator $U_f = g(\xi,\eta,\zeta) = \mathrm{e}^{-\mathrm{i}\xi K_3}\,\mathrm{e}^{-\mathrm{i}\eta K_2}\,\mathrm{e}^{-\mathrm{i}\zeta K_3}$ that will realize a desired population transfer ratio $P^k_{m_1m_2}$. Since K_3 is diagonalized in the UIR $\mathscr{D}^+(k)$, the two compact operators $\mathrm{e}^{-\mathrm{i}\xi K_3}$ and $\mathrm{e}^{-\mathrm{i}\zeta K_3}$ only affect the phase between different discrete basis states, and will not cause any population transfer. Thus the relevant population transfer ratio $P^k_{m_1m_2}$ is completely determined by the term $\mathrm{e}^{-\mathrm{i}\eta K_2}$, and

$$P^k_{m_1m_2} = |\langle m_1,k|U_f|m_2,k\rangle|^2 = |\langle m_1,k|\,\mathrm{e}^{-\mathrm{i}\eta K_2}|m_2,k\rangle|^2 = \left[V^k_{m_1m_2}(\eta)\right]^2, \tag{43}$$

where $V^k_{m_1m_2}(\eta) = (-1)^{m_2-m_1}V^k_{m_2m_1}(\eta)$ [33, 35], and for $m_1 \geqslant m_2$

$$V^k_{m_1m_2}(\eta) = (-1)^{m_2-k}\frac{1}{(2k-1)!}\left[\frac{(m_1+k-1)!(m_2+k-1)!}{(m_1-k)!(m_2-k)!}\right]\left(\tanh\frac{1}{2}\eta\right)^{m_1+m_2} \times\left(\cosh\frac{1}{2}\eta\right)^{-2k} F\left(k-m_2;\,k-m_1;\,2k;\,-\frac{1}{\sinh^2\frac{1}{2}\eta}\right). \tag{44}$$

Once the terminal population transfer ratio $P^k_{m_1m_2}$ is provided, one can immediately compute the corresponding parameter η from (44) and then carry out the procedure to design the proper control field.

As an illustration, consider the following quantum system

$$\mathrm{i}\frac{\mathrm{d}\varphi(t)}{\mathrm{d}t} = [K_3 + u(t)K_2]\varphi(t), \tag{45}$$

where the prescribed bound of the control amplitude is assumed to be $C = 0.6$. Assume the initial of system (45) is $\varphi(0) = |2,2\rangle$, and the target population transfer ratio P^2_{24} to the terminal $|4,2\rangle$ is 0.2194. From the equation $V^2_{24}(\eta) = \sqrt{P^2_{24}}$, it can be immediately calculated out to be that $\eta = 1.3333$. Thus, making use of the result presented in example 1 the required control field is immediately determined.

It should be mentioned that not all the prescribed transfer ratios $P^k_{m_1m_2} \in [0,1]$ can be realized, because the equation $V^k_{m_1m_2}(\eta) = \sqrt{P^k_{m_1m_2}}$ may have no solution. The similar circumstance may occur in the case when the superposition states are involved. A complete transition between two different states $|\varphi_1\rangle$ and $|\varphi_2\rangle$ can be realized only if they are in the same orbit of $SU(1,1)$ group, i.e., there exists a $SU(1,1)$ transformation which sends $|\varphi_1\rangle$ to $|\varphi_2\rangle$.

However, the control laws can be found to realize the complete transition between two arbitrary $SU(1,1)$ coherent states (CS's). The $SU(1,1)$ CS, a special superposition state in $\mathscr{D}^+(k)$, plays an important role in the field of nonlinear optics as it provides an example of ideal squeezed vacuum state [18]. In the laboratory, the $SU(1,1)$ CS has been realized in many systems such as trapped ions [36–38], quantum electrodynamic cavities [39–41] and solids [42]. It will be exhibited that the desired propagators, evolving over the $SU(1,1)$ Lie group, can be achieved by piecewise constant external control pulses. These switching control laws can be easily designed based on the decomposition technique.

Following Perelomov [43], in the UIR $\mathscr{D}^{+}(k)$ of $SU(1,1)$, the $SU(1,1)$ CS's are defined as

$$\begin{aligned} |\xi,k\rangle &= D(\alpha)|0,k\rangle = \exp(\alpha K_+ - \alpha^* K_-)|0,k\rangle \\ &= (1-|\xi|^2)^k \sum_{m=0}^{\infty} \left[\frac{\Gamma(m+2k)}{m!\Gamma(2k)}\right]^{\frac{1}{2}} \xi^m |m,k\rangle, \end{aligned} \tag{46}$$

where $\alpha = -(\theta/2)\,\mathrm{e}^{-\mathrm{i}\varphi}, \xi = -\tanh(\theta/2)\,\mathrm{e}^{-\mathrm{i}\varphi}$, with the parameters φ and θ obeying $-\infty < \theta < \infty, 0 \leqslant \varphi \leqslant 2\pi$. Usually, the defined Perelomov $SU(1,1)$ CS's are governed by the quantum control system with the Hamiltonian [17]

$$H(t) = A(t)K_3 + f(t)K_+ + f^*(t)K_- + B(t). \tag{47}$$

Without loss of generality, it can be assumed that $B(t) = 0$, $A(t) = 2\varpi_0$ and $f(t) \in \mathbb{R}$ [18], by which equation (47) is reduced to

$$H(t) = 2\varpi_0 K_3 + 2f(t)K_1. \tag{48}$$

The Hamiltonian (48) is also a Foldy-like Hamiltonian used to depict a Bose–Einstein condensate system [20, 44], where $f(t)$ represents the coupling constant of interbosonic interactions.

To realize Perelomov $SU(1,1)$ CS's in a realistic quantum system, a special realization of the Lie algebra is required. In the frame work of bosonic operators, the Hamiltonian (47) can be used to describe the parametric down conversion process as well [45]. There are two different kinds of realizations which are familiar in the literatures [18, 23]. The first kind is the single-mode case, which is used to describe a degenerate parametric amplifier [18, 46]. In this case, the $su(1,1)$ generators are given by

$$K_+ = \tfrac{1}{2}(a^\dagger)^2, \qquad K_- = \tfrac{1}{2}(a)^2, \qquad K_3 = \tfrac{1}{4}(a^\dagger a + aa^\dagger). \tag{49}$$

The second kind of realization is the two-mode case, which describes the non-degenerate parametric amplifier [18, 47]. The corresponding $su(1,1)$ generators are

$$K_+ = a^\dagger b^\dagger, \qquad K_- = ab, \qquad K_3 = a^\dagger a + b^\dagger b + 1. \tag{50}$$

No matter which kind of realization is involved, however, will the structured decomposition be affected. To find a feasible control $u(t)$ that steers the system (48) from the vacuum state $|0,k\rangle$ to the target $|\xi,k\rangle$, we may rewrite the propagator $D(\alpha)$ defined in (46) as

$$\begin{aligned} D(\alpha) &= \exp(\alpha K_+ - \alpha^* K_-) \\ &= \exp\{-\mathrm{i}[-2\,\mathrm{Im}(\alpha)K_1 - 2\,\mathrm{Re}(\alpha)K_2]\} \\ &= \exp(\mathrm{i}t_1 K_3)\exp(-\mathrm{i}2|\alpha|K_2)\exp(\mathrm{i}t_3 K_3), \end{aligned} \tag{51}$$

where $t_1 = \arccos[\mathrm{sign}(\mathrm{Im}(\alpha))\mathrm{Re}(\alpha)/|\alpha|] + \frac{3}{2}[1 + \mathrm{sign}(\mathrm{Im}(\alpha))]\pi$, $t_3 = 4\pi - t_1$.

Suppose the prescribed bound of the external control pulses $C > \varpi_0$. From proposition 1, the factor $\exp(-\mathrm{i}2|\alpha|K_2)$ can be decomposed as

$$\exp(-\mathrm{i}2|\alpha|K_2) = \exp(-\mathrm{i}t_{21}K_3)\exp\{-\mathrm{i}t_{22}[2\varpi_0 K_3 + uK_2]\}\exp(-\mathrm{i}t_{23}K_3), \tag{52}$$

where

$$\begin{cases} t_{21} = 2\,\mathrm{arccot}\left[\dfrac{u}{2\varpi_0}\coth|\alpha| - \sqrt{\left(\dfrac{u}{2\varpi_0}\right)^2\coth^2|\alpha| - 1}\right] + 2n\pi, \\ t_{22} = \dfrac{1}{\varpi_0\sqrt{(u/2\varpi_0)^2-1}}\mathrm{arcoth}\sqrt{\dfrac{(u/2\varpi_0)^2\coth^2|\alpha|-1}{(u/2\varpi_0)^2-1}}, \\ t_{23} = 2\,\mathrm{arccot}\left[\dfrac{u}{2\varpi_0}\coth|\alpha| - \sqrt{\left(\dfrac{u}{2\varpi_0}\right)^2\coth^2|\alpha| - 1}\right] + 2m\pi, \end{cases} \tag{53}$$

where $2\varpi_0 < u < 2C$, and $n = 0, 1, 2, \ldots, \infty, m - n = 0, \pm 2, \pm 4, \ldots, \pm\infty$. The two integers m and n are chosen to assure that $t_{21}, t_{23} > 0$. On the other hand, it can be observed that $\exp\{-\mathrm{i}[2\varpi_0 K_3 + uK_2]\} = \exp(-\mathrm{i}5\pi/2K_3)\exp\{-\mathrm{i}[2\varpi_0 K_3 + uK_1]\} \times \exp(-\mathrm{i}3\pi/2K_3)$, thus we have

$$\begin{aligned} D(\alpha) &= \exp(\mathrm{i}t_1K_3)\exp(-\mathrm{i}t_{21}K_3)\exp(-\mathrm{i}5\pi/2K_3)\exp[-\mathrm{i}t_{22}(2\varpi_0K_3 + uK_1)] \\ &\quad \times \exp(-\mathrm{i}3\pi/2K_3)\exp(-\mathrm{i}t_{23}K_3)\exp(\mathrm{i}t_3K_3) \\ &= \exp[-\mathrm{i}(-t_1 + t_{21} + 5\pi/2)K_3]\exp[-\mathrm{i}t_{22}(2\varpi_0K_3 + uK_1)] \\ &\quad \times \exp[-\mathrm{i}(-t_3 + t_{23} + 3\pi/2)K_3]. \end{aligned} \tag{54}$$

For example, if u is selected to be $3\varpi_0$, we can immediately realize the propagator $D(-2\,\mathrm{e}^{-\mathrm{i}\frac{\pi}{6}})$ by (setting $\hbar = 1$)

$$\begin{aligned} D\big(-2\,\mathrm{e}^{-\mathrm{i}\frac{\pi}{6}}\big) &= \exp\left(-\mathrm{i}\frac{3.8398}{\varpi_0}\times 2\varpi_0K_3\right)\exp\left[-\mathrm{i}\frac{1.5386}{\varpi_0}(2\varpi_0K_3 + 3\varpi_0K_1)\right] \\ &\quad \times \exp\left(-\mathrm{i}\frac{1.7454}{\varpi_0}\times 2\varpi_0K_3\right). \end{aligned} \tag{55}$$

It indicates that the target Perelomov $SU(1,1)$ CS $|-\tanh(2)\,\mathrm{e}^{-\mathrm{i}\frac{\pi}{6}}, k\rangle$ can be achieved from the original state $|0, k\rangle$ by the following control pulses:

$$f(t) = \begin{cases} 0, 0 \leqslant t \leqslant \dfrac{1.7454}{\varpi_0}; \\ \dfrac{3}{2}\varpi_0, \dfrac{0.8727}{\varpi_0} < t \leqslant \dfrac{3.284}{\varpi_0}; \\ 0, \dfrac{1.642}{\varpi_0} < t \leqslant \dfrac{7.1239}{\varpi_0}. \end{cases} \tag{56}$$

Similarly, the structured decomposition also offers control laws for complete transitions between two arbitrary $SU(1,1)$ CS's.

5. Conclusions

This paper presented constructive algorithms to achieve the structured decomposition for an arbitrary $SU(1,1)$ matrix under the restrictions **O1** and **O2**. It is shown that any element in $SU(1,1)$ can be achieved in this way if the total Hamiltonian $H_0 + uH_I$ can be adjusted to be the generator of some compact subgroup of $SU(1,1)$. Recall that for quantum systems evolving on the $SU(2)$ Lie group, since the corresponding total Hamiltonians are always compact, the structured decomposition exists for arbitrary H_0 and H_I as long as they are linearly independent [2]. We believe that the structured decomposition method can be extended to controlling quantum systems with more complex noncompact symmetry groups such as the control of $SU(m, n)$ CS's [48, 49].

Acknowledgments

This research was supported in part by the National Natural Science Foundation of China under grant number 60433050 and 60274025. TJ Tarn would also like to acknowledge partial support from the U.S. Army Research Office under grant W911NF-04-1-0386. The authors would like to thank anonymous referees for their constructive suggestions.

Appendix A. The proof of proposition 1

(1) For the case of $|C| > 1$. Let $-1 < u_1, u_3 < 1$ and $1 < |u_2| < C$. Then from the matrix equation $\prod_{k=1}^{3} \mathrm{e}^{-\mathrm{i}t_k(K_3+u_kK_2)} = \mathrm{e}^{-\mathrm{i}\theta K_2}$ (or equivalently, $\mathrm{e}^{-\mathrm{i}t_1(K_3+u_1K_2)}\,\mathrm{e}^{-\mathrm{i}t_2(K_3+u_2K_2)} = \mathrm{e}^{-\mathrm{i}\theta K_2}\,\mathrm{e}^{\mathrm{i}t_3(K_3+u_3K_2)}$), we can get the following four equations (actually, only three of them are independent) by equating the entries of the matrices on both sides:

$$\cos\left(\frac{t_1r_1}{2}\right)\cosh\left(\frac{t_2r_2}{2}\right) - \frac{1-u_1u_2}{r_1r_2}\sin\left(\frac{t_1r_1}{2}\right)\sinh\left(\frac{t_2r_2}{2}\right) = \cos\left(\frac{t_3r_3}{2}\right)\cosh\frac{\theta}{2} - \frac{u_3}{r_3}\sin\left(\frac{t_3r_3}{2}\right)\sinh\frac{\theta}{2}, \tag{A.1}$$

$$\frac{1}{r_1}\sin\left(\frac{t_1r_1}{2}\right)\cosh\left(\frac{t_2r_2}{2}\right) + \frac{1}{r_2}\cos\left(\frac{t_1r_1}{2}\right)\sinh\left(\frac{t_2r_2}{2}\right) = -\frac{1}{r_3}\sin\left(\frac{t_3r_3}{2}\right)\cosh\frac{\theta}{2}, \tag{A.2}$$

$$\frac{u_1}{r_1}\sin\left(\frac{t_1r_1}{2}\right)\cosh\left(\frac{t_2r_2}{2}\right) + \frac{u_2}{r_2}\cos\left(\frac{t_1r_1}{2}\right)\sinh\left(\frac{t_2r_2}{2}\right) = -\frac{u_3}{r_3}\sin\left(\frac{t_3r_3}{2}\right)\cosh\frac{\theta}{2} + \cos\left(\frac{t_3r_3}{2}\right)\sinh\frac{\theta}{2}, \tag{A.3}$$

$$\frac{u_1-u_2}{r_1r_2}\sin\left(\frac{t_1r_1}{2}\right)\sinh\left(\frac{t_2r_2}{2}\right) = -\frac{1}{r_3}\sin\left(\frac{t_3r_3}{2}\right)\sinh\frac{\theta}{2}, \tag{A.4}$$

where $r_1 = \sqrt{1-u_1^2}, r_2 = \sqrt{u_2^2-1}, r_3 = \sqrt{1-u_3^2}$. Equations (A.1)–(A.4) can be further recast as

$$\begin{cases} \cot\left(\frac{t_1r_1}{2}\right) = \frac{u_1-u_2}{r_1}\coth\frac{\theta}{2} \pm \frac{1}{r_1}\sqrt{\triangle}, \\ \coth\left(\frac{t_2r_2}{2}\right) = \mp\frac{1}{r_2}\sqrt{\triangle}, \\ \cot\left(\frac{t_3r_3}{2}\right) = \frac{u_3-u_2}{r_3}\coth\frac{\theta}{2} \pm \frac{1}{r_3}\sqrt{\triangle}, \end{cases} \tag{A.5}$$

where $\triangle = (u_2-u_1)(u_2-u_3)\left(\coth^2\frac{\theta}{2}-1\right) + \left(u_2^2-1\right)$. As $-1 < u_1, u_3 < 1$ and $|u_2| > 1$, it can be concluded that $\triangle > 0$ and $\frac{1}{r_2}\sqrt{\triangle} > 1$. Therefore, equation (A.5) is solvable for

(t_1, t_2, t_3), and the corresponding positive solutions can be written as

$$\begin{cases} t_1 = \dfrac{2}{r_1}\left[\operatorname{arccot}\left(\dfrac{u_1-u_2}{r_1}\coth\dfrac{\theta}{2} - \dfrac{1}{r_1}\sqrt{\triangle}\right) + m\pi\right], \\ t_2 = \dfrac{2}{r_2}\operatorname{arcoth}\left(\dfrac{1}{r_2}\sqrt{\triangle}\right), \\ t_3 = \dfrac{2}{r_3}\left[\operatorname{arccot}\left(\dfrac{u_3-u_2}{r_3}\coth\dfrac{\theta}{2} - \dfrac{1}{r_3}\sqrt{\triangle}\right) + n\pi\right], \end{cases} \tag{A.6}$$

where the integers m and n are introduced to keep t_1 and t_3 positive, and they have an identical parity when $(u_2 - u_1)\theta \geqslant 0$, otherwise they have different parities.

(2) For the case of $C \leqslant 1$. Let $|u_1|, |u_3| < C$, and then from $\mathrm{e}^{-\mathrm{i}t_1(K_3+u_1K_2)}\,\mathrm{e}^{-\mathrm{i}t_2(K_3+u_2K_2)} = \mathrm{e}^{-\mathrm{i}\theta K_2}\,\mathrm{e}^{\mathrm{i}t_3(K_3+u_3K_2)}$ it can be deduced that

$$\cos\left(\frac{t_1r_1}{2}\right)\cosh\left(\frac{t_2r_2}{2}\right) - \frac{1-u_1u_2}{r_1r_2}\sin\left(\frac{t_1r_1}{2}\right)\sin\left(\frac{t_2r_2}{2}\right) = \cos(t_3r_3)\cosh\frac{\theta}{2} - \frac{u_3}{r_3}\sin\left(\frac{t_3r_3}{2}\right)\sinh\frac{\theta}{2}, \tag{A.7}$$

$$\frac{1}{r_1}\sin\left(\frac{t_1r_1}{2}\right)\cos\left(\frac{t_2r_2}{2}\right) + \frac{1}{r_2}\cos\left(\frac{t_1r_1}{2}\right)\sin\left(\frac{t_2r_2}{2}\right) = -\frac{1}{r_3}\sin\left(\frac{t_3r_3}{2}\right)\cosh\frac{\theta}{2}, \tag{A.8}$$

$$\frac{u_1}{r_1}\sin\left(\frac{t_1r_1}{2}\right)\cos\left(\frac{t_2r_2}{2}\right) + \frac{u_2}{r_2}\cos\left(\frac{t_1r_1}{2}\right)\sin\left(\frac{t_2r_2}{2}\right) = -\frac{u_3}{r_3}\sin\left(\frac{t_3r_3}{2}\right)\cosh\frac{\theta}{2} + \cos\left(\frac{t_3r_3}{2}\right)\sinh\frac{\theta}{2}, \tag{A.9}$$

$$\frac{u_1-u_2}{r_1r_2}\sin\left(\frac{t_1r_1}{2}\right)\sin\left(\frac{t_2r_2}{2}\right) = -\frac{1}{r_3}\sin\left(\frac{t_3r_3}{2}\right)\sinh\frac{\theta}{2}, \tag{A.10}$$

where $r_k = \sqrt{1-u_k^2}$, $(k = 1, 2, 3)$. With a few calculations, one can rewrite equations (A.7)–(A.10) as

$$\begin{cases} \cot\left(\dfrac{t_1r_1}{2}\right) = \dfrac{u_1-u_2}{r_1}\coth\dfrac{\theta}{2} \pm \dfrac{1}{r_1}\sqrt{\triangle}, \\ \cot\left(\dfrac{t_2r_2}{2}\right) = \mp\dfrac{1}{r_2}\sqrt{\triangle}, \\ \cot\left(\dfrac{t_3r_3}{2}\right) = \dfrac{u_3-u_2}{r_3}\coth\dfrac{\theta}{2} \pm \dfrac{1}{r_3}\sqrt{\triangle}, \end{cases} \tag{A.11}$$

where $\triangle = (u_2-u_1)(u_2-u_3)\left(\coth^2\frac{\theta}{2} - 1\right) + \left(u_2^2 - 1\right)$. It is easy to verify that $\triangle \geqslant 0$ iff

$$\begin{cases} (u_2-u_1)(u_2-u_3) > 0 \\ \coth^2\dfrac{\theta}{2} \geqslant \dfrac{1-u_1u_2+u_1u_3-u_2u_3}{(u_2-u_1)(u_2-u_3)}. \end{cases} \tag{A.12}$$

Thus if

$$|\theta| \leqslant 2\max\left[\operatorname{arcoth}\sqrt{\frac{1+u_1u_3-Cu_1-Cu_3}{(C-u_1)(C-u_3)}}, \operatorname{arcoth}\sqrt{\frac{1+u_1u_3+Cu_1+Cu_3}{(C+u_1)(C+u_3)}}\right], \tag{A.13}$$

and u_2 satisfies

$$\begin{cases} (u_2-u_1)(u_2-u_3)>0, \\ \dfrac{1}{2\cosh^2\dfrac{\theta}{2}}\left|\sqrt{u_1^2+u_3^2-2u_1u_3\cosh\theta+\sinh^2\theta}+u_1+u_3\right| \leqslant |u_2| \leqslant C, \end{cases} \tag{A.14}$$

from equations (A.11) one can immediately obtain the corresponding positive solutions for t_1, t_2 and t_3 as

$$\begin{cases} t_1 = \dfrac{2}{r_1}\left[\operatorname{arccot}\left(\dfrac{u_1-u_2}{r_1}\coth\dfrac{\theta}{2} \pm \dfrac{1}{r_1}\sqrt{\triangle}\right)+m\pi\right] \\ t_2 = \dfrac{2}{r_2}\operatorname{arccot}\left[\left(\mp\dfrac{1}{r_2}\sqrt{\triangle}\right)+l\pi\right] \\ t_3 = \dfrac{2}{r_3}\left[\operatorname{arccot}\left(\dfrac{u_3-u_2}{r_3}\coth\dfrac{\theta}{2} \pm \dfrac{1}{r_3}\sqrt{\triangle}\right)+n\pi\right] \end{cases} \tag{A.15}$$

where the integers m, n and l are introduced to keep t_1, t_2 and t_3 positive, and m and n have an identical parity when $(u_2-u_1)\theta \geqslant 0$, otherwise they have different parities. This completes the proof of proposition 1.

Appendix B. The proof of proposition 2

Let $-1 < u_1, u_3 < 1$ and $1 < |u_2| < C$, and (u_1, u_2, u_3) is constrained in the set $\Xi_+ = \{(u_1,u_2,u_3)|\triangle \geqslant 0; \frac{u_k-u_2}{r_k}\cot\frac{\theta}{2}+\frac{1}{r_k}\sqrt{\triangle} \geqslant 1, k=1,3\}$, where $\triangle = (u_2-u_1)(u_2-u_3)\left(\cot^2\frac{\theta}{2}+1\right)+\left(1-u_2^2\right)$. It is easy to verify that such (u_1,u_2,u_3) always exists when $C>1$. From the matrix equation $\mathrm{e}^{-\mathrm{i}t_1(K_2+u_1K_3)}\,\mathrm{e}^{-\mathrm{i}t_2(K_2+u_2K_3)} = \mathrm{e}^{-\mathrm{i}\theta K_3}\,\mathrm{e}^{\mathrm{i}t_3(K_2+u_3K_3)}$, we may obtain the following equations (actually, they are equivalent to three independent equations) by equating entries of the matrices on both sides:

$$\begin{aligned} &\cosh\left(\frac{t_1r_1}{2}\right)\cos\left(\frac{t_2r_2}{2}\right)+\frac{1-u_1u_2}{r_1r_2}\sinh\left(\frac{t_1r_1}{2}\right)\sin\left(\frac{t_2r_2}{2}\right) \\ &\qquad = \cosh\left(\frac{t_3r_3}{2}\right)\cos\frac{\theta}{2}+\frac{u_3}{r_3}\sinh\left(\frac{t_3r_3}{2}\right)\sin\frac{\theta}{2}, \end{aligned} \tag{B.1}$$

$$\begin{aligned} &\frac{u_1}{r_1}\sinh\left(\frac{t_1r_1}{2}\right)\cos\left(\frac{t_2r_2}{2}\right)+\frac{u_2}{r_2}\cosh\left(\frac{t_1r_1}{2}\right)\sin\left(\frac{t_2r_2}{2}\right) \\ &\qquad = \cosh\left(\frac{t_3r_3}{2}\right)\sin\frac{\theta}{2}-\frac{u_3}{r_3}\sinh\left(\frac{t_3r_3}{2}\right)\cos\frac{\theta}{2}, \end{aligned} \tag{B.2}$$

$$\begin{aligned} &\frac{1}{r_1}\sinh\left(\frac{t_1r_1}{2}\right)\cos\left(\frac{t_2r_2}{2}\right)+\frac{1}{r_2}\cosh\left(\frac{t_1r_1}{2}\right)\sin\left(\frac{t_2r_2}{2}\right) \\ &\qquad = -\frac{1}{r_3}\sinh\left(\frac{t_3r_3}{2}\right)\cos\frac{\theta}{2}, \end{aligned} \tag{B.3}$$

$$\frac{u_1-u_2}{r_1r_2}\sinh\left(\frac{t_1r_1}{2}\right)\sin\left(\frac{t_2r_2}{2}\right) = -\frac{1}{r_3}\sinh\left(\frac{t_3r_3}{2}\right)\sin\frac{\theta}{2}, \tag{B.4}$$

where $r_1 = \sqrt{1-u_1^2}$, $r_2 = \sqrt{u_2^2-1}$, $r_3 = \sqrt{1-u_3^2}$. With a few calculations, equations (B.1)–(B.4) can be simplified to

$$\coth\left(\frac{t_1r_1}{2}\right) = \frac{u_1-u_2}{r_1}\cot\frac{\theta}{2} \pm \frac{1}{r_1}\sqrt{\triangle}, \tag{B.5}$$

$$\cot\left(\frac{t_2 r_2}{2}\right) = \mp\frac{1}{r_2}\sqrt{\triangle}, \tag{B.6}$$

$$\coth\left(\frac{t_3 r_3}{2}\right) = \frac{u_3 - u_2}{r_3}\cot\frac{\theta}{2} \pm \frac{1}{r_3}\sqrt{\triangle}. \tag{B.7}$$

Therefore, the positive solutions for t_1, t_2 and t_3 in equations (B.5)–(B.7) are

$$\begin{cases} t_1 = \dfrac{2}{r_1}\left[\operatorname{arcoth}\left(\dfrac{u_1 - u_2}{r_1}\cot\dfrac{\theta}{2} + \dfrac{1}{r_1}\sqrt{\triangle}\right)\right], \\ t_2 = \dfrac{2}{r_2}\left[\operatorname{arccot}\left(-\dfrac{1}{r_2}\sqrt{\triangle}\right) + 2l\pi\right], \\ t_3 = \dfrac{2}{r_3}\left[\operatorname{arcoth}\left(\dfrac{u_3 - u_2}{r_3}\cot\dfrac{\theta}{2} + \dfrac{1}{r_3}\sqrt{\triangle}\right)\right]. \end{cases} \tag{B.8}$$

where $l = 0, 1, 2, \ldots, \infty$, which is introduced to keep t_2 positive. This completes the proof.

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