Chapter 1

Phase Space methods

The language of statistical moments and covariance matrices we introduced in the previous chapter offers a compact and efficient formalism to deal with Gaussian states. A much more general approach may be taken to describe any quantum state in a setting which is reminiscent of classical phase space. This is the framework of characteristic functions and quasi-probability distributions, which goes back to seminal work by Wigner on quantum corrections to classical statistical mechanics, and bloomed in the sixties with the rise of theoretical quantum optics and the emergence of a general unifying picture.

Conceptually, the phase space description of quantum states hinges on the completeness of the set of displacement operators, which we shall prove in the form a Fourier-Weyl relation between density matrices and characteristic functions. This relationship will constitute the bridge between phase space and Hilbert space descriptions, which will be useful in several applications to quantum technologies. It will be convenient to handle most of the proofs and mathematical arguments concerning characteristic functions and quasi-probability distributions on a single mode of the system. Because displacement operators of multimode systems are just tensor products of local displacement operators, the extension of the formalism to systems with many degrees of freedom will be straightforward. Nevertheless, we shall always take care of explicitly linking the single-mode formulae which will appear in this chapter to the general multimode description adopted in the previous one.

1.1 Coherent states

The coherent state $|\alpha\rangle$ is the eigenvector of the operator a with eigenvalue α. If one defines $\alpha = \frac{x+ip}{\sqrt{2}}$, $\mathbf{r} = (x,p)^\mathsf{T}$ and the operator $\hat{D}_\alpha$ as

$$\hat{D}_\alpha = \hat{D}_{-\mathbf{r}} = \mathrm{e}^{-i\mathbf{r}^\mathsf{T}\Omega\hat{\mathbf{r}}} = \mathrm{e}^{\alpha a^\dagger - \alpha^* a}, \tag{1.1}$$

convalpha

in keeping with the convention adopted in the previous chapter, one finds $\hat{D}_\alpha^\dagger a \hat{D}_\alpha = a + \alpha$. The eigenstate $|\alpha\rangle$ is then easily determined as

$$\hat{D}_\alpha |0\rangle \, . \tag{1.2}$$

alphadef

In fact, one has

$$a\hat{D}_\alpha|0\rangle = \hat{D}_\alpha \hat{D}_\alpha^\dagger a \hat{D}_\alpha |0\rangle = \hat{D}_\alpha (a+\alpha)|0\rangle = \alpha \hat{D}_\alpha |0\rangle \, . \tag{1.3}$$

The expression of $|\alpha\rangle$ in Fock basis is also readily established, and reads

$$|\alpha\rangle = \mathrm{e}^{-\frac{|\alpha|^2}{2}} \sum_{m=0}^{\infty} \frac{\alpha^m}{\sqrt{m!}} |m\rangle \, . \tag{1.4}$$

fockalpha

Problem. Prove Eq. (1.4).

Two further relationships we will use throughout the chapter are the expression of the CCR for Weyl operators with complex variables

$$\hat{D}_\alpha \hat{D}_\beta = \mathrm{e}^{\frac{1}{2}(\alpha\beta^* - \alpha^*\beta)} \hat{D}_{\alpha+\beta} \tag{1.5}$$

weyls

and the overlap between two coherent states:

$$\begin{aligned} \langle\beta|\alpha\rangle &= \langle 0|\hat{D}_{-\beta}\hat{D}_\alpha|0\rangle = \langle 0|\hat{D}_{\alpha-\beta}|0\rangle \, \mathrm{e}^{\frac{1}{2}(\alpha\beta^*-\alpha^*\beta)} \\ &= \langle 0|\alpha-\beta\rangle \, \mathrm{e}^{\frac{1}{2}(\alpha\beta^*-\alpha^*\beta)} \; = \; \mathrm{e}^{-\frac{1}{2}|\alpha-\beta|^2} \, \mathrm{e}^{\frac{1}{2}(\alpha\beta^*-\alpha^*\beta)} \, . \end{aligned} \tag{1.6}$$

alphalap

While not orthogonal, the coherent states form a complete set, in that the identity operator can be represented as a weighted integral of projectors on coherent states:

$$\frac{1}{\pi}\int_{\mathbb{C}} |\alpha\rangle\langle\alpha| \, \mathrm{d}^2\alpha = \mathbb{1} \, . \tag{1.7}$$

idalpha

This can be seen by using the Fock basis decomposition (1.4):

$$\begin{aligned} \frac{1}{\pi}\int_{\mathbb{C}} \mathrm{d}^2\alpha \, |\alpha\rangle\langle\alpha| &= \frac{1}{\pi}\sum_{m,n=0}^{\infty} \int_{\mathbb{C}} \mathrm{d}^2\alpha \, \frac{\alpha^m \alpha^{*n}}{\sqrt{m!n!}} \, \mathrm{e}^{-|\alpha|^2} |m\rangle\langle n| \\ &= \frac{1}{\pi}\sum_{m,n=0}^{\infty} \int_0^\infty \mathrm{d}\rho \int_0^{2\pi} \mathrm{d}\varphi \, \mathrm{e}^{i(m-n)\varphi} \, \frac{\mathrm{e}^{-\rho^2}\rho^{m+n+1}}{\sqrt{m!n!}} |m\rangle\langle n| \\ &= \sum_{m}^{\infty} 2\int_0^\infty \mathrm{d}\rho \, \frac{\mathrm{e}^{-\rho^2}\rho^{2m+1}}{m!} |m\rangle\langle m| = \sum_{m=0}^{\infty} |m\rangle\langle m| = \mathbb{1} \, , \end{aligned} \tag{1.8}$$

alphaid

where we used $\int_0^{2\pi} \mathrm{e}^{i(m-n)\varphi} \, \mathrm{d}\varphi = 2\pi\delta_{mn}$ and $\int_0^\infty \mathrm{e}^{-\rho^2}\rho^{2m+1} \, \mathrm{d}\rho = \frac{m!}{2}$.

Since they are not orthogonal and yet they resolve the identity as an integral of one-dimensional projectors, the coherent states form actually an *overcomplete* set, that is a set which is still complete after removal of any one element.

Eq. (1.8) implies that the trace of any trace-class operator $\hat{O}$ can be determined as an integral over coherent states:

$$\begin{aligned}\mathrm{Tr}\left[\hat{O}\right] &= \sum_{m=0}^{\infty}\langle m|\hat{O}|m\rangle = \frac{1}{\pi}\int_{\mathbb{C}}\mathrm{d}^2\alpha\sum_{m=0}^{\infty}\langle m|\alpha\rangle\langle\alpha|\hat{O}|m\rangle \\ &= \frac{1}{\pi}\int_{\mathbb{C}}\mathrm{d}^2\alpha\sum_{m=0}^{\infty}\langle\alpha|\hat{O}|m\rangle\langle m|\alpha\rangle = \frac{1}{\pi}\int_{\mathbb{C}}\mathrm{d}^2\alpha\,\langle\alpha|\hat{O}|\alpha\rangle\,. \end{aligned} \tag{1.9}$$

alphatr

In terms of the language we developed in the previous chapter, it is immediately clear from the definition (1.2) that a coherent state $|\alpha\rangle$ is a Gaussian state with covariance matrix $\boldsymbol{\sigma} = \mathbb{1}$ and first moments $x = \mathrm{Re}(\alpha)/\sqrt{2}$ and $p = \mathrm{Im}(\alpha)/\sqrt{2}$ (recall that $\hat{D}_\alpha = \hat{D}^\dagger_{\mathbf{r}}$ with the definitions above for x and p).

1.2 A Fourier Weyl relation

We shall now establish a direct connection between density matrices on $L^2(\mathbb{R}^n)$ and functions of $2n$ variables, which stems from the fact that the displacement operators $\hat{D}_{\mathbf{r}}$ form an orthogonal complete set on the space of operators on $L^2(\mathbb{R}^n)$ with respect to the Hilbert space scalar product. To this aim, we shall essentially follow the classical treatment due to Glauber. For simplicity, we will prove the statement explicitly for a single mode, adopting the complex single-mode notation $\hat{D}_\gamma$ for the Weyl operators. As anticipated above, the extension of such a result to the multimode case is straightforward.

First note that, due to the decomposition of the identity (1.8), any bounded operator $\hat{O}$, for which $\langle\alpha|\hat{O}|\beta\rangle$ may be decomposed as follows in term soy coherent states:

$$\hat{O} = \frac{1}{\pi^2}\int_{\mathbb{C}^2}\mathrm{d}\alpha\,\mathrm{d}\beta\,\langle\alpha|\hat{O}|\beta\rangle\,|\alpha\rangle\langle\beta|\,, \tag{1.10}$$

so that one just needs to prove that the operator $|\alpha\rangle\langle\beta|$ can be expanded in terms of displacement operators to extend the proof to all bounded operators. We intend to show that

$$|\alpha\rangle\langle\beta| = \frac{1}{\pi}\int_{\mathbb{C}}\mathrm{d}^2\gamma\,\mathrm{Tr}\left[|\alpha\rangle\langle\beta|\hat{D}_\gamma\right]\hat{D}^\dagger_\gamma\,. \tag{1.11}$$

fweyl

Eq. (1.11) is equivalent to

$$\begin{aligned}|0\rangle\langle 0| &= \frac{1}{\pi}\int_{\mathbb{C}}\mathrm{d}^2\gamma\,\mathrm{Tr}\left[|\alpha\rangle\langle\beta|\hat{D}_\gamma\right]\hat{D}_{-\alpha}\hat{D}_{-\gamma}\hat{D}_\beta \\ &= \frac{1}{\pi}\int_{\mathbb{C}}\mathrm{d}^2\gamma\,\mathrm{Tr}\left[|\alpha\rangle\langle\beta-\gamma|\right]\mathrm{e}^{\frac{1}{2}(\gamma\beta^*-\gamma^*\beta)}\hat{D}_{-\alpha}\hat{D}_{-\gamma}\hat{D}_\beta \\ &= \frac{1}{\pi}\int_{\mathbb{C}}\mathrm{d}^2\gamma\,\langle\beta-\gamma|\alpha\rangle\mathrm{e}^{\frac{1}{2}(\gamma\beta^*-\gamma^*\beta)}\hat{D}_{-\alpha}\hat{D}_{-\gamma}\hat{D}_\beta \\ &= \frac{1}{\pi}\int_{\mathbb{C}}\mathrm{d}^2\gamma\,\mathrm{e}^{-\frac{1}{2}|\beta-\alpha-\gamma|^2}\hat{D}_{\beta-\alpha-\gamma} = \frac{1}{\pi}\int_{\mathbb{C}}\mathrm{d}^2\gamma\,\mathrm{e}^{-\frac{1}{2}|\gamma|^2}\hat{D}_\gamma\,,\end{aligned}$$

where we used Eqs. (1.5) and (1.6). We are thus left with having to prove the following relationship:

$$|0\rangle\langle 0| = \frac{1}{\pi}\int_{\mathbb{C}} \mathrm{d}^2\gamma\, \mathrm{e}^{-\frac{1}{2}|\gamma|^2}\hat{D}_\gamma \ . \qquad (1.12) \quad \boxed{\texttt{fweyl0}}$$

In order to do that, let us apply the operator on the right hand side on the Fock basis vector $|m\rangle$:

$$\begin{aligned}
\frac{1}{\pi}\int_{\mathbb{C}} \mathrm{d}^2\gamma\, \mathrm{e}^{-\frac{1}{2}|\gamma|^2}\hat{D}_\gamma\, |m\rangle &= \frac{1}{\pi}\int_{\mathbb{C}} \mathrm{d}^2\gamma\, \mathrm{e}^{-\frac{1}{2}|\gamma|^2}\hat{D}_\gamma \frac{a^{\dagger m}}{\sqrt{m!}}\,|0\rangle \\
&= \frac{1}{\pi}\int_{\mathbb{C}} \mathrm{d}^2\gamma\, \mathrm{e}^{-\frac{1}{2}|\gamma|^2}\frac{(a^\dagger-\gamma^*)^m}{\sqrt{m!}}\,|\gamma\rangle \\
&= \frac{1}{\pi}\int_{\mathbb{C}} \mathrm{d}^2\gamma \sum_{n=0}^{\infty} \mathrm{e}^{-|\gamma|^2}\frac{(a^\dagger-\gamma^*)^m}{\sqrt{m!}}\frac{\gamma^n}{\sqrt{n!}}\,|n\rangle \\
&= \sum_{n=0}^{\infty}\sum_{j=0}^{m}\frac{1}{\pi}\int_{\mathbb{C}} \mathrm{d}^2\gamma\, \mathrm{e}^{-|\gamma|^2}\binom{m}{j}(-1)^j\frac{\gamma^{*j}\gamma^n}{\sqrt{m!n!}}a^{\dagger(m-j)}\,|n\rangle \\
&= \sum_{j=0}^{m}\binom{m}{j}(-1)^j|m\rangle = \delta_{m0}|0\rangle \ , \qquad (1.13) \quad \boxed{\texttt{prfweyl}}
\end{aligned}$$

where we inserted the following integral, already employed in (1.8):

$$\frac{1}{\pi}\int_{\mathbb{C}} \mathrm{d}^2\gamma\, \mathrm{e}^{-|\gamma|^2}\gamma^{*j}\gamma^n = n!\,\delta_{jn} \ . \qquad (1.14)$$

Eq. (1.13) is equivalent to (1.12) and hence to (1.11).

Given a state ϱ, we are thus led to define $\chi(\alpha) = \mathrm{Tr}[\hat{D}_\alpha\varrho]$ such that

$$\varrho = \frac{1}{\pi}\int_{\mathbb{C}} \mathrm{d}^2\alpha\, \chi(\alpha)\hat{D}_{-\alpha} \ . \qquad (1.15)$$

As we will see in the next section, the function $\chi(\alpha)$ is known as the symmetrically ordered characteristic function associated to the quantum state ϱ. Clearly, complete knowledge of $\chi(\alpha)$ provides one with complete information about ϱ.

For a system of n modes, in the notation of the previous chapter, the Fourier Weyl relation we just derived reads

$$\varrho = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^{2n}} \mathrm{d}^{2n}r\, \chi(\mathbf{r})\hat{D}_{\mathbf{r}} \ , \qquad (1.16) \quad \boxed{\texttt{multifw}}$$

with $\mathrm{d}^{2n}r = \mathrm{d}x_1\mathrm{d}p_1\ldots\mathrm{d}x_n\mathrm{d}p_n$ and

$$\chi(\mathbf{r}) = \mathrm{Tr}\left[\hat{D}_{-\mathbf{r}}\varrho\right] \ . \qquad (1.17) \quad \boxed{\texttt{multichi}}$$

Notice the change of sign due to the convention in defining $\hat{D}_\alpha$ set by Eq. (1.1). The additional factor $(1/2)^n$ is due to the change of measure associated with the change of variables $\alpha = \frac{x+ip}{\sqrt{2}}$.

Note also the orthogonality of the Weyl operators with respect to the Hilbert Schmidt norm:

$$\mathrm{Tr}\left[\hat{D}_\alpha \hat{D}_{-\beta}\right] = \pi\delta(\alpha-\beta)\ , \tag{1.18}$$

or, with $2n$ real variables,

$$\mathrm{Tr}\left[\hat{D}_{\mathbf{r}} \hat{D}_{-\mathbf{s}}\right] = (2\pi)^n \delta^{2n}(\mathbf{r}-\mathbf{s})\ . \tag{1.19}$$

multinorm

1.3 Characteristic functions and quasi-probability distributions

Given a quantum state ϱ of a single mode, the corresponding s-ordered characteristic function $\chi_s(\alpha)$ may be defined as

$$\chi_s(\alpha) = \mathrm{Tr}\left[\hat{D}_\alpha \varrho\right] \mathrm{e}^{\frac{s}{2}|\alpha|^2} \tag{1.20}$$

chip1

The significance of the exponential factor becomes manifest if one considers the evaluation of expectation values of ordered products of ladder operators. By recalling that $\hat{D}_\alpha = \mathrm{e}^{\alpha a^\dagger}\mathrm{e}^{-\alpha^* a}\mathrm{e}^{-\frac{1}{2}|\alpha|^2} = \mathrm{e}^{-\alpha^* a}\mathrm{e}^{-\alpha a^\dagger}\mathrm{e}^{+\frac{1}{2}|\alpha|^2}$ and inserting such expressions in the definition of the s-ordered characteristic function, one obtains

$$\langle a^{\dagger m} a^n\rangle_1 = \mathrm{Tr}\left[a^{\dagger m} a^n \varrho\right] = \left(\frac{\partial}{\partial\alpha}\right)^m \left(-\frac{\partial}{\partial\alpha^*}\right)^n \chi_1(\alpha)\Big|_{\alpha=0}\ , \tag{1.21}$$

$$\langle a^{\dagger m} a^n\rangle_0 = \left(\frac{\partial}{\partial\alpha}\right)^m \left(-\frac{\partial}{\partial\alpha^*}\right)^n \chi_0(\alpha)\Big|_{\alpha=0}\ , \tag{1.22}$$

0ord

$$\langle a^{\dagger m} a^n\rangle_{-1} = \mathrm{Tr}\left[a^n a^{\dagger m} \varrho\right] = \left(\frac{\partial}{\partial\alpha}\right)^m \left(-\frac{\partial}{\partial\alpha^*}\right)^n \chi_{-1}(\alpha)\Big|_{\alpha=0}\ . \tag{1.23}$$

Differentiating the s-ordered characteristic function in $\alpha = 0$ allows one to retrieve the expectation value of s-ordered products of creation and annihilation operators, with $s = 1$, $s = 0$ and $s = -1$ corresponding to normal, symmetric and anti-normal ordering respectively.[1]

Notice that the following relationship holds for all values of p:

$$\chi_s(0) = \mathrm{Tr}\left[\varrho\right] = 1\ . \tag{1.24}$$

By taking the complex Fourier transform of $\chi_s(\alpha)$ one may define the s-ordered "quasi-probability" distribution $W_s(\alpha)$:

$$W_s(\alpha) = \frac{1}{\pi^2}\int_{\mathbb{C}} \mathrm{d}^2\beta\, \mathrm{e}^{(\alpha\beta^* - \alpha^*\beta)}\chi_s(\beta)\ , \tag{1.25}$$

ws

[1] In Eq. (0ord 1.22), we have implicitly defined the symmetric ordering of a product of $a^{\dagger m}$ and a^n as the normalised sum of products of m $a^\dagger$ and n a in all possible orders. For instance,

$$\langle a^{\dagger 2} a\rangle_0 = \mathrm{Tr}\left[\frac{1}{3}(a^{\dagger 2}a + a^\dagger a a^\dagger + a a^{\dagger 2})\varrho\right]\ .$$

which is normalised:

$$\int_{\mathbb{C}} \mathrm{d}^2\alpha \, W_s(\alpha) = \chi_s(0) = 1 \tag{1.26}$$

wpnorm

and has the property that

$$\begin{aligned}
\int_{\mathbb{C}} \mathrm{d}^2\alpha \, \alpha^{*m}\alpha^n W_s(\alpha) &= \frac{1}{\pi^2}\int_{\mathbb{C}} \mathrm{d}^2\alpha \int_{\mathbb{C}} \mathrm{d}^2\beta \left(\left(-\frac{\partial}{\partial\beta}\right)^m \left(\frac{\partial}{\partial\beta^*}\right)^n \mathrm{e}^{(\alpha\beta^*-\alpha^*\beta)} \right) \chi_s(\beta) \\
&= \int_{\mathbb{C}} \mathrm{d}^2\beta \left(\left(-\frac{\partial}{\partial\beta}\right)^m \left(\frac{\partial}{\partial\beta^*}\right)^n \delta(\beta) \right) \chi_s(\beta) \\
&= \left(\frac{\partial}{\partial\beta}\right)^m \left(-\frac{\partial}{\partial\beta^*}\right)^n \chi_s(\beta)\Big|_{\beta=0} = \langle a^{\dagger m} a^n \rangle_s \,, \qquad (1.27)
\end{aligned}$$

wpmom

where we applied the representation of the derivative of the delta function given by the integral over α. Eqs. (1.26) and (1.27) justify the terminology 'quasi-probability' distributions, with the "quasi-" there to remind one that the quantity $W_s(\alpha)$ is in general not positive, and in fact it may not even be a proper function at all, as we shall see in what follows.

Although they allow for a simple, unified treatment, the quasi-probability distributions for different values of s emerged historically at different times in different contexts, responding to different demands.

For $s = 1$, the quantity $W_1(\alpha)$ is the celebrated Glauber-Sudarshan P-representation, and is commonly denoted with $P(\alpha)$. One has the remarkable property:

$$\varrho = \int_{\mathbb{C}} \mathrm{d}^2\alpha \, P(\alpha)|\alpha\rangle\langle\alpha| \,, \tag{1.28}$$

prep

which may be easily proven by applying the operator $\hat{D}_\alpha$ by similarity on both sides of Eq. (1.12), getting

$$|\alpha\rangle\langle\alpha| = \frac{1}{\pi}\int_{\mathbb{C}} \mathrm{d}^2\gamma \, \mathrm{e}^{-\frac{1}{2}|\gamma|^2} \mathrm{e}^{(\alpha\gamma^*-\alpha^*\gamma)} \hat{D}_{-\gamma} \,, \tag{1.29}$$

and then by inserting this expression in the right hand side of Eq. (1.28), to obtain

$$\begin{aligned}
\int_{\mathbb{C}} \mathrm{d}^2\alpha \, P(\alpha)|\alpha\rangle\langle\alpha| &= \frac{1}{\pi}\int_{\mathbb{C}} \mathrm{d}^2\alpha \int_{\mathbb{C}} \mathrm{d}^2\gamma \, P(\alpha)\mathrm{e}^{-\frac{1}{2}|\gamma|^2} \mathrm{e}^{(\alpha\gamma^*-\alpha^*\gamma)} \hat{D}_{-\gamma} \qquad (1.30) \\
&= \frac{1}{\pi}\int_{\mathbb{C}} \mathrm{d}^2\gamma \, \chi_1(\gamma)\mathrm{e}^{-\frac{1}{2}|\gamma|^2} \hat{D}_{-\gamma} = \frac{1}{\pi}\int_{\mathbb{C}} \mathrm{d}^2\gamma \, \chi_0(\gamma)\hat{D}_{-\gamma} = \varrho \,,
\end{aligned}$$

where we solved the integral over α through the inverse Fourier transform relation

$$\chi_1(\gamma) = \int_{\mathbb{C}} \mathrm{d}^2\alpha \mathrm{e}^{(\alpha\gamma^*-\alpha^*\gamma)} P(\alpha) \,, \tag{1.31}$$

and applied the Fourier-Weyl relation (1.11) in the last identity.

Due to their over-completeness, coherent states allow for a diagonal decomposition of any density matrix. However, this decomposition is given in terms of the quasi-probability $P(\alpha) = W_1(\alpha)$, which may not be a distribution but not a proper function. For instance, the P-representation of a coherent state $|\alpha\rangle\langle\alpha|$ is the delta function $\delta(\gamma - \alpha)$, while the P-representation of a number state may only be expressed in term of derivatives of the delta function.

States with a P-representation "not more singular" than a delta function, including coherent states, are often referred to as 'classical' states. Within the subset of Gaussian states, it is easy to see that such states are those for which the smallest eigenvalue of the covariance matrix $\boldsymbol{\sigma}$ is greater than or equal to 1. We shall not discuss this notion further but we will employ it later on for a remarkable consequence it has when studying the separability of Gaussian states.

Problem. Show that a Gaussian state is classical, according to the definition given above, if the smallest eigenvalue of its density matrix is greater than or equal to 1.

The Fourier-Weyl relation allows one to shed light on the nature of the anti-symmetrically ordered quasi-probability function $W_{-1}(\alpha)$ too. Such a function is commonly referred to as the 'Husimi' Q-function, and denoted as $Q(\alpha)$. One has

$$\begin{aligned}\frac{1}{\pi}\langle\alpha|\varrho|\alpha\rangle &= \frac{1}{\pi^2}\int_{\mathbb{C}} \mathrm{d}^2\beta\, \chi_0(\beta)\langle\alpha|\hat{D}_{-\beta}|\alpha\rangle = \frac{1}{\pi^2}\int_{\mathbb{C}} \mathrm{d}^2\beta\, \mathrm{e}^{\frac{1}{2}(\alpha\beta^*-\alpha^*\beta)}\chi_0(\beta)\langle\alpha|\alpha-\beta\rangle \\ &= \frac{1}{\pi^2}\int_{\mathbb{C}} \mathrm{d}^2\beta\, \mathrm{e}^{\alpha\beta^*-\alpha^*\beta}\chi_0(\beta)\mathrm{e}^{-\frac{|\beta|^2}{2}} = W_{-1}(\alpha) = Q(\alpha)\,. \qquad (1.32)\end{aligned}$$

qfunction

The Q-function is therefore always positive and does not diverge. On account of these properties, it found wide application in the study of quantum dynamical systems, whenever a well defined probability distribution is desirable. As we shall see, the value of the Q-function at a point α represents the probability that the heterodyne measurement of the system yields the outcome α (heterodyne measurements give complex outcomes).

As for the symmetrically ordered $W_0(\alpha)$, it was historically the first quasi-probability to be introduced, by Wigner, and goes under the name of Wigner function. In the following of the book, we shall denote it as $W(\alpha)$, omitting the subscript. Likewise, χ will represent the symmetrically ordered characteristic function χ_0. It is instructive to express the integral (ws 1.25), that defines the Wigner function, in terms of real variables through the identifications $\alpha = \frac{x+ip}{\sqrt{2}}$ and $\beta = \frac{x'+ip'}{\sqrt{2}}$, obtaining

$$\begin{aligned}W(x,p) &= \frac{1}{\pi^2}\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{\mathrm{d}x'\,\mathrm{d}p'}{2}\,\mathrm{e}^{i(px'-xp')}\chi_0(x',p') \\ &= \frac{1}{2\pi^2}\int_{\mathbb{R}}\int_{\mathbb{R}} \mathrm{d}x'\,\mathrm{d}p'\,\mathrm{e}^{i(px'-xp')}\int_{\mathbb{R}}\mathrm{d}q\,\langle q|\hat{D}_{-\frac{\mathbf{r}'}{2}}\varrho\hat{D}_{-\frac{\mathbf{r}'}{2}}|q\rangle \\ &= \frac{1}{2\pi^2}\int_{\mathbb{R}^3}\mathrm{d}q\,\mathrm{d}x'\,\mathrm{d}p'\,\mathrm{e}^{ipx'}\mathrm{e}^{ip'(q-x)}\langle q+\frac{x'}{2}|\varrho|q-\frac{x'}{2}\rangle \qquad (1.33) \\ &= \frac{1}{\pi}\int_{\mathbb{R}}\mathrm{d}x'\,\mathrm{e}^{ipx'}\langle x+\frac{x'}{2}|\varrho|x-\frac{x'}{2}\rangle = \frac{2}{\pi}\int_{\mathbb{R}}\mathrm{d}x'\,\mathrm{e}^{i2px'}\langle x+x'|\varrho|x-x'\rangle\,,\end{aligned}$$

wigpq

where we expressed the trace, as well as the final result, in terms of the improper quadrature eigenvalues $|q\rangle$, as per Eq. (??), and $\hat{D}_{-\frac{\mathbf{r}'}{2}} = \mathrm{e}^{-i\frac{x'}{2}\hat{p}}\mathrm{e}^{i\frac{p'}{2}\hat{p}}\mathrm{e}^{i\frac{p'x'}{8}} = \mathrm{e}^{i\frac{p'}{2}\hat{p}}\mathrm{e}^{-i\frac{x'}{2}\hat{p}}\mathrm{e}^{-i\frac{p'x'}{8}}$ (after ordering via Baker-Campbell-Hausdorff). We also applied the representation of the delta function (??).

Eq. (1.33) implies that

$$\frac{1}{2}\int_{-\infty}^{+\infty} \mathrm{d}p\, W(x,p) = \langle x|\varrho|x\rangle\ . \tag{1.34}$$

Up to a factor $\frac{1}{2}$, the integral of the Wigner function over a certain phase space quadrature, gives the probability of measuring the conjugate quadrature. This statement is clearly phase invariant, in the specific sense that, if one defines the generalised canonical operators $\hat{x}_\theta = \cos\theta\,\hat{x} - \sin\theta\,\hat{p}$ and associated canonical variables x_θ, one has

$$\frac{1}{2}\int_{-\infty}^{+\infty} \mathrm{d}x_{\theta-\frac{\pi}{2}}\, W(x,p) = \langle x_\theta|\varrho|x_\theta\rangle\ , \tag{1.35}$$

since $[\hat{x}_\theta, \hat{x}_{\theta-\frac{\pi}{2}}] = i$. The marginal of the Wigner function along any phase space direction is hence, up to a factor $\frac{1}{2}$, a positive probability distribution which describes the statistics of quadrature measurements. Such measurements, corresponding to position or momentum measurements for a material particle in the cases $\theta = 0, -\frac{\pi}{2}$, is implemented thorugh 'homodyne' detection in quantum optics.

1.4 Characteristic function of a Gaussian state

In order to link the phase space approach back to the material we introduced in the previous chapter, let us now determine the (symmetrically ordered) characteristic function χ_G of the most general Gaussian state, which was parametrised in Eq. (??) in terms of a covariance matrix $\boldsymbol{\sigma} = S\boldsymbol{\nu}S^{\mathsf{T}}$ (with normal form $\boldsymbol{\nu}$) and a vector of first moments which we will denote here with $\mathbf{r}'$. We will start off with the general case with any number of variables, as in Eq. (1.17), and then reduce it to single-mode problem, for which we will adopt more convenient complex variables.

By inserting Eq. (??) into Eq. (1.17) one gets

$$\begin{aligned}\chi_G(\mathbf{r})) &= \left(\prod_{j=1}^{n}\left(1-\mathrm{e}^{-\xi_j}\right)\right)\mathrm{Tr}\left[\left(\bigotimes_{j=1}^{n}\left(\sum_{m=0}^{\infty}\mathrm{e}^{-\xi_j m}|m\rangle_{jj}\langle m|\right)\right)\hat{S}\hat{D}_{\mathbf{r}'}D_{\mathbf{r}}\hat{D}^{\dagger}_{\mathbf{r}'}\hat{S}^{\dagger}\right]\\ &= \left(\prod_{j=1}^{n}\left(1-\mathrm{e}^{-\xi_j}\right)\right)\mathrm{Tr}\left[\left(\bigotimes_{j=1}^{n}\left(\sum_{m=0}^{\infty}\mathrm{e}^{-\xi_j m}|m\rangle_{jj}\langle m|\right)\right)D_{S^{-1}\mathbf{r}}\right]\mathrm{e}^{i\mathbf{r}'^{\mathsf{T}}\Omega\mathbf{r}},\end{aligned} \tag{1.36}$$

red1m

where we applied Eq. (??) and the projective representation of the symplectic group through the indentity

$$\hat{S}\hat{D}_{\mathbf{r}}\hat{S}^{\dagger} = \mathrm{e}^{i\mathbf{r}^{\mathsf{T}}\Omega S\hat{\mathbf{r}}} = \mathrm{e}^{i\mathbf{r}^{\mathsf{T}}S^{\mathsf{T}-1}\Omega\hat{\mathbf{r}}} = \hat{D}_{S^{-1}\mathbf{r}}\ . \tag{1.37}$$

Since the displacement operators are tensor products of local operators, the problem of determining χ_G has been reduced to finding the characteristic function of the single mode operator $\sum_{m=0}^{\infty}\mathrm{e}^{-\xi_j m}|m\rangle\langle m|$, where $|m\rangle$ is a Fock state. In fact, the characteristic function of a tensor product of operators is the product of their individual characteristic functions, so that the characteristic function of the Gaussian state may be determined, up to normalisation, by taking the product of such single-mode characteristic functions, multiplying it by the phase factor we determined above, and applying the symplectic transformation S^{-1} on the variable $\mathbf{r}$.

It is more expedient to evaluate the single-mode characteristic function of $\sum_{m=0}^{\infty}\mathrm{e}^{-\xi_j m}|m\rangle\langle m|$ by switching temporarily to complex variables:

$$\begin{aligned}
\mathrm{Tr}\left[\sum_{m=0}^{\infty}\mathrm{e}^{-\xi_j m}|m\rangle\langle m|\hat{D}_{\alpha}\right] &= \frac{1}{\pi}\int_{\mathbb{C}}\mathrm{d}^2\gamma\ \sum_{m=0}^{\infty}\mathrm{e}^{-\xi_j m}\langle\gamma|m\rangle\langle m|\hat{D}_{\alpha}|\gamma\rangle \\
&= \frac{1}{\pi}\int_{\mathbb{C}}\mathrm{d}^2\gamma\,\mathrm{e}^{-\frac{1}{2}(|\gamma|^2+|\alpha+\gamma|^2+\alpha^*\gamma-\alpha\gamma^*)}\sum_{m=0}^{\infty}\frac{\left(\gamma^*(\alpha+\gamma)\mathrm{e}^{-\xi_j}\right)^m}{m!} \\
&= \frac{\mathrm{e}^{-\frac{|\alpha|^2}{2}}}{\pi}\int_{\mathbb{C}}\mathrm{d}^2\gamma\,\mathrm{e}^{-|\gamma|^2\left(1-\mathrm{e}^{-\xi_j}\right)}\mathrm{e}^{\alpha\gamma^*\mathrm{e}^{-\xi_j}-\alpha^*\gamma}.
\end{aligned} \tag{1.38}$$

The latter is a Gaussian integral, which can be reduced to a particularly simple form by setting $\gamma = (x+iy)/\sqrt{1-\mathrm{e}^{-\xi_j}}$ and yields, upon use of Eq. (??)

$$\mathrm{Tr}\left[\sum_{m=0}^{\infty}\mathrm{e}^{-\xi_j m}|m\rangle\langle m|\hat{D}_{\alpha}\right] = \frac{\mathrm{e}^{-\frac{|\alpha|^2}{2}\left(\frac{1+\mathrm{e}^{-\xi_j}}{1-\mathrm{e}^{-\xi_j}}\right)}}{1-\mathrm{e}^{-\xi_j}} = \frac{\mathrm{e}^{-\frac{|\alpha|^2}{2}\nu_j}}{1-\mathrm{e}^{-\xi_j}}\ , \tag{1.39}$$

where we have inserted the symplectic eigenvalue of the covariance matrix $\boldsymbol{\sigma}$ according to Eq. (??). Combining Eqs. (1.36) and (1.39) leads to the following expression for the characteristic function of a multimode Gaussian state, after the substitution $\alpha = (x_j + ip_j)/\sqrt{2}$ for each different j:

$$\begin{aligned}
\chi_G(\mathbf{r}) &= \mathrm{e}^{-\frac{1}{4}\mathbf{r}^{\mathsf{T}}S^{-1\mathsf{T}}\left(\bigoplus_{j=1}^{n}\nu_j\mathbb{1}_2\right)S^{-1}\mathbf{r}}\mathrm{e}^{i\mathbf{r}^{\mathsf{T}}\Omega^{\mathsf{T}}\mathbf{r}'} = \mathrm{e}^{-\frac{1}{4}\mathbf{r}^{\mathsf{T}}S^{-1\mathsf{T}}\left(\bigoplus_{j=1}^{n}\nu_j\mathbb{1}_2\right)S^{-1}\mathbf{r}}\mathrm{e}^{i\mathbf{r}^{\mathsf{T}}\Omega^{\mathsf{T}}\mathbf{r}'} \\
&= \mathrm{e}^{-\frac{1}{4}\mathbf{r}^{\mathsf{T}}S^{-1\mathsf{T}}\Omega^{\mathsf{T}}\left(\bigoplus_{j=1}^{n}\nu_j\mathbb{1}_2\right)\Omega S^{-1}\mathbf{r}}\mathrm{e}^{i\mathbf{r}^{\mathsf{T}}\Omega^{\mathsf{T}}\mathbf{r}'} = \mathrm{e}^{-\frac{1}{4}\mathbf{r}^{\mathsf{T}}\Omega^{\mathsf{T}}S\left(\bigoplus_{j=1}^{n}\nu_j\mathbb{1}_2\right)S^{\mathsf{T}}\Omega\mathbf{r}}\mathrm{e}^{i\mathbf{r}^{\mathsf{T}}\Omega^{\mathsf{T}}\mathbf{r}'} \\
&= \mathrm{e}^{-\frac{1}{4}\mathbf{r}^{\mathsf{T}}\Omega^{\mathsf{T}}\boldsymbol{\sigma}\Omega\mathbf{r}}\mathrm{e}^{i\mathbf{r}^{\mathsf{T}}\Omega^{\mathsf{T}}\mathbf{r}'}\ ,
\end{aligned} \tag{1.40}$$

where we exploited the invariance of $\left(\bigoplus_{j=1}^{n}\nu_j\mathbb{1}_2\right)$ under the action of Ω by congruence (the latter represents a product of local rotations on 2×2 subspaces, and thus preserves local identity matrices), and also repeatedly employed $S\Omega S^{\mathsf{T}} = \Omega$.

Summarising, we have found that the characteristic function χ_G of a Gaussian state with covariance matrix $\boldsymbol{\sigma}$ and vector of first moments $\mathbf{r}'$ is given by:

$$\chi_G(\mathbf{r}) = \mathrm{e}^{-\frac{1}{4}\mathbf{r}^\mathsf{T}\Omega^\mathsf{T}\boldsymbol{\sigma}\Omega\mathbf{r}}\,\mathrm{e}^{i\mathbf{r}^\mathsf{T}\Omega^\mathsf{T}\mathbf{r}'}\,. \tag{1.41}$$

gauchar

Notice how the vector variable $\mathbf{r}$ always enters this expression after multiplication by the symplectic form Ω.

We are thus led to yet another general characterisation of Gaussian states, as the quantum states with a Gaussian characteristic function. The Wigner function W_G of a Gaussian state, obtained by taking the complex Fourier transform of Eq. (gauchar 1.41), is promptly evaluated by applying (gauint **??**), obtaining

$$\begin{aligned} W_G(\mathbf{r}) &= \frac{1}{2^n\pi^{2n}}\int_{\mathbb{R}^{2n}} \mathrm{d}^{2n}\mathbf{r}''\,\mathrm{e}^{-\frac{1}{4}\mathbf{r}''^\mathsf{T}\Omega^\mathsf{T}\boldsymbol{\sigma}\Omega\mathbf{r}''}\,\mathrm{e}^{i\mathbf{r}''^\mathsf{T}\Omega^\mathsf{T}(\mathbf{r}'-\mathbf{r})} \\ &= \frac{2^n}{\pi^n\sqrt{\mathrm{Det}\,\boldsymbol{\sigma}}}\,\mathrm{e}^{-(\mathbf{r}-\mathbf{r}')^\mathsf{T}\boldsymbol{\sigma}^{-1}(\mathbf{r}-\mathbf{r}')}\,. \end{aligned} \tag{1.42}$$

gauwig

The latter is, with respect to the measure $\frac{\mathrm{d}^{2n}\mathbf{r}}{2^n}$, a Gaussian probability distribution centred in $\mathbf{r}'$ and with covariances described by $\boldsymbol{\sigma}$.

As we saw above, the marginal Wigner function along any phase space direction describes the probability distribution of the quantum measurement of the associated quadrature operator. Hence, as far as Gaussian states are concerned, the Wigner function provides one with a local, 'realistic' model to describe quadrature measurements. *If one restricts to quadrature measurements*, such systems may be mimicked by multivariate classical Gaussian distributions and will never show any signature of quantum non locality, such as a violation of Bell or CHSH inequalities. Clearly, quantum Gaussian states admit, besides the phase space description akin to classical distributions, an underlying Hilbert space description: general quantum measurements on the Hilbert space do allow for quantum non locality to become manifest with Gaussian states. It is however clear from the analogy with classical distributions that, whilst Gaussian states are entirely described by first and second moments of their Wigner distributions, any analysis regarding genuinely quantum features, such as quantum entanglement, requires one to look beyond the phase space formalism.