Systems & Control Letters 5 (1984) 187–190
North-Holland

December 1984

Controllability of right invariant systems on real simple Lie groups

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Received 9 April 1984

We exhibit some classes of Lie groups, and a set of open assumptions on these groups, such that, under these assumptions, the 'controllability rank condition' becomes a necessary and sufficient condition for controllability of right invariant systems.

AMS Subject Classification: Primary 93B05, 17B20; Secondary 93C10, 22E46.

Keywords: Simple real Lie groups, Controllability, Invariant vector fields.

1. Introduction

We deal with controllability of families of right invariant vector fields on a Lie group G, of the form

$$\Gamma = \{A + uB \mid u \in \mathbb{R}\}. \tag{1}$$

What we mean by controllability is 'strong controllability', that is to say:

- The *sub-semigroup* of G generated by the elements $\exp tX$, $X \in \Gamma$, $t \geq 0$, is all of G.

Our purpose is to improve on the results of [3] for some classes of simple groups, and to generalize the results obtained in [2] when $G = \mathrm{Sl}(n, R)$.

We say that Γ satisfies the *controllability rank condition* when $\mathscr{L}(A, B)$, the Lie algebra generated by A and B, is equal to L, the Lie algebra of G.

For a family of vector fields of the form (1), we say that a set of assumptions is open if the set of couples (A, B) for which they are satisfied is an open subset of $L \times L$.

We will exhibit some classes of *Lie groups*, and a set of *open assumptions* on these groups, such that the *controllability rank condition* becomes, under these assumptions, a *necessary and sufficient condition* for controllability.

In Section 2 we state a few definitions, and the main result. In Section 3 we give the proof of the result.

As suggested by the referee, in our appendix, we explain some of the Lie-theoretic concepts we are using in our paper.

2. Definitions, statement of the result

Let L be a real simple Lie algebra, which is the real *normal* form (in the sense of [5], Th. 1.1.1.9, p. 6, for example) of a complex simple Lie algebra. (Any complex simple Lie algebra has a real normal form, which is unique up to an isomorphism.)

An element B in L is said to be *real, strongly regular* iff:

- *B is regular*: B is semi-simple, and $\mathrm{Ker}(\mathrm{ad}\, B)$ has minimal dimension.

- All the nonzero eigenvalues of ad B are *real*, and the corresponding eigenspaces are *one-dimensional*.

The set of real, strongly regular elements in a real *normal* form of a complex simple Lie algebra is *open, nonempty*.

Denote by $\mathrm{Sp}(B)$ the set of non-zero eigenvalues of ad B, and by $L(\alpha)$ the eigenspace

$\mathrm{Ker}(\mathrm{ad}\, B - \alpha I)$, $\alpha \in \mathrm{Sp}(B)$,

$L(0) = \mathrm{Ker}(\mathrm{ad}\ B)$.

One has

$$L = L(0) \oplus \sum_{\alpha \in \mathrm{Sp}(B)}^{\oplus} L(\alpha).$$

Denote by $A(\alpha)$ the component of $A \in L$ on $L(\alpha)$ in the preceding decomposition:

$$A = A(0) + \sum_{\alpha \in \mathrm{Sp}(B)} A(\alpha).$$

Our result is the following:

Theorem 1. *Let G be a real connected Lie group, with* ***finite center***, *whose Lie algebra L is the real* ***normal*** *form of a complex simple Lie algebra L_c, of one of the following types*:

A_r, D_r, E_6, E_7, E_8.

Let Γ be a subset of L of the form (1) *such that*:

(1) *B is real, strongly regular.*

(2) *With $s = \sup\{a \mid a \in \mathrm{Sp}(B)\}$,*

$\mathrm{trace}(\mathrm{ad}\ A_{(s)} \circ \mathrm{ad}\ A_{(-s)}) < 0$.

Then, the controllability rank condition is a necessary and sufficient condition for the controllability of Γ.

3. Proof of Theorem 1

Let L be a real normal form of a complex simple Lie algebra L_c, $\Gamma \subset L$, let $\mathrm{LS}(\Gamma)$ be the *Lie-saturated* cone of Γ, as defined in [2] (Definition 6, p. 20).

Lemma 1. *Assume that $\pm B \in \mathrm{LS}(\Gamma)$, for B real, strongly regular. For any $A \in L$ such that $\pm A \in \mathrm{LS}(\Gamma)$, then $L(\alpha) \subset \mathrm{LS}(\Gamma)$, $\forall \alpha \in \mathrm{Sp}(B)$ such that $A(\alpha) \neq 0$.*

Proof. Let $s = \sup\{\alpha \mid A(\alpha) \neq 0\}$. From [1], Prop. 5.4, p. 164, it follows that

$$X_s(v) = \pm \exp(v\ \mathrm{ad}\ B)(A)/e^{vs} \in \mathrm{LS}(\Gamma), \quad \forall v \in R.$$

Since $\mathrm{LS}(\Gamma)$ is closed ([3], Prop. 5.2, p. 164),

$$\lim_{v \to +\infty} X_s(v) \in \mathrm{LS}(\Gamma),$$

and $\pm A(s) \in \mathrm{LS}(\Gamma)$.

Then one considers $\hat{A} = A - A(s)$, and iterates the procedure to get any $L(\alpha)$, $\alpha > 0$, such that $A(\alpha) \neq 0$. The same applies for $\alpha < 0$, beginning with $-s = \inf\{\alpha \mid A(\alpha) \neq 0\}$. □

Lemma 2. *Assume that L_c has one of the types A_r, D_r, E_6, E_7, E_8. Then, $\forall \alpha, \beta \in \mathrm{Sp}(B)$, the α-string of β (see* [4,3]) *contains at most the values* -1, 0, $+1$.

Proof. If $\alpha = a(B)$, $\beta = b(B)$, a, b roots of $L_c(0)$, the α-string of β is the set of all integers l such that $b + la$ is a root of $L_c(0)$. An inspection of the Dynkin diagrams shows that for A_r, D_r, E_6, E_7, E_8, and only for these, the primitive roots have the same length. Any root being conjugate to a primitive root, all the roots have the same length. The absolute value of l in the α-string of β is bounded by

$$\max\left(\frac{\|b\|^2}{\|a\|^2}, \frac{\|a\|^2}{\|b\|^2}\right).$$

Hence it is at most 1 in our case. (See Th. 1.1.1.1, p. 3 and §1.1.2, pp. 9–10 of [5] for example.) □

For the other simple Lie algebras (B_r, C_r, F_4, G_2), it is easy to check that there exist strings, between primitive roots, such that $l = 2$ or $l = 3$.

For all the properties of the roots used above one may consult [4,5] for example (pp. 9–10 of [5], pp. 38–49 of [4]).

Let Γ be of the form (1), satisfying the assumptions of Theorem 1. Denote by $H(\alpha)$ the space $[L(\alpha), L(-\alpha)]$, $\alpha \in \mathrm{Sp}(B)$.

Let S be the linear space generated by

(a) the $L(\alpha)$ such that $L(\alpha) \subset \mathrm{LS}(\Gamma)$,

(b) the $H(\alpha)$ such that $L(\alpha)$ and $L(-\alpha) \subset \mathrm{LS}(\Gamma)$.

From the general theory developed in [3], it is obvious that S is a Lie subalgebra of L.

Claim. *S is an ideal of $\mathscr{L}(A, B)$, the Lie algebra generated by A and B.*

$\mathscr{L}(A, B)$ is generated by S,

$$S_1 = \sum^{\oplus} \{L(\beta) \mid A(\beta) \neq 0,\ L(\beta) \not\subset S\},$$

and $L_0 \subset L(0)$, with $L_0 = \mathrm{span}\{A(0), B\}$.

(a) *As $B \in L_0$, $A \in S + S_1 + L_0$, then $\mathscr{L}(A, B)$ is included in the Lie algebra generated by S, S_1 and L_0.*

(b) *From Lemma* 1 (*and from* [3], *Prop*. 5.3, *p*. 164), *the Lie algebra generated by* S, S_1 *and* L_0 *is contained in* $\mathrm{LS}(\{\pm A, \pm B\}) = \mathscr{L}(A, B)$.

Proof. Trivially, $[S, L(0)] \subset S$, so, to prove the claim, it is sufficient to prove

$$[S, S_1] \subset S. \tag{2}$$

This is implied by

$$\begin{gathered}[L(\alpha), L(\beta)] \subset S,\\ \forall L(\alpha) \subset S, \forall L(\beta) \not\subset S \text{ and } A(\beta) \neq 0\end{gathered} \tag{3}$$

(since the $L(\alpha) \subset S$ generate S).
Proof of (3). Assume that $[L(\alpha), L(\beta)] \neq 0$. Consider

$$\Phi(v, X) = \exp(v \operatorname{ad} X)(A)$$

for $X \in L(\alpha)$, $\forall v \in \mathbb{R}$. One has $\Phi(v, X) \in \mathrm{LS}(\Gamma)$, but

$$\begin{aligned}\Phi(v, X) &= \sum_i v^i \frac{\operatorname{ad}^i X}{i!} A\\ &= A + \lambda X + \sum_{\substack{i \geq 1\\ \gamma \in \mathrm{Sp}(B)}} \frac{v^i}{i!} \operatorname{ad}^i X[A(\gamma)]\end{aligned}$$

for real λ. From Lemma 2,

$$\operatorname{ad}^i X[A(\gamma)] = 0 \text{ or } \lambda X \quad \text{for } i \geq 2.$$

Then

$$\pm \sum_{\gamma \in \mathrm{Sp}(B)} [X, A(\gamma)] \in \mathrm{LS}(\Gamma).$$

With Lemma 1, this proves (3), and (2). The claim is proved. □

Necessity of Theorem 1 is obvious. For the sufficiency, suppose that $\mathscr{L}(A, B) = L$. Since L is simple, it follows that $S = L$ or $\{0\}$ from the claim above; from assumption (2) in the statement of the theorem, S cannot be $\{0\}$: It contains at least $L(s)$, $s = \sup\{a \mid a \in \mathrm{Sp}(B)\}$ (this has been proved in [3], Lemma 8, p. 177). So, $S = L$, and $\mathrm{LS}(\Gamma) = L$, Γ is controllable.

4. Appendix

Let L be a semi-simple real Lie algebra. The killing form $B: L \times L \to \mathbb{R}$ is the symmetric bilinear form

$$B(X, Y) = \operatorname{trace}(\operatorname{ad} X \circ \operatorname{ad} Y).$$

B is non-degenerate.

Example. In the case of $\mathrm{sl}(n)$, $B(X, Y) = \operatorname{trace}(XY)$, the trace of the product matrix XY.

4.1. *Cartan decomposition.* Let us denote by K (resp. P) the set of all $Z \in L$ such that ad Z is skew symmetric (resp. symmetric) with respect to B:

$$B(\operatorname{ad} Z(X), Y) + B(X, \operatorname{ad} Z(Y)) = 0,$$

$$B(\operatorname{ad} Z(X), Y) - B(X, \operatorname{ad} Z(Y)) = 0.$$

It is clear that $K \cap P = 0$, and it can be shown that $L = K \oplus P$.

It is easy to see that K is a *subalgebra* of L and $[K, P] \subset P$, $[P, P] \subset K$.

Example. $\mathrm{sl}(n)$: ad $X \in K$ (resp. P) if the matrix X is skew symmetric (resp. symmetric).

4.2. A Cartan space $a \in P$ is a maximal commutative subalgebra of L contained in P.

The elements of a are simultaneously diagonalisable.

All Cartan spaces are conjugated under the action of the subgroup of $\mathrm{Int}(L)$ generated by K.

4.3. L is called a *normal* real form, or a *split* real form if the normalizer of a in L is a itself.

An equivalent definition is: in the complexification $L_c = L \otimes_{\mathbb{R}} \mathbb{C}$, $a \otimes_{\mathbb{R}} \mathbb{C}$ is a Cartan subalgebra.

4.4. Given a normal real form L and a Cartan space a of L, there exists a finite set R of nonzero real linear forms on L such that:

(a) $-R = R$,

(b) $L = a \oplus \sum^{\oplus}_{\phi \in R} L(\phi)$ ($\Sigma^{\oplus}$ denotes direct sum), where $\dim_{\mathbb{R}} L(\phi) = 1$ and $\forall X \in a$

$$\operatorname{ad} X | L(\phi) = \phi(X) \cdot \text{Identity}.$$

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