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SOME COMMENTS ON POSITIVE ORTHANT CONTROLLABILITY OF BILINEAR SYSTEMS*

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Abstract. Consider the bilinear system $\dot{x}=(A+uB)x$, $x\in R^n$, and u unrestricted. The system has the property that any solution $x(t)$ with $x(0)\geqq 0$ (i.e., all components of x nonnegative) will remain in the positive orthant, $R_+^n=\{x\in R^n|x\geqq 0\}$ for $0\leqq t<\infty$ if and only if B is diagonal and $A=(a_{ij})$ has the property that $a_{ij}\geqq 0$ if $i\neq j$. In this note the controllability of solutions from a point of R_+^n to another such point is studied. Some results are given for arbitrary $n>0$ and detailed results are presented for the case $n=2$.

1. Introduction. In what follows we consider bilinear systems

$$(*)\qquad \dot{x}=(A+uB)x,$$

with $x\in R^n$ and A, B real $n\times n$ matrices with u denoting an admissible control function, which here will mean a piecewise constant or piecewise continuous function on $[0,\infty)$ into R.

We will let $x\geqq 0$, $(x>0)$, $A\geqq 0$, $(A>0)$, etc., mean that the given vector or matrix has only nonnegative components (resp. only positive components). The set $R_+^n=\{x\in R^n|x\geqq 0\}$ is called the *positive orthant*; its interior is $\mathring{R}_+^n=\{x\in R^n|x>0\}$. An $n\times n$ matrix A which satisfies $A\geqq 0$ clearly has the property $A(R_+^n)\subset R_+^n$, but what is more important for us is the following proposition, which is well known and is easy to verify.

PROPOSITION 1.1. *Let $A=(a_{ij})$ be an $n\times n$ matrix such that $a_{ij}\geqq 0$ when $i\neq j$, and let $x(t)$ be a solution of $\dot{x}=Ax$. If $x(0)\in R_+^n$, then for $t\geqq 0$ $x(t)\in R_+^n$. Conversely, if some off-diagonal element is negative, there exists a solution which leaves the positive orthant.*

We will find it convenient to call such a matrix A *essentially nonnegative*, or, if the inequalities are strict, *essentially positive*. It is well known that $e^{At}\geqq 0$ (resp. >0) for all $t\geqq 0$ if and only if A is essentially nonnegative (resp. positive), see, for example, Bellman [1, p. 176].

We are interested in those bilinear systems $(*)$ which have the property that no matter how the controls are chosen, any solution $x(t)$ which lies in R_+^n at $t=0$ will remain in R_+^n for all $t\geqq 0$. If the controls are unrestricted, e.g., u can take on negative values, then it is clear that a necessary and sufficient condition is as follows.

(1.2). *For unrestricted controls a necessary and sufficient condition for the statement:*

$$x(0)\geqq 0 \textit{ implies } x(t)\geqq 0 \textit{ for } t\geqq 0$$

to hold is that A be essentially nonnegative and B be a diagonal matrix.

There are many examples of such systems; several are given in the next section. The question we wish to investigate is the following: given a system $(*)$ with A essentially nonnegative and B diagonal and given $x_0\neq 0$, $x_1>0$ in R_+^n, can we choose controls so that there is a solution of $(*)$ with $x(0)=x_0$ and $x(T)=x_1$ for some $T>0$? If so, we call this property *positive orthant controllability*. We shall henceforth restrict ourselves to the subclass of systems $(*)$ for which this question is meaningful, i.e., A essentially nonnegative and B diagonal. In fact, for the most part we consider only

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the generic case, an open subset of these (A, B), for which A is essentially positive, B is nonsingular and $b_i - b_j$ are distinct for $i \neq j$. This latter is a natural condition on B in terms of Lie theory—it is a requirement on ad(B). It also appears in the Jurdjevic–Kupka study of bilinear systems [5].

2. An example. Given the facts that (1) a very large class of systems are locally approximable by bilinear systems (Krener [7]) (which are in fact the simplest nonlinear systems) and that (2) in many situations it is natural to assume that the variables which describe the state of the system are positive (populations of various species, prices of goods, masses of chemical reagents, probabilities, etc., see Luenberger [9], for example), the questions studied in this note seem to be quite natural. There is also some theoretical interest which stems from the fact that the situation considered here—given local accessibility—is in contrast to that studied in the important work of Jurdjevic and Kupka [5]. There the matrix B satisfied conditions similar to those above ($b_i - b_j$ all distinct), but rather different conditions on A, in particular $a_{1n}a_{n1} < 0$, which preclude $a_{ij} > 0$ for $i \neq j$ but result in controllability on all of $R^n - \{0\}$. This, of course, is in marked contrast to the case considered here where the accessible set of a point in the positive orthant must lie in the positive orthant. This might cast light on necessary conditions for controllability of bilinear systems on R^n.

Probably the most important model which is well known and which satisfies the conditions we have imposed arises in the case in which one is studying estimation theory of various stochastic processes. In [2], for example, Brockett and Clark have shown the importance of determining the accessibility set for the control problem

$$\dot{p} = (A - \tfrac{1}{2}B^2)p + uBp, \qquad p(0) = p_0$$

in studying an unnormalized conditional density equation

$$dp = Ap\,dt + Bp\,dy$$

rewritten in Fisk–Stratonovich form as

$$\dot{p} = (A - \tfrac{1}{2}B^2)p\,dt + bp\,d^+y.$$

Here, the vector $p(t)$ takes values in the positive orthant.

We also mention that in [3] Brockett studied in § 3 the reachability set for bilinear systems rather closely related to, but not quite the same as those studied here. Finally, it was pointed out to the author by L. Markus that matrices satisfying conditions similar to those discussed here are widely used in economics, see Markus [10], for example. However, control problems do not seem to have been considered in this context.

3. Some generalities. It is well known from the work of several authors that a necessary condition for controllability of any system of the form $\dot{x} = f(x, u)$, where $f(x, u)$ is a vector field on a manifold depending analytically on controls u as well as the point x, can be given in terms of the rank of Lie algebra $\mathfrak{g}$ generated by these vector fields. The necessary condition is that for each point x of the manifold the vectors of the Lie algebra must span the tangent space at x. The condition is not usually sufficient; however, it is enough to insure that the set of points accessible from x in positive time has a nonempty interior, in whose closure it is contained [6], [12].

For our case where the manifold may be taken to be $\mathring{R}^n_+$ and $f(x, u) = (A + uB)x$, $u \in R$, it follows easily from Lie theory that generically this maximum rank condition is satisfied, i.e., for a very large (open and dense) subset of pairs A, B.

We require that A be essentially positive, i.e., all off-diagonal elements of A be positive and that the diagonal matrix B satisfy the following condition for all $i \neq j$,

(3.1). *$b_i - b_j = b_k - b_1$ implies $i = k$ and $j = 1$ (in particular $b_i - b_j \neq 0$).*

We then easily see that the following is true.

PROPOSITION 3.2. *If A, B are $n \times n$ matrices with A essentially positive and B a diagonal matrix satisfying (3.1), then the Lie algebra $\mathfrak{g}$ generated by the set $\{A + uB \mid u \in R\}$ is $\mathfrak{sl}(n, R)$ if $\operatorname{tr} A = 0$ and $\mathfrak{gl}(n, R)$ otherwise.*

This follows from the facts that B is regular and that A has a nonzero component in each eigenspace belonging to a nonzero eigenvalue of $\operatorname{ad} B$; hence A, B generate $\mathfrak{sl}(n, R)$. From this it is clear that if either $\operatorname{tr} B \neq 0$ or $\operatorname{tr} A \neq 0$, then the algebra generated has dimension n^2 and is thus all of $\mathfrak{gl}(n, R)$.

4. Characteristic roots and vectors. If C is an essentially positive $n \times n$ matrix, let d be its smallest diagonal element. If $d > 0$, then $C > 0$, i.e., C is a positive matrix. If $d \leqq 0$, then $C + pI, p > |d|$, is a positive matrix whose invariant subspaces, if any, are the same as those of C and whose spectrum is the translate by p of that of C, i.e., λ is a characteristic value of C with characteristic vector v if and only if $\lambda + p$ is a characteristic value of $C + pI$ with the same characteristic vector v. It follows from the Perron–Frobenius theory of positive matrices that the following facts hold:

(4.1). *C has a real characteristic value r to which there corresponds a positive characteristic vector $v > 0$. The value r is a simple root of the characteristic equation. If $w > 0$ is any other positive characteristic vector, then w is a multiple of v. The root r corresponds to the dominant root $r + p$ of $C + pI$; hence any other characteristic root λ satisfies the inequality $r + p > |\lambda + p|$.*

Since for each real $u, A + uB$ is essentially positive if A is essentially positive, then (4.1) holds. As noted earlier, we assume throughout this note that A is essentially positive, i.e., all off-diagonal elements are strictly positive. This is the generic case (and already involves enough problems!). Although $A + uB$ is essentially positive for each choice of u, there may or may not be choices of u such that $A + uB$ is a positive (or even nonnegative) matrix. Let $a_{ii} + ub_i$ denote the ith diagonal element of $A + uB$. Graphically we consider n lines $v = a_{ii} + ub_i$ on the uv-plane. Whether or not there is a nonempty set of values of u such that all diagonal elements are positive depends on the intercept a_{ii} and slope b_i of these lines. Of course, for u large, it is the slopes b_i which are important. Thus, if $b_1, \cdots, b_n$ all have the same sign, then for some choice of u, $A + uB > 0$. In this case, as we can see from Gershgorin's theorem, for suitable choice of u, we can make the (dominant) characteristic value r associated with the characteristic vector $v > 0$ mentioned in (4.1) take on a positive value, $r > 0$, or a negative value, $r < 0$. To see this we first quote the theorem.

THEOREM 4.2 (Gershgorin). *Let $C = (c_{ij})$ be an $n \times n$ matrix and for each i, $1 \leqq i \leqq n$, let $\rho_i = \sum_{j \neq i} |c_{ij}|$. Let $\mathscr{D}_i$ given by $\mathscr{D}_i = \{z \in C \mid |z - a_{ii}| \leqq \rho_i\}$ be a disk in the complex plane. Then each characteristic root is in one of the disks $\mathscr{D}_i$; and if the disks are pairwise disjoint, then there is exactly one in each $\mathscr{D}_i$. In particular, for a real matrix, if the $\mathscr{D}_i$ are pairwise disjoint, the characteristic roots are real and distinct.*

From this we can derive the following proposition for our case.

PROPOSITION 4.3. *If the b_i, $i = 1, \cdots, n$, are pairwise distinct, then we may find values of u such that $A + uB$ has n distinct real characteristic values. If the b_i additionally all have the same sign, say positive, then for all sufficiently large u (resp. sufficiently negative u), the characteristic values are all positive (resp. all negative).*

Proof. If $\rho_i(C)$ denotes the radius, $\rho_i = \sum_{i \neq j} c_{ij}$ of the ith disk $\mathscr{D}_i$ of the matrix C, then note that $\rho_i(A + uB) = \rho_i(A)$ for $i = 1, \cdots, n$. On the other hand, the center of the disk $\mathscr{D}_i$ for $A+uB$ is $a_{ii}+ub_i$, and hence, $|(a_{ii}+a_{jj})+u(b_i-b_j)|$ is the distance between centers. This distance is at least equal to $|u||b_i-b_j|-|a_{ii}-a_{jj}|$ and can be made as large as we wish by choosing $|u|$ large enough. Since the radii ρ_i are independent of u, we see that taking $|u|>|u_0|$ for some $u_0>0$ will guarantee that for all $i\neq j$, $\mathscr{D}_i \cap \mathscr{D}_j=\varnothing$. This proves the first statement. The second follows from this since we can make all $a_{ii}+ub_i$ positive, or all of these diagonal elements negative, by a suitable choice of u.

We remark that if the b_i do not all have the same sign, even though they are pairwise unequal, we may not be able in some cases to change the sign of the dominant characteristic value v by our choice of u.

Example 4.4.

$$A=\begin{pmatrix}1 & 2\\ 4 & -1\end{pmatrix},\quad B=\begin{pmatrix}-1 & 0\\ 0 & +1\end{pmatrix},\quad A+uB=\begin{pmatrix}1-u & 2\\ 4 & -1+u\end{pmatrix}.$$

A has characteristic values $\lambda_1=3$ and $\lambda_2=-3$ for which corresponding characteristic vectors are, say $(1,1)'$ and $(-1,2)'$. In general, the characteristic polynomial for $A+uB$ is

$$f(t)=t^2-[(u-1)^2+8],$$

and hence, the dominant characteristic root is positive for all u. However, the slope of the corresponding positive characteristic vector or direction is given by $\frac{1}{2}\{\sqrt{(u-1)^2+8}+(u-1)\}$, which goes from 0 to ∞ as u goes from $-\infty$ to $+\infty$. Thus, for a fixed value of u, a typical picture of the flow lines of $x=(A+uB)x$ is shown in Fig. 1. By varying u the characteristic vector v can be made to take on any direction in the open positive quadrant. It would appear that since the flow always tends outward in the quadrant, positive quadrant controllability would not be possible in this case. However, it is possible by varying u to make the vector $(A+uB)x$ for x on the x_1-axis or on the x_2-axis take *any* direction pointing into the quadrant, which makes it somewhat less clear that controllability cannot be achieved in this case. This is settled by constructing a Lyapunov function in § 6.

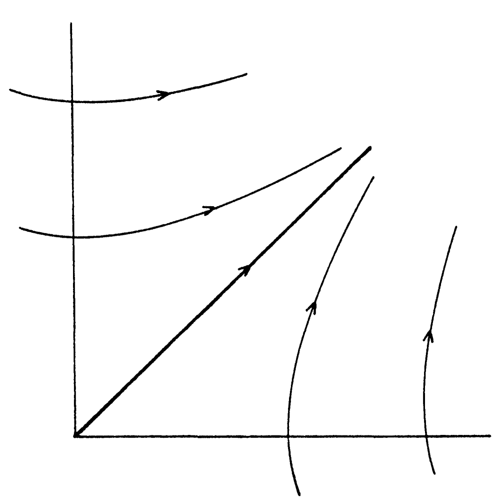


FIG. 1

5. The two-dimensional case. As examples of some possible qualitatively distinct types of behavior of solutions of bilinear systems satisfying our conditions, we consider the $n=2$ case, where it is possible to distinguish several different patterns determined by open conditions on the values of $A=(a_{ij})$ and b_1, b_2. We shall suppose that $a_{12}>0$ and $a_{21}>0$ (A essentially positive) and B nonsingular with $b_1 \neq b_2$. For convenience of notation, let $A_u = A+uB$, then $A_0=A$. The characteristic polynomial of A_u is then

$$f_u(t)=t^2-(\operatorname{tr} A_u)t+\det A_u, \tag{5.1}$$

and its discriminant is $D=(\operatorname{tr} A_u)^2-4\det A_u$, which reduces to

$$D=[(b_1+b_2)u+(a_{11}+a_{22})]^2+4a_{12}a_{21}. \tag{5.2}$$

Thus $D>0$, and for each u A_u has two distinct real roots, say λ_1, λ_2 with $\lambda_1>\lambda_2$. In fact, $\lambda_1, \lambda_2=\frac{1}{2}[a_{11}+a_{22}+u(b_1+b_2)\pm\sqrt{D}]$ with the $+$ corresponding to the dominant root λ_1. An easy computation shows that the characteristic vectors belonging to these roots lie on the two lines

$$x_2=\frac{1}{2a_{12}}[(a_{22}-a_{11})+u(b_2-b_1)\pm\sqrt{D}]x_1. \tag{5.3}$$

Let $a_{22}-a_{11}+u(b_2-b_1)=p(u)$, then $D=(p(u))^2+4a_{12}a_{21}$ so the slope of these lines is

$$\frac{1}{2a_{12}}[p(u)\pm\sqrt{(p(u))^2+4a_{12}a_{21}}], \tag{5.4}$$

with $+$ corresponding to the dominant root λ_1 and $-$ to λ_2. It is clear that the slope of the line corresponding to λ_1 is positive and the slope of the line corresponding to λ_2 is negative. Moreover, by varying u from $-\infty$ to $+\infty$, the slope of the first line goes through all values in the interval $(0,\infty)$.

Even in the two-dimensional case, we cannot give a universal criterion for controllability—our more modest goal is to show the existence of an open (i.e., large) subset in the space of pairs (A, B) in which we have positive orthant controllability and another open subset on which we do not have controllability. For the latter case the ideas of this section suggested a result which holds for arbitrary n (Theorem 6.1), so the discussion is postponed to § 6.

Intuitively, it appears that there is a greater possibility of controllability if the dominant root λ_1 changes sign at least once as u varies from $-\infty$ to $+\infty$. This can only happen if the quadratic polynomial $\det(A+uB)=\lambda_1\lambda_2$ has at least one real root. For this reason we will restrict to the open subset of (A, B) on which its discriminant $\mathscr{D}$ is positive, where

$$\mathscr{D}=(a_{11}b_2+a_{22}b_1)^2-4b_1b_2\det A. \tag{5.5}$$

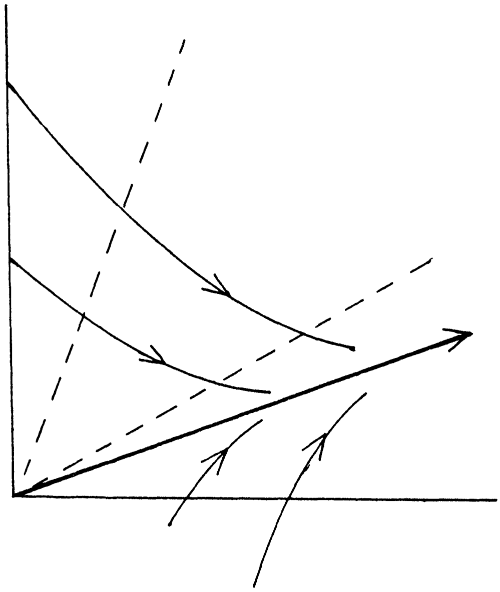
When $\mathscr{D}>0$ it is easily verified that exactly one of the following cases occurs as u goes from $-\infty$ to $+\infty$.

(a) λ_1 is always positive, λ_2 goes from negative to positive, then back to negative;

(b) λ_2 is always negative but λ_1 goes from positive to negative, then back to positive;

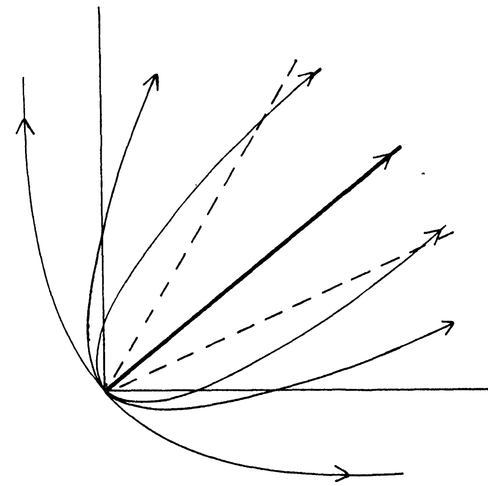
(c) λ_1 and λ_2 have the same sign and both change to the opposite sign.

In all cases let u_1, u_2 denote the u-values at which the sign change occurs, i.e., at which $\lambda_1\lambda_2=\det(A+uB)=0$. We illustrate these cases with figures in which the characteristic vectors corresponding to λ_1 when $u=u_1$ and $u=u_2$ are shown by dashed



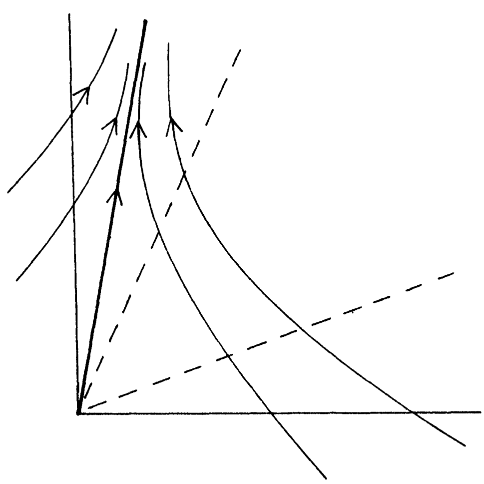
$u < u_1$

a) *Saddle*, $\lambda_2 < 0 < \lambda_1$



$u_1 < u < u_2$

b) *Unstable node*, $0 < \lambda_2 < \lambda_1$.



$u_2 < u$

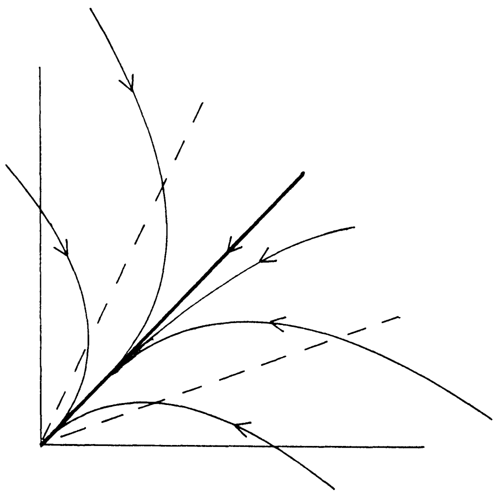
c) *Saddle*, $\lambda_2 < 0 < \lambda_1$.

FIG. 2

lines. The flow patterns corresponding to typical u in the intervals $(-\infty, u_2)$, (u_1, u_2) and $(u_2, +\infty)$ are shown in each case, the first case (a) in Figs. 2a, 2b and 2c. Of the three cases, this one seems to be the poorest candidate for positive controllability. However, I do not have a proof that it is not controllable.

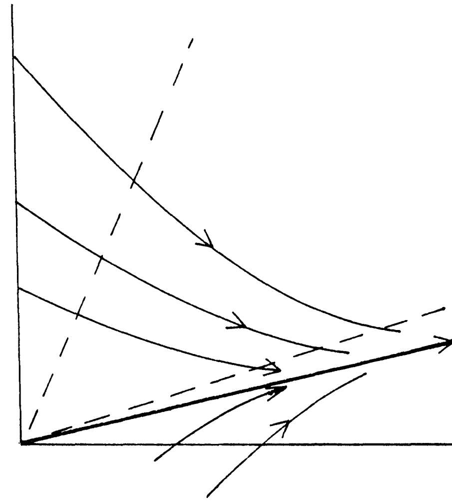
In case (b) we have two positive characteristic vectors (on the dashed lines) for which $\lambda_1 = 0$ with $\lambda_1 < 0$ for characteristic vectors between them. The pattern of flow lines for $u < u_1$, $u_1 < u < u_2$ and $u_2 < u$ are shown in Figs. 3a, 3b, 3c.

In this case it can be seen that we have controllability in the first quadrant. To see this we must show that $x_0 \in R^2_+$, $x_0 \neq 0$, can be steered to any interior point x_1 of the first quadrant. The dotted lines divide the quadrant into three sectors with flow patterns as shown in Fig. 3, where u is chosen so that the dominant characteristic



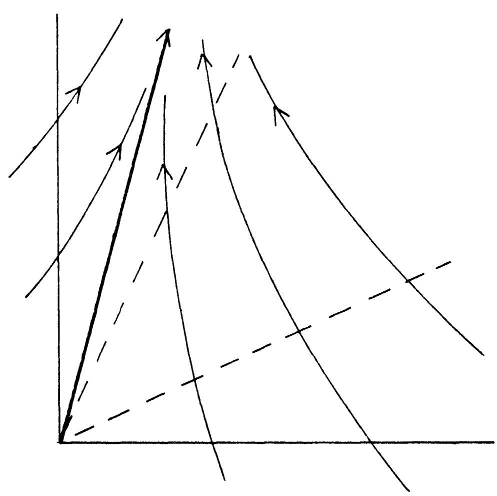
$u < u_1$

a) *Saddle*, $\lambda_2 < 0 < \lambda$.



$u_1 < u < u_2$

b) *Stable node*, $\lambda_2 < \lambda_1 < 0$.



$u_2 < u$

c) *Saddle*, $\lambda_2 < 0 < \lambda_1$.

FIG. 3

vector v is in the first, second or third sector (numbered counterclockwise). Suppose x_1 is in sector III, then it will lie on a flow line passing through sector II (Fig. 4a) which in turn will be cut by a flow line corresponding to λ_1 negative which can be chosen to cut the first flow line and carry x_0 to it, at least if x_0 is far enough out to lie on the opposite side from the origin of the first flow line (Fig. 4b). If it is not, we can easily move it out to such a position in a first step by choosing u so that $\lambda_1 > 0$ as in Fig. 3c, for example.

If x_1 does not, in fact, lie in sector III, it does lie on a flow line (taking $\lambda_1 > 0$ as in Fig. 3a or 3c) which comes in from sector III, so we can add one more step to the above path. Thus, we must switch controls at most four times to pass from x_0 to x_1.

Finally, we wish to look at case (c) in which both λ_1 and λ_2 change sign as u goes through its range of values. The corresponding flow patterns are illustrated in Figs. 5a, 5b, 5c with the dashed lines corresponding to the values $u = u_1$ and $u = u_2$.

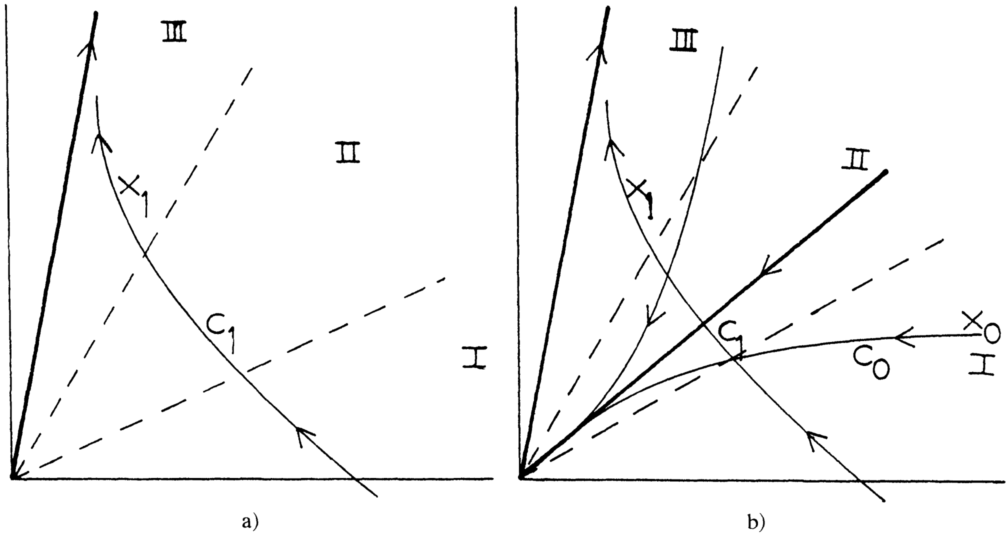


FIG. 4

By arguments similar to the previous case, we can establish controllability in the first quadrant in this case also. Roughly speaking, there is a flow line of type shown in Fig. 4b or 4c passing to any x_1 and we can carry any x_0, using $u < u_1$ as in Fig. 4a, to a point on this flow line which moves along it to x_1.

In summary, there do exist fairly large, i.e., open, subsets of our generic pairs A, B for which we have controllability in the first octant. In fact, the space of pairs A, B is divided by $\mathscr{D} = 0$ into two open sets: $\mathscr{D} > 0$ and $\mathscr{D} < 0$. On the former we have controllability in cases (b) and (c)—thus, in an open subset. When $\mathscr{D} < 0$ we shall see below that the system is not controllable. (Corollary 6.9).

It would, of course, be interesting to see if something similar is true in the case $n = 3$. However, a geometric analysis of the type above is much more difficult, and the author knows of no analytic approach which could be used here.

6. Some noncontrollable cases. In this section we will establish the existence of an open subset of pairs A, B with A essentially positive and B diagonal which can be shown to be noncontrollable. We do this by establishing the existence of a function $\Phi(x) = \Phi(x_1, \cdots, x_n)$ on R^n which is monotone along each solution curve of $\dot{x} = Ax + uBx$ for a special class of A, B.

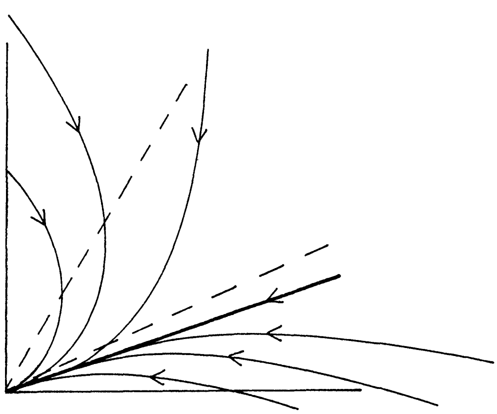
THEOREM 6.1. *Suppose that A, B are real $n \times n$ matrices, B diagonal with entries $b_1, \cdots, b_n$ and $A = (a_{ij})$ such that $a_{ij} \geqq 0$ for $i \neq j$. If there exists a nonnegative vector $p = (p_1, \cdots, p_n) \geqq 0$ such that* (i) $\sum_{i=1}^n p_i b_i = 0$ *and* (ii) $\sum_{i=1}^n p_i a_{ii} \geqq 0$, *then $\dot{x} = (A + uB)x$ is not controllable in the positive orthant.*

Proof. Define $\Phi(x) = \Phi(x_1, \cdots, x_n)$ on R^n by $\Phi(x) = \prod_{i=1}^n x_i^{p_i}$, with $p_i, \cdots p_n$ satisfying (i) and (ii). We show that

$$\langle \operatorname{grad} \Phi, (A + uB)x \rangle \geqq 0$$

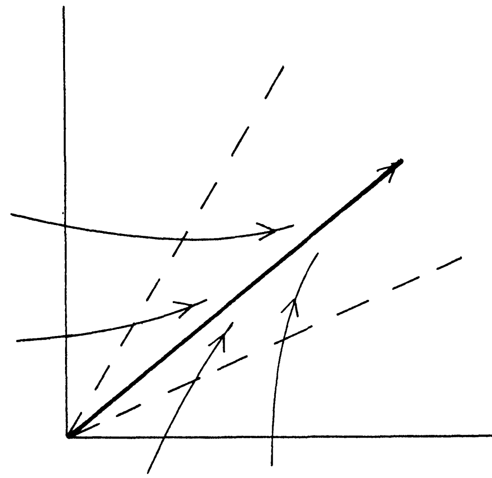
for all $x \neq 0$ such that $x \geqq 0$. This implies that Φ is monotone nondecreasing along any solution curve $x(t), t \geqq 0$, of $\dot{x}(t) = (A + u(t)B)x(t)$ such that $x(0) > 0$. Thus, we do not have controllability on R^n_+. Computing we obtain

$$\left(\frac{\partial \Phi}{\partial x_1}, \cdots, \frac{\partial \Phi}{\partial x_n}\right)(A + uB)(x_1, \cdots, x_n)' = \sum_{i,j} \Phi_i a_{ij} x_j + u \sum_i \Phi_i b_i x_i.$$



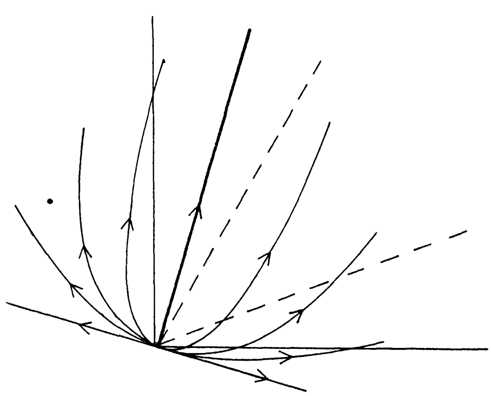
$u < u_1$

a) *Stable node,* $\lambda_2 < \lambda_1 < 0$.



$u_1 < u < u_2$

b) *Saddle,* $\lambda_2 < 0 < \lambda_1$.



$u_2 < u$

c) *Unstable node,* $0 < \lambda_2 < \lambda_1$.

FIG. 5

However, $\Phi_i = \partial\Phi/\partial x_i = p_i x_i^{p_i-1}\prod_{j\neq i} x_j^{p_j} = p_i x_1^{-1}\Phi$ if $x_i > 0$ for all i. Thus

$$\text{(6.2)}\qquad \begin{aligned}\langle \operatorname{grad}\Phi, (A+uB)x\rangle &= \sum_{i,j} p_i x_i^{-1}\Phi a_{ij}x_j + u\sum_i p_i b_i \Phi \\ &= \Phi\left[\sum_{i=1}^{n} p_i a_{ii} + \sum_{i\neq j}(p_i x_i^{-1})a_{ij}x_j\right] \geqq 0\end{aligned}$$

for all $x > 0$. This completes the proof.

Remark 6.3. If A, B are as in the hypothesis but no $b_i = 0$ (this is the generic case) and $\sum_i p_i a_{ii} > 0$, then, clearly conditions (i) and (ii) are satisfied for matrices $\tilde{A}, \tilde{B}$ near A and B, i.e., in an open subset of pairs A, B of the same type (A essentially positive and rank $B = n$).

Remark 6.4. The conditions obviously do not specify $p = (p_1, \cdots, p_n)$ uniquely. Also note that we could equally well search for $(p_1, \cdots, p_n) \leqq 0$ satisfying (i) and (ii′) $\sum p_i a_{ii} \leqq 0$. Finally, as we see by studying the $n = 2$ case, it is possible to weaken the requirement that all components of p have the same sign.

Consider the case $n=2$. Given A, B with $a_{12}>0$ and $a_{21}>0$, $b_1\neq 0\neq b_2$, we let $\Phi(x)=x_1^{p_1}x_2^{p_2}$.

(6.5)

$$\langle \text{grad}\ \Phi, (A+uB)x\rangle=\left\{p_1a_{11}+p_2a_{22}+\left(\frac{p_1}{x_1}\right)a_{12}x_2+\left(\frac{p_2}{x_2}\right)a_{21}x_1+(p_1b_1+p_2b_2)u\right\}\Phi.$$

For $x>0$, $\Phi>0$, and hence the sign of the expression is determined by the terms in parenthesis. If we wish the sign to be the same for all $x>0$, we clearly must require p_1, p_2 to be chosen so that $p_1b_1+p_2b_2=0$. The solutions are all of the form $p_1=kb_2$, $p_2=-kb_1$. We also note that the sign of the expression is not altered by multiplying by x_1x_2. Thus, the sign of $\langle \text{grad}\ \Phi, (A+uB)x\rangle$ for $x>0$ is always the same if and only if that of the quadratic form $Q=-(b_2a_{11}-b_1a_{22})x_1x_2-b_2a_{12}x_2^2+b_1a_{21}x_1^2$ or

$$Q(x_1, x_2)=a_{21}b_1x_1^2+(b_1a_{22}-b_2a_{11})x_1x_2-a_{12}b_2x_2^2 \tag{6.6}$$

stays the same for $x>0$.

A sufficient condition for this is that the discriminant $\mathscr{D}$ of $Q(x)$ be negative:

$$\mathscr{D}=(a_{22}b_1-a_{11}b_2)^2+4b_1b_2a_{12}a_{21}<0. \tag{6.7}$$

If the discriminant is positive, we have as the graph of $z=Q(x_1, x_2)$ a saddle surface, and only the case $Q(x_1, x_2)=cx_1x_2$ would guarantee that the sign of Q does not change in the first quadrant. But we have assumed a_{12}, a_{21}, b_1, b_2 are all nonzero. Hence $\mathscr{D}>0$ implies that the sign of $Q(x)$ changes in the first quadrant, and we are not able to draw the conclusion of noncontrollability in this case. The case $\mathscr{D}=0$ is somewhat special and occurs only if

$$(a_{22}b_1-a_{11}b_2)^2=-4b_1b_2a_{12}a_{21}, \tag{6.8}$$

which, in particular, requires b_1 and b_2 to have opposite signs. We will eliminate it from consideration as "nongeneric".

Finally, note that the expression $\mathscr{D}$ above was considered earlier, it is exactly the discriminant of the quadratic polynomial in u given by $\det(A+uB)$. Hence, $\mathscr{D}<0$ implies that for all u the characteristic roots λ_1, λ_2 of the $A+uB$ satisfy $\lambda_2<0<\lambda_1$. Thus we have what was conjectured earlier as a corollary to our theorem. (See (4.4) and subsequent discussion.)

COROLLARY 6.9. *If $\mathscr{D}<0$, or equivalently, $\lambda_2<0<\lambda_1$, for all $u\in \mathbf{R}$, then the system $\dot{x}=(A+uB)x$ is not controllable on $\mathbf{R}_+^2$.*

Note that p_1 and p_2 might be of opposite sign (but we must have $p_1b_1+p_2b_2=0$). This shows that it is possible to weaken the requirement $p\geqq 0$ in the theorem.

7. Increasing the number of controls. The following theorem is very easy to prove, but it answers a natural question and is worth stating as indicating some possibilities.

THEOREM. *Suppose $A=(a_{ij})$ is an $n\times n$ real matrix with $a_{ij}\geqq 0$ for $i\neq j$ and that $B_1, \cdots, B_n$ are n linearly independent diagonal matrices. Consider the system on $\mathbf{R}_+^n$,*

$$\dot{x}=(A+u_1B_1+\cdots+u_nB_n)x, \tag{**}$$

where $u=(u_1, \cdots, u_n)$ are piecewise continuous control functions with values in $\mathbf{R}^n$. If $x^{(0)}$ and $x^{(1)}$ are points of $\mathring{\mathbf{R}}_+^n$ and $x(t)$, $0\leqq t\leqq T$ is any piecewise differentiable path in $\mathring{\mathbf{R}}_+^n$, then there exist controls such that $x(t)$ is a solution of (**).

Proof. It is enough to show that for each vector $y=(y_1, \cdots, y_n)$ at $x=(x_1, \cdots, x_n)>0$ there exists a vector $u=(u_1, \cdots, u_n)$ such that $y=(A+\sum u_iB_i)x$.

There is no loss of generality in supposing $B_k = (\delta_{ik}\delta_{kj})$ the diagonal matrix with $+1$ in the kth row and column and zeros elsewhere. Thus, for x given, we have

$$(A+\sum u_iB_i)x = \begin{pmatrix} a_{11}+u_1 & a_{12} & a_{1n} \\ a_{21} & a_{22}+u_2 & a_{2n} \\ & \cdots & \\ a_{n1} & a_{n2} & a_{nn}+u_n \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} u_1x_1+\sum a_{1j}x_j \\ \vdots \\ u_nx_n+\sum a_{nj}x_j \end{pmatrix},$$

where $x_1, \cdots, x_n$ are all positive and fixed as is $\sum_j a_{ij}x_j$ for $i = 1, \cdots, n$. Obviously we need only choose $u_1, \cdots, u_n$ according to the rule

$$u_i = \frac{1}{x_i}\Big(y_i - \sum_j a_{ij}x_j\Big), \qquad i = 1, \cdots, n,$$

in order to get $(A+\sum u_iB_i)x = y$. This assures that we have total controllability in a very strong sense. In particular, given any x and a neighborhood U of x, there exists $V \subset U$ a neighborhood of x such that we have controllability in V without leaving U.

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