The Controllability of Infinite Quantum Systems and Closed Subspace Criteria

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Abstract

Quantum phenomena of interest in connection with quantum computation and communication often deal with transfers between eigenstates, and their linear superpositions. For systems having only a finite number of states, the quantum evolution equation (the Schrödinger equation) is finite-dimensional and the results on controllability on Lie groups as worked out decades ago [1] provide most of what is needed insofar as controllability of non-dissipative systems is concerned. However, for infinite-dimensional evolution of quantum systems, many difficulties, both conceptual and technical, remain. In this paper we organize some recent results from the physics literature in control-theoretic terms and emphasize the type of analysis needed to go beyond what basic differential geometry can provide. In particular, we analyze the problem of controllability of quantum systems subject to the constraint that the trajectories must lie in pre-defined subspaces, and discuss from a controllability viewpoint the important results of Law and Eberly [2].

I. Introduction

Over the last decade, there has been a steady stream of papers in the physics literature beginning with the work of [3], [4], [5], [6] that describe new experiments and new ways of thinking in which control-theoretic ideas are of central importance. In many cases, the driving

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force has been the desire to manipulate quantum states in ways that would make possible quantum computation or quantum communication, see for example [7], [8], [9], [10], [11]. Phenomena involving the interaction between electromagnetic radiation (light) and matter (e.g. ions, spin states, etc.) are especially interesting because they are possible paradigms of future quantum computing devices [12]. Many of the exciting ideas are related to the control of these systems. In this paper, we attempt to isolate some of the mathematical features of such models with the expectation that such a treatment will be helpful in furthering both the communication between control researchers and physicists working in this area. This is related to earlier work in control of trapped-ion quantum states (see [2], [13], [14] and references therein), and extends earlier works of the authors in [15], [16], [17].

II. Infinite-Dimensional Controllability

Controllability problems for infinite-dimensional systems are seldom just straightforward extensions of the finite-dimensional results, and this is especially true for bilinear systems. Some of the difficulties have to do with the nature of e^{At} (e.g., see [18]) when A is an unbounded operator; and others have to do with the fact that when A is unbounded, its domain of definition is seldom such that the operators $A^2, A^3, ...$ can be used without some strong restrictions on their domain of definition. This means, for example, that any criterion involving a Lie algebra generated by a pair of operators, say A and B, must be prefaced with some statement about the domains of the operators.

In quantum mechanics, one is often concerned with skew-hermitian operators defined on a complex Hilbert space which we denote by $\mathcal{H}$. One says that an operator mapping a domain $\mathcal{D} \subset \mathcal{H}$ into $\mathcal{H}$ is *closed* if for any Cauchy sequence x_i such that Ax_i is Cauchy, the limits of x_i and Ax_i are such that $\lim_{i\to\infty} x_i \in \mathcal{D}$ and

$$A \lim_{i\to\infty} x_i = \lim_{i\to\infty} Ax_i.$$

To avoid trivialities, we always assume that $\mathcal{D}$ is dense in $\mathcal{H}$. According to standard results from the theory of semigroups (e.g., see [18]), if $A : \mathcal{D} \subset \mathcal{H} \to \mathcal{H}$ is a densely defined, closed, skew-adjoint operator then there exists a one-parameter group of bounded operators

$$\{T | T = e^{At} \; ; \; -\infty < t < \infty\} \tag{1}$$

satisfying the usual composition law, $e^{AT}e^{A\tau} = e^{A(t+\tau)}$ and such that e^{At} maps $\mathcal{D}$ into itself and such that for all x in $\mathcal{D}$,

$$\lim_{h\to 0} \frac{1}{h}\left(e^{Ah}x - x\right) = Ax. \tag{2}$$

A. Example

Consider an evolution equation in the Hilbert space l_2, given by

$$\dot{x} = u(t)Ax + v(t)Bx, \tag{3}$$

with B being bounded and A being unbounded, closed and densely defined. Let the domain of definition of A be $\mathcal{D}$ and assume that in addition to B being bounded, $\mathcal{D}$ is an invariant subspace for B.

Under this hypothesis, we can assert that products of the form

$$T = e^{Bt_1}e^{At_2}...e^{Bt_k}e^{At_{k+1}} \tag{4}$$

are well-defined and represent the solution of

$$\dot{x} = (u(t)A + v(t)B)x \tag{5}$$

as long as u and v are never nonzero at the same time. The space of bounded operators on a Hilbert space is a Banach algebra and so any combination of sums and products of e^{At_i} and B is a bounded operator. Thus, we can legitimately refer to the linear span of the set of operators obtained from A and B by combining B and $e^{At_i}Be^{-At_i}$ (even though one cannot talk about the Lie algebra generated by A and B without some discussion of the domain of definition of an operator with a formal representation such as $[B,[A,B]]$). In particular, it makes sense to speak about the Lie algebra generated by $e^{-At_1}Be^{At_1}$ and $e^{-At_2}Be^{At_2}$. (Recall that e^{At} is bounded even if A is not!)

B. Resonant Control

We want to establish a suitable language for discussing a recurring idea showing up in the control of quantum systems. Let A be a closed, skew-hermitian operator. Suppose that $B_1, B_2, ..., B_k$ are bounded. Consider

$$\dot{\psi} = \left(A + \sum_i U_i B_i\right)\psi. \tag{6}$$

Let $c = e^{-At}\psi$ so that

$$\dot{c} = \left(\sum_i U_i e^{-At} B_i e^{At} \right) c. \tag{7}$$

If the spectrum of A is discrete, the entries in any one of the operators $e^{-At}B_i e^{At}$ are almost periodic functions.

$$\begin{aligned} \dot{c} &= \left(\sum_i U_i \sum_m |m\rangle\langle m| e^{-At} B_i e^{At} \sum_n |n\rangle\langle n| \right) c. & (8) \\ &= \left(\sum_i U_i \sum_m |m\rangle e^{-A_m t} \langle m| B_i \sum_n |n\rangle e^{A_n t} \langle n| \right) c. & (9) \\ &= \left(\sum_i U_i(t) \sum_{m,n} e^{-(A_m - A_n)t} \langle m|B_i|n\rangle |n\rangle\langle m| \right) c. & (10) \end{aligned}$$

Let $U_i(t)$ be a single frequency (monochromatic) field that is resonant with a the transition between a single pair of eigenstates of A; i.e., $U_i(t) = u_i e^{i\omega_i t} + c.c$, where $\imath\omega_i = A_m - A_n$. Under this hypothesis, and if the eigenvalues of A are separated such that the differences between the eigenvalues are all unique, it is reasonable to approximate $U_i(t)e^{-At}B_i e^{At}$ by

$$u_i \left(|m\rangle B_i^{mn} \langle n| + |n\rangle B_i^{nm} \langle m| \right). \tag{11}$$

(This is known as the rotating wave approximation, and formally assumes that $u_i B_i^{mn} \ll \omega_i$ and that $|E_m - E_n - \omega_i| \ll E_m - E_n, \omega_i$ [19].) This produces a type of projection of B_i onto the sum of rank one terms formed from those eigenvectors of A that correspond to the eigenvalues that resonate with u. Different choices of u yield different projections making it possible to obtain different "effective" B's even if there is just one control term. The realization of a controlling sequence consists of a sequence of sinusoidal pulses of different frequencies whose duration is determined as in the usual bilinear control theory.

A useful concept that we will use later is that of π-pulses. It is well known that a single-frequency field resonant with the transition between two quantum states drives the population back and forth between them sinusoidally in time, with the frequency of transfer being proportional to $u_i B_i^{mn}$ [19]. If the initial state is one of the two levels, as time progresses, a superposition of the two states will be formed, and after half a period, the population will entirely be driven to the other state. Thus, the duration of the field required to transfer population completely from one to the other will depend inversely on the amplitude of the field and the transition matrix coupling between them. Such a pulse is called a π-pulse. Experimentally, the intensity of the laser is kept fixed, therefore, the duration of the laser (pulse duration) is varied depending on the strength of the transition (B_i^{mn}) that is to be driven, and the superposition that is desired.

In the following two sections, we analyze the controllability of two infinite-dimensional quantum systems, namely, the quantum harmonic oscillator and a spin-half system coupled to a harmonic oscillator.

III. Case 1: Quantum Harmonic oscillator

The problem of controlling the harmonic oscillator has been discussed many times [20]. We review a few aspects here on our way to the the problem of controlling a spin-$\frac{1}{2}$ particle in a quadratic potential, which is the subject of the next section. Recall that the harmonic oscillator leads to the Schrödinger equation in dimensionless units [21]:

$$i\frac{\partial\psi}{\partial t} = \left(-\frac{1}{2}\frac{\partial^2\psi}{\partial x^2} + \frac{1}{2}x^2\right)\psi. \tag{12}$$

If the control is a sinusoidal resonant driving field as shown in Fig. III, then the evolution is via

$$i\frac{\partial\psi}{\partial t} = \left(-\frac{1}{2}\frac{\partial^2}{\partial x^2} + \frac{1}{2}x^2 + u(t)x\right)\psi. \tag{13}$$

Here, the control term $u(t)x$ arises because of the dipole interaction between the field and harmonic oscillator. Dividing by i we see that again there are two operators of interest,

$$A = \frac{i}{2}\left(\frac{\partial^2\psi}{\partial x^2} - x^2\right), \tag{14}$$

and

$$B = -ix. \tag{15}$$

The domain of definition of A can be taken to be those elements of $L_2(-\infty,\infty)$ having square integrable second derivatives and growth at infinity such that multiplication by x^2 does not take them out of $L_2(\infty,\infty)$. Any function with this property is such that multiplication by ix results in an element in $L_2(\infty,\infty)$. Thus B is defined on the domain of A. Furthermore, there is dense subset of $L_2(-\infty,\infty)$ for which both $A(B\psi)$ and $B(A\psi)$ are well-defined and on this set

$$A(B\psi) - B(A\psi) = -i\frac{\partial\psi}{\partial x}. \tag{16}$$

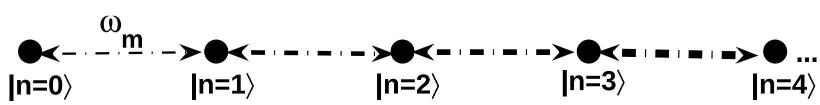

Fig. 1. Graphical representation of the quantum harmonic oscillator driven by a sinusoidal resonant field. The strengths of the transition couplings increase as the square root of the quantum number n.

In this sense

$$[A, B] = C = \frac{\partial}{\partial x}. \tag{17}$$

Finally, in analogous way, we are led to $[B, C] = D = -iI$, and to the conclusion that A and B generate a Lie algebra of skew-hermitian operators that is just four-dimensional. Each operator generates a one-parameter group of bounded transformations e^{At}, e^{Bt}, e^{Ct} and e^{Dt}, which act on $L_2(-\infty, \infty)$. As is explained in most introductory books on quantum mechanics [21], the spectrum of A is discrete. If we choose a basis for $L_2(\infty, \infty)$ consisting of appropriately scaled Hermite functions represented by $|n\rangle$, the description of the operator A (in dimensionless form) takes the form of a diagonal matrix

$$\tilde{A} = i \ \mathrm{diag}(1, 2, 3, ...) = \mathrm{iN}. \tag{18}$$

In this same basis B becomes

$$\tilde{B} = \frac{i}{\sqrt{2}} \begin{bmatrix} 0 & \sqrt{1} & 0 & 0 & ... \\ \sqrt{1} & 0 & \sqrt{2} & 0 & ... \\ 0 & \sqrt{2} & 0 & \sqrt{3} & ... \\ ... & ... & ... & ... & ... \end{bmatrix} = ix_N, \tag{19}$$

and C takes the form

$$\tilde{C} = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & \sqrt{1} & 0 & 0 & ... \\ -\sqrt{1} & 0 & \sqrt{2} & 0 & ... \\ 0 & -\sqrt{2} & 0 & \sqrt{3} & ... \\ ... & ... & ... & ... & ... \end{bmatrix}. \tag{20}$$

Thus, in the basis defined by the eigenfunctions of A, this particular controlled Schrödinger equation takes the form

$$\dot{\psi} = -iN\psi - iu(t)x_N\psi. \tag{21}$$

Let us introduce the differential operators

$$a = \frac{1}{\sqrt{2}}\left(x + \frac{\partial}{\partial x}\right), \tag{22}$$

and

$$a^{\dagger} = \frac{1}{\sqrt{2}}\left(x - \frac{\partial}{\partial x}\right). \tag{23}$$

These are the familiar creation and annihilation operators of the quantized harmonic oscillator [21].

$$\dot{\psi} = -ia^{\dagger}a\psi - iu(t)\frac{1}{\sqrt{2}}(a + a^{\dagger})\psi. \tag{24}$$

In the eigenbasis of A, viz. $|n\rangle$,

$$\tilde{B} = \frac{i}{\sqrt{2}}(a + a^{\dagger}), \tag{25}$$

and

$$\tilde{C} = \frac{i}{\sqrt{2}}(a - a^{\dagger}). \tag{26}$$

It is possible to be quite explicit about the reachable wave functions under the application of a control. For example, if we agree to call any function of t and x which has the form

$$\psi(0,x) = k(t)e^{i(a(t)x^2 + b(t)x)} \tag{27}$$

i-gaussian , then if the initial value of ψ is i-gaussian the solution will be i-gaussian for all time, regardless of the choice of $u(t)$. (This is directly analogous to the fact that the solution of the conditional density equation of estimation theory remains gaussian if it has a gaussian initial condition.) Thus if we describe the evolution in terms of an eigenfunction expansion, with the basis being the eigenfunctions of $\partial^2/\partial x^2 - x^2$, then the evolution is via

$$\dot{x}_n = -inx_n - u(t)(\sqrt{n-1}x_{n-1} - \sqrt{n}x_{n+1}). \tag{28}$$

An example of such a function is the well-known coherent state [22]

$$|\alpha\rangle = \exp(-|\alpha|^2/2)\sum_{n=0}^{\infty}\frac{\alpha^n}{n!}|n\rangle. \tag{29}$$

We see that it is not possible to transfer $x(0) = |0\rangle$ to $x(T) = |i\rangle$ for $i > 1$ because $|i\rangle$ is not i-gaussian, but $|0\rangle$ is, and is in fact a coherent state.

IV. Case II: Spin-half particle in a quadratic potential

The model of a spin-half particle coupled to a harmonic oscillator is a good representation of a ion with two essential internal states trapped in a quadratic potential. The spin-$\frac{1}{2}$model represents a two-level atom with an energy splitting $\hbar\omega_0$, where the frequency $\omega_0/2\pi$ is in the several GHz range. The atomic levels are coupled to the motion of the ion in a harmonic trap [23]. These quantized vibrational energy levels are separated by a frequency $\omega_m/2\pi$ in the MHz range.

In a frequently cited paper, Law and Eberly [2] showed that when properly interpreted, this system has interesting controllability properties, quite different from the properties of the harmonic oscillator alone. In fact, by coupling the harmonic oscillator with a two-level system it is possible to arrive at a system which is much more controllable than the harmonic oscillator. At an intuitive level, this can be seen simply as a consequence of the fact that the addition of a spin degree of freedom breaks the infinite degeneracy associated with the harmonic oscillator and allows the system to resonate with more than one frequency. This allows the transfer of population from any eigenstate to any other eigenstate by sequentially applying the two frequencies. We now analyze this system from a controllability viewpoint.

As a start toward describing this model, we introduce the three Pauli matrices

$$\sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \;;\; \sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \;;\; \sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \tag{30}$$

Consider the description of a particle with two spin states in a quadratic potential field. The Hamiltonian now includes terms that reflect both the linear momentum of the particle in the potential field, and the spin angular momentum of the particle. The Schrödinger equation can be written as a two-component vector equation

$$\begin{bmatrix} i\frac{\partial\psi_+}{\partial t} \\ i\frac{\partial\psi_-}{\partial t} \end{bmatrix} = \frac{1}{2}\begin{bmatrix} \left(-\frac{\partial^2\psi}{\partial x^2}+x^2\right)+\omega_0 & 0 \\ 0 & \left(-\frac{\partial^2\psi}{\partial x^2}+x^2\right)-\omega_0 \end{bmatrix}\begin{bmatrix} \psi_+ \\ \psi_- \end{bmatrix}. \tag{31}$$

The subscripts $+$ and $-$ refer to the two levels of the atom (modelled in physics literature as the spin-up and spin-down states of a spin-$1/2$ particle).

A spin-0 particle in a quadratic potential being excited by a traveling wave of central frequency ω with controllable amplitude might be described by an equation of the form

$$i\frac{\partial\psi}{\partial t} = \frac{1}{2}\left(-\frac{\partial^2\psi}{\partial x^2}+x^2\right)\psi + e(t)\cos(kx-\omega t)\psi. \tag{32}$$

Expanding $\cos(kx-\omega t)$ as $\cos kx\cos\omega t+\sin(kx)\sin(\omega t)$ we see that we can think of controlling

$$i\frac{\partial\psi}{\partial t} = \left(-\frac{\partial^2\psi}{\partial x^2}+x^2\right)\psi + \left(e(t)\cos\omega t\cos(kx)+e(t)\sin\omega t\sin kx\right)\psi. \tag{33}$$

Introducing the differential operators corresponding the position of the particle in the harmonic potential, Eq. 33 becomes

$$\begin{aligned} i\frac{\partial\psi}{\partial t} &= (N+\frac{1}{2})\omega_0\psi \\ &\quad + \left(e(t)\cos\omega t\cos(kx_0(a+a^\dagger))+e(t)\sin\omega t\sin kx_0(a+a^\dagger)\right)\psi. \end{aligned} \tag{34}$$

The product of k, the wavelength of the light, and x_0, which is amplitude of the zero-point motion of the particle in the harmonic potential (or the spatial extent of the ground state harmonic oscillator wave function) is called variously the Lamb-Dicke parameter η_0.

Returning to the system described by Eq. 31, an applied field causes transitions between the eigenstates of the coupled spin-oscillator system. A monochromatic field of frequency ω_0 causes resonant transitions between states $|\downarrow,n\rangle$ and $|\uparrow,n\rangle$ (carrier transitions). This “spin-flip” is described by the operator $\sigma_+=\sigma_x+\imath\sigma_y$ (and its hermitian conjugate σ_-). A monochromatic field of frequency $\omega_0-\omega_m$ causes resonant transitions between states $|\downarrow,n\rangle$ and $|\uparrow,n-1\rangle$ (red sideband transitions). As pointed out in Ref. [17], when both fields are applied *simultaneously*, the eigenstates of the system are sequentially connected. In this paper, we deal with the case when these two fields are applied *sequentially*.

Denoting the "spin-states" of the system by $|S\rangle$, a matrix element of the control Hamiltonian in the field-free eigenbasis of the coupled system in the interaction picture (in the resonant case) can be written as [16]

$$\begin{aligned}\langle S'n'|H_I'|Sn\rangle &= \Omega(t)2\mathrm{Re}\left[\langle S'|\sigma_+|S\rangle\right.\\ &\left.\otimes\langle n'|\exp(\imath(\eta(a+a^\dagger)))|n\rangle\right], \end{aligned} \tag{35}$$

The harmonic oscillator part of this matrix element [23] is written as

$$\begin{aligned}&\langle n'|\exp(\imath(\eta(a+a^\dagger)))|n\rangle \;=\\ &\exp(-\eta/2)\sqrt{\frac{n_<!}{n_>!}}\,(\imath\eta)^{|n'-n|}\quad L_{n<}^{|n'-n|}(\eta^2).\end{aligned} \tag{36}$$

The symbol $n_>$ refers to the larger of n and n', and $n_<$ refers to the smaller of n and n'. $L_n^\alpha(x)$ is the associated Laguerre polynomial. When the applied field connects states $|\downarrow, n\rangle$ and $|\uparrow, n\rangle$ (carrier transitions), $n' = n$, and when the applied field connects states $|\downarrow, n\rangle$ and $|\uparrow, n-1\rangle$ (red sideband transitions), $n' = n-1$. The matrix elements are zero for all other values of n'. These transitions are graphically depicted in Fig. IV with the thickness of the edges qualitatively representing the strength of the coupling between the states.

The electric field corresponding to the frequencies that cause the carrier and red transitions are dubbed E_c and E_r respectively. The eigenstates can be ordered as $|\uparrow, 0\rangle,\ |\uparrow, 1\rangle,\ \ldots, |\downarrow, 0\rangle,\ |\downarrow, 1\rangle,\ \ldots$. The drift Hamiltonian H_0 of this system can be written in block matrix form as:

$$\left(\begin{array}{c|c}\frac{\omega_0}{2}+(N+\frac{1}{2})\omega_m & 0\\ \hline 0 & \frac{-\omega_0}{2}+(N+\frac{1}{2})\omega_m\end{array}\right), \tag{37}$$

where, N is the previously defined number operator.

In the interaction picture, the Schrödinger equation is written as

$$\dot{Y} \;=\; (u(t)B_c + v(t)B_r)Y. \tag{38}$$

Since we consider the case when E_c and E_r are never nonzero at the same time, we can neglect

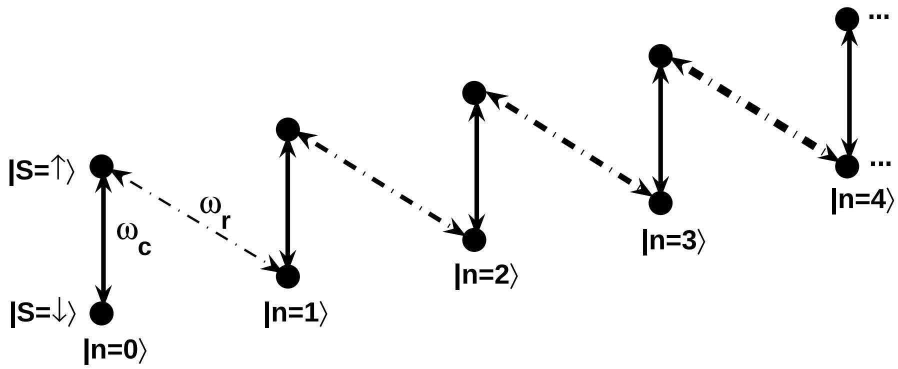

Fig. 2. Graphical representation of the coupled spin-half quantum harmonic oscillator system driven by sinusoidal resonant fields ω_c and ω_r as shown. The strengths of the ω_c transition couplings are independent of the harmonic oscillator quantum number n, whereas the strengths of the ω_r transition couplings increase as the square root of n.

any phase difference between them. Then,

$$u(t) = c_1 E_c(t) = 0.25\mu_0 \exp(-\eta^2/2) E_c(t), \tag{39}$$

$$v(t) = c_2 E_r(t) = 0.25\eta\mu_0 \exp(-\eta^2/2) E_r(t), \tag{40}$$

$$B_c = \left(\begin{array}{c|c} 0 & \imath L_0 \\ \hline \imath L_0^T & 0 \end{array}\right). \tag{41}$$

$$B_r = \left(\begin{array}{c|c} 0 & L_1 \\ \hline -L_1^T & 0 \end{array}\right). \tag{42}$$

The upper-triangular matrices L_0 and L_1 are defined as

$$L_0 = \begin{pmatrix} L_0(\eta^2) & 0 & 0 & \dots \\ 0 & L_1(\eta^2) & 0 & \dots \\ 0 & 0 & L_2(\eta^2) & \ddots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{43}$$

$$L_1 = \begin{pmatrix} 0 & L_0^{(1)}(\eta^2) & 0 & \dots \\ 0 & 0 & L_1^{(1)}(\eta^2) & \dots \\ 0 & 0 & 0 & \ddots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{44}$$

The control of this system can be analyzed in two limits - one in which the extent of zero-point motion of the spin-half particle in the harmonic potential ξ is much smaller than the wavelength of the applied light $2\pi/k$, and the other in which $\eta \simeq 1$. In the first case, the long-wavelength approximation can be made and this is called the Lamb-Dicke limit. The latter case must be treated exactly.

A. Lamb-Dicke limit

When the zero-point motion of the spin-half particle in the harmonic potential is much smaller than the wavelength of the applied light, the Lamb-Dicke parameter $\eta \ll 1$. Making the long-wavelength approximation, the terms in equations 43 and 44 can be expanded to first order in η. The control Hamiltonians can then be expressed in operator form as

$$B_c = \imath(\sigma_+ \otimes I + \sigma_- \otimes I), \text{ and} \tag{45}$$

$$B_r = \eta(\sigma_+ \otimes \hat{a} - \sigma_- \otimes \hat{a}^\dagger). \tag{46}$$

The latter is the same Hamiltonian as obtained from the well-known Jaynes-Cummings model [24] that describes the interaction between a quantized cavity field and a two-level atom. Unlike the situation encountered in the analysis of the oscillator algebra, here the formal commutation of the operators B_1 and B_2 does not lead to a finite-dimensional algebra, suggesting that the model with spin is much more controllable. This is the case, as will be explored in the next section.

B. Controllability: Lie Algebra

It is interesting to look at this problem from a Lie theoretic point of view. The first thing to do is to determine the formal structure of the Lie algebra, which we now consider. Let T be an operator acting on a complex Hilbert space. We associate with T a skew-hermitian operator acting on $\mathcal{H} \oplus \mathcal{H}$ defined by

$$J(T) = \begin{bmatrix} 0 & T \\ -T^{\dagger} & 0 \end{bmatrix}. \tag{47}$$

For convenience, let $K(T)$ be another operator defined in a similar way as

$$K(T) = \begin{bmatrix} T & 0 \\ 0 & -T \end{bmatrix}. \tag{48}$$

Of course, $K(T)$ is skew-hermitian if and only if T is.

The control operators we are interested in are given by $B_c = J(iI)$ and $B_r = J(a)$. We have

Lemma : The Lie algebra generated by $J(iI)$ and $J(T)$ includes the operators

$$J(W^{2p}) \; ; \; p = 1, 2, 3, \cdots \quad ; \quad K(W^{2p+1}) \; ; \; p = 0, 1, 2, \cdots \; , \tag{49}$$

where, $W = i(T + T^{\dagger})$.

Proof: A calculation shows that $[J(T), J(iI)] = K(W)$ and further, $[J(iI, K(W)] = -2iJ(W)$. We can then check that

$$ad^{p}_{J(W)}(K(W)) = (-2)^{p} \left\{ \begin{array}{ll} J(W^{p+1}), & \text{if p is odd} \\ K(W^{p+1}), & \text{if p is even} \end{array} \right\}. \tag{50}$$

These calculations make it clear that if the powers of W are independent then $J(iI)$ and $J(T)$ do not generate a finite-dimensional algebra. Thus if T is nonzero only on the diagonal immediately above the main diagonal (which is true for the operator a), and if every term on this upper-diagonal is nonzero, then the successive powers of W are independent and the algebra is infinite-dimensional. This is the case for the coupled spin-half harmonic oscillator system. Of course, this calculation only shows that this system, unlike the harmonic oscillator, does not generate a finite-dimensional controllability Lie algebra. More work is required to say with precision exactly what the reachable states are.

Note: In the case where the Lamb-Dicke limit does not apply, the Lie algebra will still be infinite-dimensional but the terms are more complicated.

V. A Generalized Subspace Controllability Theorem

Let $\mathcal{H}$ be a complex Hilbert space, let $\mathcal{D}$ be a dense subspace and let $0 \subset \mathcal{V}_1 \subset \mathcal{V}_2 \subset ... \subset \mathcal{D}$ be an infinite chain of finite dimensional subspaces of $\mathcal{H}$. Let $\mathcal{G} = \{G_1, G_2, ...G_m\}$ be a finite set of skew-Hermitian operators mapping $\mathcal{D}$ to $\mathcal{H}$. Assume that each of the G_α has an infinite set of invariant subspaces $S_1^\alpha \subset S_2^\alpha \subset ...$ with each of the subspaces belonging to the set $\{\mathcal{V}_i\}$. Matters being so, we will say that the family $\{G_\alpha\}$ has *property* Γ if there exists an integer γ such that for all α and i, $G_\alpha : \mathcal{V}_i \to \mathcal{V}_{i+\gamma}$. Such a family is rather special in that they must have an infinite set of eigenvalues, etc. If $\mathcal{V}_i$ is an invariant subspace for G_α let $\pi_i(G_\alpha)$ denote the restriction G_α to that subspace. If $\mathcal{G}$ is such a collection, then for any triple of positive integers i, μ, ν, the operator $[\pi(G_\mu), \pi(G_\nu)]$ is well-defined as an operator from $\mathcal{V}_i$ to $\mathcal{V}_{i+2\gamma}$. Define $[G_\alpha, G_\beta]$ to be the operator which restricts to $[\pi_i(G_\alpha), \pi_j(G_\beta)]$ for all integers i and j. We define $\mathcal{G}_{LA}$ to be the set of all operators which can be expressed as linear combinations of brackets of the form $[G_i, [G_j, ...[G_k, G_l]...]]$ and brackets of depth p will yield operators that map $\mathcal{V}_i$ into $\mathcal{V}_{i+p\gamma}$.

Theorem: Let $\mathcal{H}$, $\mathcal{D}$, $\mathcal{G}$ and $0 \subset \mathcal{V}_1 \subset \mathcal{V}_2 \subset ... \subset \mathcal{D}$ be as above and assume that there exists a function $\alpha : \mathbb{Z}^+ \to \{1, 2, ..., m\}$ be such that

1) $\mathcal{V}_i$ is invariant for $G_{\alpha(i)}$
2) if $\bar{G}_{\alpha(i)}$ denotes the restriction of $G_{\alpha(i)}$ to $\mathcal{V}_i$. Then for each $x \in \mathcal{V}_i$ there is $T \in exp\{\bar{\mathcal{G}}_i\}_{LA}$ such that $Tx \in \mathcal{V}_{i-1}$.
3) exp $\{\bar{\mathcal{G}}_1\}_{LA}$ acts transitively on the set of unit vectors in $\mathcal{V}_1$.

Then if B_i contains a basis for $\mathcal{G}$, the system

$$\dot{x} = (B_i u_i + B_2 u_2 + ... + B_m u_m)\, x$$

is controllable in the sense that for any two positive integers i and j, any unit vector in $\mathcal{V}_i$ can be transferred to any unit vector in $\mathcal{V}_j$ in finite time using a piecewise constant control while generating a trajectory lying in $\mathcal{V}_{\text{max i,j}}$.

Proof: Observe that because of the skew-Hermitian property of the G_i

$$e^{G_\alpha u_\alpha} = \left(e^{-G_\alpha u_\alpha}\right)^{-1}$$

it follows that if x is transferred to y by the action $e^{G_\alpha t}x$ then y can be transferred to x by the action $e^{-G_\alpha t}y$. In words, if there exists a transfer that involves just one nonzero u_i then the transfer in the reverse direction is also possible. This property is exploited by Law and Eberly [2] and Kneer and Law [13] in order to devise a scheme for the production of a finite superposition of eigenstates from another finite superposition in the control of a spin-half particle coupled to a harmonic oscillator. It shows that if x can be transferred to y by a series of such "single nonzero u_i" moves then the transfer from y to x is also possible.

VI. Example 1: The Law-Eberly method for controlling a trapped-ion

An example of the generalized subspace controllability theorem outlined above is the Law-Eberly method of controlling the two-level ion trapped in a harmonic potential presented in Section 4. Law and Eberly [2] present an argument that shows that any eigenstate $|i\rangle$ can be transferred to any other eigenstate $|j\rangle$. Their argument is both easy to appreciate and constructive. It involves the alternate use of transitions generated by spin reversal (π-pulses of E_c) and transitions generated by π-pulses of E_r which convert from a state in which the oscillator has energy E_i and spin down to a state in which the energy of the oscillator in altered by one unit and the spin is flipped as well (see equation (46).

Note that the π-pulses of E_c are all of the same time duration because in the Lamb-Dicke limit, all the carrier transitions are equally strong. However, the coupling strengths of the red-sideband transitions are proportional to $\sqrt{n}$, and therefore the π-pulses of E_r are shorter in duration as eigenstates of higher n are addressed. In a subsequent paper, Kneer and Law [13] show that this method can be used to generate an arbitrary superposition of a finite number of eigenstates, starting from another arbitrary superposition. The additional trick there is to go through the ground state of the system which acts as a "pass state" [25].

A. *Law-Eberly and Kneer-Law schemes for state transfer*

We have shown that the span of the Lie algebra in this control problem is infinite-dimensional. As mentioned, this is of course not sufficient to prove controllability. Nor do we expect complete controllability in this setting. With the Law-Eberly algorithm, it can be shown that it is possible to reach any eigenstate from any other eigenstate. With the Kneer-Law extension, any superposition of a finite number of eigenstates can be reached from any other such superposition. We note that the Kneer-Law scheme makes essential use of a ground state. Since it is possible to provide an explicit algorithm which will drive the system from any finite superposition to any other finite superposition, we obtain a form of controllability provided by the Law-Eberly algorithm described above and its generalization by Kneer and Law. We give some further details below.

The easiest case to analyze is passage from an eigenstate to any other eigenstate. The idea is alternate π pulses in the carrier and red (or blue) sidebands. For example suppose we wish to drive a state from the $|\downarrow, n\rangle$ to $|\uparrow, n-2\rangle$ (see Fig. IV).

This can be done using B_r to drive the system from $|\downarrow, n\rangle$ to $|\uparrow, n-1\rangle$, B_c to drive the system from $|\uparrow, n-1\rangle$ to $|\downarrow, n-1\rangle$ and finally B_r to go from $|\downarrow, n-1\rangle$ to $|\uparrow, n-2\rangle$.

To prepare an arbitrary finite superposition the simplest path is to take the system through the ground state. One assumes that the initial state is the desired state and then designs a sequence of alternating pulses of the E_c and E_r fields that would take this state to the ground state $|\downarrow, 0\rangle$ [13]. The actual sequence is the time-reversed sequence that was designed.

For example, if the desired superposition is $(\uparrow, 3\rangle + \downarrow, 2\rangle)/\sqrt{(}2)$, the sequence of pulses that will transfer this state to the ground state is

$$E_c^{(1)}(\pi) E_r^{(2)}(\phi_2) E_c^{(3)}(\phi_3) E_r^{(4)}(\phi_4) E_c^{(5)}(\phi_5) E_r^{(6)}(\phi_6) E_c^{(7)}(\phi_7). \tag{51}$$

The action of each pulse is the following: $E_c^{(1)}$ is a π pulse of the carrier field that moves the state $|\uparrow, 3\rangle$ to $|\downarrow, 3\rangle$. (Simultaneously, the population in $|\downarrow, 2\rangle$ is transferred to $|\uparrow, 2\rangle$). $E_r^{(2)}$ is a pulse of the red-sideband field that moves between the states $|\downarrow, 3\rangle$ and $|\uparrow, 2\rangle$. Since there is already a superposition of the two states, the duration of the red-sideband field is shorter than that of a π-pulse. Simultaneously, a superposition of $|\downarrow, 2\rangle$ and $|\uparrow, 1\rangle$ is created. The next transition $E_c^{(3)}(\phi_3)$ transfers population between $|\uparrow, 2\rangle$ and $|\downarrow, 2\rangle$, and again is shorter than a π pulse. This sequence progresses till all the population is in $|\downarrow, 0\rangle$. The actual sequence is the time-reversed sequence of the one that is described above — this creates the desired superposition from the initial ground state.

If one were to transfer an arbitrary initial superposition to an arbitrary final superposition of eigenstates, one employs the above algorithm twice. The sequences A and B that take the system from the initial and final superpositions respectively to the ground state are first calculated. Then the sequence A is first applied taking all the population to the ground state. The time time-reversed sequence of B is then applied which takes the population to the desired final superposition. In this way, the ground state is used as the "pass state", although strictly speaking, there does not seem to be a theoretical reason for using this route other than convenience. Clearly, this scheme works in finite time only if the initial and final states are both superpositions of a *finite* number of states.

Note that finite superpositions are dense in the Hilbert space of all possible states. Hence from our Lie algebra analysis and the use of the Law-Eberly algorithm we have

Theorem 6.1: The span of the Lie algebra generated by the operators B_c and B_r for the quantum control system in Eq. (38) is infinite-dimensional and the reachable set, which is dense in the Hilbert states of all states, includes all finite superpositions.

Note that this provides an explicit dense subspace controllability results which is hard to prove by abstract methods (see [26] and [27]).

B. Closed Subspaces and the Law-Eberly algorithm

In this subsection, we demonstrate why the dense subspace controllability results works in the context of the Law-Eberly algorithm even though the control Lie algebra spans states outside the dense subspace of finite superpositions.

This is best understood by writing the control matrices B_c and B_r in a re-ordered basis as follows: The eigenstates can be ordered as $|\uparrow,0\rangle,\ |\uparrow,1\rangle,\ \ldots,|\downarrow,0\rangle,\ |\downarrow,1\rangle,\ \ldots$. The drift Hamiltonian H_0 of this system can be written in matrix form as:

$$\left(\begin{array}{cc|cc|cc|c|cc}
0 & 0 & 0 & 0 & 0 & 0 & \ldots & 0 & 0 \\
0 & \omega_0 & 0 & 0 & 0 & 0 & \ldots & 0 & 0 \\
\hline
0 & 0 & \omega_m & 0 & 0 & 0 & \ldots & 0 & 0 \\
0 & 0 & 0 & \omega_0+\omega_m & 0 & 0 & \ldots & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 2\omega_m & 0 & \ldots & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \omega_0+2\omega_m & \ldots & 0 & 0 \\
\hline
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & & \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & \ldots & (\frac{N}{2}-1)\omega_m & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \ldots & 0 & \omega_0+(\frac{N}{2}-1)\omega_m
\end{array}\right). \qquad (52)$$

In the interaction picture, the Schrödinger equation written as

$$\dot{Y} = (u(t)B_c + v(t)B_r)Y, \qquad (53)$$

where $u(t)$ and $v(t)$ are defined as before. Then

$$B_c = \imath \left(\begin{array}{cc|cc|cc|c} 0 & L_0(\eta^2) & 0 & 0 & 0 & 0 & \ldots \\ L_0(\eta^2) & 0 & 0 & 0 & 0 & 0 & \ldots \\ \hline 0 & 0 & 0 & L_1(\eta^2) & 0 & 0 & \ldots \\ 0 & 0 & L_1(\eta^2) & 0 & 0 & 0 & \ldots \\ \hline 0 & 0 & 0 & 0 & 0 & L_2(\eta^2) & \ldots \\ 0 & 0 & 0 & 0 & L_2(\eta^2) & 0 & \ldots \\ \hline \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right). \tag{54}$$

$$B_r = \left(\begin{array}{c|cc|cc|cc} 0 & 0 & 0 & 0 & 0 & 0 & \ldots \\ \hline 0 & 0 & L_0^{(1)}(\eta^2) & 0 & 0 & 0 & \ldots \\ 0 & -L_0^{(1)}(\eta^2) & 0 & 0 & 0 & 0 & \ldots \\ \hline 0 & 0 & 0 & 0 & L_1^{(1)}(\eta^2) & 0 & \ldots \\ 0 & 0 & 0 & -L_1^{(1)}(\eta^2) & 0 & 0 & \ldots \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right). \tag{55}$$

All finite-dimensional subspaces are closed in the usual topology but infinite-dimensional subspaces need not be and there are a number of issues involving infinite-dimensional controllability that can be clarified by giving more attention to the distinction. Let l_2 denote the Hilbert space of infinite vectors whose entries are square summable. Let l_0 denote the subspace consisting of those elements with only a finite number of nonzero entries. Of course, this subspace is not closed.

These ideas are key to understanding the Law-Eberly algorithm in its infinite setting. We recall below some general ideas from operator theory and discuss their relationship to the Law-Eberly algorithm.

1) If e^A and e^B are bounded operators mapping a Hilbert space $\mathcal{H}$ into itself and if $\mathcal{V}$ is an invariant subspace for e^A and e^B, then $\mathcal{V}$ is invariant for the product $e^A e^B$.

   Example: In our setting e^{B_c} and e^{B_r} are unitary and bounded.

2) If A is the infinitesimal generator of a semigroup mapping a Hilbert space $\mathcal{H}$ into itself and if $\mathcal{V}$ is a closed, invariant subspace for A, then $\mathcal{V}$ is invariant subspace for e^{At}.

Example: To show that this is not true in general, consider the operator A on l_2 that maps the i^{th} unit basis vector e_i into e_{i+1} for all $i = 1, 2, ...$ This operator clearly sends l_0 into itself but e^{At} for $t \neq 0$ sends e_1 into the element $\sum_i e_i t^i / i!$ which is not in l_0.

3) If A and B are infinitesimal generators of a semigroups mapping a Hilbert space $\mathcal{H}$ into itself and if $\mathcal{V}$ is a closed, invariant subspace for A and B, then $\mathcal{V}$ is invariant subspace for $e^{(A+B)t}$.

Example: Again, the assumption that $\mathcal{V}$ is closed cannot be dispensed with. For example the operators

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & ... \\ -1 & 0 & 0 & 0 & 0 & 0 & ... \\ 0 & 0 & 0 & 1 & 0 & 0 & ... \\ 0 & 0 & -1 & 0 & 0 & 0 & ... \\ 0 & 0 & 0 & 0 & 0 & 1 & ... \\ 0 & 0 & 0 & 0 & -1 & 0 & ... \\ ... & ... & ... & ... & ... & ... & ... \end{bmatrix},$$

$$B = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & ... \\ 0 & 0 & 1 & 0 & 0 & 0 & ... \\ 0 & -1 & 0 & 0 & 0 & 0 & ... \\ 0 & 0 & 0 & 0 & 1 & 0 & ... \\ 0 & 0 & 0 & -1 & 0 & 0 & ... \\ 0 & 0 & 0 & 0 & 0 & 0 & ... \\ ... & ... & ... & ... & ... & ... & ... \end{bmatrix}.$$

leave l_0 invariant but the exponential of their sum does not.

The Law-Eberly model highlights the existence of important examples for which it is desirable for the evolution to occur on a non-closed subspace of a Hilbert space, i. e., the space of finitely nonzero elements. In this model, this non-closed subspace consists of vectors in the oscillator representation with finitely many nonzero elements. The B_c and B_r operators and their one parameter groups, leave invariant the subspace of l_2 consisting of finitely nonzero sequences. The semigroup $e^{\alpha B_c + \beta B_r}$ will not, however, typically have any nontrivial invariant subspace. In Law-Eberly one never in fact turns on both operators simultaneously. Further, as we have seen, the key to controllability is that each operator has different invariant subspaces within the set of finite superpositions. Thus, the proof of controllability that Law and Eberly give of what they term “arbitrary control” might be more accurately described as demonstrating that any state in l_0 can be mapped to any other state in l_0, staying within l_0. This points out the need for additional analysis concerning the treatment of non-closed subspaces.

VII. Example 2: Control of an N-level atom trapped in a quadratic potential

A more realistic model of a trapped-ion is that of an N-level system coupled to a quantum harmonic oscillator. Without loss of generality, it can be assumed that the energy levels in the ion are not equally spaced. If $N-1$ monochromatic, resonant fields are available to couple every pair of adjacent energy levels, the N-level system itself is transitively connected. It is necessary to have one more control field in order to make the $\mathcal{V}_i$ to $\mathcal{V}_{i-1}$ transition (ladder transition). There are multiple control schemes that are in keeping with the spirit behind the Generalized Subspace Controllability theorem. Consider the specific case when $N = 3$ (the generalization to higher N is fairly obvious). The eigenstates of the coupled system are shown graphically below in Fig. VII. The resonant fields ω_{c1} and $\omega_c 2$ that transitively connect the eigenstates of the ion are also shown.

Consider an additional field that accomplishes the ladder transition by connecting the $|1, n\rangle$ and the $|N, n-1\rangle$ states. In this case, the eigenstates of the coupled system are sequentially connected, and l_0 subspace control proceeds exactly as in the Law-Eberly and Kneer-Law schemes. The control matrices also look very similar. As before, in the interaction picture, the Schrödinger equation written as

$$\dot{Y} \;\; = \;\; (u_1(t)B_{c1} + u_2(t)B_{c2} + v(t)B_r)Y, \tag{56}$$

where $u_{1,2}(t)$ and $v(t)$ are defined as before. Then qualitatively, with ‘X’, ‘Y’, ‘Z_i’s denoting

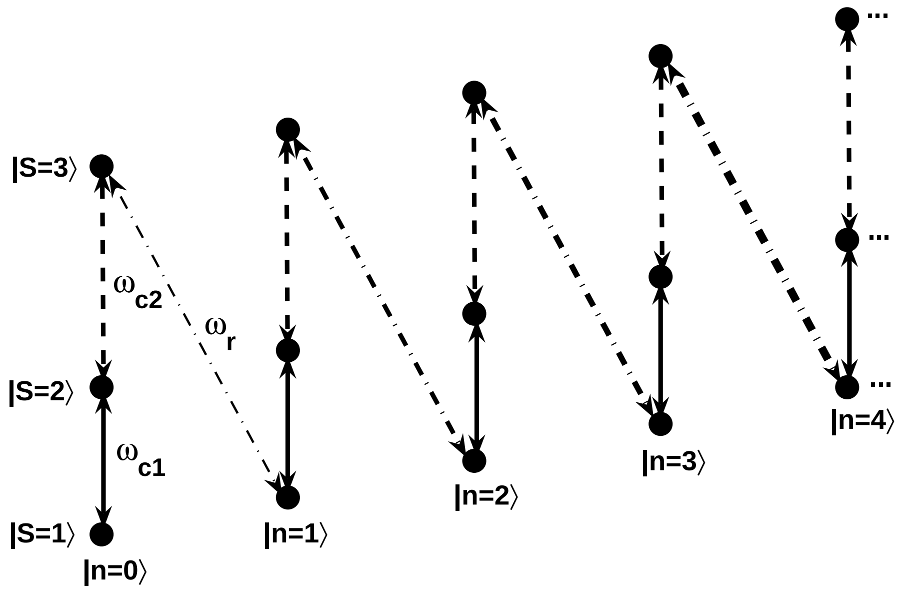

Fig. 3. Graphical representation of the N-level ion coupled to quantum harmonic oscillator driven by sinusoidal resonant fields ω_{c1}, ω_{c2} and ω_r as shown. The strengths of the ω_{c1} and ω_{c2} transition couplings are independent of the harmonic oscillator quantum number n, whereas the strengths of the ω_r transition couplings depend on n.

non-zero matrix elements,

$$B_{c1} = \imath \left(\begin{array}{ccc|ccc|cc} 0 & X & 0 & 0 & 0 & 0 & 0 & \dots \\ X & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ \hline 0 & 0 & 0 & 0 & X & 0 & 0 & \dots \\ 0 & 0 & 0 & X & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array} \right) . \tag{57}$$

$$B_{c2} = \imath \left(\begin{array}{ccc|ccc|cc} 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots \\ 0 & 0 & Y & 0 & 0 & 0 & 0 & \ldots \\ 0 & Y & 0 & 0 & 0 & 0 & 0 & \ldots \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots \\ 0 & 0 & 0 & 0 & 0 & Y & 0 & \ldots \\ 0 & 0 & 0 & 0 & Y & 0 & 0 & \ldots \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right). \tag{58}$$

$$B_{r} = \left(\begin{array}{ccc|ccc|cc} 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots \\ 0 & 0 & 0 & Z_1 & 0 & 0 & 0 & \ldots \\ \hline 0 & 0 & Z_2 & 0 & 0 & 0 & 0 & \ldots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots \\ 0 & 0 & 0 & 0 & 0 & 0 & Z_2 & \ldots \\ \hline 0 & 0 & 0 & 0 & 0 & Z_2 & 0 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right). \tag{59}$$

Another control scheme that exemplifies this theorem is an additional field that accomplishes the ladder transition by connecting the $|2,n\rangle$ and the $|N,n-1\rangle$ states, as shown in Fig. VII. Control in this case is a little more complicated, because the fields that transitively connect the N-level system must be correctly applied in order to bring the population to the $|2,n\rangle$ states before applying the ladder transition. The control matrices do not look as elegant as those in the first case, and practically this is a weaker scheme. As N increases, it is clear that the number of such schemes increases, however, the sequentially connected scheme is the one that is easiest to implement.

VIII. SUMMARY

Many of the novel questions that arise in laying the ground work for quantum computing can be thought of as questions about the controllability of Schrödinger's equation. Among the more interesting results in this direction are the theoretical analysis of Law and Eberly and its recent experimental verification by Ben-Kish et al. In this paper, we discuss a general setting for

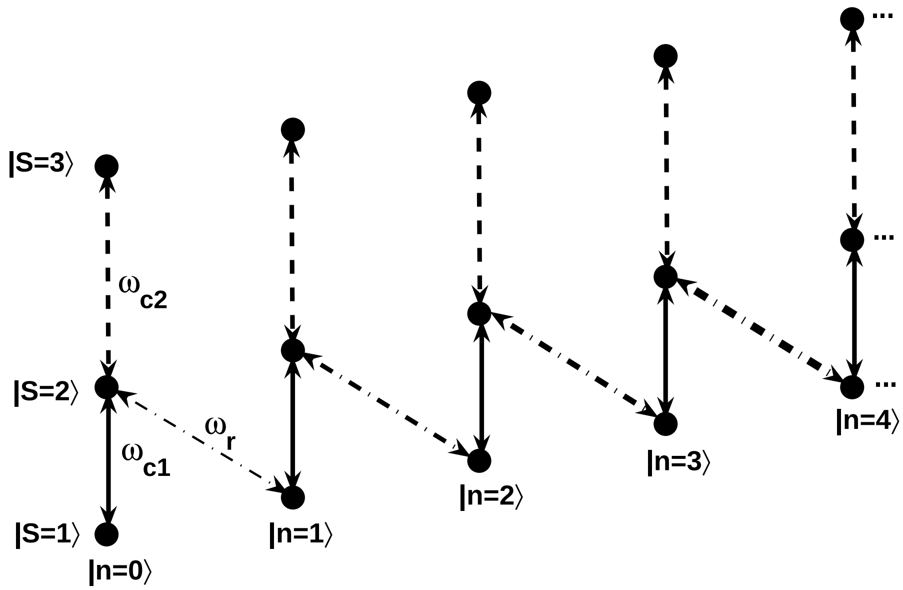

Fig. 4. Graphical representation of the N-level ion coupled to quantum harmonic oscillator driven by sinusoidal resonant fields ω_{c1}, ω_{c2} and ω_r as shown. This control scheme, although not very intuitive or elegant, is based on the Generalised subspace controllability theorem.

this type of problem based on infinite-dimensional differential equations and Lie groups acting on a Hilbert space. This allows us to explore the Lie algebraic approach to controllability in this setting. In particular, we show that even though the formal Lie algebra associated with the Jaynes-Cummings model is infinite-dimensional, explicit dense subspace controllability results can be determined. We also establish generalized subspace-controllability criteria for improving the controllability of some infinite-dimensional quantum systems.

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