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A Transitivity Problem from Control Theory

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1. CONTROLLABILITY—BILINEAR SYSTEMS OF DIFFERENTIAL EQUATIONS

Let $X_1, \dots, X_k$ be a family of C^∞ vector fields on a connected manifold M of dimension n, and let Ω be a family of functions on the real line with values in R^k, i.e., functions $u(t) = (u^1(t), \dots, u^k(t))$, $-\infty < t < \infty$, satisfying suitable conditions. In this paper for example, we shall suppose Ω to consist of *piecewise constant* functions. On M consider systems of differential equations determined by these data as follows:

$$dx/dt = u_1(t)\, X_1(x) + \cdots + u_k(t)\, X_k(x) \tag{1}$$

where $x \in M$ and $X_i(x)$ denotes the value of X_i at x.

A solution of Eq. (1) is a piecewise C^∞ curve $t \to x(t)$ on M whose tangent vector dx/dt coincides with the vector on the right-hand side for each $x = x(t)$. The system (1) is said to be *controllable* with respect to the family of control functions Ω if for two points $p, q \in M$ and $T > 0$, there is a $u(t) \in \Omega$ and a solution $x(t)$ of (1)—as determined by $u(t)$—such that $x(0) = p$ and $x(T) = q$. A well-known sufficient condition for controllability, which stems from a paper of W. L. Chow [3], is that the vector fields $X_1, \dots, X_k$ together with all vector fields obtainable from them by taking brackets, $[X_i, X_j]$, $[X_i[X_j, X_k]]$, etc., span the tangent space at each $x \in M$. Under conditions which apply to the case we consider, namely when the data are real analytic, D. L. Elliott [4] has proved a converse to this theorem.

A special case of the general system (1) described above is the *bilinear system* on $M = R^n - \{0\}$, punctured Euclidean space, with a system of ordinary differential equations written as follows:

$$dx/dt = (u_1(t)\, A_1 + \cdots + u_r(t)\, A_r)x. \tag{2}$$

Here $A_1, \dots, A_r$ is a set of real (constant) $n \times n$ matrices; $u(t) = (u_1(t), \dots, u_r(t))$ are piecewise constant control functions on R; and $X_i(x)$ is the vector with components $A_i x$ in the standard basis of R^n.

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Adapting a result of J. Kucera [7], D. L. Elliott and T. J. Tarn in an unpublished note [5] have shown that in the case of bilinear systems a necessary and sufficient condition for controllability is that the Lie algebra $\mathfrak{g} \subset \mathfrak{gl}(n, R)$ generated by $A_1, \ldots, A_r$ be the algebra of an analytic group $G \subset Gl(n, R)$ whose natural linear action on R^n is transitive on $R^n - \{0\}$. They listed a number of groups known to have this property and raised the question as to whether their list[1] was complete.

The purpose of this note is to answer this question and thus to classify the possible controllable bilinear systems. We have added several more groups to the tentative list of Elliott and Tarn and have shown that there are no others. Further, a clarification is made of the possibilities in their case[1] $R^+ \times F$; and finally some comments are included on the number r of matrices needed in Eq. (2).

The author wishes to thank Professor Elliott for communicating to him the contents of the note mentioned above and for raising this question to him.

2. Transitive Action on Punctured Euclidean Space

We will denote by $R_*{}^n$ the punctured Euclidean space $R^n - \{0\}$, and we consider connected Lie subgroups G of $Gl(n, R)$ with their natural action on R^n. For convenience we say that G is *transitive* if it is transitive on $R_*{}^n$ in this action. The following two theorems will enable us to determine all transitive G to within conjugacy in $Gl(n, R)$, i.e., up to choice of basis of R^n.

Theorem A. *Suppose that G is transitive and K is a maximal compact subgroup of G, and let $\langle x, y \rangle$ be a K-invariant inner product on R^n. Then K is transitive on the unit sphere $\tilde{S}^{n-1} = \{x \in R^n \mid \langle x, x \rangle = 1\}$.*

If G is bounded in $Gl(n, R)$, then its orbits are bounded in R^n, thus a necessary condition for G to be transitive is that G be unbounded. With this being so, a converse to the above theorem is the following—recall that $G \subset Gl(n, R)$ was assumed connected.

Theorem B. *If G is unbounded and has a maximal compact subgroup K whose action as a subgroup of $Gl(n, R)$ is transitive on $\tilde{S}^{n-1}$, the unit sphere of a K-invariant inner product, then G is transitive on $R_*{}^n$.*

Theorem A will enable us to use the fact that the groups transitive on spheres (as well as their actions) were completely determined by Montgomery

[1] The list of Elliott and Tarn reads as follows: $Gl(n, R)$, $Gl(n/2, C)$, $Sl(n, R)$, $Sl(n/2, C)$, and $R^+ \times F$, R^+ the multiplicative group of positive real numbers and F any group transitive on S^{n-1}.

and Samelson [10] and Borel [1], thus limiting the possible maximal compact subgroups K of G to certain very special cases. Theorem B will enable us to verify that our candidates for transitivity are indeed transitive on R_*^n. This may be done by showing that they are unbounded subsets of $Gl(n, R)$ either by simple direct verification, or by using the following fact: If the Lie algebra of a real, simple group is not a compact Lie algebra, then no nontrivial representation is bounded. Thus, if the Lie algebra $\mathfrak{g}$ of G contains a noncompact, simple component, then G is not bounded. This is a consequence of the facts that (i) a connected simple or semisimple subgroup of $Gl(n, R)$ is closed (an easy consequence of a result of Mostow) and (ii) simple or semisimple groups have compact Lie algebra if and only if they are themselves compact groups (see Helgason [8, pp. 120–128]).

The proofs of these two theorems are simple and direct. They could probably also be derived from much deeper theorems of Mostow [11].

Proof of Theorem A. Given the hypotheses of Theorem A, we may define a transitive, differentiable action of G on the sphere S^{n-1} of the usual inner product (x, y) on R^n as follows. We define a map $\theta\colon G \times S^{n-1} \to S^{n-1}$ by $\theta(A, x) = (1/\| Ax \|)\, Ax$ where $\| x \| = (x, x)^{1/2}$. For simplicity we let $A * x = \theta(A, x)$. This map is clearly differentiable since $Ax \neq 0$ for any $x \neq 0$ and $(A, x) \to Ax$ ($n \times n$ matrix times $n \times 1$ matrix) is differentiable. Moreover, θ defines a left action of G on S^{n-1}: let $y = B * x = (1/\| Bx \|)\, Bx$, then

$$A * (B * x) = \frac{1}{\| Ay \|} Ay = \frac{1}{\| Ay \|} A \left(\frac{Bx}{\| Bx \|} \right) = \frac{ABx}{\| Ay \| \, \| Bx \|}$$

$$(AB) * x = \frac{ABx}{\| ABx \|}.$$

However $\| Ay \| = \| A(Bx/\| Bx \|)\| = \| ABx \|/\| Bx \|$, so we have

$$A * (B * x) = (AB) * x,$$

as required.

Quite obviously, this action consists of letting $A \in G$ act on $x \in S^{n-1}$ by matrix multiplication and then defining $A * x$ to be the unit vector lying on Ax. It follows that G is transitive on S^{n-1}, which is simply connected (for $n > 2$) and compact. According to a theorem of Montgomery [9], these facts imply that G has a maximal compact subgroup K_1, which is transitive on S^{n-1} relative to this action $*$. But this is equivalent to stating that K_1 acting on R^n by matrix multiplication as a subgroup of $G \subset Gl(n, R)$ is transitive on *directions*, i.e., rays from the origin. Clearly any group K which is conjugate to K_1 in G or in $Gl(n, R)$ is then also transitive on directions. It is known that all maximal compact subgroups of G are conjugate in G (Iwasawa [15]);

thus, as was claimed, any maximal compact subgroup K of G is transitive on the unit-sphere $\tilde{S}^{n-1}$ of any K-invariant inner product $\langle x, y \rangle$ on R^n.

Remark. It is easily verified that if G, G' are conjugate in $Gl(n, R)$, then G is transitive on $R_*{}^n$ if and only if G' is also transitive. The same remark applies to the property of transitivity on directions. Thus if K is transitive on $\tilde{S}^{n-1}$, the unit sphere of the K-invariant inner product $\langle x, y \rangle$, then any conjugate K' of K which is in $O(n)$ is transitive on the *standard* unit sphere S^{n-1} of R^n. Combining this with the fact [15] that every compact subgroup K of $Gl(n, R)$ is conjugate in $Gl(n, R)$ to a subgroup $K' \subset O(n)$, we see that no generality is lost in assuming the subgroup K of Theorem B to lie in $O(n)$—replacing it and the group G by conjugate groups if necessary. This clarifies, too, the meaning of our earlier statement that the classification we give of transitive groups is only to "within conjugacy."

Proof of Theorem B. Spheres $S_r = \{x \in R^n \mid \| x \| = r\}$ are the orbits of K. Combining this with our assumption that G is connected, we see that the orbits of G consist of connected subsets of R^n which are the unions of spheres. Thus it is sufficient to show that the orbit of G contains vectors of arbitrarily large norm and vectors of arbitrarily small norm.

Each $A \in G \subset Gl(n, R)$ decomposes uniquely in $Gl(n, R)$ into a product $A = US$ where U is orthogonal and S is symmetric positive definite. The characteristic values of S are real and positive, say, $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n > 0$. If $A = (a_{ij})$, then $\sum_{i,j} a_{ij}^2$ may be expressed as the trace of ${}^tAA$, the transpose of A times A; giving

$$\sum a_{ij}^2 = \operatorname{Tr}({}^tAA) = \operatorname{Tr}({}^t(US)(US)) = \operatorname{Tr} S^2 = \sum_{u=1}^{n} \lambda_i^2.$$

It follows that the characteristic values of the symmetric parts of matrices of G are bounded only if there is a uniform bound N for the entries of all matrices in G, i.e., G lies in a bounded set of matrices, contrary to hypothesis. Therefore, given any $N > 0$, there is an $A \in G$ such that $\lambda_1 > N$, λ_1 being the largest characteristic value of the symmetric part S of A. Let x be a unit vector such that $Sx = \lambda_1 x$ and note that

$$\| Ax \| = \| USx \| = \| Sx \| = \lambda_1 > N.$$

If $e_1 = (1, 0, \ldots, 0)$, then $x \in Ke_1$ and $Ax \in Ge_1$, so the orbit of e_1 contains a vector of length greater than N.

A parallel argument, using the fact that the characteristic value of S^{-1} are $\lambda_1^{-1} \leqslant \lambda_2^{-1} \leqslant \cdots \leqslant \lambda_n^{-1}$, shows that Ge_1 contains vectors of length arbitrarily close to zero. Thus the orbit of e_1 is all of $R_*{}^n$ as claimed.

3. Reformulation of the Problem—First Cases

In order to achieve our objective of determining all transitive subgroups of $Gl(n, R)$, we may answer the following more general question: what are the connected Lie groups $\tilde{G}$ which have a representation ρ on R^n which is transitive on $R_*{}^n$ and, for each such $\tilde{G}$, what are the representations with this property? For each pair $\tilde{G}$, ρ the group $G = \rho(\tilde{G})$ is a subgroup of $Gl(n, R)$ which is transitive on $R_*{}^n$; conversely each G transitive on $R_*{}^n$ is so given.

We note that a representation (ρ, R^n) of $\tilde{G}$ which is transitive in this sense is necessarily irreducible. It is known [6] that this implies the following facts about the Lie algebra $\tilde{\mathfrak{g}}$ of $\tilde{G}$:

(3.1) $\tilde{\mathfrak{g}}$ *is reductive, i.e.,* $\tilde{\mathfrak{g}} = \tilde{\mathfrak{g}}_1 \oplus \cdots \oplus \tilde{\mathfrak{g}}_r \oplus \tilde{\mathfrak{c}}$ *where* $\tilde{\mathfrak{g}}_i$ *are simple and* $\tilde{\mathfrak{c}}$ *is the center of* $\tilde{\mathfrak{g}}$. *Moreover, the matrices representing elements of* $\tilde{\mathfrak{c}}$ *are semi-simple.*

One consequence of (3.1) is that we may restrict our attention to *faithful* representations ρ of $\tilde{G}$ in R^n. For let ρ denote the Lie algebra representation of $\tilde{\mathfrak{g}}$ and note that reductivity of $\tilde{\mathfrak{g}}$ implies the existence of an ideal complementary to the kernel of ρ. Not only is ρ determined by its restriction to this ideal, on which it is faithful, but the image of this ideal is the same as the image of $\mathfrak{g}$. Therefore from the Lie algebra point of view it is no loss of generality to suppose ρ is faithful. This implies that $\rho\colon \tilde{G} \to G$ is a covering homomorphism. However, once G is determnied, we may determine its coverings if we need them—which we do not if our program is merely to determine the transitive subgroups $G \subset Gl(n, R)$.

We now remark that Theorem A and the remarks on conjugate groups allow us to suppose that the maximal compact subgroup of G is transitive on the unit sphere S^{n-1}. Montgomery and Samelson [10] and Borel [1] (see also Poncet [12] have determined all groups $\tilde{K}$ which act transitively on S^{n-1}. They all act linearly, i.e., by a representation θ of $\tilde{K}$ on R^n with $\theta(\tilde{K}) = K \subset O(n)$. The groups $\tilde{K}$ are the following:

(3.2) (i) $SO(n)$ for all n, and θ the defining representation, $\tilde{K} = K$.

(ii) $SU(m)$ and $U(m)$ for $n = 2m$, θ the standard imbedding in $O(n)$.

(iii) (a) $Sp(k)$, $n = 4k$; θ the standard imbedding in $O(n)$ and (b) $Sp(k) \cdot Sp(1)$, $n = 4k$; θ to be explained later.

(iv) three exceptional groups: (a) $G_{2(-14)}$, $n = 7$; (b) Spin(5), $n = 8$; and (c) Spin(7), $n = 16$. (The representations θ in the cases are unique and well known, see [1, 11]; we do not need their precise form.)

We let α denote the standard representation of $Gl(m, C)$ as a subgroup of $Gl(2m, R)$ and β the standard representation of $Gl(k, H)$—the H-linear automorphisms of the right-quaternion space H^k ($H =$ quaternions)—as a subgroup of $Gl(4k, R)$. Finally, each $q = x + yi + uj + vk \in H$ acts on H^k by right multiplication and as such determines an R-linear transformation of H^k. Since H^k with scalars restricted to R is isomorphic to R^{4k}, this action of q determines an element $\gamma(q) \in Gl(4k, R)$ (which commutes with the matrices in the image of β). The matrices so determined are given as follows:

(3.3)
$$\alpha(A + iB) = \begin{pmatrix} A & B \\ -B & A \end{pmatrix}$$

$$\beta(P + Qi + Rj + Sk) = \begin{pmatrix} P & -Q & R & -S \\ Q & P & -S & R \\ R & S & P & -Q \\ S & -R & Q & P \end{pmatrix}$$

$$\gamma(q) = \begin{pmatrix} xI & -yI & -uI & -vI \\ yI & xI & vI & -uI \\ uI & -vI & xI & yI \\ vI & uI & -yI & xI \end{pmatrix}$$

A, B $m \times m$ and P, Q, R, S $k \times k$ *real* matrices, $I = k \times k$ identity. With this notation we have

(3.4) *In* (ii) θ *is the restriction of* α *to* $\tilde{K} = SU(m)$ *or* $U(m)$ *and in* (iii)(a) θ *is the restriction of* β *to* $\tilde{K} = Sp(k)$, *which describes* $K = \theta(\tilde{K})$.

We denote the Lie algebras of $\tilde{G}$, G, K by $\tilde{\mathfrak{g}}$, $\mathfrak{g}$, $\mathfrak{k}$ respectively and recall that $\rho: \tilde{G} \to G$ is a covering so the induced $\rho: \tilde{\mathfrak{g}} \to \mathfrak{g}$ is an isomorphism. Using (3.1) we see that k must decompose into $\mathfrak{k} = \mathfrak{k}_1 \oplus \cdots \oplus \mathfrak{k}_r \oplus \mathfrak{c}_1$ with $\mathfrak{k}_i \subset \mathfrak{g}_i$, $i = 1,..., r$ and $\mathfrak{c}_1 \subset \mathfrak{c}$. According to the list of groups K above, $r = 1$ except in case (iii)(b) when $r = 2$. Moreover $\mathfrak{c}$ must centralize $\mathfrak{k}$ which is irreducible on R^n, and therefore by Schur's Lemma [1] is either (a) $\{\lambda I \mid \lambda \text{ real}\}$ with I the $n \times n$ identity or (b) the image under α of $\{(u + iv)I \mid u, v \text{ real}\}$ with I the $m \times m$ identity matrix in $Gl(m, C)$. Using these facts, we will now divide our problem into three cases:

(3.5) (I) $r = 1$, and $\mathfrak{g}_1 = \mathfrak{k}_1$, i.e., the semisimple part of $\mathfrak{g}$ is simple and compact.

(II) $r = 1$ and $\mathfrak{g}_1 \supsetneq \mathfrak{k}_1$, i.e., the semisimple part of $\mathfrak{g}$ is simple and noncompact.

(III) $r = 2$ and $\mathfrak{k}_1 \oplus \mathfrak{k}_2 = \mathfrak{sp}(k) \oplus \mathfrak{sp}(1)$. This corresponds to (iii)(b) above with $\tilde{K} = \mathfrak{sp}(k) \times \mathfrak{sp}(1)$; it contains subcases similar to (I) and (II).

We shall determine the groups G transitive on S^{n-1} by finding all of those in each category above. Case (I) is the simplest: We have $\mathfrak{k} = \mathfrak{k}_1 \oplus \mathfrak{c}_1$, the Lie algebra of K and $\mathfrak{g} = \mathfrak{k}_1 \oplus \mathfrak{c}$, the Lie algebra of G. The Lie algebra $\tilde{\mathfrak{g}}$, as we have agreed, is isomorphic to $\mathfrak{g}$ and $\rho: \tilde{\mathfrak{g}} \to \mathfrak{g} \subset \mathfrak{gl}(n, R)$ must be faithful. The group $\tilde{K}$ must be one of the compact simple groups transitive on spheres (or $U(m) = SU(m) \cdot T^1$) and K must be $\theta(\tilde{K})$. All that remains to be done is to determine the center of G. The center must be unbounded; and if it is, transitivity follows by Theorem B. Therefore we have these possibilities:

(3.6) Transitive groups of type (I)

(i)	$G = SO(n) \cdot C' \subset Gl(n, R)$,	$\dim C' = 1$.
(ii)	$G = SU(m) \cdot C' \subset Gl(2m, R)$,	$\dim C' = 1$ or 2.
(iii)	$G = Sp(k) \cdot C' \subset Gl(4k, R)$,	$\dim C' = 1$ or 2.
(iv)(a)	$G = G_{2(-14)} \cdot C' \subset Gl(7, R)$,	$\dim C' = 1$,
(b)	$G = \mathrm{Spin}(5) \cdot C' \subset Gl(8, R)$,	$\dim C' = 1$ or 2,
(c)	$G = \mathrm{Spin}(7) \cdot C' \subset Gl(16, R)$,	$\dim C' = 1$ or 2,

[$K \cdot C'$ denotes the subgroup of $Gl(n, R)$ generated by K and C'].

In cases (i) and (iv)(a) the full centralizer is the group $\{\lambda I \mid \lambda \text{ real}\}$ so that C' is uniquely determined—it is the full centralizer. In cases (ii), (iii), (iv)(b), and (iv)(c) the groups are the images under α of subgroups of $Gl(m, C)$, $GL(4, C)$, and $Gl(8, C)$ respectively, and their centralizers are the image of $\{zI \mid z \in C\}$ in $Gl(m, C)$. Hence C' may be all of this image or any subgroups of it of the form $\{e^{ct}I \mid t \in R, c = u + iv, u \neq 0\}$. Note that $U(m) = SU(m) \cdot T^1$, $T^1 = \{e^{it} \mid t \in R\}$.

Case II is more tedious. We know that ρ is an isomorphism of $\tilde{\mathfrak{g}}$ onto $\mathfrak{g} \subset \mathfrak{gl}(n, R)$ and we have assumed the same for $\rho: \tilde{G} \to G \subset Gl(n, R)$. Although $\tilde{G}$ and G are not known, $\tilde{G}$ may be assumed to have one of the above groups $\tilde{K}$ transitive on S^{n-1} as its maximal compact subgroup. Moreover $\rho = \theta$ [see (3.4)] on $\tilde{K}$. Our procedure for determining all possibilities is as follows:

(1) Consult the standard list of simple groups $\tilde{G}_1$ having the given group $\tilde{K}$ as maximal compact subgroup, e.g., Helgason [8, Tables I and II, Chapter IX].

(2) Find those representations ρ of $\tilde{G}_1$ which agree with θ on $\tilde{K}$. This was done with the aid of Tits [13]. The crucial question, it turns out, is to check the dimensions of the representations ρ of $\tilde{G}_1$, which in only a few special and natural cases is the same as the (given) representation $\theta = \rho \mid \tilde{K}$, as it must be. This narrows the possible ρ for each $\tilde{G}_1$ to a few possibilities which are checked individually.

(3) Finally, verify that the hypothesis of Theorem B is satisfied. This is quite simple and will be omitted, especially in view of the remarks following the statements of Theorems A and B:

Applying this program we find by procedure (1) (with $n = \dim \theta$):

(3.7) (i) For $\tilde{K} = SO(n)$, $\tilde{G}_1$ may be $Sl(n, R)$ or $O(n, C)$.

(ii) For $\tilde{K} = SU(m)$, $\tilde{G}_1$ may be $Sl(m, C)$, $n = 2m$,
For $\tilde{K} = U(m)$, $\tilde{G}_1$ may be $SO^*(n)$ or $Sp(m, R)$, $n = 2m$.

(iii) For $\tilde{K} = Sp(k)$, $\tilde{G}_1$ may be $Sp(k, C)$ or $SU^*(2k)$, $n = 4k$.

(iv) There are also possibilities corresponding to the three exceptional cases of $K = \tilde{K}$ transitive on S^6, S^7, and S^{15}, which we will not enumerate (see below).

Next we check the representations of these groups, say in [13], and we find that the real representation of lowest dimension for $O(n, C)$ is on R^{2n} and for $SO^*(n)$ is on R^{2n}, so that $\rho \mid \tilde{K}$ is not the same as θ. Therefore these groups are not transitive on $R_*{}^n$. The same reasoning eliminates all cases with $\tilde{K}$ exceptional as in (iv). These are straightforward verifications from the tables of [13]; we omit the details. There remain the following five cases with the representations indicated. In each case it is possible also to include any (connected) subgroup C' of the centralizer C if we wish. The centralizer is again the group of real or complex scalar multiples of I in $Gl(n, R)$ or $Gl(n, C)$; we indicate which by giving the bound on its dimension.

(3.8) Transitive groups of type (II)

(i) $G = Sl(n, R) \cdot C$, $\dim C \leqslant 1$.

(ii) $\tilde{G} = Sl(m, C) \cdot C$ and G is its image in $Gl(n, R)$, $r = 2m$; $\dim C \leqslant 2$.

(iii) $\tilde{G} = Sp(m, R) \cdot C$, $n = 2m$, and G is its image in $Gl(n, R)$, the subgroup of all elements of $Gl(n, R)$ leaving invariant

$$\Omega = dx^1 \wedge dx^{m+1} + \cdots + dx^m \wedge dx^{2m}, \qquad \dim C \leqslant 1.$$

(iv) $\tilde{G} = Sp(k, C) \cdot C$, $Sp(k, C)$ the subgroup of $Gl(2k, C)$ leaving invariant the form $\Omega = dz^1 \wedge dz^{k+1} + \cdots + dz^k \wedge dz^{2k}$, represented as $G \subset Gl(4k, R)$ by the representation α of (3.3), $\dim C \leqslant 2$.

(v) $\tilde{G} = Gl(k, H) \cdot C$, $Gl(k, H)$ represented in $Gl(4k, R)$ by β of (3.3), this is also the image by α of $SU^*(2k) \subset Gl(2k, C)$, $\dim C \leqslant 2$.

These are, in fact, real simple groups and their maximal compact subgroups: $Sl(n, R) \supset SO(n)$, $Sl(m, C) \supset SU(m)$, $Sp(m, R) \supset U(m)$, $Sp(k, C) \supset Sp(k)$, $SU^*(2k) \supset Sp(k)$. These representations ρ are the standard ones of the tables of [8, Chapter IX] and restrict to θ on the compact subgroups $\tilde{K}$. In the case of $Sl(n, R)$ and $Sl(m, C)$ there is a second representation: ρ^*, the contragrediant to ρ. Although these representations are inequivalent—so that the actions on $R_*{}^n$ are inequivalent—the images $G = \rho(\tilde{G})$ in $Gl(n, R)$ are the same subgroup.

4. The Remaining Cases

The only case left to consider is that with maximal compact subgroup $\tilde{K} = Sp(k) \times Sp(1)$, $n = 4k$. The action of $\tilde{K}$ on S^{n-1} or, what is the same, the representation θ is (according to [10]) obtained by considering R^n as the right-quaternion vector space H^k, with the usual norm and letting $Sp(k)$ denote the H-linear transformations of H^k which preserve the norm and $Sp(1)$ be quaternions of norm one acting as right (scalar) multiplication on H^k (see also [2]). This is a representation of $\tilde{K}$ since these operations commute. If we restrict the scalars to $R \subset H$, then $(H^k)_R = R^{4k}$ and this gives the representation θ on R^n, $n = 4k$. Restricted to $Sp(k)$, this θ is exactly β and the representation is the same as (3.7) (iii)(a). By Schur's Lemma again [2] the set of all non-singular transformations on H^k commuting with every element of $Sp(k)$ is exactly H^*, the nonzero quaternions, acting by right multiplication. Thus the centralizer of K consists of the largest commutative subalgebra of H commuting with $Sp(1)$ the quaternions of norm one in H, i.e., R^* acting by scalar multiplication. Therefore the first possibility for $\tilde{G}$ is $Sp(k) \times Sp(1) \times R$ corresponding to $\tilde{\mathfrak{g}} = \tilde{\mathfrak{k}}_1 \oplus \tilde{\mathfrak{k}}_2 \oplus \mathfrak{c}$; we will discuss G below.

Next suppose that $\tilde{\mathfrak{g}}_1 \neq \tilde{\mathfrak{k}}_1$. There appear to be two possibilities for $\tilde{G}_1$ with $\tilde{K}_1 = Sp(k)$ as maximal compact subgroup, namely by (3.7) (iii)(a) $Sl(k, H) = SU^*(2k)$ and $Sp(k, C)$. However, according to the tables of [13], $Sp(k, C)$ contains no representations of quaternion type so that $Sl(k, H)$ is the only possibility. By Schur's Lemma the multiplicative group of nonzero quaternions H^* is the only possible $\tilde{G}_2$. If we note that $\tilde{\mathfrak{k}}_2 \oplus \mathfrak{c}$ above is just the Lie algebra of H^*, i.e., H^* contains R, the centralizer of $Sp(k) \cdot Sp(1)$, then we have the following possibilities for this final case.

(4.1) Transitive groups of type (III)[2]

(i) $\tilde{G} = Sp(k) \times H^*$ and $G = \beta(Sp(k)) \cdot \gamma(H^*) \subset Gl(n, R)$.

(ii) $\tilde{G} = Sl(k, H) \times H^*$ and $G = \beta(Sl(k, H)) \cdot \gamma(H^*) \subset Gl(n, R)$.

(iii) $\tilde{G} = Sl(k, H) \times Sp(1)$ and $G = \beta(Sl(k, H)) \cdot \gamma(Sp(1) \subset Gl(n, R)$ [$n = 4k$, $k > 1$, and β, γ are as in (3.3)].

[2] In [8] $Sl(k, H)$ is denoted $SU^*(2k)$, then $\alpha(SU^*(2k)) = \beta(Sl(k, H))$.

5. A Notion of Equivalence

Let $X_1, \ldots, X_k$ and $Y_1, \ldots, Y_l$ be vector fields on a manifold M determining systems of ordinary differential equations as in Section 1:

$$\text{(1a)} \quad \frac{dx}{dt} = \sum_{i=1}^{k} u_i X_i \qquad \text{and} \qquad \text{(1b)} \quad \frac{dx}{dt} = \sum_{j=1}^{l} u_j Y_j$$

with $u_1(t), u_2(t), \ldots$ piecewise constant as before. We shall say that (1a) and (1b) are *equivalent* if at each $p \in M$ the vector fields $X_1, \ldots, X_k$ and their brackets span the same subspace of $T_p(M)$ as do $Y_1, \ldots, Y_l$ and their brackets. In this case the subspaces of M traversed by the solution curves are the same for each system. It is known that a semisimple Lie algebra over R is generated by two elements (Kuranishi [14], see also Jacobson [6, p. 150, Exercise 8]). Using this result, one obtains as an immediate consequence the following theorem for controllable bilinear systems (2) on $M = R_*^n = R^n - \{0\}$.

THEOREM C. *No matter how large n may be, a controllable bilinear system on R_*^n is equivalent to one determined by two vector fields, i.e.,* (2) *is equivalent to a system:*

$$dx/dt = (uA + vB)x$$

with u, v piecewise constant functions.

Proof. This theorem follows from the result of Kuranishi ([14, Theorem 6]) and the classification above once the following rather trivial lemma is established.

LEMMA. *Let $\mathfrak{g}$ be a reductive Lie algebra with center of dimension $\leqslant k$. If the semisimple part of $\mathfrak{g}$ is generated by k elements, then $\mathfrak{g}$ is generated by k elements.*

Proof. Let $\mathfrak{g} = \mathfrak{s} + \mathfrak{c}$, $\mathfrak{s}$ a semisimple ideal and $\mathfrak{c}$ the center. Suppose $X_1, \ldots, X_k$ generate $\mathfrak{s}$. Since $\mathfrak{s} = [\mathfrak{s}, \mathfrak{s}]$, $X_1, \ldots, X_k$ must each be linear combinations of brackets of $X_1, \ldots, X_k$. Let $C_1, \ldots, C_k$ span $\mathfrak{c}$ then any linear combination of brackets of $X_1, \ldots, X_k$ is unchanged if we replace each X_i, $i = 1, \ldots, k$, by $X_i' = X_i + C_i$. Hence $X_1', \ldots, X_k'$ generate $\mathfrak{s}$ and, in particular, $X_1, \ldots, X_k$ are linear combinations of brackets of $X_1', \ldots, X_k'$. It follows that every element of $\mathfrak{g}$ is a linear combination of $X_1', \ldots, X_k'$ and brackets of these elements.

Since the dimension of the center of $\mathfrak{g}$, the Lie algebra of G, is $\leqslant 2$ if G is transitive on R_*^n, Theorem C follows from the result of Kuranishi cited above. The proof of this theorem is very short; we include it for the sake of completeness.

THEOREM (Kuranishi). *Let $\mathfrak{g}_0$ be a real semisimple Lie algebra, then there exist two elements A, B of $\mathfrak{g}_0$ which generate $\mathfrak{g}_0$ as an algebra.*

To prove the theorem, one first obtains the result for the complexification $\mathfrak{g}$ of $\mathfrak{g}_0$. Choose a fixed Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$, and let $\Delta = \{\alpha\}$ denote the roots and $\mathfrak{g}^\alpha$ the corresponding root space. For each $\alpha \in \Delta$ let $H_\alpha \in \mathfrak{h}$ be the unique element satisfying $\alpha(H) = (H_\alpha, H)$, with (X, Y) denoting the Killing form, and let $\{E_\alpha \in \mathfrak{g}^\alpha \mid \alpha \in \Delta\}$ be a Weyl basis, so that $[E_\alpha, E_{-\alpha}] = H_\alpha$ and $ad(H) E_\alpha = [H, E_\alpha]$. Then $\mathfrak{g} = \mathfrak{h} + \sum_{\alpha\in\Delta} \mathfrak{g}^\alpha$ and we let $\tilde{\mathfrak{g}}$ denote the subspace $\sum_{\alpha\in\Delta} \mathfrak{g}^\alpha$. Note that $\tilde{\mathfrak{g}}$ generates $\mathfrak{g}$ and is invariant under $ad(H)$ for all $H \in \mathfrak{h}$. Choose $H_0 \in \mathfrak{h}$ so that the numbers $\alpha(H_0)$ over all $\alpha \in \Delta$ are distinct, then $ad(H_0)$ is cyclic on $\tilde{\mathfrak{g}}$, i.e., its minimal and characteristic polynomial are equal and so there is an $X_0 \in \tilde{\mathfrak{g}}$ such that X_0, $ad(H) X_0$, $ad(H)^2 X_0$,..., span $\tilde{\mathfrak{g}}$. Clearly, then, H_0 and X_0 generate $\mathfrak{g}$, i.e., there is a set of n ($= \dim \mathfrak{g}$) monomial expressions in brackets of H_0 and X_0, say $P_i(H_0, X_0) = [\cdots [H_0, X_0] \cdots]$, $i = 1,..., n$ which form a basis of $\mathfrak{g}$.

Now choose a fixed basis $F_1,..., F_n$ of the real Lie algebra $\mathfrak{g}_0$, it is also a basis of $\mathfrak{g}$. We define an $n \times n$ matrix of polynomials (with real coefficients) $(\lambda_i^j(z^1,..., z^n, w^1,..., w^n))$ in the $2n$ complex variables $(z, w) = (z^1,..., z^n, w^1,..., w^n)$ by

$$P_i\left(\sum_k z^k F_k, \sum_l w^l F_l\right) = \left[\cdots \left[\sum_k z^k F_k, \sum_l w^l F_l\right] \cdots\right] = \sum_{j=1}^{n} \lambda_i^j(z, w) F_j,$$

i.e., by expressing the monomials $P_i(z, w)$, $z = \sum_k z^k F_k$ and $w = \sum_1 w^1 F_1$, as linear combinations of $F_1,..., F_n$ for each $i = 1,..., n$. Consider $\det(\lambda_i^j(z, w))$, it is a polynomial in the $2n$ variables (z, w), which has real coefficients and does not vanish for the values (z_0, w_0) such that $H_0 = \sum z_0^k F_k$ and $X_0 = \sum w_0^1 F_1$. Since it is not identically zero it must be different from zero also at a set of $2n$ *real* values $(a, b) = (a^1,..., a^n, b^1,..., b^n)$ of (z, w). This means that $F_1,..., F_n$ can be written as linear combinations of $P_i(\sum_k a^k F_k, \sum_l b^l F_l)$, $i = 1,..., n$ and proves the theorem.

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