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On the Controllability of Nonlinear Systems with Applications to Polynomial Systems*

Hiroshi Kunita

Department of Applied Science, Faculty of Engineering, Kyushu University, Fukuoka 812, Japan and Department of System Science, School of Engineering and Applied Science, University of California, Los Angeles, CA. 90024

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Abstract. The purpose of this paper is to examine the controllability for a class of nonlinear control systems of the form

$$\frac{d\phi_t}{dt} = X_0(\phi_t) + \sum_{j=1}^{r} u_j(t)X_j(\phi_t)$$

Our general theorem is stated in terms of differential geometry, but it will be applied to concrete systems such that coefficients $X_0, \ldots, X_r$ are polynomials.

1. Introduction

We shall consider the control system of the form

$$\frac{d\phi_t}{dt} = X_0(\phi_t) + \sum_{j=1}^{r} u_j(t)X_j(\phi_t), \qquad \phi_0 = x_0 \in M, \tag{1.1}$$

where M is a connected C^∞-manifold over M and the controls $u_j(t)$ are in U, the class of piecewise constant functions from $[0,\infty]$ into $R=(-\infty,\infty)$. We denote the solution of (1.1), defined up to the explosion time, as $\phi_t^u(x_0)$. Here $u=u(t)=(u_1(t),\ldots,u_r(t))$. Let $x,y \in M$. We say that y is attainable from x at time $t>0$ if there exists a control u such that $\phi_t^u(x)=y$. We denote by $A_t(x)$ the attainable set from x at time t, and $A(x)=\cup A_t(x)$. The system (1.1) is called strongly completely controllable if $A_t(x)=M$ holds for all $t>0$ and $x \in M$. If $A(x)=M$ holds for all x, the system is called completely controllable.

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In this paper, we shall characterize the condition of the strong controllability by means of coefficients (=vector fields) $X_0,\dots,X_r$ of (1.1), and apply it to a nonlinear system such that $X_0,\dots,X_r$ are polynomial functions.

There are extensive studies for the controllability problem. See for example, Hirschorn [2,3], Lobry [5] and Kunita [4]. These results may be applied satisfactorily to the linear or bilinear systems together with some nice system in Lie groups. It seems to us, however, that little is known of other concrete systems except the case where Chow's theorem can be applied. Our results might be regarded as a step to these unfamiliar nonlinear systems.

2. Vector Fields with Attainable Integral Curves

Let us first introduce several notations and terminologies. We denote by $\mathbf{V}(M)$ the set of all C^∞-vector fields on the manifold M. Then $\mathbf{V}(M)$ is a Lie algebra over $C^\infty(M)$, the space of all real C^∞-functions on M. That is to say, it is a vector space over $C^\infty(M)$ with multiplication by Lie bracket $[X,Y]=XY-YX$. The linear map $\mathrm{ad}_X:\mathbf{V}(M)\to\mathbf{V}(M)$ is defined by

$$\mathrm{ad}_X Y=[X,Y],\qquad Y\in\mathbf{V}(M).$$

Given a vector field X, we denote by $X_t(x_0)$ the integral curve of X starting at x_0 at time 0. Hence it satisfies

$$\frac{d}{dt}X_t(x_0)=X(X_t(x_0)),\qquad X_0(x_0)=x_0.$$

If $X_t(x_0)$ is defined for all $t\in R$ for any $x_0\in M$, the vector field X is called complete.

Suppose now X is a complete vector field. The differential dX_t is, by definition, the isomorphism of $\mathbf{V}(M)$ such that dX_tY is a vector field whose integral curve is $X_{-t}Y_sX_t$, $s\in R$. The following formula is basic in our later discussion.

$$\frac{d}{dt}dX_tY=dX_t[Y,X]=-dX_t\mathrm{ad}_XY. \tag{2.1}$$

Given a subset $\mathbf{V}$ of $\mathbf{V}(M)$, the restriction of $\mathbf{V}$ to a point x of M, i.e., $\mathbf{V}(x)=\{Y(x);\,Y\in\mathbf{V}\}$ is a tangent subspace of M at x. Further, $\tilde{\mathbf{V}}=\{\mathbf{V}(x):x\in M\}$ is a differential system. The Lie algebra generated by $\mathbf{V}$ is denoted by $\mathbf{L}(\mathbf{V})$.

Now let us come back to the control system (1.1). We shall call that the integral curve $X_t(x)$ is attainable (by the system (1.1)) if both of $X_t(x)$ and $X_{-t}(x)$ belong to $\overline{A_t(x)}$ for any $t>0$ up to the explosion time. If only $X_t(x)$ belongs to $\overline{A_t(x)}$, it is called semi-attainable. More strongly, if the whole curve $\{X_s(x)\}$ belongs to $\overline{A_t(x)}$ for any $t>0$, the curve is called strongly attainable. Strongly semi-attainable curve is defined similarly. We denote by $\mathbf{A}$ (resp. $\mathbf{A}_s$) the set of all vector fields whose integral curves are (resp. strongly) attainable. The set $\mathbf{A}^+$ (resp. $\mathbf{A}_s^+$) corresponds to semi-attainable (resp. strongly semi-attainable) ones.

In the remainder of this section, we shall study the set $\mathbf{A}$, etc. The following properties are immediate.

Proposition 2.1. *The following four assertions are valid.*

(i) *It holds* $\mathbf{A} \supset \mathbf{A}_s$ *and* $\mathbf{A}^+ \supset \mathbf{A}_s^+$

(ii) *The vector field* X *of* $\mathbf{A}$ (*resp. of* $\mathbf{A}^+$) *belongs to* $\mathbf{A}_s$(resp. $\mathbf{A}_s^+$) *if* cX *belongs to* $\mathbf{A}$ (resp. *to* $\mathbf{A}^+$) *for any* c *of* R (resp. *of* $c \geqslant 0$)

(iii) $X \in \mathbf{A}^+$ (resp. $\mathbf{A}_s^+$) *belongs to* $\mathbf{A}$ (resp. $\mathbf{A}_s$) *if and only if* $-X \in \mathbf{A}^+$ (resp. $\mathbf{A}_s^+$).

Proposition 2.2. $\mathbf{A}_s$ *is a Lie algebra including* $\mathbf{B} = \mathbf{L}(X_1, \ldots, X_r)$

Proof. It is obvious that $\mathbf{A}_s$ is a vector space. Let $X, Y \in \mathbf{A}_s$. A geometric interpretation of Lie bracket $[X, Y]$ is

$$[X, Y](x) = \lim_{s \downarrow 0} \frac{X_s Y_s X_{-s} Y_{-s}(x) - x}{s^2}$$

Then we can show that the integral curve of $[X, Y]$ is approximated by the combination of two integral curves of X and Y, such as $X_{t_1} X_{t_2} \ldots X_{t_n} X_{t-t_n}(x)$ if $\Sigma_{k=1}^{n} t_k \leqslant t < \Sigma_{k=1}^{n+1} t_k$ (see [4]). Therefore, the integral curve of $[X, Y]$ is strongly attainable. This proves that $\mathbf{A}_s$ is a Lie algebra. It is easy to see that $\mathbf{A}_s$ contains $\mathbf{B} = \mathbf{L}(X_1, \ldots, X_r)$.

Proposition 2.3. (i) *Assume* $\dim \mathbf{A}_s(x)$ *is constant. Let* $I(x)$ *be the maximal integral manifold of differential system* $\{\mathbf{A}_s(x); x \in M\}$, *containing the point* x *of* M. *Then* $\overline{I(x)} \subset \overline{A_t(x)}$ *holds for all* x *of* M *and* $t > 0$.

(ii) *Assume* $\dim \mathbf{L}(\mathbf{A})(x)$ *is constant. Let* $J(x)$ *be the maximal integral manifold of the differential system* $\{\mathbf{L}(\mathbf{A})(x); x \in M\}$, *containing* x. *Then* $\overline{I(x)} \subset \overline{A(x)}$ *holds for all* x.

Proof. For the proof of (i), take an arbitrary point y from $I(x)$. We can choose a vector field Z in $\mathbf{A}_s$ such that $Z_t(x) = y$. Hence y belongs to $\overline{A_t(x)}$. Next, take an arbitrary point y from $J(x)$. Then by Chow's theorem, x and y are joined by integral curves of suitable $Z_1, \ldots, Z_n$ of $\mathbf{A}$, say $y = Z_{t_1}^{(1)}. \ldots Z_{t_n}^{(n)}(x)$, where $Z_t^{(i)}$ is the integral curve of Z_i. Hence y belongs to $\overline{A(x)}$. The proof is complete.

From the above proposition, we have

Theorem 2.4. (i) *Assume* $\dim \mathbf{A}_s(x) = d$ *holds for all* x *of* M, *then the system* (1.1) *is strongly completely controllable.*

(ii) *Assume* $\dim \mathbf{L}(\mathbf{A})(x) = d$ *holds for all* x *of* M, *then the system* (1.1) *is completely controllable.*

Proof. (i) It holds $I(x) = M$ because $\dim \mathbf{A}_s(x) = d$ for all x of M. Therefore $\overline{A_t(x)} = M$ is satisfied by previous proposition. Since $A_t(x) \supset \operatorname{int} \overline{A_t(x)}$ is satisfied (for the proof, see [4]), we see that $A_t(x) = M$ is satisfied for any $t > 0$ and $x \in M$. This proves assertion (i). Assertion (ii) can be proved similarly.

3. Strongly Complete Controllability

We have seen in the previous theorem that the system (1.1) is strongly completely controllable if $\dim \mathbf{A}_s(x)=d$ holds everywhere. We will express a sufficient condition of this by means of vector fields $X_0,\dots,X_r$ defining the system (1.1). We begin with two preliminary lemmas.

Lemma 3.1. *Let $Z_0, Z_1,\dots,Z_n$ be vector fields such that $[Z_i, Z_j]=0$ for $1\leqslant i,j\leqslant n$. Then*

(i) $[\dots[Z_0, Z_{i_1}],\dots]Z_{i_k}]$, $1\leqslant i_1,\dots,i_k\leqslant n$ *does not depend on the order of* $(i_1,\dots,i_n)$.

(ii) *Suppose $Z_1,\dots,Z_n$ are complete and $Z_t^{(1)},\dots,Z_t^{(n)}$ are integral curves of $Z_1,\dots,Z_n$. Then $dZ_t^{(i)}$ and ad_{Z_j} are cummutative,* i.e., *$dZ_t^{(i)}\mathrm{ad}_{Z_j}=\mathrm{ad}_{Z_j}dZ_t^{(i)}$ holds for all $i,j=1,\dots,n$ and $t\in R$.*

Proof. Jacobi identity of Lie bracket implies

$$\left[\left[Z_0, Z_{i_1}\right], Z_{i_2}\right]+\left[\left[Z_{i_1}, Z_{i_2}\right], Z_0\right]+\left[\left[Z_{i_2}, Z_0\right], Z_{i_1}\right]=0.$$

Since $[Z_{i_1}, Z_{i_2}]=0$ by assumption, we have

$$\left[\left[Z_0, Z_{i_1}\right], Z_{i_2}\right]=\left[\left[Z_0, Z_{i_2}\right], Z_{i_1}\right]$$

The general case of (i) can be proved by induction.

For the proof of (ii), note that $dZ_t^{(i)}Z_j=Z_j$ if Z_i and Z_j are commutative. Then we have

$$\begin{aligned}\mathrm{ad}_{Z_j}dZ_t^{(i)}Y&=\left[Z_j, dZ_t^{(i)}Y\right]=\left[dZ_t^{(i)}Z_j, dZ_t^{(i)}Y\right]\\&=dZ_t^{(i)}\left[Z_j, Y\right]=dZ_t^{(i)}\mathrm{ad}_{Z_j}Y\end{aligned}$$

for any Y. This proves (ii).

Lemma 3.2. *If $Z\in\mathbf{A}_s$, $Y\in\mathbf{A}^+$* (resp. $\mathbf{A}_s^+$) *and Z is complete, then $dZ_tY\in\mathbf{A}^+$* (resp. $\mathbf{A}_s^+$). *Furthermore, if $Y\in\mathbf{A}_s$, then so is dZ_tY.*

Proof. The integral curve of dZ_tY is $Z_{-t}Y_sZ_t, s\in R$. Since $Z\in\mathbf{A}_s$, $Z_{-t}Y_sZ_t$ belongs to $\overline{A_{s+\varepsilon}(x)}$ for any $\varepsilon>0$. Hence it belongs to $\overline{A_s(x)}$. In particular, if Y belongs to $\mathbf{A}_s^+$, then the integral curve $\{Z_{-t}Y_sZ_t(x); s\geqslant 0\}$ is included in $\overline{A_u(x)}$ for any $u>0$. The last assertion can be proved similarly.

We will now show our main lemma.

Lemma 3.3. *Let $Z_0\in\mathbf{A}^+$ and $Z_1,\dots,Z_n$ be commutative complete vector fields in $\mathbf{A}_s$. Assume $\mathrm{ad}_{Z_1}^{i_1}\dots\mathrm{ad}_{Z_n}^{i_n}Z_0$ belongs to $\mathbf{A}_s$ for all $i_1,\dots,i_n\geqslant 0$ such that $i_1+\dots+i_n=m+1$. Then any $(-1)^m\mathrm{ad}_{Z_1}^{i_1}\dots\mathrm{ad}_{Z_n}^{i_n}Z_0$ with $i_1+\dots+i_n=m$ belongs to $\mathbf{A}_s^+$. Furthermore, if at least one of $(i_1,\dots,i_n)$ is odd, then $\mathrm{ad}_{Z_1}^{i_1}\dots\mathrm{ad}_{Z_n}^{i_n}Z_0$ belongs to $\mathbf{A}_s$.*

Proof. Let $Z_t^{(i)}, i=1,\dots,n$ be integral curves of Z_i, respectively. Set

$$f(t_1,\dots,t_n) = dZ_{t_1}^{(1)}\dots dZ_{t_n}^{(n)}Z_0, \qquad t_1 \geqslant 0,\dots,t_n \geqslant 0.$$

It is a C^∞-function of $(t_1,\dots,t_n)$. Apply the formula (2.1) repeatedly and we get

$$\frac{\partial^m}{\partial t_i^{i_1}\dots\partial t_n^{i_n}} f(t_1,\dots,t_n) = (-1)^m dZ_{t_1}^{(1)}\,\mathrm{ad}_{Z_1}^{i_1}\dots dZ_{t_n}^{(n)}\,\mathrm{ad}_{Z_n}^{i_n} Z_0$$

where $m=i_1+\dots+i_n$. Therefore,

$$\begin{aligned} f(t_1,\dots,t_n) = Z_0 &- \sum_{i=1}^{n} (\mathrm{ad}_{Z_i} Z_0)t_i + \frac{1}{2}\sum_{i,j} (\mathrm{ad}_{Z_i}\mathrm{ad}_{Z_j} Z_0)t_it_j \\ &+\dots+ \frac{(-1)^m}{m!} \sum_{i_1+\dots+i_n=m} \left(\mathrm{ad}_{Z_1}^{i_1}\cdots\mathrm{ad}_{Z_n}^{i_n} Z_0\right)t_1^{i_1}\cdots t_n^{i_n} \\ &+ \frac{(-1)^{m+1}}{(m+1)!} \sum_{i_1+\dots+i_n=m+1} \left(dZ_{\theta_1t_1}^{(1)}\,\mathrm{ad}_{Z_1}^{i_1}\cdots dZ_{\theta_nt_n}^{(n)}\,\mathrm{ad}_{Z_n}^{i_n} Z_0\right)t_1^{i_1}\cdots t_n^{i_n} \end{aligned}$$

where $0\leqslant\theta_i\leqslant 1$ for $i=1,\dots,n$. The last member is written as

$$\frac{(-1)^{m+1}}{(m+1)!} \sum_{i_1+\dots+i_n=m+1} \left(dZ_{\theta_1t_1}^{(1)}\cdots dZ_{\theta_nt_n}^{(n)}\,\mathrm{ad}_{Z_1}^{i_1}\cdots\mathrm{ad}_{Z_n}^{i_n} Z_0\right)t_1^{i_1}\cdots t_n^{i_n} \tag{3.2}$$

by Lemma 3.1 (ii). The term $\mathrm{ad}_{Z_1}^{i_1}\cdots\mathrm{ad}_{Z_n}^{i_n} Z_0$ $(i_1+\dots+i_n=m+1)$ is in $\mathbf{A}_s$ by assumption of the lemma. Apply Lemma 3.2 repeatedly, then we see that each member of (3.2) is in $\mathbf{A}_s$. Since $f(t_1,\dots,t_n)$ belongs $\mathbf{A}^+$ for any $t_1\geqslant 0,\dots,t_n\geqslant 0$ by Lemma 3.2, we conclude that

$$Z_0 - \sum_i (\mathrm{ad}_{Z_i} Z_0)t_i + \dots + \frac{(-1)^m}{m!} \sum_{i_1+\dots+i_n=m} \left(\mathrm{ad}_{Z_1}^{i_1}\cdots\mathrm{ad}_{Z_n}^{i_n} Z_0\right)t_1^{i_1}\cdots t_n^{i_n} \tag{3.3}$$

is an element of $\mathbf{A}^+$ for any $t_i\geqslant 0$, $i=1,\dots,n$. Set now $t_i=s_is$, divide (3.3) by s^m and make s tend to infinity. Then we see that

$$(-1)^m \sum_{i_1+\dots+i_n=m} \left(\mathrm{ad}_{Z_1}^{i_1}\cdots\mathrm{ad}_{Z_n}^{i_n} Z_0\right)s_1^{i_1}\cdots s_n^{i_n}$$

is an element of $\mathbf{A}^+$. Since $s_1,\dots,s_n$ are arbitrary, each $(-1)^m\mathrm{ad}_{Z_1}^{i_1}\cdots\mathrm{ad}_{Z_n}^{i_n} Z_0$ belongs to $\mathbf{A}^+$.

The fact stated above is valid for $c_1Z_1,\dots,c_nZ_n$, where $c_1,\dots,c_n$ are real constants. Since

$$(-1)^m\mathrm{ad}_{c_1Z_1}^{i_1}\cdots\mathrm{ad}_{c_nZ_n}^{i_n} Z_0 = (-1)^m c_1^{i_1}\cdots c_n^{i_n}\mathrm{ad}_{Z_1}^{i_1},\cdots\mathrm{ad}_{Z_n}^{i_n} Z_0,$$

the right hand of the above is also in $\mathbf{A}^+$. Therefore $(-1)^m \mathrm{ad}^{i_1}_{Z_1} \cdots \mathrm{ad}^{i_n}_{Z_n} Z_0$ is in $\mathbf{A}_s^+$ by Proposition 2.1, (iii).

In cases where at least one of $(i_1,\dots,i_n)$ is odd, say i_k, then

$$\begin{aligned} &-(-1)^m \mathrm{ad}^{i_1}_{Z_1} \cdots \mathrm{ad}^{i_k}_{Z_k} \cdots \mathrm{ad}^{i_n}_{Z_n} Z_0 \\ &\quad = (-1)^m \mathrm{ad}^{i_1}_{Z_1} \cdots \mathrm{ad}^{i_k}_{-Z_k} \cdots \mathrm{ad}^{i_n}_{Z_n} Z_0 \in \mathbf{A}_s^+ \end{aligned}$$

Therefore, $\mathrm{ad}^{i_1}_{Z_1} \cdots \mathrm{ad}^{i_n}_{Z_n} Z_0$ belongs to $\mathbf{A}_s$ by Proposition 2.1, (iii). The proof is complete.

Before we state our main theorem, we shall introduce notations. Let $\mathbf{V}$ be a subset of $\mathbf{V}(M)$ and $X \in \mathbf{V}(M)$. Given a positive integer m, consider the set of vector fields $\mathrm{ad}^{i_1}_{Z_1} \cdots \mathrm{ad}^{i_n}_{Z_n} X$ such that $i_1 + \cdots + i_n = m$, where $Z_1,\dots,Z_n$ are commutative complete elements of $\mathbf{V}$. We denote by $\mathrm{ad}^{(m)}_{\mathbf{V}} X$, the collection of all such vector fields varying all possible commutative complete systems $\{Z_1,\dots,Z_n\}$ in $\mathbf{V}$. (The number n is also varying). The subset of $\mathrm{ad}^{(m)}_{\mathbf{V}} X$ such that at least one of indexes $(i_1,\dots,i_n)$ is odd, is denoted by odd $\mathrm{ad}^{(m)}_{\mathbf{V}} X$.

Theorem 3.4. *Suppose that the control system* (1.1) *is equipped with a sequence of families of vector fields* $\mathbf{V}_0 \subset \mathbf{V}_1 \subset \dots$ *satisfying following conditions* (i) *and* (ii).

(i) $\mathbf{V}_0 \subset \mathbf{L}(X_1,\dots,X_1)$

(ii) *For each non-negative integer* k, *there exists a positive integer* m_k *such that*

$$\mathrm{ad}^{(m_k+1)}_{\mathbf{V}_k} X_0 \subset \mathbf{L}(\mathbf{V}_k) \tag{3.3}$$

and

$$\mathbf{V}_{k+1} \subset \mathbf{L}\big(\mathbf{V}_k, \text{odd } \mathrm{ad}_{\mathbf{V}_k}{}^{(m_k)} X_0\big). \tag{3.4}$$

Then we have $\mathbf{L}\left(\bigcup_{k=0}^{\infty} \mathbf{V}_k\right) \subset \mathbf{A}_s$.

In particular, if $\dim \mathbf{L}\left(\bigcup_{k=0}^{\infty} \mathbf{V}_k\right)(x) = d$ everywhere, then the system (1.1) is strongly completely controllable.

Proof. It suffices to prove $\mathbf{V}_k \subset \mathbf{A}_s$ for each non-negative integer k. The case $k=0$ is obvious from Proposition 2.2 and assumption (i). Suppose it is valid for k. The condition (ii) implies that odd $\mathrm{ad}^{(m_k)}_{\mathbf{V}_k} X_0$ belongs to $\mathbf{A}_s$ by previous lemma. Hence the right hand side of (3.4) is included in $\mathbf{A}_s$, so that $\mathbf{V}_{k+1} \subset \mathbf{A}_s$. This proves the theorem.

As an application, we shall give another (simple) proof of Hirshorn's theorem.

Theorem 3.5 (c.f. [2], [3], [4]). *Let* $\mathbf{I}$ *be the ideal in* $\mathbf{L}(X_0, X_1, \dots, X_r)$ *generated by* $X_1,\dots,X_r$. *Assume that* $\dim \mathbf{I}(x) = d$ *holds everywhere and that* $[\mathbf{I}, \mathbf{B}] \subset \mathbf{B}$. *Then the control system* (1.1) *is strongly completely controllable.*

Proof. Let

$$\mathbf{V}_k = \mathbf{L}\big(\mathbf{B}, \mathrm{ad}_{X_0}\mathbf{B}, \ldots, \mathrm{ad}^k_{X_0}\mathbf{B}\big), \qquad k = 0, 1, 2, \ldots$$

Then it holds $\mathbf{I} = \mathbf{L}\left(\bigcup_{k=1}^{\infty} \mathbf{V}_k\right)$, since

$$\mathbf{I} = \mathbf{L}\big(\mathrm{ad}^k_{X_0}\mathbf{B}; k = 0, 1, 2, \ldots\big).$$

We shall first prove $[\mathbf{V}_k, \mathbf{V}_l] \subset \mathbf{V}_l$ if $k \geqslant l$. By Jacobi's identity, we have

$$\begin{aligned}\left[\mathrm{ad}^k_{X_0}X, \mathrm{ad}^l_{X_0}Z\right] &= \mathrm{ad}_{X_0}\left[\mathrm{ad}^k_{X_0}X, \mathrm{ad}^{l-1}_{X_0}Z\right] - \left[\mathrm{ad}^{k+1}_{X_0}X, \mathrm{ad}^{l-1}_{X_0}Z\right] \\ &= \ldots \\ &= \sum_{j=0}^{l} (-1)^j \binom{l}{j} \mathrm{ad}^{l-j}_{X_0}\left[\mathrm{ad}^{k+j}_{X_0}X, Z\right]\end{aligned}$$

Since $[\mathbf{I}, \mathbf{B}] \subset \mathbf{B}$, it holds $[\mathrm{ad}^{k+j}_{X_0}X, Z] \in \mathbf{B}$ if $Z \in \mathbf{B}$. Hence the last member of the above equality is in $\mathbf{V}_l$.

We shall now prove $\mathbf{V}_k, k = 0, 1, 2, \ldots$ satisfy condition (3.3) and (3.4). Since

$$\mathrm{ad}^{(2)}_{\mathbf{V}_k}X_0 = \left[\left[X_0, \mathbf{V}_k\right], \mathbf{V}_k\right] \subseteq \left[\mathbf{V}_{k+1}, \mathbf{V}_k\right] \subset \mathbf{V}_k,$$

condition (3.3) is fulfilled with $m_k = 1$ for all k. Furthermore,

$$\text{odd } \mathrm{ad}^{(1)}_{\mathbf{V}_k}X_0 = \mathrm{ad}^{(1)}_{\mathbf{V}_k}X_0 = \mathrm{ad}_{X_0}\mathbf{V}_k \supset \mathrm{ad}^{k+1}_{X_0}\mathbf{B}$$

and $\mathbf{V}_{k+1} = \mathbf{L}(\mathbf{V}_k, \mathrm{ad}^{k+1}_{X_0}\mathbf{B})$, so that condition (3.4) is also fulfilled. Hirschorn's theorem then follows from Theorem 3.4.

4. Polynomial Control System

As an application, let us consider the polynomial control system. We first consider the system on R^d represented as

$$\begin{aligned}\frac{dx_1}{dt} &= f_1(x_1, \ldots, x_d) + \sum_{j=1}^{r} g_{1j}(x_1, \ldots, x_d)u_j(t) \\ &\ldots \\ \frac{dx_r}{dt} &= f_r(x_1, \ldots, x_d) + \sum_{j=1}^{r} g_{rj}(x_1, \ldots, x_d)u_j(t) \\ \frac{dx_{r+1}}{dt} &= \sum_{0 \leqslant i_1 + \cdots + i_r \leqslant n} h^{(i_1, \ldots, i_r)}_{r+1}(x_{r+1}, \ldots, x_d)x_1^{i_1} \cdots x_r^{i_r} \\ &\ldots \\ \frac{dx_d}{dt} &= \sum_{0 \leqslant i_1 + \cdots + i_r \leqslant n} h^{(i_1, \ldots, i_r)}_{d}(x_{r+1}, \ldots, x_d)x_1^{i_1} \cdots x_r^{i_r},\end{aligned} \tag{4.1}$$

where f_i, g_{ij} and $h_k^{(\cdot)}$ are C^∞-functions

For simplicity, we shall introduce vector notations, $y=(x_1,\dots,x_r)$, $z=(x_{r+1},\dots,x_d)$ etc. and use multi-indexes $I=(i_1,\dots,i_r)$ and $y^I=x_1^{i_1}\dots x_r^{i_r}$. Then, the above (4.1) is written as

$$\begin{cases} \dfrac{dy}{dt}=f(y,z)+\displaystyle\sum_{j=1}^{r} g_j(y,z)u_j(t) \\ \dfrac{dz}{dt}=\displaystyle\sum_{0\leq|I|\leq n} h^{(I)}(z)y^I \end{cases} \tag{4.1}$$

where $|I|=i_1+\cdots+i_r$. Note that the latter equation involves n-th degree polynomials of y. We say that the index $I=(i_1,\dots,i_r)$ is odd, if at least one of $i_k, k=1,\dots,r$ is an odd integer.

Theorem 4.1. *Suppose that the system* (4.1) *satisfies conditions* (a) *and* (b) *below.*

(a) *The rank of $r\times r$-matrix*

$$(g_{ij}(y,z)), \qquad i,j=1\dots r$$

is r everywhere on R^d.

(b) *There exist $d-r$ odd indexes $\hat{I}_{r+1},\dots,\hat{I}_d$ of the highest degree n, such that the rank of $(d-r)\times(d-r)$ matrix*

$$\left(h_k^{(\hat{I}_l)}(z)\right)k,l=r+1,\dots,d$$

is $d-r$ everywhere.

Then the system is strongly completely controllable.

Proof. Let us define vector fields corresponding to the system (4.1) as

$$X_0=f_1\frac{\partial}{\partial x_1}+\cdots+f_r\frac{\partial}{\partial x_r} + \left\{\sum_I h_{r+1}^{(I)}y^I\right\}\frac{\partial}{\partial x_{r+1}}+\cdots+\left\{\sum_I h_d^{(I)}y^I\right\}\cdot\frac{\partial}{x_d}$$

and

$$X_j=\sum_{i=1}^{r} g_{ij}\frac{\partial}{\partial x_i}, \qquad j=1,\dots,r.$$

Define two families of commutative and complete vector fields as

$$\mathbf{V}_0=\left\{\frac{\partial}{\partial x_1},\dots,\frac{\partial}{\partial x_r}\right\}, \mathbf{V}_1=\left\{\frac{\partial}{\partial x_1},\dots,\frac{\partial}{\partial x_d}\right\}$$

We shall show that these vector fields satisfy conditions (i) and (ii) of Theorem

3.4. The relation $\mathbf{V}_0 \subset \mathbf{B} = \mathbf{L}(X_1,\ldots,X_r)$ is clear from condition (a). For each $k = r+1,\ldots,d$, it holds

$$\mathrm{ad}_{\mathbf{V}_1}^{\hat{I}_k} X_0 = C_{\hat{I}_k} \sum_{i=r+1}^{d} h_i^{(\hat{I}_k)}(z)\frac{\partial}{\partial x_i} + \sum_{i=1}^{r} \phi_i(y,z)\frac{\partial}{\partial x_i}, \tag{4.2}$$

where $C_{\hat{I}_k}$ are non-zero constants. From condition (b) we see that the collection of the first terms of (4.2) for $k = r+1,\ldots,d$ spans

$$\left\{ \frac{\partial}{\partial x_{r+1}}, \ldots, \frac{\partial}{\partial x_d} \right\}.$$

Since $\hat{I}_k$'s are all odd, we see that $\mathbf{V}_0$ and odd $\mathrm{ad}_{\mathbf{V}_1}^{(n)} X_0$ span $\mathbf{V}_1$.

On the other hand, we have

$$\mathrm{ad}_{\mathbf{V}_1}^{(n+1)} X_0 = \sum_{i=1}^{r} \tilde{\phi}_i(x)\frac{\partial}{\partial x_i},$$

which belongs to $\mathbf{L}(\mathbf{V}_0)$. Hence, condition (ii) of Theorem 3.4 is satisfied. Therefore, the system is strongly completely controllable by Theorem 3.4.

Example. In two dimensional system, the statement of the theorem is simpler. Consider the control system in R^2,

$$\frac{dx_1}{dt} = f(x_1,x_2) + g(x_1,x_2)u(t)$$

$$\frac{dx_2}{dt} = \sum_{m=0}^{n} h^{(m)}(x_2)x_1^m$$

If $g(x_1,x_2) \neq 0$ and $h^{(n)}(x_2) \neq 0$ hold everywhere and if n is an odd number, then the system is strongly completely controllable.

Let us next consider a more general system. It is convenient to split the coordinate $(x_1,\ldots,x_d)$ into e-components $y_1 = (x_1,\ldots,x_{d_1}), y_2 = (x_{d_1+1},\ldots,x_{d_2}),\ldots,y_e = (x_{d_{e-1}+1},\ldots,x_d)$ where $d_1 < d_2 < \cdots < d_e = d$. Suppose that the control system on R^d can be expressed as

$$\begin{aligned}
\frac{dy_1}{dt} &= f(y_1,\ldots,y_e) + \sum_{j=1}^{d_1} g_j(y_1,\ldots,y_e)u_j(t) \\
\frac{dy_2}{dt} &= \sum_{0 \le |I_1| \le n_2} h_2^{(I_1)}(y_2,\ldots,y_e)y_1^{I_1}, \\
&\cdots \\
\frac{dy_i}{dt} &= \sum_{0 \le |I_1| + \cdots + |I_{i-1}| \le n_i} h_i^{(I_1,\ldots,I_{i-1})}(y_i,\ldots,y_e)y_1^{I_1}\cdots y_{i-1}^{I_{i-1}} \\
i &= 3,\ldots,e.
\end{aligned} \tag{4.3}$$

Here f, g_j are d_1-vector functions, and $h_k^{(\cdot)}$ are $(d_k - d_{k-1})$-vector functions. I_k are multi-indexes of the length $(d_k - d_{k-1})$ and

$$y_k^{I_k} = x_{d_{k-1}+1}^{i_1} \cdots x_{d_k}^{i_{d_k - d_{k-1}}} \text{ if } I_k = (i_1, \ldots, i_{d_k - d_{k-1}}).$$

Sometimes we write $(I_1, \ldots, I_k)$ as J_k.

Concerning the degrees of polynomials of $y_1, \ldots, y_e$ we assume

Condition (c). If $j > i$, the highest order of polynomials of $y_1, \ldots, y_{i-1}$ appearing in the equation

$$\frac{dy_j}{dt} = \sum \left(h_j^{(J_{j-1})}(y_1, \ldots, y_e) y_i^{I_i} \ldots y_{j-1}^{I_{j-1}} \right) y_1^{I_1} \ldots y_{i-1}^{I_{i-1}}$$

is less than or equal to n_i, i.e., it holds

$$|I_1| + \cdots + |I_{i-1}| \leqslant n_i$$

Remark. If the order n_i coincides each other, then the above condition is obviously satisfied.

Theorem 4.2. *Suppose that the system* (4.3) *satisfies the following conditions* (a'), (b') *and the preceding condition* (c):

(a′) *The rank of $d_1 \times d_1$-matrix $(g_j(y_1, \ldots, y_e))$ is d_1 everywhere*

(b′) *For each i $(2 \leqslant i \leqslant e)$ there exist odd indexes $\hat{J}_{i-1}^{(d_{i-1}+1)}, \ldots, \hat{J}_{i-1}^{(d_i)}$ of degree n_i, such that the rank of matrix*

$$\left(h_i^{(\hat{J}_{i-1}^{(d_{i-1}+1)})}, \ldots, h_i^{(\hat{J}_{i-1}^{(d_i)})} \right)$$

is $d_i - d_{i-1}$ everywhere.

Then the system (4.3) *is strongly completely controllable.*

Proof. Let

$$\mathbf{V}_k = \left\{ \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_{d_{k+1}}} \right\}, \qquad k = 0, 1, \ldots, e-1.$$

We shall prove that this family of vector fields satisfies condition (ii) of Theorem 3.4. Note that n_k is the highest degree of polynomials of $y_1, \ldots, y_{k-1}$ for every equation involving $\frac{dy_l}{dt}$ where $l \geqslant k$. Then we see that any element of $\mathrm{ad}_{\mathbf{V}_k}^{(n_k+1)} X_0$ is written by linear sums of $\mathbf{V}_k$, i.e., we have $\mathrm{ad}_{\mathbf{V}_k}^{(n_k+1)} X_0 \subset \mathbf{L}(\mathbf{V}_k)$.

On the other hand, we see that each element of $\mathrm{ad}_{\mathbf{V}_k}^{(\hat{J}_{k-1}^{(i)})} X_0$ is written as

$$\text{linear sums of } \mathbf{V}_{k-1} + C_{\hat{J}_{k-1}^{(i)}} \sum_{l=d_k+1}^{d_{k+1}} h_{il}\left(\hat{J}_{k-1}^{(i)}\right) \frac{\partial}{\partial x_l}$$

$$+ \text{linear sums of } \left\{ \frac{\partial}{\partial x_{d_{k+1}+1}}, \ldots, \frac{\partial}{\partial x_d} \right\}.$$

Note that the collection of the second terms for $i = d_{k-1} + 1, \ldots, d_k$ spans

$$\left\{ \frac{\partial}{\partial x_{d_k + 1}}, \ldots, \frac{\partial}{\partial x_{d_{k+1}}} \right\}$$

by assumption (b′), and that $\hat{J}_{k-1}^{(i)}$ are odd indexes. Then we see easily that the relation (3.4) is also satisfied. Now the assertion of the theorem is immediate from Theorem 3.4.

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