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On the positive orthant controllability of two-dimensional bilinear systems

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In this note we solve a problem suggested by W. Boothby in a recent paper, concerning the positive orthant controllability of bilinear systems in the plane.

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In a very interesting paper recently published [3], W. Boothby considers the following controllability problem.

Let

$$M = \{x = (x_1, \ldots, x_n): x_i > 0\}$$

and let

$$\dot{x} = (A + u(t)B)x, \quad x \in M, \tag{1}$$

be a bilinear control system such that

(A) $A = \{a_{ij}\}$ is an $n \times n$ matrix essentially positive, that is, $a_{ij} > 0$ for $i \neq j$.

(B) B is an $n \times n$ diagonal matrix, with diagonal elements $b_1, \ldots, b_n$.

(C) B is non-singular.

(D) $b_i \neq b_j$ for $i \neq j$.

Conditions (A) and (B) imply that the solutions of (1) corresponding to any choice of the control function $t \mapsto u(t)$ do not leave M. Conditions (C) and (D) are open conditions; in particular, (C) implies that the Lie algebra generated by the matrices A and B has a full rank.

(1) is said to be complete controllable on M if any $x_0 \in M$ can be steered into any $x_1 \in M$ by means of a solution of (1) corresponding to a piecewise constant control $t \mapsto u(t)$. There is no bound on the values of u.

W. Boothby states in [3] some results about the controllability of (1) on M. However, his method does not lead to a satisfactory characterization of those systems of the form (1) which are completely controllable on M, not even if $n = 2$. Actually, if $n = 2$, a necessary and sufficient condition for the controllability problem of (1) on M can be found in a very simple way by using a method for the investigation of the attainable sets developed in [1], [2], [4], [5]. This condition is stated in the following Theorem; the cases (i) and (ii) of the Theorem correspond to the partial results obtained by Boothby [3, p. 641].

First of all, let us remark that it can be always assumed

(E) $b_2 > 0$.

Indeed, if b_2 is negative, we shall replace B by $-B$; this is equivalent to substituting u with $-u$ and so the system remains unchanged.

Theorem. *Consider the system* (1) *with* $n = 2$, *and assume* (A), (B), (C), (D) *and* (E). *The following three statements hold.*

(i) *If* $b_1 > 0$, *then* (1) *is completely controllable on* M.

(ii) *If* $b_1 < 0$ *but*

$$\text{(F)} \quad \mathfrak{D} = (b_2 a_{11} - b_1 a_{22})^2 + 4 b_1 b_2 a_{12} a_{21} > 0$$

and

$$\text{(G)} \quad b_1 a_{22} - b_2 a_{11} > 0,$$

then (1) *is completely controllable on* M.

(iii) *In any other case,* (1) *is not completely controllable on* M.

Proof. Let us consider the system of differential equations on M defined by

$$\dot{x} = Bx, \quad x = (x_1, x_2)'. \tag{2}$$

It is easy to verify that the family of the integral

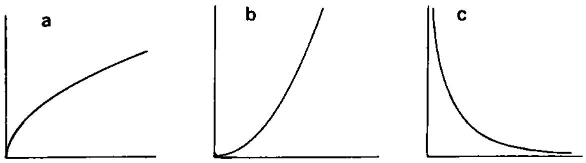

Fig. 1. Integral curves. (a) $b_1 > b_2 > 0$, (b) $b_2 > b_1 > 0$, (c) $b_2 > 0 > b_1$.

curves of (2) on M is given by the implicit equation

$$f(x) = x_2^{b_1} x_1^{-b_2} = \text{const.}$$

therefore, as shown in Figure 1, every integral curve of (2) disconnects M.

An application of Theorem 5.1 of [2] to our case shows that (1) is completely controllable on M if and only if the form

$$\langle Ax, \operatorname{grad} f(x) \rangle \tag{3}$$

changes sign on each integral curve of (2), that is, if and only if the trajectories of the vector field $x \mapsto Ax$ cross the integral curves of (2) in both directions (cf. also [5]). Since x_1 and x_2 are both positive, the sign of (3) on M is the same as the sign of

$$b_1 a_{21} x_1^2 + (a_{22} b_1 - b_2 a_{11}) x_1 x_2 - b_2 a_{12} x_2^2. \tag{4}$$

(4) changes sign on M if and only if the equation

$$b_2 a_{12} \lambda^2 - (a_{22} b_1 - b_2 a_{11}) \lambda - b_1 a_{21} = 0 \tag{5}$$

admits two distinct real roots λ_1 and λ_2, and at least one of them, say λ_1, is positive. Indeed in this case, (4) is equal to

$$-b_2 a_{12} [(x_2 - \lambda_1 x_1)(x_2 - \lambda_2 x_1)]$$

and its sign changes when x crosses the set

$$L = \{(x_1, x_2) \in M: x_2 = \lambda_1 x_1\}.$$

(See Figure 2.)

On the other hand, it is clear that for each

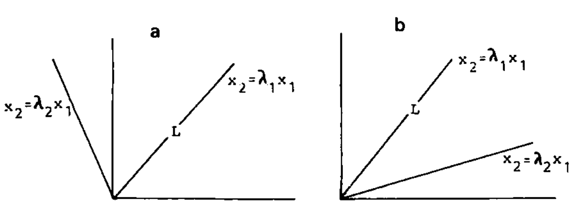

Fig. 2. The set L. (a) $\lambda_1 > 0 > \lambda_2$, (b) $\lambda_1 > 0$, $\lambda_2 > 0$.

choice of b_1 and b_2, the integral curves of (2) actually cross L. Thus, we conclude that the complete controllability of our system on M is equivalent to the following statement:

(5) has at least a positive root. $\qquad (*)$

It is easy to verify that if $b_1 > 0$, the discriminant of (5)

$$\mathscr{D} = (a_{22} b_1 - b_2 a_{11})^2 + 4 b_1 b_2 a_{12} a_{21}$$

is positive and

$$\lambda_1 \lambda_2 = -b_1 a_{21} / b_2 a_{12}$$

is negative; thus $(*)$ holds.

If $b_1 < 0$, we have to assume $\mathscr{D} > 0$. Since now $\lambda_1 \lambda_2 > 0$, the roots have the same sign and that depends on the sign of

$$\lambda_1 + \lambda_2 = (a_{22} b_1 - b_2 a_{11}) / b_2 a_{12}.$$

Thus, $(*)$ is implied by (G) and (F). □

Let us remark that our proof is based on a general method and does not require an investigation 'case by case' as in [3]. In principle, the same method can be used in order to study the complete controllability of more general bilinear systems, for instance those of the form

$$\dot{x} = \left(A + \sum_{i=1}^{n-1} u_i(t) B_i \right) x, \quad x \in M \subset \mathbb{R}^n,$$

where the matrices $B_1, \ldots, B_{n-1}$ are linearly independent.

Further, our method, contrary to that of Boothby, gives information also in the case $\mathscr{D} = 0$. Notice that $\mathscr{D}$ is the same as (5.5) in [3].

On page 638 of [3], Boothby considers three different situations (a), (b) and (c) about (1). By means of some tedious but not difficult calculations, it is possible to verify that (i) corresponds to the situation (c) in the article of Boothby, and that (ii) corresponds to the situation (b). Thus, we have proved Boothby's conjecture, namely that a system in the situation (a) is not completely controllable on M.

We conclude with a remark concerning the case $n > 2$. Proposition 4.3 of [3] and the case (i) of the previous Theorem could suggest the following conjecture for $n > 2$:

"If (A), (B), (C) and (D) are satisfied and all the b_i's have the same sign, then (1) is completely controllable on M."

This conjecture is false. Indeed, consider a system with $n = 3$,

$$A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The cone

$$\mathcal{C} = \{(x_1, x_2, x_3) \in M: 9x_1x_3 - 2x_2^2 = 0\}$$

disconnects M into two parts,

$$\mathcal{C}^+ = \{(x_1, x_2, x_3) \in M: 9x_1x_3 - 2x_2^2 > 0\}$$

and

$$\mathcal{C}^- = \{(x_1, x_2, x_3) \in M: 9x_1x_3 - 2x_2^2 < 0\}.$$

It is not difficult to see that the integral curves of $\dot{x} = Bx$ starting from points of $\mathcal{C}$ remain on $\mathcal{C}$ and that the integral curves of $\dot{x} = Ax$ cross $\mathcal{C}$ always in the direction from $\mathcal{C}^-$ to $\mathcal{C}^+$. Therefore, arguments from [1] and [2] show that the system is not completely controllable on M.

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