Chapter 1

General Gaussian Operations

1.1 Choi isomorphism and gate teleportation

The Choi isomorphism is a well known bijective mapping between quantum states and quantum completely positive (CP)-maps. Its definition is straightforward for a finite dimensional Hilbert space $\mathcal{H}$. Let us first define the maximally entangled state $|\psi\rangle = \frac{1}{\sqrt{d}} \sum_{j=1}^{d} |e_j, e_j\rangle$, where d is the dimension of the Hilbert space $\mathcal{H}$, $\{|e_j\rangle, j = 1, \ldots, d\}$ is any orthonormal basis of $\mathcal{H}$ and $|e_j, e_j\rangle = |e_j\rangle \otimes |e_j\rangle \in \mathcal{H}^{\otimes 2}$. Then, given a CP-map Φ on $\mathcal{B}(\mathcal{H})$, representing a generic quantum operation, one can define the associated Choi state $\varphi \in \mathcal{B}(\mathcal{H})$ as the output of the action of Φ on the first subsystem of the maximally entangled state $|\psi\rangle\langle\psi|$:

$$\varphi = (\Phi \otimes \mathbb{1})(|\psi\rangle\langle\psi|) = \frac{1}{d} \sum_{j,k=1}^{d} \Phi(|e_j\rangle\langle e_k|) \otimes |e_j\rangle\langle e_k| \, . \tag{1.1}$$

Notice that, since Φ is completely positive, the operator φ is certainly positive. If Φ is also trace preserving, then φ is a proper, normalised quantum state. Conversely, one may probabilistically retrieve the CP-map Φ from the Choi state φ by a gate teleportation procedure: given the Choi state φ on subsystems 1 and 2, and a generic input ϱ on subsystem 3, it is easy to show that the projection on the maximally entangled state $|\psi\rangle_{23}$ of subsystems 2 and 3 yields the quantum state $\Phi(\varrho)$ in subsystem 1:

$$\begin{aligned} {}_{23}\langle\psi|\varphi \otimes \varrho|\psi\rangle_{23} &= \frac{1}{d^2} \sum_{j,k,l,m,p,q=1}^{d} \Phi(|e_j\rangle\langle e_k|)_{23}\langle e_l, e_l|e_j\rangle_{22}\langle e_k|\varrho_{pq}|e_p\rangle_{33}\langle e_q|e_m, e_m\rangle_{23} \\ &= \frac{1}{d^2} \sum_{j,k=1}^{d} \varrho_{jk} \Phi(|e_j\rangle\langle e_k|) = \frac{1}{d^2} \Phi(\varrho) \, . \end{aligned} \tag{1.2}$$

Any trace preserving map on any input state ϱ can hence be obtained, with probability $1/d^2$, by a projective measurement in a basis containing $|\psi\rangle$ acting on the input and on half of the Choi state's subsystems. Although this retrieval is probabilistic, it has the advantage of providing one with an operational procedure to enact the CP-map.

1.2 Choi isomorphism in infinite dimension

In infinite dimension, the analogue of the maximally entangled state $|\psi\rangle$ is not normalisable, and hence the isomorphism should be handled with some care. To this aim, we will exploit the fact that a maximally entangled state can be approached by a limiting sequence of normalisable Gaussian states: consider a system of two modes, with annihilation operators a_1 and a_2, and notice that

$$\mathrm{e}^{(a_1^\dagger a_2^\dagger - a_1 a_2)r}|0,0\rangle = \frac{1}{\cosh r}\sum_{j=0}^{\infty}\tanh(r)^j|j,j\rangle \ , \tag{1.3}$$

where r is a real parameter and the notation $|j,j\rangle$ stands for a tensor product of Fock states. Eq. (1.3) can be proven by differentiating the left and right hand sides with respect to r, and then inserting the original equation into the equation for the differentials, obtaining

$$\sum_{j=0}^{\infty}\tanh(r)^j(a_1^\dagger a_2^\dagger - a_1 a_2)|j,j\rangle = \sum_{j=0}^{\infty}\left(j\tanh(r)^{j-1} - (j+1)\tanh(r)^{j+1}\right)|j,j\rangle \ , \tag{1.4}$$

which certainly holds for all values of r. We can hence define the set of states

$$|\psi_r\rangle = \mathrm{e}^{(a_1^\dagger a_2^\dagger - a_1 a_2)r}|0,0\rangle = \frac{1}{\cosh r}\sum_{j=0}^{\infty}\tanh(r)^j|j,j\rangle \tag{1.5}$$

and notice that, as $r \mapsto \infty$, the state $|\psi_r\rangle$ tends to an even superposition of tensor product of Fock states and thus approximates a maximally entangled state.

Note also that $|\psi_r\rangle$ is a Gaussian state, since it results from the action of a unitary operator with quadratic generator on the vacuum state. We can then apply the techniques detailed in chapter (**??**) to characterise such a state through its first and second moments of the canonical operators. The first moments are clearly zero, since the generating unitary operator does not include a linear part. The CM $\boldsymbol{\sigma}_r$ can instead be determined by noting that the state is pure, and hence all the symplectic eigenvalues of $\boldsymbol{\sigma}$ are equal to 1, sand then by determining the symplectic operation S_r which corresponds to $\mathrm{e}^{(a_1^\dagger a_2^\dagger - a_1 a_2)r}$. To so this, let us switch to quadrature operators and write

$$\mathrm{e}^{(a_1^\dagger a_2^\dagger - a_1 a_2)r} = \mathrm{e}^{i\frac{1}{2}\hat{\mathbf{r}}^{\mathsf{T}}J\hat{\mathbf{r}}} \ , \tag{1.6}$$

with

$$J = -\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} . \tag{1.7}$$

The symplectic transformation S_r is then obtained as $S_r = \mathrm{e}^{-\Omega J}$, which yields

$$S_r = \begin{pmatrix} \cosh r & 0 & \sinh r & 0 \\ 0 & \cosh r & 0 & -\sinh r \\ \sinh r & 0 & \cosh r & 0 \\ 0 & -\sinh r & 0 & \cosh r \end{pmatrix} . \tag{1.8}$$

Finally, the covariance matrix $\boldsymbol{\sigma}_r$ is given by

$$\boldsymbol{\sigma}_r = S_r S_r^{\mathsf{T}} = \begin{pmatrix} \cosh(2r) & 0 & \sinh(2r) & 0 \\ 0 & \cosh(2r) & 0 & -\sinh(2r) \\ \sinh(2r) & 0 & \cosh(2r) & 0 \\ 0 & -\sinh(2r) & 0 & \cosh(2r) \end{pmatrix} . \tag{1.9}$$

This Gaussian state is the celebrated two-mode squeezed state with squeezing parameter r, also known as the 'twin-beam' state. As we just saw, two-mode squeezed states approximate maximally entangled states with increasing r. In the limit $r \mapsto \infty$, the state $|r\rangle$ tends to the improper eigenvector of the conjugate quadrature operators $(\hat{x}_1 - \hat{x}_2)$ and $(\hat{p}_1 - \hat{p}_2)$ with eigenvalue 0. This can be seen directly by applying such operators on the state. This (improper) state was central to the seminal Einstein Podolski Rosen *gedanken* experiment argument concerning the reality and completeness of quantum mechanics.

Operationally, the projection of $|\psi_r\rangle$ in the limit $r \mapsto \infty$ corresponds therefore to obtaining the value 0 when measuring the pair of quadratures $(\hat{x}_1 - \hat{x}_2)$ and $(\hat{p}_1 - \hat{p}_2)$. As we shall see, such a measurement can be performed on optical systems through a scheme known as homodyne detection. Thus, we shall interchangeably refer to such measurements as quadrature or homodyne measurements in the following. It is a remarkable practical feature of optical continuous variables that a feasible scheme (homodyne detection of combined quadratures) allows one to implement a projection on a maximally entangled state.[1]

Problem. Show that $|\psi_r\rangle$ tends to the zero eigenvector of $(\hat{x}_1 - \hat{x}_2)$ and $(\hat{p}_1 - \hat{p}_2)$ in the limit $r \mapsto \infty$.

Given any CP-map Φ on the set of bounded operators $\mathcal{B}(L^2(\mathbb{R}))$, let us then define the Choi state for a given r as

$$\varphi_r = (\Phi \otimes \mathbb{1})(|\psi_r\rangle\langle\psi_r|) . \tag{1.10}$$

This is still an isomorphism between CP-maps on $\mathcal{B}(L^2(\mathbb{R}))$ and states on $L^2(\mathbb{R}^2)$, even at finite r, because the Schmidt rank of the state $|\psi_r\rangle$ is infi-

[1] At least in principle, since any continuous measurement will always be subject to a finite error.

nite.[2] Besides, given any input state $\varrho \in \mathcal{B}(L^2(\mathbb{R}))$, we can reproduce Eq. (1.2) in the limit $r \mapsto \infty$:

$$\lim_{r\to\infty} \cosh(r)^4 {}_{23}\langle\psi_r|\varphi_r \otimes \varrho|\psi_r\rangle_{23} = \lim_{r\to\infty} \Phi(\varrho) \ . \tag{1.11}$$

This convergence holds element-wise in the Fock basis, and also implies the convergence of the expectation values of linear and quadratic combinations of quadrature operators on Gaussian states, which can be approximated arbitrarily well by finite truncations of the Hilbert space in the Fock basis. Such expectation values are all that we need to characterise Gaussian maps, the main point of our agenda in the present chapter.

Eq. (1.11) is useful in two respects. In the first place, it provides one with an operational recipe, based on the availability of the Choi state and on a Gaussian measurement, to approximate arbitrarily well any trace preserving CP-map, albeit with the arbitrarily small probability $1/\cosh(r)^4$. Secondly, regardless of the probabilistic nature of the retrieval procedure, it will allow us to characterise the set of Gaussian CP-map through the set of Gaussian states, and to infer some interesting implications concerning Gaussian operations.

Notice that the isomorphism above can be straightforwardly extended to n modes by replacing $|\psi_r\rangle$ with a tensor product of two-mode squeezed states $|\psi_r\rangle^{\otimes r}$. Such an n-mode state is a Gaussian with CM $\boldsymbol{\sigma}_r^{(n)}$ given by

$$\boldsymbol{\sigma}_r^{(n)} = \begin{pmatrix} \cosh(2r)\mathbb{1}_{2n} & \sinh(2r)\Sigma_n \\ \sinh(2r)\Sigma_n & \cosh(2r)\mathbb{1}_{2n} \end{pmatrix} , \quad \text{with} \quad \Sigma_n = \bigoplus_{j=1}^{n} \sigma_z \ . \tag{1.12}$$

1.3 The most general Gaussian CP-map

By virtue of the Choi isomorphism and its reversal, described by Eqs. (1.10) and (1.11), we are now in a position to characterise the class of Gaussian CP-maps, defined as the CP-maps that send Gaussian states into Gaussian states, and that preserve the Gaussian character of the state $|\psi_r\rangle^{\otimes n}$ when acting only on its first n subsystems.[3] Clearly, the Choi state of all such maps is a Gaussian state and, conversely, each Gaussian state corresponds to one such map. Hence, we can parametrise all Gaussian CP-maps on $\mathcal{B}(L^2(\mathbb{R}^n))$ with their Choi Gaussian state φ_r for given r, which in turn can be described by its $4n \times 4n$ CM $\boldsymbol{\sigma}$ and its $4n$-dimensional vector of first moments $\mathbf{d}$.

[2] In general, this is the necessary condition for the Choi map to define an isomorphism between CP-maps and quantum states. A maximally entangled state is adopted for convenience, but is not strictly speaking necessary to obtain a bijective mapping.

[3] We added this last proviso in the interest of rigour, although the authors are not aware of any CP-map Φ that preserves the Gaussian characters of all Gaussian states and such that $(\Phi \otimes \mathbb{1})(|\psi_r\rangle\langle\psi_r|)$ is not Gaussian.

1.4 General-dyne measurements

The resolution of the identity (**??**) generalises to n modes as

$$\frac{1}{(2\pi)^n}\int_{\mathbb{R}^{2n}} \mathrm{d}^{2n}r\, \hat{D}_{-\mathbf{r}}|0\rangle\langle 0|\hat{D}_{\mathbf{r}} = \mathbb{1}\ , \tag{1.13}$$

which can be further generalised by acting by congruence with a purely quadratic unitary transformation $\hat{S}$, corresponding to the symplectic transformation S, on both sides:

$$\begin{aligned}\frac{1}{(2\pi)^n}\int_{\mathbb{R}^{2n}} \mathrm{d}^{2n}r\, \hat{S}\hat{D}_{-\mathbf{r}}|0\rangle\langle 0|\hat{D}_{\mathbf{r}}\hat{S}^\dagger &= \\ \frac{1}{(2\pi)^n}\int_{\mathbb{R}^{2n}} \mathrm{d}^{2n}r\, \hat{D}_{-S\mathbf{r}}\hat{S}|0\rangle\langle 0|\hat{S}^\dagger\hat{D}_{S\mathbf{r}} &= \\ \frac{1}{(2\pi)^n}\int_{\mathbb{R}^{2n}} \mathrm{d}^{2n}r\, \hat{D}_{-\mathbf{r}}\hat{S}|0\rangle\langle 0|\hat{S}^\dagger\hat{D}_{\mathbf{r}} &= \mathbb{1}\ ,\end{aligned} \tag{1.14}$$

where we changed the integration variables to $S\mathbf{r}$ and took advantage of the fact that $\det S = 1$. The measurement process described by these resolutions of the identity correspond, if the measurement outcome is recorded, to projections on the completely generic pure Gaussian state $\hat{D}_{-\mathbf{r}}\hat{S}|0\rangle$. Such measurements go under the name of 'general-dyne' measurements, as they include heterodyne detection for $\hat{S} = \mathbb{1}$, and can approach arbitrarily well homodyne detection in the limit where $\hat{S}$ is a squeezing operator with infinite squeezing parameter.

1.4.1 Conditional Gaussian dynamics

The projection on pure Gaussian states with the same second moments and varying first moments describes hence legitimate measurement processes. If the measurement outcome, labelled above by $\mathbf{r}$, is recorded, such measurements give rise to specific Gaussian CP-maps, which can be interpreted as the filtering of the system conditioned on recording the measurement outcome $\mathbf{r}$. It is instructive to understand how the CM of a Gaussian state is affected when a portion of the system modes is measured through general-dyne detection.

Given the initial Gaussian state of a system partitioned in subsystem A and B, with CM

$$\boldsymbol{\sigma} = \begin{pmatrix} \boldsymbol{\sigma}_A & \boldsymbol{\sigma}_{AB} \\ \boldsymbol{\sigma}_{AB}^{\mathsf{T}} & \boldsymbol{\sigma}_B \end{pmatrix}$$

and first moments

$$\mathbf{r}' = \begin{pmatrix} \mathbf{r}'_A \\ \mathbf{r}'_B \end{pmatrix}\ ,$$

let us then determine both the probability $p(\mathbf{r}_m)$ of measuring the general-dyne outcome $\mathbf{r}_m$ on the n-mode subsystem B as well as the final CM and first moments of the m-mode subsystem A given such an outcome. We need to evaluate the overlap between the initial state ϱ and the pure Gaussian state on subsystem B $|\psi_G\rangle_B$, with CM $\boldsymbol{\sigma}_m$ and first moments $\mathbf{r}_m$. Note that, while $\mathbf{r}_m$ labels the outcome of the measurement, the CM $\boldsymbol{\sigma}_m$ characterises the specific choice of general-dyne detection.

By using the Fourier-Weyl relation (**??**), noticing that $\langle\psi_G|\hat{D}_{\mathbf{r}_B}|\psi_G\rangle$ is nothing but the characteristic function of $|\psi_G\rangle\langle\psi_G|$ and applying the multivariate Gaussian integral (**??**), one obtains

$$\begin{aligned}
\langle\psi_G|\varrho|\psi_G\rangle &= \frac{1}{(2\pi)^{m+n}}\int_{\mathbb{R}^{2(m+n)}} \mathrm{e}^{-\frac{1}{4}\mathbf{r}^\mathsf{T}\Omega^\mathsf{T}\boldsymbol{\sigma}\Omega\mathbf{r}+i\mathbf{r}^\mathsf{T}\Omega\mathbf{r}'}\langle\psi_G|\hat{D}_{\mathbf{r}}|\psi_G\rangle\,\mathrm{d}\mathbf{r} \\
&= \frac{1}{(2\pi)^{m+n}}\int_{\mathbb{R}^{2(m+n)}} \mathrm{e}^{-\frac{1}{4}\mathbf{r}^\mathsf{T}\boldsymbol{\sigma}\mathbf{r}+i\mathbf{r}^\mathsf{T}\mathbf{r}'}\,\hat{D}_{\Omega^\mathsf{T}\mathbf{r}_A}\mathrm{e}^{-\frac{1}{4}\mathbf{r}^\mathsf{T}\boldsymbol{\sigma}_m\mathbf{r}_B-i\mathbf{r}_B^\mathsf{T}\mathbf{r}_m}\,\mathrm{d}\mathbf{r} \\
&= \frac{2^n\mathrm{e}^{-(\mathbf{r}_m-\mathbf{r}'_B)^\mathsf{T}\frac{1}{\boldsymbol{\sigma}_B+\boldsymbol{\sigma}_m}(\mathbf{r}_m-\mathbf{r}'_B)}}{(2\pi)^m\sqrt{\mathrm{Det}(\boldsymbol{\sigma}_B+\boldsymbol{\sigma}_m)}}\int_{\mathbb{R}^{2m}}\mathrm{e}^{-\frac{1}{4}\mathbf{r}_A^\mathsf{T}\left(\boldsymbol{\sigma}-\boldsymbol{\sigma}_{AB}\frac{1}{\boldsymbol{\sigma}_B+\boldsymbol{\sigma}_m}\boldsymbol{\sigma}_{AB}^\mathsf{T}\right)\mathbf{r}_A}\times \\
&\quad \mathrm{e}^{i\mathbf{r}_A^\mathsf{T}\left(\mathbf{r}'_A+\boldsymbol{\sigma}_{AB}\frac{1}{\boldsymbol{\sigma}_B+\boldsymbol{\sigma}_m}(\mathbf{r}_m-\mathbf{r}'_B)\right)}\hat{D}_{\Omega^\mathsf{T}\mathbf{r}_A}\,\mathrm{d}\mathbf{r}_A\,, \qquad (1.15)
\end{aligned}$$

which shows that, under general-dyne measurement of a set of modes, the initial CM $\boldsymbol{\sigma}_A$ and first moments $\mathbf{r}'_A$ of the subsystem which is not measured are mapped according to

$$\boldsymbol{\sigma}_A \mapsto \boldsymbol{\sigma}_A - \boldsymbol{\sigma}_{AB}\frac{1}{\boldsymbol{\sigma}_B+\boldsymbol{\sigma}_m}\boldsymbol{\sigma}_{AB}^\mathsf{T}\,, \qquad (1.16)$$

$$\mathbf{r}_A \mapsto \mathbf{r}'_A + \frac{1}{2}\boldsymbol{\sigma}_{AB}\frac{1}{\boldsymbol{\sigma}_m+\boldsymbol{\sigma}_B}(\mathbf{r}_m-\mathbf{r}'_B)\,, \qquad (1.17)$$

with probability [see Eq. (**??**)]

$$p(\mathbf{r}_m) = \frac{\mathrm{e}^{-(\mathbf{r}_m-\mathbf{r}'_B)^\mathsf{T}\frac{1}{\boldsymbol{\sigma}_m+\boldsymbol{\sigma}_B}(\mathbf{r}_m-\mathbf{r}'_B)}}{\pi^n\sqrt{\mathrm{Det}(\boldsymbol{\sigma}_m+\boldsymbol{\sigma}_B)}}\,. \qquad (1.18)$$

Running the risk of being tedious, let us remind that $\boldsymbol{\sigma}_m$ is the CM of a pure Gaussian state of n modes, that is a $2n\times 2n$ real matrix with all symplectic eigenvalues equal to 1, and characterises the choice of general-dyne measurement, $\boldsymbol{\sigma}_B$ is the initial CM of the measured subset of modes, while $\boldsymbol{\sigma}_{AB}$ contains the correlations between subsystem of interest and measured subsystem. Obviously, if no correlations are present (*i.e.*, if $\boldsymbol{\sigma}_{AB}=0$), the map above reduces to the identity, in that measuring subsystem B cannot have any effect on subsystem A if the two subsystems are not initially correlated.

1.5 Deterministic Gaussian CP-maps

$$S = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \qquad (1.19)$$

Simplecticity:

$$\begin{pmatrix} A\Omega_n A^\mathsf{T} + B\Omega_m B^\mathsf{T} & A\Omega_n C^\mathsf{T} + B\Omega_m D^\mathsf{T} \\ C\Omega_n A^\mathsf{T} + D\Omega_m B^\mathsf{T} & C\Omega_n C^\mathsf{T} + D\Omega_m D^\mathsf{T} \end{pmatrix} = \begin{pmatrix} \Omega_n & 0 \\ 0 & \Omega_m \end{pmatrix}\,, \qquad (1.20)$$

where Ω_x is a symplectic form of x degrees of freedom.

CP-map is given by tracing out $S(\boldsymbol{\sigma}\oplus\boldsymbol{\sigma}_E)S^{\mathsf{T}}$, such that $X=A$ and $Y=B\boldsymbol{\sigma}_E B^{\mathsf{T}}$. Also, one has $\boldsymbol{\sigma}_E+i\Omega_m\geq 0$, from which

$$B\boldsymbol{\sigma}_E B^{\mathsf{T}}+iB\Omega_m B^{\mathsf{T}}\geq 0\,. \tag{1.21}$$

Because of the simplecticity condition above one has $B\Omega_m B^{\mathsf{T}}=\Omega_n-A\Omega_n A^{\mathsf{T}}$, which can be inserted in the last expression to get

$$B\boldsymbol{\sigma}_E B^{\mathsf{T}}+i\Omega_n-iA\Omega_n A^{\mathsf{T}}\geq 0\,, \tag{1.22}$$

that is

$$Y+i\Omega_n-iY\Omega_n Y^{\mathsf{T}}\geq 0\,. \tag{1.23}$$

The action of a Gaussian CP-map Φ on Weyl operators is best described in terms of the *dual* CP-map Φ^*, that is of the CP map defined by the following relation:

$$\mathrm{Tr}\left[\sigma\Phi^*(\tau)\right]=\mathrm{Tr}\left[\Phi(\sigma)\tau\right]\quad\forall\,\sigma,\tau\in\mathcal{B}(\mathcal{H})\,. \tag{1.24}$$

Then, by considering the action of the map on a Gaussian characteristic function, if Φ is parametrised as above in terms of X and Y, one has

$$\begin{aligned}\mathrm{Tr}\left[\varrho\varphi^*\left(\hat{D}_{\Omega\mathbf{r}}\right)\right]&=\mathrm{Tr}\left[\varphi\left(\varrho\right)\hat{D}_{\Omega\mathbf{r}}\right]=\mathrm{e}^{-\frac{1}{4}\mathbf{r}^{\mathsf{T}}\left(X\boldsymbol{\sigma}X^{\mathsf{T}}+Y\right)\mathbf{r}}\\&=\mathrm{e}^{-\frac{1}{4}\mathbf{r}^{\mathsf{T}}Y\mathbf{r}}\chi\left(\Omega^{\mathsf{T}}X^{\mathsf{T}}\mathbf{r}\right)=\mathrm{e}^{-\frac{1}{4}\mathbf{r}^{\mathsf{T}}Y\mathbf{r}}\mathrm{Tr}\left[\varrho\hat{D}_{\Omega X^{\mathsf{T}}\mathbf{r}}\right]\,,\end{aligned} \tag{1.25}$$

which holds for all quantum state ϱ, such that one is led to

$$\Phi^*(\hat{D}_{\Omega\mathbf{r}})=\mathrm{e}^{-\frac{1}{4}\mathbf{r}^{\mathsf{T}}Y\mathbf{r}}\hat{D}_{\Omega X^{\mathsf{T}}\mathbf{r}}\,. \tag{1.26}$$

This relationship completely determines the dual CP-map Φ^*. In light of the above, determining the action on Weyl operators of the Gaussian CP-map Φ parametrised by X and Y is straightforward for invertible X. One may in fact assume the *ansatz* $\Phi\left(\hat{D}_{\Omega\mathbf{r}}\right)=c\,\mathrm{e}^{-\mathbf{r}^{\mathsf{T}}Y^*\mathbf{r}}\hat{D}_{\Omega X^{*\mathsf{T}}\mathbf{r}}$ for some $c\in\mathbb{R}$ and $n\times n$ matrices X^* and Y^*, and write

$$\begin{aligned}\mathrm{e}^{-\frac{1}{4}\mathbf{r}'^{\mathsf{T}}Y\mathbf{r}'}\delta^{2n}(\mathbf{r}+X^{\mathsf{T}}\mathbf{r}')&=\frac{\mathrm{e}^{-\frac{1}{4}\mathbf{r}'^{\mathsf{T}}Y\mathbf{r}'}}{(2\pi)^n}\mathrm{Tr}\left[\hat{D}_{\Omega\mathbf{r}}\hat{D}_{\Omega X^{\mathsf{T}}\mathbf{r}'}\right]=\frac{1}{(2\pi)^n}\mathrm{Tr}\left[\hat{D}_{\Omega\mathbf{r}}\Phi^*(\hat{D}_{\Omega\mathbf{r}'})\right]\\&=\frac{1}{(2\pi)^n}\mathrm{Tr}\left[\Phi(\hat{D}_{\Omega\mathbf{r}})\hat{D}_{\Omega\mathbf{r}'}\right]=c\,\mathrm{e}^{-\frac{1}{4}\mathbf{r}^{\mathsf{T}}Y^*\mathbf{r}}\delta^{2n}(X^{*\mathsf{T}}\mathbf{r}+\mathbf{r}')\,,\end{aligned} \tag{1.27}$$

which is verified for

$$c=\frac{1}{|\mathrm{Det}X|}\,, \tag{1.28}$$

$$X^*=X^{-1}\,, \tag{1.29}$$

$$Y^*=X^{-1}YX^{-1\mathsf{T}}\,. \tag{1.30}$$

Note that, since the dual map is unique, the constructive argument above ensures that the dual map of a Gaussian CP-map is Gaussian too.

The action of the dual CP-map Φ^* on a generic Gaussian state ϱ_G can also be easily determined, for invertible X, by applying Eq. (1.26) above on the Fourier-Weyl expansion of ϱ_G:

$$\begin{aligned}\Phi^*\left(\varrho_G\right) &= \int_{\mathbb{R}^{2n}} \mathrm{e}^{-\frac{1}{4}\mathbf{r}^\mathsf{T}\boldsymbol{\sigma}\mathbf{r}}\Phi^*\left(\hat{D}_{\Omega\mathbf{r}}\right)\mathrm{d}^{2n}\mathbf{r} = \int_{\mathbb{R}^{2n}} \mathrm{e}^{-\frac{1}{4}\mathbf{r}^\mathsf{T}(\boldsymbol{\sigma}+Y)\mathbf{r}}\hat{D}_{\Omega X^\mathsf{T}\mathbf{r}}\mathrm{d}^{2n}\mathbf{r} \\ &= \frac{1}{|\mathrm{Det}X|}\int_{\mathbb{R}^{2n}} \mathrm{e}^{-\frac{1}{4}\mathbf{r}^\mathsf{T}(X^{-1}\boldsymbol{\sigma}X^{-1\mathsf{T}}+X^{-1}YX^{-1\mathsf{T}})\mathbf{r}}\hat{D}_{\Omega\mathbf{r}}\mathrm{d}^{2n}\mathbf{r}\,. \end{aligned} \tag{1.31}$$

Note that $\mathrm{Tr}\left[\Phi^*\left(\varrho_G\right)\right] = \frac{1}{|\mathrm{Det}X|}$. The channel Φ^* is hence a Gaussian completely positive map, but not a trace preserving one unless $|\mathrm{Det}X| = 1$. Since a quantum channel is trace preserving if and only if its dual is unital, it follows that a trace-preserving Gaussian CP-map Φ is unital if and only if $|\mathrm{Det}X| = 1$.