A STABILITY THEOREM FOR A LINEAR HAMILTONIAN SYSTEM WITH PERIODIC COEFFICIENTS

V. A. Yakubovich

1°. Introduction

Consider the Hamiltonian system

$$\mathcal{J}\frac{dx}{dt} = H(t)x, \tag{1.1}$$

in which x is the vector solution, $\mathcal{J}$ is a complex (or real) matrix such that

$$\mathcal{J}^* = -\mathcal{J}, \quad \det\mathcal{J} \neq 0, \tag{1.2}$$

and the complex-valued matrix function $H(t)$ (called the Hamiltonian) satisfies the conditions

$$H(t)^* = H(t) \text{ almost everywhere on } [0,T], \ |H(t)| \in L(0,T). \tag{1.3}$$

The vector x and matrices $\mathcal{J}, H(t)$ have order n, and the asterisk denotes the Hermitian conjugate. A system (1.1) subject to conditions (1.2) and (1.3) is called a Hamiltonian system. We propose to consider the case of a periodic Hamiltonian:

$$H(t+T) = H(t) \quad \text{almost everywhere.} \tag{1.4}$$

Systems of this type commonly occur in applications, where $\mathcal{J}$ and $H(t)$ are usually real matrices in (1.1). One of the main problems encountered in applications is the stability of (1.1), the boundedness of all its solutions $x(t)$ on the interval $[0,\infty)$ [and so also on $(-\infty,+\infty)$]. Equation (1.1) (and its corresponding Hamiltonian) is said to be strongly stable (by the definition of M. G. Krein) if it is stable and remains stable under small [in the sense that $\|H\| = \int_0^T |H(t)|dt$] deformations of the Hamiltonian. Necessary and sufficient conditions for strong stability have been obtained by Krein [1] (sufficiency) and by Gel'fand and Lidskii [2] (necessity). This result is given below. Some definitions introduced in [1-4] are required for the discussion that follows.

Any solution $x(t)$ of the system (1.1) is uniquely determined by the specification of $x(0)$ and depends linearly on $x(0)$: $x(t) = X(t)x(0)$. The operator $X(t)$ (an $n \times n$ matrix in a fixed basis) defined

Translated from Zapiski Nauchnykh Seminarov Leningradskogo Otdeleniya Matematicheskogo Institute im. V. A. Steklova Akademii Nauk SSSR, Vol. 27, pp. 220-234, 1972.

by this relation is called the solving operator. It is an absolutely continuous function of the time and is unitary in the indefinite metric $\langle u, v\rangle = i^{-1}(\mathcal{J}u, v)$ i.e., for any t the operator $\mathcal{U} = X(t)$ satisfies the equation

$$\overset{*}{\mathcal{U}}\mathcal{J}\mathcal{U} = \mathcal{J}. \tag{1.5}$$

Operators $\mathcal{U}$ satisfying Eq. (1.5) are called $\mathcal{J}$-unitary.

It is obvious that $X(0) = I_n$ is the unit operator. The operator $X(T)$ is called the monodromy operator of Eq. (1.1). Relation (1.4) is equivalent to the relation

$$X(t+T) \equiv X(t)\cdot X(T). \tag{1.6}$$

Consequently, $X(nT) = X(T)^n$. It is readily deduced from the latter and from (1.6) that the stability of the system (1.1) is equivalent to boundedness of the powers $X(T)^n$, $n = 0, 1, 2, \cdots$. It follows at once from (1.5) that the spectrum of $X(T)$, like that of $X(t)$ for any t, is symmetric (with allowance for multiplicity of the eigenvalues and orders of the Jordan cells) with respect to the unit circle. For stability, therefore, it is necessary and sufficient that the spectrum of $X(T)$ lie on the unit circle and that first-order Jordan cells correspond to multiple eigenvalues.

The strong stability condition makes use of a concept introduced by M. G. Krein, namely, the "kind" of an eigenvalue. Let ρ be a simple eigenvalue of a $\mathcal{J}$-unitary operator $\mathcal{U}$, and let $|\rho| = 1$. Let $a \neq 0$ be the corresponding eigenvector: $\mathcal{U}a = \rho a$. It then turns out that $\langle a, a\rangle \neq 0$. If $\langle a, a\rangle > 0$, the number ρ is said to be an eigenvalue of the first kind, and if $\langle a, a\rangle < 0$, it is said to be of the second kind. Let ρ, $|\rho| = 1$ be a multiple eigenvalue of multiplicity $m > 1$. It turns out that the Gram matrix of the corresponding root subspace is nondegenerate. Suppose that it has m_1 positive and m_2 negative eigenvalues, $m_1 + m_2 = m$. We say in this case that m_1 eigenvalues of the first kind and m_2 eigenvalues of the second kind occur at the point ρ. Let ρ be an eigenvalue of a $\mathcal{J}$-unitary operator $\mathcal{U}$, and let $|\rho| \neq 1$. It proves convenient to call the number ρ an eigenvalue of the first kind if $|\rho| < 1$, and of the second kind if $|\rho| > 1$. Thus, all the eigenvalues of a $\mathcal{J}$-unitary operator $\mathcal{U}$ are classified as eigenvalues of either the first or the second kind. It is convenient to represent the corresponding points on the plane as dark (first kind) and light (second kind) points. Those points are continuously displaced under continuous deformation of the operator $\mathcal{U}$ in the group $\mathcal{G}$ of $\mathcal{J}$-unitary operators (fixed n). It might be expected that their color (dark or light) could change suddenly. However, this is not the case. Specifically, the following statements are true [3, 4]: Every $\mathcal{J}$-unitary operator $\mathcal{U}$ has p eigenvalues of the first kind and q eigenvalues of the second kind, where p and q are the numbers of positive and negative eigenvalues of the Hermitian matrix $i^{-1}\mathcal{J}$; 2) under continuous deformation of $\mathcal{U}$ in $\mathcal{G}$, its eigenvalues of the first kind (as well as its eigenvalues of the second kind) vary continuously.

The eigenvalues of the monodromy matrix $X(T)$ are called multipliers of the system (1.1). According to the foregoing discussion the system (1.1) has p multipliers of the first kind and q multipliers of the second kind. If the multipliers ρ_j ; $j = 1, 2, \ldots, p+q=n$, are distinct, the system (1.1) then has n linearly independent solutions $x_j(t)$ such that $x_j(t+T) = \rho_j x_j(t)$ and necessarily $\langle x_j(t), x_j(t)\rangle =$

$\gamma_j = \text{const}$, where $\gamma_j > 0$ if $|\rho_j| = 1$ and the multiplier ρ_j is of the first kind; $\gamma_j < 0$ if $|\rho_j| = 1$ and the multiplier ρ_j is of the second kind; and $\gamma_j = 0$, if $|\rho_j| \neq 1$.

The Krein–Gel'fand–Lidskii strong-stability theorem mentioned above is stated as follows: For the strong stability of a system (1.1) it is necessary and sufficient that all its multipliers be situated on the unit circle and that they do not include members of different kind that coincide.*

In other words, the criterion for strong stability is definiteness of the form $\langle w, v \rangle$ on each of the root subspaces of $X(T)$ [thus, a definite root subspace of $X(T)$ necessarily coincides with a proper subspace).

The stated theorem is pivotal to the theory of linear periodic Hamiltonian systems. However, it cannot be used directly to exhibit the stability of a system (1.1), because in most applications the system (1.1) is known, but the monodromy operator $X(T)$ is not. We point out that normally in applications the coefficients of the system (1.1) depend on several parameters, which must be chosen so as to ensure stability; methods associated with the computer-programmed calculation of $X(T)$ therefore have little application. In the next section we formulate a theorem by which it is possible to obtain effective (i.e., expressed in terms of $H(t)$] sufficient conditions for strong stability.

2°. A Sufficient Condition for Strong Stability

We denote by $\mathfrak{L}$ the complete space of Hamiltonians with properties (1.3)-(1.4) and norm

$$\|H\| = \int_0^T |H(t)| \cdot dt. \tag{2.1}$$

We write $H_1(t) \geqslant H_2(t)$ if for any vector $c \in E_n$ the following inequality holds almost everywhere on $[0,T]$: $(H_1(t)c, c) \geqslant (H_2(t)c, c)$.

We call a set $\mathfrak{M} \subset \mathfrak{L}$ directly unrestricted if the membership of the "segment" $H(t,s) = sH_1(t) + (1-s)H_2(t)$, $0 \leqslant s \leqslant 1$ [with "endpoints" $H_1(t)$ and $H_2(t)$] in that set implies the membership in that set of any Hamiltonian $H(t)$ for which the following holds:

$$H_1(t) \leqslant H(t) \leqslant H_2(t). \tag{2.2}$$

THEOREM 1. The set $\mathfrak{M} \subset \mathfrak{L}$ of strongly stable Hamiltonian is directionally unrestricted.

Below we give a brief explanation of how various effective stability conditions can be obtained from this theorem (see [4] for details). Theorem 1 is a straightforward consequence of Theorem 2.3 in [4], which has an exceedingly complex proof [4]. The objective of the present article is to give a simple proof of Theorem 1.

*This theorem has also been obtained by Moser [5] in a slightly different formulation. Several results pertaining to the stability domains of Hamiltonian systems (1.1) may be found in [6, 8]. Real Hamiltonian systems have been treated in [1, 2], but this case is nonessential. A simple proof of the Krein–Gel'fand–Lidskii Theorem (for the complex case) is given in [9].

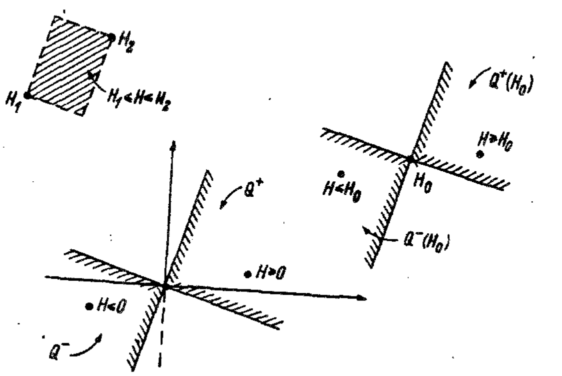

Fig. 1

All the concrete stability of [4], §3, can also be deduced from the above-stated Theorem 1.*

We now explain Theorem 1 and the definition of directionally unrestricted sets. In place of $\mathfrak{L}$ we discuss the set of points H on the plane; we identify H with a corresponding complex number. Let Q^+ be a certain closed cone with vertex at the origin (see Fig. 1). We introduce the following partial ordering on the plane. For two points H_1 and H_2 on the plane we write $H_1 \leqslant H_2$ if $H_2 - H_1 \in Q^+$. Let Q^- be the set of points symmetric to the points $H \in Q^+$ with respect to the origin: $H \in Q^-$ if $(-H) \in Q^+$ (see Fig. 1). The cone Q^+ is therefore the set of points $H \geqslant 0$; the cone Q^- is the set of points $H \leqslant 0$. The exterior to the cones Q^+ and Q^- contains points incongruent with the point $H = 0$. We impart a plane-parallel translation to the cones Q^+ and Q^- in such a way as to make their common vertex coincide with a certain point H_0, and we denote the resulting cones by $Q^+(H_0)$ and $Q^-(H_0)$ (see Fig. 1). The cone $Q^+(H_0)$ is the set of points $H \geqslant H_0$, and $Q^-(H_0)$ is the set of points $H \leqslant H_0$. The set of points H satisfying the condition $H_1 \leqslant H \leqslant H_2$ is therefore a parallelogram, with opposite corners at the points H_1 and H_2, obtained by the intersection of $Q^+(H_1)$ and $Q^-(H_2)$ (see Fig. 1). The above-defined partial ordering of the points H on the plane enables us to define directionally unrestricted sets on the plane in exactly the same words as for the space $\mathfrak{L}$: A set $\mathfrak{M}$ is called directionally unrestricted if the membership $\mathfrak{M}$ of a segment with endpoints H_1 and H_2, where $H_2 \geqslant H_1$, implies the membership in $\mathfrak{M}$ of the rectangle of points H such that $H_1 \leqslant H \leqslant H_2$.

If the segment H_1, H_2 is parallel to one side of the cone Q^+, then the rectangle $H_1 \leqslant H \leqslant H_2$ coincides with the segment H_1, H_2. The rectangle $H_1 \leqslant H \leqslant H_2$ is wider, the closer the direction of the segment H_1, H_2 comes to the direction of the bisector of the cone Q^+. Consequently, the possible "width" of a directionally unrestricted set $\mathfrak{M}$ is related to its "direction" relative to the cone $H_1 \geqslant 0$. This circumstance is illustrated by Fig. 2, which shows three directionally unrestricted sets $\mathfrak{M}_1$, $\mathfrak{M}_2$, $\mathfrak{M}_3$ and two sets $\mathfrak{M}_4$, $\mathfrak{M}_5$ that are not directionally unrestricted. We point out that the set $\mathfrak{M} = \mathfrak{M}_1 \cup \mathfrak{M}_2 \cup \mathfrak{M}_3$ in Fig. 2 is also directionally unrestricted. Therefore, a disconnected set can also be directionally unrestricted.

The directionally unrestricted property of a set of strongly stable Hamiltonians makes it a simple matter to deduce strong stability conditions for Eq. (1.1). Thus, we interpret H_1 and H_2 as sufficiently simple Hamiltonians such that: 1) the equation with the Hamiltonian $H(t, s) = sH_1(t) + (1-s)H_2(t)$ $(\forall\, 0 \leqslant s \leqslant 1)$ is explicitly integrable, for example, $H_1(t) = \mathrm{const}$, $H_2(t) = \mathrm{const}$; 2) for $0 \leqslant s \leqslant 1$ this equation

*A brief historical background of this theorem is in order. The first proposition similar to Theorem 1 (actually, a theorem somewhat stronger than Theorem 1) was set down for real second-order systems in [10]. A similar theorem (namely Theorem 2.3 of [4]) has been advanced for systems of arbitrary order, along with the basic concept of the proof, by the author in correspondence with M. G. Neigauz. The first actual proofs of Theorem 2.3 in [4] were obtained by Neigauz and Krein in 1953 (a fact that came to the author's attention through personal correspondence). Those proofs, however, remained unpublished.

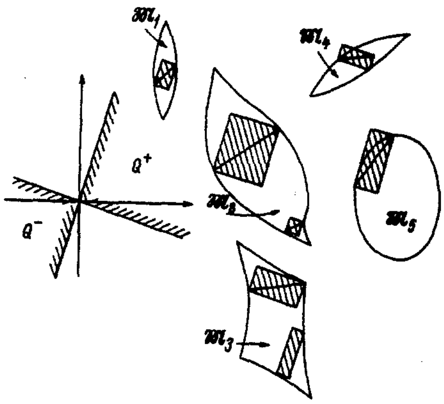

Fig. 2

is strongly stable. We then infer on the basis of the directionally unrestricted property that Eq. (1.1) is also strongly stable with any Hamiltonian $H(t)$ satisfying the inequalities $H_1(t) \leqslant H(t) \leqslant H_2(t)$.

We clarify this method for the deduction of stability conditions in the example of the scalar equation (Hill equation)

$$\ddot{\xi} + p(t)\xi = 0. \tag{2.3}$$

Here $p(t+T) = p(t)$ is a real function, and $p(t) \in L(0,T)$. We write Eq. (2.3) in the form of the system (1.1) for $n = 2$, where

$$x = \left\| \begin{matrix} \xi \\ \dot{\xi} \end{matrix} \right\|, \quad J = \left\| \begin{matrix} 0 & -1 \\ +1 & 0 \end{matrix} \right\|, \quad H(t) = \left\| \begin{matrix} p(t) & 0 \\ 0 & 1 \end{matrix} \right\|. \tag{2.4}$$

Let there correspond to the functions $p_1(t)$, $p_2(t)$ Hamiltonians $H_1(t)$, $H_2(t)$ having the above-stated form (2.4) for $p(t)=p_1(t)$ and $p(t)=p_2(t)$. It is clear that the matrix inequality $H_1(t) \leqslant H_2(t)$ is equivalent to the scalar inequality $p_1(t) \leqslant p_2(t)$, which must hold almost everywhere.

Corresponding to the Hamiltonian $sH_1(t) + (1-s)H_2(t)$, obviously, is Eq. (2.3) with the function $p(t,s) = sp_1(t) + (1-s)p_2(t)$. Consequently, the directionally unrestricted property of the set $\mathfrak{M}$ (which property is yet to be proved) is restated in terms of Eq. (2.3) as follows. If for $0 \leqslant s \leqslant 1$ Eq. (2.3) with the function $p(t,s) = sp_1(t) + (1-s)p_2(t)$ is strongly stable, then Eq. (2.3) with any function $p(t)$ satisfying the inequalities $p_1(t) \leqslant p(t) \leqslant p_2(t)$ is also strongly stable. We take the following identical constants $p_1(t)$ and $p_2(t)$ as $p_1(t) = c_1^2, p_2(t) = c_2^2$. Equation (2.3) with a constant coefficient $p(t) = c^2 > 0$ has solutions $y_1 = e^{ict}, y_2 = e^{-ict}$. According to Sec. 1°, the corresponding multipliers are $\rho_1 = e^{icT}$ $\rho_2 = e^{-icT}$. In the given case, $p = q = 1$. Consequently, one of the numbers ρ_1 , ρ_2 is a multiplier of the first kind, and the other is a multiplier of the second kind. They coincide for $cT = n\pi (n = 0, \pm 1, \ldots)$. If $cT \neq n\pi$, i.e., if for a certain integer n it is true that $\frac{n^2\pi^2}{T} < c^2 < \frac{(n+1)^2\pi^2}{T^2}$, then by the Krein–Gel'fand–Lidskii Theorem, Eq. (2.3) with $p(t) \equiv c^2$ is strongly stable. Therefore, Eq. (2.3) with the function $p(t,s) = sc_1^2 + (1-s)c_2^2$ is strongly stable for all $0 \leqslant s \leqslant 1$ if and only if

$$\frac{n^2\pi^2}{T} < c_1^2 \leqslant c_2^2 < \frac{(n+1)^2\pi^2}{T^2}$$

for some $n = 1, 2, 3, \ldots$. The following result is deduced from the directionally unrestricted property: Equation (2.3) is strongly stable if the following holds for some $n = 0, 1, 2, \ldots$:

$$\frac{n^2\pi^2}{T^2} < c_1^2 \leqslant p(t) \leqslant c_2^2 < \frac{(n+1)^2\pi^2}{T^2}. \tag{2.5}$$

Here the numbers c_1^2 , c_2^2 can be as close as we like to, respectively, the numbers $n^2\pi^2/T^2$ and $(n+1)^2\pi^2/T^2$. Thus, strong stability obtains if

$$\frac{n^2\pi^2}{T^2} \lneqq p(t) \lneqq \frac{(n+1)^2\pi^2}{T^2}. \tag{2.6}$$

Condition (2.6) is the well-known stability criterion of N. E. Zhukovskii. We have shown that the directionally unrestricted property makes it possible at once to obtain the criterion (2.5), which is very close to the Zhukovskii criterion (2.6).

It is easily shown by exact analogy that the vector equation

$$\frac{d^2y}{dt^2} + \mathcal{P}(t)y = 0$$

(of order κ), where $\mathcal{P}(t)^* = \mathcal{P}(t)$, $\mathcal{P}(t+T) = \mathcal{P}(t)$ is strongly stable if the following holds for certain integers $n_1, n_2, \ldots, n_\kappa$ and certain small numbers $\varepsilon_1 > 0, \ldots, \varepsilon_\kappa > 0$:

$$\operatorname{diag}\left[\frac{n_1^2\pi^2}{T^2} + \varepsilon_1, \ldots, \frac{n_\kappa^2\pi^2}{T^2} + \varepsilon_\kappa\right] \leqslant \mathcal{P}(t) \leqslant \operatorname{diag}\left[\frac{(n_1+1)^2\pi^2}{T^2} - \varepsilon_1, \ldots, \frac{(n_\kappa+1)^2\pi^2}{T^2} - \varepsilon\right].$$

Here $\operatorname{diag}[a_1, \ldots, a_\kappa]$ is a diagonal matrix with diagonal elements $a_1, \ldots, a_\kappa$.

Many other criteria ensuing from Theorem 1 are presented in [4], §3.

3°. Proof of Theorem 1

For brevity we drop the argument of the Hamiltonian $H(t) \in \mathfrak{L}$, i.e., we write $H, H_1 \ldots$ instead of $H(t), H_1(t), \ldots$.

We make significant use of the following important property of the multipliers, which was established by M. G. Krein: As the Hamiltonian is increased the multipliers of the first kind situated on the unit circle move counterclockwise, while those of the second kind move clockwise. This property is more precisely stated as follows. Let $H(\lambda)$ be a Hamiltonian depending analytically on λ at the point $\lambda = \lambda_0$, where λ_0 is a real number. Then the monodromy operator $X(T, \lambda)$ is also an analytic function of λ at $\lambda = \lambda_0$. At the point ρ_0, where $|\rho_0| = 1$, let $m_1 \geqslant 1$ multipliers coincide, all of the same (say, the first) kind. Let $\rho(\lambda)$ - be the algebraic function defined by the relation $\det[X(T, \lambda) - \rho I] = 0$ and the condition $\rho(\lambda_0) = \rho_0$. Then λ_0 is not a branch point of the function $\rho(\lambda)$; all m_1 values of the function $\rho(\lambda)$ which are denoted here by $\rho_j(\lambda) = \exp[i\varphi_j(\lambda)]$ are analytic functions of λ at $\lambda = \lambda_0$, and also $\varphi_j'(\lambda_0) \geqslant 0$. The analogous statement holds for multipliers of the second kind, except for them, $\varphi_j'(\lambda_0) \leqslant 0$.

The proof of this proposition may be found in [1]. We mention that it suffices to limit the ensuing discussion to the case of simple multipliers, $m_1 = 1$. In this case, the analyticity of $\rho_1(\lambda)$ is obvious; we have only to prove the inequality $\varphi_1'(\lambda_0) \geqslant 0$ for multipliers of the first kind and the inequality $\varphi_1'(\lambda_0) \leqslant 0$ for multipliers of the second kind. A straightforward proof of those relations is given in [4].

In the Banach space $\mathfrak{L}$ of Hamiltonians, we consider the triangle H_1, H, H_2 (i.e., the triangle composed of the set-theoretic sum of the Hamiltonians $\lambda_1 H_1 + \lambda_2 H_2 + \lambda_3 H_3$ when the numbers λ_j run through the set $0 \leqslant \lambda_j \leqslant 1$, $\lambda_1 + \lambda_2 + \lambda_3 = 1$). We prove that all the Hamiltonians of this triangle, H in particular, are strongly stable.

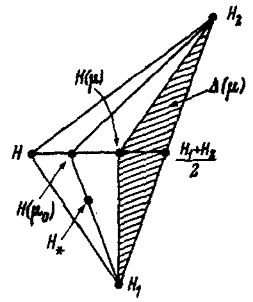

Fig. 3

We continuously deform the triangle H_1, H, H_2, shrinking it toward the base H_1, H_2 (see Fig. 3). We specifically consider the set of triangles $\Delta(\mu)$ with base H_1, H and vertex at the point

$$H(\mu) = \frac{H_1+H_2}{2} + \mu\left[H - \frac{H_1+H_2}{2}\right] \tag{3.1}$$

(see Fig. 3). We have $H(1) = H$, $H(0) = \frac{1}{2}(H_1+H_2)$. Thus, the triangle $\Delta(1)$ is congruent with the triangle H_1, H, H_2, and the triangle $\Delta(0)$ degenerates into the base H_1, H_2. By the premise of the theorem, the base H_1, H_2 consists of strongly stable Hamiltonians [the base H_1, H_2 is spanned by the Hamiltonians $H(t,s) = sH_1(t) + (1-s)H_2(t)$ for $0 \le s \le 1$]. Inasmuch as a set of strongly stable Hamiltonians is, by definition, an open set, all the triangles $\Delta(\mu)$ sufficiently close to the base (i.e., with small $\mu > 0$) are also strongly stable.*

One of two situations is possible: either all the triangles $\Delta(\mu)$, $0 \le \mu \le 1$ are strongly stable (whereupon the pertinent Hamiltonian H is also strongly stable) or there is a μ_0 ($0 < \mu_0 \le 1$) such that the triangle $\Delta(\mu_0)$ has on its lateral sides H_1, $H(\mu_0)$ and H_2, $H(\mu_0)$ at least one Hamiltonian H_* that is not strongly stable (see Fig. 3), in which case all the triangles $\Delta(\mu)$, $0 \le \mu < \mu_0$ are strongly stable; we therefore examine the second possibility.

Let us consider once again the triangle $\Delta(\mu)$, $0 \le \mu \le 1$ with sides $H_1, H(\mu)$ and $H(\mu), H_2$. We introduce the Hamiltonian $H(\lambda,\mu)$, which runs along the side $H_1, H(\mu)$ as λ is varied from $\lambda = 0$ to $\lambda = \frac{1}{2}$ and along the side $H(\mu), H_2$ as λ is varied from $\lambda = \frac{1}{2}$ to $\lambda = 1$, namely:

$$H(\lambda,\mu) = \begin{cases} H_1 + 2\lambda[H(\mu) - H_1] & (0 \le \lambda \le \frac{1}{2}), \\ H(\mu) + 2(\lambda - \frac{1}{2})[H_2 - H(\mu)] & (\frac{1}{2} \le \lambda \le 1). \end{cases}$$

We show that for each of these expressions $\frac{\partial H}{\partial \lambda} \geqslant 0$, i.e., that

$$H_1 \le H(\mu) \le H_2 \quad (0 \le \mu \le 1). \tag{3.2}$$

Replacing H by H_1 and H_2 in (3.1), we obtain on the basis of (2.2)

$$\frac{H_1+H_2}{2} - \frac{1}{2}\mu(H_2 - H_1) \le H(\mu) \le \frac{H_1+H_2}{2} + \frac{1}{2}\mu(H_2 - H_1).$$

Inasmuch as $H_1 - H_2 \geqslant 0$ the replacement of μ by unity on the left- and right-hand sides of the preceding inequality can only strengthen it. We then obtain (3.2).

We fix an arbitrary μ in the interval $0 \le \mu \le \mu_0$. By the definition of the number μ_0 the Hamiltonian $H(\lambda,\mu) = H(t,\lambda,\mu)$ is strongly stable for any $0 \le \lambda \le 1$, i.e., all the corresponding multipliers lie on the unit circle and do not include any two of different kinds that coincide.

*A triangle of Hamiltonians is called strongly stable when all its Hamiltonians are strongly stable.

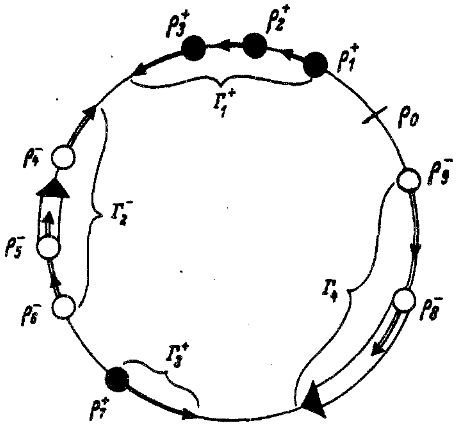

Fig. 4

We show that it is possible to isolate on the unit circle nonintersecting arcs such that only multipliers of one kind of the equation with Hamiltonian $H \in \Delta(\mu)$ are situated on one arc for any $0 \leq \mu < \mu_0$ and, accordingly, any multiplier lies in some isolated arc, i.e., the situation illustrated by Fig. 4 exists.

We consider the equation with Hamiltonian H_1.

We pick a certain point ρ_0 located on the unit circle "between" multipliers of the first and second kind, namely a point such that in moving from it along the unit circle in the direction of the increasing argument we first meet a multiplier of the first kind, while in the direction of the decreasing argument we first meet a multiplier of the second kind (see Fig. 4). We enumerate the multipliers of this equation in their order of accession as we move around the unit circle from the point ρ_0 in the direction of the increasing argument. We assign a "+" sign to multipliers of the first kind and a "-" sign to multipliers of the second kind. We partition the multipliers into groups combining consecutive multipliers of the same kind into a single group.

We first carry out the proof by a less-than-rigorous, but more transparent, procedure. For definiteness, let $n = 9$ and let the multipliers be arranged as depicted in Fig. 4. In this case we have four groups:

$$\{\rho_1^+, \rho_2^+, \rho_3^+\}, \ \{\rho_4^-, \rho_5^-, \rho_6^-\}, \ \{\rho_7^+\}, \ \{\rho_8^-, \rho_9^-\}.$$

We now deform the Hamiltonian, "progressing" along the side H_1, H_2. In other words, we consider the Hamiltonian $H(\lambda, 0)$, $0 \leq \lambda \leq 1$. In accordance with the preceding discussion, as λ is varied from $\lambda = 0$ to $\lambda = 1$ the multipliers $\rho_j^{\pm}(\lambda)$ of the equation with Hamiltonian $H(\lambda, 0)$ move around the unit circle in the direction of the increasing argument (multipliers of the first kind) and in the direction of the decreasing argument (multipliers of the second kind). Since the segment H_1, H_2 is strongly stable, multipliers of different kinds are not met along the path of motion. Consequently, $\rho_j^{\pm}(0) = \rho_j$, and the multipliers $\rho_j^{\pm}(\lambda)$ for any $0 \leq \lambda \leq 1$ are divided analogously into groups:

$$\{\rho_1^+(\lambda), \rho_2^+(\lambda), \rho_3^+(\lambda)\}, \{\rho_4^-(\lambda), \rho_5^-(\lambda), \rho_6^-(\lambda)\}, \{\rho_7^+(\lambda)\}, \{\rho_8^-(\lambda), \rho_9^-(\lambda)\}.$$

Therefore, for $0 \leq \lambda \leq 1$ the multipliers of the first group are located on a closed arc Γ_1^+, whose endpoints are $\rho_1^+(0)$ and $\rho_3^+(1)$, and the multipliers of the second group are located on an arc Γ_2^- with endpoints $\rho_4^-(1)$ and $\rho_6^-(0)$, and so on; all of these arcs are nonintersecting (see Fig. 4).

The motion of the multipliers is exactly analogous when the Hamiltonian is deformed on the sides H_1, $H(\mu)$, H_2 for $0 \leq \mu < \mu_0$ (as long as strong stability holds), i.e., in the case of multipliers of the equation with Hamiltonian $H(\lambda, \mu)$ for fixed μ, $0 \leq \mu < \mu_0$, and λ varying from $\lambda = 0$ to $\lambda = 1$. Now the arcs $\Gamma^{\pm}$ are the same, because they are determined by the "endpoints" $H(0, \mu) = H_1$, $H(1, \mu) = H_2$, which do not depend on μ. We have thus deduced that for an equation with any Hamiltonian

$H \in \Delta(\mu)$, $0 \leq \mu < \mu_0$, the multipliers of the first kind are located on the closed arcs Γ_j^+ and the multipliers of the second kind are on the closed arcs Γ_j^-, where Γ_j^+ and Γ_j^- are nonintersecting. We have thus proved the above-stated proposition.

On the other hand, passing to the limit $H \to H_*$ inside the triangle $\Delta(\mu)$, we infer that multipliers of unlike kinds must coincide. This situation is clearly impossible by the established distribution of the multipliers. The second case is therefore impossible, thus proving the theorem.

We now reason in a more rigorous context, reinstating some minor details omitted above. We write the arguments of the multipliers of the first and second kind [equation with Hamiltonian $H(\lambda, \mu)$] in the order met as the unit circle is traversed in the positive sense from the point $\rho_0 = \exp[i\varphi_0]$ around to the same point. Suppose that p_1 multipliers of the first kind are met first, followed by (if they exist) q_1 multipliers of the second kind, and so on. We denote their arguments by $\varphi_j(\lambda, \mu)$, $\psi_i(\lambda, \mu)$ (φ_j for the first kind and ψ_j for the second kind):

$$\varphi_0 < \varphi_1(\lambda,\mu) \leq \varphi_2(\lambda,\mu) \leq \ldots \leq \varphi_{p_1}(\lambda,\mu) <$$

$$< \psi_1(\lambda,\mu) \leq \ldots \leq \psi_{q_1}(\lambda,\mu) < \varphi_{p_1+1}(\lambda,\mu) \leq \ldots \varphi_0 + 2\pi.$$

It was shown above that

$$\frac{\partial H}{\partial \lambda} \geq 0 \text{ for } 0 \leq \lambda < \tfrac{1}{2}, \ \tfrac{1}{2} < \lambda \leq 1.$$

For $\lambda = \frac{1}{2}$ the function $\frac{\partial H}{\partial \lambda}$ acquires a discontinuity, but for the left and right derivatives we have $\left(\frac{\partial H}{\partial \lambda}\right)_+ \geq 0, \left(\frac{\partial H}{\partial \lambda}\right)_- \geq 0$. Then by the theorem of M. G. Krein, as λ is increased the arguments of the multipliers of the first kind do not decrease, while the arguments of those of the second kind do not increase; more precisely stated, for $0 \leq \lambda < \frac{1}{2}$, $\frac{1}{2} < \lambda \leq 1$ the derivatives $\partial\varphi_j/\partial\lambda \geq 0$, $\partial\psi_j/\partial\lambda \leq 0$, $\partial\varphi_j/\partial\lambda \geq 0$, $\partial\psi_j/\partial\lambda \leq 0$ exist; for $\lambda = \frac{1}{2}$ the latter inequalities hold separately for the right and left derivatives. Thus, $\varphi_j(\lambda, \mu)$ and $\psi_k(\lambda, \mu)$ are, respectively, nondecreasing and nonincreasing functions of λ ($0 \leq \lambda \leq 1$) for any fixed μ in the interval* $0 \leq \mu < \mu_0$. Consequently,

*We have made use here of the above-stated theorem of Krein [1] on the motion of the multipliers with increasing Hamiltonian. We can also use a simpler proposition pertaining only to simple multipliers, in which case the proof ([4], pp. 67-68) is far simpler. It is necessary to add the following simple argument in this case. We exclude from the interval $0 \leq \lambda \leq 1$ the value $\lambda = 1/2$, along with the values for which the equation with Hamiltonian $H(\lambda, \mu)$ has multiple multipliers (recalling that μ is fixed, $0 \leq \mu < \mu_0$). We show that only finitely many values of λ are so excluded. First, let $H(\lambda, \mu) = H_1 + 2\lambda [H(\mu) - H_1]$. The corresponding monodromy operator $X(T, \lambda)$ is an entire function of λ. The discriminant $\Delta(\lambda)$ of the equation $\det[X(T,\lambda) - \rho I] = 0$ is therefore also an entire function of λ. Inasmuch as the excluded values of λ in the interval $0 \leq \lambda < 1/2$ are the roots of the equation $\Delta(\lambda) = 0$, there can only be finitely many of them. The interval $1/2 < \lambda \leq 1$ is treated analogously. For all λ, $0 \leq \lambda \leq 1$ except those excluded, the derivatives (see [4], pp. 67-68) $\partial\varphi_j/\partial\lambda$, $\partial\psi_j/\partial\lambda$ and $\partial\varphi_j/\partial\lambda \geq 0$, $\partial\psi_i/\partial\lambda \leq 0$ exist. Therefore, $\varphi_j(\lambda, \mu)$ and $\psi_k(\lambda, \mu)$ are, respectively, nondecreasing and nonincreasing functions of λ for $0 \leq \lambda \leq 1$.

$$\varphi_0 < \varphi_1(0,\mu) \leq \varphi_{p_1}(1,\mu) < \psi_1(1,\mu) \leq \psi_{q_1}(0,\mu) < \varphi_{p_1+1}(0,\mu) < \ldots < \varphi_0 + 2\pi.$$

The numbers $\varphi_j(0,\mu)$, $\varphi_j(1,\mu)$, $\psi_j(0,\mu)$, $\psi_j(1,\mu)$ involved in these inequalities are the arguments of multipliers with Hamiltonians H_1 and H_2 and are therefore independent of μ.

We denote by Φ_j' and Φ_j'' the arguments of the endpoints of the arcs denoted above by Γ_j^+, and by Ψ_j' and Ψ_j'' the same for the arcs Γ_j^-:

$$\begin{gathered}
\Phi_1' = \varphi_1(0,\mu), \quad \Phi_1'' = \varphi_{p_1}(1,\mu), \\
\Psi_1' = \psi_1(1,\mu), \quad \Psi_1'' = \psi_{q_1}(0,\mu), \\
\Phi_2' = \varphi_{p_1+1}(0,\mu), \ldots \\
\ldots\ldots\ldots\ldots\ldots\ldots
\end{gathered}$$

These numbers do not depend on μ. We have thus deduced the following for $0 \leq \lambda \leq 1$, : $0 \leq \mu < \mu_0$

$$\begin{gathered}
0 < \Phi_1' \leq \varphi_1(\lambda,\mu) \leq \ldots \leq \varphi_{p_1}(\lambda,\mu) \leq \Phi_2'' < \Psi_1', \\
\Psi_1' \leq \psi_1(\lambda,\mu) \leq \ldots \leq \psi_{q_1}(\lambda,\mu) \leq \Psi_2' < \Phi_2', \\
\ldots\ldots\ldots\ldots\ldots\ldots
\end{gathered}$$

Consequently, all the multipliers for the Hamiltonians of the triangles $\Delta(\mu)$, $0 \leq \mu < \mu_0$ are arranged as follows: the arc $\Gamma_1^+ = (\Phi_1', \Phi_1'')$ contains p_1 multipliers of the first kind, the arc $\Gamma_2^- = (\Psi_1', \Psi_2'')$ contains q_1 multipliers of the second kind, and so on. The indicated arcs are nonintersecting, and there are no multipliers outside of them (see Fig. 4). Therefore, the smallest distance between multipliers of the first and second kind for equations with Hamiltonians of the triangles $\Delta(\mu)$, $0 \leq \mu < \mu_0$ is greater than some positive number ε_0 that does not depend on λ or μ.

On the other hand, the path through the Hamiltonians of this triangle can be made as close as we like to the Hamiltonian H_* (which is located on the boundary of the triangle) at which the multipliers of the first and second kind coincide. As $H(\lambda,\mu) \to H_*$ the set of multipliers for the Hamiltonian $H(\lambda,\mu)$ is continuously transformed into the set of multipliers for the Hamiltonian H_* (by the theorem on the continuous dependence of multipliers with allowance for kind). Consequently, the equation with Hamiltonian H_* cannot have any two multipliers of different kinds that coincide. The resulting contradiction eliminates the second-stated possibility, thus proving the theorem.

LITERATURE CITED

1. M. G. Krein, "The basic propositions of the theory of λ-zones of stability of a canonical system of linear differential equations with periodic coefficients," in: In Memory of A. A. Andronov [in Russian], Izd. AN SSSR, Moscow (1955), pp. 414-498.
2. I. M. Gel'fand and V. B. Lidskii, "Structure of the stability zones of canonical linear systems of differential equations with periodic coefficients," Usp. Matem. Nauk, 10, No. 1 (63), 3-40 (1955).
3. V. A. Yakubovich, "Construction of a group of simplectic matrices and structure of the set of unstable canonical systems with periodic coefficients," Matem. Sborn., 44 (86), No. 3, 313-352 (1958).

4. V. A. Yakubovich, "Convexity properties of the stability zones of linear Hamiltonian systems of differential equations with periodic coefficients," Vest. Leningrad. Gos. Univ., Ser. 3, No. 3 (1962).
5. J. Moser, "New aspects in the theory of stability of Hamiltonian systems," Commun. Pure Appl. Math., 11, No. 1, 81-114 (1958).
6. M. Segal, "Stability problems for systems of linear differential equations," Commun. Pure Appl. Math., 17, 401-414 (1964).
7. W. A. Coppel and A. Howe, "On the stability of linear canonical systems with periodic coefficients," J. Australian Math. Soc., 5, Part 2, 169-195 (1965).
8. V. A. Yakubovich, "Construction of the function space of complex canonical equations with periodic coefficients," Dokl. Akad. Nauk SSSR, 139, No. 1, 54-57 (1961).
9. V. A. Yakubovich, "Critical frequencies of quasicanonical systems," Vest. Leningrad. Gos. Univ., Ser. Matem. Mekh. Astron. (Ser. 2), No. 13 (1958).
10. V. A. Yakubovich, "Stability criteria for a system of two canonical equations with periodic coefficients," Dokl. Akad. Nauk SSSR, 78, No. 2 (1951).