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ON M.G. KREIN'S WORK IN THE THEORY OF LINEAR PERIODIC HAMILTONIAN SYSTEMS

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We present basic ideas of the theory of linear periodic Hamiltonian systems.

Introduction

This field was apparently on the periphery of M. Krein's scientific interests. Nevertheless, here, as well as in many other fields of mathematics, he was the first. Other researchers followed his lead.

The idea of this paper was not just to list all the related results of M. Krein but to give a concise and clear exposition of his basic ideas, making our presentation as coherent as possible. This is why the results were classified not chronologically but according to their subjects and many particular results of Krein and his followers were not mentioned.

A series of papers we intend to speak about is devoted to the investigation of linear Hamiltonian systems

$$\frac{dp_j}{dt} = \frac{d\mathcal{H}}{dq_j}, \quad \frac{dq_j}{dt} = -\frac{d\mathcal{H}}{dp_j}, \quad j = 1, \dots, m, \tag{1}$$

where $\mathcal{H} = \mathcal{H}(t, p, q)$ is a quadratic form of variables $p = \|p_j\| \in \mathbb{R}^m$ and $q = \|q_j\| \in \mathbb{R}^m$ with T-periodic real coefficients. By introducing a vector x with the components $p_1, \dots, p_m, q_1, \dots, q_m$, we rewrite system (1) as a vector equation

$$J\frac{dx}{dt} = H(t)x, \tag{2}$$

where J and $H(t)$ are $(2m \times 2m)$-dimensional matrices and

$$J = \begin{bmatrix} 0 & -I_m \\ I_m & 0 \end{bmatrix}, \quad H(t+T) = H(t) = H(t)^*. \tag{3}$$

[The elements of the matrix $H(t)$ are real and belong to $L_1(0, T)$.]

At first sight, the subject of these researches may seem to be quite special and narrow. This is indeed true but many engineering and physical problems lead to the study of systems of this sort. In the first place, one must mention the problems of parametric resonance; note that rocking a swing is a well-known example of parametric resonance. Let us present several more interesting examples.

1. It is quite clear that if a plane one-component spring pendulum suspended with a hinge (Fig. 1) is pulled downward, it begins to vibrate in the vertical direction. At the same time, and this is not so obvious, if its parameters satisfy the condition $3mg \approx cl$ (m is the mass, c is the stiffness of the spring, and l is the length of the pendulum in the relaxed state), vertical vibrations become unstable and turn into transverse vibrations and then back.

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2. Similarly, beams and other structures may lose the plane shape of bending if their parameters satisfy certain relations.

3. It is most likely that the collapse of the Tacoma Narrows Bridge (Tacoma, Washington, USA, 1940) was caused by the phenomenon of parametric resonance.

4. Many people who watched how water is pumped out of a well witnessed the following phenomenon: Normally the hose used for pumping remains immobile. But, at a certain frequency of pumping, it starts moving and twisting.

5. It was shown by Lyapunov that the Laplacian motion in the three-body problem is orbitally stable (in the linear approximation) if the masses of the bodies $M,\ m,$ and m' satisfy the condition

$$0 < \mu < 1/12 \quad \text{or} \quad 1/12 < \mu < 1/9, \quad \text{where} \quad \mu = \frac{3(Mm + Mm' + mm')}{(M + m + m')^2}, \tag{4}$$

and the eccentricity ε of the motion is sufficiently small $(\varepsilon < \varepsilon_0(\mu))$.

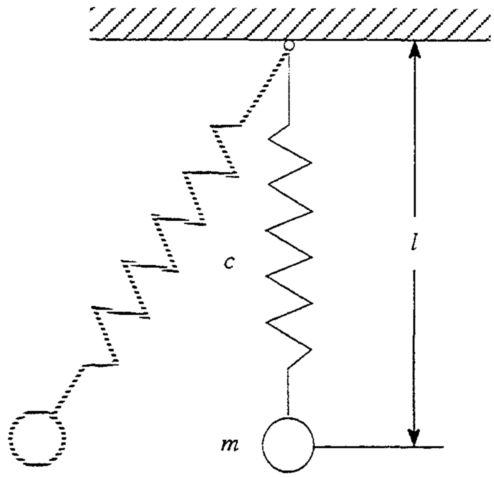

Fig. 1

These and some other problems lead researchers to the mathematical problem of finding the conditions under which system (1) is stable (all its solutions are bounded for $-\infty < t < +\infty$) or unstable.

At the end of the 19th century and at the beginning of this century, Eqs. (1) were mainly studied in connection with the needs of celestial mechanics (Bruns, Hill, Poincaré, and others); this means that the researches were mainly focused on the development of methods for integrating systems with small parameters and solutions were found in the form of (ordinary and asymptotic) series. The intense use of computers has partially diminished the importance of these methods. Being mainly interested in Krein's results, we now concentrate on the efficient stability and instability conditions imposed on the coefficients of the system.

Before M. Krein, this field was not investigated too well. The results obtained were mostly connected with the Hill equation

$$\frac{d^2y}{dt^2} + p(t)y = 0 \qquad (p(t+T)) = (p(t)), \tag{5}$$

deduced by Hill in his studies of the motion of the moon. Equation (5) reduces to (2) if we set

$$x = \begin{bmatrix} y \\ dy/dt \end{bmatrix} \quad \text{and} \quad H(t) = \begin{bmatrix} p(t) & 0 \\ 0 & 1 \end{bmatrix}. \tag{6}$$

Among the known results, one should mention the Lyapunov criteria

$$p(t) \geqq 0, \quad T\int_0^T p(t)dt \leq 4 \tag{7}$$

and the Zhukovskii criterion: For integer k,

$$\frac{k^2\pi^2}{T^2} \leqq p(t) \leqq \frac{(k+1)^2\pi^2}{T^2}. \tag{8}$$

Both these criteria guarantee the stability of the Hill equation {and the absence of T-periodic and T-semiperiodic $[y(t+T) = -y(t)]$ solutions}. There were also some other similar criteria but the constants appearing in them were in many cases wrong.

(Note the difference between these conditions: The Lyapunov criterion selects a single convex region, and the Zhukovskii criterion gives countably many regions; below, we return to this fact.)

The Mathieu equation $(p(t) = \alpha + \beta\cos 2t)$ was also well studied and the plane $\{\alpha, \beta\}$ was decomposed into countably many regions of stability and instability. Also note a series of fine results of Lyapunov and Haupt for Hill equations with a parameter. Although systems (1) with $m > 1$ (m is the number of degrees of freedom of the corresponding mechanical system) are quite important, there were almost no results in this field before Krein's work.

1. Stability Criterion for a Linear Periodic Hamiltonian Equation

Before passing to the presentation of the results of M. Krein, let us recall the required terminology.

Let $X(t)$ be a $(2m \times 2m)$-dimensional evolutionary matrix of Eq. (2) and let

$$\det[X(T) - \rho I] = 0 \tag{9}$$

be its characteristic equation. The roots of Eq. (9) are called multipliers, each multiplier is associated with a (generally speaking, complex) solution $x(t)$ of the equation $J\dot{x} = H(t)x$ with the property $x(t+T) = \rho x(t)$. The matrix $X(t)$ is symplectic for any $t > 0$. Denote by $\mathfrak{S} = \mathrm{Sp}_{2m}(\mathbb{R})$ a group of symplectic matrices ($X \in \mathfrak{S} \Leftrightarrow X^*JX = J$). The eigenvalues of the matrices $X \in \mathfrak{S}$ (with the collection of elementary divisors) are located symmetrically with respect to the real axis and with respect to the unit circle. The same is true for multipliers (the Poincaré theorem). The equation $J\dot{x} = H(t)x$ is stable [all its solutions are bounded on $(-\infty, +\infty)$] if all its multipliers lie on the circle $|\rho| = 1$ and the multiple ones correspond to simple elementary divisors; otherwise, it is unstable.

Krein wrote [1, p. 415] that the progress in the theory of equations of the type $J\dot{x} = H(t)x$ (with $m > 1$) was connected with the possibility of selecting a class of equations of *positive type*, i.e., equations for which

$$H(t) \geq 0 \quad \text{and} \quad H_{\text{av}} = \frac{1}{T}\int_0^T H(t)dt > 0. \tag{10}$$

Let us outline Krein's method [1, 2] for establishing stability conditions for the equation $J\dot{x} = H(t)x$ of positive type. Parallel with the original equation, one should consider the boundary-value problem

$$J\frac{dx}{dt} = \lambda H(t)x, \tag{11}$$

$$x(T) = \rho_0 x(0) \quad (|\rho_0| = 1). \tag{12}$$

It is easy to show that its eigenvalues are real. For small $\lambda > 0$, the multipliers $\rho_j(\lambda)$ of Eq. (11) lie on the circle $|\rho| = 1$. Let us prove this for the case where all eigenvalues $i\omega_k$ of the matrix $J^{-1}H_{\text{av}}$ are distinct. (Clearly, they are purely imaginary.) It readily follows from the equality $X(T, \lambda) = I_{2m} + J^{-1}H_{\text{av}}T\lambda + O(\lambda^2)$ that $\rho_k(\lambda) = \exp[i(\omega_k + \varepsilon_k(\lambda))\lambda]$, where $\varepsilon_k(\lambda) \to 0$ as $\lambda \to 0$. For small $\lambda > 0$, all $\varepsilon_k(\lambda)$ are real; otherwise, the symmetry of the numbers $\rho_k(\lambda)$ with respect to the unit circle is violated. Hence, for small $\lambda > 0$, all $\rho_k(\lambda)$ are distinct and $|\rho_k(\lambda)| = 1$, i.e., the equation $J\dot{x} = \lambda H(t)$ is stable.

We want to know whether it remains stable for $\lambda = +1$. Krein studied the behavior of multipliers as λ increases from zero to one and introduced an important concept of the kinds of multipliers. Consider some $\lambda_0 > 0$ and a simple multiplier $\rho(\lambda_0)$, $|\rho(\lambda_0)| = 1$. If λ_0 is shifted to the region $\operatorname{Re}\lambda > 0$, this multiplier leaves the unit circle because, otherwise, the boundary-value problem (11), (12) would have a nonreal eigenvalue λ for $\rho_0 = \rho(\lambda)$, $|\rho(\lambda)| = 1$. A multiplier $\rho(\lambda_0)$ is called a multiplier of the *first kind* if $|\rho(\lambda)| < 1$ and a multiplier of the *second kind* if $|\rho(\lambda)| > 1$. Let $\rho_0 = \rho(\lambda_0)$, $|\rho_0| = 1$, be a multiplier with multiplicity $r > 1$. It is said that r_1 multipliers of the first kind and r_2 multipliers of the second kind coincide at the point $\rho_0 = \rho(\lambda_0)$ if the shift of λ_0 to the region $\operatorname{Re}\lambda > 0$ results in the situation where the equation possesses r_1 multipliers in the region $|\rho| < 1$ and r_2 multipliers in the region $|\rho| > 1$, $r_1 + r_2 = r$.

It is also possible to give an equivalent definition of the kinds of multipliers applicable to arbitrary symplectic matrices. Let ρ_0, $|\rho_0| = 1$, be a simple eigenvalue of a symplectic matrix X, $Xa = \rho_0 a$ and $a \neq 0$. It is of the first kind if $i^{-1}(Ja, a) > 0$ and of the second kind if $i^{-1}(Ja, a) < 0$. [It can be proved that $(Ja, a) \neq 0$.] By analogy, in the case of a multiple eigenvalue ρ_0, $|\rho_0| = 1$, the numbers of eigenvalues of the first kind and of the second kind that coincide at the point ρ_0 are determined by the numbers of positive and negative eigenvalues of the Gram matrix of the root subspace. It is convenient to generalize Krein's definition by assuming that the eigenvalues ρ_k are of the first kind if $|\rho_k| < 1$ and of the second kind if $|\rho_k| > 1$. Then the symplectic $(2m \times 2m)$-dimensional matrix X has m eigenvalues ρ_k^+ of the first kind and m eigenvalues ρ_k^- of the second kind and, moreover, ρ_k^+ (and, hence, ρ_k^-) continuously depend on X. (This fact is extremely important; it follows from the Riesz formulas for projectors and from the J-orthogonality of the root spaces $\mathfrak{L}_{\rho_1}$ and $\mathfrak{L}_{\rho_2}$ of the symplectic matrix that correspond to the eigenvalues ρ_1 and ρ_2, $\rho_1\bar{\rho}_2 \neq 1$.) It can be easily proved that the eigenvalues $e^{i\omega_0}$ and $e^{-i\omega_0}$ have the opposite kinds and a multiple eigenvalue of a single kind (the corresponding Gram matrix is definite) corresponds to simple elementary divisors.

Let us return to the study of the behavior of multipliers of the equation $J\dot{x} = \lambda H(t)x$ as λ increases from zero to one. (Conditions (10) still hold.) We have showed that, for a simple multiplier of the first kind, the mapping $\rho = \rho^+(\lambda)$ transforms a semineighborhood $\operatorname{Re}\lambda > 0$ of a point $\lambda_0 \in (0, 1)$ into a semineighborhood $|\rho| < 1$ of the point $\rho_0 = \rho^+(\lambda_0)$, $|\rho_0| > 1$. Therefore, an increase in $\lambda \in (0, 1)$, results in a shift of a multiplier of the first kind along the unit circle with an increase in its argument. Similarly, a shift of a multiplier of the second kind leads to a decrease in its argument. This important rule deduced without calculations was repeatedly used in many subsequent papers. (It could also be obtained as a result of calculations.)

For $\lambda = 0$, all multipliers of the first and second kinds of the equation $J\dot{x} = \lambda H(t)x$ coincide at the point $\rho = 1$. It was shown that, for small $\lambda > 0$, all multipliers lie on the unit circle, whence we easily get that, for small $\lambda > 0$, all multipliers of the first kind lie on the upper semicircle and all multipliers of the second kind lie (symmetrically) on the lower semicircle. When multipliers of the same kind meet on the upper (lower) open semicircle, they cannot pass to the region $|\rho| \neq 1$, since, in this case, their kind would change. Moreover, the multipliers of the same kind that coincide are associated with simple elementary divisors. Thus, as λ increases from zero, the multipliers of the first kind move along the unit circle so that their arguments increase (the arguments of the multipliers of the second kind decrease) until multipliers of the opposite kinds meet. Before this meeting, the equation $J dx/dt = \lambda H(t)x$ remains stable. The multipliers may leave the unit circle only after a meeting of some multipliers of the opposite kinds, and the multipliers may meet only at the point $\rho = -1$. Let $\lambda = \Lambda_+ > 0$ be the corresponding value of λ. For $\lambda = \Lambda_+$, the equation may become unstable (due to the presence of nonsimple elementary divisors) and, for λ close to Λ_1, $\lambda > \Lambda_1$, it may become exponentially unstable (if the multipliers leave the unit circle). Thus, the original equation $J dx/dt = H(t)x$ is stable for $\Lambda_+ > 1$. The number Λ_+ is the least positive eigenvalue of the boundary-value problem (11), (12) with $\rho_0 = -1$. This means that M. Krein managed to reduce the original new problem to a relatively standard problem of estimating the eigenvalue Λ_+ from below.

A rough estimate can be obtained very easily. Assume that $x(t)$ is a solution of problem (11), (12) such that $\rho_0 = -1$, $\lambda = \Lambda_1$, and $x(t+T) = x(t)$. Let

$$|x(t_0)| = \max_{[0,T]} |x(t)|.$$

Then

$$2|x(t_0)| = |x(t_0+T) - x(t_0)| \le \int_{t_0}^{t_0+T} |\dot{x}|\,dt = \int_{t_0}^{t_0+T} |J\dot{x}|\,dt$$

$$= \Lambda_+ \int_{t_0}^{t_0+T} |Hx|\,dt \le \Lambda_+ \int_{t_0}^{t_0+T} |H|\,dt\,|x(t_0)| = \Lambda_+ \int_0^T |H|\,dt\,|x(t_0)|.$$

Therefore [2, p. 447],

$$\Lambda_+ \ge 2\left(\int_0^T |H|\,dt\right)^{-1}.$$

Consequently, the equation $J\dot{x} = H(t)x$ of positive type is stable if

$$\int_0^T |H(t)|\,dt < 2. \tag{13}$$

By analogy, one can easily obtain (see [2, Theorem 4]) the Lyapunov criterion (4) and its extension to the vector Hill equation

$$\frac{d^2y}{dt^2} + P(t)y = 0, \tag{14}$$

where $y(t) \in \mathbb{R}^m$, $P(t+T) = P(t) = P(t) \ge 0$, and $P_{av} > 0$. This equation is stable [2, p. 488] if

$$T\int_0^T |P(t)|\,dt \le 4.$$

Note that, as in the scalar case, the constant 4 on the right-hand side of the last inequality cannot be made smaller. (Nevertheless, it was possible to improve this criterion in various directions, and this was done by M. Krein.)

The use of finer methods for finding lower estimates of the number Λ_+ (one of these methods is presented in the next section) leads to more informative criteria for the stability of the equation $J\dot{x} = H(t)x$ [1] (Section 7); these criteria may be regarded as generalizations of the Lyapunov criterion (4) in various directions.

Parallel with some other results, the paper [3] contains a method for estimating the number Λ_+ that differs from the method presented in [1]. It gives the following criteria convenient for numerical evaluation:

The equation $J\dot{x} = H(t)x$ of positive type is stable if

$$T^2 \operatorname{Sp}(JH_{\mathrm{av}})^2 < 4; \tag{15}$$

if, in addition, $H(t) = H(-t)$, then the equation is stable provided that

$$T^2 \operatorname{Sp}(JH_{\mathrm{av}})^2 < 8. \tag{16}$$

In the last case, the estimate is exact, i.e., the number 8 cannot be replaced by a smaller number.

The book by Gokhberg and M. Krein [4] contains a presentation of the theory of transformers of Volterra operators developed by the authors. The problem discussed above was considered as one of the possible applications of this theory. By using the rule governing the motion of multipliers and numerous new facts in the theory of nonself-adjoint operators established by the authors, they arrived at the following criterion:

The equation under consideration is stable if

$$\sum_{i=1}^{n} \frac{\omega_j}{2j-1} < \frac{\pi}{2}, \tag{17}$$

where $\pm i\omega_j$ are imaginary eigenvalues of the matrix JH_{av} (and $\omega_j > 0$).

This criterion is exact: For any fixed n and ω_j, the constant $\pi/2$ in (17) cannot be made smaller.

As shown above, the equation $J\dot{x} = \lambda H(t)x$ is stable for $0 < \lambda < \Lambda_+$. Similarly, one can prove that it is also stable for $\Lambda_- < \lambda < 0$, where $\Lambda_- < 0$ is the largest (in modulus) negative eigenvalue of the boundary-value problem (11), (12) with $\rho_0 = -1$.. Furthermore, for $\Lambda_- < \lambda < \Lambda_+$, $\lambda \neq 0$, we have "strong" stability:

The equation $J\dot{x} = H(t)x$ is called *strongly stable* if it is stable and this property is preserved under small (in the L-norm) perturbations of the Hamiltonian $H(t)$.

Krein proved that this equation is strongly stable if it has no multiple multipliers of different kinds. (This follows from the continuity of the eigenvalues $\rho_j^{\pm}(X)$, $X \in \mathfrak{S}$, with regard to their kinds.) Somewhat later, Gel'fand and Lidskii [5] showed that this is also a necessary condition. All the criteria presented above are criteria of strong stability.

The criteria of stability described above guarantee the estimate $\Lambda_+ > 1$; all the multipliers of the first kind of the corresponding equation lie on the upper semicircle of the unit circle and the multipliers of the second kind lie (symmetrically) on the lower semicircle. However, there exist strongly stable equations with arbitrary distributions of multipliers on the unit circle. By using the concept of stability regions introduced by Gel'fand and Lidskii in [5] (see Section 4), one can prove that the criteria of the indicated type ensuring the estimate $\Lambda_+ > 1$ cut out the Hamiltonians from a single "central" stability region, whereas there exist countably many stability regions. In this sense, all criteria of this type are analogous to the Lyapunov criterion (4) but not to the Zhukovskii criterion (5). Being aware of this fact, M. Krein indicated in [3] (Section 3) the methods for transforming the criteria for the central stability region into the criteria for arbitrary stability regions.

2. Investigation of the Boundary-Value Problem (11), (12)

In connection with the discovered relationship between the problem under consideration and the self-adjoint boundary-value problem (11), (12) with $\rho_0 = -1$, M. Krein undertook serious research into this and more general self-adjoint problems, which can be characterized by the application of diverse analytical tools.

Let us describe, for example, that unexpected set of tools from complex analysis used by Krein to establish (see [1, Theorem 6.2]) the existence of eigenvalues $\Lambda_+ > 0$ and $\Lambda_- < 0$ in the indicated problem. (Without the condition $H_{av} > 0$, these eigenvalues may be absent.)

The eigenvalues λ_k of problem (11), (12) with $\rho_0 = -1$ are the poles of the matrix function $[X(T, \lambda) + I_m]^{-1}$, where $X(T, \lambda)$ is the monodromy matrix of the equation $J\dot{x} = \lambda H(t)x$. Let $\ldots \le \lambda_{-2} \le \lambda_{-1} < 0 < \lambda_1 \le \lambda_2 < \ldots$. By using the Chebotarev theorem on the Mittag-Leffler decomposition of the functions $f(\lambda)$ such that $J_m f(\lambda) > 0$ for $J_m \lambda > 0$, Krein proved [1, Theorem 3.5] that the following absolutely convergent decomposition is true [1, p. 460][1]:

$$2\,[X(T, \lambda) + I_m]^{-1} = I_{2n} + J\left\{-\lambda A_1 + \lambda \sum_{j=0} \frac{B_j}{\lambda_j(\lambda_j \lambda)}\right\}, \tag{18}$$

where A_j and B_j are Hermitian matrices such that $A_1 \ge 0$, $B_j \ge 0$, and

$$\sum |B_j| (\lambda_j^2)^{-1} < \infty.$$

Assume, for example, that there are no numbers $\lambda_j < 0$ (i.e., that $\Lambda_- = -\infty$). Then, as shown above, for any $\lambda < 0$, all $|\rho_j^{\pm}(\lambda)| = 1$. Since

$$\Delta(\lambda) = \det[X(T, \lambda) + I_{2m}] = \Pi[-1 - \rho_j^{\pm}(\lambda)],$$

we have $|\Delta(\lambda)| \le \text{const}$ for $\lambda < 0$.

On the other hand, it follows from (18) with $\lambda_j = \zeta_j^2$ that

$$2\,[X(T, \lambda^2) + I]^{-1} = I_{2m} - JA_1\lambda^2 + \frac{\lambda^2}{2} \sum_{j=0} \frac{JB_j}{\zeta_j}\left(\frac{1}{\zeta_j - \lambda} + \frac{1}{\zeta_j + \lambda}\right). \tag{19}$$

Somewhat earlier, in [6], M. Krein showed that an entire matrix function $W(\lambda)$ admitting a decomposition

$$W(\lambda)^{-1} = \sum_{j=0}^{p} A_j \lambda^j + \lambda^p \sum_{j=1}^{\infty} \frac{c_k}{\lambda - \mu_k}, \quad \text{where} \quad \sum_{j=1}^{\infty} \frac{|c_k|}{|\mu_k|} < \infty, \tag{20}$$

and all μ_k are real, has at most exponential type, i.e., $|W(\lambda)| \le \alpha e^{\beta|\lambda|}$, where $\alpha > 0$, $\beta > 0$, and, moreover,

$$\int_{-\infty}^{+\infty} \frac{\log^+ |F(\lambda)|}{1 + \lambda^2}\, d\lambda < \infty. \tag{21}$$

[1] In [1], Krein also considered a more general boundary-value problem: The matrix $(-I_{2m})$ was replaced by an arbitrary J-unitary matrix ε. For the simplicity, we restrict ourselves to the boundary-value problem (11), (12) and present the decomposition for this case.

By applying this assumption to the function $W(\lambda) = X(T, \lambda^2) + I_{2m}$, by virtue of (19), we obtain

$$|\Delta(\lambda)| \le \alpha_1 e^{\beta_1\sqrt{|\lambda|}}, \quad \alpha_1 > 0, \quad \beta_1 > 0.$$

By the Hadamard theorem,

$$\Delta(\lambda) = 2^{2m}\prod_{j=1}^{\infty}\left(1 - \frac{\lambda}{\lambda_j}\right).$$

Since all $\lambda_j > 0$, we have $\Delta(\lambda) \to +\infty$ as $\lambda \to -\infty$. The contradiction demonstrates that there exists at least one eigenvalue $\lambda_j < 0$.

The same reasoning leads us to the following statement [4, Theorem 2]:

If $H(t) \ge 0$ and $H_{av} > 0$, then the monodromy matrix $X(T, \lambda)$ of the equation $J\dot{x} = \lambda H(t)x$ satisfies the conditions

$$h(\varphi) = \overline{\lim_{r\to\infty}}\frac{\log\left|X\left(T, re^{i\varphi}\right)\right|}{r} < \infty \quad \text{and} \quad \int_{-\infty}^{+\infty}\frac{\log^+|X(T, \lambda)|}{1+\lambda^2}\,d\lambda < \infty.$$

By using these relations, M. Krein proved [4, p. 170] that the boundary-value problem (11), (12) possesses an infinite sequence of eigenvalues $\ldots \le \lambda_{-2} \le \lambda_{-1} < \lambda_0 \le \lambda_1 \le \lambda_2 \le \ldots$ and

$$\lim_{k\to\infty}\frac{\lambda_k}{k} = \lim_{k\to\infty}\frac{|\lambda_{-k}|}{k} = 2\pi\left[h\left(\frac{\pi}{2}\right) + h\left(-\frac{\pi}{2}\right)\right]^{-1}.$$

In [6], he also indicated relationships of the theory of monodromy matrices of equations $J\dot{x} = \lambda H(t)x$ with his theory of entire Hermitian operators (see, e.g., [7, 8]) and the problem of extension of Hermitian positive functions (see, e.g., [9]).

In the paper [10], the expansions of the multipliers $\rho_j(\lambda)$ of Eq. (11) of positive type into the series in fractional powers

$$\rho_j(\lambda) = \rho_0 + \sum_{k=1}^{\infty} c_{kj}(\lambda - \lambda_0)^{k/q} \tag{22}$$

were considered and their unexpectedly simple properties were established. Namely, let κ be a multiplicity of a multiplier ρ_0 and let d be the number of branches of (22), $q_1 + q_2 + \ldots + q_d = \kappa$. Then (see [6, Theorem 1]) the numbers $q_1, \ldots, q_d$ coincide with the orders of the Jordan boxes corresponding to the eigenvalue ρ_0 in the Jordan form of the matrix $X(T, \lambda_0)$ and all $c_{1j} \ne 0$.

Important facts about the boundary-value problem (11), (12) with $\rho_0 = -1$ were discovered in [3]. Namely, it was showed that

$$\lim_{r\to\infty}\sum_{|\lambda_j|<r}\frac{1}{\lambda_j} = 0 \quad \text{and} \quad \sum_j\frac{1}{\lambda_j^2} = \frac{T^2}{4}\,\mathrm{Sp}\,(JH_{av})^2. \tag{23}$$

The condition $H_{av} > 0$ implies that $\mathrm{Sp}\,(JH_{av})^2 > 0$. This means that [3] contains another proof of the existence of the numbers $\lambda_1 = \Lambda_+$ and $\lambda_{-1} = \Lambda_-$, which is much simpler than in [1]. The second relation in (23) (the "trace formula") yields the estimate $\Lambda_+^2 > ((T^2/4)\,\mathrm{Sp}\,(JH_{av})^2)^{-1}$ and criteria (15) and (16).

3. Parametric Resonance

Consider a stable Hamiltonian equation

$$J\frac{dx}{dt} = H_0 x \tag{24}$$

with a constant Hamiltonian (all solutions of (24) are bounded) and a "perturbed" Hamiltonian equation

$$J\frac{dx}{dt} = H(\theta t)x, \tag{25}$$

where $H(t+2\pi) = H(t)$. A frequency θ is called critical for system (24) if, for any $\delta > 0$, there exists a Hamiltonian $H(t)$, $\|H(t)-H_0\| < \delta$, such that Eq. (25) is unstable (there exist unbounded solutions). The appearance of unboundedly growing oscillations caused by arbitrarily small periodic perturbations of the parameters of a system is called parametric resonance. Let $i\omega_k$, $k = 1, \dots, m$, be eigenvalues of the matrix $J^{-1}H_0$ of the first kind, then $(-i\omega_k)$ are its eigenvalues of the second kind. (Their kinds are determined just as the kinds of eigenvalues of a symplectic matrix; the kinds of the numbers $(\pm i\omega_k)$ coincide with the kinds of eigenvalues $\exp(\pm i\omega_k)$ of the matrix $\exp(J^{-1}H_0) \in \mathcal{S}$.) Assume that all $\omega_j + \omega_k \neq 0$. (This is certainly true if $H_0 > 0$ because, in this case, all $\omega_k > 0$.) It follows from M. Krein's results supplemented by Gel'fand and Lidskii [5] that the frequencies

$$\theta = (\omega_j+\omega_k)/N, \quad j, k = 1, \dots, m, \quad N = \pm 1, \pm 2, \dots, \tag{26}$$

and only these frequencies are critical. The frequencies $\theta = 2\omega_k/N$ are called the frequencies of principal resonance and the frequencies $\theta = (\omega_j+\omega_k)/N$ with $j \neq k$ are called the frequencies of combination resonance. In applications, one can most often assume that the Hamiltonian in (25) has the form $H(t) = H_0 + \varepsilon H_1(t) + \varepsilon^2 H_2(t) + \dots$, where ε is a small parameter. The regions in the plane $\{\varepsilon, \theta\}$ that correspond to unbounded solutions are called the regions of dynamic instability. In ordinary cases they have the form of **small wedges** with vertices at the points $(0, (\omega_j+\omega_k)/N)$. Although the phenomenon of parametric resonance is quite important for applications (see, e.g., [11]), the presence of combination resonance was never taken into account in numerous applied works prior to the publication of Krein's results. At the same time, subsequent studies demonstrated that just combination resonance played a key role in many problems under consideration. (The width of the region of combination resonance had the order ε, whereas the width of the region of principal resonance had the order ε^p, $p \geq 2$.) This was connected with the absence of combination resonance in systems with one degree of freedom $(m = 1)$ and with the fact that the boundaries of the regions of dynamic instability are found from the existence conditions for the $2\pi/\theta$-periodic or $2\pi/\theta$-antiperiodic solutions.

Similar engineering calculations were also carried out for systems with $m > 1$ degrees of freedom but, in view of the fact that the indicated property holds only for the boundaries of the regions of principal resonance, the regions of combination resonance were missing. M. Krein's results stimulated the development of the methods for finding the boundaries of the regions of dynamic instability in the case of combination resonance. It was also discovered that the principal (and important for applications) numerical characteristics of the phenomenon of parametric resonance can be found from simple relations that enable one to judge the emerging phenomena by using just the coefficients of system (25) (see, e.g., [12–14]). New relations for critical frequencies were obtained for narrower classes of perturbations [14].

In [3], M. Krein studied the phenomenon of high-frequency stabilization. This phenomenon is, in a certain sense, opposite to parametric resonance. The simplest example of high-frequency stabilization is well known: The unstable upper position of a pendulum becomes stable if its suspension point (a hinge) vibrates with a sufficiently high frequency. It turns out that almost all mechanical systems admit high-frequency stabilization of this sort. Namely, let $P(t)$ and $R(t)$ be real symmetric 2π-periodic matrix functions such that $P(t) \geq 0$ and $P_{av} > 0$ and let

$$\frac{d^2y}{dt^2} + [R(\theta t) + \varepsilon\theta^2 P(\theta t)]y = 0 \tag{27}$$

be the Lagrange equation for the system under consideration. (This system can be unstable for $\varepsilon = 0$.) Then [3, p. 679], there exist $\varepsilon_0 > 0$ and a function $\Theta(\varepsilon)$ defined on $(0, \varepsilon_0)$ such that Eq. (27) is stable for $0 < \varepsilon < \varepsilon_0$ and $\theta > \Theta(\varepsilon)$.

4. Other Papers of M. Krein in the Theory of Hamiltonian Equations

Almost all results described above were extended by M. Krein to the case of infinite-dimensional (Banach or Hilbert) spaces of states (see, e.g., [4], Chap. 5, [15], Chap. 6, and [16]). The theory of linear Hamiltonian equations is tightly bound with the theory of operators in spaces with indefinite metrics (see, e.g., [15–17]). It is related to the inverse Sturm–Liouville problem [18-20], to the theory of waveguides [21], to quadratic operator equations, and to the damped oscillations of infinite-dimensional systems [22].

It seems probable that these investigations of M. Krein were provoked by the papers of Lyapunov devoted to linear periodic systems. All Lyapunov's important results in this field found new proofs based on absolutely new ideas in Krein's works. Indeed, Lyapunov's proofs were mainly based on difficult skillful calculations while Krein applied modern methods in the theory of operators, theory of integral equations, and theory of functions [1–3, 6, 23–29]. Moreover, in many cases, Lyapunov's results were made more accurate and the proofs became much shorter and clearer. As far as the Lyapunov stability criteria for the Hill equation are concerned, note that some of them were considerably strengthened [23] and all of them were extended by M. Krein to the multi-dimensional and infinite-dimensional cases [24]. (In many cases, there exist several generalizations of this sort.)

Paper [23] deals with the problems of maximum and minimum for the characteristic numbers—the boundaries of the stability regions for the Hill equation. In other words, this means that [23] contains various stability criteria for the Hill equation similar to the Zhukovskii criterion (5) but much stronger. In modern interpretation, these are special problems of optimum control with uniform restrictions imposed on a control. They are now solved by applying the Pontryagin's "maximum principle." However, it is worth noting that this principle appeared five years later than the paper [23]. At the same time, even the maximum principle does not make solution of such problems much simpler (see, e.g., the solution of a similar problem due to Kazarinov [30, Appendix]).

5. Influence of the Ideas and Results of M. Krein on Further Development in the Theory of Linear Hamiltonian Systems

The concept of the kinds of multipliers introduced by Krein was extensively used by Gel'fand and Lidskii [5] in studying the structure of a set of Hamiltonians of stable systems. Let us clarify the basic idea of [5] and subsequent works in this field. Let $L = \{H(\cdot)\}$ be a Banach space of Hamiltonians (3) with the L_1-norm and let O be a set of Hamiltonians of strongly stable equations. Also let $X \in \mathcal{S}$ be a symplectic matrix such that all its eigenvalues lie on the circle $|\rho| = 1$ and no eigenvalues of different kinds coincide. (According to the Krein–Gel'fand–Lidskii theorem, the monodromy matrices and the equations with $H(\cdot) \in \mathrm{O}$ possess this property.) Let us move along the upper semicircle of the unit circle from the point $(+1)$ to the point (-1) writing a sequence of pluses and minuses corresponding to encountered eigenvalues of the first and second kinds, respectively. (For an eigenvalue with multiplicity r, we write either r pluses or r minuses.) A sequence $\mu = (+, +, -, \ldots, -)$ of m pluses and m minuses obtained as a result is called a spectral type of the matrix X. Obviously, the total number of different spectral types of $H(\cdot) \in \mathrm{O}$ is 2^m. Let $\mathrm{O}^{(\mu)}$ be a set of all Hamiltonians $H(\cdot) \in \mathrm{O}$ with the monodromy matrix of spectral type

μ. Clearly, any two Hamiltonians from O with different spectral types cannot be connected by a continuous curve in O because a perturbation of this sort leads to the situation where multipliers of different kinds coincide. However, it turns out that each set $O^{(\mu)}$ also splits into disjoint subsets. Let us clarify this situation and point out the difference in the properties of the Hamiltonians from different regions.

A Hamiltonian $H(\cdot) \in L$ is associated in a unique manner with an evolutionary operator $X(t)$, $0 \le t \le T$, i.e., with a curve in the group $\mathfrak{S}$ with the origin at $X(0) = I_{2m}$. It is known [5] that the group $\mathfrak{S}$ is homeomorphic to the topological product of a connected and simply connected topological space $\mathfrak{S}_0$ by the circle $\mathbb{T}$. The homeomorphism $\mathfrak{S} \sim \mathfrak{S}_0 \times \mathbb{T}$ induces the correspondence $X \to e^{i\varphi}$ (where $X \in \mathfrak{S}$ and $e^{i\varphi} \in \mathbb{T}$) and the many-valued function $\varphi = \operatorname{Arg} X$. For the continuous curve $X(\cdot)|_0^T$ in $\mathfrak{S}$, the increment $\Delta \operatorname{Arg} X(\cdot)|_0^T$ is determined uniquely and if $X(\cdot)|_0^T$ is a closed curve, then $\Delta \operatorname{Arg} X(\cdot)|_0^T = 2\pi n$, where n is an integer called the index of this curve ($n = \operatorname{ind} X(\cdot)$). The curves with index zero and only these curves contract into a point in $\mathfrak{S}$.

Different homeomorphisms $\mathfrak{S} \sim \mathfrak{S}_0 \times \mathbb{T}$ are associated with different arguments $\operatorname{Arg} X$. It is convenient to have explicit formulas for arguments in $\mathfrak{S}$. It was shown [31] that the following functions can be treated as arguments in $\mathfrak{S}$:

$$\operatorname{Arg}_* X = \sum_{X=1}^{m} \operatorname{Arg} \rho_k^+(X), \tag{28}$$

where $\rho_k^+(X)$ is an eigenvalue of the first kind of the matrix $X \in \mathfrak{S}$ and

$$\operatorname{Arg}_j X = \operatorname{Arg} \det\,[U_j - i V_j], \quad j = 1, 2, \quad \text{where} \quad X = \begin{bmatrix} U_1 & U_2 \\ V_1 & V_2 \end{bmatrix}. \tag{29}$$

Here, U_j and V_j are matrices of the order $m \times m$. We can also present other explicit formulas for the arguments in $\mathfrak{S}$ (see [31]). Arguments (28), (29), and those presented in [31] are equivalent in the sense that the increments of the arguments coincide for closed curves and differ at most by a fixed constant for any nonclosed curves.

Let $\tilde{\mathfrak{m}} \subset \mathfrak{S}$ be a connected and simply connected set in $\mathfrak{S}$. (The last statement means that any closed curve in $\tilde{\mathfrak{m}}$ has index zero.) Let $\mathfrak{m} \subset L$ be a set of all Hamiltonians with monodromy matrices from $\tilde{\mathfrak{m}}$. The set $\mathfrak{m}$ can be identified with the set of corresponding evolutionary operators, i.e., with the curves $X(\cdot)|_0^T$ in $\mathfrak{S}$ such that $X(0) = I_{2m}$ and $X(T) \in \tilde{\mathfrak{m}}$. We say that curves $X_1(\cdot)$ and $X_2(\cdot)$ (where $X_1(0) = X_2(0) = I_{2m}$, $X_1(T) \in \tilde{\mathfrak{m}}$, and $X_2(T) \in \tilde{\mathfrak{m}}$) are homotopic modulo $\tilde{\mathfrak{m}}$ (and write $X_1(\cdot) \sim X_2(\cdot) \bmod \tilde{\mathfrak{m}}$) if one of the curves admits a continuous transformation into the other such that its end remains in $\tilde{\mathfrak{m}}$. This condition is obviously equivalent to the existence of a continuous curve that lies in $\mathfrak{m} \subset L$ and joins the corresponding Hamiltonians $H_1(\cdot) \in \mathfrak{m}$ and $H_2(\cdot) \in \mathfrak{m}$. The curves $X_1(\cdot)$ and $X_2(\cdot)$ that are not homotopic modulo $\tilde{\mathfrak{m}}$ correspond to Hamiltonians from different connected components of the set $\mathfrak{m}$ (Fig. 2). Thus, $\mathfrak{m} \subset L$ splits into a sum of disjoint connected subsets $\mathfrak{m}_n$, $n = 0, \pm 1, \ldots$, corresponding to the curves homotopic modulo $\tilde{\mathfrak{m}}$. Let $X_* \in \tilde{\mathfrak{m}}$ and let $X_*(\cdot)$ be an arbitrary fixed curve with the origin at X_* and the end at I_{2m}. We associate the curve $X(\cdot)$ with the endpoint $X(T) \in \tilde{\mathfrak{m}}$ with a "complemented curve" $\hat{X}(\cdot)$ composed consecutively of the curve $X(\cdot)$, a curve $Y(\cdot)$ joining (arbitrarily) the matrix $X(T)$ with X_* in $\tilde{\mathfrak{m}}$, and the curve $X_*(\cdot)$. (For $X_1(\cdot)$ and $X_2(\cdot)$, the complemented curves are depicted in Fig. 2.) Since $\hat{X}(\cdot)$ is a closed curve, we have

$$\Delta \operatorname{Arg} \hat{X}(\cdot) = 2n\pi. \tag{30}$$

Since $\tilde{\mathfrak{m}}$ is simply connected in $\mathfrak{S}$, the integer number n does not depend on the choice of the curve $Y(\cdot)$; it is completely determined by the curve $X(\cdot)$ (and the fixed curve $X_*(\cdot)$). It is evident that the set $\mathfrak{m}_n$ can be defined as the set of all Hamiltonians whose evolutionary operators satisfy the conditions: $X(T) \in \tilde{\mathfrak{m}}$ and (30).

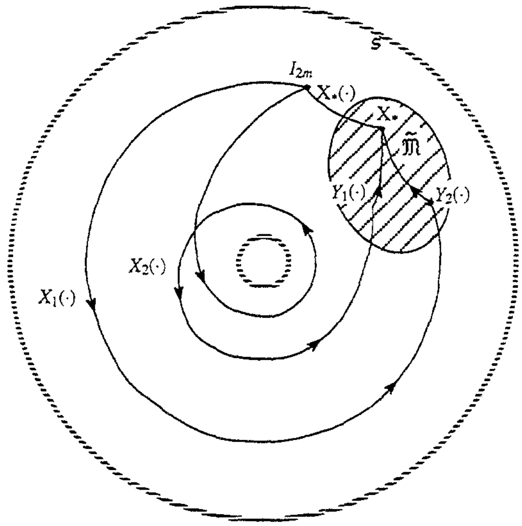

Fig. 2

The arguments presented above remain true in the case where the matrix X_* is taken on the boundary of the set $\tilde{\mathfrak{m}}$ provided that $\tilde{\mathfrak{m}} \cup X_*$ is a simply connected set in $\mathfrak{S}$.

Now assume that $\tilde{\mathfrak{m}} = \tilde{O}^{(\mu)}$, i.e., that $\tilde{\mathfrak{m}}$ is the set of all symplectic matrices of the above-mentioned form with the spectral type μ. It is easy to show that $\tilde{O}^{(\mu)}$ is a connected set. Clearly, for a closed curve $Y(T) \in \tilde{O}^{(\mu)}$, we have $\Delta \operatorname{Arg}_* Y(\cdot) = 0$, i.e., $\operatorname{ind} Y(\cdot) = 0$. Indeed, the eigenvalues of the first kind $\rho_j^+(Y(t))$ situated on the upper (lower) semicircle $|\rho| = 1$ can move only along this semicircle, and the initial position differs from the final one only by transposition. (The points $(+1)$ and (-1) are forbidden because multipliers of different kinds coincide there.) Hence, $\tilde{O}^{(\mu)}$ is simply connected in $\mathfrak{S}$. The same is true for $\tilde{O}^{(\mu)} \cup I_{2n}$ and, therefore, X_* can be chosen as follows: $X_* = I_{2m}$. Then $X_*(t) \equiv I_{2m}$. By virtue of the reasoning presented above, the corresponding set of Hamiltonians splits into the regions $O_n^{(\mu)}$ and condition (30) holds for $H(\cdot) \in O_n^{(\mu)}$. Let

$$\theta_k = \arg \rho_k^+ [X(T)] \quad (-\pi < \theta_k < +\pi) \tag{31}$$

be the arguments of multipliers of the first kind of equation with $H(\cdot) \in O_n^{(\mu)}$ and let $Y(\cdot)$ be a curve joining the matrix $X(T)$ with I_{2m} in $\tilde{O}^{(\mu)}$. Then

$$\Delta \operatorname{Arg}_* Y(\cdot) = -\sum_k \theta_k.$$

Therefore, for $\operatorname{Arg} = \operatorname{Arg}_*$, relation (30) takes the form

$$\sum_{k=1}^{m} \Delta \operatorname{Arg} \rho_j^+ [X(t)] \big|_0^T - \sum_{k=1}^{m} \theta_k = 2\pi n. \tag{32}$$

To make our discussion more precise, we present the following final assertion [5]:

The set O *of the Hamiltonians* $H(\cdot)$ *of strongly stable equations* $J\dot{x} = H(t)x$ *consists of countably many disjoint regions* $\mathrm{O}_n^{(\mu)}$ $(\mu = \mu_1, \dots, \mu_{2^k},\ n = 0, \pm 1, \pm 2, \dots)$ *each of which is the set of all Hamiltonians with spectral type* μ *satisfying condition (32).*

The use of different arguments may lead to different formulas for the index n. (In particular, a somewhat different formula is given in [5].) It can be easily shown that the regions $\mathrm{O}^{(\mu)}$ with the same μ are affine homeomorphic.

It has relatively recently become known that some linear-quadratic problems of optimum control and problems of absolute stability of nonlinear systems also lead to the investigation of the Hamiltonian equations $J\dot{x} = H(t)x$. In this connection, it became necessary to study the structure of the set $\mathcal{H}$ of Hamiltonians of completely unstable equations. (Complete instability means that there are no multipliers on the unit circle.) Arguments similar to that presented above enable us to conclude [33] that $\mathcal{H}$ consists of countably many affine homeomorphic regions $\mathcal{H}_n$, $n = 0, \pm 1, \dots$, and, moreover, $H(\cdot) \in \mathcal{H}_n$ if, in addition to complete instability, the relation

$$\sum_{j=1}^{m} \Delta \operatorname{Arg}_* \rho_j^+ [\mathrm{X}(t)] \Big|_0^T = n\pi \tag{33}$$

holds. By using the argument defined by (29) (e.g., for $j = 1$), we arrive at the equivalent condition [33]

$$\Delta \operatorname{Arg} \det [\, U(t) - i\, V(t)] \,\Big|_0^T = n\pi, \tag{34}$$

where the columns of the matrix $\begin{bmatrix} U(t) \\ V(t) \end{bmatrix}$ form a basis in a stable subspace $\Lambda(t)$ of the equation with $H(\cdot) \in \mathcal{H}$. (It is evident that $\Lambda(t)$ is a Lagrange subspace.) Since $U(t+T) = U(t)K$ and $V(t+T) = V(t)K$, we conclude that $\Lambda_+(t)$ is a closed curve in the Lagrangian Grassmannian; relation (34) indicates that n is its Maslov index.

Similarly, the Gel'fand–Lidskii index of the stability region $\mathrm{O}_n^{(\mu)}$ in (32) (and in relation (6.6) in [5]) can be treated as the Maslov index of a certain curve in the Lagrangian Grassmannian. {As is known, the Maslov indices appeared in an absolutely different situation (see, e.g., [34]).}

The description of the structure of the set O given in [5] clarified and ordered the activities aimed at establishing the stability criteria. It is easy to show that the criterion ensuring the estimate $\Lambda_+ > 1$ corresponds to the Hamiltonians $H(\cdot) \in \mathrm{O}_n^{(\mu)}$, where $\mu = (+, +, \dots, +)$, $n = 0$. (This makes the presentation more precise.) Various transformations (see [3] and [30, Chap. 3, Sec. 6]) made it possible to extend these criteria to the regions with arbitrary μ and n. Also note that special methods were developed to transform any criterion valid for the systems with $m = 1$ into criteria of various types for the systems with $m > 1$ in any region $\mathrm{O}_n^{(\mu)}$.

Let us describe one of these methods. A set $\mathfrak{m} \subset L$ is called *directedly broad* if, for any $H_1(t)$, $H_2(t)$, and $H(t)$ from L such that $H_1(t) \le H(t) \le H_2(t)$, the condition

$$H(t, s) = sH_1(t) + (1 - s)H_2(t) \in \mathfrak{m} \qquad (\forall s \in [0, 1])$$

implies that $H(t) \in \mathfrak{m}$.

Theorem [30, Chap. 3, Sec. 6.2]. *All regions* $\mathrm{O}_n^{(\mu)}$ *are directedly broad.*

Let $H_1(t)$ and $H_2(t)$ be Hamiltonians of equations reducible to equations with $m = 1$. In this case, we can extend the criteria established for the case $m = 1$ to the case of $m > 1$. As a result, we get criteria for an arbitrary region $\mathrm{O}_n^{(\mu)}$. Constant Hamiltonians can also be taken to play the role $H_1(t)$ and $H_2(t)$.

Consider another application of this theorem, namely, the problem of orbital stability of periodic Laplacian motion mentioned in the introduction. Lyapunov's result [32] formulated there[2] (its proof occupies 65 pages in [32]) contains the words "... if the eccentricity is sufficiently small." The application of the theory of directed convexity for constant Hamiltonians $H_1(t)$ and $H_2(t)$ enables one to get, after simple calculation, an explicit estimate for ε and μ [33]: Stability takes place if one of the following conditions is satisfied:

$$4\varepsilon/3 + 4\varepsilon^2/3 < \mu < 1/12 - \varepsilon/2 + 5\varepsilon^2/12 \quad \text{or} \quad 1/12 + \varepsilon/2 + 5\varepsilon^2/12 < \mu < 1/9 - 8\varepsilon/9 + 16\varepsilon^2/9.$$

(By using more complicated analysis [35], one can slightly extend these regions; instability takes place for $\mu > 1/9$ and sufficiently small ε.)

It is worth noting that Lyapunov's result [32] formulated in the introduction [see relation (9)] follows almost without calculations from the Krein formula for the critical frequencies $\theta = (\omega_1 + \omega_k)/N$. This is one of the numerous fascinating examples of the situation where new theories immediately yield the results that were obtained earlier by using complicated skillful computations. Another similar example: The important inequality $(n-1)A_n^2 > nA_{n-1}A_{n-1}$ for the coefficients of expansion of the characteristic function $A(\lambda) = 1 - A_1\lambda + A_2\lambda^2 - \ldots$ of the Hill equation $\ddot{y} + \lambda p(t)y = 0$, $p(t) \geq 0$, was established by Lyapunov in [36] as result of direct and extremely cumbersome computations [36, pp. 421–435]. This is the basic inequality of Lyapunov's method [36] for studying the Hill equation $\ddot{y} + p(t)y = 0$ with $p(t) \geq 0$. M. Krein [23] gave a surprisingly simple proof of this inequality based on the Hadamard theorem and the above-mentioned relationship with a self-adjoint boundary-value problem. Krein's method makes it possible to extend Lyapunov's method to the case of the Hill equation with alternating coefficients $p(t)$ [30, Chap. 7, Sec. 2] and to some other cases.

In [36], Lyapunov proved (also by direct complicated calculations) that the estimate $A_n \leq (1/(2n)!)(T^2 P_{\mathrm{av}})$ holds for $n = 2, 3$ and $p(t) \geq 0$. He made a conjecture [35, p. 419] that this estimate is true and exact for all n and wrote "but we failed to prove this." M. Krein [26] found an unexpectedly simple proof of this hypothesis of Lyapunov and showed that it is a consequence of his general theorem on kernel operators in Hilbert spaces.

Let us return to the Lyapunov (7) and Zhukovskii (8) criteria. The Zhukovskii criterion selects the Hamiltonians from all stability regions [that may contain Hamiltonian (6)]. The Lyapunov criterion selects them only in the region $O_0^{(+)}$. Various tests for stability, which strengthen the Lyapunov criterion (7) and extend it to other stability regions, can be found in [30, Chap. 8, Sec. 4]. Here, we mention the criterion of Neigauz and Lidskii [37]

$$p(t) \geq \frac{k^2\pi^2}{T^2}, \qquad \int_0^T \left|\sin\frac{k\pi t}{T}\right| \left[p(t-\tau) - \frac{k^2\pi^2}{T^2}\right] dt < \frac{2k\pi}{T} \tag{35}$$

(k is an integer number) and the criteria of Borg [38] and Komlenko [39].

By using the theorem on directed convexity of stability regions, these and other criteria for the scalar Hill equation can be generalized for the vector equations $\ddot{y} + P(t)y = 0$, $y(t) \in \mathbb{R}^m$, $P(t+T) = P(t) = P(t)^*$.

The structure of stability regions of the complex Hamiltonian equations $\tilde{J}\dot{x} = \tilde{H}(t)x$, where $\tilde{H}(t+T) = \tilde{H}(t) = \tilde{H}(t)^*$ and $\tilde{J}^* = -\tilde{J}$ are complex matrices with $\det \tilde{J} \neq 0$, was studied in [40, 41]; note that both papers contain insignificant inaccuracies.

For more general equations $Q(t)dx/dt = (S(t) - (1/2)\dot{Q}(t))x$ with $Q(t+T) = Q(t) = -Q(t)^*$, $\det Q(t) \neq 0$, and $S(t+T) = S(t) = S(t)^*$, the structure of stability sets was studied by Lidskii and Frolov [42]. They obtained an unexpected result that, both in complex and real cases, there are only finitely many stability regions and established their characteristics.

Many of Krein's results considered above were later extended to infinite-dimensional Hamiltonian equations [4, 15, 16, 43, 44]. The first results in this field are due to Derguzov; these results, as Krein mentioned in [16, p. 47], stimulated him to further research into the theory of infinite-dimensional systems.

[2] For simplicity, we restrict ourselves to the case of Newton's gravity. Similar assertions are true in the general case.

There exist various interrelations between the results in this field and other branches of mathematics, namely, with variational calculus, operator theory, symplectic geometry, the theory of absolute stability, and the theory of adaptive systems. Some of them are described in [45].

Lastly, note the following: The readers of the works of M. Krein will always be impressed by his incredibly wide range of analytical tools. It may even seem that Krein sometimes chose one or another weapon at random, using the first idea that came into his mind. He was often surprised at the necessity of using such strong analytical tools [1, p. 462] and, in his subsequent papers, returned to the corresponding problems, applied completely different arguments and technique, and found much simpler proofs. Sometimes, the reader may even discover simpler proofs of intermediate results. But just the diversity and unexpectedness of the approaches and tools used by Krein made it possible to reveal deep relations between absolutely different (at first sight) branches of mathematics and this is why M. Krein's mathematical heritage is so attractive.

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