JOURNAL OF DIFFERENTIAL EQUATIONS 39, 186–211 (1981)

Control Systems Subordinated to a Group Action: Accessibility

V. JURDJEVIC AND I. KUPKA

Department of Mathematics, University of Toronto, Toronto, Ontario, Canada

Received January 24, 1979; revised April 16, 1980

INTRODUCTION

The main objective in this paper is to give transitivity (controllability) criteria for a certain class of polysystems which evolve on Lie groups. Since this paper will focus only on linear groups, only the superficial knowledge of Lie groups is required. The generalizations of these results to the case of general Lie groups will appear in [2].

The first part of the paper deals with the theoretical basis for transitivity. Following the recent point of view we consider generalizations of control systems which we call polysystems. A polysystem on a manifold M is simply a collection $\mathscr{F}$ of vector fields on M. A trajectory of $\mathscr{F}$ is a continuous curve x from an interval $[0, T]$, $T \geqslant 0$ of the real line into M such that for some partition of $[0, T]$ the restriction of x to any interval of the partition is an integral curve of some element from $\mathscr{F}$. We will call $\mathscr{F}$ transitive if for each pair (p, q) of points in M there is a trajectory x of $\mathscr{F}$ such that $x(0) = p$ and $x(T) = q$. The usual terminology in the literature on control theory is to call such an $\mathscr{F}$ controllable. But since the accessibility set of a point q in M is the orbit of q of a certain semi-pseudo group, and in our cases, semigroup of diffeomorphisms associated to $\mathscr{F}$, it seems that such a choice of nomenclature is more compatible with the general mathematical terminology.

In general it is difficult to find simple conditions for a polysystem to be transitive since it seems that this question depends on too many parameters. Nevertheless, in the first part of the paper we present a general method of finding sufficient conditions for the transitivity of polysystems. We outline certain simple enlarging operations on polysystems which do not essentially alter its accessibility sets. By applying a succession of these operations to a given polysystem, we can, in certain cases, enlarge the initial polysystem so much that it is easy to see that the final polysystem is transitive.

0022-0396/81/020186-26\$02.00/0

It turns out that this method is successful when applied to polysystems which satisfy certain finiteness conditions and hence are rigid. The prototype of these polysystems are the so called "bilinear systems": the manifold M is the complement V_0 of the origin in some finite dimensional real vector space V. The polysystem $\mathscr{F}$ is of the form $\{A + \mu\mathbf{B}: \mu \in \mathbb{R}\}$, where $\mathbf{A}$ and $\mathbf{B}$ are the vector fields on V_0 induced by the endomorphisms A, B of V such that $\mathbf{A}(p) = Ap$ and $\mathbf{B}(p) = Bp$ for $p \in V_0$.

As is customary in the theory of linear differential equations, we can associate to such polysystem $\mathscr{F}$, a "matrix" polysystem $\mathscr{F}'$ on the group $GL_+(V)$ of all automorphisms on V with positive determinant (to make it connected) as follows: $\mathscr{F}' = \{\tilde{A}_r + u\tilde{B}_r : u \in \mathbb{R}\}$, where $\tilde{A}_r$ and $\tilde{B}_r$ are the right invariant vector fields on $GL(V)$ associated to A and B: If $X \in GL(V)$, $\tilde{A}_r(X) = A \circ X$, $\tilde{B}_r(X) = B \circ X$, where $\circ$ denotes the composition of linear mappings. Such systems are particular cases of the following situation: A connected real Lie group G acts smoothly on M. Then any element X of $L(G)$ the Lie algebra of G induces canonically a vector field $\tilde{X}$ on M. Hence any subset Γ of $L(G)$ defines a polysystem $\mathscr{F}_\Gamma = \{\tilde{X}: X \in \Gamma\}$ on M. We call such polysystems subordinated to a group action. In particular, if A, B are elements in $L(G)$, and if $\Gamma = \{A + uB: u \in \mathbb{R}\}$, then $\mathscr{F}_\Gamma$ is the generalization of the "bilinear systems." The extension of our results to general Lie groups will appear in [2].

The second part of our paper contains the main transitivity results. As we mentioned earlier, we study the right invariant polysystems $\mathscr{F}$ on $GL(V)$ on the form $\mathscr{F} = \{\tilde{A}_r + u\tilde{B}_r : u \in R\}$ associated to $\Gamma = \{A + uB: u \in \mathbb{R}$, $A \in \mathrm{End}(V)$, $B \in \mathrm{End}(V)\}$. Theorem 1 deals with the case where the spectrum of B is real. This Theorem is a substantial generalization of the well known case where $\{e^{At}: t \in \mathbb{R}\}$ is relatively compact [3]. Contrary to such a case, our transitivity condition is an open condition.

When the spectrum of B is purely complex, the situation is very different. Theorem 2 deals with that situation. It turns out that the rotations of B improve the transitivity conditions. In that situation, the set of all $A \in \mathrm{End}(V)$ such that $\{\tilde{A}_r + \mu\tilde{B}_r : u \in \mathbb{R}\}$ is open and dense in $\mathrm{End}(V)$.

Finally, Theorem 3 deals with the general case where the spectrum of B has both real and comples eigenvalues.

I. Notation and the Basic Concepts

(A) *Notation*

Throughout this paper M will denote an n-dimensional real C^∞ manifold countable at ∞. TM will denote the tangent bundle, and T_qM will denote the tangent space of M at q. $\mathrm{Diff}^\infty(M)$ will be used to denote the group of all C^∞ diffeomorphisms on M.

$F^{\infty}(M)$ will denote the set of all C^{∞} vector fields endowed with the C^{∞} topology of uniform convergence on compact subsets of M; $F^{\infty}(M)$ is a separable Fréchet space. If X and Y are elements of $F^{\infty}(M)$, we will write $[X, Y]$ for their Lie bracket. As is well known with this Lie bracket $F^{\infty}(M)$ becomes a topological infinite-dimensional Lie algebra.

If $\mathscr{F} \subset F^{\infty}(M)$ we will denote by $\mathscr{L}(\mathscr{F})$ the Lie subalgebra of $F^{\infty}(M)$ generated by the elements of $\mathscr{F}$. For any point $q \in M$, and any $\mathscr{F} \subset F^{\infty}(M)$ we will denote by $\mathscr{F}_q$ the set $\{X(q): X \in \mathscr{F}\}$. $\mathscr{F}_q \subset T_qM$, and $\mathscr{L}_q(\mathscr{F})$ is a linear subspace of T_qM for each $q \in M$.

In view of the nature of the main results it will be convenient to reduce the generalities to vector fields on $F^{\infty}(M)$ which are complete. We will denote the set of such vector fields by $CF^{\infty}(M)$.

If $X \in CF^{\infty}(M)$, we shall write Φ_X for the C^{∞} mapping on $M \times \mathbb{R}$ into M which satisfies

(i) $\Phi_X(q, s+t) = \Phi_X(\Phi_X(q, s), t)$ for all $q \in M$, and s, t in $\mathbb{R}$.

(ii) $\partial\phi_X/\partial t = X \circ \Phi_X$.

We shall denote by e^{tX} the mapping $t \to \phi_X \mid M \times \{t\}$. As t varies over $\mathbb{R}$, the mappings e^{tX} form a one-parameter group of diffeomorphisms on M.

(B) *Polysystems: The Accessibility Mapping*

$\mathscr{F}$ will denote a subset of $CF^{\infty}(M)$. We shall refer to $\mathscr{F}$ as a polysystem. The set of polysystems on M we will denote by $\mathrm{Pol}(M)$. $S(\mathscr{F})$ will denote the semigroup of $\mathrm{Diff}^{\infty}(M)$ generated by $\bigcup_{X \in \mathscr{F}} \{e^{tX}: t \geqslant 0\}$, and $S(\mathscr{F})(q)$ will denote the orbit of q under $S(\mathscr{F})$. That is, $S(\mathscr{F})(q) = \{\Phi(q): \Phi \in S(\mathscr{F})\}$. For any $q \in M$, *the accessibility set of q with respect to* $\mathscr{F}$ is the closure of $S(\mathscr{F})(q)$ in M. We denote by $A_{\mathscr{F}}(q)$ such a set. The mapping $\mathscr{A}_{\mathscr{F}}: M \to \mathscr{C}(M)$, the class of all closed subsets of M will be called the accessibility mapping of $\mathscr{F}$.

Two polysystems $\mathscr{F}$ and $\mathscr{F}'$ will be called *equivalent* if $\mathscr{A}_{\mathscr{F}} = \mathscr{A}_{\mathscr{F}'}$. It follows directly that the relation just defined is an equivalence relation; its equivalence classes are the fibres of $\mathscr{A}$ (Here, $\mathscr{A}: \mathrm{Pol}(M) \to \mathrm{Map}(M, \mathscr{C}(M))$ is given by $\mathscr{F} \to \mathscr{A}_{\mathscr{F}}$.).

Associated with a polysystem $\mathscr{F}$ we shall call the polysystem $\bigcup_{\mathscr{F} \sim \mathscr{F}'} \mathscr{F}'$ *the saturate of* $\mathscr{F}$. We shall denote by $\mathrm{Sat}(\mathscr{F})$ such a polysystem. It follows that $\mathrm{Sat}(\mathscr{F})$ is the largest (in the sense of inclusion) polysystem on M which is equivalent to $\mathscr{F}$.

(C) *Invariance of the Accessibility Mapping*

In this section we describe certain types of enlargements of a polysystem which do not change the accessibility mapping.

(C1) *Enlargement to cones.* If $\mathscr{F} \subset \mathrm{Pol}(M)$ we denote by $\mathbb{R}_+$ the set

$\{\lambda X: \lambda \geqslant 0, X \in \mathscr{F}\}$. Since the trajectories of X and those of λX, for $\lambda > 0$, are the same, it follows that $\mathbb{R}_+\mathscr{F} \subset \mathrm{Pol}(M)$, and that $\mathbb{R}_+\mathscr{F} \sim \mathscr{F}$.

(C2) *Enlargement under the normalizer.* If $\mathscr{F} \subset \mathrm{Pol}(M)$, *the normalizer of* $\mathscr{F}$ is the set of all $\Phi \in \mathrm{Diff}^\infty(M)$ such that for any $q \in M$ the orbit of q under $\{\Phi^n: n \in \mathbb{Z}\}$ belongs to $\mathscr{A}_{\mathscr{F}}(q)$. We shall use $N^\infty(\mathscr{F})$ to denote the normalizer of $\mathscr{F}$. It is obvious that $\Phi \in N^\infty(\mathscr{F})$ if and only if both $\Phi(q)$ and $\Phi^{-1}(q)$ belong to $\mathscr{A}_{\mathscr{F}}(q)$ for all $q \in M$.

If $\Phi \in \mathrm{Diff}^\infty(M)$, and if $X \in CF^\infty(M)$ we shall write $\Phi_*(X)$ for the vector field $q \to d\Phi(X(\Phi^{-1}(q)))$, where $d\Phi$ is the differential of Φ at $\Phi^{-1}(q)$. If $\Phi \in N^\infty(\mathscr{F})$, we shall write $\Phi_*(\mathscr{F})$ for the set $\{\Phi_*(X): X \in \mathscr{F}\}$.

Since $e^{t\Phi_*(X)} = \Phi \circ e^{tX} \circ \Phi^{-1}$ it follows that $\Phi_*(X) \in CF^\infty(M)$ for each $X \in CF^\infty(M)$. Thus, $\Phi_*(\mathscr{F}) \subset \mathrm{Pol}(M)$. Moreover, we have that $\bigcup_{\Phi \in N^\infty(\mathscr{F})} \Phi_*(\mathscr{F}) \sim \mathscr{F}$.

(C3) *Enlargement under the closure.* If $\mathscr{F} \in \mathrm{Pol}(M)$, and if $\mathrm{cl}(\mathscr{F})$ is the topological closure of $\mathscr{F}$, then it follows from elementary considerations that compact portions of the trajectories of elements of $\mathrm{cl}(\mathscr{F})$ are in $\mathscr{A}_{\mathscr{F}}$. However, the closure of complete vector fields need not be complete. Hence, we have that $\mathrm{cl}(\mathscr{F}) \cap CF^\infty(M) \sim \mathscr{F}$.

(C4) *Enlargement under convex combinations.* If $\mathscr{F} \subset \mathrm{Pol}(M)$, let $\mathrm{Co}(\mathscr{F})$ be the convex envelope generated by the elements of $\mathscr{F}$. Since the vector sums of complete vector fields need not be complete, $\mathrm{Co}(\mathscr{F}) \subset F^\infty(M)$, but it need not be a subset of $CF^\infty(M)$.

It is reasonalby well known, however, that the positive integral trajectories of convex combinations of elements of $\mathscr{F}$ remain in $A_{\mathscr{F}}(q)$. (For instance, this can be found in the work of J. Warga on "chattering controls," or in [1, Sect. 20].) Yet another proof, more directly related to this setting, will appear in [2]. Since in this paper we only consider polysystems in $CF^\infty(M)$, the above remarks imply that $\mathrm{Co}(\mathscr{F}) \cap CF^\infty(M) \in \mathrm{Pol}(M)$ and that $\mathrm{Co}(\mathscr{F}) \cap CF^\infty(M)$ is equivalent to $\mathscr{F}$.

(D) *Transitivity of Polysystems*

A polysystem $\mathscr{F}$ is termed *transitive* if $\mathscr{A}_{\mathscr{F}}(q) = M$ for each $q \in M$. As we mentioned in the Introduction, the main objective of this paper is to find conditions on $\mathscr{F}$ which will ensure that $\mathscr{F}$ is transitive. Before discussing polysystems which are generated by a group action, we recall some well known general facts about the accessibility sets. Also, we describe the general procedure which will be used to demonstrate transitivity.

If $\mathscr{F} \subset \mathrm{Pol}(M)$ is such that $\mathscr{L}_q(\mathscr{F}) = T_qM$, then $S(\mathscr{F})(q)$ has a non-empty interior in M, and moreover, the interior of $S(\mathscr{F})(q)$ is dense in $S(\mathscr{F})(q)$. Thus, in such a case $A_{\mathscr{F}}(q) = \mathrm{cl}(\mathrm{int}\, S(\mathscr{F})(q))$. If in addition, $\mathscr{F}$ is a subset of analytic vector fields, then $\mathscr{L}_{\mathscr{F}}(q) = T_qM$ is a necessary and sufficient condition for the preceding equality to hold ([4]).

If $\mathscr{L}_{\mathscr{F}}(q) = T_qM$ for each $q \in M$, then $\mathscr{L}_{-\mathscr{F}}(q) = T_qM$. Hence, $\mathscr{A}_{-\mathscr{F}}(q) = \mathrm{cl}(\mathrm{int}(S(-\mathscr{F}))(q))$. As shown in [3], this easily implies that if $A_{\mathscr{F}}(q) = M$ and if $\mathscr{L}_q(\mathscr{F}) = T_qM$ for each $q \in M$, then $S(\mathscr{F})(q) = M$ for each $q \in M$. Since in all the cases we consider $\mathscr{L}_q(\mathscr{F}) = T_qM$, it thus follows that $A_{\mathscr{F}}(q) = M$ for each $q \in M$ is a sufficient condition for transitivity.

Our method of proving transitivity will consist of the following procedure. Let $\mathscr{F} \in \mathrm{Pol}(M)$ be given.

Step 1. Let $\mathscr{F}_1 =$ (the closed conical envelope of $\mathscr{F}$) $\cap CF^\infty(M)$. By (C1), (C3) and (C4), $\mathscr{F}_1$ is equivalent to $\mathscr{F}$.

Step 2. Suppose that there are vector fields in $\mathscr{F}_1$ such that their entire trajectories are contained in $\mathscr{A}_{\mathscr{F}_1}(q)$ for all $q \in M$. If X is such a vector field, then $e^{tX} \in N^\infty(\mathscr{F})$ for each $t \in \mathbb{R}$. Let $\mathscr{F}_2 = \bigcup e^{tX}_*(\mathscr{F})$, where the union is taken over all $t \in \mathbb{R}$ and over elements X of $\mathscr{F}_1$ whose trajectories are in $\mathscr{A}_{\mathscr{F}}(q)$ for all $q \in M$. By (C2), $\mathscr{F}_2$ is equivalent to $\mathscr{F}_1$.

$\mathscr{F}_2$ may not be closed under positive linear combinations. In such a case repeat Steps 1 and 2.

A repetitive use of Steps 1 and 2 gives an increasing sequence of polysystems $\mathscr{F} \subset \mathscr{F}_1 \subset \mathscr{F}_2 \subset \cdots \subset \mathscr{F}_n$ such that they are all equivalent. In certain instances, to be described subsequently, $\mathscr{F}_n$ will contain a subset F', $F' \subset \mathscr{L}(\mathscr{F})$ such that

(i) if $X \in \mathscr{F}'$, then $-X \in \mathscr{F}'$,

(ii) $\mathscr{L}_q(\mathscr{F}') = T_qM$ for all $q \in M$.

It is well known, from a theorem of W. L. Chow, that in such a case, $\mathscr{A}_{\mathscr{F}'}(q) = M$, and that hence, $\mathscr{F}$ is transitive.

II. Polysystems Subordinated to a Group Action

In this section we shall assume that G is a real connected Lie group which acts smoothly on M. Recall that this means that there exists a C^∞ mapping $\theta: G \times M \to M$ such that for each $g \in G$, the induced mapping $\tilde{\theta}_g: M \to M$ defined by $\tilde{\theta}_g(q) = \theta(g, q)$ satisfies:

(1) $\tilde{\theta}_{g_1} \circ \tilde{\theta}_{g_2} = \tilde{\theta}_{g_1 g_2}$ for any g_1 and g_2 in G.

(2) $\tilde{\theta}_e = \mathrm{Id} =$ identity mapping on M, where e is the identity of G.

From this it follows immediately that for any $g \in G$, $\tilde{\theta}_g \in \mathrm{Diff}^\infty(M)$, and that the mapping $G \to \mathrm{Diff}^\infty(M)$ given by $g \to \tilde{\theta}_g$ is a group homomorphism.

A group action θ induces canonically a continuous linear mapping Λ_θ

from $L(G)$, the Lie algebra of G, into $CF^{\infty}(M)$ as follows: If $X \in L(G)$, then $t \to \tilde{\theta}_{e^{tX}}$ is a one-parameter group of diffeomorphisms on M.

If $\tilde{X}$ is its infinitesimal generator, then $\Lambda_\theta(X) = \tilde{X}$. It is well known that Λ_θ is continuous and that $\Lambda_\theta(L(G)) \subset CF^{\infty}(M)$.

We will say that a polysystem $\mathscr{F}$ is *subordinated to a group action* if there exists an action θ from a group G such that $\Lambda_\theta(\Gamma) = \mathscr{F}$ for some subset Γ of $L(G)$.

In such a case we will say that $\mathscr{F}$ is *associated* with Γ.

(A) *Right Invariant Vector Fields*

In the special case where $M = G$, and $\theta: G \times G$ is the left (resp. right) translation action of G onto itself, then Λ_θ is the well known bijective correspondence between $L(G)$ and the space of right (resp. left) invariant vector fields on G. If $X \in L(G)$, then $\Lambda_\theta(X) = \tilde{X}$ is the unique right (resp. left) invariant vector field taking the value X (resp. $-X$) at the identity of G. Associated with any subset Γ of $L(G)$, we will denote by $\tilde{\Gamma}_r$ (resp. $\tilde{\Gamma}_l$) the set $\Lambda_\theta(\Gamma)$ generated by the left (resp. right) translations on G.

If $\Gamma \subset L(G)$, then $S(\Gamma)$ is the semigroup generated by $\bigcup \{e^{tX}: t \in \mathbb{R}_+, X \in \Gamma\}$. It follows immediately from the basic definitions that $S(\tilde{\Gamma}_r)$ is the set of all left translations by the elements of $S(\Gamma)$, and that $S(\tilde{\Gamma}_l)$ is the set of all right translations by the elements of $S(\Gamma)^{-1}$. The orbit under $\tilde{\Gamma}_r$ of $g \in G$ is the set $S(\Gamma)\, g = \{hg: h \in S(\Gamma)\}$, and the corresponding orbit of $\tilde{\Gamma}_l$ is $gS(\Gamma)^{-1} = \{gh: h \in S(\Gamma)^{-1}\}$. In particular, the orbit of the identity under $\tilde{\Gamma}_r$ is $S(\Gamma)$, and under $\tilde{\Gamma}_l$ it is $S(\Gamma)^{-1}$.

In what follows we will consider only the right invariant polysystems. In view of the obvious duality between left and right invariant vector fields, the subsequent results can be stated for left invariant polysystems as well.

(B) *Equivalence of Right Invariant Vector Fields*

Two subsets Γ_1 and Γ_2 of $L(G)$ will be called equivalent if the corresponding right invariant vector fields $\tilde{\Gamma}_{r1}$ and $\tilde{\Gamma}_{r2}$ are equivalent. Recall that this means that $A_{\tilde{\Gamma}_{r1}}(g) = A_{\tilde{\Gamma}_{r1}}(g)$ for all $g \in G$, or in view of the preceding section this means that

$$\mathrm{cl}(S(\Gamma_1)) = \mathrm{cl}(S(\Gamma_2)).$$

Given a subset Γ of $L(G)$, the union of all subsets of $L(G)$ which are equivalent to Γ will be denoted by $\mathrm{Sat}(\Gamma)$ and will be called the saturation of Γ. It is the largest subset of $L(G)$ to Γ.

$\mathrm{Sat}(\Gamma)$ admits also the following two characterizations:

(1) $\mathrm{Sat}(\Gamma)$ is the set of all $X \in L(G)$ such that $\{e^{tX}: t \geq 0\} \subset \mathrm{cl}\, S(\Gamma)$.

(2) $\mathrm{Sat}(\Gamma)$ is such that $\widetilde{\mathrm{Sat}}(\Gamma)_r$ is the largest right invariant polysystem contained in $\mathrm{Sat}(\tilde{\Gamma}_r)$.

These facts will be used subsequently; we assembly them in the following.

PROPOSITION 1. (a) *If* $\Gamma_1 \subset \Gamma_2$ *then* $\mathrm{Sat}(\Gamma_1) \subset \mathrm{Sat}(\Gamma_2)$.

(b) *For any* $\Gamma \subset L(G)$, $\mathrm{Sat}(\Gamma)$ *is the closed convex cone. In particular, any saturated set is a closed convex cone.*

(c) *Let* V *be a subset of* $L(G)$ *such that if* $X \in V$, *then* $-X \in V$. *If* V *is contained in the saturated set* Γ, *then the Lie subalgebra of* $L(G)$ *generated by the elements of* V *is contained in* Γ.

(d) *If* X *and* $-X$ *belong to* $\mathrm{Sat}(\Gamma)$, *then* $e^{\mathrm{ad}\,X}(\mathrm{Sat}(\Gamma)) \subset \mathrm{Sat}(\Gamma)$.

(e) *If* $P\colon L(G) \to L(G)$ *is a projection, that is,* $P^2 = P$, *and if* $\mathrm{Ker}\, P \subset \mathrm{Sat}(\Gamma)$, then $P(\mathrm{Sat}(\Gamma)) \subset \mathrm{Sat}(\Gamma)$.

(f) *If* $X \in \mathrm{Sat}(\Gamma)$ *is such that* $\{e^{tX} : t \in \mathbb{R}\}$ *is relatively compact, then* $\mathbb{R}X \in \mathrm{Sat}(\Gamma)$.

Proof. We start proving (b) since (a) is obvious.

Call $\mathrm{Co}(\mathrm{Sat}(\Gamma))$ (resp. $\mathrm{Co}\,\widetilde{\mathrm{Sat}}(\Gamma)_r$) the convex closed cone generated by $\mathrm{Sat}(\Gamma)$ (resp. $\widetilde{\mathrm{Sat}}(\Gamma)_r$) in $\Gamma(G)$ (resp. in $F^{\infty}(G)$). If $\theta\colon G \times G \to G$ denotes the left translation action of G, then $\Lambda_\theta(\mathrm{Sat}(\Gamma)) = \widetilde{\mathrm{Sat}}(\Gamma)_r$. Since Λ_θ is linear and continuous, $\Lambda_\theta(\mathrm{Co}(\mathrm{Sat}(\Gamma))) = \mathrm{Co}(\widetilde{\mathrm{Sat}}(\Gamma)_r)$, and thus $\mathrm{Co}(\widetilde{\mathrm{Sat}}(\Gamma)_r)$ is right invariant. As it is contained in $\mathrm{Sat}(\tilde{\Gamma}_r)$, $\mathrm{Co}(\mathrm{Sat}(\Gamma)) \subset \mathrm{Sat}(\Gamma)$. Hence, $\mathrm{Co}(\mathrm{Sat}(\Gamma)) = \mathrm{Sat}(\Gamma)$. This proves (b).

Since V is a symmetric subset of $L(G)$, the semi-group $S(V)$ is a group. Being arc-wise connected, it is a Lie subgroup of G. Its Lie algebra $L(S(V))$ contains V, and hence it contains $\mathrm{Lie}(V)$, the Lie algebra generated by V. If Q is the integral subgroup of G corresponding to $\mathrm{Lie}(V)$, then $S(V) = Q$. Hence, $\mathrm{Lie}(V) = L(S(V))$. Since $V \subset \mathrm{Sat}(\Gamma)$, $S(V) \subset S(\mathrm{Sat}(\Gamma))$. Thus, $S(\mathrm{Lie}(V)) \subset S(\mathrm{Sat}(\Gamma))$. Hence, $\mathrm{Lie}(V) \subset \mathrm{Sat}(\Gamma) = \Gamma$, as Γ is saturated. This proves (c).

To prove (d) note that $\mathbb{R}X \subset \mathrm{Sat}(\Gamma)$ since both X and $-X$ belong to $\mathrm{Sat}(\Gamma)$. This means that $\{e^{tX} : t \in \mathbb{R}\} \subset S(\Gamma)$. Hence, $e^X e^{tY} e^{-X} \in S(\Gamma)$ for any $Y \in \mathrm{Sat}(\Gamma)$ and any $t \geqslant 0$. Thus, $\mathrm{ad}(\exp X)(Y) \in \mathrm{Sat}(\Gamma)$, and since $\mathrm{ad}(\exp X)(Y) = \exp \mathrm{ad}\, X(Y)$, the result follows. (e) follows from the fact that for any $L \in \mathrm{Sat}(\Gamma)$, $P(L) = L - (L - P(L))$. Since $-(L - P(L)) \in \mathrm{Ker}(P) \subset \mathrm{Sat}(\Gamma)$, and since $\mathrm{Sat}(\Gamma)$ is a convex cone, it follows that $P(L) \in \mathrm{Sat}(\Gamma)$.

Finally, if X is compact, then for each $t > 0$, $e^{-tX} \in S(\Gamma)$; hence $-X \in \mathrm{Sat}(\Gamma)$ whenever $X \in \mathrm{Sat}(\Gamma)$. Therefore, $\mathbb{R}X \in \mathrm{Sat}(\Gamma)$. This concludes our proof.

(C) *The LS Sets and the Criteria of Transitivity for the Right Invariant Systems*

For any subset Γ of $L(G)$ we define $LS(\Gamma)$ by $LS(\Gamma) = \mathrm{Sat}(\Gamma) \cap \mathrm{Lie}(\Gamma)$, where $\mathrm{Lie}(\Gamma)$ is the Lie algebra generated by Γ. It turns out that $LS(\Gamma)$, and not just Sat(Γ), is the relevant set for studying transitivity of right invariant systems. To see this, consider the well known case of the torus T_2, where Γ consists of one vector field with the irrational slope. Then, $S(\Gamma)$ is a one-dimensional submanifold of T_2 which is dense in T_2. However, Γ is *not* transitive. In this case $LS(\Gamma) = \mathbb{R}_t \Gamma$.

The following proposition justifies the preceding definition.

PROPOSITION 2. *A right invariant polysystem $\tilde{\Gamma}_r$ associated to a subset Γ of the Lie algebra $L(G)$ of a connected real Lie group G is transitive if and only if $LS(\Gamma) = L(G)$.*

In view of this proposition it is clear that the LS sets are crucial for transitivity. For that reason and for convenience to the reader we quote this obvious analogue of Proposition 1.

PROPOSITION 3. (a) *If $\Gamma_1 \subset \Gamma_2$, then $LS(\Gamma_1) \subset LS(\Gamma_2)$.*

(b) *For any $\Gamma \subset L(G)$, $LS(\Gamma)$ is a closed cone.*

(c) *Let $V \subset L(G)$ be such that if $X \in V$, then $-X \in V$. If V is contained in $LS(\Gamma)$, then the Lie subalgebra of $L(G)$ generated by the elements of V is contained in $LS(\Gamma)$.*

(d) *If X and $-X$ belong to $LS(\Gamma)$, then $e^{\operatorname{ad} X}(LS(\Gamma)) \subset LS(\Gamma)$.*

(e) *If $P: L \to L$ is a projection and* Ker $P \subset LS(\Gamma)$ *then $P(LS(\Gamma)) \subset LS(\Gamma)$.*

(f) *If $X \in \Gamma$ is compact, then $\mathbb{R}X \subset LS(\Gamma)$.*

We end this section with the following:

PROPOSITION 4. *If $\theta: G \times M \to M$ is a transitive action on the manifold M, and if Γ is a subset of $L(G)$ such that the right invariant polysystem $\tilde{\Gamma}_r$ is transitive on G, then the polysystem $\mathcal{F}$ on M associated to Γ and subordinated to θ, is transitive on M.*

(D) *Matrix Groups*

Let V be a finite-dimensional real vector space. We denote by End(V) the vector space of all endomorphisms on V, and we will let $GL(V)$ be the group of all automorphisms on V. As is well known, End(V) is a Lie algebra with the Lie bracket $[X, Y] = X \circ Y - Y \circ X$ for X, Y in End(V). We will specialize our main results to $GL_+(V)$, the group of all automorphisms on V with a positive determinant. Thus, in the context of the preceding formalism, $G = GL_+(V)$ and $L(G) = \mathrm{End}(V)$.

As is well known, End(V) splits as $\mathbb{R}\,\mathrm{Id} \oplus sl(V)$, where Id denotes the identity map, and where $sl(V)$ is the space of all endomorphisms of V having zero trace. $sl(V)$ is an ideal of End(V), and $\mathbb{R}$ Id is the center of End(V).

$sl(V)$ is the Lie algebra of the semi-simple Lie group $SL(V)$ the group of all automorphisms on V having determinant equal to 1.

III. The Main Results

In order to state and prove our main results we will need the following additional ideas.

(A) *Canonocal Splitting of V Corresponding to Elements of* $\mathrm{End}(V)$.

Let B be a fixed element of $\mathrm{End}(V)$. Since B may have no real eigenvalues, it is necessary to complexify V in order to describe the necessary ideas. Let $V^{\mathbb{C}}$ be the complexification of V, and let $B^{\mathbb{C}}$ be the complexification of B. Let $\lambda_1,\ldots,\lambda_n$ be the eigenvalues of $B^{\mathbb{C}}$. We shall assume that the eigenvalues of $B^{\mathbb{C}}$ are distinct. Since B is a real endomorphism, the complex eigenvalues of $B^{\mathbb{C}}$ occur in conjugate pairs. We shall write $\alpha_j = \mathrm{Re}\,\lambda_j$, and $\beta_j = \mathrm{Im}\,\lambda_j$ for $j = 1, 2,\ldots, n$. We let $\sigma_{\mathbb{C}}$ be the set of all eigenvalues λ of B such that $\mathrm{Im}\,\lambda \neq 0$, σ_r be the set of all eigenvalues λ of B such that $\mathrm{Im}\,\lambda = 0$, and we write $\sigma = \sigma_{\mathbb{C}} \cup \sigma_r$. If $\lambda \in \sigma_{\mathbb{C}}$, then $\bar{\lambda}$, the complex conjugate of λ, also belongs to $\sigma_{\mathbb{C}}$. Thus, the cardinality of $\sigma_{\mathbb{C}}$ is even. For each $j = 1, 2,\ldots, n$, let $V^{\mathbb{C}}(\lambda_j)$ be the eigenspace of $B^{\mathbb{C}}$ corresponding to the eigenvalue λ_j. $V^{\mathbb{C}} = \bigoplus_{j=1}^{n} V^{\mathbb{C}}(\lambda_j)$. Corresponding to this splitting of $V^{\mathbb{C}}$ we define the following splitting of V.

(i) If $\lambda_j = \alpha_j$, i.e., if $\lambda_j \in \sigma_r$, let e_j be any nonzero vector in $V^{\mathbb{C}}(\lambda_j)$. Since B is real, $e_j \in V$. We denote by E_j the subspace of V generated by e_j.

(ii) If $\lambda_j \in \sigma_{\mathbb{C}}$, let v be any nonzero vector in $V^{\mathbb{C}}(\lambda_j)$. Then, $\bar{v}$, the conjugate vector of v, belongs to $V^{\mathbb{C}}(\bar{\lambda}_j)$. We define $e_{j1} = \frac{1}{2}(v + \bar{v})$, and we define $e_{j2} = (1/2i)(v - \bar{v})$. Here, $i^2 = -1$. It is well known that e_{j1} and e_{j2} are linearly independent vectors in V. We write E_j for the vector subspace of V generated by e_{j1} and e_{j2}. It follows that $Be_{j1} = \alpha_j e_{j1} - \beta_j e_{j2}$, and that $Be_{j2} = \beta_j e_{j1} + \alpha_j e_{j2}$. Thus, $B(E_j) \subset E_j$.

If $\mathrm{card}\,\sigma_{\mathbb{C}} = 2k$, then V splits as: $V = \bigoplus_{j=1}^{n-k} E_j$. Corresponding to this splitting of V, $\mathrm{End}(V)$ splits as $\mathrm{End}(V) = \bigoplus_{i,j=1}^{n-k} \mathrm{End}(E_i, E_j)$, where $\mathrm{End}(E_i, E_j)$ stands for the vector space of all endomorphisms from E_i into E_j. If $X \in \mathrm{End}(V)$, we shall write X_{ij} for the projection of X onto $\mathrm{End}(E_i, E_j)$.

We will let W be the set of basis vectors described in (i) and (ii) above. Occasionally, it will be convenient to work with matrices of elements of $\mathrm{End}(V)$ corresponding to the basis W. If $X \in \mathrm{End}(V)$ we will write $M(X)$ for the matrix of X with respect to W. $M_{ij}(X)$ will stand for the matrix of X_{ij} with respect to W.

In this notation, $M_{ij}(B)=0$ if $i \neq j$. If E_j is one dimensional, the only nonzero entry of $M_{jj}(B)$ is α_j in the (j, j)th position, and if E_j is two dimensional, then the nonzero entries of $M_{jj}(B)$ are given by the 2×2 matrix $\begin{pmatrix} \alpha_j & -\beta_j \\ \beta_j & \alpha_j \end{pmatrix}$.

(B) *Strongly Regular Elements of* End(V)

We will say that $B \in \operatorname{End}(V)$ is *strongly regular* if

(B1) the eigenvalues $\lambda_1,..., \lambda_n$ are all distinct and

(B2) the real parts $\alpha_1,..., \alpha_{n-k}$ of the eigenvalues satisfy $\alpha_p - \alpha_q \neq \alpha_s - \alpha_t$ for all p, q, s, t such that $\{p, q\} \neq \{s, t\}$.

It is obvious that the set of strongly regular elements of End(V) is open and dense in End(V).

We are now ready to start with the development of the main results. We shall, for the remainder of the paper, work *with a fixed strongly regular element B* of End(V), and we shall always adhere to the notation established in Part (A) of this section without further explicit mention. In addition, we shall always assume that the real parts $\alpha_1,..., \alpha_{n-k}$ of the eigenvalue of B are ordered such that $\alpha_1 < \alpha_2 < \cdots \alpha_{n-k}$.

(C) *Transitivity in the Case Where the Spectrum of B is real*

In this section we will assume that each of the subspaces E_i is one dimensional, $i = 1, 2,..., n$. For simplicity of notation, we shall write E_{ij} for the transformation $X \in \operatorname{End}(V)$ such that $X(e_i) = e_j$, $X(e_k) = 0$ for $k \neq i$. We start with the following:

LEMMA 1. *Let p and q be arbitrary integers such that $p \neq q$ and $1 \leqslant p$, $q \leqslant n$. If Γ is the positive convex cone generated by αE_{pq} and βE_{qp}, where $\alpha\beta < 0$, then $LS(\Gamma) = sl(p, q)$, where $sl(p, q)$ is the Lie algebra generated by E_{pq} and E_{qp}. Thus, $S(\Gamma) = SL(p, q)$, where $SL(p, q)$ is the Lie group whose Lie algebra is $sl(p, q)$.*

Proof. If $X \in \Gamma$, then $X = \lambda E_{pq} + \mu E_{qp}$, where $\lambda\mu \leqslant 0$. When $\lambda\mu < 0$, then by direct computation,

$$e^{tX} = (\cos wt)\, E_{pp} + \left(\sqrt{\left|\frac{\lambda}{\mu}\right|} \sin wt\right) E_{pq} - \sqrt{\left|\frac{\mu}{\lambda}\right|} \sin wt E_{qp}$$

$$+ \ (\cos wt)\, E_{pp} + \mathrm{Id}_{pq},$$

where $w = \sqrt{|\lambda\mu|}$, and where Id_{pq} is the identity element of the space $\bigoplus_{i \notin \{p,q\}} E_{ii}$.

In this expression we have without any loss of generality assumed that $\lambda > 0$ and that $\mu < 0$. Thus, $\{e^{tX} : t \in \mathbb{R}\}$ is compact, and hence, by

Proposition 3(f), $\mathbb{R}X \in LS(\Gamma)$. Therefore, for each $t \geqslant 0$, $e^{-tX} \in S(\Gamma)$, and hence, $S(\Gamma)$ is a group. Being arc-wise connected, $S(\Gamma)$ is a Lie group. It is easy to see that $\mathscr{L}(\Gamma) = sl(p, q)$. For instance, if $X_1 = E_{pq}$, $X_2 = E_{qp}$, then $[X_1, X_2] = E_{pp} - E_{qq}$. Thus, $\mathscr{L}(\Gamma)$ is a three-dimension subalgebra of $sl(p, q)$ and hence, it must be equal to $sl(p, q)$.

By Proposition 1(c), $sl(p, q) \subset \mathrm{Sat}(\Gamma) \cap \mathrm{Lie}(\Gamma)$. Since $\Gamma \subset sl(p, q)$, $LS(\Gamma) \subset sl(p, q)$, and hence $sl(p, q) = LS(\Gamma)$. Thus, we have proved the lemma.

The following lemma, although very simple, is needed for technical reasons, and very likely should be ignored by the reader until its use in the main theorem.

LEMMA 2. *Let a_{11}, a_{12}, a_{21} and a_{22} be any real numbers such that $|a_{11}| + |a_{22}| \neq 0$.*

If $y_1 = \alpha a_{12} + \gamma a_{11}$, and if $y_2 = \delta a_{21} - \beta a_{22}$, then there exist numbers α, β, γ and δ with $\alpha\delta - \gamma\beta = 1$ such that $y_1 y_2 < 0$.

Proof. If $a_{21} = a_{12} = 0$, the statement is obvious.

Assume that $a_{12} \neq 0$, and let $\alpha = \lambda a_{12} a_{22}$, $\beta = +\lambda$, $\gamma = -1/\lambda$ and $\delta = 0$. For such a choice of α, β, γ and δ, $y_1 y_2 = (\alpha a_{12} + \gamma a_{11})(\delta a_{21} - \beta a_{22}) = a_{12} a_{21} + \beta\gamma(a_{12} a_{21} - a_{11} a_{22}) + \gamma\delta a_{11} a_{21} - \alpha\beta a_{12} a_{22} = a_{12} a_{21} - (a_{12} a_{21} - a_{11} a_{21}) - \lambda^2 (a_{12} a_{22})^2$. $y_1 y_2$ becomes negative as $\lambda \to \infty$. If $a_{21} \neq 0$, let $\delta = -\lambda a_{11} a_{21}$, $\beta = -1/\lambda$, $\gamma = \lambda$ and $\alpha = 0$. For such a choice of α, β, γ and δ, an argument analogous to the preceding one shows that $y_1 y_2$ becomes negative for sufficiently large values of λ. Thus, the proof is finished.

We now state and prove

THEOREM 1. *Let A and B be any elements of $sl(V)$, such that B is strongly regular, and such that A satisfies:*

(A1) $A_{ij} \neq 0$ *for all* $1 \leqslant i, j \leqslant n$ *such that* $|i - j| = 1$.

(A2) $A_{1n} \cdot A_{n1} < 0$.

Then, the right invariant polysystem $\mathscr{F} = \{\tilde{A}_r + u\tilde{B}_r : u \in \mathbb{R}\}$ is transitive on $SL(V)$.

Proof. Let $\Gamma = \{A + uB : u \in \mathbb{R}\}$. Our proof will follow if we show that $LS(\Gamma) = sl(V)$. As before, let $sl(i, j)$ be the Lie subalgebra of $sl(V)$ generated by $\mathrm{End}(E_i, E_j)$ and $\mathrm{End}(E_j, E_i)$. Since $\mathscr{L}(\bigcup sl(i, j)) = sl(V)$, in view of Proposition 3(c), it would suffice to show that each $i \neq j$, $1 \leqslant i$, $j \leqslant n$, $sl(i, j) \subset \mathrm{Sat}(\Gamma)$.

For each pair of integers (i, j), $1 \leqslant i$, $j \leqslant n$, let $\Delta_{ij} = \alpha_i - \alpha_j$. We write $\Delta = \{\Delta_{ij} : 1 \leqslant i, j \leqslant n\}$. Δ is a symmetric set, $-\Delta_{ij} = \Delta_{ji}$. Also, by the strong regularity property of B the correspondence $(i,j)\to\Delta_{ij}$ is one–one from the set $\mathbb{Z}_n=\{(i,j),\ 1\leqslant i,j\leqslant n,\ i\neq j\}$ onto Δ. For each $(p,q)\in Z_n$, we will write

$$V^{(p,q)}=\bigoplus_{|\Delta_{ij}|\geqslant|\Delta_{pq}|}\operatorname{End}(E_i,E_j).$$

Our basic aim is to show that for each $(p,q)\in Z_n$, $V^{(p,q)}\subset LS(\Gamma)$. For, then by Proposition 3(c), $\mathscr{L}(V^{(p,q)})\subset LS(\Gamma)$, and since $sl(p,q)\subset\mathscr{L}(V^{(p,q)})$ our proof would follow.

We proceed to prove that $V^{(p,q)}\subset LS(\Gamma)$ by induction on the set of positive elements of Δ. We will start with the largest element of Δ and proceed in descending order. We first show that $V^{(n,1)}\subset LS(\Gamma)$. $\operatorname{Lim}_{n\to\infty}(1/n)(A\pm nB)=\pm B$. Hence, $\pm B\in LS(\Gamma)$. If $v\in\mathbb{R}$, then by Proposition 3(d), $A_v=e^{-Bv}Ae^{Bb}\in LS(\Gamma)$. Since $LS(\Gamma)$ is a closed convex cone, $\lim_{v\to\infty}e^{-\Delta_{n1}v}A_v\in LS(\Gamma)$. In view of the equality $A_v=\bigoplus_{i,j=1}^n e^{-\Delta_{ij}v}A_{ij}$, it is easy to see that the above limit is equal to A_{n1}. A similar reasoning applied to the limit $\lim_{v\to-\infty}e^{\Delta_{n1}v}A_v$ shows that $A_{1n}\in LS(\Gamma)$. Thus, C the positive convex cone spanned by A_{1n} and A_{n1} is contained in $LS(\Gamma)$. Our assumption (A2) implies that $A_{1n}=a_{1n}E_{1n}$ and $A_{n1}a_{n1}E_{n1}$, where $a_{1n}a_{1n}<0$. Lemma 1 shows that $LS(C)=sl(1,n)$, and by Proposition 3(a) it follows that $sl(1,n)\subset LS(\Gamma)$. Since $V^{(n,1)}\subset sl(1,n)$ we have proved that our induction hypothesis holds for the maximal element of Δ.

Assume now that for our induction hypothesis holds for all pairs $(i,j)\in Z_n$ such that $|\Delta_{ij}|\geqslant|\Delta_{pq}|$. Let $\Delta_{lm}=\operatorname{Max}\{\Delta_{ij}:\Delta_{ij}<|\Delta_{pq}|\}$. Obviously, $l>m$. We want to show that $V^{(l,m)}\subset LS(\Gamma)$.

Case I. $l<n$. Then $\Delta_{(l+1)m}\geqslant|\Delta_{pq}|$, and hence $V^{(l+1,m)}\subset LS(\Gamma)$.

Since $\mathscr{L}(V^{(l+1,m)})\supset sl(l+1,m)$, it follows that $sl(l+1,m)\subset LS(\Gamma)$. Either $\Delta_{(l+1)l}\geqslant|\Delta_{pq}|$ or $\Delta_{(l+1)l}<|\Delta_{pq}|$.

Assume that $\Delta_{(l+1)l}\geqslant|\Delta_{pq}|$. In such a case, our induction hypothesis, along with an argument similar to the above, shows that $sl(l+1,l)\subset LS(\Gamma)$. If $X\in sl(l+1,l)$, and if $g\in SL(l+1,m)$, then by Proposition 3(d), $Y=g^{-1}Xg\in LS(\Gamma)$. We want to choose X and g such that $Y_{ml}\circ Y_{lm}<0$. If $X=x_{(l+1),l}E_{(l+1)l}+x_{l(l+1)}E_{l(l+1)}$, and if $g=\alpha E_{mm}+\beta E_{m(l+1)}+\gamma E_{(l+1)m}+\delta E_{(l+1)(l+1)}+\operatorname{Id}_{(l+1)m}$, with $\alpha\delta-\beta\gamma=1$ and, where $\operatorname{Id}_{(l+1)m}$ is the identity on $\bigoplus_{\{i\notin\{l+1,m\}\}}\operatorname{End}(E_i,E_i)$, then we have that

$$Y_{lm}=\gamma x_{l(l+1)}E_{lm}\qquad\text{and that}\qquad Y_{ml}=-\beta x_{(l+1)l}E_{ml}.$$

Choose any X and g such that $-\gamma x_{l(l+1)}\cdot\beta x_{(l+1)l}<0$. Let $P^{(p,q)}$ be the projection of $\operatorname{End}(V)$ onto $V^{(p,q)}$, and let $A^{(m,l)}=Y-P^{(p,q)}(Y)$. $A_{(m,l)}\in LS(\Gamma)$, and additionally:

(i) $A_{ij}^{(m,l)}=A_{ji}^{(m,l)}=0$ for all $(i,j)\in Z_n$ such that $|\Delta_{ij}|\geqslant|\Delta_{pq}|$.

(ii) $A_{lm}^{(m,l)}\circ A_{ml}^{(m,l)}<0$.

We show that the positive closed cone generated by $A_{ml}^{(m,l)}$ and $A_{lm}^{(m,l)}$ belongs to $LS(\Gamma)$. But this follows from the following limits:

$$\lim_{v\to\infty} e^{-\Delta_{lm}v}(e^{-Bv}A^{(m,l)}e^{Bv}) = A_{lm}^{(m,l)},$$

and

$$\lim_{v\to\infty} e^{\Delta_{lm}v}(e^{-Bv}A^{(m,l)}e^{Bv}) = A_{ml}^{(m,l)}.$$

An application of Lemma 1 to the preceding cone shows that $sl(l, m) \subset LS(\Gamma)$. Hence, $V^{(l,m)} \subset LS(\Gamma)$.

Consider now the case where $\Delta_{(l+1)l} < |\Delta_{pq}|$. As before, let $P^{(p,q)}$ be the projection map onto $V^{(p,q)}$. Let $X = A - P^{(p,q)}(A)$, $X \subset LS(\Gamma)$. Additionally, $X_{l(l+1)} \neq 0$, and $X_{l(l+1)} \neq 0$. We next want to choose an element $g \in SL(l+1, m)$ such that $U = g^{-1}Xg$ satisfies $Y_{ml} \circ Y_{lm} < 0$. As in the preceding computation we let g be given by α, β, γ and δ such that $\alpha\delta - \beta\gamma = 1$. If $M(X) = (x_{ij})$ and $M(Y) = (y_{ij})$, then: $y_{lm} = \alpha x_{lm} + \gamma x_{l(l+1)}$, and $y_{ml} = \delta x_{ml} - \beta x_{(l+1)l}$. Since $|x_{l(l+1)}| + |x_{(l+1)l}| \neq 0$, we have by Lemma 2 that there exist α, β, γ and δ with $\alpha\delta - \beta\gamma = 1$ such that $\gamma_{ml} \cdot y_{lm} < 0$. For such a choice of g, let $A^{(m,l)} = X - P^{(p,q)}(X)$. Since both X and $-P^{(p,q)}(X)$ belong to $LS(\Gamma)$ so does $A^{(m,l)}$. $A^{(m,l)}$ satisfies:

(i) $A_{ij}^{(m,l)} = 0$ if $|\Delta_{ij}| \geqslant |\Delta_{pq}|$.

(ii) $A_{ml}^{(m,l)} \circ A_{lm}^{(m,l)} < 0$.

By an argument similar to that of the preceding paragraph we show that the convex cone generated by $A_{ml}^{(m,l)}$ and $A_{lm}^{(m,l)}$ belongs to $LS(\Gamma)$. Hence, in this case as well, $V^{(m,l)} \subset LS(\Gamma)$.

Case II. $l = n$. Then $m > 1$ (since our induction hypothesis is already verified for $m = 1$). In view of the ordering used, $\Delta_{l(m-1)} \geqslant |\Delta_{pq}|$. Thus, $V^{(l,m-1)} \subset LS(\Gamma)$, and hence $sl(l, m-1) \subset LS(\Gamma)$.

We will omit the remaining part of the arguments, since they are completely analogous to those of Case I except with the roles of $l+1$ and $m-1$ reversed. Thus, our induction hypothesis is true for (m, l), and hence, it must be true for all elements of Z_n. This proves the theorem.

COROLLARY 1. *Let A and B be elements of* End(V), *where B is strongly regular, and where A satisfies* (A1) *and* (A2) *of the previous theorem. If the trace of B is not equal to zero, then $\mathscr{F} = \{\tilde{A}_r + u\tilde{B}_r : u \in \mathbb{R}\}$ is transitive on $GL(V)$.*

Proof. We use the splitting $\mathrm{End}(V) = \mathbb{R}\,\mathrm{Id} \oplus sl(V)$. In this splitting $B = \beta\,\mathrm{Id} \oplus B_0$, where $\beta = \mathrm{tr}(B)$, and where $B_0 \in sl(V)$. From Theorem 1, $sl(V) \subset LS(\Gamma)$, where $\Gamma = \{A + uB: u \in \mathbb{R}\}$. Thus, $\pm B_0 \in LS(\Gamma)$. Since $\pm B \in LS(\Gamma)$, it follows that $\pm\beta\,\mathrm{Id} \in LS(\Gamma)$. Hence, $\mathbb{R}\,\mathrm{Id} \in LS(\Gamma)$. Thus, $LS(\Gamma) = \mathrm{End}(V)$, and therefore, $S(\Gamma) = GL_+(V)$. Our proof is now finished.

COROLLARY 2. *Let $\theta: GL_+(V) \times V \to V$ be given by $\theta(G, v) = G \cdot v$. If $\mathscr{F}$ is a polysystem on V associated with $\Gamma = \{A + uB: u \in \mathbb{R}\}$, where A and B satisfy the conditions of Theorem* 1, *then $\mathscr{F}$ is transitive on $V_0 = V - \{0\}$.*

Proof. $SL(V)$ acts transitively on V_0.

It is not difficult to show [3] that condition (A2) of Theorem 1 is also a necessary condition in the case where $\dim V = 2$. In higher dimensions, the situation is not so clear. For instance, when A and B are symmetric endomorphisms on V then condition (A2) is never satisfied. We have managed to show that in a case when $\dim V \leqslant 3$, no polysystem on V_0 associated with $\{A + uB: u \in \mathbb{R}\}$ is transitive. Although, the method of proof braks down in higher dimensions, it seems that polysystems of the form $\{A + uB: u \in \mathbb{R}\}$, where A and B are symmetric, never give rise to transitive polysystems on V_0.

(D) *Transitivity in the Case Where the Spectrum of B is Complex*

In this section we will assume that all the eigenvalues of B are complex. This implies that $\dim E_i = 2$ for $i = 1, 2, \ldots, k$, $\dim V = 2k$, and that $\dim \mathrm{End}(E_i, E_j) = 4$ for all $1 \leqslant i, j \leqslant n$. As before we will write $sl(i, j)$ for the Lie subalgebra of $sl(V)$ generated by $\mathrm{End}(E_i, E_j) \oplus \mathrm{End}(E_j, E_i)$. If B is a strongly regular element, then there is a bijective correspondence $(i, j) \to \Delta_{ij} = \alpha_i - \alpha_j$ between the sets $Z_k = \{(i, j), 1 \leqslant i \neq j \leqslant k\}$ and $\Delta = \{\Delta_{ij}: (i, j) \in Z_k\}$.

LEMMA 3. *$sl(V)$ is generated by the vector space*

$$\bigoplus_{1 \leqslant i \neq j \leqslant k} (\mathrm{End}(E_i, E_j) \oplus \mathrm{End}(E_j, E_i)).$$

We will omit the proof.

If $X = X_{pq}$ for some $1 \leqslant p \neq q \leqslant k$, then we will say that X is *exceptional* provided that X_{pq} belongs to the union $W^1_{pq} \cup W^2_{pq}$ of the following linear subspaces of $\mathrm{End}(E_p, E_q)$. Let L_1, L_2, L_3 and L_4 be elements of $\mathrm{End}(E_p, E_q)$ defined as follows: $L_1(e_{p1}) = e_{q1}$, $L_1(e_{p2}) = -e_{q2}$; $L_2(e_{p1}) = e_{q2}$, $L(e_{p2}) = e_{q1}$; $L_3(e_{p1}) = e_{q1}$, $L_3(e_{p2}) = e_{q2}$; and $L_4(e_{p1}) = e_{q2}$, $L_4(e_{p2}) = -e_{q1}$. Then W^1_{pq} is the linear space generated by L_1 and L_2, while W^2_{pq} is the linear space generated by L_3 and L_4.

LEMMA 4. *Let $X = X_{pq}$ for some pair of integers $(p, q) \in Z_k$.*
If B is strongly regular element of $\operatorname{End}(V)$, *and if X is not exceptional, then C, the closed convex cone generated by $\{e^{-Bv}Xe^{Bv} : v \in \mathbb{R}\}$, is equal to* $\operatorname{End}(E_p, E_q)$.

Proof. For each pair of integers (p, q), $1 \leqslant p,\ q \leqslant k$, the space $\operatorname{End}(E_p, E_q)$ is isomorphic with $M_2(\mathbb{R})$, the space of all 2×2 matrices. If $L(e_{p1}) = \alpha e_{q1} + \gamma e_{q2}$, and if $L(e_{p2}) = \beta e_{q1} + \delta e_{q2}$, then we represent L by the matrix

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}.$$

In this representation:

$$e^{B_{pp}v} = e^{\alpha_p v} \begin{pmatrix} \cos \beta_p v & -\sin \beta_p v \\ \sin \beta_p v & \cos \beta_p v \end{pmatrix},$$

and

$$e^{B_{qq}v} = e^{\alpha_q v} \begin{pmatrix} \cos \beta_q v & -\sin \beta_q v \\ \sin \beta_q v & \cos \beta_q v \end{pmatrix}.$$

In this representation the exceptional spaces are

$$W^1_{pq} = \mathbb{R} \begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix} + \mathbb{R} \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix},$$

and

$$W^2_{pq} = \mathbb{R} \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} + \mathbb{R} \begin{vmatrix} 0 & 1 \\ -1 & 0 \end{vmatrix}.$$

Thus, if X_{pq} is represented by

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = M$$

we require that the closed convex cone C generated by

$$e^{-\Delta_{pq}v} \begin{vmatrix} \cos \beta_p v & \sin {}_p v \\ -\sin \beta_p v & \cos {}_p v \end{vmatrix} \begin{vmatrix} \alpha & \beta \\ \gamma & \delta \end{vmatrix} \begin{vmatrix} \cos \beta_q v & -\sin \beta_q v \\ \sin \beta_q v & \cos \beta_q v \end{vmatrix}$$

$$= e^{-\Delta_{pq}v} \begin{vmatrix} x_{11}(v) & x_{12}(v) \\ x_{21}(v) & x_{22}(v) \end{vmatrix}, \qquad v \in \mathbb{R},$$

be equal to $M_2(\mathbb{R})$. By expanding the preceding expression we get:

$$\begin{aligned}
x_{11}(v) &= \alpha\cos\beta_q v\cos\beta_p v - \beta\cos\beta_q v\sin\beta_p v - \gamma\sin\beta_q v\cos\beta_p v \\
&\quad + \delta\sin\beta_q v\sin\beta_p v. \\
x_{12}(v) &= \alpha\cos\beta_q v\sin\beta_p v + \beta\cos\beta_q v\cos\beta_p v - \gamma\sin\beta_q v\sin\beta_p v \\
&\quad - \delta\sin\beta_q v\cos\beta_p v. \\
x_{21}(v) &= \alpha\sin\beta_q v\cos\beta_p v - \beta\sin\beta_q v\sin\beta_p v + \gamma\cos\beta_q v\cos\beta_p v \\
&\quad - \delta\cos\beta_q v\sin\beta_p v. \\
x_{22}(v) &= \alpha\sin\beta_q v\sin\beta_p v + \beta\sin\beta_p v\cos\beta_p v + \gamma\cos\beta_q v\sin\beta_p v \\
&\quad + \delta\cos\beta_q v\cos\beta_p v.
\end{aligned}$$

If $f(t)$ is any positive function, then $(1/v)\int_0^v f(t)\, e^{-B_{pp}t}Xe^{B_{qq}t}\,dt \in C$ for each $v \in \mathbb{R}^+$. Furthermore, since C is closed, $\lim_{v\to\infty}(1/v)\int_0^t f(t)\, e^{-B_{pp}t}Xe^{B_{qq}t}\,dt \in C$ whenever such a limit exists.

Making use of the elementary trigonometric identities we get, after a simple computation that

$$\varepsilon\begin{pmatrix} \alpha+\beta & -\gamma+\beta \\ -\beta+\gamma & \alpha+\delta \end{pmatrix}$$

$$= \varepsilon L_1 = \lim_{v\to\infty}\int_0^v e^{-\Delta t}(1 + 4\varepsilon\cos(\beta_p+\beta_q)t)\, e^{-B_{pp}t}Xe^{B_{qq}t}\,dt.$$

Since for $|\varepsilon| \leqslant \frac{1}{4}$, $e^{-\Delta v}(1 + 4\varepsilon\cos(\beta_q+\beta_p)v) \geqslant 0$ for all v, $\varepsilon L_1 \in C$, and hence $\mathbb{R}L_1 \in C$.

Similarly,

$$\lim_{v\to\infty}\frac{1}{v}\int_0^v e^{-\Delta t}(1 + 4\varepsilon\cos(\beta_q-\beta_p)t)\, e^{-B_{pp}t}Xe^{B_{qq}t}\,dt$$

$$= \varepsilon\begin{vmatrix} \alpha-\delta & \beta+\gamma \\ \gamma+\beta & -\alpha+\delta \end{vmatrix} = \varepsilon L_2,$$

$$\lim_{v\to\infty}\frac{1}{v}\int_0^v e^{-\Delta t}(1 + 4\varepsilon\sin(\beta_q+\beta_p)t)\, e^{-B_{pp}t}Xe^{B_{qq}t}\,dt$$

$$= \varepsilon\begin{vmatrix} -\beta-\gamma & \alpha-\delta \\ \alpha-\delta & \beta+\gamma \end{vmatrix} = \varepsilon L_3$$

and finally,

$$\lim_{v\to\infty} \frac{1}{v}\int_0^v e^{-\Delta t}(1 + 4\varepsilon \sin(\beta_q - \beta_p)t)\, e^{-B_{pp}t} X e^{B_{qq}t}\, dt$$

$$= \varepsilon \begin{vmatrix} \beta-\gamma & -\alpha-\delta \\ \alpha+\delta & \beta-\gamma \end{vmatrix} = \varepsilon L_4.$$

Since in each of the above cases $|\varepsilon| \leqslant \frac{1}{4}$ it follows that C includes the vector space spanned by L_1, L_2, L_3, and L_4.

We denote by

$$M_1 = \tfrac{1}{2}(L_1 + L_2) = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \qquad M_2 = \tfrac{1}{2}(L_1 - L_2) = \begin{pmatrix} \delta & -\gamma \\ -\beta & \alpha \end{pmatrix},$$

$$M_3 = \tfrac{1}{2}(L_3 + L_4) = \begin{vmatrix} -\gamma & -\delta \\ \alpha & \beta \end{vmatrix} \quad \text{and} \quad M_4 = \tfrac{1}{2}(L_3 - L_4) = \begin{vmatrix} -\beta & \alpha \\ -\delta & \gamma \end{vmatrix}.$$

The vector space generated by M_1, M_2, M_3 and M_4 will be four-dimensional, and hence equal to $M_2(\mathbb{R})$ whenever

$$\det\begin{vmatrix} \alpha & \delta & -\gamma & -\beta \\ \beta & -\gamma & -\delta & \alpha \\ \gamma & -\beta & \alpha & -\delta \\ \delta & \alpha & \beta & \gamma \end{vmatrix} =$$

$$= -\left[(\alpha^2-\delta^2)^2 + (\beta^2-\gamma^2) + 2(\alpha^2+\delta^2)(\beta^2+\gamma^2) + 8\alpha\beta\gamma\delta\right] \neq 0.$$

It is easy to check that the minimum value of the above expression in parentheses is equal to zero, and that it occurs when $\alpha = \delta$ and $\beta = -\gamma$, or $\alpha = -\delta$ and $\beta = \gamma$.

In the first case

$$X = \begin{vmatrix} \alpha & \beta \\ -\beta & \alpha \end{vmatrix} = \alpha \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} + \beta \begin{vmatrix} 0 & 1 \\ -1 & 0 \end{vmatrix},$$

and in the second case,

$$X = \begin{vmatrix} \alpha & \beta \\ \beta & -\alpha \end{vmatrix} = \alpha \begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix} + \beta \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}.$$

But, by our assumption X is excluded from these possibilities, and hence we have proved the lemma.

Using the notations of the previous lemma we get:

LEMMA 5. *If $X = X_{pq} \neq 0$ is exceptional, then $W^1_{pq} \subset C$ when $X \in W^1_{pq}$, and $W^2_{pq} \subset C$ when X^2_{pq}.*

Proof. We shall use the same notations as those developed in the proof of the previous lemma. Let $X = \left(\begin{smallmatrix} \alpha & -\beta \\ \beta & \alpha \end{smallmatrix}\right)$. Then,

$$M_1 = X, \quad M_2 = \begin{vmatrix} \alpha & -\beta \\ \beta & \alpha \end{vmatrix}, \quad M_3 = \begin{vmatrix} +\beta & \alpha \\ \alpha & \beta \end{vmatrix}, \quad M_4 = \begin{vmatrix} -\beta & \alpha \\ -\alpha & -\beta \end{vmatrix}.$$

If either $\alpha \neq 0$ or $\beta = 0$, then it is easy to see that the above matrices span W^1_{pq}. In the remaining case,

$$X = \begin{vmatrix} \alpha & \beta \\ \beta & -\alpha \end{vmatrix} = M_1.$$

Then,

$$M_2 = \begin{vmatrix} -\alpha & -\beta \\ -\beta & \alpha \end{vmatrix}, \qquad M_3 = \begin{vmatrix} -\beta & \alpha \\ \alpha & \beta \end{vmatrix}, \qquad M_4 = \begin{vmatrix} -\beta & \alpha \\ -\alpha & \beta \end{vmatrix}.$$

Hence, when either $\alpha \neq 0$, or $\beta \neq 0$, $W^2_{pq} \subset C$. Thus we have proved the lemma.

Before stating the next lemma it will be convenient to introduce the following additional terminology. If $X_{pq} \in \mathrm{End}(E_p, E_q)$ and if $X_{qp} \in \mathrm{End}(E_q, E_p)$, then we will say that X_{pq} and X_{qp} *do not belong to the same exceptional type* if either $X_{pq} \notin W^1_{pq}$ and $X_{qp} \notin W^2_{pq}$, or $X_{pq} \notin W^2_{pq}$ and $X_{qp} \notin W^1_{qp}$.

LEMMA 6. *Let $X_{pq} \in \mathrm{End}(E_p, E_q)$, and let $X_{qp} \in \mathrm{End}(E_q, E_p)$ for some pair of integers (p, q), $p \neq q$, $1 \leqslant p$, $q \leqslant k$. If B is strongly regular, then $Sl(p, q) \subset LS(\{X_{pq}, X_{qp}, B\}) = LS(\Gamma)$ provided that* (i) *neither X_{pq} nor X_{qp} are equal to zero, and* (ii) *X_{pq} and X_{qp} do not belong to the same exceptional type.*

Proof. The Lie algebra $sl(p, q)$ is isomorphic with $sl(\mathbb{R}^4)$ via the matrix representation with respect to the basis $\{e_{p1}, e_{p2}, e_{q1}, e_{q2}\}$. For the sake of convenience but at the risk of notational abuse, we will regard the elements of $\mathrm{End}(E_p, E_q) \oplus \mathrm{End}(E_q, E_p)$ as elements of $sl(\mathbb{R}^4)$. By Lemma 4 it follows that $\mathrm{Sat}(\Gamma)$ includes $\mathrm{End}(E_q, E_q) \oplus \mathrm{End}(E_q, E_p)$ provided that neither X_{pq} nor X_{qp} is exceptional. Obviously, the statement of the lemma will follow if we prove it for the exceptional case. Without any loss of generality assume that $X_{pq} \in W^1_{pq}$ and that $X_{qp} \in W^2_{qp}$. By Lemma 5, $W^1_{pq} \subset LS(\Gamma)$, and $W^2_{qp} \subset LS(\Gamma)$.

If $L_1 \in W^1_{pq}$ then we will identity L_1 with

$$\begin{vmatrix} 0 & 0 & \alpha & \beta \\ 0 & 0 & -\beta & \alpha \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix},$$

and if $L_2 \in W^2_{qp}$ then we write

$$L_2 = \begin{vmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \gamma & \delta & 0 & 0 \\ \delta & -\gamma & 0 & 0 \end{vmatrix}.$$

By varying α, β, γ and δ we will show that $[L_1, L_2]$ includes all the matrices of the form

$$(1) \begin{vmatrix} \alpha & \beta & 0 & 0 \\ \gamma & -\alpha & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix} \quad \text{and} \quad (2) \begin{vmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha & \beta \\ 0 & 0 & \gamma & -\alpha \end{vmatrix},$$

where α, β, γ, and δ are arbitrary.

$$[L_1 L_2] = (\alpha\gamma + \beta\delta) D_1 + (\alpha\delta - \beta\gamma) D_2 - (\gamma\alpha - \beta\delta) D_3 - (\alpha\delta + \beta\gamma) D_4,$$

where

$$D_1 = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix}, \quad D_2 = \begin{vmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix},$$

$$D_3 = \begin{vmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{vmatrix}, \quad D_4 = \begin{vmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{vmatrix}.$$

(i) $\alpha\gamma = 1$, and $\beta = \delta = 0$ implies that $[L_1, L_2] = D_1 - D_3$,

(ii) $\beta\delta = 1$, and $\alpha = \gamma = 0$ implies that $[L_1, L_2] = D_1 + D_3$,

(iii) $\alpha\delta = 1$, and $\beta = \gamma = 0$ implies that $[L_1, L_2] = D_2 - D_4$,

(iv) $\beta\gamma = 1$, and $\alpha = \delta = 0$ implies that $[L_1, L_2] = -D_2 - D_4$.

It is now easy to see that the vector space of $[L_1, L_2]$ as α, β, γ and δ vary includes the vector space containing D_1, D_2, D_3 and D_4. $[D_1, D_2]$ is linearly independent from D_1 and D_2, and thus the vector space generated by D_1, D_2

and $[D_1, D_2]$ produces matrices of type (1), while D_3, D_4 and $[D_3, D_4]$ generate matrices of type (2).

It is easy to check that the commutators of matrices of type (1) with those of W^1_{pq} generate matrices of the form

$$(3) \quad \begin{vmatrix} 0 & 0 & \alpha & \beta \\ 0 & 0 & \gamma & \delta \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix},$$

while commutators of matrices type (2) with those of W^2_{qp} generate matrices of the form

$$(4) \quad \begin{vmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \alpha & \beta & 0 & 0 \\ \gamma & \delta & 0 & 0 \end{vmatrix}$$

(α, δ, γ and δ are arbitrary).

It is now easy to check that the Lie algebra generated by the matrices of types (3) and (4) is equal to $sl(\mathbb{R}^4)$, and hence its verification will be omitted. The proof of the lemma is now complete.

We are now ready to state and prove the following.

THEOREM 2. *Let B be a strongly regular element such that its spectrum contains no real eigenvalues, and let $A \in \operatorname{End}(V)$ be such that A both*:

(i) *$A_{pq} \neq 0$ for all pairs of integers (p, q), $p \neq q$, $1 \leqslant p, q \leqslant k$, and*

(ii) *there exists at least one pair of integers (p, q), $p \neq q$, $1 \leqslant p, q \leqslant k$,*

such that X_{pq} and X_{qp} do not both belong to the same exceptional type.

Then, if both A and B belong to $sl(V)$, $LS(\{A + uB: u \in \mathbb{R}\}) = sl(V)$. Thus $X = \{\tilde{A}_r + u\tilde{B}_r: u \in \mathbb{R}\}$ is transitive on $SL(V)$.

Proof. In the course of the proof we will show that $T = \bigoplus_{1 \leqslant i \neq j \leqslant k} \operatorname{End}(E_i, E_j) \subset LS(\Gamma)$, where $\Gamma = \{A + uB: u \in \mathbb{R}\}$. For then, by Property 2(c), the Lie algebra $L(T)$, generated by T, is contained in $LS(\Gamma)$, and hence by Lemma 4, $L(T) = sl(V)$. Thus, we would have $sl(V) = LS(\Gamma)$, and that implies the theorem.

As before let, $V^{(p,q)} = \bigoplus_{|\Delta_{ij}| \geqslant |\Delta_{pq}|} \operatorname{End}(E_i, E_j)$. The proof will basically consist of two parts. In the first part we will show that for each pair of integers $(p, q) \in Z_k$, $\pm A_{pq} \in LS(\Gamma)$, while in the second part we will show that $\operatorname{End}(E_p, E_q) \subset LS(\Gamma)$.

Part 1. We make the inductive hypothesis that for each $(p, q) \in Z_k$,

$\pm A_{pq} \in LS(\Gamma)$. We proceed in the descending order on the element of Δ. Let $\Delta_{pq} = \text{Max}\{\Delta_{ij} : \Delta_{ij} \in \Delta\}$. Then, $\Delta_{qp} = \text{Min}\,\Delta$. For each $v \in \mathbb{R}$,

$$e^{-Bv}Ae^{Bv} = \bigoplus_{1 \leqslant i,j \leqslant k} e^{-\Delta_{ij}v}\, O_i^{-1}(v)\, A_{ij}\, O_j(v),$$

where $O_l(v) = e^{-\alpha_l v} e^{B_l v}$ for each integer $l = 1, 2,..., k$. Each $O_l(v)$ is compact; in fact $O_l(2\pi/\beta_l) = \text{Id}$.

If $f(t)$ is any positive and bounded function then for each $v > 0$, $(1/v) \int e^{\Delta_{pq}t} f(t)\, e^{-Bt}Ae^{Bt}\, dt \in LS(\Gamma)$, and $(1/v) \int_0^v e^{\Delta_{qp}t} f(t)\, e^{-Bt}Ae^{Bt}\, dt \in LS(\Gamma)$.

In view of the maximality of Δ_{pq}, $\lim_{v\to\infty} (1/v) \int_0^{-v} e^{\Delta_{pq}t} f(t)\, e^{-Bt}A_{ij}\, e^{Bt}\, dt = 0$ for each $(i, j) \neq (p, q)$, and similarly $\lim_{v\to\infty} (1/v) \int_0^v e^{+\Delta_{qp}t} f(t)\, e^{-Bt}A_{ij}\, e^{Bt}\, dt = 0$ for $(i, j) \neq (q, p)$. Hence, for each function f which is positive and bounded on $[0, \infty)$, $\lim_{v\to\infty} (1/v) \int_0^{-v} f(t)\, e^{-Bt}A_{pq}\, e^{Bt}\, dt \in LS(\Gamma)$, and $\lim_{v\to\infty} (1/v) \int_0^v f(t)\, e^{-Bt}A_{qp}\, e^{Bt}\, dt \in LS(\Gamma)$ whenever the above limits exist.

But, by Lemmas 4 and 5, the set of above limits for various choices of f always include $\pm A_{pq}$, and $\pm A_{pq}$.

Assume now that our inductive hypothesis is true for all pairs $(i,j) \in Z_k$ such that $|\Delta_{ij}| \geqslant |\Delta_{pq}|$, where (p, q) is a fixed element of Z_k.

Let $A^{(p,q)} = A - \bigoplus_{|\Delta_{ij}| \geqslant |\Delta_{pq}|} A_{ij}$. We have that $A_{ij}^{(p,q)} = 0$ for $|\Delta_{ij}| \geqslant |\Delta_{pq}|$.

Let $\Delta_{lm} = \text{Max}(\Delta - \{\Delta_{ij} : |\Delta_{ij}| \geqslant |\Delta_{pq}|\})$.

An argument similar to the preceding one shows that for each function f such that $f(t) \geqslant 0$ for $t \geqslant 0$. $\lim_{v\to\infty} (1/v) \int_0^v e^{-\Delta_{lm}t} f(t)\, e^{-Bt}A^{(p,q)}\, e^{Bt}\, dt \in LS(\Gamma)$, and that $\lim_{v\to\infty} (1/v) \int_0^{-v} e^{\Delta_{lm}t} f(-t)\, e^{-Bt}A^{(p,q)}\, e^{Bt}\, dt \in LS(\Gamma)$. Again, by Lemmas 4 and 5, the above limits include $\pm A_{ml}$ and $\pm A_{lm}$. Thus, our inductive hypothesis must be true for all pairs $(p, q) \in Z_k$. This proves the first part.

Part 2. Let now $(p, q) \in Z_k$ be the pair which satisfies condition (ii) of the theorem. By Lemma 6, we have that $sl(p, q) \subset LS(\Gamma)$. Let $s \notin \{p, q\}$. If $g \in SL(p, q)$, then $X = g^{-1}A_{sp}\, g$ belongs to $LS(\Gamma)$ (Proposition 3(d)). We shall now choose such a g such that X_{sq} is not exceptional. If $g = \bigoplus_{1 \leqslant i,j \leqslant k} g_{ij}$, then $X = X_{sp} + X_{sq}$, where $X_{sp} = A_{sp}\, g_{pq}$ and where $X_{sq} = A_{sp}\, g_{pq}$. Since g_{pq} can be any element of $\text{End}(E_p, E_q)$, it follows that there exists $g \in SL(p, q)$ such that X_{sq} is not exceptional. Lemmas 5 and 6 now imply that $sl(s, q) \subset LS(\Gamma)$. It is clear now that this precedure extends to all pairs $(i, j) \in Z_k$. Thus, we have proved the theorem.

It is clear that the obvious analogues of the corollaries of Theorem 1 hold true as corollaries of Theorem 2, and therefore, we will not state them.

Contrary to the case where B has only real eigenvalues, the set of endomorphisms A such that $LS(\{A + uB : u \in \mathbb{R}\}) = sl(V)$, where B is strongly regular having no real eigenvalues, is open and dense in $\text{End}(V)$.

(E) *The General Case*

In this section we will assume that B has $2k$ complex eigenvalues. As we described before, corresponding to these eigenvalues there are $E_1,..., E_{n-k}$ linear subspaces which split V. In this section we shall assume that $E_1,..., E_{n-k}$ are numbered such that $\alpha_1 < \alpha_2 < \cdots \alpha_{n-k}$. Here, $\alpha_1,..., \alpha_{n-k}$ are the real parts of the eigenvalue of B. Thus, if α_i belongs to σ_B the spectrum of B, then E_i is one dimensional. Otherwise, E_i is two dimensional. Recall from Section (A) that in such a case e_{i1} and e_{i2} are the distinguished basis vectors in E_i. As before, we will use the splitting $\mathrm{End}(V) = \bigoplus_{1 \leqslant i,j \leqslant n-k} \mathrm{End}(E_i, E_j)$. If $X \in \mathrm{End}(V)$, then X_{ij} will be its projection on $\mathrm{End}(E_i, E_j)$. As we mentioned earlier, B will always denote the fixed element $\mathrm{End}(V)$. In this case, it will be convenient to write $Z_{n-k} = \{(i,j): i \neq j, 1 \leqslant i, j \leqslant n-k\}$. When B is strongly regular, the correspondence $(i,j) \to \alpha_i - \alpha_j = \Delta_{ij}$ is one-one from Z_{n-k} onto Δ.

We start with the following lemma.

LEMMA 7. *Let $(i,j) \in Z_{n-k}$ be such that exactly one element of $\{E_i, E_j\}$ is two dimensional. Let $X \in \mathrm{End}(V)$ be such that $X = X_{ij} + X_{ji}$, where neither $X_{ij} = 0$, nor $X_{ji} = 0$. Then, $LS(\{X, \pm B\})$ includes $sl(i,j)$, the nine-dimensional Lie subalgebra of $sl(V)$ spanned by* $\mathrm{End}(E_i, E_j) \oplus \mathrm{End}(E_j, E_i)$.

Proof. Let $\Gamma = \{X, B, -B\}$. For each $V \in \mathbb{R}$, let

$$X(v) = e^{-Bv} X e^{Bv} = e^{-B_{ii}v} X_{ij} e^{B_{jj}v} \oplus e^{-B_{jj}v} X_{ji} e^{B_{ii}v} = X_{ij}(v) \oplus X_{ji}(v).$$

For each $v \in \mathbb{R}$, $X(v) \in LS(\Gamma)$.

Without any loss of generality assume that E_i is two dimensional. Let e_{i1} and e_{i2} be the basis vectors for E_i, as described previously, and let e_j be a basis vector for E_j. If $X_{ij}(e_{i1}) = a_{13} e_j$, $X_{ij}(e_{i2}) = a_{23} e_j$, and if $X_{ji}(e_j) = a_{31} e_{i1} + a_{32} e_{i2}$, then by our assumptions $|a_{13}| + |a_{23}| \neq 0$, and $|a_{31}| + |a_{32}| \neq 0$.

After a simple computation we get that

$$X_{ij}(v)(e_{i1}) = e^{(\alpha_j - \alpha_i)v}(a_{13} \cos \beta_i v + a_{23} \sin \beta_i v)\, e_j,$$

and that

$$X_{ij}(v)(e_{i2}) = e^{(\alpha_j - \alpha_i)v}(-a_{13} \sin \beta_i v + a_{23} \cos \beta_i v)\, e_j.$$

Similarly,

$$X_{ji}(v)(e_j) = e^{(\alpha_i - \alpha_j)v}\{(a_{31} \cos \beta_j v + a_{32} \sin \beta_j v)\, e_{i1} + (a_{32} \cos \beta_j v - a_{31} \sin \beta_j v)\, e_{i2}\}.$$

Let $f(t) = e^{\Delta_{ji}t}(1 + \varepsilon \cos \beta_j t)$ for $t \in \mathbb{R}$, and $|\varepsilon| < 1/2$, then $f(t) \geqslant 0$. Hence, by Property 3(b), for each $v \in \mathbb{R}$, $v \neq 0$, $|1/v| \int_0^v f(t) X(t)\, dt \in LS(\Gamma)$. If $\Delta_{ij} > 0$, then $\lim_{v \to +\infty} (1/|v|) \int_0^v f(t) X(t)\, dt \in LS(\Gamma)$. If X_1 denotes such a limit, then it is easy to verify that $X_1 \in \mathrm{End}(E_j, E_i)$ and that $X_1(e_j) = \varepsilon(a_{31}e_{i1} + a_{32}e_{i2})$. By computing the similar limit when $f(t) = e^{\Delta_{ji}t}(1 + \varepsilon \sin \beta_j t)$, $t \in \mathbb{R}$, we get that $X_2 = \lim(1/v) \int_0^v f(t) X(t)\, dt \in LS(\Gamma)$, where $X_2(e_j) = \varepsilon(a_{32}e_{i1} - a_{32}e_{i2})$.

Since $LS(\Gamma)$ is a closed convex cone, it follows that $\Gamma S(\Gamma)$ includes the vector space generated by $(1/\varepsilon) X_1$ and $(1/\varepsilon) X_2$. Such a space is two dimensional, and is hence equal to $\mathrm{End}(E_j, E_i)$. To show that $\mathrm{End}(E_i, E_j) \subset LS(\Gamma)$ consider the limits of the type $\lim_{v \to \infty} (1/v) \int_0^{-v} f(t) X(t)\, dt$, where $f(t)$ varies over the functions of the form $f(t) = e^{\Delta_{ij}t} g(t)$ for g bounded and positive. The details of these arguments are quite analogous to those of the preceding paragraph and therefore will be omitted.

Finally, if $\Delta_{ij} < 0$, then all the preceding arguments are the same except that the limits where $v \to \infty$ become the limits as $v \to -\infty$, and conversely. Thus, in either case, $\mathrm{End}(E_i, E_j) \oplus \mathrm{End}(E_j, E_i) \subset LS(\Gamma)$, and hence by Proposition 3(c), $sl(i, j)$, the Lie algebra generated by it, also belongs to $LS(\Gamma)$. Hence, we have proved the lemma.

We now state and prove the main theorem of this paper.

THEOREM 3. *Let A and B be elements of $sl(V)$ such that B is strongly regular. Assume that A satisfies*:

(A1) *$A_{pq} \neq 0$ for all $(p, q) \in Z_{n-k}$ such that E_p and E_q are both one dimensional, and such that $|p - q| = 1$.*

(A2) *If E_1 and E_{n-k} are one dimensional, then we require that $A_{1(n-k)} \cdot A_{(n-k)1} < 0$.*

(A3) *$A_{pq} \neq 0$ for all pairs $(p, q) \in Z_{n-k}$ such that at least one of E_p or E_q is two dimensional.*

(A4) *If $n = 2k$, then we require that there exist an element $(p, q) \in Z_{n-k}$ such that X_{pq} and X_{qp} do not both belong to the same exceptional type.*

Then, the right invariant polysystem $\mathscr{F} = \{\tilde{A}_r + u\tilde{B}_r : u \in \mathbb{R}\}$ associated with $\Gamma = \{A + uB : u \in \mathbb{R}\}$ is transitive on $SL(V)$.

Proof. If $k = 0$, then our theorem reduces to Theorem 1, and if $k = 2n$, then it reduces to Theorem 2. Hence, we will assume that $0 < 2k < 2n$. Our method of the proof will be to show that for each pair of integers $(i, j) \in Z_{n-k}$, $sl(i, j) \subset LS(\Gamma)$.

We shall prove this by induction on the set of positive elements of Δ. Our inductive hypothesis is that for each $\Delta pq > 0$ in Δ $sl(i, j) \subset LS(\Gamma)$ for all $(i, j) \in Z_{n-k}$ with $|\Delta ij| \geqslant \Delta pq$. We first verify that our induction hypothesis is true for $\Delta_{(n-k)1}$ the largest element of Δ.

Case 1. If both α_1 and α_{n-k} are the eigenvalues of B, then E_1 and E_{n-k} are one dimensional. In such a case, an argument completely analogous to that in Theorem 1 shows that $sl(1, n-k) \subset LS(\Gamma)$.

Case 2. Suppose that one (but not both) of α_1 and α_{n-k} is an eigenvalue of B. This corresponds to the case where one of E_1 and E_{n-k} is one dimensional and the other is two dimensional. As we have seen before, $e^{-Bv}Ae^{Bv} = \bigoplus_{1 \leqslant i,j \leqslant n-k} e^{-B_{ii}v}A_{ij}e^{B_{jj}v}$ belongs to $LS(\Gamma)$ for each $v \in \mathbb{R}$. Since Γ is a closed cone, $A_{1(n-k)} = \lim_{m\to\infty} e^{-\Delta_{(n-k)1}v_m}e^{-Bv_m}Ae^{Bv_m} \in LS(\Gamma)$, where

$$\begin{aligned} v_m &= \frac{2\pi m}{\beta_1} && \text{if } E_1 \text{ is two dimensional} \\ &= \frac{2\pi m}{\beta_{n-k}} && \text{if } E_{n-k} \text{ is two dimensional.} \end{aligned}$$

Similarly, $A_{(n-k)1} = \lim_{m\to\infty} e^{-\Delta_{1(n-k)}v_m}e^{-Bv_m}Ae^{Bv_m} \in LS(\Gamma)$, where v_m in this case is the negative of the previous one. Now, Lemma 7 applied to $X = A_{(n-k)1} \oplus A_{1(n-k)}$ shows that $sl(1, n-k) \subset LS(\Gamma)$.

Case 3. Suppose that neither of α_1 and α_{n-k} is an eigenvalue of B. This means that E_1 and E_{n-k} are two dimensional.

Let $(p, q) \in Z_{n-k}$ be such that:

(i) Either α_p or α_q (but not both) is an eigenvalue of B, and

(ii) if $\Delta_{ij} > |\Delta_{pq}|$, then neither α_i nor α_j is an eigenvalue of B; $\dim E_i = \dim E_j = 2$ for all such (i, j). By Step 1 of the proof of Theorem 2, $\pm A_{ij}$, $\pm A_{ji}$ are elements of $LS(\Gamma)$ for all $(i, j) \in Z_{n-k}$ such that $\Delta_{ij} < |\Delta_{pq}|$. Consider, $A^{(p,q)} = A - (\bigoplus_{\Delta ij > |\Delta pq|} (A_{ij} + A_{ji})$. $A^{(p,q)}_{ij} = 0$ for $|\Delta_{ij}| > |\Delta_{pq}|$ and neither $A^{(p,q)}_{pq}$ nor $A^{(p,q)}_{qp}$ is equal to zero by condition (A3). $A^{(p,q)} \in LS(\Gamma)$.

An argument analogous to that in Case 2, with $\Delta_{(n-k)1}$ replaced by Δ_{pq} shows that A_{pq} and A_{qp} belong to $LS(\Gamma)$. Lemma 7 then implies that $sl(p, q) \subset LS(\Gamma)$. Without any loss of generality assume α_p is an eigenvalue of B. Then, E_q is two dimensional. Since $\mathrm{End}(E_q, E_q) \subset sl(p, q)$, we get that $\mathrm{End}(E_q, E_q) \subset LS(\Gamma)$, or that $sl(q, q)$, the Lie algebra generated by $\mathrm{End}(E_q, E_q)$, is contained in $LS(\Gamma)$. Assume that $\Delta_{pq} < 0$. Either $p < n-k$ or $q > 1$. When $p < n-k$, then $\Delta_{(p+1)q} > \Delta pq$. Hence, by our assumption $\dim E_{p+1} = 2$. This means that $\pm A_{(p+1)q}$ and that $\pm A_{q(p+1)}$ belong to $LS(\Gamma)$ (by Step 1 of Theorem 2). If $X \in sl(q, q)$, then $[A_{(p+1)q}, X] = A_{(p+1)q}X \in LS(\Gamma)$. Hence $sl(q, q)A_{(p+1)q} \subset LS(\Gamma)$. It is an easy computation to show that there exists an element $X \in sl(q, q)$ such that $Y_{(p+1)q}$ of $Y = XA_{(p+1)q}$ is not exceptional. (Definition of exceptional precedes Lemma 5.) An argument identical to that of Step 2 in Theorem 2 shows that $sl(p+1, q) \subset LS(\Gamma)$, and hence by the same argument, we show that $sl(i,j) \subset LS(\Gamma)$ for all $(i,j) \in Z_{n-k}$ such that $|\Delta_{ij}| \geqslant |\Delta_{pq}|$. In particular, $sl(1, n-1) \subset LS(\Gamma)$.

Thus, our induction hypothesis is true for the pair $(n-k, 1)$. Assume now that it holds true for all pairs $(i,j) \in Z_{n,k}$ such that $|\Delta_{ij}| > \Delta_{pq}$ for some $\Delta_{pq} < \Delta_{(n-k)1}$. We want to prove that it holds true for Δ_{pq}; i.e., we want to show that $sl(p,q) \subset LS(\Gamma)$. Again we need to examine separate cases according to whether α_p and α_q are eigenvalues of B or not.

Case 1. α_p and α_q are eigenvalues of B. Assume that $p < n-k$. Otherwise, $q < 1$ and the subsequent argument is symmetrical. $\Delta_{(p+1)q} > \Delta_{pq}$, and hence by our hypothesis, $sl(p+1, q) \subset LS(\Gamma)$. If α_{p+1} is an eigenvalue of B, then $A_{p(p+1)} \neq 0$, $A_{(p+1)p} \neq 0$ (condition (A1)). Hence, an argument analogous to that of Theorem 1 shows that $sl(p,q) \subset LS(\Gamma)$. If α_{p+1} is not an eigenvalue of B, then $\dim E_{p+1} = 2$. If $g \in SL(p+1, q)$, let $X = g^{-1}Ag$. $X \in LS(\Gamma)$, and after a simple computation we get that $X_{pq} = A_{pq} \circ g_{qq} \oplus A_{p(p+1)} \circ g_{(p+1)q}$ and that $X_{qp} = g_{qq}^{-1} \oplus g_{q(p+1)}^{-1} \circ A_{(p+1)p}$.

By condition (A3), $A_{p(p+1)} \neq 0$, and $A_{(p+1)p} \neq 0$. It is an easy verification that there exists $g \in SL(p+1, q)$ such that $X_{pq} \circ X_{qp} < 0$. For such X, let $A^{(p,q)} = X - \bigoplus_{|\Delta_{ij}| > |\Delta_{pq}|} (X_{ij} + X_{ji})$). Since $\pm(X_{ij} + X_{ji}) \in sl(i,j) \subset LS(\Gamma)$ for all (i,j) such that $|\Delta_{ij}| > |\Delta_{pq}|$, $A^{(p,q)} \in LS(\Gamma)$. An argument identical to that of Theorem 1 shows that $sl(p,q) \subset LS(\Gamma)$.

Case 2. Either α_p or α_q (but not both) is an eigenvalue of B. In such a case let, $A^{(p,q)} = A - (\bigoplus_{|\Delta_{ij}| > |\Delta_{pq}|} (A_{ij} \oplus A_{ji}))$. By taking the limits $\lim_{m \to -\infty} e^{\Delta_{pq} v_m} e^{-Bv_m} A^{(p,q)} e^{Bv_m}$ and $\lim_{m \to \infty} e^{-\Delta_{pq} v_m} e^{-Bv_m} A^{(p,q)} e^{Bv_m}$ where

$$
\begin{aligned}
v_m &= \frac{2\pi m}{\beta_p} \qquad \text{if} \quad \dim E_p = 2 \\
&= \frac{2\pi m}{\beta_q} \qquad \text{if} \quad \dim E_q = 2,
\end{aligned}
$$

we show that A_{pq} and A_{qp} belong to $LS(\Gamma)$. Hence, Lemma 7 shows that $sl(p,q) \subset LS(\Gamma)$.

Case 3. Neither α_p nor α_q is an eigenvalue of B. Let

$$
A^{(p,q)} = A - \left(\bigoplus_{|\Delta_{ij}| > |\Delta_{pq}|} (A_{ij} \oplus A_{ji}) \right).
$$

Arguments similar to those of Theorem 2 show that $\pm A_{pq}$, and $\pm A_{qp}$ belong $LS(\Gamma)$. It is an easy computation to verify that there exist elements g and h in $SL(p+1, q)$ such that Y_{pq} and Z_{qp} corresponding to $Y = g^{-1} A_{pq} g$ and $Z = h^{-1} A_{qp} h$ are not exceptional. Lemma 6 then implies that $sl(p,q) \subset LS(\Gamma)$.

Hence, our induction hypothesis holds for (p, q), and it therefore holds true for all elements of Z_{n-k}.

Thus, the proof of this theorem is finished.

COROLLARY 3. *If* $A \in \mathrm{End}(V)$ *satisfies* (A1) *through* (A4), *and if* B *is a strongly regular element of* $\mathrm{End}(V)$ *such that its trace is not equal to zero, then the right invariant polysystem* $\{\tilde{A}_r + u\tilde{B}_r : u \in \mathbb{R}\}$ *associated to* $\Gamma = \{A + uB : u \in \mathbb{R}\}$ *is transitive on* $GL_+(V)$.

COROLLARY 4. *Let* $\theta: GL_+(V) \times V \to V$ *be given by* $\theta(G, V) = G \cdot v$. *If* $\mathscr{F}$ *is the polysystem on* $V_0 = V - \{0\}$, *associated with* $\Gamma = \{A + uB : u \in \mathbb{R}\}$, *where* A *and* B *satisfy the conditions of Theorem* 3, *then* $\mathscr{F}$ *is transitive on* V_0.

The proofs of these corollaries are identical to those of Corollaries 1 and 2, and will therefore be omitted.

REFERENCES

1. H. HERMES AND J. LASALLE, "Functional Analysis and Time-Optimal Control," Academic Press, New York, 1969.
2. V. JURDJEVIC AND I. KUPKA, Control systems: Accessibility on semi-simple Lie groups and their homogenous spaces, to appear.
3. V. JURDJEVIC AND J. P. QUINN, Controllability and stability, *J. Differential Equations* **28** (1976), 381–389.
4. H. J. SUSSMAN AND V. JURDJEVIC, Controllability of non-linear systems, *J. Differential Equations* **12** (1972), 95–116.