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Quantum Nondemolition Filters[1]

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Abstract. This is the second of two papers concerned with the formulation of a continuous-time quantum-mechanical filter. In the first paper, the invertibility of a quantum system coupled to a weak time-dependent classical field was studied. The physical system is modelled as an infinite-dimensional bilinear system. Necessary and sufficient conditions for invertibility were derived under the assumption that the output observable is a quantum nondemolition observable (QNDO), characterized by the classical property that its expected value is equal to its measured value. In this paper necessary and sufficient conditions are developed for an observable to qualify as a QNDO; if in addition the criteria for invertibility are met, the given observable defines a quantum nondemolition filter (QNDF). The associated filtering algorithm thus separates cleanly into the choice of output observable (a QNDO) and the choice of procedure for processing the measurement outcomes. This approach has the advantage over previous schemes that no optimization is necessary. Applications to demodulation of optical signals and to the detection and monitoring of gravitational waves are envisioned.

1. Introduction

In an earlier paper [1] we studied the invertibility of quantum-mechanical control systems described by

$$i\hbar\frac{d\psi(t)}{dt} = \left[H_0 + u(t)H_1\right]\psi(t), \tag{1}$$

$$y(t) = \langle\psi(t)|C(t)\psi(t)\rangle. \tag{2}$$

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The state of the system is represented by a vector ψ in a suitable Hilbert space $\mathscr{H}$. The time evolution of ψ (in the Schrödinger picture [2] of quantum dynamics) is governed by a Hamiltonian operator $H = H_0 + u(t)H_1$ acting in $\mathscr{H}$. This Hamiltonian is made up of self-adjoint operators H_0 and H_1 in $\mathscr{H}$, the coefficient $u(t)$ being a bounded, real, analytic function of the time t. One may interpret H_0 as the unperturbed Hamiltonian of the system and H_1 as the system observable which couples to an external classical field whose strength is represented by $u(t)$. Thus $u(t)$ plays the role of (a) a signal to be extracted or, alternatively, (b) a control to be utilized to manipulate the time-development of the system. In the present work (as in [1]) we shall focus attention on role (a).

By virtue of the self-adjointness of the Hamiltonian operator, the evolution of $\psi(t)$ is confined to a manifold of constant norm, chosen for convenience to be the unit sphere $S_{\mathscr{H}}$. The time-development of the system is monitored via the function $y(t)$, taken as the Hilbert-space inner product of the vector $C(t)\psi(t)$ with the state representative $\psi(t)$. The system observable C, generally time-varying, is also a self-adjoint operator in $\mathscr{H}$. It is a well-known principle of quantum theory [2] that the inner product $y(t)$ gives the *expected value* of the dynamical quantity $\mathscr{C}(t)$ associated with the operator $C(t)$, for the system in state $\psi(t)$. As usual, one must imagine that, at time t, the quantity $\mathscr{C}$ is measured for a large number of copies of the system, all in quantum state $\psi(t)$. In [1], we derived necessary and sufficient conditions for invertibility of the control system (1)–(2) under the assumption that the chosen output $y(t)$ yields not merely the expected value of $\mathscr{C}(t)$ but in fact the *actual* measurement result. Clearly, this assumption is not justified unless the dispersion in measurement-results obtained for the aforementioned ensemble is zero. The latter requirement implies in turn (see e.g. [2]) that the state $\psi(t)$ evolves on an eigenmanifold of C, once an initial precise measurement of $\mathscr{C}$ has been performed. Accordingly, the results of [1] were obtained under the assumption that the output observable C is a very special sort of operator, known in the physics literature [3] as a *quantum nondemolition observable* (QNDO).

For invertibility to hold it is further required that C be such as to permit the input $u(t)$ to be calculated from a knowledge of the output $y(t)$. A QNDO for which this is the case will be called a *quantum nondemolition filter* (QNDF) [4]. The aim of the present article is to explore the concepts of QNDO and QNDF within the systems-theoretic framework. Appealing to the existence of an analytic domain in the sense of Nelson [5], we shall, in particular, establish necessary and sufficient conditions for an observable to be a QNDF, making use of the results obtained in [1]. Such criteria may prove useful in the application of quantum nondemolition techniques to the detection and accurate reconstruction of signals which are so weak that the quantum nature of the detection apparatus becomes important.

Some essential aspects of quantum theory are reviewed in Section 2. In Section 3 we sketch the physical background of quantum nondemolition measurement and present a formal definition of quantum nondemolition observable (QNDO). Similarly, in Section 4 we lay the intuitive and formal basis for the concept of quantum nondemolition filter (QNDF). In Section 5 the main results of the paper—incisive criteria for a QNDF—are presented and proved. Several elementary examples of QNDF's are provided in Section 6, and connections with earlier work on quantum-mechanical filtering [6, 7] are briefly discussed. A point of special interest is that the two aspects of the filtering problem, namely the choice of output variable (choice of output observable) and the choice of procedure for processing the measurement outcomes, separate quite naturally in the QND approach.

2. Quantum-Mechanical Preliminaries

An understanding of the axioms of quantum theory and their operational interpretation, at the level of such textbooks as Messiah [2] and Cohen-Tannoudji, Diu and Laloë [8], will be needed to follow some of our discussion and argumentation. For deeper treatments of the structure and meaning of quantum mechanics, the reader is referred to von Neumann [9], Jauch [10] and d'Espagnat [11]. The most essential elements for our work are the laws of quantum dynamics, the statistical interpretation of the state vector ψ (including the notion of "collapse of the wave packet"), and the uncertainty principle for canonically conjugate dynamical quantities.

2.1. Quantum Dynamics

The two most prominent descriptions of quantum dynamics are the *Schrödinger picture* and the *Heisenberg picture*.

In the Schrödinger picture, which we have adopted in setting up the control system (1)–(2), the time-development of the system is described in terms of the time-dependence of the state vector $\psi(t)$, which obeys the Schrödinger equation (1). The self-adjointness of the Hamiltonian operator $H = H_0 + u(t)H_1$ implies that the time-development of $\psi(t)$ can be expressed in terms of a *unitary* operator $U(t)$, the evolution operator, which solves

$$i\hbar\frac{dU(t)}{dt} = HU(t), \qquad U(0) = I. \tag{3}$$

In the Schrödinger picture, the state vector moves around in Hilbert space, but the observables (i.e., the operators representing dynamical quantities) are stationary—unless they are assigned some explicit time-dependence.

Conversely, in the Heisenberg picture, it is the observables which embody the dynamical evolution of the system, while the state vector sits still in Hilbert space. The time-dependence of a given observable A is governed by the Heisenberg equation

$$i\hbar\frac{dA}{dt} = [A, H] + i\hbar\frac{\partial A}{\partial t}, \tag{4}$$

where the last term arises from the explicit time-dependence assigned to A.

In cases where confusion might arise, one indicates that a given state vector or operator is being viewed in the Schrödinger picture (Heisenberg picture) by a subscript S (subscript H). The two pictures are unitarily equivalent. Indeed, the connection between them may be expressed, in terms of the evolution operator U, by

$$\begin{aligned}\psi_S(t) &= U(t)\psi_H, \qquad \psi_H = \text{const. vector},\\ A_H(t) &= U^{\dagger}(t)A_S(t)U(t).\end{aligned}\tag{5}$$

Obviously, there are any number of unitarily equivalent pictures "intermediate" between Schrödinger and Heisenberg.

We insert the following remarks to allay possible confusion as to the meaning of the last term in (4) and the partial-time-derivative notation for quantum operators more generally. In Eq. (4), $\partial A/\partial t = \partial A_H/\partial t$ is used as a shorthand for the Heisenberg transform of the time derivative of the Schrödinger version A_S of A, i.e.,

$$\frac{\partial A_H}{\partial t} \triangleq \left(\frac{dA_S(t)}{dt}\right)_H = U^{\dagger}(t)\frac{dA_S(t)}{dt}U(t).$$

In the Schrödinger picture, where the evolution of an operator A is unaffected by the Hamiltonian dynamics, there is strictly no need to distinguish between total and partial time derivatives of A, so

$$\frac{\partial A_S}{\partial t} = \frac{dA_S}{dt}.$$

It will be convenient in the present paper to adhere throughout to the partial-derivative notation when differentiating with respect to the explicit (or "non-dynamical") time-dependence of A. This is the common practice in the quantum literature. (Note, however, that in [1]—where only the Schrödinger picture was considered—we found it more convenient to use the total derivative notation.) For further detail and examples clarifying this technical point, the reader may consult [2].

2.2. *Statistical Interpretation*

In our work, all measurements are assumed to be "of the first kind" [10, 12]. The properties of such a measurement of $\mathscr{C}$ are the following: (i) Suppose the system is in an eigenstate of C, with eigenvalue c_i, at the time of observation (i.e., $C\psi = c_i\psi$); then the outcome of the measurement is precisely equal to that eigenvalue. (ii) Let the system be in an *arbitrary* state ψ at the time of measurement, and suppose the outcome c_n (necessarily one of the eigenvalues of C) is obtained; then the measurement leaves the system in an eigenstate of C corresponding to that outcome ("collapse of the wave packet").

Generally the outcome in situation (ii) is uncertain and can be predicted only in a statistical sense. Thus, by a basic hypothesis of quantum mechanics, the proposed result c_n is realized with probability $\langle\psi|P_n|\psi\rangle$, where P_n is the projection operator onto the subspace of $\mathscr{H}$ of eigenvalue c_n of C. There is evidently a fundamental acausality (an "uncontrollable disturbance") endemic to quantum observation. Consequently, one is forced to a nondeterministic formulation of the systems-theoretic problems posed by (1), (2)—*except* in the special circumstance that ψ is an eigenstate of C and remains one through subsequent evolution and observation. This "escape clause" leads directly to the idea of quantum nondemolition measurement [12].

2.3. *Heisenberg Uncertainty Principle*

Suppose two observables A, B have commutator $[A, B] = iK$, where K is again self-adjoint. For a given quantum state ψ the dispersions $\triangle A = [\langle \psi | A^2 \psi \rangle - \langle \psi | A \psi \rangle^2]^{1/2}$, $\triangle B = [\langle \psi | B^2 \psi \rangle - \langle \psi | B \psi \rangle^2]^{1/2}$ characterizing independent sets of measurements of A, B obey the uncertainty relation

$$(\triangle A)(\triangle B) \geq \tfrac{1}{2} |\langle \psi | K \psi \rangle| . \tag{6}$$

If $K = \hbar$, the two dynamical quantities represented by A, B are said to be canonically conjugate, and one has the Heisenberg uncertainty relation in its usual form.

If A and B commute, the corresponding dynamical variables are compatible, and no such restriction as (6) is placed on the dispersions $\triangle A, \triangle B$. It is said that A and B can, in that case, be measured simultaneously with arbitrary precision. The proper formal statement has been given (and proved) by von Neumann [9]: There exists at least one basis (orthonormal set of vectors complete in $\mathscr{H}$) made up of common eigenstates of A and B *if and only if* $[A, B] = 0$. (Note the inclusion of the converse statement.)

3. Quantum Nondemolition Measurement

3.1. *Back-Action-Evading Measurements and Quantum Nondemolition Observables*

Quantum nondemolition (QND) measurement consists of a time sequence of precise, instantaneous measurements of a suitably chosen observable. Such measurements were first proposed by physicists as a means for monitoring a weak classical force acting on a one-dimensional quantum-mechanical harmonic oscillator [12–15]. The force is presumed to be so weak that it changes the amplitude of the oscillator by an amount less than the amplitude of zero-point fluctuations. (The latter measure is given by the dispersion, or variance, $\triangle x = (\hbar / 2 m \omega)^{1/2}$ of the position observable x in the ground state of the oscillator, ω being the angular frequency and m the mass of the oscillator.) For some time the conventional wisdom was that a force $F(t) = F_0 \sin \omega t$ ($= u(t)$ in our notation) can be detected reliably only if its amplitude F_0 exceeds the so-called standard quantum limit, $F_0 \geq (2/\tau)(2 \hbar m \omega)^{1/2}$, where τ is the duration of measurement. This limit is encountered when one decides to measure the observables

$$\begin{aligned} X_1 &= x \cos \omega t - (p/m\omega) \sin \omega t, \\ X_2 &= x \sin \omega t + (p/m\omega) \cos \omega t \end{aligned} \tag{7}$$

with equal accuracy, p being the momentum observable of the oscillator. Such a measurement strategy is called "amplitude-and-phase measurement". (The complex amplitude of the oscillator is given by the operator $X_1 + iX_2$. Thus one attempts to evaluate—approximately—the (real) amplitude and the phase of the oscillator). Since $[X_1, X_2] = i\hbar/m\omega$, the uncertainty principle restricts the accuracy of amplitude-and-phase measurement to $\Delta X_1 = \Delta X_2 \geq (\hbar/2m\omega)^{1/2}$. This restriction implies the aforementioned standard quantum limit on the detection of a sinusoidal classical force. (For a complete argument, see [15], where the analysis for an arbitrary $F(t)$ is also given.)

The force $F(t)$ is assumed to be classical; strictly, quantum mechanics should not impose any absolute limit on its measurement. Indeed, one can circumvent the standard quantum limit as follows: Simply carry out an accurate measurement of X_1 alone (actually, a series of such measurements), with $\Delta X_1 \ll (\hbar/2m\omega)^{1/2}$. Then, of course, X_2 will be rendered highly uncertain, with $\Delta X_2 \gg (\hbar/2m\omega)^{1/2}$, but that is the price one must pay. The crucial point is that the Heisenberg evolution of X_1 may be completely uncoupled from any uncertainty in X_2. This can be seen by forming from (4) the Heisenberg equation of motion for $X_1 + iX_2$ in the presence of a coupling $u(t)H_1 = -xF(t)$ to a classical field, the unperturbed oscillator Hamiltonian being $H_0 = p^2/2m + m\omega^2x^2/2$. Explicitly, we find

$$\frac{d}{dt}(X_1 + iX_2) = -\frac{i}{\hbar}[X_1 + iX_2, -xF(t)] = \frac{F(t)}{m\omega}(-\sin\omega t + i\cos\omega t)I. \tag{8}$$

Thus there is no "back-action" on X_1 of disturbances in X_2. In the absence of $F(t)$, the results of subsequent measurements of X_1 are exactly predictable from the first such measurement; moreover, when $F(t)$ is acting, this function can be determined with arbitrary accuracy from the results of a sequence of arbitrarily accurate measurements of X_1. (See [12, 15] for details.) We note that a strategy of accurate measurement of X_2 (while not caring about X_1) would have worked equally well.

This situation of back-action evading measurement of X_1 (alternatively, X_2) for the quantum oscillator with linear coupling to the classical field may be contrasted with that in which one carries out repeated measurements of the position x at arbitrary times. Measurement of x disturbs p through the usual Heisenberg uncertainty principle, and the consequent uncertainty in p feeds back into the Heisenberg evolution of x via the commutator $[x, H] = i\hbar p/m$. Thus, future measurement results for x will be unpredictable because of the back-action of disturbances of p. Clearly, x-measurement is unsuited to the task of precise determination of the time course of a classical force applied to a quantum oscillator. Momentum-measurement is likewise unsuitable for the oscillator. However, for a *free* particle the momentum operator p (but not the position operator x) obviously does qualify as a back-action-evading observable, since $[p, H] = [p, p^2/2m] = 0$.

The back-action-evading measurements discussed above are examples of QND measurements, of particular QND observables. The key feature of a QND observable is that its Heisenberg time-development proceeds independently of observables with which it does not commute. Accordingly, if the Hamiltonian governing the evolution is known, repeated precise measurement of the QNDO may be carried out with results which are free from uncertainty and completely predictable.

These ideas have been formalized by Unruh [16] and Caves *et al.* [12]. The formal definition of QNDO to be adopted here is:

Definition 1. Given the system Hamiltonian $H = H_0 + u(t)H_1$, an observable C is *a quantum nondemolition observable* (QNDO) if, in a sequence of precise measurements of C, the result of each measurement of C after the first is uniquely determined by the outcome of the first measurement of C together with the outcomes $b_0^{(1)}, \dots, b_0^{(g)}$ of measurements of an additional set of g quantities, $0 \le g < \infty$, performed (say) at the initial time.

This is actually what Caves *et al.* [12] call a *generalized* QNDO. The additional g measurements are required to deal with the case that C depends, in its Heisenberg evolution, on g observables $B_1, \dots, B_g$ which commute with one another and with the initial value of C. It is assumed that there has been no preparation of the state of the system prior to the first measurement.

Remark 1. From Definition 1 and the preceding discussion, we see immediately that a necessary and sufficient condition for $C(t)$ to be a QNDO is that after the initial measurement(s) the system remains in an eigenstate of C, the eigenvalue $c = c(c_0, b_0^{(1)}, \dots, b_0^{(g)}; t, t_0)$ being determined uniquely as a function of time by the outcome $(c_0, b_0^{(1)}, \dots, b_0^{(g)})$ of the first measurement(s). Accordingly we may state this essential property of a QNDO: there is no dispersion in any subsequent measurement of C, the expected value $y(t) = \langle \psi(t) | C(t) \psi(t) \rangle$ being coincident with the certain outcome $c(c_0, b_0^{(1)}, \dots, b_0^{(g)}; t, t_0)$. In this sense a QNDO may be thought of as a "quasiclassical" observable. We note that in the case of QNDO's, measurements may be performed and predictions given for a *single* system, without reference to an ensemble.

Alternative statements of necessary and sufficient conditions have been given by Caves *et al.* [12], in terms of the Heisenberg picture of quantum dynamics: The observable C is a (generalized) QNDO if and only if

$$\left[C_H(t), C_H(t')\right] = 0, \qquad \forall t, t', \tag{9}$$

or, by Taylor expansion of this condition, if and only if

$$\left[C_H(t), \frac{d^n C_H(t)}{dt^n}\right] = 0, \qquad \forall t, \qquad \forall n = 0,1,2,\dots. \tag{9'}$$

For the case that $\partial H / \partial t = 0$, thus $u(t) \equiv 0$, the dynamical law (4) may be invoked to reduce (9′) to the explicit form

$$\left[C(t), \sum_{l=0}^{n}\left(\frac{i}{\hbar}\right)^{n-l}\binom{n}{l}\mathrm{ad}_{H_0}^{n-l}\frac{\partial^l C(t)}{\partial t^l}\right] = 0, \qquad \forall t, \qquad \forall n, \tag{10}$$

which, by virtue of the unitarity of the transformation (5), applies in both Schrödinger and Heisenberg pictures. (As in [1], we use the common notation $\mathrm{ad}_X^\kappa Y = [X, \mathrm{ad}_X^{\kappa-1} Y]$, where κ is a positive integer and $\mathrm{ad}_X^0 Y = Y$.) Specializing further to the case that $\partial C(t)/\partial t = 0$, relation (10) becomes

$$\left[C, \mathrm{ad}_{H_0}^n C\right] = 0, \qquad \forall n, \tag{11}$$

which is just the QNDO condition given by Unruh.

For C to be a QNDO it is evidently sufficient (but not necessary!) that C be a constant of motion of the system, $dC_H(t)/dt = 0$.

We have already encountered examples of QNDO's for the free particle (momentum p) and the linearly-coupled oscillator (real and imaginary parts X_1, X_2 of the complex amplitude). Another example for the oscillator problem (but with *quadratic* coupling of the field to the coordinate x) is the number of quanta [17, 18], $N = a^\dagger a$, where $a^\dagger$ and a are, respectively, the creation and annihilation operators for oscillator quanta, with $a \triangleq (m\omega/2\hbar)^{1/2}(x + ip/m\omega)$. All of these examples are in fact constants of the motion for the corresponding *isolated* ($u(t) = 0$) systems.

3.2. Interaction of the System with the Measuring Apparatus

So far we have said nothing about the measuring apparatus and how it interacts with the system. In [12] Caves *et al.* showed that, so far as the formal development is concerned, this interaction can be ignored—provided only that the following demands on the measuring apparatus are met.

(a) The measuring instrument responds to C and hence the interaction Hamiltonian H_I depends on C as well as one or more dynamical quantities referring to the measuring apparatus alone.

(b) The measuring instrument does not respond to observables of the system other than C.

Most simply, we might assume $H_I = kCM$, where k is a coupling constant and M is some observable of the measuring apparatus.

Proposition 1 [12]. *Given that conditions* (a) *and* (b) *are fulfilled, the evolution of a* QNDO C *in the Heisenberg picture is* completely unaffected *by the interaction with the measuring instrument.*

What this proposition means is that the expectation and variance of C evolve during and after measurement just as if the measuring apparatus is disconnected. Appealing to this proposition, we shall suppress any details of the interaction H_I in all further discussion. Of course, in practice these details are extremely important. Given an arbitrary H_I satisfying (a) and (b) it may be difficult or impossible to realize, in hardware, a suitable measuring apparatus—as for example when the coupling is to a polynomial in a and $a^\dagger$.

4. Quantum Nondemolition Filters

Quantum nondemolition measurements were first conceived as a possible means for detecting gravitational waves, propagated through space from energetic cosmic sources (supernovae, neutron-star binaries, etc.). The existence of gravity waves is predicted by Einstein's general relativity; the difficulty of detecting them lies in the weakness of the coupling to material detectors and the great distances of proposed sources. The considerations of Section 3 may be brought to bear if one imagines that the gravitational wave couples (weakly) to a quantum harmonic oscillator (in practice, a resonant bar of macroscopic size and suitable material). The Hamiltonian for this detection system takes the standard form $H = H_0 + u(t)H_1$, where $u(t)$ represents the unknown classical gravitational signal, H_0 is the Hamiltonian for the free one-dimensional oscillator, and H_1 is the oscillator coordinate x. Another practical context in which QND ideas may be useful is optical communication. In that example the components of the standard form of H may assume the following interpretations: $H_0 = \hbar\omega(a^\dagger a + \frac{1}{2})$ is the Hamiltonian of the unperturbed, single-mode radiation field, H_1 is a function of a and $a^\dagger$ depending on the modulation scheme adopted, and $u(t)$ is the impressed optical signal.

Consider now the problem of signal detection and reconstruction for one or another such system, in terms of the results of measurement of a dynamical quantity $\mathscr{C}$. If, in the absence of any signal ($u(t) \equiv 0$) the corresponding observable C is a QNDO, then by definition the results of a sequence of precise measurements of $\mathscr{C}$ will be exactly predictable from the results of an initial set of measurements. If C remains a QNDO in the presence of the $u(t)$ term in the system Hamiltonian, then there is the possibility of determining $u(t)$ by the changes it produces in the otherwise precisely predictable results of a series of $\mathscr{C}$-measurements. Clearly, not all QNDO's will permit $u(t)$ to be monitored in this way. QNDO's for which this possibility *can* be realized are embraced by the following definition.

Definition 2. An observable C is a *quantum nondemolition filter* (QNDF) iff in the presence of an arbitrary analytic signal a sequence of measurements of C can reveal with arbitrary accuracy the time-dependence of the signal.

We may already observe—trivially—that a QNDF must be a QNDO in the presence (or absence) of the external force or signal. A further requirement cited by Caves *et al.* [12] is that measurements of C can be carried out at arbitrarily closely-spaced times. This latter requirement may actually be more stringent than necessary, since interpolating functions can be used to reconstruct the output $y(t)$ provided $y(t)$ has a finite spectrum and its Nyquist rate does not exceed the sampling rate [19]. Given that C is a QNDF, the signal $u(t)$ can be recovered by suitably processing the measurement outcomes of C furnished by $y(t)$; usually one expresses $u(t)$ in terms of an appropriate derivative of $y(t)$. This and other aspects of the invertibility problem were fully considered in [1].

5. Main Results: Conditions for a Quantum Nondemolition Filter

The QNDO condition (9) stated by Caves *et al.* [12], together with the invertibility criteria (Theorem 1 and Corollary 1.1) established in [1], provide necessary and/or sufficient conditions for an observable C to be a QNDF. We prefer here to replace (9) by a QNDO condition framed in terms of the operators C_k which played such an important role in our treatment [1] of the invertibility problem.

Unless otherwise noted, we shall adhere to the Schrödinger picture of dynamics. To conform with [1] and with the systems-theory literature generally, it is convenient to define skew-adjoint operators $\hat{H}_0 = -iH_0$ and $\hat{H}_1 = -iH_1$. Our state-evolution equation (1) is then rewritten as

$$\frac{d\psi(t)}{dt} = [\hat{H}_0 + u(t)\hat{H}_1]\psi(t), \qquad \psi(t) \in S_{\mathscr{H}}. \tag{$\hat{1}$}$$

(Here, and henceforth, Dirac's constant $\hbar$ is set equal to unity.) From now on we shall always work with the new operators $\hat{H}_0$ and $\hat{H}_1$ (rather than H_0 and H_1); *for notational simplicity the carets will be omitted.*

The assumptions on control system ($\hat{1}$)–(2) which were listed in Section 3 of [1] will again be adopted. In summary, these run as follows: The observable C is supposed to have the structure $C(t) = \sum_{r=1}^{q<\infty} \gamma_r(t) iQ_r$, where the functions $\gamma_r(t)$ are real analytic in t and the Q_r are time-independent skew-adjoint operators. To ensure a well-defined output, we take $\operatorname{dom} C \supset \operatorname{dom} H_0 \cap \operatorname{dom} H_1$. We assume that there exists an analytic domain $\mathscr{D}_\omega$ (as defined by Nelson [5]) for the Lie algebra $\mathscr{A}'$ generated by $H_0, H_1, Q_1, \ldots, Q_q$ under the bracket operation $[A, B]$. It is further assumed that the tangent space $\mathscr{A}(\phi) \triangleq \{X(\phi), X \in \mathscr{A}\}$ of the Lie algebra $\mathscr{A}$ generated by H_0, H_1 has constant, finite dimension for all $\phi \in \mathscr{D}_\omega$.

Herein we shall in fact impose the stronger restriction that the dimension of the Lie algebra $\mathscr{A}'$ is finite.

Let $\mathscr{E}_C$ be the subspace of $\mathscr{H}$ spanned by the union over all $t \in \mathbb{R}$ of the set of all eigenstates of $C(t)$. According to Remark 1, an essential prerequisite for $C(t)$ to be a QNDO is that (after the prescribed initializing measurement of C and other observables), the state $\psi(t)$ of the system remain in the subspace $\mathscr{E}_C$. Further, since the analytic domain $\mathscr{D}_\omega$ is dense in the unit sphere $S_{\mathscr{H}}$, it is sufficient to carry out the analysis on $\mathscr{D}_\omega \cap \mathscr{E}_C$. For $\psi(t) \in \mathscr{D}_\omega$, the map $t \to C(t)\psi(t)$ is analytic in $\mathbb{R}$ and therefore the function $y(t) = \langle \psi(t) | C(t)\psi(t) \rangle$ is also analytic for all real t. If $\psi(t)$ is *outside* $\mathscr{D}_\omega$, we *approximate* it by some $\psi_\omega(t) \in \mathscr{D}_\omega$ (see [1] for details, especially Remark 2 of that paper), in such a way that we may always regard $y(t)$ as analytic in t, $\forall t \in \mathbb{R}$. This ensures that complete information for constructing the output at some future time t is contained in the time derivatives of the output function, evaluated at the fiducial time t_0.

Again referring to Remark 1: for C to be a QNDO, the variance associated with $y(t)$ must be zero for all t. This in turn implies certain commutation relations between $H_0, H_1, C(t)$ and the partial derivatives of $C(t)$. In explicating these relations we make use of the following definition and lemmas.

Definition 3. For any positive integer k, a family $\Lambda^{[k]}$ of operators is specified recursively by

$$\Lambda^{[k]} = \begin{cases} \text{the single element } C_k, & 0 \le k < \mu, \\ \{\dot{L} + [L, H_0], [L, H_1]\colon L \in \Lambda^{[k-1]}\}, & k \ge \mu, \end{cases}$$

where

$$C_k = [C_{k-1}, H_0] + \frac{\partial C_{k-1}(t)}{\partial t}, \qquad C_0(t) = C(t).$$

The index μ is the relative order of the system as defined in [1], i.e., the least positive integer k such that $[C_{k-1}(t), H_1] \neq 0$ (for almost all t).

We also construct the set Λ as the union of all the $\Lambda^{[k]}$. The elements of Λ, being self-adjoint operators in $\mathscr{H}$, are presumed to define physically measurable quantum-mechanical observables [2].

Lemma 1. *Given that $y(t) = \langle \psi(t) | C(t) \psi(t) \rangle$ is analytic in t, the nth time derivative of $y(t)$ is expressible as*

$$y^{(n)}(t) = \left\langle \psi(t) \middle| \sum_{k \le n} \sum_{p_k} \lambda(n, k, p_k; t) L_{p_k}^{[k]}(t) \psi(t) \right\rangle,$$

where $L_{p_k}^{[k]} \in \Lambda^{[k]}$ and the coefficients $\lambda(n, k, p_k)$ with $p_k = 1, 2, \ldots, m_k$, are real. The $\lambda(n, k, p_k; t)$ depend on $u(t)$ and its derivatives when $n \ge \mu$.

Proof. (By induction.) The assumed analyticity of $y(t)$ implies that all its derivatives exist. There are two cases to examine, the case $n = 0$ being trivial.

Case 1. $1 \le n < \mu$. Noting that $\dot{C} \equiv dC/dt$ is simply $\partial C/\partial t$ in the Schrödinger picture, we have

$$\begin{aligned} y^{(1)}(t) &= \langle \psi | \dot{C} \psi \rangle + \langle \dot{\psi} | C \psi \rangle + \langle \psi | C \dot{\psi} \rangle \\ &= \langle \psi | C_1 \psi \rangle. \end{aligned}$$

Assume $y^{(n-1)} = \langle \psi | C_{n-1} \psi \rangle$. Then

$$\begin{aligned} y^{(n)}(t) &= \langle \psi | \dot{C}_{n-1} \psi \rangle + \langle \dot{\psi} | C_{n-1} \psi \rangle + \langle \psi | C_{n-1} \dot{\psi} \rangle \\ &= \langle \psi(t) | C_n(t) \psi(t) \rangle. \end{aligned}$$

This certainly has the desired structure, since $C_n \in \Lambda^{[n]}$.

Case 2. $n \geq \mu$. We have $y^{(\mu-1)} = \langle \psi | C_{\mu-1} \psi \rangle$ from Case 1. Then

$$y^{(\mu)}(t) = \left\langle \psi | \left\{ C_\mu + u(t)\left[C_{\mu-1}, H_1 \right] \right\} \psi \right\rangle,$$

which is also of the required form. Next assume

$$y^{(n-1)}(t) = \left\langle \psi(t) \middle| \sum_{k \leq n-1} \sum_{p_k} \lambda(n-1, k, p_k; t) L_{p_k}^{[k]}(t) \psi(t) \right\rangle, \quad n-1 \geq \mu.$$

Then, differentiating and appealing to the Schrödinger equation,

$$\begin{aligned} y^{(n)}(t) = {} & \left\langle \psi \middle| \sum_{k \leq n-1} \sum_{p_k} \dot{\lambda}(n-1, k, p_k) L_{p_k}^{[k]} \psi \right\rangle \\ & + \left\langle \psi \middle| \sum_{k \leq n-1} \sum_{p_k} \lambda(n-1, k, p_k) \right. \\ & \left. \times \left\{ \dot{L}_{p_k}^{[k]} + \left[L_{p_k}^{[k]}, H_0 \right] + u(t)\left[L_{p_k}^{[k]}, H_1 \right] \right\} \psi \right\rangle. \end{aligned}$$

With proper choice of the $\lambda(n, k, p_k)$, this is expressible as

$$y^{(n)}(t) = \left\langle \psi(t) \middle| \sum_{k \leq n} \sum_{p_k} \lambda(n, k, p_k; t) L_{p_k}^{[k]}(t) \psi(t) \right\rangle, \qquad \square$$

and the lemma is established.

Lemma 2. *If* $[C, L] = 0, \forall L \in \Lambda, \forall t,$ *then*

$$[L, M] = 0, \qquad \forall L, M \in \Lambda.$$

Proof. Again we give a case-by-case analysis.

Case 1. $i, j \leq \mu - 1$. Let $f(t) = \langle \psi(t) | [C, C_j] | \psi(t) \rangle = 0 \ \forall j$ and for arbitrary $\psi(t)$ satisfying the Schrödinger equation. Since $\psi(t)$ obeys Eq. (1), its first derivative is guaranteed to exist, and

$$f'(t) = \left\langle \psi(t) | \left\{ \left[C_1, C_j \right] + \left[C, C_{j+1} \right] \right\} \psi(t) \right\rangle = 0.$$

However, $[C, C_{j+1}] = 0$ by assumption, since $C_{j+1} \in \Lambda^{[j+1]} \subset \Lambda$. Thus, upon appealing to the fact that the (unnormalized) Schrödinger solutions are dense in $\mathscr{H}$ at any t, we may conclude that

$$\left[C_1, C_j \right] = 0.$$

Suppose next that $[C_i, C_j] = 0 \ \forall j \leq \mu - 1$ and fixed $i \leq \mu - 1$. Then, differentiating in the strong sense for $\psi(t)$ specified as above,

$$\left[C_{i+1}, C_j \right] + \left[C_i, C_{j+1} \right] = 0.$$

If either $i+1$ or $j+1$ is equal to μ, see Case 2; otherwise $[C_i, C_{j+1}]=0$ by assumption and hence

$$\left[C_{i+1}, C_j\right] = 0.$$

By induction, $[C_i, C_j]=0\ \forall i, j \le \mu-1$.

Case 2. $i \le \mu-1,\ j \ge \mu$. By assumption, $[C, L]=0\ \forall L \in \Lambda^{[j]}$. Differentiating in the strong sense, we have

$$[C_1, L]+[C, \dot{L}+[L, H_0]+u(t)[L, H_1]] = 0.$$

Since $\dot{L}+[L, H_0]$ and $[L, H_1]$ belong to $\Lambda^{[j+1]}$, we have

$$[C_1, L] = 0,$$

initializing the induction process. Assuming $[C_i, L]=0\ \forall L \in \Lambda^{[j]}$ and i arbitrary (but $\le \mu-1$), similar considerations establish that

$$[C_{i+1}, L] = 0, \qquad \forall i+1 \le \mu-1.$$

For $i+1 \ge \mu$, refer to Case 3. By induction, $[C_i, L]=0$ holds with $C_i \in \Lambda^{[i]}$ for any $i \le \mu-1$, $L \in \Lambda^{[j]}$ for any $j \ge \mu$.

Case 3. $i, j \ge \mu$. From Case 2, we have $[C_{\mu-1}, L]=0$ for $L \in \Lambda^{[k]}$, any $k \ge \mu$. Differentiating in the strong sense,

$$\left[C_\mu + u(t)\left[C_{\mu-1}, H_1\right], L\right] + \left[C_{\mu-1}, \dot{L}+[L, H_0]+u(t)[L, H_1]\right] = 0.$$

The second term is zero according to Case 2. By Jacobi's identity,

$$\left[\left[C_{\mu-1}, H_1\right], L\right] = \left[C_{\mu-1}, [H_1, L]\right] + \left[H_1, \left[L, C_{\mu-1}\right]\right].$$

But $[H_1, L] \in \Lambda^{[j+1]}$ and $[L, C_{\mu-1}]=0$ by Case 2. Therefore

$$\left[\left[C_{\mu-1}, H_1\right], L\right] = 0.$$

A similar argument shows that $[C_{\mu+1}, L]=0$.

Next we suppose that $[L, M]=0\ \forall L \in \Lambda^{[i]}$, $\forall M \in \Lambda^{[j]}$, for fixed i and any $j \ge \mu$. Then computations similar to those above establish that

$$[L, M] = 0, \qquad \forall L \in \Lambda^{[i+1]}, \qquad \forall M \in \Lambda^{[j]} \quad \text{for any } j \ge \mu.$$

Therefore

$$[L, M] = 0, \qquad \forall L, M \in \Lambda^{[i]}, \quad \text{any } i \ge \mu,$$

by induction. The proof is now complete. □

Lemma 3. *Let* $C(s)=\sum_{n=0}^{\infty} s^n C_{(n)}$, *where to be definite we assume that* $C_{(n)}$, *for* $n=1,2,\ldots$, *has the same domain as the "unperturbed" self-adjoint operator* $C_{(0)}$. *Then* $(\triangle C)^2=0$ iff $(\triangle C_{(n)})^2=0$, $n=0,1,2,\ldots$.

Proof. We shall use $\langle A\rangle$ to denote the expected value of an observable A when measurements are carried out on an ensemble of identical systems prepared in state ψ. The associated standard deviation, or variance, is denoted $\triangle A$.

We begin the proof by writing out

$$\begin{aligned}(\triangle C)^2 &= \langle C^2\rangle - \langle C\rangle^2\\ &= \sum_n s^{2n}\left(\langle C_{(n)}^2\rangle - \langle C_{(n)}\rangle^2\right) + \sum_{\substack{i,j\\ i\neq j}} s^{i+j}\left(\langle C_{(i)}C_{(j)}\rangle - \langle C_{(i)}\rangle\langle C_{(j)}\rangle\right)\\ &= \sum_n s^{2n}\left\{\langle C_{(n)}^2\rangle - \langle C_{(n)}\rangle^2 + \sum_{i+j=2n} \langle C_{(i)}C_{(j)}\rangle - \langle C_{(i)}\rangle\langle C_{(j)}\rangle\right\}\\ &\quad + \sum_{n\,\mathrm{odd}} s^n\left\{\sum_{i+j=n} \langle C_{(i)}C_{(j)}\rangle - \langle C_{(i)}\rangle\langle C_{(j)}\rangle\right\}.\end{aligned}$$

(Sufficiency). If all $(\triangle C_{(n)})^2=0$, $n=0,1,2,\ldots$, then ψ is a simultaneous eigenstate of $C_{(0)}, C_{(1)}, C_{(2)},\ldots$, i.e., $C_{(n)}\psi=c_{(n)}\psi\ \forall n$. Therefore all cross-correlation terms disappear,

$$\langle C_{(i)}C_{(j)}\rangle - \langle C_{(i)}\rangle\langle C_{(j)}\rangle = c_{(i)}c_{(j)} - c_{(i)}c_{(j)} = 0 \qquad \forall i,j,$$

and

$$(\triangle C)^2 = 0.$$

(Necessity). If $(\triangle C)^2=0$, then the coefficients of s^n must vanish $\forall n$. Equating the coefficient of s^0 to zero implies that $(\triangle C_{(0)})^2=0$; therefore ψ is an eigenstate of $C_{(0)}$. This in turn implies that the coefficient of s^1 vanishes, since

$$\begin{aligned}&\langle C_{(1)}C_{(0)}\rangle - \langle C_{(1)}\rangle\langle C_{(0)}\rangle + \langle C_{(0)}C_{(1)}\rangle - \langle C_{(0)}\rangle\langle C_{(1)}\rangle\\ &\quad = c_{(0)}\langle C_{(1)}\rangle - c_{(0)}\langle C_{(1)}\rangle + c_{(0)}\langle C_{(1)}\rangle - c_{(0)}\langle C_{(1)}\rangle = 0.\end{aligned}$$

A similar computation shows that the coefficient of s^2 is then given by $(\triangle C_{(1)})^2$, which must of course vanish.

In general, $(\triangle C_{(m)})^2=0$ for $m\le N$ implies

(i) $(\triangle C_{(N+1)})^2=0$,

(ii) $\langle C_{(i)}C_{(j)}\rangle - \langle C_{(i)}\rangle\langle C_{(j)}\rangle = 0$ for $i\neq j$, $i+j\le 2N+1$.

To establish (i), recall that the coefficient of $s^{2(N+1)}$ in $(\triangle C)^2$ is given by

$$(\triangle C_{(N+1)})^2 + \sum_{\substack{i+j=2N+2\\ i\neq j}} \left(\langle C_{(i)}C_{(j)}\rangle - \langle C_{(i)}\rangle\langle C_{(j)}\rangle\right).$$

Since $i+j=2N+2$ and $i \neq j$, at least one of the indices i or j must be less than or equal to N. But ψ is an eigenstate of $C_{(k)}$, $k \leq N$. Therefore the sum of the cross-correlation terms vanishes and $(\triangle C_{(N+1)})^2 = 0$. We establish (ii) by a similar argument: $i+j \leq 2N+1$ implies that at least one of the indices i or j is less than or equal to N and hence the cross-correlation terms disappear. □

Theorem 1. *The observable C qualifies as a* QNDF *for the quantum control system* (1)–(2) *iff*

(a) *the system is invertible, and*
(b) *the commutation relations*

$$[C(t), L(t)] = 0 \tag{12}$$

hold for all $L \in \Lambda$, $\forall t$.

Proof. (Necessity). If the system is *not* invertible, then treating it as a classical system, it is not possible to retrieve $u(t)$ from the output $y(t)$. Treating it as a quantum system, the measurement process is irreversible [9], and we cannot hope to do better than in the classical case.

Now consider that, according to Lemma 1,

$$y^{(n)}(t) = \left\langle \psi(t) \middle| \sum_{k \leq n} \sum_{p_k} \lambda(n,k,p_k;t) L_{p_k}^{[k]}(t) \psi(t) \right\rangle,$$

where $L_{p_k}^{[k]} \in \Lambda^{[k]}$ and $\lambda(n,k,p_k;t)$ generally depends on $u(t)$ and its derivatives. Suppose C is a QNDO but condition (b) of Theorem 1 is *not* met. To be explicit, suppose there is some $L_{p_k}^{[k]}(t)$, for some $k \leq n$ and some p_k, which does not commute with $C(t)$, at some time t which we may take as the time t_0 of our initial measurement of C. It follows that, in general (cf. Remark 4 below),

$$\left[C(t_0), \sum_{k \leq n} \sum_{p_k} \lambda(n,k,p_k;t_0) L_{p_k}^{[k]}(t_0) \right] \neq 0.$$

Consequently, there will exist eigenstates of $C(t_0)$ which are not simultaneous eigenstates of

$$\sum_{k \leq n} \sum_{p_k} \lambda(n,k,p_k;t_0) L_{p_k}^{[k]}(t_0)$$

(cf. Remark 5 below). For such states the dispersion in the results of measurement of the latter observable, performed at time t_0, will not vanish. Therefore we may achieve a contradiction by appealing to Lemma 3, which applies to the power series

$$C(t) = \sum_{n=0}^{\infty} (t-t_0)^n C_{(n)}, \qquad t > t_0, \tag{13}$$

where

$$C_{(n)} = \frac{1}{n!} \sum_{k \leq n} \sum_{p_k} \lambda(n, k, p_k; t_0) L_{k_p}^{[k]}(t_0).$$

This series corresponds to the Taylor expansion

$$y(t) = \sum_{n=0}^{\infty} \frac{y^{(n)}(t_0)}{n!}(t-t_0)^n, \qquad t > t_0.$$

Since there exists a term in (13) for which the dispersion of measurement results at t_0 does not vanish, we must conclude that measurements of $C(t)$ itself are subject to dispersion at $t > t_0$. Hence the future values of $y(t)$ are merely expected values of C and C cannot be a QNDO.

(Sufficiency). If condition (b) does hold, then by Lemma 2, the observable C and the other members of the set Λ form a family of mutually commuting observables. Moreover, since the Lie algebra $\mathscr{A}'$ is finite-dimensional by assumption, we may construct any member of this family as a linear combination of the elements of a finite set $\tilde{\Lambda}$ of mutually commuting observables including C. By a theorem of von Neumann [9, pp. 173–175], there exists a basis of eigenstates common to all of the members of $\tilde{\Lambda}$, and hence common to all elements of Λ. Thus a finite number of measurements carried out at the initial instant t_0 will suffice to prepare the system in a simultaneous eigenstate ψ of C and the other observables belonging to $\tilde{\Lambda}$. In such a state there would, in particular, be no dispersion in the results of measurements of the $C_{(n)}$, i.e., $(\Delta C_{(n)})^2 = 0$, $n = 0, 1, 2, \ldots$. By Lemma 3, future measurements of C would then be dispersion-free; their results may be computed from the power series (13), the derivatives $y^{(n)}(t_0)$ being given by the scalar products $\langle \psi | n! C_{(n)} | \psi \rangle$, which in this case are just the eigenvalues of the $n! C_{(n)}$. It is then clear that C is a QNDO.

If in addition to (b) the system is invertible, then by Theorem 1 of [1] there exists an inverse system such that with $y(t)$ as input, $u(t)$ is obtained as the output of the inverse system. Consequently C is a QNDF. □

Corollary 1.1. *If C is a time-independent observable, it is a* QNDF iff

(a) $[C, \mathrm{ad}_{H_0}^{k-1} H_1] \neq 0$ *for some positive integer* $k < \infty$,

(b) $\left[C, \mathrm{ad}_{H_0}^{k-1} C\right] = 0$ *and* $\left[C, \left[\mathrm{ad}_{H_0}^{k-1} C, H_1\right]\right] = 0 \ \forall k > 0.$ (14)

Proof. This result follows from Theorem 1 above together with Corollary 1.2 of [1], after simplifying Definition 3. □

Remark 2. The requirement that the relative order of the system be finite ensures that C does respond to $u(t)$. When $u \equiv 0$, conditions (12) and (14) reduce, respectively, to condition (10) of Caves *et al.* and Unruh's condition (11). (Of course, the general condition (12) is ultimately equivalent to (9).)

Remark 3. As we mentioned earlier, if $\psi(t)$ is not in the analytic domain $\mathscr{D}_\omega$, it can be approximated by $\psi_\omega(t) \in \mathscr{D}_\omega$. (This possibility rests on the fact [1, 20] that $\psi(t)$ belongs to the closure of $R_p^t(\phi)$, the reachable set of $\psi(0) = \phi$ at time t for piecewise-constant controls.) The corresponding approximation $y_\omega(t) = \langle \psi_\omega(t) | C(t) \psi_\omega(t) \rangle$ to the measurement result $y(t) = \langle \psi(t) | C(t) \psi(t) \rangle$ converges uniformly to $y(t)$. This can be seen from the following series of manipulations, the Cauchy-Schwarz inequality being invoked in the last step:

$$\begin{aligned}
|y - y_\omega| &= |\langle \psi | C\psi \rangle - \langle \psi_\omega | C\psi_\omega \rangle| \\
&= |\langle \psi | C\psi \rangle - \langle \psi_\omega | C\psi \rangle + \langle \psi_\omega | C\psi \rangle - \langle \psi_\omega | C\psi_\omega \rangle| \\
&= |\langle \psi - \psi_\omega | C\psi \rangle + \langle \psi_\omega | C(\psi - \psi_\omega) \rangle| \\
&\leq |\langle \psi - \psi_\omega | C\psi \rangle| + |\langle \psi_\omega | C(\psi - \psi_\omega) \rangle| \\
&\leq \|\psi - \psi_\omega\|^2 \|C\psi\|^2 + \|C\psi_\omega\|^2 \|\psi - \psi_\omega\|^2.
\end{aligned}$$

Since $\|C\psi\|$ and $\|C\psi_\omega\|$ are bounded, and since $\psi_\omega \to \psi$ uniformly, $y_\omega \to y$ uniformly.

Remark 4. In the proof of the theorem (necessity of the stated conditions), we did not consider the possibility that, with special choices of u, it may happen that $\Sigma_{k \leq n} \Sigma_{p_k} \lambda(n, k, p_k)[C, L_{p_k}^{[k]}] = 0$ even if $[C, L_{p_k}^{[k]}] \neq 0$ for some k, p_k. We exclude this case since we are seeking a set of *practical* criteria which may be applied for arbitrary $u(t)$ in the given class. The indicated possibility can be ruled out with another choice of $u(t)$.

Remark 5. Again referring to the necessity proof, note that $[C, L_{p_k}^{[k]}] \neq 0$ does not exclude the possibility that the observables C and $L_{p_k}^{[k]}$ share common eigenstates, only that these common eigenstates do not constitute a complete set. Obviously, if one prepares the system in a state which is an eigenstate of C but not $L_{p_k}^{[k]}$, dispersion in future measurement results of C will occur. On the other hand, if the system is prepared as a simultaneous eigenstate of C and all the L's including the particular $L_{p_k}^{[k]}$ in question (assuming that is ever possible in the face of Lemma 2), then indeed observations will be QND—and observable C qualifies as a partial QNDO. It is clear that satisfaction of the conditions needed to implement such a partial QNDO is highly problematic.

6. Examples

Next we take up some simple examples which illuminate our main results.

Example 1. (Electrooptic Amplitude Modulation.) Consider the Hamiltonian

$$H = \omega a^\dagger a + iu(t)(a^\dagger - a)$$

defined on the Schwarz space $\mathscr{S}(\mathbb{R})$ of infinitely differentiable functions, where a

and $a^\dagger$ are the annihilation and creation operators introduced previously. The set of finite linear combinations of the Hermite functions

$$\psi_n(x') = \pi^{-1/4}(n!)^{1/2}(-1)^n 2^{-n/2}\exp(-x'^2/2)h_n(x'),$$
$$x' \in \mathbb{R}, \qquad n = 0,1,2,\ldots,$$

where $h_n(x')$ is the nth-order Hermite polynomial, provides a dense set of analytic vectors invariant under a, $a^\dagger$ and $a^\dagger a$ (see [21] for details). We note that $iH_0 = \omega a^\dagger a$ is just the unperturbed oscillator Hamiltonian, apart from an additive constant $\omega/2$, while $iH_1 = (2/m\omega)^{1/2}p$. (The reader should recall that H_0 and H_1 have been redefined as the skew-adjoint caret operators of Eq. ($\hat{1}$), Section 5.)

For the Hamiltonian H, the observable

$$C = ae^{i\omega t} + a^\dagger e^{-i\omega t} \tag{15}$$

is a QNDF according to the criteria laid down in Theorem 1 of Section 5. Explicitly:

(i) We have $[C, H_1] = e^{i\omega t} + e^{-i\omega t} = 2\cos\omega t \neq 0$ except at $t = \ell\pi/2$, $\ell = 1,3,5,\ldots$. Consequently the invertibility condition (a) of Theorem 1 is met (see [1]).

(ii) We have $C_1 = [C, H_0] + \partial C/\partial t = 0$, and therefore $C_j = 0$ $\forall j \geq 1$. From this it is seen that condition (b), i.e., Eq. (12), is satisfied.

As a check on the claim that C is a QNDO, we may integrate the Heisenberg equation of motion for $C_H(t)$, beginning at time t_0. The result is

$$C_H(t) = 2C_H(t_0)\cos\omega t_0 + 2I\int_{t_0}^{t} u(s)\cos\omega s\, ds.$$

Hence an eigenstate of $C_H(t_0)$ remains an eigenstate of $C_H(t)$, $\forall t > t_0$, and C is indeed a QNDO. The input $u(t)$ is obtained trivially as

$$u(t) = \frac{dy(t)}{dt}(2\cos\omega t)^{-1};$$

avoiding the zeros of $\cos\omega t$, we can reconstruct $u(t)$ from $y(t)$. We therefore conclude that C is a QNDF as well as a QNDO.

For the given Hamiltonian, Baras [7] arrived at the following observable as a representation of the receiver structure:

$$\tfrac{1}{2}(a + a^\dagger). \tag{16}$$

Working in the discrete-time case, this form was obtained after making assumptions which permit the choice of optimal quantum observables for measurement to be separated from the choice of optimal classical post-processing of the measurement outcomes (in this case Kalman filtering). Baras *et al.* [6] call this *filter separation*. Within the QND approach, filter separation is quite natural.

Choosing an optimal quantum observable corresponds to choosing a QNDF, and classical postprocessing is the invertibility procedure. Note that (16), aside from a multiplicative constant, is a special case of (15) at times $t = 0, \pi, 2\pi, \ldots$.

Example 2. Next consider the Hamiltonian

$$H = J^0 + u(t)(J^+ + J^-),$$

where (suppressing dimensional constants)

$$J^0 = (-d^2/dx^2 + x^2)/2, \qquad J^{\pm} = \pm(2)^{-1/2}(d/dx \mp x).$$

The operators $J^0, J^{\pm}$ have the following commutation relations:

$$[J^0, J^{\pm}] = \pm J^{\pm}, \qquad [J^+, J^-] = -I.$$

Moreover, they share a common dense invariant domain of analytic vectors—indeed, just that involved in the first example. As in Example 1, the given Hamiltonian may be interpreted as that of a simple harmonic oscillator coupled to an external classical force; however, in the present example the coupling is through the position operator x rather than the momentum operator p.

From the commutation relations, we claim that either J^+ or J^- qualifies as a QNDF if we widen the definition of QNDF to allow C to be non-selfadjoint. (Of course, such operators do not in general correspond to physically measurable quantities, and accordingly are not termed observables.) Making the obvious identifications,

$$H_0 = -iJ^0, \qquad H_1 = -i(J^+ + J^-), \qquad C = J^{\pm},$$

we readily find that

$$\text{(i) } [C, H_1] = -i[J^{\pm}, J^+ + J^-] = \pm iI$$

and

$$\text{(ii) } [C, H_0] = -i[J^{\pm}, J^0] = \pm iJ^{\pm}.$$

From (i) it follows that the relative order is finite and the system is invertible; from (i) and (ii) it follows that $[C, \mathrm{ad}_{H_0}^{k-1}C] = 0\ \forall k$ and $[C, [\mathrm{ad}_{H_0}^{k-1}C, H_1]] = 0\ \forall k$. Therefore according to Corollary 1.1 of the last section, C is a QNDF (in the wider sense).

In the Heisenberg picture, $C = J^{\pm}$, denoted $C_H(t)$, evolves according to

$$\begin{aligned}\frac{dC_H(t)}{dt} &= i[J_H^0, C_H] + iu(t)[J_H^+ + J_H^-, C_H] \\ &= iC_H(t) + iu(t)I.\end{aligned}$$

Hence the input may be (formally) retrieved as

$$u(t) = y(t) - i\frac{dy(t)}{dt}.$$

Example 3. As another example, consider the Hamiltonian (again with dimensional constants suppressed)

$$H = p^2/2 + u(t)(px + xp), \tag{17}$$

where p and x are respectively the momentum and position operators for a one-dimensional quantum system. The latter operators possess a common, dense, invariant domain of analytic vectors, which may be constructed according to the same recipe as in Examples 1 and 2 [21]. Here $H_0 = -ip^2/2$, $H_1 = -i(px + xp)$, and we assert that $C = p^2$ is a QNDF. The invertibility condition is satisfied by virtue of

$$[C, H_1] = -i[p^2, px + xp] = -4p^2 \neq 0.$$

Furthermore, $[C, H_0] = 0$ and hence $[C, \mathrm{ad}_{H_0}^{k-1}C] = 0$, $[C, [\mathrm{ad}_{H_0}^{k-1}C, H_1]] = 0$, $\forall k$. In the Heisenberg picture p^2 evolves according to

$$\frac{dp_H^2(t)}{dt} = -4u(t)p_H^2.$$

Thus the input can be extracted via

$$u(t) = -\frac{1}{4}\frac{1}{y(t)}\frac{dy(t)}{dt}. \tag{18}$$

Note that the choice of $C = p$ would have provided an equally good QNDF, leading to (18) with a factor $\frac{1}{2}$ in place of $\frac{1}{4}$.

Example 4. We may elaborate on Example 3 by adding a term $+\lambda x$ to H of (17), where λ is a real constant; the modified version of iH_0 corresponds for example to a particle in a uniform gravitational field. Again both p and p^2 qualify as QNDF's. These operators remain QNDF's if the interaction $u(t)(px + xp)$ with the external field is replaced by $u(t)x$.

Example 5. (Dual inputs.) The Hamiltonian

$$H = \omega a^\dagger a + \omega b^\dagger b + \alpha(ab^\dagger + ba^\dagger) + iu_1(t)(a^\dagger - a) + u_2(t)(a + a^\dagger) \tag{19}$$

may be used to describe a physical system consisting of two interacting one-dimensional oscillators, with the same angular frequency ω and with respective pairs of annihilation and creation operators $(a, a^\dagger)$ and $(b, b^\dagger)$. As usual $[a, a^\dagger] = [b, b^\dagger] = I$, and each of $(a, a^\dagger)$ is supposed to commute with each of $(b, b^\dagger)$. There is an interaction between the two oscillators, represented by the term $\alpha(ab^\dagger + ba^\dagger)$, where α is a real constant. The "a" oscillator is coupled to two external fields (or two signals) $u_1(t)$ and $u_2(t)$, via its momentum and coordinate operators, respectively.

It is claimed that the observables

$$C = ae^{i\omega t} + a^\dagger e^{-i\omega t} - i(be^{i\omega t} - b^\dagger e^{-i\omega t}),$$
$$D = \omega^{-1}\partial C/\partial t$$

are commuting QNDF's for the Hamiltonian (19). That C and D commute is easily verified by direct computation. To see that C is a QNDF, let us construct its Heisenberg equation of motion. We have

$$\begin{aligned}[H,C] = {} & iu_1(t)(e^{-i\omega t} + e^{i\omega t}) + u_2(t)(e^{i\omega t} - e^{-i\omega t}) + \omega[a^\dagger a, C] \\ & + \omega[b^\dagger b, C] + \alpha(be^{i\omega t} - b^\dagger e^{-i\omega t} - iae^{i\omega t} - ia^\dagger e^{-i\omega t}).\end{aligned}$$

Since

$$i\omega[a^\dagger a, C] + i\omega[b^\dagger b, C] + \partial C/\partial t = 0,$$

we arrive at

$$dC_H/dt = 2Iu_1(t)\cos\omega t - 2Iu_2(t)\sin\omega t + \alpha C_H$$

and conclude that C is a QNDO. For $D = \omega^{-1}\partial C/\partial t$ we obtain straightaway the Heisenberg equation

$$dD_H/dt = -2Iu_1(t)\sin\omega t - 2Iu_2(t)\cos\omega t + \alpha D_H,$$

which implies that D is likewise a QNDO. The measurement results for $C(t)$ and $D(t)$, denoted respectively by $c(t)$ and $d(t)$, evidently satisfy

$$\dot{c} = 2u_1(t)\cos\omega t - 2u_2(t)\sin\omega t + \alpha c,$$
$$\dot{d} = -2u_1(t)\sin\omega t - 2u_2(t)\cos\omega t + \alpha d.$$

Knowing $c(t)$ and $d(t)$, these two equations can be solved simultaneously for $u_1(t)$ and $u_2(t)$. The explicit result is

$$u_1(t) = \tfrac{1}{2}[(\dot{c} - \alpha c)\cos\omega t - (\dot{d} - \alpha d)\sin\omega t],$$
$$u_2(t) = -\tfrac{1}{2}[(\dot{c} - \alpha c)\sin\omega t + (\dot{d} - \alpha d)\cos\omega t].$$

This last example is not offered as a direct illustration of the formal results of Section 5. Rather, it is intended to show in concrete terms that the QNDF scheme is not restricted to the single-input case. Work continues on a general treatment of the multi-signal problem.

7. Conclusions

In this paper we have examined the conditions under which an observable qualifies as a quantum nondemolition filter, i.e., the conditions under which a series of measurements allows an external classical force affecting a quantum system to be monitored in continuous time. The conditions for quantum nondemolition filtering separate naturally into a condition of quantum nondemolition and a condition of invertibility. We have not addressed the question of filter sensitivity or robustness. Some discussion of this aspect of quantum nondemolition filtering may be found in [12].

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