Controllability of a system evolving under the symplectic group $Sp(2n, \mathbb{R})$

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One can recast the evolution of a system evolving under the action of a quadratic Hamiltonian with $2n$ continuous degrees of freedom in the symplectic representation:

$$\frac{dS}{dt} = A(t)S \quad S(0) = \mathbb{1}. \tag{1}$$

The $A(t)$ are determined by the control functions. By manipulating the control functions let's assume we can obtain a set

$$\mathcal{E} = \{\tilde{A}_1, \ldots, \tilde{A}_m\} \tag{2}$$

of linearly independent generators of a Lie algebra $\mathcal{L}$, where $\mathcal{G} = e^{\mathcal{L}}$. As a result, any $K \in \mathcal{G}$ can be written

$$K = e^{\tilde{A}_1 t_1} e^{\tilde{A}_2 t_2} \ldots e^{\tilde{A}_m t_m} \quad \text{with } \tilde{A}_j \in \mathcal{E} \text{ and } t_j \in \mathbb{R} \tag{3}$$

If the group G is compact then for $t < 0$ there exists a sequence of positive times $t_k > 0$ such that

$$\lim_{k\to\infty} e^{\tilde{A} t_k} = e^{\tilde{A} t} \tag{4}$$

Hence we can generate any element of $\mathcal{G}$ using only positive times and we call the system controllable.

The above is directly taken from the 2012 Dynamical recurrence paper. Do you mean $\lim_{t_k \to \infty}$? Otherwise $\lim_{k\to\infty} t_k$ where the sequence converges is just some finite time and the system is exactly periodic.

If $\mathcal{G}$ is noncompact then we look for a condition on $\tilde{A}$ such that for $t_1 > 0$ there exists a $t_2 > 0$ such that

$$e^{\tilde{A}(-t_1)} = e^{\tilde{A} t_2} \tag{5}$$

which would act as an extra 'compact-like' condition on our generators so that we are able to claim that the group is controllable.

This is equivalent to a periodicity condition on $\tilde{A}$ as:

$$\begin{aligned} e^{\tilde{A}(-t_1)} &= e^{\tilde{A}t_2} \\ \implies e^{\tilde{A}(t_1+t_2)} &= \mathbb{1} \\ \implies e^{\tilde{A}2(t_1+t_2)} &= \mathbb{1} \end{aligned} \tag{6}$$

So $e^{\tilde{A}t}$ has period of $T = t_1 + t_2$ in this case.

For the constant matrix $\tilde{A}$, periodicity is equivalent to it having pure imaginary eigenvalues and being diagonalisable. (as shown in the other pdf).

1 Almost periodicity

The statement that one only requires almost periodicity is equivalent to one only needing a transformation arbitrarily close to the symplectic transformation required.

For g' in the set of all possibly reachable transformations with positive time and $g \in Sp(2n, \mathbb{R})$ we require

$$\exists g' \text{ such that } ||g' - g|| < \epsilon \tag{7}$$

for any given ϵ (presumably set by the experimenter).

This sounds fine as a definition of controllability but is certainly broader than that defined by d'Alessandro and the Lie Algebra Rank Criterion.