Multi-input Schrödinger equation: controllability, tracking, and application to the quantum angular momentum *

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Abstract

We present a sufficient condition for approximate controllability of the bilinear discrete-spectrum Schrödinger equation exploiting the use of several controls. The controllability result extends to simultaneous controllability, approximate controllability in H^s, and tracking in modulus. The result is more general than those present in the literature even in the case of one control and permits to treat situations in which the spectrum of the uncontrolled operator is very degenerate (e.g. it has multiple eigenvalues or equal gaps among different pairs of eigenvalues). We apply the general result to a rotating polar linear molecule, driven by three orthogonal external fields. A remarkable property of this model is the presence of infinitely many degeneracies and resonances in the spectrum preventing the application of the results in the literature.

Keywords: Quantum control; bilinear Schrödinger equation; Galerkin approximations, quantum angular momentum.

1 Introduction

In this paper we study the controllability and the tracking problem for a multi-input bilinear Schrödinger equation

$$i\frac{d\psi}{dt}(t) = (H_0 + u_1(t)H_1 + \ldots + u_p(t)H_p)\psi(t) \tag{1}$$

where $H_0, \ldots, H_p$ are self-adjoint operators on a Hilbert space $\mathcal{H}$ and the drift Schrödinger operator H_0 (the *internal Hamiltonian*) has discrete spectrum. The control functions $u_1(\cdot), \ldots, u_p(\cdot)$, representing external fields, are real-valued and $\psi(\cdot)$ takes values in the unit sphere of $\mathcal{H}$.

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The controllability of system (1) is a well-established topic when the state space $\mathcal{H}$ is finite-dimensional (see for instance [D'A08] and reference therein), thanks to general controllability methods for left-invariant control systems on compact Lie groups ([Bro72, JS72, JK81, GB82, EAGK96]).

When $\mathcal{H}$ is infinite-dimensional, it is known that the bilinear Schrödinger equation is not exactly controllable (see [BMS82, Tur00]). Hence, one has to look for weaker controllability properties as, for instance, approximate controllability or controllability between eigenstates of the internal Hamiltonian H_0 (which are the most relevant physical states). In certain cases, when the space variable is one-dimensional, a description of reachable sets has been provided (see [BC06, BL10]). In dimension larger than one or for more general situations, the exact description of the reachable set appears to be more difficult and at the moment only approximate controllability results are available. Most of them are for the single-input case (see, in particular, [CMSB09, Mir09, Ner09, Ner10, BN10, BCCS12, NN12]). Such results are based on sufficient conditions for controllability that are generic [PS10, MS10, Ner10] even in the case $p = 1$. Nevertheless, in many examples interesting for applications these conditions cannot be directly applied or controllability fails to hold, as a consequence of the symmetries of the system. Symmetries can induce degeneracies in the spectrum (e.g. multiple eigenvalues or presence of equal spectral gaps) and reduce the coupling of eigenstates via the control. This happens, for instance, in a planar rotating molecule controlled by only one control [BCCS12, Section 8] which is not (approximately) controllable.

The use of more than one control opens new controllability horizons.

Controllability results with more than one input have been obtained for specific systems [EP09, BBR10] and some general approximate controllability results between eigenfunctions have been proved via adiabatic methods [AB05, BCMS12]. The first multi-input result via Lie-algebraic methods is given in [BCCS12, Section 8] where the problem of the spectral degeneracies in the planar rotating molecule has been overcome associating with every 1-dimensional slice of the set of admissible controls an invariant subspace of the state space $\mathcal{H}$ on which the single-input controllability result applies. Anyhow, such a technique does not apply for more general cases. In the case of a rotating rigid symmetric 3D molecule this application of this method is obstructed by the fact that eigenspaces may have arbitrarily large dimension, since at every higher energy level new degeneracies appear. In some sense the strategy of [BCCS12, Section 8] does not fully exploit the potentialities of the geometric method based on the controllability of the Galerkin approximations.

In this paper, we present a sufficient condition for controllability of the discrete-spectrum bilinear Schrödinger equation which can be applied to cases in which the spectrum of the internal Hamiltonian H_0 is very degenerate. The results fully exploit the use of more than one control and extend to simultaneous controllability, approximate controllability in H^s, and stalking. Proving that a system is a stalker (i.e. it permits to track in modulus any given trajectory; for precise definitions see Section 2) is a crucial issue when describing systems containing dissipative levels (and the dissipation is not taken into account in the mathematical model). In this case, a strategy is to keep the population of the dissipative levels as low as possible during the transitions in order to minimize the effects of the dissipation (see for instance the STIRAP model [CH90, VHBB01, BCG$^+$02]).

The result presented in this paper is more general than those present in the literature even in the single-input case. For instance it applies to the Laplace–Dirichlet operator on a compact interval of $\mathbb{R}$ with a control term of the type $u(t)x$. Let us mention that in [BCCS12], approximate simultaneous controllability of this model has been proved breaking the degeneracy between gaps among eigenstates through perturbation techniques. Here we prove the approximate simultaneous controllability and stalking without perturbation arguments. In this framework, proving a controllability result without perturbation arguments allows to translate directly the constructive proof of the main result in an algorithm that provides explicit expressions of controls [CBCS11].

1.1 Brief description of the general results

The main result of the paper is a sufficient condition for approximate simultaneous controllability which we call the Lie–Galerkin Control Condition (see Definition 2.5).

Roughly speaking, both the sufficient condition proposed in [BCCS12] and the one presented here are based on the idea of driving the system with control laws in resonance with spectral gaps of the internal Hamiltonian H_0. However, whereas in [BCCS12] the only actions on the system obtained by resonance which are exploited for the controllability are those corresponding to elementary transitions between two eigenstates, no such a restriction is imposed in the Lie–Galerkin Control Condition (see Section 2.5).

The Lie–Galerkin Control Condition ensures strong controllability properties for the Galerkin approximations. Indeed it provides controllability for a fixed Galerkin approximation while avoiding the transfer of population to higher energy levels. This allows estimates on the difference between the dynamics of the finite-dimensional Galerkin approximation and the ones of the original infinite-dimensional system. The Lie–Galerkin Control Condition also ensures a bound on the L^1 norm of the control achieving controllability which is uniform with respect to the prescribed tolerance, when the required transfer is between finite combinations of eigenstates.

Under the Lie–Galerkin Stalking Condition, a slight modification of the Lie–Galerkin Control Condition, we can prove that any trajectory can be tracked in modulus (see Theorem 2.8).

The Lie–Galerkin Control Condition under the additional assumption that the system is s-weakly coupled (see Definition 2.11) as introduced in [BCC], allows to conclude that the system is controllable in $H^{s/2}$ (see Theorem 2.12).

1.2 Application to the quantum angular momentum

Rotational molecular dynamics is one of the most important examples of quantum systems with an infinite-dimensional Hilbert space and a discrete spectrum. Molecular orientation and alignment are well-established topics in the quantum control of molecular dynamics both from the experimental and the theoretical point of view (see [SS03, SKA$^+$04, SH06] and references therein). For linear molecules driven by linearly polarized laser fields in gas phase, alignment means an increased probability direction along the polarization axis whereas orientation requires in addition the same (or opposite) direction as the polarization vector. A large amount of numerical simulations have been done in this domain but the mathematical part is not yet fully understood. From this perspective, the controllability problem is a necessary step towards comprehension.

We focus in this paper on the control by external fields of the rotation of a rigid linear molecule in $\mathbb{R}^3$. This control problem corresponds to the control of the Schrödinger equation on the unit sphere S^2. We show that the system driven by three fields along the three axes is approximately controllable for arbitrarily small controls.

Up to normalization of physical quantities (in particular, in units such that $\hbar = 1$), the dynamics are governed by the equation

$$i\frac{\partial \psi(\theta,\varphi,t)}{\partial t} = -\Delta\psi(\theta,\varphi,t) + (u_1(t)\sin\theta\cos\varphi + u_2(t)\sin\theta\sin\varphi + u_3(t)\cos\theta)\psi(\theta,\varphi,t) \quad (2)$$

where θ, φ are the spherical coordinates, which are related to the Euclidean coordinates through the identities

$$x = \sin\theta\cos\varphi, \quad y = \sin\theta\sin\varphi, \quad z = \cos\theta,$$

while Δ is the Laplace–Beltrami operator on the sphere S^2 (called in this context the *angular momentum operator*), i.e.,

$$\Delta = \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}.$$

The wavefunction ψ evolves in the unit sphere $\mathcal{S}$ of $\mathcal{H} = L^2(S^2, \mathbb{C})$.

As a consequence of the general multi-input result presented in Section 2 we have that (2) is approximately controllable with arbitrarily small controls. A stronger statement, including simultaneous controllability in H^s and stalking, is given in Section 3.

Theorem 1.1. *For every ψ^0, ψ^1 belonging to $\mathcal{S}$ and every $\delta > 0$, there exist $T > 0$ and $u \in L^\infty([0,T],[0,\delta]^3)$, such that the solution $\psi(\cdot)$ of equation* (2)*, corresponding to the control u and with initial condition ψ_0, satisfies $\|\psi^1 - \psi(T)\| < \varepsilon$.*

There are two main difficulties preventing the application to this system results previously in the literature. Firstly, we deal here with several control parameters, while those general results were specifically conceived for the single-input case. Notice that, because of symmetry obstructions, equation (2) is not controllable with only two of the three controls u_1, u_2, u_3. Secondly, the general theory developed in [CMSB09, Ner10, BCCS12] is based on nonresonance conditions on the spectrum of the internal Hamiltonian. The Laplace–Beltrami operator on S^2, however, has a severely degenerate spectrum, since the ℓ-th eigenvalue $-i\ell(\ell+1)$ has multiplicity $2\ell+1$. In [CMSB09] we proposed a perturbation technique in order to overcome resonance relations in the spectrum of the drift. This technique was applied in [BCM$^+$09] to the case of the orientation of a molecule confined in a plane driven by one control. The planar case is already technically challenging and a generalization of the same technique to the case of three controls in 3D seems hard to achieve.

1.3 Structure of the paper

The structure of the paper is the following: in the next section we present the general multi-input abstract framework and the main abstract results. In Section 3 we apply them to system (2). The proofs of the abstract results are contained in Sections 4, 5, and 6.

2 Framework and main results

Let $p \in \mathbb{N}$, $\delta > 0$, and $U = U_1 \times \cdots \times U_p$ with either $U_j = [0, \delta]$ or $U_j = [-\delta, \delta]$.

Definition 2.1. Let $\mathcal{H}$ be an infinite-dimensional Hilbert space with scalar product $\langle \cdot, \cdot \rangle$ and $A, B_1, \dots, B_p$ be (possibly unbounded) skew-adjoint operators on $\mathcal{H}$, with domains $D(A)$, $D(B_1), \dots, D(B_p)$. Let us introduce the controlled equation

$$\frac{d\psi}{dt}(t) = (A + u_1(t)B_1 + \cdots + u_p(t)B_p)\psi(t), \quad u(t) \in U. \tag{3}$$

We say that A satisfies ($\mathbb{A}1$) if the following assumption is true:

($\mathbb{A}1$) A has discrete spectrum with infinitely many distinct eigenvalues (possibly degenerate).

Denote by Φ a Hilbert basis $(\phi_k)_{k\in\mathbb{N}}$ of $\mathcal{H}$ made of eigenvectors of A associated with the family of eigenvalues $(i\lambda_k)_{k\in\mathbb{N}}$ and let $\mathcal{L}$ be the set of finite linear combinations of eigenstates, that is,

$$\mathcal{L} = \bigcup_{k\in\mathbb{N}} \operatorname{span}\{\phi_1, \dots, \phi_k\}.$$

We say that $(A, B_1, \dots, B_p, U, \Phi)$ satisfies ($\mathbb{A}$) if A satisfies ($\mathbb{A}1$) and the following assumptions hold:

($\mathbb{A}2$) $\phi_k \in D(B_j)$ for every $k \in \mathbb{N}, j = 1, \dots, p$;

($\mathbb{A}3$) $A + u_1B_1 + \cdots + u_pB_p : \mathcal{L} \to \mathcal{H}$ is essentially skew-adjoint for every $u \in U$.

If $(A, B_1, \dots, B_p, U, \Phi)$ satisfies ($\mathbb{A}$) then, for every $(u_1, \dots, u_p) \in U$, $A + u_1B_1 + \cdots + u_pB_p$ generates a subgroup $e^{t(A+u_1B_1+\cdots+u_pB_p)}$ of the group of unitary operators $\mathbf{U}(\mathcal{H})$. It is therefore possible to define the propagator Υ^u_T at time T of system (2) associated with a p-uple of piecewise constant controls $u(\cdot) = (u_1(\cdot), \dots, u_p(\cdot))$ by composition of flows of the type $e^{t(A+u_1B_1+\cdots+u_pB_p)}$. If, moreover, $B_1, \dots, B_p$ are bounded operators then the definition can be extended by continuity to every L^∞ control law (see [BMS82, Theorem 2.5]).

Definition 2.2. Let $(A, B_1, \dots, B_p, U, \Phi)$ satisfy ($\mathbb{A}$). We say that (3) is *approximately controllable* if for every ψ_0, ψ_1 in the unit sphere of $\mathcal{H}$ and every $\varepsilon > 0$ there exists a piecewise constant control function $u : [0, T] \to U$ such that $\|\psi_1 - \Upsilon^u_T(\psi_0)\| < \varepsilon$.

Definition 2.3. Let $(A, B_1, \dots, B_p, U, \Phi)$ satisfy ($\mathbb{A}$). We say that (3) is *approximately simultaneously controllable* if for every r in $\mathbb{N}$, $\psi_1, \dots, \psi_r$ in $\mathcal{H}$, $\hat{\Upsilon}$ in $\mathbf{U}(\mathcal{H})$, and $\varepsilon > 0$ there exists a piecewise constant control $u : [0, T] \to U$ such that

$$\left\| \hat{\Upsilon}\psi_k - \Upsilon^u_T\psi_k \right\| < \varepsilon, \qquad k = 1, \dots, r.$$

If, moreover, for every $\psi_1, \dots, \psi_r \in \mathcal{L}$ and $\hat{\Upsilon} \in \mathbf{U}(\mathcal{H})$ such that $\hat{\Upsilon}\psi_1, \dots, \hat{\Upsilon}\psi_r \in \mathcal{L}$, there exists $K > 0$ (not depending on ε) such that u can be chosen to satisfy, in addition, $\|u\|_{L^1} \leq K$, we say that (3) is *L^1-bounded approximately simultaneously controllable.*

This last definition of controllability with *a priori* bound on the L^1-norm of the control achieving controllability has been observed in preceding works [BCCS12, Cha12]. It implies a stronger controllability property as shown in Section 2.4.

Due to presence of the internal Hamiltonian and the boundedness of the controls, it is not possible in general to track, with arbitrarily precision, an unfeasible curve in $\mathcal{S}$. We introduce, then, the notion of *stalker*, that is a system for which it is possible to track any given curve up to phases (both for a single initial condition and in the spirit of simultaneous control). This definition makes sense from the physical point of view, since tracking up to phases means imposing the population of all energy levels of H_0 along the evolution.

The identification up to phases of elements of $\mathcal{H}$ in the basis $\Phi = (\phi_k)_{k\in\mathbb{N}}$ can be introduced through the projection

$$\mathfrak{M} : \psi \mapsto \sum_{k\in\mathbb{N}} |\langle \phi_k, \psi\rangle| \phi_k.$$

Definition 2.4. Let $(A, B_1, \dots, B_p, U, \Phi)$ satisfy $(\mathbb{A})$. We say that (3) is a *stalker* if for every r in $\mathbb{N}$, $\psi_1, \dots, \psi_r$ in $\mathcal{H}$, $\hat{\Upsilon} : [0, T] \to \mathbf{U}(\mathcal{H})$ continuous, with $\hat{\Upsilon}_0 = \mathrm{Id}_{\mathcal{H}}$, and $\varepsilon > 0$ there exist an invertible continuous function $\tau : [0, T] \to [0, T_\tau]$ and a piecewise constant control $u : [0, T_\tau] \to U$ such that

$$\left\| \mathfrak{M}(\hat{\Upsilon}_t \psi_k) - \mathfrak{M}(\Upsilon^u_{\tau(t)} \psi_k) \right\| < \varepsilon, \qquad k = 1, \dots, r,$$

for every $t \in [0, T_\tau]$.

2.1 Notation

For every n in $\mathbb{N}$, define the orthogonal projection

$$\pi_n : \mathcal{H} \ni \psi \mapsto \sum_{k=1}^{n} \langle \phi_k, \psi\rangle \phi_k \in \mathcal{H}.$$

Given a linear operator Q on $\mathcal{H}$ we identify the linear operator $\pi_n Q \pi_n$ preserving $\mathrm{span}\{\phi_1, \dots, \phi_n\}$ with its $n \times n$ complex matrix representation with respect to the basis $(\phi_1, \dots, \phi_n)$. We define

$$A^{(n)} = \pi_n A \pi_n \quad \text{and} \quad B_j^{(n)} = \pi_n B_j \pi_n,$$

for every $j = 1, \dots, p$.

Let us introduce the set Σ_n of spectral gaps associated with the n-dimensional Galerkin approximation as

$$\Sigma_n = \{|\lambda_l - \lambda_k| \mid l, k = 1, \dots, n\}.$$

For every $\sigma \geq 0$, every $m \in \mathbb{N}$, and every $m \times m$ matrix M, let

$$\mathcal{E}_\sigma(M) = (M_{l,k} \delta_{\sigma, |\lambda_l - \lambda_k|})_{l,k=1}^{m}.$$

The $n \times n$ matrix $\mathcal{E}_\sigma(B_j^{(n)})$, $j = 1, \dots, p$, corresponds to the "activation" of the spectral gap σ: it reflects the action of the convexification procedure detailed in the following sections, which annihilates all the matrix elements $(B_j^{(n)})_{l,k}$ such that $|\lambda_l - \lambda_k| \neq \sigma$.

Define

$$\Xi_n = \left\{ (\sigma, j) \in \Sigma_n \times \{1, \dots, p\} \mid \exists M \in \mathfrak{u}(n) \text{ s.t. } \mathcal{E}_\sigma(B_j^{(N)}) = \left(\begin{array}{c|c} M & 0 \\ \hline 0 & * \end{array} \right) \text{ for every } N > n \right\}. \tag{4}$$

The matrices $\mathcal{E}_\sigma(B_j^{(n)})$ for $(\sigma, j) \in \Xi_n$ correspond to "compatible dynamics" for the n-dimensional Galerkin approximation (*compatible*, that is, with higher dimensional Galerkin approximations).

2.2 Controllability results

Let

$$\mathcal{V}_n^0 = \left\{ A^{(n)} \right\} \cup \left\{ \mathcal{E}_\sigma(B_j^{(n)}) \mid (\sigma, j) \in \Xi_n \text{ and } j \text{ is such that } (0, j) \in \Xi_n \right\}$$
$$\cup \left\{ \mathcal{E}_\sigma(B_j^{(n)}) \mid (\sigma, j) \in \Xi_n, \sigma \neq 0, U_j = [-\delta, \delta] \right\}.$$

Definition 2.5. Let $(A, B_1, \dots, B_p, U, \Phi)$ satisfy $(\mathbb{A})$. We say that the *Lie–Galerkin Control Condition* holds if for every $n_0 \in \mathbb{N}$ there exists $n > n_0$ such that

$$\mathrm{Lie}\mathcal{V}_n^0 \supseteq \mathfrak{su}(n). \tag{5}$$

Theorem 2.6 (Abstract multi-input controllability result)**.** *Assume that* $(\mathbb{A})$ *holds true. If the Lie–Galerkin Control Condition holds then the system*

$$\dot{x} = (A + u_1 B_1 + \dots + u_p B_p)x, \quad u \in U,$$

is L^1-bounded approximately simultaneously controllable.

2.3 Stalking results

For every $\xi \in S^1 \subset \mathbb{C}$, consider the matrix operator J_ξ such that

$$(J_\xi(M))_{j,k} = \begin{cases} \xi M_{j,k} & \text{if } \lambda_j < \lambda_k \\ 0 & \text{if } \lambda_j = \lambda_k \\ \bar{\xi} M_{j,k} & \text{if } \lambda_j > \lambda_k. \end{cases}$$

Let

$$\mathcal{V}_n = \left\{ J_\xi(\mathcal{E}_\sigma(B_j^{(n)})) \mid (\sigma, j) \in \Xi_n, \sigma \neq 0, \xi \in S^1 \right\}.$$

Notice that $\mathcal{V}_n \subset \mathfrak{su}(n)$.

Definition 2.7. Let $(A, B_1, \dots, B_p, U, \Phi)$ satisfy $(\mathbb{A})$. We say that the *Lie–Galerkin Stalking Condition* holds if for every $n_0 \in \mathbb{N}$ there exists $n > n_0$ such that

$$\mathrm{Lie}\mathcal{V}_n = \mathfrak{su}(n). \tag{6}$$

Theorem 2.8 (Abstract multi-input tracking result)**.** *Let $U_j = [-\delta, \delta]$ for some $\delta > 0$ and every $j = 1, \dots, p$. Assume that $(\mathbb{A})$ holds true. If the Lie–Galerkin Stalking Condition holds then the system*

$$\dot{x} = (A + u_1 B_1 + \cdots + u_p B_p)x, \quad u \in U,$$

is a stalker.

Remark 2.9. If $U_j = [-\delta, \delta]$ for every $j = 1, \dots, p$, then the Lie–Galerkin Stalking Condition implies the Lie–Galerkin Control Condition, as it follows from the relation

$$\left[A^{(n)}, \mathcal{E}_\sigma(B_j^{(n)})\right] = \sigma J_i(\mathcal{E}_\sigma(B_j^{(n)})).$$

2.4 Controllability in higher norms

We define for $s > 0$,

$$|A|^s \psi = \sum_{n \in \mathbb{N}} |\lambda_n|^s \langle \psi, \phi_n \rangle \phi_n$$

for every ψ belonging to

$$D(|A|^s) = \left\{ \psi \in \mathcal{H} \mid \sum_{n \in \mathbb{N}} |\lambda_n|^{2s} |\langle \psi, \phi_n \rangle|^2 < +\infty \right\}.$$

For every $\psi \in D(|A|^s)$ we can define the $|A|^s$-norm (or simply s-norm) of ψ by $\|\psi\|_s = \||A|^s \psi\|$. If A is the Laplace–Dirichlet operator on some bounded domain of $\mathbb{R}^n$ then the s-norm is equivalent to the H^{2s}-norm on $D(|A|^s)$.

Definition 2.10. Let $(A, B_1, \dots, B_p, U, \Phi)$ satisfy Assumption $(\mathbb{A})$ and let $s > 0$. System (3) is *approximately simultaneously controllable* (respectively *approximately controllable*) *for the s-norm* if for every $\varepsilon > 0$, $r \in \mathbb{N}$ (respectively $r = 1$), $\psi_1, \dots, \psi_r$ in $D(|A|^s)$, and $\hat{\Upsilon} \in \mathbf{U}(\mathcal{H})$ such that $\hat{\Upsilon}\psi_1, \dots, \hat{\Upsilon}\psi_r \in D(|A|^s)$ there exists a piecewise constant function $u_\varepsilon : [0, T_\varepsilon] \to \mathbb{R}$ such that

$$\|\hat{\Upsilon}\psi_j - \Upsilon^{u_\varepsilon}_{T_\varepsilon} \psi_j\|_s < \varepsilon,$$

for every $j = 1, \dots, r$.

We say that $(A, B_1, \dots, B_p, U, \Phi)$ satisfies $(\mathbb{A}')$ if it satisfies $(\mathbb{A})$ and the following additional assumptions hold:

$(\mathbb{A}4)$ the operator $i(A + u_1 B_1 + \cdots + u_p B_p)$ is bounded from below for every $u \in \mathbb{R}^p$;

$(\mathbb{A}5)$ the sequence $(\lambda_k)_{k \in \mathbb{N}}$ is non-increasing and unbounded.

Definition 2.11. Let $(A, B_1, \dots, B_p, U, \Phi)$ satisfy Assumption $(\mathbb{A}')$ and let $s > 0$. Then $(A, B_1, \dots, B_p)$ is *s-weakly-coupled* if $D(|A + u_1 B_1 + \cdots + u_p B_p|^{s/2}) = D(|A|^{s/2})$ for every $u \in \mathbb{R}^p$ and there exists C such that

$$|\Re\langle |A|^s \psi, B_l \psi \rangle| \le C |\langle |A|^s \psi, \psi \rangle|$$

for every $l = 1, \dots, p$, $\psi \in D(|A|^s)$.

The following result is a consequence of [BCC, Proposition 2] and can be obtained by adapting the arguments proposed in [BCC, Proposition 5]. We provide its proof in Section 6.

Theorem 2.12. *Let $(A, B_1, \ldots, B_p, U, \Phi)$ satisfy Assumption $(\mathbb{A}')$ and $(A, B_1, \ldots, B_p)$ be s-weakly coupled for some $s > 0$. If (3) is L^1-bounded approximately simultaneously controllable then it is approximately simultaneously controllable for the $s/2$-norm.*

As a direct consequence we have the following result generalizing [BCC, Proposition 5].

Corollary 2.13. *Let $(A, B_1, \ldots, B_p, U, \Phi)$ satisfy Assumption $(\mathbb{A}')$ and $(A, B_1, \ldots, B_p)$ be s-weakly coupled for some $s > 0$. If the Lie–Galerkin Control Condition holds then system (3) is approximately simultaneously controllable for the $s/2$-norm.*

2.5 Example: the infinite potential well

We present the case of a particle confined in the interval $(-1/2, 1/2)$ as a toy model to compare the result in [BCCS12] and Theorem 2.8 on a single-input system. The model has been extensively studied by several authors in the last decade and it has been the first quantum system for which a positive controllability result has been obtained (see [BC06]). In [BCCS12] an approximate simultaneous controllability has been obtained with geometric methods and using perturbations techniques. Indeed this model presents several resonances preventing the direct application of the results in [BCCS12].

The Schrödinger equation writes

$$i\frac{\partial\psi}{\partial t} = -\frac{1}{2}\frac{\partial^2\psi}{\partial x^2} - u(t)x\psi(x,t) \tag{7}$$

with the boundary conditions $\psi(-1/2, t) = \psi(1/2, t) = 0$ for every $t \in \mathbb{R}$. We consider controls $u(\cdot)$ piecewise constant with values in $U = [-\delta, \delta]$ for some $\delta > 0$.

In this case $\mathcal{H} = L^2((-1/2, 1/2), \mathbb{C})$ endowed with the Hermitian product $\langle\psi_1, \psi_2\rangle = \int_{-1/2}^{1/2} \overline{\psi_1(x)}\psi_2(x)dx$. The operators A and $B = B_1$ are defined by $A\psi = i\frac{1}{2}\frac{\partial^2\psi}{\partial x^2}$ for every ψ in $D(A) = (H_2 \cap H_0^1)((-1/2, 1/2), \mathbb{C})$ and $B\psi = ix\psi$. A complete set of eigenfunctions of A is

$$\phi_k(x) = \begin{cases} \sqrt{2}\cos(k\pi x) & \text{when } k \text{ is odd,} \\ \sqrt{2}\sin(k\pi x) & \text{when } k \text{ is even,} \end{cases} \qquad k \in \mathbb{N},$$

associated with the eigenvalues $i\lambda_k = -i\frac{k^2\pi^2}{2}$, $k \in \mathbb{N}$. Notice that

$$\langle\phi_j, B\phi_k\rangle \neq 0$$

if and only if $j + k$ is odd. In particular $\mathcal{V}_n \subset \mathfrak{su}(n)$ for every n.

We prove by induction on n that $\mathrm{Lie}\mathcal{V}_n = \mathfrak{su}(n)$, and hence that the Lie–Galerkin Stalking Condition is fulfilled. Notice that the matrices $\mathcal{E}_{2k-1}(B^{(N)})$, for $k \leq N$, have only zero elements in the positions (j, l) for $j \leq N$ and $l \geq N + 1$, since

$$l^2 - j^2 \geq (N + 1)^2 - N^2 = 2N + 1 > 2k - 1.$$

Hence $\mathcal{E}_{2k-1}(B^{(N)}) \in \mathcal{V}_N$ for $k = 1, \ldots, N$. We prove the claim by showing that

$$\mathrm{Lie}\left(\{\mathcal{E}_{2k-1}(B^{(n)}) \mid k = 1, \ldots, n\}\right) = \mathfrak{su}(n). \tag{8}$$

For $n=2$, $\mathcal{E}_3(B^{(2)}) = \begin{pmatrix} 0 & b_{12} \\ -\bar{b}_{12} & 0 \end{pmatrix}$ generates $\mathfrak{su}(2)$ because $b_{12} \neq 0$.

Now assume that

$$\mathrm{Lie}\left(\{\mathcal{E}_{2k-1}(B^{(n-1)}) \mid k=1,\dots,n-1\}\right) = \mathfrak{su}(n-1),$$

and let us prove (8). The matrices $\mathcal{E}_{2k-1}(B^{(n)})$ are in $\mathfrak{su}(n)$ for every $k=1,\dots,n-1$ and generate the subalgebra of matrices in $\mathfrak{su}(n)$ with zero elements in the nth row and nth column. In particular there exists $M \in \mathrm{Lie}\left(\{\mathcal{E}_{2k-1}(B^{(n)}) : k=1,\dots,n-1\}\right)$ such that $M + \mathcal{E}_{2n-1}(B^{(n)})$ has only two nonzero elements in the positions $(n-1,n)$ and $(n,n-1)$. So

$$\begin{aligned} &\mathrm{Lie}(\{\mathcal{E}_{2k-1}(B^{(n)}) \mid k=1,\dots,n\}) \\ &\qquad \supset \mathrm{Lie}\left(\{\mathcal{E}_{2k-1}(B^{(n)}) \mid k=1,\dots,n-1\} \cup \{M + \mathcal{E}_{2n-1}(B^{(n)})\}\right) = \mathfrak{su}(n). \end{aligned}$$

Therefore, thanks to Theorems 2.6 and 2.8, system (7) is approximately simultaneously controllable and a stalker.

3 The 3D molecule

Let us go back to the system presented in the introduction for the orientation of a linear molecule, that is,

$$i\hbar\dot{\psi} = -\Delta\psi + (u_1\cos\theta + u_2\cos\varphi\sin\theta + u_3\sin\varphi\sin\theta)\psi, \tag{9}$$

where $\psi(t) \in \mathcal{H} = L^2(S^2,\mathbb{C})$.

A basis of eigenvectors of the Laplace–Beltrami operator Δ is given by the spherical harmonics $Y_\ell^m(\theta,\varphi)$, which satisfy

$$\Delta Y_\ell^m(\theta,\varphi) = -\ell(\ell+1)Y_\ell^m(\theta,\varphi).$$

The spectrum of $A = i\Delta$ is $\{-i\ell(\ell+1) \mid \ell \in \mathbb{N}\}$. Each eigenvalue $-i\ell(\ell+1)$ is of finite multiplicity $2\ell+1$. Therefore A satisfies Assumptions ($\mathbb{A}1$) and ($\mathbb{A}5$). Using the notations of the preceding sections we set B_1, B_2, B_3 to be the multiplication operators by $-i\cos\varphi\sin\theta$, $-i\sin\varphi\sin\theta$, $-i\cos\theta$ respectively. Being B_1, B_2, B_3 bounded, conditions ($\mathbb{A}2$), ($\mathbb{A}3$), and ($\mathbb{A}4$) hold. Hence ($\mathbb{A}'$) is satisfied. Moreover, as proved in [BCC, Proposition 8], (9) is s-weakly coupled for every $s>0$. The main goal of this section is to prove that system (9) satisfies the Lie–Galerkin Stalking Condition. As a consequence we obtain the following result, whose corollary is Theorem 1.1.

Theorem 3.1. *System* (9) *is:*

(i) *L^1-bounded approximately simultaneously controllable,*

(ii) *approximately simultaneously controllable in H^s for every $s>0$,*

(iii) *a stalker.*

Using classical identities for Legendre polynomials and trigonometric relations one can prove that

$$\langle Y_\ell^m, B_j Y_\ell^{m'}\rangle = 0$$

for every $j = 1, 2, 3$, and $m, m' \in \{-\ell - 1, \dots, \ell + 1\}$.

Moreover

$$\langle Y_\ell^m, B_j Y_{\ell'}^{m'}\rangle = 0$$

with $|\ell - \ell'| \geq 2$ for every $m \in \{-\ell - 1, \dots, \ell + 1\}$, $m' \in \{-\ell' - 1, \dots, \ell' + 1\}$, $j = 1, 2, 3$. In order to prove that the Lie–Galerkin Stalking Condition is satisfied, we choose a reordering $(\phi_k)_{k\in\mathbb{N}}$ of the spherical harmonics in such a way that

$$\{\phi_k \mid k = 1, \dots, 4\ell + 4\} = \{Y_\ell^{-\ell}, \dots, Y_\ell^{\ell}, Y_{\ell+1}^{-\ell-1}, \dots, Y_{\ell+1}^{\ell+1}\},$$

and we are left to prove that

$$\mathrm{Lie}\mathcal{V}_{4\ell+4} = \mathfrak{su}(4\ell + 4).$$

The characteristics spectral gap of the space $\mathcal{H}_\ell$ is $(\ell + 1)(\ell + 2) - \ell(\ell + 1) = 2(\ell + 1)$. In particular $(2(\ell + 1), 1), (2(\ell + 1), 2)$, and $(2(\ell + 1), 3)$ are in $\Xi_{4\ell+4}$.

3.1 Matrix representations

Denote by $\mathcal{J}_\ell$ the set of integer pairs $\{(r, m) \mid r = \ell, \ell + 1,\ m = -j, \dots, j\}$. Consider the lexicographic ordering $\varrho : \{1, \dots, 4\ell+4\} \to \mathcal{J}_\ell$. For $j, k = 1, \dots, 4\ell+4$, let $e_{j,k}$ be the $(4\ell+4)$-square matrix whose entries are all zero, but the one at line j and column k which is equal to 1. Define

$$E_{j,k} = e_{j,k} - e_{k,j},\ F_{j,k} = ie_{j,k} + ie_{k,j},\ D_{j,k} = ie_{j,j} - ie_{k,k}.$$

By a slight abuse of language, also set $e_{\varrho(j),\varrho(k)} = e_{j,k}$. The analogous identification can be used to define $E_{\varrho(j),\varrho(k)}, F_{\varrho(j),\varrho(k)}, D_{\varrho(j),\varrho(k)}$. Note that

$$J_i(E_{(\ell,m),(\ell+1,n)}) = -F_{(\ell,m),(\ell+1,n)}, \quad \text{and} \quad J_i(F_{(\ell,m),(\ell+1,n)}) = E_{(\ell,m),(\ell+1,n)}. \tag{10}$$

Thanks to this notation we can conveniently represent the matrices corresponding to the controlled vector field (projected on $\mathcal{H}_\ell$). A computation shows that the control potentials in the x and y directions, $-i\cos\varphi\sin\theta$ and $-i\sin\varphi\sin\theta$ respectively, projected on $\mathcal{H}_\ell$, have the matrix representations

$$\begin{aligned} B_1^{(4\ell+4)} &= \sum_{m=-\ell}^{\ell} (-q_{\ell,m} F_{(\ell,m),(\ell+1,m-1)} + q_{\ell,-m} F_{(\ell,m),(\ell+1,m+1)}) \\ B_2^{(4\ell+4)} &= \sum_{m=-\ell}^{\ell} (q_{\ell,m} E_{(\ell,m),(\ell+1,m-1)} + q_{\ell,-m} E_{(\ell,m),(\ell+1,m+1)}), \end{aligned}$$

where

$$q_{\ell,m} = \sqrt{\frac{(\ell - m + 2)(\ell - m + 1)}{4(2\ell + 1)(2\ell + 3)}}.$$

Similarly, we associate with the control potential in the z direction, $-i\cos\theta$ the matrix representation

$$B_3^{(4\ell+4)} = \sum_{m=-\ell}^{\ell} p_{\ell,m} F_{(\ell,m),(\ell+1,m)},$$

with

$$p_{\ell,m} = -\sqrt{\frac{(\ell+1)^2 - m^2}{(2\ell+1)(2\ell+3)}}.$$

3.2 Useful bracket relations

From the identity

$$[e_{j,k}, e_{n,m}] = \delta_{kn} e_{j,m} - \delta_{jm} e_{n,k}$$

we get the relations

$$[E_{j,k}, E_{k,n}] = E_{j,n}, \qquad [F_{j,k}, F_{k,n}] = -E_{j,n}, \qquad [E_{j,k}, F_{k,n}] = F_{j,n}, \tag{11}$$

and

$$[E_{j,k}, F_{j,k}] = 2D_{j,k}, \quad \text{and} \quad [F_{j,k}, D_{j,k}] = 2E_{j,k}. \tag{12}$$

The relations above can be interpreted following a "triangle rule": the bracket between an operator coupling the states j and k and an operator coupling the states k and n couples the states j and n. On the other hand, the bracket is zero if two operators couple no common states, that is,

$$[Y_{j,k}, Z_{j',k'}] = 0 \quad \text{if } \{j,k\} \cap \{j',k'\} = \emptyset, \tag{13}$$

with $Y, Z \in \{E, F, D\}$.

3.3 Controllability in $\mathfrak{su}(4\ell+4)$

Lemma 3.2. *The Lie algebra L generated by $B_1^{(4\ell+4)}, B_2^{(4\ell+4)}, B_3^{(4\ell+4)}$, $J_i(B_1^{(4\ell+4)})$, $J_i(B_2^{(4\ell+4)})$, $J_i(B_3^{(4\ell+4)})$ is equal to $\mathfrak{su}(4\ell+4)$.*

Proof. The first step of the proof consists in showing that the Lie algebra L contains the elementary matrices

$$E_{(\ell,k),(\ell+1,k+j)} \quad \text{for } k = -\ell, \dots, \ell, \ j = -1, 0, 1. \tag{14}$$

With a slight abuse of notation and for the sake of readability, let us write $B_j = B_j^{(4\ell+4)}$, $j = 1,2,3$. Let us also write $\operatorname{ad}_\alpha \beta$ for $[\alpha, \beta]$ and $\operatorname{ad}_\alpha^{j+1} \beta$ for $[\alpha, \operatorname{ad}_\alpha^j \beta]$.

Notice that

$$J_i(B_3) = \sum_{\ell=-m}^{m} p_{\ell,m} J_i(F_{(\ell,m),(\ell+1,m)}) = \sum_{\ell=-m}^{m} p_{\ell,m} E_{(\ell,m),(\ell+1,m)}.$$

By induction on $j \geq 0$ and using the bracket relations (12), we have

$$\begin{aligned}\operatorname{ad}_{B_3}^{2j} J_i(B_3) &= [B_3, [B_3, \operatorname{ad}_{B_3}^{2j-2} J_i(B_3)]] \\ &= (-1)^j 2^{2j} \sum_{m=-\ell}^{\ell} p_{\ell,m}^{2j+1} E_{(\ell,m),(\ell+1,m)}.\end{aligned}$$

By invertibility of the Vandermonde matrix and since $p_{\ell,m} \neq p_{\ell,n}$ for every $n \neq m, -m$, it follows

$$E_{(\ell,-m),(\ell+1,-m)} + E_{(\ell,m),(\ell+1,m)} \in L \qquad \text{for } m = 0, \dots, \ell. \tag{15}$$

In particular $E_{(\ell,0),(\ell+1,0)} \in L$. The double bracket of

$$\frac{B_2 - J_i(B_1)}{2} = \sum_{m=-\ell}^{\ell} q_{\ell,m} E_{(\ell,m),(\ell+1,m-1)} \in L, \tag{16}$$

with $E_{(\ell,0),(\ell+1,0)}$ is easily computed using (11) and (13) and gives

$$\begin{aligned}\Big[\Big[\sum_{m=-\ell}^{\ell} q_{\ell,m} E_{(\ell,m),(\ell+1,m-1)}, E_{(\ell,0),(\ell+1,0)}\Big], E_{(\ell,0),(\ell+1,0)}\Big] &= -q_{\ell,1}[E_{(\ell,0),(\ell,1)}, E_{(\ell,0),(\ell+1,0)}] \\ &\quad - q_{\ell,0}[E_{(\ell+1,-1),(\ell+1,0)}, E_{(\ell,0),(\ell+1,0)}] \\ &= q_{\ell,0} E_{(\ell,0),(\ell+1,-1)} + q_{\ell,1} E_{(\ell,1),(\ell+1,0)} \in L.\end{aligned}$$

Define $Q_0 = q_{\ell,0} E_{(\ell,0),(\ell+1,-1)} + q_{\ell,1} E_{(\ell,1),(\ell+1,0)}$ and, similarly, $Q_m = q_{\ell,-m} E_{(\ell,-m),(\ell+1,-m-1)} + q_{\ell,m+1} E_{(\ell,m+1),(\ell+1,m)}$ for $0 < m < \ell$, $Q_\ell = q_{\ell,-\ell} E_{(\ell,-\ell),(\ell+1,-\ell-1)}$. In particular $B_2 - J_i(B_1) = 2\sum_{m=0}^{\ell} Q_m$.

Using again (11) and (13), we have

$$\Big[\Big[\sum_{m=k}^{\ell} Q_m, E_{(\ell,-k),(\ell+1,-k)} + E_{(\ell,k),(\ell+1,k)}\Big], E_{(\ell,-k),(\ell+1,-k)} + E_{(\ell,k),(\ell+1,k)}\Big] = Q_k,$$

for $k = 1, \dots, \ell$. By recurrence on k and because of (15), it follows that $Q_k \in L$ for $k = 0, \dots, \ell$.

Now, since $Q_\ell / q_{\ell,-\ell} = E_{(\ell,-\ell),(\ell+1,-\ell-1)}$ is in L, then

$$\operatorname{ad}^2_{E_{(\ell,-\ell),(\ell+1,-\ell-1)}}(E_{(\ell,-\ell),(\ell+1,-\ell)} + E_{(\ell,\ell),(\ell+1,\ell)}) = -E_{(\ell,-\ell),(\ell+1,-\ell)} \in L,$$

which, in turns, implies that

$$\operatorname{ad}^2_{E_{(\ell,-\ell),(\ell+1,-\ell)}}(Q_{\ell-1}) = -q_{\ell,-\ell} E_{(\ell,-\ell+1),(\ell+1,-\ell)} \in L.$$

Iterating the argument, $E_{(\ell,m),(\ell+1,m)}$ and $E_{(\ell,m),(\ell+1,m-1)}$ are in L for every $m = -\ell, \dots, \ell$.

Developing the same argument as above replacing (16) by

$$\frac{B_2 + J_i(B_1)}{2} = \sum_{m=-\ell}^{\ell} q_{\ell,-m} E_{(\ell,m),(\ell+1,m+1)} \in L,$$

we have that also $E_{(\ell,m),(\ell+1,m+1)}$ is in L for every $m = -\ell, \dots, \ell$, proving (14). It then follows from (11) that each $E_{j,k}$ is in L.

If now we replace (16) by

$$\frac{B_1 + J_i(B_2)}{2} = -\sum_{m=-\ell}^{\ell} q_{\ell,m} F_{(\ell,m),(\ell+1,m-1)} \in L,$$

or

$$\frac{B_1 - J_i(B_2)}{2} = \sum_{m=-\ell}^{\ell} q_{\ell,-m} F_{(\ell,m),(\ell+1,m+1)} \in L,$$

we obtain from the arguments above that $F_{(\ell,m),(\ell+1,m-1)}$ and $F_{(\ell,m),(\ell+1,m+1)}$ are in L for every $m = -\ell, \ldots, \ell$. The relations (11) and (12) allow then to conclude that $L = \mathfrak{su}(4\ell + 4)$. $\square$

4 Proof of Theorem 2.6

4.1 Time-reparametrization

Up to replacing each B_j by δB_j, we can assume that $\delta = 1$.

For every piecewise constant function z such that $z(t) \geq 1$ for every t, we consider the time-reparametrization

$$\frac{d\psi}{dt}(t) = (z(t)A + u_1(t)z(t)B_1 + \cdots + u_p(t)z(t)B_p)\psi(t) \tag{17}$$

of system (3). Each $u_j(t)z(t)$ belongs to the time-varying set $z(t)U_j$.

If $u_1, \ldots, u_p$ are control laws in (17) then the corresponding controls in (3) are their time-reparametrizations $\tilde{u}_j(s) = u_j(t(s))$ with $t(s) = \int_0^s z(\tau)d\tau$, $j = 1, \ldots, p$. By restricting the range of available controls and setting $v_j(t) = u_j(t)z(t)$, we can focus our attention to trajectories of

$$\frac{d\psi}{dt}(t) = (z(t)A + v_1(t)B_1 + \cdots + v_p(t)B_p)\psi(t), \quad z(t) \geq 1, \quad v(t) = (v_1(t), \ldots, v_p(t)) \in U. \tag{18}$$

Each solution of (18) with z and v piecewise constant is the time-reparametrization of a solution of (3) with piecewise constant controls (but the converse is not necessarily true, since we restricted the set of admissible controls). Hence, the approximate simultaneous controllability of (18) implies the approximate simultaneous controllability of (3). Moreover

$$\|\tilde{u}_j\|_{L^1} = \int_0^{t^{-1}(T)} |\tilde{u}_j(\tau)|d\tau = \int_0^T |u_j(t)|z(t)dt = \int_0^T |v_j(t)|dt \leq T,$$

for $j = 1, \ldots, p$. The last inequality holds since either $U_j = [0, 1]$ or $U_j = [-1, 1]$. Hence the approximate simultaneous controllability in $\mathcal{L}$ of (18) with a bound on the controllability time uniform with respect to the tolerance implies, in fact, the L^1-bounded approximate simultaneous controllability of (3).

4.2 Interaction framework

Given a solution $\psi(\cdot)$ of (18) with controls $z(\cdot), v_1(\cdot), \dots, v_p(\cdot)$ and a piecewise constant function $\alpha(\cdot)$ with values in $\{0,1\}$, let us define

$$\omega(t) = \int_0^t (z(s) - \alpha(s))ds$$

and

$$y(t) = e^{-\omega(t)A}\psi(t).$$

In particular

$$|\langle \phi_k, y(t)\rangle| = |\langle \phi_k, \psi(t)\rangle|, \quad k \in \mathbb{N}, \tag{19}$$

for every t. For $\omega, v_1, \dots, v_p \in \mathbb{R}$ set $\Theta(\omega, v_1, \dots, v_p) = e^{-\omega A}(v_1 B_1 + \cdots + v_p B_p)e^{\omega A}$. Note that

$$\Theta(\omega, v_1, \dots, v_p)_{jk} = \langle \phi_k, \Theta(\omega, v_1, \dots, v_p)\phi_j\rangle = e^{i(\lambda_k - \lambda_j)\omega}\left(v_1 (B_1)_{jk} + \cdots + v_p (B_p)_{jk}\right), \tag{20}$$

and that $y(\cdot)$ satisfies

$$\dot{y}(t) = (\alpha(t)A + \Theta(\omega(t), v_1(t), \dots, v_p(t)))y(t), \qquad \alpha \in \{0,1\}, v \in U, \dot{\omega} + \alpha \geq 1. \tag{21}$$

Conversely, each solution of (21) with $\alpha \in \{0,1\}$ and $v \in U$ piecewise constant and ω continuous and piecewise affine, with $\dot{\omega} + \alpha = z \geq 1$ almost everywhere, is, up to a time-dependent change of coordinates preserving the modulus of each component with respect to the basis Φ, a solution of (18) with u piecewise constant. In particular, each solution of

$$\dot{y}(t) = (\alpha(t)A + \Theta(\omega(t), v_1(t), \dots, v_p(t)))y(t), \qquad \alpha \in \{0,1\}, v \in U, \dot{\omega} \geq 1, \tag{22}$$

with α, v piecewise constant and ω continuous and piecewise affine is, up to a time-dependent change of coordinates preserving the modulus of each component, a solution of (18) with u piecewise constant (but the converse is not necessarily true).

Proposition 4.1. *Approximate simultaneous controllability of* (22) *implies approximate simultaneous controllability of* (3). *If, moreover, approximate simultaneous controllability in* $\mathcal{L} = \bigcup_{k\in\mathbb{N}} \operatorname{span}\{\phi_1, \dots, \phi_k\}$ *of* (22) *is achieved with a uniform bound on time then* (3) *is* L^1*-bounded approximate simultaneous controllable.*

Proof. The strategy of the proof follows the idea of the proof of [BCCS12, Proposition 6.1].

It follows from (19) that approximate simultaneous controllability of (22) implies approximate simultaneous controllability of (18) in modulus.

Moreover, because of the unitarity of the evolution, the approximate simultaneous controllability of (22) is equivalent to the approximate simultaneous controllability of the system

$$\dot{y}(t) = -(\alpha(t)A + \Theta(\omega(t), v_1(t), \dots, v_p(t)))y(t), \qquad \alpha \in \{0,1\}, \quad v \in U,$$

which implies approximate simultaneous controllability in modulus of the time-reversed of (18).

Take r orthonormal initial conditions $\psi_0^1, \dots, \psi_0^r$ and r orthonormal final conditions $\psi_1^1, \dots, \psi_1^r$. Since the the spectrum of A is infinite by Assumption ($\mathbb{A}1$) we can apply [BCCS12, Lemma 6.3] so that for every tolerance $\eta > 0$ there exist $k_1, \dots, k_r$ such that

$$\mathcal{C} = \{e^{tA}\phi_{k_1} + \cdots + e^{tA}\phi_{k_r} \mid t \in \mathbb{R}\}.$$

is η-dense in the torus

$$\mathcal{T} = \{e^{\theta_1 A}\phi_{k_1} + \cdots + e^{\theta_r A}\phi_{k_r} \mid \theta_1, \ldots, \theta_r \in \mathbb{R}\}.$$

By approximate simultaneous controllability in modulus of (18) it follows that there exists an admissible control (z, v) steering simultaneously each ψ_0^j, for $j = 1, \ldots, r$, η-close to $e^{\theta_j A}\phi_{k_j}$ for some $\theta_1, \ldots, \theta_r \in \mathbb{R}$.

Similarly, by approximate simultaneous controllability in modulus of the time-reversed of (18) there exists an admissible control $(\tilde{z}, \tilde{v})$ steering system (18) simultaneously, for some $\tilde{\theta}_1, \ldots, \tilde{\theta}_r \in \mathbb{R}$, from $e^{\tilde{\theta}_1 A}\phi_{k_1}, \ldots, e^{\tilde{\theta}_r A}\phi_{k_r}$ to an η-neighborhood of $\psi_1^j, \ldots, \psi_1^r$.

Finally the concatenation of the control (z, v), a control constantly equal to $(1, 0)$ on a time interval of suitable length, and $(\tilde{z}, \tilde{v})$ steers system (18) simultaneously from $\psi_0^j, \ldots, \psi_0^r$ to a 3η-neighborhood of $\psi_1^j, \ldots, \psi_1^r$.

According to the conclusion of Section 4.1, the approximate simultaneous controllability of (18) implies approximate simultaneous controllability of (3). □

4.3 Galerkin approximation

Definition 4.2. Let $N \in \mathbb{N}$. The *Galerkin approximation* of (22) of order N is the system

$$\dot{x} = (\alpha A^{(N)} + \Theta^{(N)}(\omega, v_1, \ldots, v_p))x, \quad x \in \mathcal{H}, \tag{23}$$

where $\Theta^{(N)}(\omega, v_1, \ldots, v_p) = \pi_N \Theta(\omega, v_1, \ldots, v_p)\pi_N$. The controls v are piecewise constant with values in U, while ω is continuous and piecewise affine, with $\dot{\omega} \geq 1$ almost everywhere.

In the following section we recall a convexification result whose role is to identify the matrices that can be obtained by convexification of matrices of the form $\Theta^{(N)}(\omega, v_1, \ldots, v_p)$. Recall that the elements of $\Theta^{(N)}(\omega, v_1, \ldots, v_p)$ are described by (20).

4.4 Convexification

The following technical result has been proved in [BCCS12].

Lemma 4.3. *Let κ be a positive integer and $\gamma_1, \ldots, \gamma_\kappa \in \mathbb{R} \setminus \{0\}$ be such that $|\gamma_1| \neq |\gamma_j|$ for $j = 2, \ldots, \kappa$. Let*

$$\varphi(t) = (e^{it\gamma_1}, \ldots, e^{it\gamma_\kappa}).$$

Then, for every $\tau_0 \in \mathbb{R}$, we have

$$\overline{\mathrm{conv}\varphi([\tau_0, \infty))} \supseteq \varpi S^1 \times \{(0, \ldots, 0)\},$$

where

$$\varpi = \prod_{k=2}^{\infty} \cos\left(\frac{\pi}{2k}\right) > 0. \tag{24}$$

Moreover, for every $R > 0$ and $\xi \in S^1$ there exists a sequence $(\tau_k)_{k=1}^{\infty}$ such that $\tau_1 \geq \tau_0$, $\tau_{k+1} - \tau_k > R$, and

$$\lim_{K\to\infty} \frac{1}{K} \sum_{k=1}^{K} \varphi(\tau_k) = (\varpi\xi, 0, \ldots, 0).$$

4.5 Choice of the order of the Galerkin approximation

In order to prove approximate simultaneous controllability, we should take r in $\mathbb{N}$, $\psi_1, \dots, \psi_r$ in $\mathcal{H}$, $\hat{\Upsilon}$ in $\mathbf{U}(\mathcal{H})$, and $\varepsilon > 0$ and prove the existence of a piecewise constant control $u : [0, T] \to U$ such that

$$\left\| \hat{\Upsilon}\psi_k - \Upsilon^u_T \psi_k \right\| < \varepsilon, \qquad k = 1, \dots, r.$$

Notice that for n_0 large enough there exists $g \in SU(n_0)$ such that

$$|\langle \phi_j, \hat{\Upsilon}\psi_k \rangle - \langle \pi_{n_0}\phi_j, g\pi_{n_0}\psi_k \rangle| < \varepsilon \tag{25}$$

for every $1 \leq k \leq r$ and $j \in \mathbb{N}$. This simple fact suggests to prove approximate simultaneous controllability by studying the controllability of the lift of (22) in the Lie group $SU(n_0)$.

4.6 Control in $SU(n)$

Let $n \geq n_0$ be chosen, in accord with the statement of Theorem 2.6, such that hypothesis (5) holds true. Define the set of matrices

$$\begin{aligned} \mathcal{W}_n = \left\{ A^{(n)} \right\} &\cup \left\{ \mathcal{E}_0(B_j^{(n)}) \mid (0, j) \in \Xi_n \right\} \\ &\cup \left\{ \mathcal{E}_0(B_j^{(n)}) + \varpi \mathcal{E}_\sigma(B_j^{(n)}) \mid (\sigma, j) \in \Xi_n \text{ and } \sigma, j \text{ are such that } (0, j) \in \Xi_n, \sigma \neq 0 \right\} \\ &\cup \left\{ \varpi \mathcal{E}_\sigma(B_j^{(n)}) \mid (\sigma, j) \in \Xi_n,\ \sigma \neq 0,\ \text{and } U_j = [-1, 1] \right\}, \end{aligned}$$

where Ξ_n and ϖ are defined as in (4) and (24), respectively. (Recall that by rescaling we are assuming $\delta = 1$.)

Notice that $\mathrm{Lie}(\mathcal{W}_n) = \mathrm{Lie}(\mathcal{V}_n^0)$.

Consider the auxiliary control system

$$\dot{x} = M(t)x, \quad M(t) \in \mathcal{W}_n, \tag{26}$$

where M plays the role of control. It follows from (5) and standard controllability results on compact Lie groups (see [JS72]) that for every $g \in SU(n)$ there exists a piecewise constant function $M : [0, T] \to \mathcal{W}_n$ such that

$$\overrightarrow{\exp} \int_0^T M(s)\, ds = g,$$

where the chronological notation $\overrightarrow{\exp} \int_0^t V_s\, ds$ is used for the flow from time 0 to t of the time-varying equation $\dot{q} = V_s(q)$, $q \in \mathbb{C}^n$ (see [AS04]).

4.7 System reduction by convexification

Let n be fixed as in the previous section. For every $N \geq n$ let

$$\begin{aligned} \mathcal{W}_{n,N} = \left\{ A^{(N)} \right\} &\cup \left\{ \mathcal{E}_0(B_j^{(N)}) \mid (0, j) \in \Xi_n \right\} \\ &\cup \left\{ \mathcal{E}_0(B_j^{(N)}) + \varpi \mathcal{E}_\sigma(B_j^{(N)}) \mid (\sigma, j) \in \Xi_n \text{ and } \sigma, j \text{ are such that } (0, j) \in \Xi_n, \sigma \neq 0 \right\} \\ &\cup \left\{ \varpi \mathcal{E}_\sigma(B_j^{(N)}) \mid (\sigma, j) \in \Xi_n,\ \sigma \neq 0,\ \text{and } U_j = [-1, 1] \right\}. \end{aligned}$$

Lemma 4.4. *For every $N \geq n$ and for every piecewise constant $M : [0,T] \to \mathcal{W}_{n,N}$ there exist $\alpha : [0,T] \to \{0,1\}$, $v : [0,T] \to U$ piecewise constant and a sequence $(\omega_h(\cdot))_{h\in\mathbb{N}}$ of continuous and piecewise affine functions from $[0,T]$ to $[0,\infty)$ with $\dot{\omega}_h \geq 1$ almost everywhere, such that*

$$\left\| \int_0^t (\alpha(s)A^{(N)} + \Theta^{(N)}(\omega_h(s), v_1(s), \ldots, v_p(s)))ds - \int_0^t M(s)ds \right\| \to 0$$

uniformly with respect to $t \in [0,T]$ as h tends to infinity.

Proof. Let $N \geq n$. Let us fix α and $v_1, \ldots, v_p$ at each $t \in [0,T]$ as follows: if $M(t) = A^{(N)}$ then $\alpha(t) = 1$ and $v_1(t) = \cdots = v_p(t) = 0$; otherwise, if $M(t) = \mathcal{E}_0(B_j^{(N)})$, $M(t) = \mathcal{E}_0(B_j^{(N)}) + \varpi\mathcal{E}_\sigma(B_j^{(N)})$, or $M(t) = \varpi\mathcal{E}_\sigma(B_j^{(N)})$ for some j, then take such a j minimal and set $v_j(t) = 1$ and $\alpha(t) = v_k(t) = 0$ for $k \neq j$.

We are going to apply Lemma 4.3 for every interval on which $M(\cdot)$ is constant. Fix $\omega_h(0) = 0$ for every h. Take an interval (t_0, t_1) on which $M(\cdot)$ is constant and assume that $\omega_h(t_0)$ has been computed. We next extend ω_h on (t_0, t_1).

If $\alpha = 1$ on (t_0, t_1) then take $\omega_h(\tau) = \omega_h(t_0) + \tau - t_0$ for every $\tau \in (t_0, t_1)$.

Otherwise, let $v_j = 1$ on (t_0, t_1) and assume for now that $M(t) = \mathcal{E}_0(B_j^{(N)}) + \varpi\mathcal{E}_\sigma(B_j^{(N)})$. Apply Lemma 4.3 with $\gamma_1 = \sigma$, $\{\gamma_2, \ldots, \gamma_\kappa\} = \Sigma_N \setminus \{\sigma\}$, $\xi = 1$, $R = T$, and $\tau_0 = \omega_h(t_0)$. Then there exists a sequence $(\tau_k)_{k=1}^\infty$ such that $\tau_1 \geq \omega_h(t_0)$, $\tau_{k+1} - \tau_k > T$, and

$$\lim_{K\to\infty} \frac{1}{K} \sum_{k=1}^K (e^{i\tau_k\gamma_1}, \ldots, e^{i\tau_k\gamma_\kappa}) = (\varpi, 0, \ldots, 0).$$

In particular there exists $K = K(h)$ such that

$$\left| \frac{1}{K} \sum_{k=1}^K e^{i(\lambda_l - \lambda_m)\tau_k} b_{ml}^{(j)} - \left(\mathcal{E}_0(B_j^{(N)}) + \varpi\mathcal{E}_\sigma(B_j^{(N)})\right)_{m,l} \right| < \frac{1}{h},$$

for every $1 \leq l, m \leq N$.

Consider the piecewise constant function $Y : (t_0, t_1) \to \mathbb{R}$ defined as follows: set $s_\alpha = t_1 + (t_1 - t_0)\alpha/K$, $\alpha = 0, \ldots, K$, and let

$$Y(t) = \omega_h(t_0) + \sum_{\alpha=1}^K \tau_\alpha \chi_{[s_{\alpha-1}, s_\alpha)}(t)\,.$$

Following the smoothing procedure of [BCCS12, Proposition 5.5] one can construct a continuous piecewise affine approximation $\omega_h : [t_0, t_1] \to \mathbb{R}$ of Y with $\dot{\omega}_h \geq 1$ almost everywhere such that

$$\left\| \int_{t_0}^t (\Theta^{(N)}(\omega_h(s), v_1(s), \ldots, v_p(s)))ds - \int_{t_0}^t M(s)ds \right\| \to 0 \tag{27}$$

uniformly with respect to $t \in [t_0, t_1]$ as h tends to infinity.

The same argument can be carried out in the case in which $M(t) = \mathcal{E}_0(B_j^{(N)})$ by applying Lemma 4.3 with γ_1 in $(0,\infty) \setminus \Sigma_N$, $\{\gamma_2, \ldots, \gamma_\kappa\} = \Sigma_N$, $\xi = 1$, $R = T$, and $\tau_0 = \omega_h(t_0)$.

The final case to be considered is when $M(t) = \varpi \mathcal{E}_\sigma(B_j^{(N)})$ with $\sigma \neq 0$, $(\sigma, j) \in \Xi_n$, and $U_j = [-1, 1]$. Notice that

$$\varpi \mathcal{E}_\sigma(B_j^{(N)}) = \frac{(\mathcal{E}_0(B_j^{(N)}) + \varpi \mathcal{E}_\sigma(B_j^{(N)})) - (\mathcal{E}_0(B_j^{(N)}) + \varpi J_{-1}(\mathcal{E}_\sigma(B_j^{(N)})))}{2}. \tag{28}$$

The argument above can be easily adapted to matrices $M(t)$ of the type $v_j(\mathcal{E}_0(B_j^{(N)}) + \varpi J_\xi(\mathcal{E}_\sigma(B_j^{(N)})))$, with $v_j \in U_j$, $\xi \in S^1$ (just not imposing $\xi = 1$ while applying Lemma 4.3), and in particular to $-(\mathcal{E}_0(B_j^{(N)}) + \varpi J_{-1}(\mathcal{E}_\sigma(B_j^{(N)})))$.

It suffices then to introduce a sequence $(M^h)_{h\in\mathbb{N}}$ of piecewise constant functions with values in

$$\{v_j(\mathcal{E}_0(B_j^{(N)}) + \varpi J_\xi(\mathcal{E}_\sigma(B_j^{(N)}))) \mid v_j \in U_j,\ \xi \in S^1\}$$

such that $\int_{t_0}^t M^h(s)ds$ converges uniformly for $t \in [t_0, t_1]$ to $\int_{t_0}^t M(s)ds$ as h tends to infinity and to apply a diagonal procedure based on the approximation introduced above. $\square$

As a consequence of the lemma above and thanks to [AS04, Lemma 8.2], we have

$$\left\| \overrightarrow{\exp} \int_0^t \left(\alpha(s)A^{(N)} + \Theta^{(N)}(\omega_h(s), v_1(s), \dots, v_p(s))\right) ds - \overrightarrow{\exp} \int_0^t M(s)\, ds \right\| \to 0 \tag{29}$$

uniformly with respect to $t \in [0, T]$ as h tends to infinity.

4.8 Control of the infinite-dimensional system

Next proposition states that we can pass to the limit as N tends to infinity without losing the controllability property (29). Its proof is based on the special sparsity structure of the matrices in $\mathcal{W}_{n,N}$, guaranteeing that the difference between the dynamics of the infinite-dimensional system and the dynamics of the Galerkin approximations is small.

We introduce the following notation: given $n \in \mathbb{N}$ and a bounded linear transformation L of $\mathcal{H}$, let $\mathrm{Crop}_n(L)$ be the $n \times n$ matrix $(\langle \phi_j, L\phi_k \rangle)_{j,k=1}^n$. We use the same symbol Crop_n to denote the similar cropping operation acting on the space of $N \times N$ matrices, with $N \geq n$.

Proposition 4.5. *Let $n \in \mathbb{N}$ and $M : [0, T] \to \mathcal{W}_n$ be piecewise constant. Then, for every $\varepsilon > 0$, there exist piecewise constant controls $z : [0, T] \to [1, \infty)$ and $v : [0, T] \to U$ and a continuous piecewise affine function ω with $\dot{\omega} \geq 1$ such that the propagator Ψ of (22) satisfies*

$$\left\| \overrightarrow{\exp} \int_0^t M(s)\, ds - \mathrm{Crop}_n(\Psi_t) \right\| < \varepsilon,$$

for every $t \in [0, T]$.

Proof. Consider $\mu > 0$ to be fixed later. For every $j \in \mathbb{N}$ the hypothesis that ϕ_j belongs to $D(B_l)$ implies that the sequence $((B_l)_{jk})_{k\in\mathbb{N}}$ is in ℓ^2 for every $l = 1, \dots, p$. It is therefore possible to choose $N \geq n$ such that $\|((B_l)_{jk})_{k>N}\|_{\ell^2} < \mu$ for every $j = 1, \dots, n$ and $l = 1, \dots, p$.

Let $\hat{M}$ be a piecewise constant function from $[0, T]$ to $\mathcal{W}_{n,N}$ such that $\mathrm{Crop}_n \hat{M}(t) = M(t)$ for every t in $[0, T]$. Because of the definition of Ξ_n and of the classes $\mathcal{W}_{n,N}$ and $\mathcal{W}_n$ we have

$$\overrightarrow{\exp} \int_s^t \hat{M}(\tau)\, d\tau = \left(\begin{array}{c|c} \overrightarrow{\exp} \displaystyle\int_s^t M(\tau)\, d\tau & 0 \\ \hline 0 & * \end{array} \right).$$

By Lemma 4.4, for every $\eta > 0$, there exist piecewise constant controls $\alpha : [0,T] \to \{0,1\}$, $v : [0,T] \to U$ and a continuous piecewise affine function ω with $\dot{\omega} \geq 1$ such that

$$\left\| \overrightarrow{\exp} \int_s^t \left(\alpha(\tau) A^{(N)} + \Theta^{(N)}(\omega(\tau), v_1(\tau), \ldots, v_p(\tau))\right) d\tau - \overrightarrow{\exp} \int_s^t \hat{M}(\tau)\, d\tau \right\| < \eta$$

for every s, t in $[0,T]$.

Consider the solution Ψ of (22) associated with α, ω and v. Set, for $k \in \mathbb{N}$,

$$Q_t^{(k)} = \operatorname{Crop}_k \Psi_t.$$

Now

$$\dot{Q}_t^{(N)} = \left(\alpha A^{(N)} + \Theta^{(N)}(\omega, v_1, \ldots, v_p)\right) Q_t^{(N)} + R_t^{(N)},$$

and the choice of N is such that

$$|(R_t^{(N)})_{j,k}| \leq \mu, \tag{30}$$

for every $j = 1, \ldots, n$ and $k = 1, \ldots, N$. Notice, moreover, that the norm of $R_t^{(N)}$ can be uniformly bounded by a positive constant C independent of η (possibly depending on N and hence on μ).

By the variation formula and since $Q_t^{(n)} = \operatorname{Crop}_n(Q_t^{(N)})$ we have

$$\begin{aligned} Q_t^{(n)} = \operatorname{Crop}_n \Bigg[& \overrightarrow{\exp} \int_0^t \left(\alpha(\tau) A^{(N)} + \Theta^{(N)}(\omega(\tau), v_1(\tau), \ldots, v_p(\tau))\right) d\tau + \\ & \int_0^t \left(\overrightarrow{\exp} \int_s^t \left(\alpha(\tau) A^{(N)} + \Theta^{(N)}(\omega(\tau), v_1(\tau), \ldots, v_p(\tau))\right) d\tau \right) R_s^{(N)} ds \Bigg], \end{aligned}$$

so that

$$\begin{aligned} & \left\| \operatorname{Crop}_n \left(\Psi_t - \overrightarrow{\exp} \int_0^t \left(\alpha(\tau) A^{(N)} + \Theta^{(N)}(\omega(\tau), v_1(\tau), \ldots, v_p(\tau))\right) d\tau \right) \right\| \\ & \qquad \leq t\eta C + \left\| \int_0^t \left(\overrightarrow{\exp} \int_s^t M(\tau)\, d\tau \right) \operatorname{Crop}_n \left(R_s^{(N)}\right) ds \right\|. \end{aligned}$$

The norm of the matrix product

$$\overrightarrow{\exp} \int_s^t M(\tau)\, d\tau \operatorname{Crop}_n \left(R_s^{(N)}\right)$$

is equal to

$$\| \operatorname{Crop}_n R_s^{(N)} \|.$$

The *max* norm of $\operatorname{Crop}_n R_s^{(N)}$ is smaller than μ as it follows from (30). Hence

$$\begin{aligned} & \left\| \operatorname{Crop}_n \left(\Psi_t - \overrightarrow{\exp} \int_0^t \left(\alpha(\tau) A^{(N)} + \Theta^{(N)}(\omega(\tau), v_1(\tau), \ldots, v_p(\tau))\right) d\tau \right) \right\| \\ & \qquad \leq T(\eta C + \sqrt{n}\mu). \end{aligned}$$

The constant $T(\eta C + \sqrt{n}\mu)$ can be made arbitrarily small by choosing μ small with respect to n and T and then η small with respect to $C = C(\mu)$ and T. $\square$

Proof of Theorem 2.6. Let r in $\mathbb{N}$, $\psi_1, \dots, \psi_r$ in $\mathcal{H}$, $\hat{\Upsilon}$ in $\mathbf{U}(\mathcal{H})$, and $\varepsilon > 0$. Let n be as in Section 4.6 and let $g \in SU(n)$ satify (25). Notice that if $\psi_1, \dots, \psi_r, \hat{\Upsilon}(\psi_1), \dots, \hat{\Upsilon}(\psi_r)$ are in $\mathcal{L}$ then n can be taken independently of ε.

From Section 4.6, there exists $M : [0, T] \to \mathcal{W}_n$ such that

$$\overrightarrow{\exp} \int_0^T M(s)\, ds = g.$$

Proposition 4.5 ensures the existence of two piecewise constant functions z and v and of a continuous piecewise affine function ω with $\dot{\omega} \geq 1$ almost everywhere such that the associated propagator Ψ of (22) satisfies

$$\left\| \overrightarrow{\exp} \int_0^T M(s)\, ds - \mathrm{Crop}_n(\Psi_T) \right\| < \varepsilon.$$

If $\psi_1, \dots, \psi_r, \hat{\Upsilon}(\psi_1), \dots, \hat{\Upsilon}(\psi_r)$ are in $\mathcal{L}$ then T is independent of ε. By Proposition 4.1 system (3) is L^1-bounded approximately simultaneously controllable. $\square$

5 Proof of Theorem 2.8

In order to prove Theorem 2.8 we adapt the proof of Theorem 2.6. The key point of the argument is the following: it has been proved in Proposition 4.5 that system (18) can track every trajectory of (26). The idea is to replace (26) by a system which can track with arbitrary precision every trajectory in $SU(n)$. The crucial property, beyond the Lie bracket generating condition, that the new version of (26) should satisfy in order to achieve this goal is that it is a driftless system (i.e., the time-reversal of each of its admissible trajectories is itself admissible).

The same time-reparameterization and time-dependent change of coordinates as in Section 4.1 allows to consider the tracking problem for system (22) instead of system (3). As in the previous section we consider δ to be renormalized to 1.

We can then base our argument on the following analogue of Proposition 4.1.

Proposition 5.1. *If* (22) *is a stalker then* (3) *is a stalker as well.*

The following proposition allows to reduce a tracking problem in the space of unitary operators of $\mathcal{H}$ into a tracking problem in $SU(n)$ for n large enough. Its proof can be found in [BCCS12, Proposition 5.7].

Proposition 5.2. *Let $\hat{\Upsilon} : [0, T] \to \mathbf{U}(\mathcal{H})$ be a continuous curve. Take $\varepsilon > 0$ and $m \in \mathbb{N}$. Then for $n \geq m$ sufficiently large there exists a continuous curve $g : [0, T] \to SU(n)$ such that $|\langle \phi_j, \hat{\Upsilon}(t)\phi_k \rangle - \langle e_j, g(t)e_k \rangle| < \varepsilon$ for every t in $[0, T]$, $1 \leq k \leq m$, and $j = 1, \dots, n$, where $e_1, \dots, e_n$ denotes the canonical basis of $\mathbb{R}^n$.*

Let n be chosen as in Proposition 5.2. In accord with the Lie–Galerkin Stalking Condition, we can assume, without loss of generality, that $\mathrm{Lie}(\mathcal{V}_n) = \mathfrak{su}(n)$.

The roles played in Sections 4.6 and 4.7 by $\mathcal{W}_n$ and $\mathcal{W}_{n,N}$ are now played by $\varpi\mathcal{V}_n$ and $\varpi\mathcal{V}_{n,N}$ with

$$\mathcal{V}_{n,N} = \left\{ J_\xi(\mathcal{E}_\sigma(B_j^{(N)})) \mid (\sigma, j) \in \Xi_n, \sigma \neq 0, \xi \in S^1 \right\}.$$

In particular, we consider as auxiliary control system

$$\dot{x} = M(t)x, \qquad M(t) \in \varpi\mathcal{V}_n, \tag{31}$$

M being the matrix-valued control parameter. It follows from the equality $\mathrm{Lie}(\mathcal{V}_n) = \mathfrak{su}(n)$ and Rashevski–Chow's theorem that every trajectory on $SU(n)$ can be tracked with arbitrarily precision (up to time-reparameterization) by a trajectory of (31).

The relation between the trajectories of (23) and those of (31) (or, more precisely, $\dot{x} = M(t)x$, $M(t) \in \varpi\mathcal{V}_{n,N}$), is described by the following lemma.

Lemma 5.3. *For every $N \geq n$ and for every piecewise constant $M : [0,T] \to \varpi\mathcal{V}_{n,N}$ there exist $\alpha : [0,T] \to \{0,1\}$, $v : [0,T] \to U$ piecewise constant and a sequence $(\omega_h(\cdot))_{h\in\mathbb{N}}$ of continuous and piecewise affine functions from $[0,T]$ to $[0,\infty)$ with $\dot{\omega}_h \geq 1$ almost everywhere, such that*

$$\left\| \int_0^t \left(\alpha(s)A^{(N)} + \Theta^{(N)}\left(\omega_h(s), v_1(s), \ldots, v_p(s)\right)\right) ds - \int_0^t M(s)ds \right\| \to 0$$

uniformly with respect to $t \in [0,T]$ as h tends to infinity.

Proof. The proof is almost identical to that of Lemma 4.4 in the case $M(t) = \varpi\mathcal{E}_\sigma(B_j^{(N)})$. The only difference is in replacing (28) by

$$\varpi J_\xi(\mathcal{E}_\sigma(B_j^{(N)})) = \frac{(\mathcal{E}_0(B_j^{(N)}) + \varpi J_\xi(\mathcal{E}_\sigma(B_j^{(N)}))) - (\mathcal{E}_0(B_j^{(N)}) + \varpi J_{-\xi}(\mathcal{E}_\sigma(B_j^{(N)})))}{2}.$$

We then apply the same convexification argument. □

As in the previous section, the lemma above and [AS04, Lemma 8.2] imply that

$$\left\| \overrightarrow{\exp} \int_0^t \left(\alpha(s)A^{(N)} + \Theta^{(N)}(\omega_h(s), v_1(s), \ldots, v_p(s))\right) ds - \overrightarrow{\exp} \int_0^t M(s)\, ds \right\| \to 0$$

uniformly with respect to $t \in [0,T]$ as h tends to infinity.

In analogy with Section 4.8 we can conclude the proof of Theorem 2.8 thanks to the proposition below, which states that we can pass to the limit as N tends to infinity without losing the tracking property of the finite-dimensional Galerkin approximations. Its proof is basically the same as that of Proposition 4.5.

Proposition 5.4. *Let $n \in \mathbb{N}$ and $M : [0,T] \to \varpi\mathcal{V}_n$ be piecewise constant. Then, for every $\varepsilon > 0$, there exist piecewise constant controls $z : [0,T] \to [1,\infty)$ and $v : [0,T] \to U$ and a continuous piecewise affine function ω with $\dot{\omega} \geq 1$ almost everywhere such that the propagator Ψ of (22) satisfies*

$$\left\| \overrightarrow{\exp} \int_0^t M(s)\, ds - \mathrm{Crop}_n(\Psi_t) \right\| < \varepsilon,$$

for every $t \in [0,T]$.

Remark 5.5. The hypothesis that each U_j contains 0 in its interior can be relaxed. Indeed, up to reordering, let p' be such that $U_j = [0, \delta]$ if $j = 1, \dots, p'$ and $U_j = [-\delta, \delta]$ for $j > p'$. Assume that, for every $j \in \{1, \dots, p'\}$, if $l \neq k$ are such that $\lambda_l = \lambda_k$, then $\langle \phi_l, B_j \phi_k \rangle = 0$. Assume, moreover, that the Lie–Galerkin Control Condition is satisfied with $\mathcal{V}_n$ replaced by the set of all matrices $J_\xi(\mathcal{E}_\sigma(B_j^{(n)}))$ with $(\sigma, j) \in \Xi_n$, $\xi \in S^1$, $\sigma \neq 0$, and either $j > p'$ or the following holds: if $l, k, l', k' \in \{1, \dots, n\}$ satisfy $\lambda_l - \lambda_k = \lambda_{l'} - \lambda_{k'} = \sigma$ and

$$\langle \phi_l, B_j \phi_k \rangle \neq 0 \neq \langle \phi_{l'}, B_j \phi_{k'} \rangle$$

then

$$\langle \phi_l, B_j \phi_l \rangle - \langle \phi_k, B_j \phi_k \rangle = \langle \phi_{l'}, B_j \phi_{l'} \rangle - \langle \phi_{k'}, B_j \phi_{k'} \rangle.$$

In this case the proof of Lemma 5.3 becomes more technically involved. The point is that, even if $\mathcal{E}_0(B_j^{(N)})$ cannot be eliminated by convexification, it is a diagonal matrix by hypothesis. Hence, it can be used to define a new interaction framework. The sequence $(\omega_h(\cdot))_h$ can then be constructed by following [BCCS12, Proposition 5.5].

6 Proof of Theorem 2.12

First, let us prove L^1-bounded approximate simultaneous controllability in $s/2$-norm for initial and final data in $\mathcal{L}$. Namely we want to prove that, for $r \in \mathbb{N}$, $\psi_1, \dots, \psi_r \in \mathcal{L}$, and $\hat{\Upsilon} \in \mathbf{U}(\mathcal{H})$ with $\hat{\Upsilon}\psi_1, \dots, \hat{\Upsilon}\psi_r \in \mathcal{L}$, there exists $K > 0$ such that the following holds: For every $\varepsilon > 0$ there exists a control u, with $\|u\|_{L^1} \leq K$ such that

$$\|\psi_j - \Upsilon_T^u \psi_j\|_{s/2} < \varepsilon \qquad j = 1, \dots, r. \tag{32}$$

Let N be such that $\psi_1, \dots, \psi_r$ and $\hat{\Upsilon}\psi_1, \dots, \hat{\Upsilon}\psi_r$ are in $\operatorname{span}\{\phi_1, \dots, \phi_N\}$. Note that on $\operatorname{span}\{\phi_1, \dots, \phi_N\}$ we have

$$\|\psi\|_{s/2} = \left(\sum_{k=1}^{N} |\lambda_k|^s |\langle \phi_k, \psi \rangle|^2 \right)^{1/2} \leq (\max\{|\lambda_1|, |\lambda_N|\})^{s/2} \|\psi\|. \tag{33}$$

Since the system is L^1-bounded approximately simultaneously controllable there exists $K > 0$ such that for every $\varepsilon > 0$ there exists a piecewise constant control u, with $\|u\|_{L^1} \leq K$ such that

$$\|\psi_j - \Upsilon_T^u \psi_j\| < \varepsilon \qquad j = 1, \dots, r.$$

Hence, by (33), we deduce L^1-bounded approximate simultaneous controllability in $s/2$-norm in $\mathcal{L}$.

Now let $\psi_1, \dots, \psi_r \in D(|A|^{s/2})$, and $\hat{\Upsilon} \in \mathbf{U}(\mathcal{H})$ be such that $\hat{\Upsilon}\psi_1, \dots, \hat{\Upsilon}\psi_r \in D(|A|^{s/2})$. Let $\varepsilon > 0$ and consider $\psi_1^0, \dots, \psi_r^0$ and $\psi_1^1, \dots, \psi_r^1$ in $\mathcal{L}$ such that

$$\|\psi_j^0 - \psi_j\|_{s/2} < \varepsilon \quad \text{and} \quad \|\psi_j^1 - \hat{\Upsilon}\psi_j\|_{s/2} < \varepsilon,$$

for $j = 1, \dots, r$. As proved above (see (32)), there exist K, independent on ε, and a piecewise constant control u with $\|u\|_{L^1} \leq K$ such that

$$\|\psi_j^1 - \Upsilon_T^u \psi_j^0\|_{s/2} < \varepsilon \qquad j = 1, \dots, r.$$

By [BCC, Proposition 2], since the system is s-weakly coupled, there exists a constant C depending only on s and $A, B_1, \dots, B_p$ such that

$$\|\Upsilon_T^u \psi\|_{s/2} \leq C^K \|\psi\|_{s/2},$$

for every $\psi \in D(|A|^{s/2})$. Therefore

$$\begin{aligned}\|\Upsilon_T^u(\psi_j) - \hat{\Upsilon}(\psi_j)\|_{s/2} &\leq \|\Upsilon_T^u(\psi_j - \psi_j^0)\|_{s/2} + \|\Upsilon_T^u \psi_j^0 - \psi_j^1\|_{s/2} + \|\psi_j^1 - \hat{\Upsilon}\psi_j\|_{s/2} \\ &\leq (C^K + 2)\varepsilon,\end{aligned}$$

for $j = 1, \dots, r$.

Remark 6.1. Using arguments similar to those of the proof of Theorem 2.6 and of Theorem 2.12 it is possible to prove a finer statement than Corollary 2.13. Indeed it is possible to prove that a system satisfying Assumptions $(\mathbb{A}')$, the Lie–Galerkin Stalking Condition, and which is s-weakly coupled is a stalker for the $s/2$-norm. This is due to the fact that, actually, the Lie–Galerkin Stalking Conditionimplies stalking in $\mathcal{L}$ with a uniform bound on the L^1 norm of the control.

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