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Mathematics of Control, Signals, and Systems

Controllability of Right Invariant Systems on Real Simple Lie Groups of Type F_4, G_2, C_n, and B_n*

R. El Assoudi† and J. P. Gauthier‡

Abstract. We deal with controllability of right-invariant systems for some real simple Lie groups of F_4, G_2, C_n, and B_n types. We prove that the so-called *controllability rank condition* is a necessary and sufficient condition for controllability for an open class of systems. In other papers, analogous results were obtained for Lie groups of the remaining types (i.e., E_6, E_7, E_8, A_n, and D_n) using a special property of the root systems of their Lie algebras.

Key words. Simple Lie groups, Controllability, Invariant vector fields, Root systems.

1. Introduction

We consider control-affine right-invariant systems $\tilde{\Sigma}$ on a real simple Lie group, G, of the form

$$\tilde{\Sigma}\colon \dot{g} = A(g) + uB(g), \qquad g \in G, \quad A, B \in \mathscr{L}(G).$$

A system $\tilde{\Sigma}$ is *controllable* if for any g_0 and g_1 in G there exists a control function u defined on some interval $[0, T]$, satisfying the following:

(a) $g\colon [0, T] \to G$ is an absolutely continuous solution of $(\tilde{\Sigma})$.
(b) $g(0) = g_0$ and $g(T) = g_1$.

More generally, a family Γ of right-invariant vector fields on a Lie group G is *controllable* if for any g_0 and g_1 in G there is a sequence $(X_r, t_r), \ldots, (X_1, t_1)$, $X_i \in \mathscr{L}(G)$, $t_i \in \mathbb{R}^+$, such that $g_1 = \exp(t_r X_r) \ldots \exp(t_1 X_1)(g_0)$.

By invariance of the vector fields, controllability is equivalent to the condition that the subsemigroup S_Γ generated by the elements $\exp(tX)$, $X \in \Gamma$, and $t \geq 0$ be exactly G. Our work was motivated by some results obtained in [JK], [GB], and [GKS]. It is a consequence of the general theory developed in [JK] that a system $\tilde{\Sigma}$ is controllable if and only if the family of right-invariant vector fields $\Sigma = (A, \pm B)$ is controllable.

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† Laboratoire d'Automatique de Grenoble, BP 46, 38402 Saint-Martin-D'Heres, France.

‡ Université Claude Bernard—Lyon 1, Laboratoire d'Automatique UER de Physique, 43 boulevard du 11 Novembre 1918, 69022 Villeurbanne, France.

Let Γ be a family of right-invariant vector fields on G and let $\mathscr{L}(\Gamma)$ be the Lie subalgebra of $\mathscr{L}(G)$ generated by the elements of Γ. We say that $\mathscr{L}(\Gamma)$ satisfies the *controllability rank condition for* Γ if $\mathscr{L}(\Gamma)$ is equal to $\mathscr{L}(G)$. It follows from some refinements of Frobenius's theorem that the controllability rank condition is a necessary condition for controllability. When G is compact it is also sufficient.

In [JK] a sufficient condition for controllability of $(\tilde{\Sigma})$ was obtained for the noncompact case. This condition was improved in [GKS] for some particular cases. The following theorem was obtained in [GKS].

Theorem 1. *Let G be a real connected Lie group, with finite center. Assume its Lie algebra $\mathscr{L}$ is the split real form of a complex simple Lie algebra $\mathscr{L}_c$ of one of the following types: A_r, D_r, E_6, E_7, or E_8. Let $\tilde{\Sigma}$ be such that:*

(1) *B is real and strongly regular.*
(2) *If $s = \sup\{\alpha | \alpha \in \mathrm{Sp}(B)\}$ and $A = A(0) + \sum_{\alpha \in \mathrm{Sp}(B)} A(\alpha)$ then* $\operatorname{trace}(\operatorname{ad} A(s) \circ \operatorname{ad} A(-s)) < 0$.

(For notations see the Appendix). Then the controllability rank condition is a necessary and sufficient condition for the controllability of $\tilde{\Sigma}$.

Remarks. (a) A_r, D_r, E_6, E_7, and E_8 are simple Lie algebras satisfying the following, known as property (K): the absolute value of l in any string $\hat{\alpha} + l\hat{\beta}$ is at most 1 (see notations in the Appendix). Property (K) is a key point used in the proof of Theorem 1 [GKS].

(b) The fact that G has a finite center implies that there is a finite covering relationship between G and Ad G (the adjoint group of G). Based on this relationship, the controllability of $\tilde{\Sigma}$ on G is equivalent to that on Ad G.

(c) Condition (2) of Theorem 1 was first introduced in [JK]. It implies that some vector field of the form $C = \alpha A(s) + \beta A(-s)$, for $\alpha, \beta > 0$, is a compact vector field (i.e., $C \in K$) in a Cartan decomposition. Since a compact vector field is recurrent, we have that $-A(s)$ and $-A(-s)$ are *directions* in which it is possible to *control*. This is another key point in the proof of Theorem 1.

(d) The set of pairs (A, B) satisfying conditions (1) and (2) of Theorem 1 is open and nonempty in $\mathscr{L}(G) \times \mathscr{L}(G)$.

Theorem 1 was first proved in [GB] in the particular case of $G = Sl(n, R)$. In [SC] some refinements of Theorem 1 are given in the cases of A_r and D_r; specifically, the case where B is not strongly regular is examined. Our aim in this paper is to prove that Theorem 1 is still true for split real forms of Lie algebras of the other types. (The remaining types are B_r, C_r, F_4, and G_2; see [W] or [S].) For these Lie algebras, property (K) does not hold, so the method of proof used in [GKS] does not apply.

This paper is organized as follows. Section 2 contains the statement of our main theorem and reviews some facts about controllability of right-invariant systems. Section 3 is devoted to technical lemmas and the proof of our main result. The Appendix describes some basic results on Lie algebras and defines the notations that are used in this paper.

2. Controllability of Right-Invariant Systems

The following is the main result obtained in this paper:

Theorem 2. *Let $\mathscr{L}_c$ be of type B_r, C_r, F_4, or G_2. Assume that the hypotheses of Theorem 1 hold. Then the controllability rank condition is a necessary and sufficient condition for controllability of the system $\tilde{\Sigma}$.*

Let Γ be a family of right-invariant vector fields on a Lie group G. Following Definition 6, p. 164, of [JK] we write $LS(\Gamma)$ for the Lie saturated cone of Γ, i.e., the set of all $X \in \mathscr{L}(\Gamma)$ such that $\{\exp tX | t \geq 0\} \subset \mathrm{Cl}(S_\Gamma)$ where Cl denotes *closure*.

The following properties of $LS(\Gamma)$ can be found in Section I.3 of [JK]:

(P1) $LS(\Gamma)$ is a closed convex cone.

(P2) The edge of $LS(\Gamma)$ (i.e., the greatest vector subspace of $\mathscr{L} = \mathscr{L}(G)$ contained in $LS(\Gamma)$) is a subalgebra that normalizes $LS(\Gamma)$. That is, using $\mathbb{R}$ to denote the set of real numbers and $\mathbb{R}X = \{\lambda X | \lambda \in \mathbb{R}\}$, if $\mathbb{R}X \subset LS(\Gamma)$, then for all $Y \in LS(\Gamma)$ and for all $v \in \mathbb{R}$, $\exp(v \operatorname{ad} X)(Y) \in LS(\Gamma)$.

A necessary and sufficient condition for controllability is that $LS(\Gamma) = \mathscr{L}(G)$. This is the basic fact behind the method developed in [JK]. We start with some elements in $LS(\Gamma)$ (A, $\pm B$ in the case of $\tilde{\Sigma}$) and try to "enlarge" $LS(\Gamma)$ by using properties (P1) and (P2) until $LS(\Gamma) = \mathscr{L}(G)$.

3. Proof of the Main Results

Throughout this section we use the notations of the Appendix.

3.1. Technical Lemmas

Lemma 1 was proved on p. 188 of [GKS]. We include it here for the sake of completeness. The other results are new.

Lemma 1. *Let B be real and strongly regular and $A \in \mathscr{L}$. Assume that $\mathbb{R}B \subset LS(\Gamma)$ and $\mathbb{R}A \subset LS(\Gamma)$. Then, for all $\alpha \in \mathrm{Sp}(B)$ such that $A(\alpha) \neq 0$, $L(\alpha) \subset LS(\Gamma)$.*

Lemma 2. *Let $\hat{\beta}$ be a root such that $\hat{\beta} + \hat{s}$ or $\hat{\beta} - \hat{s}$ is also a root. If $A(\beta) \neq 0$, then $L(\beta) \subset LS(\Gamma)$.*

Proof. Assume that $\hat{\beta} \in S$ and $\hat{\beta} + \hat{s} \in S$. Clearly, $L(s) \subset LS(\Gamma)$. This is a direct consequence of assumption (2) in Theorems 1 and 2 and of property (P2). (For details see Lemma 8 of [JK].) It follows from (P2) that $\Phi(v, X_s) = \exp(v \operatorname{ad} X_s)(A) \in LS(\Gamma)$ for $X_s \in L(s)$ and any $v \in \mathbb{R}$. Since $\hat{s}$ is maximal we have that $2\hat{s} + \hat{\gamma} \in S$ if and only if $\gamma = -\hat{s}$. Then $\mathbb{R} \operatorname{ad}^2 X_s(A(\gamma))$ is either 0 or contained in $LS(\Gamma)$. Expanding $\Phi(v, X_s)$, we get

$$\Phi(v, X_s) = A + \lambda X_s + \sum_{\substack{1 \leq i \leq 3 \\ \gamma \in \mathrm{Sp}(B)}} \frac{v^i}{i!} \operatorname{ad}^i X_s(A(\gamma)), \qquad \lambda \in \mathbb{R},$$

since $|l| \leq 3$ in any string. Note that the terms corresponding to $i = 3$ vanish since $\hat{s}$ is the maximal root and for $i = 2$, the term $\mathrm{ad}^2\, X_s(A(\gamma)) \in \mathbb{R}X_s$. Then it follows that

$$\Phi(v, X_s) = A + \mu X_s + \sum_{\gamma \in \mathrm{Sp}(B)} v[X_s, A(\gamma)], \qquad \mu \in \mathbb{R}.$$

Also

$$\mathbb{R} \sum_{\gamma \in \mathrm{Sp}(B)} [X_s, A(\gamma)] \subset LS(\Gamma).$$

So, applying Lemma 1, we have $L(\beta + s) \subset LS(\Gamma)$ if $A(\beta) \neq 0$. But $L(\beta) = [L(-s), L(\beta + s)]$, therefore $L(\beta) \subset LS(\Gamma)$. The case when $\hat{\beta} - \hat{s}$ is a root is analogously obtained by considering $\Phi(v, X_{-s})$. ■

We now split S into subsets S_1, S_2, and $\{\hat{s}, -\hat{s}\}$ where

$$S_1 = \{\hat{\beta} \in S \mid \hat{\beta} + \hat{s} \in S \text{ or } \hat{\beta} - \hat{s} \in S\}$$

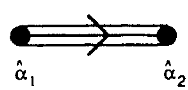
and

$$S_2 = S - S_1 \cup \{\hat{s}, -\hat{s}\}.$$

Clearly, $S_1 = -S_1$. If $\hat{\beta}$ is a positive root in S_1, $\hat{\beta} + \hat{s} \notin S$ since $\hat{s}$ is maximal. But $\hat{\beta} - \hat{s}$ is a root, $\hat{s} - \hat{\beta}$ is a root, and $-\hat{\beta} \in S_1$. Analogously we obtain that $S_2 = -S_2$. This splitting will be useful for the proof of Lemma 3.

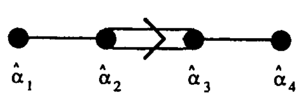
Let us describe the sets S_1, S_2 in our cases (i.e., $\mathscr{L}_c = F_4$, G_2, C_n, or B_n). (See [B], [S], and [W].)

Case of G_2. Primitive roots: $\hat{\alpha}_1, \hat{\alpha}_2$. Dynkin's diagram:



$$\begin{aligned} S^+ &= \{\hat{\alpha}_1, \hat{\alpha}_2, \hat{\alpha}_1 + \hat{\alpha}_2, \hat{\alpha}_1 + 2\hat{\alpha}_2, \hat{\alpha}_1 + 3\hat{\alpha}_2, \hat{s} = 2\hat{\alpha}_1 + 3\hat{\alpha}_2\}, \\ S_1^+ &= \{\hat{\alpha}_1, \hat{\alpha}_1 + 3\hat{\alpha}_2, \hat{\alpha}_1 + \hat{\alpha}_2, \hat{\alpha}_1 + 2\hat{\alpha}_2\}, \\ S_2^+ &= \{\hat{\alpha}_2\}. \end{aligned}$$

Case of F_4. Primitive roots: $\hat{\alpha}_1, \hat{\alpha}_2, \hat{\alpha}_3, \hat{\alpha}_4$. Dynkin's diagram:

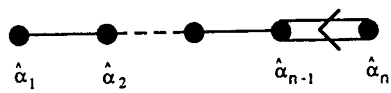


Notation. A root $\hat{\gamma}$ is of the form $\hat{\gamma} = a\hat{\alpha}_1 + b\hat{\alpha}_2 + c\hat{\alpha}_3 + d\hat{\alpha}_4$ where a, b, c, d are integers of the same sign. We denote $\hat{\gamma} = abcd$ and then we have $\hat{s} = 2342$,

$$S_1^+ = \left\{\begin{matrix} 1000, 1100, 1110, 1111, 1120, 1121, 1122, \\ 1342, 1242, 1232, 1231, 1222, 1221, 1220 \end{matrix}\right\},$$

$$S_2^+ = \left\{\begin{matrix} 0100, 0010, 0110, 0120, 0122, \\ 0001, 0011, 0111, 0121 \end{matrix}\right\}.$$

Case of C_n. Primitive roots: $\hat{\alpha}_1, \hat{\alpha}_2, \ldots, \hat{\alpha}_n$. Dynkin's diagram:



The positive roots are of the form:

$$\sum_{i \le k \le j} \hat{\alpha}_k \qquad \text{for} \quad 1 \le i \le j \le n,$$

$$\sum_{i \le k < j} \hat{\alpha}_k + 2 \sum_{j \le k < n} \hat{\alpha}_k + \hat{\alpha}_n \qquad \text{for} \quad 1 \le i < j < n,$$

$$2 \sum_{i \le k < n} \hat{\alpha}_k + \hat{\alpha}_n \qquad \text{for} \quad 1 \le i < n,$$

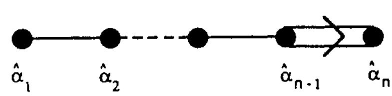
$$S_1^+ = \left\{ \hat{\beta} = \sum_{i \le i \le n} n_i \hat{\alpha}_i \in S \mid n_1 = 1 \right\},$$

$$S_2^+ = \left\{ \hat{\beta} = \sum_{i \le i \le n} n_i \hat{\alpha}_i \in S \mid n_1 = 0 \right\},$$

$$\hat{s} = 2\hat{\alpha}_1 + \cdots + 2\hat{\alpha}_{n-1} + \hat{\alpha}_n.$$

Proof. *Let* $\hat{\gamma} \in S$, $\hat{\gamma} = \sum_{1 \le i \le n} n_i \hat{\alpha}_i$ such that $n_1 = 1$. If $\hat{\gamma}$ is of the form $\hat{\gamma} = \sum_{1 \le k \le j} \hat{\alpha}_k$, $j < n$, then we have $\hat{s} - \hat{\gamma} = \sum_{1 \le k \le j} \hat{\alpha}_k + 2 \sum_{j < k < n} \hat{\alpha}_k + \hat{\alpha}_n$, which is a root, so that $\hat{\gamma} \in S_1^+$ and $\hat{s} - \hat{\gamma} \in S_1^+$. Let us consider $\hat{\gamma} \in S$, $\hat{\gamma} = \sum_{1 \le i \le n} n_i \hat{\alpha}_i$ such that $n_1 = 0$. Then $\hat{s} - \hat{\gamma}$ is of the form $2\hat{\alpha}_1 + \sum_{2 \le j \le n} m_j \hat{\alpha}_j$. Thus $\hat{s} - \hat{\gamma} \notin S$ since $\hat{s}$ is the unique root such that $n_1 = 2$. ∎

Case of B_n. Primitive roots: $\hat{\alpha}_1, \hat{\alpha}_2, \ldots, \hat{\alpha}_n$ Dynkin's diagram:



The positive roots are

$$\sum_{i \le k \le j} \hat{\alpha}_k \qquad \text{for} \quad 1 \le i < j \le n,$$

$$\sum_{i \le k < j} \hat{\alpha}_k + 2 \sum_{j \le k < n} \hat{\alpha}_k \qquad \text{for} \quad 1 \le i < j < n.$$

As for C_n, we observe that

$$S_1^+ = \left\{ \hat{\beta} = \sum_{1 \le i \le n} n_i \hat{\alpha}_i \in S \mid n_2 = 1 \right\},$$

$$S_2^+ = \left\{ \hat{\beta} = \sum_{1 \le i \le n} n_i \hat{\alpha}_i \in S \mid n_2 = 0 \right\},$$

$$\hat{s} = \hat{\alpha}_1 + 2\hat{\alpha}_2 + \cdots + \hat{\alpha}_n$$

(and $\hat{s}$ is the unique root such that $n_2 = 2$).

Remark. We observe that, in the cases of G_2, F_4, and C_n a root $\hat{\beta}$ satisfies:

$\hat{\beta} \in S_1$ if and only if its coefficient with respect to the primitive root $\hat{\alpha}_1$ is ± 1.
$\hat{\beta} \in S_2$ if and only if its coefficient is zero.
$\hat{\beta} = \pm \hat{s}$ if and only if its coefficient is ± 2.

In the case of B_n, we get the same result, by permutation of $\hat{\alpha}_1$ and $\hat{\alpha}_2$. However, we prefer to present the root system of B_n under its usual form. This property is not true for the root systems of the other simple Lie algebras. It will be the key point for the proof of Lemma 3. In [GKS] this was replaced by the fact that any string is of length 1 at most. That is to say, in any expression of the form

$$\exp(v \operatorname{ad} X_\alpha)(X_\beta) = \sum \frac{v^i}{i!} \operatorname{ad}^i X_\alpha(X_\beta)$$

the terms in the summation vanish for $i > 1$.

Lemma 3. *Let $\hat{\beta}$ be a root in S_1. If $L(\beta) \subset LS(\Gamma)$ then $[L(\beta), L(\gamma)] \subset LS(\gamma)$ for any γ such that $A(\gamma) \neq 0$.*

Proof. It is sufficient to prove that if $\hat{\beta} \in S_1$ and $L(\beta) \subset LS(\Gamma)$, then $\mathbb{R} \operatorname{ad}^2 X_\beta(A) \subset LS(\Gamma)$ for $X_\beta \in L(\beta)$ because, if that holds, it will follow that $\mathbb{R} \sum_{i \geq 2} \operatorname{ad}^i X_\beta(A)(v^i/i!) \subset LS(\Gamma)$ and then

$$\Phi(v, X_\beta) - \sum_{i \geq 2} \frac{v^i}{i!} \operatorname{ad}^i X_\beta(A) = A + \lambda X_\beta + v \sum_{\gamma \in \operatorname{Sp}(B)} [X_\beta, A(\gamma)] \in LS(\Gamma)$$

for some real λ and any real v. Applying property (P1) and Lemma 1 the result follows. So now it remains to prove that $\hat{\beta} \in S_1$ and $L(\beta) \subset LS(\Gamma)$ imply that $\mathbb{R} \operatorname{ad}^2 X_\beta(A) \subset LS(\Gamma)$. We have that

$$\mathbb{R} \operatorname{ad}^2 X_\beta(A) = \mathbb{R} \sum_{\gamma \in \operatorname{Sp}(B)} [X_\beta[X_\beta, A(\gamma)]],$$

where $[X_\beta[X_\beta, A(\gamma)]]$ is different from zero when $2\hat{\beta} + \hat{\gamma}$ is a root and $A(\gamma) \neq 0$. Assume that $2\hat{\beta} + \hat{\gamma}$ is a root. If $\hat{\gamma} \in S_1$, by Lemma 2, we know that either $A(\gamma) = 0$ or $L(\gamma) \subset LS(\Gamma)$. Thus $\mathbb{R} \operatorname{ad}^2 X_\beta(A(\gamma)) \subset LS(\Gamma)$. If $\hat{\gamma} \in S_2$, the coefficient of $\hat{\gamma}$, with respect to the primitive root $\hat{\alpha}_1$ (in its decomposition with respect to primitive roots) is zero. Then, for the root $2\hat{\beta} + \hat{\gamma}$, the coefficient is twice the one of $\hat{\beta}$. But $\hat{\beta}$ is in S_1 and its coefficient with respect to $\hat{\alpha}_1$ is ± 1. So the coefficient of $2\hat{\beta} + \hat{\gamma}$ is ± 2. According to the remark, this is possible only for $\pm \hat{s}$. Thus $L(2\beta + \gamma) \subset LS(\Gamma)$. If $\hat{\gamma} = \pm \hat{s}$, the result is obvious. ∎

Now, let $\mathscr{C}_1$ be the Lie algebra generated by $L(s)$, $L(-s)$, and the $L(\alpha) \subset LS(\Gamma)$ with $\hat{\alpha} \in S_1$. Let $\mathscr{C}_2$ be the vector space generated by the $L(\gamma)$ with $A(\gamma) \neq 0$ and $\hat{\gamma} \in S_2$. Let $L_0 = \operatorname{span}\{A(0), B\}$. It follows that $L_0 \subset L(0)$. Let $\tilde{\mathscr{L}}$ be the Lie algebra generated by L_0, $\mathscr{C}_1$, and $\mathscr{C}_2$.

Lemma 4. $\mathscr{L}(A, B) = \tilde{\mathscr{L}}$.

Proof. To show that $\mathscr{L}(A, B) \subset \tilde{\mathscr{L}}$, it is sufficient to note that (a) $B \in L_0$ (by definition) and (b) $A \in L_0 + \mathscr{C}_1 + \mathscr{C}_2$, since

$$A = A(0) + A(s) + A(-s) + \sum_{\hat{\beta} \in S_1} A(\beta) + \sum_{\hat{\gamma} \in S_2} A(\gamma)$$

and, by Lemma 2,

$$\mathbb{R} \sum_{\hat{\beta} \in S_1} A(\beta) \subset \mathscr{C}_1.$$

To show that $\tilde{\mathscr{L}} \subset \mathscr{L}(A, B)$ we prove that L_0, $\mathscr{C}_1$, and $\mathscr{C}_2$ are all contained in $\mathscr{L}(A, B)$. First, since $LS(\Gamma) \subset \mathscr{L}(A, B)$, then $\mathscr{C}_1 \subset \mathscr{L}(A, B)$. Next, by Lemma 1, $\mathscr{C}_2 \subset LS(\pm A, \pm B) = \mathscr{L}(A, B)$, because the lemma shows that, for all $\alpha \in \mathrm{Sp}(B)$ with $A(\alpha) \neq 0$, $L(\alpha) \subset LS(\pm A, \pm B) = \mathscr{L}(A, B)$. Hence $A(0) \in \mathscr{L}(A, B)$. Finally, B and $A(0) \in \mathscr{L}(A, B)$ and thus $L_0 \subset \mathscr{L}(A, B)$. ■

Lemma 5. *$\mathscr{C}_1$ is an ideal in $\mathscr{L}(A, B)$.*

Proof. Since $\mathscr{C}_1$ is a Lie algebra and $[L_0, L(\alpha)] \subset L(\alpha)$ for any $\hat{\alpha} \in S$, it is sufficient to prove that $[\mathscr{C}_1, \mathscr{C}_2] \subset \mathscr{C}_1$. Consider $\hat{\beta} \in \{\hat{s}, -\hat{s}\} \cup \{\hat{\alpha} | \hat{\alpha} \in S_1, L(\alpha) \subset LS(\Gamma)\}$ (i.e., the set of generators of $\mathscr{C}_1$), and $\hat{\gamma} \in S_2$ such that $L(\gamma) \subset \mathscr{C}_2$. It is sufficient to prove that $[L(\beta), L(\gamma)] \subset \mathscr{C}_1$. Clearly, if $\hat{\beta} = \pm \hat{s}$, we have $[L(\beta), L(\gamma)] = 0$ for any $\hat{\gamma}$ in S_2. (See the definition of S_2.) If $\hat{\beta} \in S_1$, according to the remark above, $\hat{\beta} + \hat{\gamma}$ is not a root or is in S_1 and, by Lemma 3, $[L(\beta), L(\gamma)] \subset LS(\Gamma)$. Then $[L(\beta), L(\gamma)] \subset \mathscr{C}_1$. ■

3.2. Proof of the Main Result

Assume that $\mathscr{L}(A, B) = \mathscr{L}$. $\mathscr{C}_1$ is an ideal in $\mathscr{L}(A, B)$ (from Lemma 5). Then $\mathscr{C}_1$ is zero or $\mathscr{L}$. But $\mathscr{C}_1$ is not zero; we already noted in the proof of Lemma 2 that, as a consequence of assumption (2) in Theorems 1 and 2, $\mathscr{C}_1$ contains at least $L(s)$ and $L(-s)$. Since by definition $\mathscr{C}_1$ is contained in $LS(\Sigma)$, we have that $LS(\Sigma) = \mathscr{L}$. By Section 2.2., Σ and $\hat{\Sigma}$ are controllable. The necessity of the result is obvious. ■

Appendix

Let $\mathscr{L}$ be a simple real Lie algebra. Let $\mathrm{Kil}(X, Y) = \mathrm{trace}(\mathrm{ad}\, X \circ \mathrm{ad}\, Y)$ be the Killing form on $\mathscr{L}$. Then $\mathrm{Kil}(X, Y)$ is symmetric and nondegenerate.

A.1. Cartan Decomposition

Let K (resp. P) be the set of all $Z \in \mathscr{L}$ such that $\mathrm{ad}\, Z$ is skew-symmetric (resp. symmetric) with respect to Kil. Then for $Z \in K$

$$\mathrm{Kil}(\mathrm{ad}\, ZX, Y) + \mathrm{Kil}(X, \mathrm{ad}\, ZY) = 0$$

and for $Z \in P$

$$\mathrm{Kil}(\mathrm{ad}\, ZX, Y) - \mathrm{Kil}(X, \mathrm{ad}\, ZY) = 0.$$

$\mathscr{L} = K \oplus P$, where K is a subalgebra, $[K, P] \subset P$, and $[P, P] \subset K$. A *Cartan* *space* $\mathscr{A}$ is a maximal commutative subalgebra of $\mathscr{L}$ contained in P. The elements of $\mathscr{A}$ are simultaneously diagonalizable. All Cartan spaces are conjugated under the action of the subgroup of $\mathrm{Int}(\mathscr{L})$ generated by K.

A.2. Split Real Forms

Let $\mathscr{L}_c = \mathscr{L} \oplus_{\mathbb{R}} \mathbb{C}$ and $\mathscr{A}_c = \mathscr{A} \oplus_{\mathbb{R}} \mathbb{C}$ be the complexification of $\mathscr{L}$ and $\mathscr{A}$, respectively. A *Cartan subalgebra* is a maximal commutative subalgebra of $\mathscr{L}_c$ which is its own normalizer in $\mathscr{L}_c$. $\mathscr{L}$ is called a *split real form of* $\mathscr{L}_c$ if $\mathscr{A}_c$ is a Cartan subalgebra of $\mathscr{L}_c$ for any Cartan space $\mathscr{A}$ of $\mathscr{L}$. Hereafter, $\mathscr{L}$ is assumed to be a split real form of $\mathscr{L}_c$ and $\mathscr{A}$ a Cartan space of $\mathscr{L}$.

A.3. Regular and Strongly Regular Elements

$B \in \mathscr{L}$ is *regular* if and only if B is semisimple and $\mathrm{Ker}(\mathrm{ad}\, B)$ is of minimal dimension.

B is *strongly regular* if and only if all the eigenspaces corresponding to the nonzero eigenvalues of ad B are one-dimensional in $\mathscr{L}_c$.

B is *real* and *strongly regular* if and only if B is strongly regular and the nonzero eigenvalues of ad B are real.

The set of strongly regular elements is open and dense in $\mathscr{L}$. The set of real, strongly regular elements is open and nonempty.

Let B be real and strongly regular. We will denote by $\mathrm{Sp}(B)$ the set of nonzero eigenvalues of ad B and by $L(\alpha)$ the eigenspace $\mathrm{Ker}(\mathrm{ad}\, B - \alpha I)$, where $\alpha \in \mathrm{Sp}(B)$. Let $L(0) = \mathrm{Ker}\, \mathrm{ad}\, B$. Then we have

$$\mathscr{L} = L(0) \oplus \sum_{\alpha \in \mathrm{Sp}(B)} L(\alpha),$$

and $L(0)$ is a Cartan space of $\mathscr{L}$ and $B \in L(0)$. Also, any $X \in \mathscr{L}$ can be written in a unique way,

$$X = X(0) + \sum_{\alpha \in \mathrm{Sp}(B)} X(\alpha),$$

where $X(0) \in L(0)$ and $X(\alpha) \in L(\alpha)$.

A.4. Root Systems

Let B be real and strongly regular. Assume $B \in \mathscr{A}$ (so that $L(0) = \mathscr{A}$). For any $\alpha \in \mathrm{Sp}(B)$, there is a unique linear form $\hat{\alpha}$ on $\mathscr{A}$, such that $\mathrm{ad}\, hX = \hat{\alpha}(h)X$, for $X \in L(\alpha)$, for all $h \in \mathscr{A}$. $\hat{\alpha}$ is called a *root of* $\mathscr{A}_c$. Also, there is a unique $H_\alpha \in L(0)$ such that $\mathrm{Kil}(h, H_\alpha) = \hat{\alpha}(h)$, for all $h \in \mathscr{A}$. Define $\langle \hat{\alpha}, \hat{\beta} \rangle = \mathrm{Kil}(H_\alpha, H_\beta)$, where $\langle \cdot, \cdot \rangle$ is a positive definite bilinear form on $\mathscr{A}$. Let $\|\hat{\alpha}\| = \sqrt{\langle \hat{\alpha}, \hat{\alpha} \rangle}$ and denote by S the set of roots of $\mathscr{A}_c$. We can choose $X_\alpha \in L(\alpha)$ such that:

(a) $[X_\alpha, X_{-\alpha}] = H_\alpha$.

(b) $[X_\alpha, X_\beta] = 0$ if $\hat{\alpha} + \hat{\beta} \neq 0$ and $\hat{\alpha} + \hat{\beta} \notin S$, whereas $[X_\alpha, X_\beta] = N_{\alpha\beta} X_{\alpha+\beta}$ if $\hat{\alpha} + \hat{\beta} \in S$.

(c) Let $\hat{\beta}, \hat{\alpha} \in S$. Consider the set of integers l such that $\hat{\beta} + l\hat{\alpha}$ is 0 or a root of $\mathscr{A}_c$. This set of integers is called *the $\hat{\alpha}$-string of $\hat{\beta}$*. Any string of integers l with

$p \leq l \leq q$ is *unbroken* when

$$p + q = -2\frac{\hat{\beta}(H_\alpha)}{\hat{\alpha}(H_\alpha)},$$

$$l \leq \max\left(\frac{\|\hat{\alpha}\|^2}{\|\hat{\beta}\|^2}, \frac{\|\hat{\beta}\|^2}{\|\hat{\alpha}\|^2}\right).$$

A.5. Ordering of roots

S can be identified with $\mathrm{Sp}(B)$ and totally ordered as a subset of $\mathbb{R}$. Also, $S = -S$.

(a) $\hat{\alpha}$ is *positive* (resp. *negative*) if $\alpha > 0$ (resp. $\alpha < 0$), or, equivalently, if $\hat{\alpha}(B) > 0$ (resp. $\hat{\alpha}(B) < 0$).

(b) S splits into S^+ and S^-, the sets of positive and negative roots.

(c) A root $\hat{\alpha}$ of S^+ is *primitive* if $\hat{\alpha}$ cannot be written as the sum of two other positive roots. The set of primitive roots is a basis of the dual space of $\mathscr{A}$, denoted $\mathscr{A}^*$. Any root $\hat{\alpha}$ can be written as a sum $\hat{\alpha} = \sum n_i \hat{\alpha}_i$ where $\hat{\alpha}_i$ is primitive and all the n_i are integers with the same sign.

For the proofs of the root properties stated above, we refer to pp. 9 and 10 of [JK] or pp. 38–49 of [S].

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