Notes on the equivalence between controllability of $SU(1,1)$ and $Sp(2,\mathbb{R})$

I. ISOMORPHISM BETWEEN $SU(1,1)$ AND $Sp(2,\mathbb{R})$

An generic element of the Lie group $su(1,1)$ can be written as

$$O = \begin{pmatrix} \alpha & \beta \\ \beta^* & \alpha^* \end{pmatrix} \tag{1}$$

where $\alpha, \beta \in \mathbb{C}$ satisfying $|\alpha|^2 - |\beta|^2 = 1$. One can prove that there is an isomorphism between $SU(1,1)$ and $Sp(2,\mathbb{R})$, such that the corresponding element of the symplectic group can be written as

$$S = \begin{pmatrix} \alpha_r - \beta_r & \beta_i - \alpha_i \\ \beta_i + \alpha_i & \alpha_r + \beta_r \end{pmatrix} \tag{2}$$

where $\alpha = \alpha_r + i\alpha_i$ and $\beta = \beta_r + i\beta_i$.
One can also find the isomorphism between the corresponding Lie algebra elements. Let us write a generic element $M \in su(1,1)$ as

$$M = \begin{pmatrix} ia & b + ic \\ b - ic & -ia \end{pmatrix}, \tag{3}$$

where $a, b, c \in \mathbb{R}$. The corresponding element $A \in sp(2,\mathbb{R})$, can be written as

$$A = \begin{pmatrix} -b & c - a \\ c + a & b \end{pmatrix}. \tag{4}$$

II. CONTROLLABILITY OF THE LIE GROUP SU(1,1)

In Ref. [1] they consider the controllability problem for the $SU(1,1)$ group with a single *input* (that is with a single *control*), which reads

$$\begin{aligned} \dot{X}(t) &= (M_0 + u(t)M_1)\, X(t)\,, \\ X(0) &= \mathbb{1}_2\,, \end{aligned} \tag{5}$$

where M_0 and M_1 are elements of the Lie algebra $su(1,1)$.
They proved the following theorem:
Theorem. The system is controllable if and only is the set

$$\Omega = \left\{ u \in \mathbb{R} \mid \langle\, M(u), M^\dagger(u)\,\rangle < 0 \right\} \tag{6}$$

is nonempty, where

$$M(u) = M_0 + uM_1 \tag{7}$$

$$\langle A, B\rangle = \mathrm{Tr}[AB^\dagger]\,. \tag{8}$$

By considering the generic element of the $su(1,1)$ algebra, as in Eq. (3),

$$M(u) = \begin{pmatrix} ia & b + ic \\ b - ic & -ia \end{pmatrix}, \tag{9}$$

the condition $\langle\, M(u), M^\dagger(u)\,\rangle < 0$, simply corresponds to

$$b^2 + c^2 - a^2 < 0. \tag{10}$$

III. CONTROLLABILITY OF THE LIE GROUP $Sp(2,\mathbb{R})$

By considering the isomorphism between $SU(1,1)$ and $Sp(2,\mathbb{R})$, we can rephrase the controllability problem 5, in terms of quadratic Hamiltonians for a single-mode with a single control Hamiltonian

$$\begin{aligned} \dot{\hat{U}}(t) &= \left(\hat{H}_0 + u(t)\hat{H}_1\right) U(t)\,, \\ \hat{U}(0) &= \mathbb{1}\,. \end{aligned} \tag{11}$$

Notice that every Hamiltonian operator can be written as $\hat{H} = (1/2)\hat{R}^\mathsf{T} H \hat{R}$, where $\hat{R} = (\hat{x}, \hat{p})^\mathsf{T}$ and H is a 2×2 symmetric matrix.
In [2], we proved that a sufficient condition for the *symplectic* controllability of this system is to find a linear combination $\hat{H}(u) = \hat{H}_0 + u\hat{H}_1$, such that the corresponding Hamiltonian matrix is positive (negative) definite. By using the isomorphism described above, one can easily check that the condition is equivalent to the one in Eq. (10). In detail, the corresponding generator of $sp(2,\mathbb{R}$ can be written as

$$A(u) = \begin{pmatrix} -b & c - a \\ c + a & b \end{pmatrix}. \tag{12}$$

and the corresponding Hamiltonian matrix (such that $\hat{H}(u) = (1/2)\hat{R}^\mathsf{T} H(u) \hat{R}$) can be obtained as

$$H(u) = -\Omega A(u) = \begin{pmatrix} -c - a & -b \\ -b & c - a \end{pmatrix}. \tag{13}$$

The determinant of the matrix reads $\det(H(u)) = a^2 - b^2 - c^2$, and its positivity corresponds to Eq. (10).
Because of the result in [1], we now know that the condition we obtained is not only sufficient, but also necessary for controllability. Moreover Eq. 10 is also enough to fulfil the conditions required by the Lie algebra rank criterion (see the paper [1] for details).
Other results that follows from the paper [1]:

- The system described by is never *strongly controllable* [3].
- The system is *small time local controllable* [3] if the Hamiltonian matrix corresponding to H_1 is positive (negative) definite.

Moreover if we consider two control Hamiltonians, that is the control problem

$$\begin{aligned} \dot{\hat{U}}(t) &= \left(\hat{H}_0 + u_1(t)\hat{H}_1 + u_2(t)\hat{H}_2\right) \hat{U}(t)\,, \\ \hat{U}(0) &= \mathbb{1}\,. \end{aligned} \tag{14}$$

where H_1 and H_2 are linearly independent. Then, (tutte queste cose vanno controllate meglio!)

- if H_0 can be written as a linear combination of H_1 and H_2, then the system is uncontrollable if the Hamiltonian matrix corresponding to $[\hat{H}_1, \hat{H}_2]$ has a zero eigenvalue. Otherwise the system is strongly controllable.
- If H_0, H_1, and H_2 are linearly independent, the system is controllable. Moreover if $[\hat{H}_1, \hat{H}_2]$ has no zero eigenvalues, the system is strongly controllable.

[1] J.-W. Wu, C.-W. Li, J. Zhang and T.-J. Tarn, arXiv:0708:3147v1 [math.OC] (2007).

[2] M. G. Genoni, A. Serafini, M. S. Kim and D. Burgarth, Phys. Rev. Lett. (2012).

[3] A system is said to be *controllable* if, for any element of the group, there is a time such that this element can be obtained.
A system is said to be *strongly controllable* if at any time, any element of the group can be obtained.
A system is said to be *small time local controllable* if the *identity* is an interior point of the reachable set $R(t)$, for any $t > 0$.

[4] D. D'Alessandro, *Introduction to Quantum Control and Dynamics*, (Taylor & Francis, Boca Raton, 2008).