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Some Properties of Vector Field Systems That Are Not Altered by Small Perturbations

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1. INTRODUCTION

In the study of systems of vector fields from the point of view of Control Theory, the properties of *accessibility* and *complete controllability* (defined in Section 2) are of fundamental importance. The purpose of this paper is to prove (a) that both properties are stable under small perturbations and (b) that the first one is generic.

More precisely, let ${}_rV(M)^k$ denote the set of all k-tuples of C^r vector fields on the C^{r+1} n-dimensional separable manifold M ($1 \leqslant r \leqslant \infty$). Let ${}_rA_k$, ${}_rC_k$ denote the subsets of ${}_rV(M)^k$ consisting, respectively, of those k-tuples that have the accessibility property, and those that are completely controllable. We show that ${}_rA_k$ and ${}_rC_k$ are *open* in the fine C^1 topology of ${}_rV(M)^k$, and that ${}_rA_k$ is dense in the fine C^r topology if $k \geqslant 2$. Our results contain as particular cases those of Lobry [4, 5] and Stefan [7], but they improve them in various ways.

We now describe the precise connection between our work and that of Lobry and Stefan. Both authors dealt with the property that we call "controllability in Lobry's sense," and abbreviate as c.L.s. (cf. Section 8). In [4], Lobry proved genericity in the C^r topology for $r \geqslant n^2 + n$, and he later improved this to $r \geqslant 2n$ (cf. [5]). Stefan [7, Theorem 2.6] proves the same result for M compact, $r \geqslant 1$. (It is clear that [5, 7] do not improve the *density* part of the result of [4], but only the openess part.) The work of Lobry and Stefan proves that the set ${}_rL_k$ of c.L.s. systems is dense and contains an open set. We prove that ${}_rA_k$ is open and dense (Theorems 5.3 and 7.1) and this trivially implies that ${}_rL_k$ is open and dense (Theorem 8.1). Thus, the main novelty of our work is the result that ${}_rL_k$ (and ${}_rA_k$) *are actually open* in the fine C^1 topology. Since ${}_rA_k$ is a proper subset of ${}_rL_k$, our density theorem is slightly better than those of [4, 5, 7].

In order to prove the openness result, it becomes necessary to isolate a

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a property which is equivalent to accessibility and is demonstrably stable under small perturbations (in the fine C^1 topology). We define such a property in Section 2, and we call it "normal accessibility." A point y is ***normally reachable*** from x if, roughly, there is a control μ which steers x into y and which can be embedded in a family $\{\mu(\lambda)\}$ of controls, depending *smoothly* on the m-dimensional parameter λ, in such a way that $\mu(\lambda_0) = \mu_0$, and that the map $\lambda \to y(\lambda)$ has rank n at λ_0 (here $y(\lambda)$ is the point to which x is steered by $\mu(\lambda)$). A system has the ***normal accessibility property*** if for every x there is a y which is normally accessible from x. In Section 4 we prove that normal accessibility is equivalent to accessibility (Theorem 4.1). This result is of importance in its own right. We also define normal controllability, we prove its equivalence to controllability (Theorem 4.3), and we use this to prove the openness of ${}_rC_k$ (Theorem 5.3). The density of ${}_rA_k$ is proved as Theorem 7.2. The main advantage of our proof is that it is completely elementary, in that it makes no use of "Chow's Theorem" and the Lie algebra machinery of previous work on accessibility [1, 2, 6, 8], nor of the theory of orbits of vector field systems. Also, we do not assume that M is compact. (For M compact the result follows trivially from our much easier Theorem 7.1.)

Since the openness of ${}_rA_k$ is available to us, we can prove some new density theorems, such as Theorem 8.3.

Finally, we conclude this introduction with a brief sketch of an important application of Theorem 5.3. It has been proved by N. Levitt and this author that on every connected paracompact finite-dimensional C^∞ manifold M there exists a completely controllable pair (X, Y) of C^∞ vector fields. If M is real analytic, it follows from Theorem 5.3 (plus the fact that C^∞ vector fields can be approximated in the fine C^∞ topology by real analytic ones), that X and Y *can be taken to be real analytic* (cf. [3]).

2. Notations and Basic Definitions

The word "manifold" will always mean a finite-dimensional, separable, differentiable manifold of class C^r, where $1 \leqslant r \leqslant \infty$ or $r = \omega$. As usual, we let $r - 1 = r = r + 1$ when $r = \infty$ or $r = \omega$. The real line is denoted by $\mathbb{R}$, and n-dimensional Euclidean space by $\mathbb{R}^n$. We will be working with an arbitrary but fixed manifold, which we shall denote by M, and the letter n will always denote the dimension of M.

If M is of class C^r, $r \geqslant 1$, then the tangent bundle $T(M)$ is a manifold of class C^{r-1}. If $k \leqslant r - 1$, then the set of all vector fields of class C^k on M will be denoted by ${}_kV(M)$. On ${}_kV(M)$ we can consider, for each $j \leqslant k, j \neq \omega$, two topologies, namely, the C^j and the fine C^j topology, whose definition we now recall.

If K is a compact set which is contained in the domain U of a C^r chart $\{x_1, \ldots, x_n\}$, then every vector field $X \in {}_kV(M)$ can be expressed on U as a linear combination

$$X = \sum_{i=1}^{n} (X_{xi})(\partial/\partial_{xi}).$$

We then associate with K the topology in which a sequence $\{X^m\}_{m=1}^{\infty}$ of vector fields converges to X if and only if $X^m x_i \to X x_i$ for $i = 1, \ldots, n$, uniformly on K, together with all partial derivatives $\partial^{j_1+\cdots+j_n}(Xx_i)/(\partial x_1^{j_1} \cdots \partial x_n^{j_n})$ of order $j_1 + \cdots + j_n \leqslant j$. This is called the *topology of C^j convergence* on K, and it is easily seen to be independent of the choice of U and the chart $\{x_1, \ldots, x_n\}$. The set ${}_kV(M)$ with the topology of C^j convergence on K will be denoted by ${}_kV(K)_j$. It is clear that ${}_kV(K)_j$ is not Hausdorff. If $K = K_1 \cup \cdots \cup K_s$, then the topology of ${}_kV(K)_j$ is the smallest that makes all the identity maps ${}_kV(K)_j \to {}_kV(K_l)_j$ continuous. Hence, if J is an arbitrary compact set, and if $J = J_1 \cup \cdots \cup J_p$, where the J_i are contained in domains of coordinate charts, it follows that the weakest topology on ${}_kV(M)$ which makes all the identity maps ${}_kV(M) \to {}_kV(J_i)_j$ is independent of how J is expressed as a finite union as above. We call it the *topology of C^j convergence on J*. When ${}_kV(M)$ is equipped with this topology, we shall use ${}_kV(J)_j$ to denote it. If J itself is contained in the domain of a coordinate chart, it is clear that this topology coincides with the one that was defined before. We remark that ${}_kV(J)_j$ is not Hausdorff unless $J = M$.

The *C^j topology on ${}_kV(M)$* is the weakest topology that makes all the maps ${}_kV(M) \to {}_kV(K)_j$ continuous. The space ${}_kV(M)$, with this topology, is denoted by ${}_kV(M)_j$. Then ${}_kV(M)_j$ is clearly Hausdorff, and metrizable (because M is a union of countably many compact sets). If $j = k$, then ${}_kV(M)_j$ is complete.

The *fine C^j topology* on ${}_kV(M)$ is the topology in which a fundamental system of neighborhoods of zero is given by all the sets V that are of the form

$$V = \bigcap_{\alpha \in A} V_\alpha,$$

where $\{V_\alpha\}_{\alpha \in A}$ is such that there is a locally finite family $\{K_\alpha\}_{\alpha \in A}$ of compact sets such that V_α is a neighborhood of zero in ${}_kV(K_\alpha)_j$ for each $\alpha \in A$. The set ${}_kV(M)$, with the fine C^j topology will be denoted by ${}_kV(M)_{j,\mathrm{fine}}$. It is clear that ${}_kV(M)_{j,\mathrm{fine}}$ is not metrizable (unless M is compact, in which case ${}_kV(M)_{j,\mathrm{fine}} = {}_kV(M)_j$). Indeed, a *sequence* $\{X^m\}_{m=1}^{\infty}$ of vector fields converges to X in the fine C^j topology if and only if $X^m \to X$ in the C^j topology and if, for sufficiently large m, the supports of $X^m - X$ are contained in a fixed compact set. Hence the set Σ of vector fields with noncompact support is such that no sequence in Σ converges to zero while, clearly, zero is in the closure of Σ.

If l is a positive integer, then the set of all l-tuples of C^k vector fields on M is denoted by ${}_kV(M)^l$. We will topologize it as a product of l copies of ${}_kV(M)_j$, in which case we shall denote it by ${}_kV(M)_j^l$, or as a product of l copies of ${}_kV(M)_{j,\text{fine}}$, in which case we shall denote it by ${}_kV(M)_{j,\text{fine}}^l$.

If X is a vector field of class C^k on the C^{k+1} manifold M (with $k \geqslant 1$), it follows from the well-known properties of ordinary differential equations that through every $x \in M$ there exists a unique *integral curve* $t \to \gamma(t)$ of X, such that $\gamma(0) = x$ and that the interval of definition of γ is the largest possible. To make the dependence on x explicit, we write $X_t(x)$ rather than $\gamma(t)$. Then it is well known that $(t, x) \to X_t(x)$ is a C^k map from an open subset of $\mathbb{R} \times M$ into M. More generally, if $\xi = (X^1,..., X^l)$ is a finite sequence of vector fields, and if $T = (t_1,..., t_l)$ is a finite sequence of real numbers, we use ξ_T to denote the map $X_{t_1}^1 \circ X_{t_2}^2 \circ \cdots \circ X_{t_l}^l$. It follows that $(T, x) \to \xi_T(x)$ is a C^k map from an open subset of $\mathbb{R}^l \times M$ into M. For each $x \in M$, we shall use $D_x(\xi)$ to denote the set of all $T \in \mathbb{R}^l$ for which $\xi_T(x)$ is defined. Therefore $D_x(\xi)$ is open in $\mathbb{R}^l$, and $0 \in D_x(\xi)$.

Let S be a set of vector fields on the manifold M. A subset A of M is called *S-invariant* if $X_t(x) \in A$ whenever $x \in A$, $X \in S$, $t \in \mathbb{R}$, are such that $X_t(x)$ is defined. Similarly, if it is only required that $X_t(x)$ belong to A when $x \in A$, $X \in S$, and $t \geqslant 0$, then A is called *forward S-invariant*. The smallest forward S-invariant set that contains the point $x \in M$ is called the *positive S-orbit* of S from x, and is denoted by $O_+(x, S)$. Similarly, the *orbit* of S through x is the smallest S-invariant set that contains x. Orbits have been studied in [9].

An alternative description of the positive orbits can be given as follows. A sequence $\xi = (X^1,..., X^l)$ for which $X^i \in S$, $i = 1,..., l$ will be called an *S-sequence*. Let $D_x^+(\xi)$ denote the set of all l-tuples $T = (t_1,..., t_l)$ of positive numbers such that $\xi_T(x)$ is defined. Then the positive orbit $O_+(x, S)$ is the set of all the points $\xi_T(x)$, where ξ is an S-sequence and $T \in D_x^+(\xi)$, plus the point x.

Still a third description of the positive orbits can be given by means of trajectories. Let $\xi = (X^1,..., X^l)$ be an S-sequence, and let $T \in D_x^+(\xi)$. Let $T = (t_1,..., t_l)$, and define, for $0 \leqslant j \leqslant l$,

$$T_j = t_{l+1-j} + t_{l+2-j} + \cdots + t_l$$

(so that $T_0 = 0$, $T_1 = t_l$, $T_2 = t_{l-1} + t_l$, etc.). Define $\eta(t)$ for $0 \leqslant t \leqslant T_l$ by $\eta(t) = (0,..., t - T_j, t_{l+1-j}, t_{l+2-j},..., t_l)$ for $T_j \leqslant t \leqslant T_{j+1}$. Then the curve $t \to \xi_{\eta(t)}(x)$ is said to be a *trajectory* of S. More precisely, the expression ξ_T is called, for obvious reasons, a piecewise constant control, and $t \to \xi_{\eta(t)}(x)$ is the trajectory of ξ_T from the initial state x. It is clear that $O_+(x, S)$ is the set of all points of M that belong to some trajectory of S from x. A point y is said to be *S-reachable* or S-accessible) from x if $y \in O_+(x, S)$. The relation of S-reachability is reflexive and transitive but, in general, it is not symmetric.

If S is a set of vector fields on M, and if $x \in M$, we say that S has the *accessibility property* from x if the positive orbit $O_+(x, S)$ has a nonempty interior. We say that S has the *accessibility property from a set* A if it has the accessibility property from everty point of A, and we say that S simply has the *accessibility property* if it has the accessibility property from M.

The set S is called *controllable*, or *completely controllable* if, for every $x \in M$, $O_+(x, S) = M$.

A point y is S-reachable from x, if and only if there exists an S-sequence ξ and a $T^0 \in D_x^+(\xi)$ such that $\xi_T(x) = y$. If T^0 can be chosen so that the map $T \rightarrow \xi_T(x)$ has rank n at T^0, we say that y is *normally S-reachable* (or *normally S-accessible*) *from* x. The following trivial facts will be used frequently:

(a) If y is normally reachable from x, x reachable from z and w reachable from y, then w is normally reachable from z.

(b) If y is normally reachable from x, then $O_+(x, S)$ contains a neighborhood of y.

We say that S has the property of *normal reachability* (or *normal accessibility*) *from* x if some point $y \in M$ is normally S-reachable from x. Similarly, we define *normal reachability from a set*, and *normal reachability*.

Finally, S is said to be *normally controllable* if y is normally reachable from x for every $x \in M$, $y \in M$.

3. Some Lemmas

In this section, we prove some lemmas, of which the most important ones are Lemmas 3.8 and 3.9. These, when combined with Theorems 4.1 and 4.3, will establish our basic openess results, namely, that accessibility and complete controllability are preserved under small perturbations in the fine C^1 topology.

In the following lemmas, we will use the concept of "reachability within Ω." If Ω is an open subset of M, we shall say that a system S of vector fields on M has a certain property *within* Ω if the system S_Ω whose elements are the restrictions to Ω of the vector fields in S has the given property. Thus, for instance, if x and y belong to Ω, y is S-reachable from x within Ω if and only if y is S-reachable from x by a trajectory which is entirely contained in Ω.

LEMMA 3.1. *Let Ω be open in M, with compact closure K. Let $k \geqslant 1$ be an integer. Then the set of all $(x, y, S) \in \Omega \times \Omega \times {}_1V(M)^k$ such that y is normally S-reachable from x within Ω is open in $\Omega \times \Omega \times {}_1V(K)_1^k$.*

Proof. Assume that $\{x^m\}, \{y^m\}$ are sequences of points of Ω, and $\lim x^m = x^0$, $\lim y^m = y^0$ as $m \to \infty$, where $x^0 \in \Omega, y^0 \in \Omega$. Assume that S, S^m $(m = 1, 2, ...)$ are k-tuples of C^1 vector fields on M, and that $S^m \to S$ in the topology of C^1 convergence on K. We shall establish our conclusion by proving that, if y^0 is normally S-reachable from x^0 within Ω, then y^m is normally S^m-reachable from x^m within Ω, if m is sufficiently large. Since y^0 is normally S-reachable from x^0 within Ω, there is an S-sequence ξ and a $T^0 \in D_{x^0}^+(\xi)$ such that $\xi_{T^0}(x^0) = y^0$, and that the map F defined by $F(T) = \xi_T(x^0)$ has rank n at T^0. Choose an n-dimensional submanifold P of $D_{x^0}^+(\xi)$ which contains T^0 and is such that F, restricted to P, has rank n at T^0. Then choose a closed ball B in P, having T^0 in its interior, and such that F maps a neighborhood of B (in P) diffeomorphically onto a neighborhood of y^0. Let $S = (X^1, ..., X^k)$, and let

$$S^m = ({}^mX^1, ..., {}^mX^k).$$

If $\xi = (X^{i_1}, ..., X^{i_l})$, define ξ^m to be $({}^mX^{i_1}, ..., {}^mX^{i_l})$. It follows from standard properties of ordinary differential equations that the following facts are true if m_0 is sufficiently large:

(a) $B \subseteq D_{x^m}^+(\xi^m)$ for $m \geqslant m_0$.

(b) $F_m \to F$ in the topology of C^1 convergence on B.

Here F_m is the map $T \to \xi_T{}^m(x^m)$, and F_m is defined on B, if $m \geqslant m_0$. It follows from (a) and (b) that, if m is sufficiently large, then F_m has rank n at every $T \in B$. Hence every point in $F_m(B)$ is normally S^m-reachable from x^m. Since the map F is a homeomorphism from B onto a neighborhood of y^0, and $F_m \to F$ uniformly, it follows by a standard topological argument that, for m sufficiently large, there is a *fixed* neighborhood U of y^0 such that $U \subseteq F_m(B)$. Hence, if m is sufficiently large, $y^m \in U$ and therefore y^m is normally S^m-reachable from x^m. This completes the proof.

LEMMA 3.2. *Let Ω be open in M with compact closure K. Let k be a positive integer, and let J be a compact subset of $\Omega \times \Omega$. Then the set of all $S \in {}_1V(M)^k$ such that for every $(x, y) \in J$ y is normally S-reachable from x within Ω is open in ${}_1V(K)_1^k$.*

Proof. Let $S \in {}_1V(M)^k$ be such that y is normally S-reachable from x within Ω whenever $(x, y) \in J$. For each $(x, y) \in J$ there are, by Lemma 3.1, neighborhoods $U_{x,y}$ of (x, y) in $\Omega \times \Omega$ and $W_{x,y}$ of S in ${}_1V(K)_1^k$ such that, if $(x', y') \in U_{x,y}$, $S' \in W_{x,y}$, then y' is normally S'-reachable from x' within Ω. Choose a finite set F of points (x, y) such that the corresponding $U_{x,y}$ cover J, and let

$$W = \bigcap_{(x,y) \in F} W_{x,y}.$$

Then W is a neighborhood of S in ${}_1V(K)_1^k$ and, if $S' \in W$, it follows that y' is normally S'-reachable from x' for every $(x', y') \in J$.

LEMMA 3.3. *Let Ω be open in M with compact closure K. Let k be a positive integer and let J be a compact subset of Ω. Then the set of all $S \in {}_1V(M)^k$ such that S has the property of normal accessibility from J within Ω is open in ${}_1V(K)_1^k$.*

Proof. Let $S \in {}_1V(M)^k$ have the property of normal accessibility from J within Ω. For each $x \in J$ there is a $y \in \Omega$ such that y is normally S-reachable from x within Ω. By Lemma 3.1, we can choose neighborhoods U_x of x in Ω and W_x of S in ${}_1V(K)_1^k$ such that y is normally S'-reachable from x' within Ω whenever $x' \in U_x$, $S' \in W_x$. In particular, it follows that every $S' \in W_x$ has the property of normal accessibility from U_x within Ω. Now choose a finite set F of points such that the corresponding U_x cover J, and let

$$W = \bigcap_{x \in F} W_x .$$

Then every $S' \in W$ has the property of normal accessibility from J.

LEMMA 3.4. *Let Ω be open in M with compact closure K, and let V be an open subset of Ω. Let k be a positive integer. Then the set of all $(x, S) \in \Omega \times {}_1V(M)^k$ such that x can be steered by a trajectory of S to some point of V within Ω is open in $\Omega \times {}_1V(K)_1^k$.*

Proof. Assume $x \in \Omega$, $S \in {}_1V(M)^k$, $y \in V$ are such that x can be steered to y, within Ω, by a trajectory of S. Let ξ be an S-sequence, and let $T \in D(\xi)$ be such that $\xi_T(x) = y$ and that the corresponding trajectory lies entirely within Ω. If S' is sufficiently close to S in ${}_1V(K)_1^k$ and if x' is close to x, it follows by continuity that $\xi_T'(x') \in V$ and that the trajectory from x' to $\xi_T'(x')$ lies in Ω, where ξ' is the S'-sequence that corresponds to ξ in an obvious way. This completes the proof.

LEMMA 3.5. *Let Ω be open in M with compact closure K. Let J, V be subsets of Ω, J compact and V open. Let k be a positive integer. Then the set of all $S \in {}_1V(M)^k$ such that every $x \in J$ can be steered to some $y \in V$ by a trajectory of S within Ω is open in ${}_1V(K)_1^k$.*

Proof. Assume $S \in {}_1V(M)^k$ is such that every $x \in J$ can be steered to some $y \in V$ by an S-trajectory in Ω. By Lemma 3.4, for each $x \in J$ there are neighborhoods U_x of x in Ω, W_x of S in ${}_1V(K)_1^k$ such that, if $x' \in U$, $S' \in W_x$ then x' can be steered within Ω to some $y \in V$ by an S'-trajectory. Choose finitely many x whose U_x cover J, and let W be the intersection of the corresponding W_x. Then W is a neighborhood of S in ${}_1V(K)_1^k$, and every

$S' \in {}_1V(K)_1^k$ has the property that, if $x \in J$, then x can be steered to some $y \in V$, within Ω, by an S'-trajectory.

LEMMA 3.6. *Let J be a compact subset of M, and let $S \in {}_1V(M)^k$ have the property of normal accessibility from J. Then there exists an open set Ω with compact closure such that S has the property of normal accessibility from J within Ω.*

Proof. For each $x \in J$ there is $y \in M$ normally S-accessible from x. An S-trajectory which steers x to y normally is contained in an open set Ω_x with compact closure. Then y is normally S-reachable from x within Ω_x and, by Lemma 3.1, there is a neighborhood U_x of x such that y is normally S-reachable from x' within Ω_x for every $x' \in U_x$. In particular, S has the property of normal accessibility from U_x within Ω_x. Now choose a finite set F of points x such that the U_x cover J, and take Ω to be the union of the Ω_x, $x \in F$. Then Ω satisfies the desired conditions.

LEMMA 3.7. *Let $S \in {}_1V(M)^k$ and let J be a compact subset of $M \times M$ such that, if $(x, y) \in J$, then y is normally S-reachable from x. Then there exists Ω open with compact closure such that, if $(x, y) \in J$, then y is normally S-reachable from x within Ω.*

Proof. For $(x, y) \in J$ an S-trajectory which normally steers x to y is contained in an open set $\Omega_{x,y}$ with compact closure. By Lemma 3.1 there is a neighborhood $U_{x,y}$ of (x, y) such that, if $(x', y') \in U_{x,y}$ then y' is normally S-reachable from x' within $\Omega_{x,y}$. Choose finitely many (x, y) whose $U_{x,y}$ cover J, and take Ω to be the union of the corresponding $\Omega_{x,y}$. Then Ω satisfies the desired conditions.

LEMMA 3.8. *Let k be a positive integer. Then the set of all $S \in {}_1V(M)^k$ which satisfy the normal accessibility property is open in ${}_1V(M)^k_{1,\mathrm{fine}}$.*

Proof. Take a sequence $K_1, K_2, \ldots$, of compact sets such that $K_i \subseteq \operatorname{int} K_{i+1}$ for all i, and such that $M = \bigcup_{i=1}^{\infty} K_i$. Let $S \in {}_1V(M)^k$ have the normal accessibility property. We shall find a neighborhood W of S in ${}_1V(M)^k_{1,\mathrm{fine}}$, such that every $S' \in W$ also has that property.

Using Lemma 3.6 choose, for each i, an open set Ω_i with compact closure L_i such that S has the property of normal accessibility from K_i within Ω_i. Define a new sequence Ω_i' of open sets by letting $\Omega_i' = \Omega_i$ for $i = 1, 2$, and then

$$\Omega_i' = \Omega_i - K_{i-2} \qquad \text{for} \quad i > 2.$$

Let L_i' denote the closure of Ω_i', so that the L_i' are compact. In addition, *the family of sets L_i' is locally finite.*

We now define a sequence W_i of neighborhoods of S in ${}_1V(L_i')_1^k$. For $i = 1, 2$, we use the fact that S has the property of normal accessibility from K_i within Ω_i'. By Lemma 3.3, there exists a neighborhood W_i of S in ${}_1V(L_i')_1^k$ such that every $S' \in W_i$ has the property of normal accessibility from K_i within Ω_i'.

For $i > 2$, we first define two subsets P_i, Q_i of Ω_i' as follows: P_i is the set of all $x \in \Omega_i'$ such that S has the property of normal accessibility from x within Ω_i'. As for Q_i, it is the set of those $x \in \Omega_i'$ that can be steered by S, within Ω_i', to some point in the open set $A_i = \Omega_i' \cap \operatorname{int} K_{i-1}$.

Both P_i and Q_i are open. For P_i, this follows from Lemma 3.1, and for Q_i from Lemma 3.4. In addition, *the compact set* $K_i - \operatorname{int} K_{i-1}$ *is contained in* $P_i \cup Q_i$. Indeed, if $x \in K_i - \operatorname{int} K_{i-1}$ then there is a $y \in \Omega_i$ which is normally S-reachable from x within Ω_i. If the S-trajectory that steers x to y normally never hits K_{i-2}, then it is entirely contained in Ω_i', and $x \in P_i$. Otherwise there is a first time t_0 for which this trajectory hits K_{i-2}. If $0 < t_1 < t_0$, then the restriction of our trajectory to $[0, t_1]$ lies in Ω_1'. Since the trajectory is continuous and K_{i-2} is contained in $\operatorname{int} K_{i-1}$, it follows that, if t_1 is sufficiently close to t_0, the above-mentioned restriction steers x to a point in A_i, and then $x \in Q_i$.

Let H_i, G_i be compact sets such that $H_i \subseteq P_i$, $G_i \subseteq Q_i$, and that $K_i - \operatorname{int} K_{i-1} \subseteq H_i \cup G_i$.

By Lemma 3.3, there is a neighborhood W_i^1 of S in ${}_1V(L_i')_1^k$ such that every $S' \in W_i^1$ has the property of normal accessibility from H_i within Ω_i'. By Lemma 3.5, there is a neighborhood W_i^2 of S in ${}_1V(L_i')_1^k$ such thât if $S' \in W_i^2$ then every point of G_i can be steered by an S'-trajectory within Ω_i' to some point of A_i.

We now let $W_i = W_i^1 \cap W_i^2$. We have defined, for each i, a neighborhood W_i of S in ${}_1V(L_i')_1^k$. Since the family $\{L_i'\}$ is locally finite, the intersection $W = \bigcap_{i=1}^{\infty} W_i$ is a neighborhood of S in ${}_1V(M)_{1,\text{fine}}^k$.

To conclude our proof, we show that every S' in W has the normal accessibility property. Let $S' \in W$. Then $S' \in W_1 \cap W_2$ and it follows from the way W_1 and W_2 were defined, that S' has the property of normal accessibility from K_1 and K_2. Assuming that S' has the property of normal accessibility from K_m, we show that it also has this property from K_{m+1}. If $x \in K_{m+1}$ then either $x \in K_m$ or $x \in H_{m+1}$ or $x \in G_{m+1}$. In the first case, the inductive hypothesis tells us that S' has the property of normal accessibility from x. In the second case we use the fact that $S' \in W_{m+1}^1$. Then S' has the property of normal accessibility from H_{m+1} within Ω_{m+1}', so that some $y \in M$ is normally S'-reachable from x. Finally, in the third case, we use the fact that $S' \in W_{m+1}^2$. Then x can be steered by S' to some $y \in A_{m+1}$. In particular, y is in K_m, so that, by the inductive hypothesis, some $z \in M$ is normally reachable from y, and hence from x.

The proof of our statement is now complete.

LEMMA 3.9. *Let k be a positive integer. Then the set of all normally completely controllable $S \in {}_1V(M)^k$ is open in ${}_1V(M)^k_{1,\text{fine}}$.*

Proof. Let $S \in {}_1V(M)^k$ be normally completely controllable. Let $K_1, K_2, \ldots,$ be a sequence of compact sets with $K_i \subseteq \operatorname{int} K_{i+1}$ and $M = \bigcup_{i=1}^{\infty} K_i$.

For each K_i, S has the property that, if x and y are in K_i, then y is normally reachable from x. By Lemma 3.7 (with $J = K_i \times K_i$) there are open sets Ω_i with compact closure L_i such that, if $(x, y) \in K_i \times K_i$, then y is normally S-reachable from x within Ω_i. Let $\Omega_i' = \Omega_i$ for $i = 1, 2$ and $\Omega_i' = \Omega_i - K_{i-2}$ for $i > 2$. Let L_i' = closure of Ω_i'. Then the family of compact sets L_i' is locally finite. We define a sequence W_i of neighborhoods of S in ${}_1V(L_i')^k_1$.

For $i = 1, 2$, choose W_i such that, if $S' \in W_i$, then y is normally S'-reachable from x within Ω_i' for all $x \in K_i$, $y \in K_i$ (the existence of such W_i follows from Lemma 2 with $\Omega = \Omega_i$, $J = K_i \times K_i$).

For $i > 2$, observe that the set $F_i = K_i - \operatorname{int} K_{i-1}$ is compact and contained in Ω_i'. Moreover, every $x \in F_i$ can be steered by an S-trajectory within Ω_i' to some point in the open set $B_i = \Omega_i' \cap \operatorname{int} K_{i-1}$. To see this, take an arbitrary $y \in K_{i-2}$. By the construction of Ω_i, there is a trajectory of S within Ω_i which goes from x to y. Since $x \notin K_{i-2}$, there is a first time $t_0 > 0$ in which the trajectory enters K_{i-2}. Since $\Omega_i' = \Omega_i - K_{i-2}$, it follows that the restriction to $[0, t_1]$, $t_1 < t_0$, is an S-trajectory in Ω_i'. By continuity, the inclusion $K_{i-2} \subseteq \operatorname{int} K_{i-1}$ implies that, if t_1 is close enough to t_0, then our S-trajectory lies in Ω_i' and ends at a point of B_i.

Since F_i and B_i are subsets of Ω_i', F_i is compact and B_i open, it follows from the preceding paragraph that we can apply Lemma 3.5 and find a neighborhood W_i^1 of S in ${}_1V(L_i')^k_1$ such that, if $S' \in W_i^1$, $x \in F_i$, then x can be steered by an S'-trajectory within Ω_i' to some $y \in B_i$.

By a similar reasoning it follows that every $x \in F_i$ can be reached by an S-trajectory within Ω_i' from some $y \in B_i$. This says that every $x \in F_i$ can be steered to some $y \in B_i$, within Ω_i', by a trajectory of the system $-S$ whose vector fields are the negatives of those of S. Applying Lemma 3.5 again, we get a neighborhood W_i^3 of $-S$ in ${}_1V(L_i')^k_1$ such that for every system in W_i^3 and every $x \in F_i$ there is a trajectory within Ω_i' from x to some $y \in B_i$. If we let W_i^2 be the neighborhood of S in $V_1(L_1')^k_1$ whose elements are the S' such that $-S' \in W_i^3$, it follows that, if $S' \in W_i^2$ and $x \in F_i$ then x can be reached from some $y \in B_i$ by an S'-trajectory within Ω_i'.

We now take $W_i = W_i^1 \cap W_i^2$. We have defined a sequence W_i of neighborhoods of S in ${}_1V(L_i')_1^k$. By the local finiteness of the L_1', the set

$$W = \bigcap_{i=1}^{\infty} W_i$$

is a neighborhood of S in ${}_1V(M)_{1,\text{fine}}^k$.

To complete our proof, we show that every $S' \in W$ is normally completely controllable. We prove by induction on m that, if $S' \in W$, $x \in K_m$, $y \in K_m$, then y is normally S' = reachable from x.

For $m = 1, 2$ this follows from the fact that $S' \in W_m$ and the way W_1, W_2 were defined. Assume that our assertion is true for m, and let x, y belong to K_{m+1}. If both x and y are in the interior of K_m then y is normally S'-reachable from x by the inductive hypothesis. If y is in int K_m but $x \notin \text{int } K_m$, then $x \in F_{m+1}$. Since $S' \in W_{m+1}^1$ then x can be steered by an S'-trajectory to some point $z \in B_{m+1}$. In particular, $z \in K_m$ so that, by the inductive hypothesis, y is normally S'-reachable from z, and hence from x.

A similar reasoning, using $S' \in W_{m+1}^2$, works if $x \in \text{int } K_m$, $y \notin \text{int } K_m$. Finally, the case when neither x nor y are in int K_m is handled in the same way, using both $S' \in W_{m+1}^1$ and $S' \in W_{m+1}^2$.

This concludes our proof.

4. The Equivalence of Accessibility and Normal Accessibility

In this section we prove that accessibility implies normal accessibility (Theorem 4.1) and a similar result for controllability (Theorem 4.3). We conjecture that a result stronger than Theorem 4.1 is true, but we have only been able to prove it under more restrictive assumptions (cf. Theorem 4.2 and the remarks that precede it).

Theorem 4.1. *Let S be a set of C^r vector fields on the C^{r+1} manifold M, $r \geqslant 1$. Then S has the accessibility property if and only if it has the normal accessibility property.*

Proof. It is clear that normal accessibility implies accessibility. To prove the converse, let x be a point of M from which the normal accessibility property does not hold. We will find $y \in M$ such that the positive S-orbit from y has an empty interior.

For each S-sequence ξ and each $T^0 \in D_x^+(\xi)$, let $k(\xi, T^0)$ denote the rank at T^0 of the map $T \to \xi_T(x)$. Let k denote the largest value of $k(\xi, T^0)$, as ξ varies over all S-sequences, and T^0 over all elements of $D_x^+(\xi)$. Our assumption about x is that $k < n$. If ξ, T^0 are such that $k(\xi, T^0) = k$, then $T \to \xi_T(x)$

maps a neighborhood $U_{T^0,\xi}$ of T^0 onto a k-dimensional C^r submanifold $N_{T^0,\xi}$. It follows that $N_{T^0,\xi}$ is entirely contained in the positive orbit of x.

Let $\mathscr{F}$ denote the set of all connected k-dimensional submanifolds N of M with the property that every $y \in N$ has a neighborhood (relative to N) of the form $N_{T^0,\xi}$. Then $\mathscr{F}$ is partially ordered by the relation $\leqslant$ defined as follows: $N_1 \leqslant N_2$ if N_1 is an open submanifold of N_2. We have seen that $\mathscr{F}$ is nonempty, and it is clear that, if $\mathscr{F}_0 \subseteq \mathscr{F}$ is totally ordered, then the union N of all the elements of $\mathscr{F}_0$ has a unique C^r structure which makes it into a connected manifold that contains all the elements of $\mathscr{F}_0$ as open submanifolds. Hence $N \in \mathscr{F}$.

It follows from Zorn's Lemma that $\mathscr{F}$ contains a maximal element N. We now show that N is forward S-invariant, i.e., that

(#) if $y \in N$, $X \in S$, $t \geqslant 0$, and $X_t(y)$ is defined, then $X_t(y) \in N$.

To prove this, we first show that every $X \in S$ is actually tangent to N. If $y \in N$, let $y = \xi_{T^0}(x)$, where ξ is an S-sequence, $T^0 \in D_x^+(\xi)$ and $k(\xi, T^0) = k$. Let η denote the S-sequence (X, ξ). If the rank of $(t, T) \to \eta_{(t,T)}(x)$ at $(0, T^0)$ were larger than k, it would follow that $k(\eta, (t^0, T^0)) > k$ for $t^0 > 0$ sufficiently small. This obviously contradicts the definition of k. Hence $(t, T) \to \eta_{(t,T)}(x)$ has rank k at $(0, T^0)$, so that $X(y)$ belongs to the image of the tangent space of $D_x^+(\xi)$ at T^0 under the differential of $T \to \xi_T(x)$. Hence $X(y)$ is tangent to N at y, as asserted.

Now let $X \in S$. For each $t \geqslant 0$, X_t is a C^r diffeomorphism of an open submanifold N_t of N onto a C^r manifold $\tilde{N}_t$. The intersection $\tilde{N}_t \cap N$ is the set of all points $y \in N$ for which $X_{-t}(y)$ is defined and belongs to N. Since X is tangent to N, it follows from standard properties of ordinary differential equations that, if $y \in \tilde{N}_t \cap N$, then X_{-t} maps a neighborhood U of y in N diffeomorphically to a neighborhood V (in N) of $X_{-t}(y)$. Therefore $\tilde{N}_t \cap N$ is an open submanifold of both N and $\tilde{N}_t$. If $t_1 \geqslant t_2 \geqslant 0$ then $\tilde{N}_{t_1} = X_{t_1-t_2}(\tilde{N}_{t_2})$ so that, by the same reasoning, it follows that $\tilde{N}_{t_1} \cap \tilde{N}_{t_2}$ is an open submanifold of both $\tilde{N}_{t_1}$ and $\tilde{N}_{t_2}$. Form a manifold $\tilde{N}$ by requiring that all the $\tilde{N}_t$, $t \geqslant 0$, be open submanifolds of $\tilde{N}$. Since X is tangent to N and hence to $\tilde{N}$, it follows that the integral curve $t \to X_t(y)$, $t \geqslant 0$, is, for each $y \in N$, a continuous mapping into $\tilde{N}$. Hence every point of $\tilde{N}$ can be joined by a continuous curve in $\tilde{N}$ to a point of N. Since N is connected, so is $\tilde{N}$.

If $\tilde{y} \in \tilde{N}$, then $\tilde{y} = X_{t^0}(y)$ for some $t^0 \geqslant 0$, $y \in N$. Then $y = \xi_{T^0}(x)$ for an S-sequence ξ, $T^0 \in D_x^+(\xi)$ and $k(\xi, T^0) = k$. Moreover, we can assume that $T \to \xi_T(x)$ maps a neighborhood U of T^0 in $D_x^+(\xi)$ to a neighborhood V of y in N. Then $(t^0, T) \to X_{t^0}\xi_T(x)$ maps a neighborhood of (t^0, T^0) in $\{t^0\} \times D_x^+(\xi)$ to a neighborhood of $\tilde{y}$ in $\tilde{N}_{t^0}$, and hence in $\tilde{N}$. Since X is tangent to $\tilde{N}$, $(t, T) \to X_t\xi_T(x)$ maps a neighborhood of (t^0, T^0) in $D_x^+(X, \xi)$ onto a neighborhood of $\tilde{y} \in \tilde{N}$.

This shows that $\tilde{N} \in \mathscr{F}$. Since N is an open submanifold of $\tilde{N}$, it follows that $N = \tilde{N}$. Hence $X_t(y) \in N$ for all $y \in N$ and all $X \in S$, $t \geqslant 0$, such that $X_t(y)$ is defined. This proves ((#). To complete the proof of our theorem, take an arbitrary $y \in N$. It follows from (#) that the positive orbit $O_+(y, S)$ is entirely contained in N, which is a connected submanifold of M of strictly positive codimension. Hence $O_+(y, S)$ has no interior points, and S does not have the accessibility property from y.

The preceding theorem was proved by showing that, if the normal accessibility property from x does not hold, then there is some y in the positive orbit $O_+(x, S)$ from which the accessibility property does not hold. It would be nice to prove a stronger result, namely, that accessibility from x implies normal accessibility from x. If S is a countable set of C^∞ vector fields on the C^∞ manifold M, this can be proved easily, as shown in Theorem 4.2 below. We remark, however, that our proof of this result requires the countability of S, and that the infinite differentiability is also needed in order to apply Sard's theorem to a family of mappings from open subsets of spaces $\mathbb{R}^k$ into M, because we have no upper bound on k. We do not know whether the result is true for uncountable S or for vector fields of class C^1.

THEOREM 4.2. *Let S be a finite or countable set of C^∞ vector fields on the C^∞ manifold M. Assume that the positive orbit from a point $x \in M$ has a nonempty interior. Then S has the normal accessibility property from x.*

Proof. For each S-sequence ξ, let Φ_ξ denote the map $T \to \xi_T(x)$, $T \in D_x^+(\xi)$. If the normal accessibility property from x does not hold, then the differential of Φ_ξ has rank $< n$ for all $T \in D_x^+(\xi)$. Then, by Sard's theorem, the image $\Phi_\xi(D_x^+(\xi))$ has measure zero in M. The positive orbit $O_+(x, S)$ is the union of these images, as ξ ranges over all S-sequences. Since S is countable, there are countably many S-sequences. Therefore $O_+(x, S)$ has measure zero and, in particular, its interior is empty.

THEOREM 4.3. *Let S be a system of C^r vector fields on the C^{r+1} manifold M ($1 \leqslant r \leqslant \infty$). Then the followingconditions are equivalent:*

(i) *S is controllable,*

(ii) *S is normally controllable, and*

(iii) *M is connected and, for every $x \in M$, x is normally accessible from x.*

Proof. The implication (ii) ⇒ (iii) is trivial. We prove (i) ⇒ (ii) and (iii) ⇒ (i).

(i) ⇒ (ii). Assume S is controllable. Then, by Theorem 4.1, S has the normal accessibility property. If x and y are in M, then there is some $z \in M$

$z \in M$ such that z is normally reachable from x. Since S is controllable, y is reachable from z and therefore y is normally reachable from x.

(iii) $\Rightarrow$ (i). Let $x \in M$ and let Ω denote the positive S-orbit from x. We first show that Ω is open. If $y \in \Omega$, there is an S-sequence ξ and a $T \in D_x^+(\xi)$ such that $\xi_T(x) = y$. Since (iii) holds, there is a neighborhood U of x which is S-reachable from x. By shrinking U, if necessary, we can assume that $\xi_T(x')$ is defined for all $x' \in U$. Then ξ_T maps U diffeomorphically onto a neighborhood V of y, which is entirely contained in Ω. Therefore Ω is open.

We now prove that Ω is closed. Let y belong to the closure of Ω. Since y is normally reachable from y, it follows from Lemma 3.1 that y is normally reachable from z for all z in some neighborhood U of y. In particular, U contains some $z \in \Omega$. Therefore z is reachable from x, and y is reachable from z, so that $y \in \Omega$.

Since M is connected, $\Omega = M$. Since x was an arbitrary point of M, our proof is complete.

5. The Openness Theorems

Let M be a manifold of class C^{r+1}, and let $k \geqslant 1$ be an integer. Use ${}_rA_k$ to denote the set of all k-tuples of C^r vector fields that have the accessibility property, and ${}_rC_k$ to denote the set of those k-tuples that are completely controllable. If K is a compact subset of M, let ${}_rA_k(K)$ denote the set of k-tuples that have the property of *normal* accessibility from every $x \in K$. Similarly, let ${}_rC_k(K)$ denote the set of those k-tuples such that y is normally reachable from x for every $(x, y) \in K \times K$. Let $1 \leqslant j \leqslant r$.

THEOREM 5.1. *${}_rA_k(K)$ and ${}_rC_k(K)$ are open in ${}_rV(M)_j^k$.*

THEOREM 5.2. *${}_rA_k$ and ${}_rC_k$ are G_δ subsets of ${}_rV(M)_j^k$. If M is compact, both sets are actually open. If M is not compact, both sets have empty interiors in ${}_rV(M)_j^k$.*

THEOREM 5.3. *${}_rA_k$ and ${}_rC_k$ are open in ${}_rV(M)_{j,\mathrm{fine}}^k$.*

Proof of Theorems 5.1, 5.2, 5.3. Let $S \in {}_rA_k(K)$. By Lemma 3.6, there is an open set Ω with compact closure L such that S has the property of normal accessibility from K within Ω. By Lemma 3.3 there is a neighborhood U of S in ${}_1V(L)_1^k$ such that $U \subseteq {}_1A_k(K)$. Then $U \cap {}_rV(M)^k$ is a neighborhood of S in ${}_rV(M)_j^k$ and is contained in ${}_rA_k(K)$. Therefore ${}_rA_k(K)$ is open in ${}_rV(M)_j^k$. The proof that ${}_rC_k(K)$ is open is similar, using Lemma 3.7 (with $J = K \times K$) and Lemma 3.2. By Theorem 4.1, ${}_rA_k$ is exactly the set of all k-tuples of C^r

vector fields that have the normal accessibility property. By Lemma 3.8, ${}_1A_k$ is open in ${}_1V(M)^k_{1,\mathrm{fine}}$. Hence ${}_rA_k = {}_1A_k \cap {}_rV(M)^k$ is open in ${}_rV(M)^k_{1,\mathrm{fine}}$. Since the topology of ${}_rV(M)^k_{j,\mathrm{fine}}$ is finer, ${}_rA_k$ is open in ${}_rV(M)^k_{j,\mathrm{fine}}$.

The proof that ${}_rC_k$ is open in ${}_rV(M)^k_{j,\mathrm{fine}}$ is similar, using Theorem 4.3 and Lemma 3.9. Theorem 4.1 also implies that ${}_rA_k$ is the intersection of all the sets ${}_rA_k(K)$, where K ranges over all compact subsets of M. Since M is separable, we can limit ourselves to countably many compact sets, and then we conclude that ${}_rA_k$ is a G_δ set. The proof that ${}_rC_k$ is a G_δ is similar.

If M is not compact, the set ${}_rV_0(M)^k$ of k-tuples of C^r vector fields with compact support is dense in ${}_rV(M)^k_j$. Clearly, ${}_rV_0(M)^k$ and ${}_rA_k$ are disjoint. Hence the interior of ${}_rA_k$ in ${}_rV(M)^k_j$ is empty. Since ${}_rC_k \subseteq {}_rA_k$, the same is true for ${}_rC_k$. This completes the proof of Theorems 5.1, 5.2, 5.3.

6. More Lemmas

The lemmas of this section will enable us to prove, in Section 7, the density theorems.

Lemma 6.1. *Let M be a C^{r+1} n-dimensional manifold, $r \geqslant 1$. Let l be an integer such that $1 \leqslant l \leqslant n$. Let $k > l - 1 + n/(n + 1 - l)$. Then the set of k-tuples $X^1,..., X^k$ of C^r vector fields for which* rank $\{X^1(m),..., X^k(m)\} \geqslant l$ *for all $m \in M$ is dense in ${}_rV(M)^k_r$.*

Proof. For each compact K, let $S(K)$ denote the set of all k-tuples $(X^1,..., X^k)$ of C^r vector fields such that rank $\{X^1(m),..., X^k(m)\} \geqslant l$ for all $m \in K$. Then $S(K)$ is clearly open in ${}_rV(M)^k_r$. Since M is second countable, the conclusion will follow from the category theorem if we prove that the sets $S(K)$ are dense. It is clearly sufficient to prove this when K is contained in the domain of a coordinate chart, so that we may assume that M is an open subset of $\mathbb{R}^n$. But then ${}_rV(M)^k$ can be identified with the set of C^r mappings from M to the space $\mathbb{R}^{n\times k}$ of all $n \times k$ matrices. If $\mathbb{R}^{n\times k}_j$ denotes the set of those $n \times k$ matrices whose rank is j, an easy computation shows that $\mathbb{R}^{n\times k}_j$ is a finite union of manifolds of dimension $j(k + n - j)$. If $\Sigma^{n\times k}_l$ denotes the union of all the $\mathbb{R}^{n\times k}_j$, $j < l$, then $\Sigma^{n\times k}_l$ is a finite union of manifolds of dimension not larger than $(l - 1)(k + n + 1 - l)$. If $F: M \to \mathbb{R}^{n\times k}$ is a smooth function, then $(x, A) \to F(x) - A$ is a mapping from the set $M \times \Sigma^{n\times k}_l$ into an nk-dimensional space. Moreover, $M \times \Sigma^{n\times k}_l$ is a finite union of manifolds of dimension not greater than $n + (l - 1)(k + n + 1 - l)$, and our mapping is smooth on each of these manifolds. Our choice of k implies that $n + (l - 1)(k + n + 1 - l) < nk$.

By Sard's theorem, the set of all matrices of the form $F(x) - A$, $x \in M$, $A \in \Sigma_l^{n\times k}$ is of measure zero in $\mathbb{R}^{n\times k}$. Therefore there exists a sequence $\{P_j\}_{j=1,2,\ldots}$ of matrices such that $P_j \to 0$ as $j \to \infty$ and $F(x) - A - P_j \neq 0$ for all $x \in M$ and all $A \in \Sigma_l^{n\times k}$. But this says precisely that, if we put

$$F_j(x) = F(x) - P_j$$

then $F_j(x)$ has rank $\geqslant k$ for all j and all $x \in M$, and $F_j \to F$ in the C^r topology. This concludes our proof.

We now state as a separate corollary the two particular cases of Lemma 6.1 that will be of use to us later.

COROLLARY 6.2. *Let M be a manifold, $r \geqslant 1$. Then* (a) *the set of all pairs (X, Y) of C^r vector fields such that X and Y never vanish simultaneously is dense in ${}_rV(M)_r^2$;* (b) *the set of all $2n$-tuples $(X^1,\ldots, X^{2n})$ of C^r vector fields such that* rank $\{X^1(m),\ldots, X^{2n}(m)\} = n$ *for all $m \in M$ is dense in ${}_rV(M)_r^{2n}$.*

Proof. For (a), take $l = 1$, $k = 2$, and apply Lemma 6.1. For (b) take $l = n$, $k = 2n$.

LEMMA 6.3. *Let M be a C^{r+1} manifold, $r \geqslant 1$, and let X be a C^r vector field on M. Let $m \in M$ be such that $X(m) \neq 0$. Then there exists a compact neighborhood K of m such that the set of all $Y \in {}_rV(M)$ for which (X, Y) has the accessibility property from K is dense in ${}_rV(M)_r$. Moreover, if Ω is a neighborhood of m, we can choose K so that the set of Y for which (X, Y) has the normal accessibility property from K within Ω is dense in ${}_rV(M)_r$.*

Proof. Let V be the domain of a coordinate chart $(x_1,\ldots, x_n)$ about m, so chosen that, on V, X is the vector field $\partial/\partial x_1$, and that V is a cube of side $\alpha > 0$ centered at m. Clearly, we can take V so that $V \subseteq \Omega$.

Let W denote the "strip" consisting of those points in V for which $0 < x_1 < \beta$, where

$$\beta = \alpha/2n.$$

Let $k = 2n$ and let C denote the set of all k-tuples $(Z^1,\ldots, Z^k)$ of C^r vector fields on W with the property that rank $\{Z^1(m'),\ldots, Z^k(m')\} = n$ for every $m' \in W$. It follows from Corollary 6.2 that C is dense in ${}_rV(W)_r^k$.

Let K be a compact neighborhood of m, contained in W. We show that K has the desired property. Let $Y \in V(M)$. For $1 \leqslant i \leqslant k$, let W_i denote the strip defined by $(i-1)\beta < x_1 < i\beta$. Let Y_i denote the restriction of Y to W_i, and let $\hat{Y}_i$ be the vector field on W defined by

$$\hat{Y}_i = (dX_{(i-1)\beta})^{-1} Y_i .$$

(Notice that $X_{(i-1)\beta}$ is a diffeomorphism from W onto W_i.)

By the density of C in ${}_rV(W)_r^k$, there exist sequences $\tilde{Z}_i^j$ such that $\tilde{Z}_i^j \to \hat{Y}_i$ as $j \to \infty$, for $1 \leqslant i \leqslant k$, and such that $(\tilde{Z}_1^j,..., \tilde{Z}_k^j) \in C$ for all j. Let ϕ be a C^r function on W, whose support is compact, and which is equal to one on K. Let

$$\tilde{P}_i^j(m') = \phi(m')(\tilde{Z}_i^j(m') - \hat{Y}_i(m'))$$

so that $\tilde{P}_i^j$ is a vector field on W, with compact support, that coincides with $\tilde{Z}_i^j - \hat{Y}_i$ on K.

We now let P_i^j be the vector field on W_i which is the image of $\tilde{P}_i^j$ under the diffeomorphism $X_{(i-1)\beta}$. Then P_i^j has compact support. Let P^j be the vector field on M defined by $P^j(x) = P_i^j(x)$ if $x \in W_i$, $P^j(x) = 0$ otherwise.

Then the vector fields P^j converge to 0 in the C^r topology. Let $Z^j = Y + P^j$.

We show that $\{X, Z^j\}$ has the normal accessibility property from m' within Ω for every $m' \in K$ and every j. Choose an $m' \in K$ and a j, and define

$$F(t_1 ,..., t_k) = Z^j_{t_k} X_\beta Z^j_{t_{k-1}} X_\beta ,..., X_\beta Z^j_{t_1}(m').$$

Then F is a well-defined C^r mapping from a neighborhood B of the origin of $\mathbb{R}^k$ into W_k. We show that the rank of F at the origin is equal to n.

Let $m'' = X_{(k-1)\beta}(m') = F(0,..., 0)$. More generally, put $m_i = X_{(i-1)\beta}(m')$, so that $m_1 = m'$, $m_k = m''$. The tangent vector to the curve $t_i \to F(0,..., 0, t_i, 0,..., 0)$ at $t_i = 0$ is then $dX_{(k-i)\beta}Z^j(m_i)$. To show that these vectors have rank n we prove that their pullbacks v_i under the diffeomorphism $X_{(i-1)\beta}$, which are tangent vectors at m', have rank n. But

$$\begin{aligned} v_i &= [dX_{(i-1)\beta}]^{-1}Z^j(m_i) \\ &= [dX_{(i-1)\beta}]^{-1}[Y(m_i) + P_i^j(m_i)] \\ &= \hat{Y}_i(m') + \tilde{P}_i^j(m'). \end{aligned}$$

Since $m' \in K$, $\tilde{P}_i^j(m') = \tilde{Z}_i^j(m') - \hat{Y}_i(m')$ and therefore

$$v_i = \tilde{Z}_i^j(m').$$

By construction, the set $\{Z_i^j(m'),\ i = 1,..., k\}$ has rank n. Therefore the mapping F has rank n at the origin of $\mathbb{R}^k$. If $t_1^0,..., t_k^0$ are sufficiently small positive numbers, it follows that F has rank n at $(t_1^0,..., t_k^0)$. Since the trajectory from m' that corresponds to the control $Z^j_{t_k^0} X_\beta Z^j_{t_{k-1}^0} X_\beta \cdots X_\beta Z^j_{t_1^0}$ is clearly contained in V, and hence in Ω, we conclude that $\{X, Z^j\}$ has the property of normal accessibility from m' within Ω. Since m' is an arbitrary point of K, our proof is complete.

LEMMA 6.4. *Let M be a C^{r+1} manifold, $1 \leqslant r \leqslant \infty$, and let K be a compact subset of M. Let $K \subseteq \Omega$, where Ω is open. Then the set of all pairs (X, Y) that have the accessibility property from K within Ω is dense in ${}_rV(M)_r^2$.*

Proof. Let X, Y belong to ${}_rV(M)$, and let U_X, U_Y be open neighborhoods of X, Y, respectively, in the C^r topology. We want to show that there exist $\hat{X} \in U_X$, $\hat{Y} \in U_Y$ such that $\{\hat{X}, \hat{Y}\}$ has the accessibility property from K within Ω.

To begin with, apply Corollary 6.2(a) to conclude that there are $X^0 \in U_X$, $Y^0 \in U_Y$ such that X^0 and Y^0 do not vanish simultaneously on K.

Since U_X, U_Y are also neighborhoods of X^0, Y^0, it follows that it is sufficient to assume that X and Y do not vanish simultaneously on K.

We can now write

$$K = K_X \cup K_Y$$

where K_X, K_Y are compact sets, X vanishes nowhere on K_X and Y vanishes nowhere on K_Y.

For each compact $J \subseteq \Omega$, let Q_J denote the set of all $Y' \in {}_rV(M)$ such that $\{X, Y'\}$ has the normal accessibility property within Ω from every $m \in J$. By Lemma 3.3, Q_J is open in ${}_rV(M)_r$. By Lemma 6.3, every $m \in K_X$ has a neighborhood $J(m)$ such that $Q_{J(m)}$ is dense. Let $m_1, \ldots, m_k$ be such that $J(m_1), \ldots, J(m_k)$ cover K_X. Then, if

$$Q_{J(m_1)} \cap \cdots \cap Q_{J(m_1)} = Q$$

it follows that Q is a finite intersection of open dense sets, and therefore Q is open and dense. Let V_Y denote the set of all $Y' \in U_Y$ which vanish nowhere on K_Y. Then V_Y is an open neighborhood of Y, and $V_Y \subseteq U_Y$. Since Q is open and dense, it follows that $V_Y \cap Q$ is nonempty. Let $\hat{Y}$ be an element of $V_Y \cap Q$. Then $\hat{Y}$ has the following properties:

(a) $\hat{Y} \in U_Y$,

(b) $\hat{Y}$ vanishes nowhere on K_Y,

(c) $\{X, \hat{Y}\}$ has the normal accessibility property from K_X within Ω.

Now define, for a compact set $J \subseteq \Omega$, $\tilde{Q}_J$ to be the set of all $X' \in {}_rV(M)$ such that $\{X', \hat{Y}\}$ has the normal accessibility property from J within Ω. As before, we conclude that $\tilde{Q}_J$ is open, and that K_Y can be covered by finitely many sets $J(m_1), \ldots, J(m_k)$ such that the $\tilde{Q}_{J(m_i)}$ are dense. Let

$$\tilde{Q} = \tilde{Q}_{J(m_1)} \cap \cdots \cap \tilde{Q}_{J(m_k)}.$$

Then $\tilde{Q}$ is open and dense. Let V_X denote the set of all $X' \in U_X$ such that $\{X', \hat{Y}\}$ has the normal accessibility property from K_X within Ω. Then $X \in V_X$ and, by Lemma 3.3, V_X is open. Therefore V_X is a nonempty open set and hence $\tilde{Q} \cap V_X$ is nonempty. Let $\hat{X} \in \tilde{Q} \cap V_X$. It follows that $\hat{X} \in \hat{U}_X$, $\hat{Y} \in U_Y$. Since $\hat{X} \in Q$, then $\{X, \hat{Y}\}$ has the normal accessibility property within Ω from every $m \in K_Y$. Since $\hat{X} \in V_X$, it follows that this is also true for $m \in K_X$. Therefore $\{\tilde{X}, \hat{Y}\}$ has the accessibility property from every $m \in K$, and the proof is complete.

COROLLARY 6.5. *Let M, K, Ω be as in the statement of Lemma 6.4. If $(X, Y) \in {}_rV(M)^2$, then there are sequences $\{X^m\}_{m=1}^{\infty}$, $\{Y^m\}_{m=1}^{\infty}$ such that* (i) *$X^m - X$, $Y^m - Y$ have supports contained in a fixed compact subset of Ω;* (ii) *$X^m \to X$ and $Y^m \to Y$ in ${}_rV(M)_r$; and* (iii) *$\{X^m, Y^m\}$ has the normal accessibility property from K within Ω for $m = 1, 2,\ldots$.*

Proof. Let Ω' be open, and such that $K \subseteq \Omega' \subseteq L \subsetneq \Omega$, where L is compact. By Lemma 6.4, there are sequences ${}^1X^m$, ${}^1Y^m$ which converge to X, Y in ${}_rV(M)_r$, as $m \to \infty$, and are such that ${}^1X^m$, ${}^1Y^m$ has the property of normal accessibility from K within Ω'. Let ϕ be a C^r function whose support is a compact subset of Ω. In addition, let $\phi = 1$ on L. Let $X^m = X + \phi({}^1X^m - X)$, $Y^m = Y + \phi({}^1Y^m - Y)$. Then conditions (i) and (ii) hold. Moreover, on Ω' the vector fields X^m and Y^m coincide with ${}^1X^m$, ${}^1Y^m$. Hence condition (iii) holds, and the proof is complete.

7. THE DENSITY THEOREMS

In this section we prove that the set of k-tuples that have the accessibility property is dense, if $k \geqslant 2$. We use the notations ${}_rA_k$, ${}_rA_k(K)$ as in Section 5. The density in the C^j topology $(1 \leqslant j \leqslant r)$ is, of course, easier to prove, and we deal with it first.

THEOREM 7.1. *Let M be a C^{r+1} manifold, $1 \leqslant r \leqslant \infty$. Let k be an integer, $k \geqslant 2$. Then ${}_rA_k$ is dense in ${}_rV(M)_j^k$ for $1 \leqslant j \leqslant r$.*

Proof. It is clearly sufficient to consider the case $k = 2$, $j = r$. By Lemma 6.4, ${}_rA_2(K)$ is dense in ${}_rV(M)_r^2$, and by Theorem 5.1, it is open. Now ${}_rA_2 = \bigcap_{m=1}^{\infty} {}_rA_2(K_m)$, where $\{K_m\}_{m=1}^{\infty}$ is a sequence of compact sets whose union is M. The conclusion that ${}_rA_2$ is dense in ${}_rV(M)_r^2$ follows from the category theorem.

THEOREM 7.2. *Let M be a C^r manifold, $1 \leqslant r \leqslant \infty$. Let k, j be integers such that $k \geqslant 2$, $1 \leqslant j \leqslant r$. Then ${}_rA_k$ is dense in ${}_rV(M)_{j,\mathrm{fine}}^k$.*

Proof. As in the proof of Theorem 7.1, it is sufficient to consider the case $k = 2, j = r$.

Let X and Y be C^r vector fields on M, and let U be a neighborhood of zero for the fine C^r topology of ${}_rV(M)$. We will show that there exist X', Y' having the accessibility property and such that $X' - X \in U$ and $Y' - Y \in U$.

We can assume that there is a locally finite sequence $\{K_m\}_{m=1}^{\infty}$ of compact sets such that

$$U = \bigcap_{m=1}^{\infty} U_m$$

where U_m is a neighborhood of zero in ${}_rV(K_m)_r$. Let V_m be an open neighborhood of zero in ${}_rV(K_m)_r$ such that the closure of V_m relative to ${}_rV(K_m)_r$ is contained in U_m. Put $V = \bigcap_{m=1}^{\infty} V_m$.

If $K \subseteq \Omega$, K compact, Ω open, $\bar{\Omega}$ compact, let $A(K, \Omega)$ denote the set of all the pairs P, Q of vector fields in ${}_rV(M)$ such that $(X + P, Y + Q)$ has the normal accessibility property from K within Ω.

Since each of the topologies of ${}_rV(M)_r^2$, ${}_rV(M)_1^2$, ${}_rV(\bar{\Omega})_1^2$ is finer than the preceding one, it follows from Lemma 3.3 that $A(K, \Omega)$ is open in ${}_rV(M)_r^2$.

We now show that $A(K, \Omega) \cap (V \times V)$ is dense in $V \times V$ in the fine C^r topology. By Corollary 6.5 there exist, for $(P, Q) \in V \times V$, sequences P_m, Q_m such that (i) the supports of $P_m - P$, $Q_m - Q$ are contained in $\bar{\Omega}$, (ii) $P_m \to P$, $Q_m \to Q$ in ${}_rV(M)_r$, and (iii) $(X + P_m, Y + Q_m)$ has the normal accessibility property from K within Ω, for all m.

It follows from (i) and (ii) that the convergence of P_m to P and of Q_m to Q actually holds in the *fine* C^r topology. Therefore, for sufficiently large m, P_m and Q_m belong to V. Also, by (iii), $(P_m, Q_m) \in A(K, \Omega)$. Thus $A(K, \Omega) \cap (V \times V)$ is indeed dense in $V \times V$ in the C^r topology.

Now let W denote the closure of V in ${}_rV(M)_r$. Then $W \times W$, being a closed subset of the complete metric space ${}_rV(M)_r^2$, is a complete metric space, so that the category theorem applies.

Let $A(K)$ denote the set of all (P, Q) such that $(X + P, Y + Q)$ has the normal accessibility property from K. It follows from Lemma 3.6 that $A(K)$ is the union of the sets $A(K, \Omega)$, where Ω ranges over all open neighborhoods of K with compact closure. Therefore $A(K)$ is open in ${}_rV(M)_r^2$, and $A(K) \cap (V \times V)$ is dense in $V \times V$ in the C^r topology. Therefore: (a) $A(K) \cap (W \times W)$ is relatively open and dense in $W \times W$, considered as a subspace of ${}_rV(M)_r^2$.

It follows from (a) and the category theorem that the intersection

$$\left[\bigcap_{m=1}^{\infty} A(K_m)\right] \cap (W \times W)$$

is nonempty. Since $M = \bigcup_{m=1}^{\infty} K_m$, it follows that there exist $P \in W$, $Q \in W$ such that $(X + P, Y + Q)$ has the normal accessibility property. Therefore, our proof will be complete if we show that $W \subseteq U$. Let W_m denote the closure of V_m in ${}_rV(K_m)_r$. Then $W_m \subseteq U_m$. Moreover, W_m is closed in ${}_rV(K_m)_r$, and hence in ${}_rV(M)_r$. Therefore the intersection $W' = \bigcap_{m=1}^{\infty} W_m$

is closed and, clearly $V \subseteq W' \subseteq U$. Since W is, by definition, the closure of V in ${}_rV(M)_r$, we conclude that $W \subseteq W'$, and hence $W \subseteq U$, completing the proof.

8. Some Easy Extensions

Assume that M is connected. Let us call a system S *controllable in Lobry's sense* (c.L.s.) if the orbit of some (and hence every) point $x \in M$ is all of M. Clearly every system with the accessibility property is c.L.s. Hence the c.L.s. k-tuples of C^r vector fields are dense in ${}_rV(M)^k_{r,\text{fine}}$ if $k \geqslant 2$. On the other hand, a k-tuple $(X^1,..., X^k)$ is c.L.s. if and only if the $2k$-tuple $(X^1,..., X^k, -X^1,..., -X^k)$ has the accessibility property. Hence the c.L.s. k-tuples form an open set. We have proved the following.

Theorem 8.1. *Let M be a connected C^{r+1} manifold, $1 \leqslant r \leqslant \infty$. Let k be an integer $\geqslant 2$. Then the set of all k-tuples of C^r vector fields on M that are controllable in Lobry's sense is open in ${}_rV(M)^k_{1,\text{fine}}$ and dense in ${}_rV(M)^k_{r,\text{fine}}$.*

Now assume that $r = \infty$. The *algebraic accessibility condition* (a.a.c) is defined as follows. If S is a set of C^∞ vector fields on M, let $\Lambda(S)$ denote the smallest Lie algebra of vector fields such that $S \subseteq \Lambda(S)$. For $x \in M$, let

$$\Lambda(S)(x) = \{X(x) \colon X \in \Lambda(S)\}.$$

We say that S satisfies the a.a.c. if $\dim \Lambda(S)(x) = n$ for all $x \in M$. It is well known that the a.a.c. implies the accessibility property (cf. for instance [2]) and that, for systems of *real analytic* vector fields on a real analytic manifold, it is actually equivalent to it [8, Theorem 3.1].

Theorem 8.2. *If $k \geqslant 2$, the k-tuples of C^∞ vector fields which satisfy the a.a.c. form a set which is open and dense in ${}_\infty V(M)^k_{\infty,\text{fine}}$.*

Proof. The openness is trivial. The density can be proved in at least two ways. We refer the reader to Lobry [4] for the first one. The second way is as follows. The manifold M carries a unique real-analytic structure compatible with its C^∞ structure. Real analytic vector fields are dense in ${}_\infty V(M)_{\infty,\text{fine}}$.

Since the k-tuples of C^∞ vector fields which satisfy the accessibility property are open and dense in ${}_\infty V(M)^k_{\infty,\text{fine}}$ it follows that the k-tuples of real analytic vector fields with the accessibility property are dense. But, for real analytic vector fields, the accessibility property is equivalent to the a.a.c. The proof is complete.

The following result has been proved in the course of proving Theorem 8.2.

THEOREM 8.3. *Let M be a real analytic manifold. If $k \geqslant 2$, then the k-tuples of real analytic vector fields that have the accessibility property are dense in ${}_rV(M)^k_{r,\mathrm{fine}}$ for $1 \leqslant r \leqslant \infty$.*

9. A COUNTEREXAMPLE

In this section we discuss a property, which we shall denote by (P). Our aim is to show that properties that appear very similar to accessibility and controllability may actually behave very differently. Property (P) will serve to illustrate this assertion.

We say that a system S has property (P) if, for every $x \in M$, the positive orbit $O_+(x, S)$ contains a neighborhood of x. It is easy to see that S satisfies (P) if and only if all positive orbits are open. By analogy with the theorems of Section 4, it might be expected that (P) implies (Q), where (Q) is the property that every point is normally reachable from itself. Also, it might be expected that (P) is "stable" in the same way as accessibility and controllability are stable, namely, that the set of systems for which (P) holds is open.

The following example shows that these expectations are not justified. We shall define six C^∞ vector fields $X^1,..., X^6$ on the two-dimensional plane $\mathbb{R}^2$ (coordinates x, y) for which (P) holds but (Q) does not, and we shall see that the system $(X^1,..., X^6)$ can be approximated arbitrarily close in the fine C^∞ topology by systems for which (P) does not hold.

Let ψ be a real-valued C^∞ function defined on the real line, and such that $\psi(t) > 0$ for $t \neq 0$, and that ψ vanishes of infinite order at $t = 0$. We define the vector fields X^1, X^2, X^3 as follows:

$$X^1(x, y) = \psi(x + 1)(\partial/\partial x),$$
$$X^2(x, y) = -\psi(x - 1)(\partial/\partial x),$$

and

$$X^3(x, y) = \psi(y)(\partial/\partial x).$$

Also, let ρ be a C^∞ function such that $\rho(t) > 0$ for $|t| > 1$ and $\rho(t) = 0$ for $|t| \leqslant 1$. We define $X^5(x, y) = \rho(x)(\partial/\partial y)$. Finally, we put $X^4 = -X^3$ and $X^6 = -X^5$.

Let C denote the closed segment $-1 \leqslant x \leqslant 1$, $y = 0$, and let Ω denote the complement of C. It is clear that X^3 and X^4 allow us to move freely along any horizontal line other than the x-axis. Moreover, X^1 and X^2 make it possible to move horizontally in both directions within the segment $-1 < x < 1$, $y = 0$, and also within the half lines $-\infty < x < -1$, $y = 0$, and $1 < x < \infty$, $y = 0$. Also, it is possible to move vertically, up and down, along any line $y = c$, as long as $|c| > 1$. On the other hand, no trajectory that starts at a point of Ω can enter C, but it is possible to leave C through either of its endpoints. Therefore, the positive orbit of a point (x_0, y_0) is

(a) Ω if $(x_0, y_0) \in \Omega$, and

(b) the whole plane if $(x_0, y_0) \in C$.

In particular, it is clear that (P) holds. On the other hand, (Q) is not satisfied. This can be seen directly, or it can be deduced from the implication (iii) $\Rightarrow$ (i) of Theorem 4.3, together with the fact that our system is not completely controllable.

We now show that there are systems that do not satisfy (P) and are arbitrarily close to the given one in the fine C^∞ topology. Let $\psi = \psi_1 + \psi_2$, where ψ_1, ψ_2 are C^∞ functions, $\psi_1(t) = 0$ for $t \leqslant 0$ and $\psi_2(t) = 0$ for $t \geqslant 0$. For $\epsilon > 0$ put

$$\psi^\epsilon(t) = \psi_1(t - \epsilon) + \psi_2(t + \epsilon).$$

Then ψ^ϵ is a C^∞ function which vanishes at t if and only if $|t| \leqslant \epsilon$. Moreover, $\psi^\epsilon \to \psi$ as $\epsilon \to 0$, the convergence being in the usual C^∞ topology.

Let ϕ be a C^∞ function on $\mathbb{R}^2$ such that: (1) $\phi(x, y) \geqslant 0$ for all (x, y), (2) $\phi \equiv 1$ for $|x| \leqslant 2$, $|y| \leqslant 2^0$ and (3) ϕ has compact support.

Put ${}^\epsilon X^3(x, y) = X^3(x, y) + \phi(x, y)[\psi^\epsilon(y)(\partial/\partial x) - X^3(x, y)]$. It is clear that ${}^\epsilon X^3 \to X^3$ as $\epsilon \to 0$. Moreover, the vector fields ${}^\epsilon X^3 - X^3$ have their supports in a *fixed* compact set. Hence the convergence is actually in the fine topology. Defining ${}^\epsilon X^4 = -{}^\epsilon X^3$, we see that the systems

$$S_\epsilon = (X^1, X^2, {}^\epsilon X^3, {}^\epsilon X^4, X^5, X^6)$$

approximate the given one arbitrarily closely. We show that S_ϵ does not satisfy (P). To see this consider, for $0 < |\delta| \leqslant \epsilon$, the segment C_δ defined by $y = \delta$, $-1 \leqslant x \leqslant 1$. Then it is possible to leave C_δ by a trajectory of S_ϵ, but it is not possible to enter it. Hence, in particular, no point of C_δ is reachable from the origin. Since δ is any number such that $0 < |\delta| \leqslant \epsilon$, we conclude that the positive orbit of S^ϵ from the origin is not open.

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