Chapter 1

Notation and conventions

When not dealing with tensor products in Hilbert spaces, the outer product of two real vectors $\mathbf{a}$ and $\mathbf{b}$ is defined as customary according to standard matrix multiplication, and denoted by $\mathbf{a}\mathbf{b}^\mathsf{T}$. In components: $(\mathbf{a}\mathbf{b}^\mathsf{T})_{jk} = a_j b_k$.

Vectors and matrices of operators (where each entry is an operator acting on a Hilbert space) will be used extensively throughout the book. In general, for a vector of hermitian operators $\hat{\mathbf{a}}$, one has $\hat{\mathbf{a}}\hat{\mathbf{a}}^\mathsf{T} \neq (\hat{\mathbf{a}}\hat{\mathbf{a}}^\mathsf{T})^\mathsf{T}$, as the operators in the the vector might not commute with each other. It is hence useful to define the symmetrized and anti-symmetrized forms of such products, as

$$\left\{\hat{\mathbf{a}}, \hat{\mathbf{a}}^\mathsf{T}\right\} = \hat{\mathbf{a}}\hat{\mathbf{a}}^\mathsf{T} + (\hat{\mathbf{a}}\hat{\mathbf{a}}^\mathsf{T})^\mathsf{T} \,, \tag{1.1}$$

$$\left[\hat{\mathbf{a}}, \hat{\mathbf{a}}^\mathsf{T}\right] = \hat{\mathbf{a}}\hat{\mathbf{a}}^\mathsf{T} - (\hat{\mathbf{a}}\hat{\mathbf{a}}^\mathsf{T})^\mathsf{T} \,. \tag{1.2}$$

In components, one has

$$\left\{\hat{\mathbf{a}}, \hat{\mathbf{a}}^\mathsf{T}\right\}_{jk} = \hat{a}_j\hat{a}_k + \hat{a}_j\hat{a}_k \,, \tag{1.3}$$

$$\left[\hat{\mathbf{a}}, \hat{\mathbf{a}}^\mathsf{T}\right]_{jk} = \hat{a}_j\hat{a}_k - \hat{a}_j\hat{a}_k \,. \tag{1.4}$$

Also note that

$$\left\{\hat{\mathbf{a}}, \hat{\mathbf{a}}^\mathsf{T}\right\} + \left[\hat{\mathbf{a}}, \hat{\mathbf{a}}^\mathsf{T}\right] = 2\hat{\mathbf{a}}\hat{\mathbf{a}}^\mathsf{T} \,. \tag{1.5}$$

The Dirac delta function $\delta(x)$ admits a representation as

$$\delta(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \mathrm{e}^{ipx}\mathrm{d}p \,, \tag{1.6}$$

which can be turned into the following relation for complex variables:

$$\frac{1}{4\pi^2}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \mathrm{e}^{ipx'}\mathrm{e}^{-ixp'}\mathrm{d}x'\mathrm{d}p' = \frac{1}{\pi^2}\int_{\mathbb{C}} \mathrm{e}^{\alpha\beta^*-\alpha^*\beta}\mathrm{d}^2\alpha = \delta^2(\alpha) \,, \tag{1.7}$$

where we the changes of variable $\sqrt{2}\alpha = (x+ip)$ and $\sqrt{2}\beta = (x'+ip')$ have been performed, and $\delta^2(\alpha) = \delta\left(\mathrm{Re}(\alpha)\right)\delta\left(\mathrm{Im}(\alpha)\right)$.

Most of the mathematics in the book will take place in the Hilbert space of square integrable functions $L^2(\mathbb{R}^n)$, with n finite. We will refer to a *trace-class* operator $\hat{O}$ as a linear operator on $L^2(\mathbb{R}^n)$ for which the trace may be defined, and is independent from the choice of the basis. Trace-class operators are always bounded.

Throughout the book, we will refer to the $n \times n$ identity matrix with the symbol $\mathbb{1}_n$, while the generic $\mathbb{1}$ will denote the identity operator on $L^2(\mathbb{R}^n)$.

Gaussian integrals with many variables will occur very often throughout the book. Given a symmetric, real, positive definite matrix $n \times n$ A, and an n-dimensional vector $\mathbf{b}$, one has the equality:

$$\int_{\mathbb{R}^n} \mathrm{d}^n\mathbf{r}\, \mathrm{e}^{-\mathbf{r}^\mathsf{T} A\mathbf{r}+\mathbf{r}^\mathsf{T}\mathbf{b}} = \frac{\pi^{\frac{n}{2}}}{\sqrt{\mathrm{Det}\, A}}\, \mathrm{e}^{\frac{1}{4}\mathbf{b}^\mathsf{T} A^{-1}\mathbf{b}} , \tag{1.8}$$

where $\mathrm{d}^n\mathbf{r}$ indicates the product of differential of the n integration variables that compose the vector $\mathbf{r}$. Eq. (**??**) may be proven by switching to the eigenbasis of A, which can always be done by an orthogonal transformation since A is symmetric and decouples the integral, completing the squares at the exponent, and then inserting $\int_{\mathbb{R}} \mathrm{e}^{-x^2}\mathrm{d}x = \sqrt{\pi}$. The determinant appears as the product of the eigenvalues of A. Note that Eq. (**??**) also holds for a complex $\mathbf{b}$.