Single input, 2 mode counterexample to Alessio's conjecture using Jurdjevic and Kupka

Uther Shackeley-Bennett

March 19, 2015

1 Jurdjevic amd Kupka's Theorem

From Elliott p95 and p117.

Definition 1 (Strong regularity). *The matrix $B \in \mathfrak{gl}(n, \mathbb{R})$ is strongly regular if its eigenvalues $\lambda_k = \alpha_k + i\beta_k$, $k \in 1, \ldots, n$ are distinct, including $2m$ conjugate-complex pairs, and the real parts $\alpha_1 < \ldots < \alpha_{n-m}$ satisfy $\alpha_i - \alpha_j \neq \alpha_p - \alpha_q$ unless $i = p$ and $j = q$.*

Theorem 2 (Jurdjevic and Kupka (note the capitalisation typo in Elliott)). *Assume that $\mathrm{Tr}(A) = 0 = \mathrm{Tr}(B)$ and B is strongly regular. Choose coordinates so that $B = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$. If A satisfies*

$$A_{i,j} \neq 0 \quad \forall i, j \text{ such that } |i - j| = 1, \tag{1}$$

$$A_{1,n}A_{n,1} < 0 \tag{2}$$

then with $\Omega = \mathbb{R}$, $\dot{x} = (A + uB)x$ is controllable on $\mathbb{R}^n_*$.

Proposition 3 (From Meyer). *The characteristic polynomial of a real Hamiltonian matrix is an even polynomial. Thus, if λ is an eigenvalues of a Hamiltonian matrix, then so are $-\lambda, \lambda^*, -\lambda^*$. The trace of a Hamiltonian matrix id always 0.*

Note that any matrix with dinstinct eigenvalues is diagonalisable.

I am wondering about why we are told to diagonalise B; I don't think the alphas in the theorem are the same as those in the definition.. So it just seems like a poor choice of notation in Elliott. It's annoying because it looks like we could be requiring that B have real eigenvalues. I have changed my notation but I want this to be at the back of my mind.

2 Two mode, Hamiltonians matrix with the Jurdjevic-Kupka conditions

In two modes a general Hamiltonian matrix is of the form

$$A = \begin{pmatrix} a & b & c & d \\ e & f & d & h \\ k & l & -a & -e \\ l & g & -b & -f \end{pmatrix}. \tag{3}$$

Setting this as the form of our drift field we may then add the Jurdjevic conditions: $b, d, e, l \neq 0$ and $dl < 0$.

With its distinct eigenvalues B can be diagonalised to one of three forms:

$$B_1 = \begin{pmatrix} x+iy & 0 & 0 & 0 \\ 0 & x-iy & 0 & 0 \\ 0 & 0 & -x+iy & 0 \\ 0 & 0 & 0 & -x-iy \end{pmatrix}, \tag{4}$$

$$B_2 = \begin{pmatrix} x & 0 & 0 & 0 \\ 0 & -x & 0 & 0 \\ 0 & 0 & y & 0 \\ 0 & 0 & 0 & -y \end{pmatrix}, \tag{5}$$

$$B_3 = \begin{pmatrix} ix & 0 & 0 & 0 \\ 0 & -ix & 0 & 0 \\ 0 & 0 & iy & 0 \\ 0 & 0 & 0 & -iy \end{pmatrix}, \tag{6}$$

for $x, y \in \mathbb{R}$. The conditions that it must satisfy to be strongly regular are trivially satisfied in two modes.

Therefore a general $A + uB$ matrix looks like one of

$$G_1 = \begin{pmatrix} a+u(x+iy) & b & c & d \\ e & f+u(x-iy) & d & h \\ k & l & -a+u(-x+iy) & -e \\ l & g & -b & -f+u(-x-iy) \end{pmatrix}, \tag{7}$$

$$G_2 = \begin{pmatrix} a+ux & b & c & d \\ e & f-ux & d & h \\ k & l & -a+uy & -e \\ l & g & -b & -f-uy \end{pmatrix}, \tag{8}$$

$$G_3 = \begin{pmatrix} a+iux & b & c & d \\ e & f-iux & d & h \\ k & l & -a+iuy & -e \\ l & g & -b & -f-iuy \end{pmatrix}. \tag{9}$$

3 Neutrality

$$\dot{X} = AX + \sum_1^m u_i(t)B_iX, \quad X(0) = I_n, \quad u(\cdot) = \Omega, \quad t \geq 0. \tag{10}$$

Definition 4 (Elliott p97)**.** *Neutrality of matrix C is equivalent to each of the following properties:*

- *C is similar over $\mathbb{R}$ to a skew-symmetric matrix.*
- $\operatorname{spec}(C)$ *lies in the imaginary axis and C is diagonalisable over C.*
- *The closure of $e^{\mathbb{R}C}$ is compact.*
- *There exists $\sigma > 0$ and a sequence of times $\{t_k\}_1^\infty$ with $t_k \geq \sigma$ such that $\lim_{k\to\infty} ||e^{t_kC} - I|| = 0$.*
- $|\operatorname{Tr}(C)| = 0$ *and $||e^{tC}||$ are bounded on $\mathbb{R}$.*

Theorem 5. *Let $\Omega = \mathbb{R}^m$*

- *If there are constants $\delta_i \in \Omega$ such that $A_\delta := A + \sum_1^m \delta_i B_i$ is neutral then $S = G_\alpha$;*
- *If, also, the Lie rank of $\mathfrak{g}_\alpha$ is n then (10) is controllable.*

Proof. see Elliott p97. □

3.1 Neutrality and Hamiltonian matrices