Single input control and Jurdjevic's theorem

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1 Jurdjevic amd Kupka's Theorem

From Elliott p95 and p117.

Definition 1 (Strong regularity)**.** *The matrix $B \in \mathfrak{gl}(n, \mathbb{R})$ is strongly regular if its eigenvalues $\lambda_k = \alpha_k + i\beta_k$, $k \in 1, \dots, n$ are distinct, including $2m$ conjugate-complex pairs, and the real parts $\alpha_1 < \dots < \alpha_{n-m}$ satisfy $\alpha_i - \alpha_j \neq \alpha_p - \alpha_q$ unless $i = p$ and $j = q$.*

Theorem 2 (Jurdjevic and Kupka)**.** *Assume that $\mathrm{Tr}(A) = 0 = \mathrm{Tr}(B)$ and B is strongly regular. Choose coordinates so that $B = \mathrm{diag}(\alpha_1, \dots \alpha_n)$. If A satisfies*

$$A_{i,j} \neq 0 \quad \forall i, j \text{ such that } |i - j| = 1, \tag{1}$$

$$A_{1,n}A_{n,1} < 0 \tag{2}$$

then with $\Omega = \mathbb{R}$, $\dot{x} = (A + uB)x$ is controllable on $\mathbb{R}^n_*$.

Note that our A and B are always traceless. I am wondering about why we are told to diagonalise B; I don't think the alphas in the theorem are the same as those in the definition.

2 Examples

2.1 Example 1

$$A = \begin{pmatrix} 0 & a & 0 & x \\ b & 0 & c & 0 \\ 0 & d & 0 & e \\ y & 0 & f & 0 \end{pmatrix} = \begin{pmatrix} 0 & a & 0 & c \\ b & 0 & c & 0 \\ 0 & d & 0 & -b \\ d & 0 & -a & 0 \end{pmatrix} = \begin{pmatrix} 0 & 2 & 0 & -3 \\ 4 & 0 & -3 & 0 \\ 0 & 5 & 0 & -4 \\ 5 & 0 & -2 & 0 \end{pmatrix} \tag{3}$$

$xy < 0$. The letters are non zero.

The above is recursive.

$$A = \begin{pmatrix} 0 & a & 0 & c \\ b & 0 & c & 0 \\ 0 & d & 0 & -b \\ d & 0 & -a & 0 \end{pmatrix} = \begin{pmatrix} 0 & 7 & 0 & -3 \\ 7 & 0 & -3 & 0 \\ 0 & 5 & 0 & -7 \\ 5 & 0 & -7 & 0 \end{pmatrix} \tag{4}$$

This example has all real eigenvalues. If we construct B such that this cannot be made imag. then we are OK....

$$B = \begin{pmatrix} a+ib & 0 & 0 & 0 \\ 0 & a-ib & 0 & 0 \\ 0 & 0 & c+id & 0 \\ 0 & 0 & 0 & c-id \end{pmatrix} = \begin{pmatrix} 2+i3 & 0 & 0 & 0 \\ 0 & 2-i3 & 0 & 0 \\ 0 & 0 & -2+i3 & 0 \\ 0 & 0 & 0 & -2-i3 \end{pmatrix} \tag{5}$$

The eigenvalues are all distinct and $a-c$ are not equal to any other real parts subtracted.

What I want is an example such that a compact drift field is not constructible. Presumably no point trying in 1 mode because of Wu's theorem.

$$A + uB = \begin{pmatrix} u(2+i3) & 7 & 0 & -3 \\ 7 & u(2-i3) & -3 & 0 \\ 0 & 5 & u(-2+i3) & -7 \\ 5 & 0 & -7 & u(-2-i3) \end{pmatrix} \tag{6}$$

This example is not recursive for $u = 1$. I need to show that this is never recursive for any value of u. On MATLAB I find that there are values of u for which this has imaginary eigenvalues.

So trying for a more general example

$$A = \begin{pmatrix} 0 & a & 0 & c \\ b & 0 & c & 0 \\ 0 & d & 0 & -b \\ d & 0 & -a & 0 \end{pmatrix} = \begin{pmatrix} 0 & 7 & 0 & -3 \\ 7 & 0 & -3 & 0 \\ 0 & 5 & 0 & -7 \\ 5 & 0 & -7 & 0 \end{pmatrix} \tag{7}$$

So what I want is to construct some Hamiltonian matrix $A + uB$ satisfying all the condtion on A and B such that there is no value of u that will give it imaginary eigenvalues. If there is then we could just set this as our drift field. I want to construct a controllable counter-example to Alessio's conjecture.

2.2 Master equation

General A matrix:

$$A = \begin{pmatrix} a & b & c & d \\ e & f & d & h \\ k & l & -a & -e \\ l & g & -b & -f \end{pmatrix} \tag{8}$$

such that $b, d, e, l \neq 0$ and $dl < 0$. (note that without these conditions this is just a general Hamiltonian matrix).

General B matrix:

$$B = \begin{pmatrix} x+iy & 0 & 0 & 0 \\ 0 & x-iy & 0 & 0 \\ 0 & 0 & -x+iy & 0 \\ 0 & 0 & 0 & -x-iy \end{pmatrix}. \tag{9}$$

General $A + uB$ matrix:

$$A = \begin{pmatrix} a+u(x+iy) & b & c & d \\ e & f+u(x-iy) & d & h \\ k & l & -a+u(-x+iy) & -e \\ l & g & -b & -f+u(-x-iy) \end{pmatrix}. \tag{10}$$

I would like the eigenvalue equations of $A + uB$. From this my question is: are there values of $a, b, c, d, e, f, g, h, k, l, x, y$ such that there is no value of u to make the eigenvalue expression less than 0. I can have this function in matlab and then have a load of inputs so I can play around. Ensuring that I stick to the few conditions: that some can't be zero and the dl condition.