Functions

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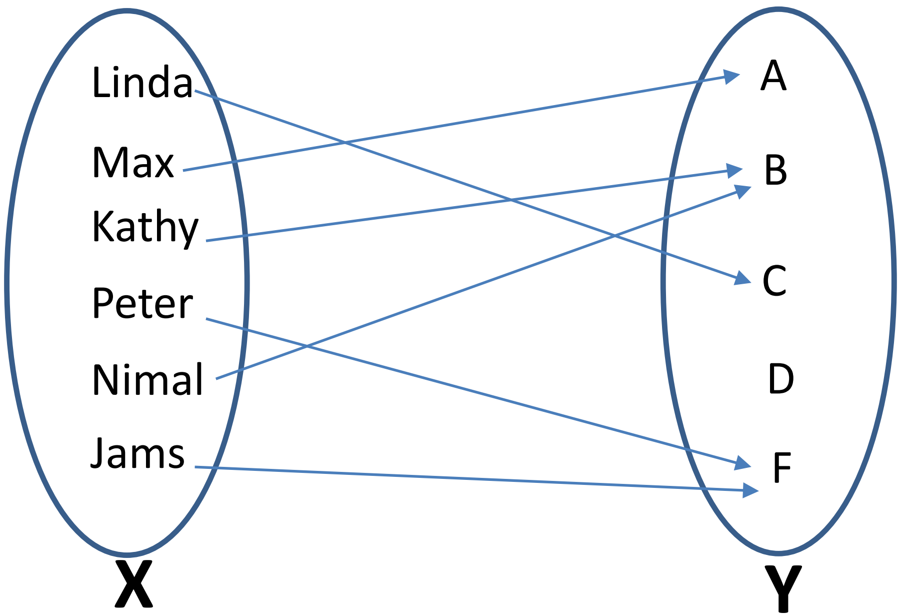
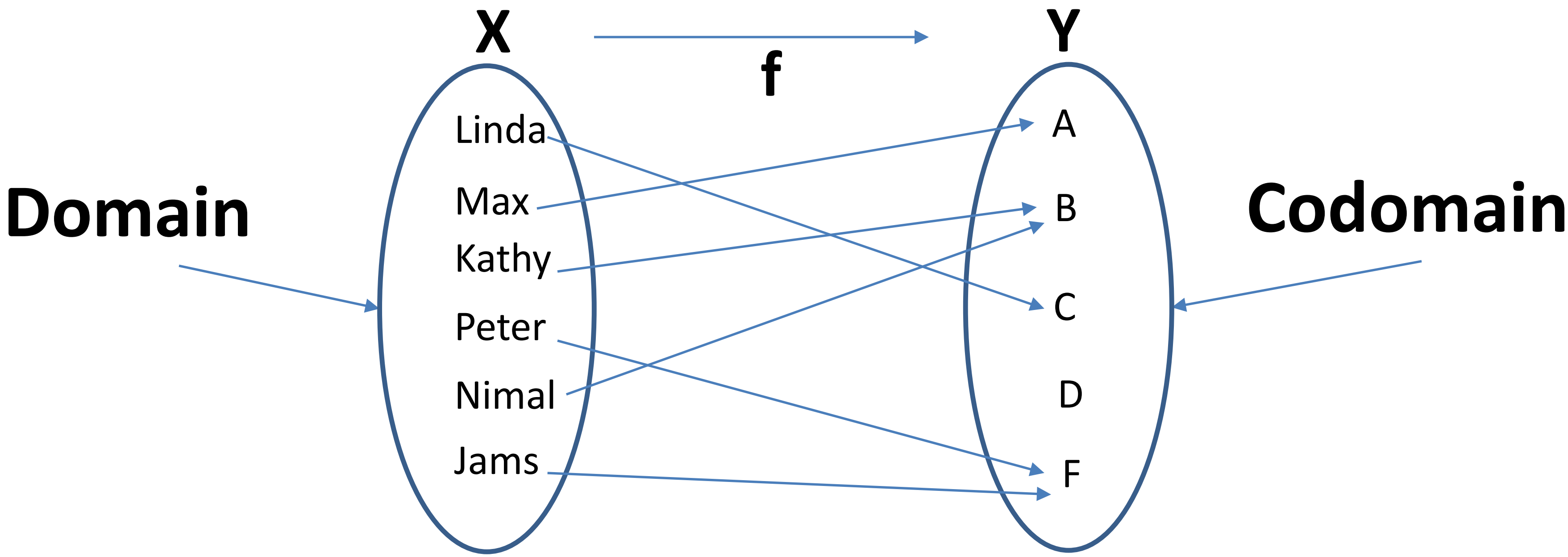
Functions



Definition: Let X and Y be nonempty sets. A function f from X to Y is an assignment of exactly one element of Y to each element of X. We write $f(a) = b$ if b is the unique element of Y assigned by the function f to the element a of X. If f is a function from X to Y, we write $f : X \to Y$.

Functions are sometimes also called mappings or transformations.

Terminology



If $f(a) = b$, we say that b is the **image** of a and a is a **pre-image** of b.

The **range**, or image, of f is the set of all images of elements of X and is denoted by Range(f).

$$Range(f) = \{y \in Y \mid y = f(x) \; for \; some \; x \in X\}.$$

Examples

Find the domain, codomain and range of the following functions:

1. Let f be the function that assigns the last two bits of a bit string of length 2 or greater to that string. For example, $f(11010) = 10$.
   - The domain of f is the set of all bit strings of length 2 or greater.
   - The codomain and range are the set $\{00,01,10,11\}$.

2. Let $f: Z \rightarrow Z$ assign the square of an integer to this integer. Then, $f(x) = x^2, for\ all\ x \in Z$.

One to one function

Some functions never assign the same value to two different domain elements. These functions are said to be one-to-one.

Definition: A function f is said to be one–to-one, or an injunction, if and only if $f(a) = f(b)$ implies that $a = b$ for all *a* and *b* in the domain of *f*.

Remark: We can express that f is one-to-one using quantifiers as follows:

$$f \text{ is one} - \text{to} - \text{one} \iff \forall a, \forall b\, (f(a) = f(b) \rightarrow a = b)$$

$$\iff \forall a, \forall b\, (a \neq b \rightarrow f(a) \neq f(b))$$

where the universe of discourse is the domain of the function.

Examples

1. Determine whether the function f from $\{a, b, c, d\}$ to $\{1,2,3,4,5\}$ with $f(a) = 4, f(b) = 5, f(c) = 1,$ $and\ f(d) = 3$ is one-to-one.
2. Determine whether the function $f(x) = x + 1$ from the set of real numbers to itself is one-to-one.
3. Determine whether the function $f(x) = x^2$ from the set of integers to the set of integers is one-to-one.

Onto Functions

Definition: A function f from A to B is called onto, or a surjection, if and only if for every element $b \in B$ there is an element $a \in A$ such that $f(a) = b$.

That is;

$$f: A \rightarrow B \text{ is onto} \Leftrightarrow \forall y \in B \; \exists x \in A \text{ such that } f(x) = y.$$

Examples:

1. Let f be the function from $\{a, b, c, d\}$ to $\{1, 2, 3\}$ defined by $f(a) = 3, f(b) = 2, f(c) = 1$, and $f(d) = 3$. Is f an onto function?
2. Is the function $f(x) = x + 1$ from the set of integers to the set of integers onto?
3. Is the function $f(x) = x^2$ from the set of integers to the set of integers onto?

Bijection

Definition: A function f: $A \rightarrow B$ is a one-to-one correspondence, or a bijection, if and only if it is both one-to-one and onto.

Example:

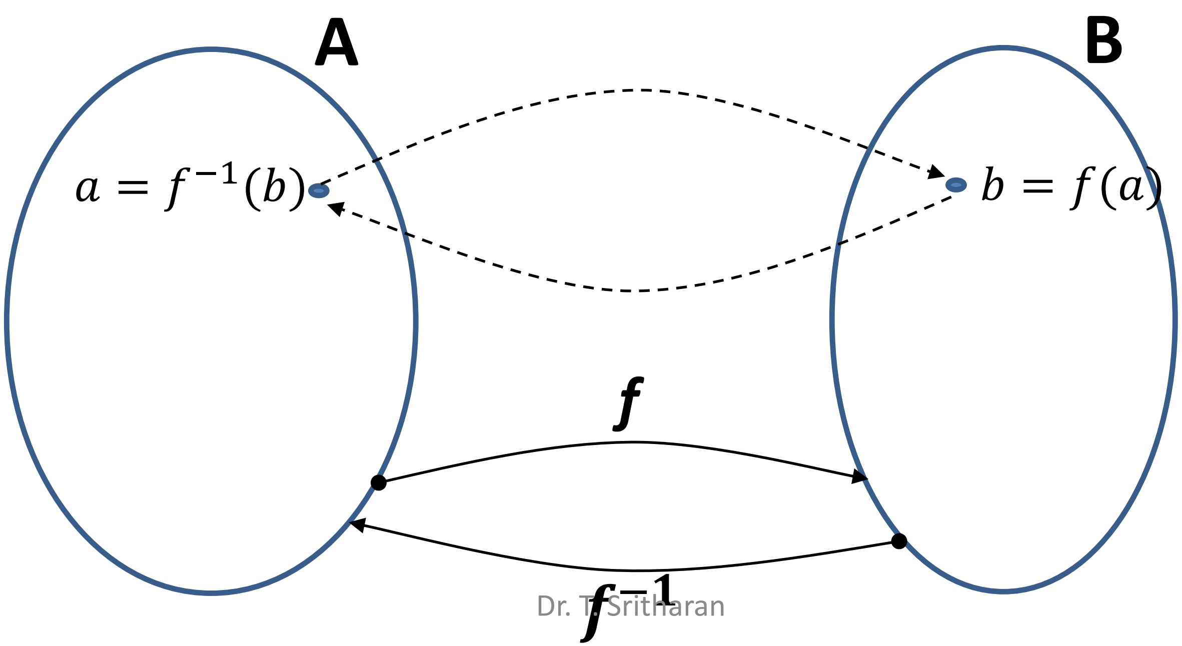
1. Let f be the function from $\{a, b, c, d\}$ to $\{1,2,3,4\}$ with $f(a) = 4, f(b) = 2, f(c) = 1,$ and $f(d) = 3$. Is f a bijection?

2. Give an example of a function from $\mathbb{N}$ to $\mathbb{N}$ that is
   a. one-to-one but not onto
   b. onto but not one-to-one
   c. both onto and one-to-one (but different from the identity function)
   d. neither one-to-one nor onto.

Inverse Functions

An interesting property of bijections is that they have an **inverse function**.

Definition: Let f be a one-to-one correspondence (bijection) from the set A to the set B. The inverse function of f is the function that assigns to an element b belonging to B the unique element a in A such that $f(a) = b$. The inverse function of f is denoted by f^{-1}. Hence, $f^{-1}(b) = a$ when $f(a) = b$.



Examples

1. Let f be the function from $\{a, b, c\}$ to $\{1,2,3\}$ such that $f(a) = 2, f(b) = 3$, and $f(c) = 1$.
   Is f invertible, and if it is, what is its inverse?

2. Let $f : Z \to Z$ be such that $f(x) = x + 1$.
   Is f invertible, and if it is, what is its inverse?

3. Let f be the function from R to R with $f(x) = x^2$.
   Is f invertible?

Compositions of Functions

The **composition** of two functions $g: A \to B$ and $f: B \to C$, denoted by $f \circ g$, is defined by

$$(f \circ g)(a) = f\big(g(a)\big) \text{ for all } a \in A.$$

Note that the composition $f \circ g$ cannot be defined unless the range of g is a subset of the domain of f.

Example 1: Let g be the function from the set $\{a, b, c\}$ to itself such that $g(a) = b, g(b) = c,$ and $g(c) = a$. Let f be the function from the set $\{a, b, c\}$ to the set $\{1,2,3\}$ such that $f(a) = 3, f(b) = 2,$ and $f(c) = 1$. What is the composition of f and g, and what is the composition of g and f?

Example 2: Let f and g be the functions from the set of integers to the set of integers defined by $f(x) = 2x + 3$ and $g(x) = 3x + 2$. What is the composition of f and g? What is the composition of g and f ?

More Examples

Example: Let $f\colon\ A \ \rightarrow\ B$ and $g\colon\ B \ \rightarrow\ C$ be both one-to-one functions. Show that $g \circ f: A \rightarrow C$ is one-to-one.

Solution: Assume that $(g \circ f)(a) = \ (g \circ f)(b)$ for $a, b \in A$.

$$\Longrightarrow g\big(f(a)\big) = g\big(f(b)\big) \text{ (}\because \text{ Def. of } g \circ f\text{)}$$

$$\Longrightarrow f(a) = f(b) \quad (\because\ g\ is\ 1 - 1\ )$$

$$\Longrightarrow a = b \quad (\because\ g\ is\ 1 - 1)$$

Hence, $\forall a, b \in A\ [(g \circ f)(a) = \ (g \circ f)(b) \ \rightarrow a = b]$

Therefore $g \circ f$ is one-to-one.

Example: Let $f\colon\ A \ \rightarrow\ B$ and $g\colon\ B \ \rightarrow\ C$ be both onto functions. Show that $g \circ f: A \rightarrow C$ is onto.

Identity Function

Definition: Let A be a non-empty set. The **identity function** on A is the function i_A: $A \to A$, where $i_A(x) = x$ for all $x \in A$. In other words, the identity function i_A is the function that assigns each element to itself.

The function i_A is one-to-one and onto, so it is a bijection.

When the composition of a function $f: A \to B$ and its inverse $f^{-1}: B \to A$ is formed, in either order, an identity function is obtained.

$$(f^{-1} \circ f)(a) = f^{-1}\big(f(a)\big) = i_A \text{ for all } a \in A, and$$

$$(f \circ f^{-1})(b) = f\big(f^{-1}(a)\big) = i_B \text{ for all } b \in B.$$

Hence $f^{-1} \circ f = i_A$, $f \circ f^{-1} = i_B$ and $(f^{-1})^{-1} = f$.

The Graphs of Functions

Definition: Let f be a function from the set A to the set B. The graph of the function f is the set of ordered pairs $\{(a, b) \mid a \in A \text{ and } f(a) = b\}$.

Example-1: Display the graph of the function $f(n) = 2n + 1$ from the set of integers to the set of integers.

Example-2: Display the graph of the function $f(x) = 2x + 1$ from the set of real numbers to the set of real numbers.

Example-3: Display the graph of the function $f(x) = x^2$ from the set of real numbers to the set of real numbers.

The Image of a Subset of the Domain

Let $f: A \to B$ be a function and let $S \subseteq A$. The image of S under f is the subset of B that consists of the images of the elements of S, and is denoted by $f(S)$.

$$f(S) = \{\, t \,|\, \exists s \in S\,(t = f(s))\} = \{f(s) \mid s \in S\}$$

Example: Let $f(x) = 2x$ where the domain is the set of real numbers. What is

a. $f(Z)$?
b. $f(N)$?
c. $f(R)$?

Example: Let $f: A \to B$ be a function. Let $\mathrm{S}, \mathrm{T} \subseteq A$. Show that

a. $f(S \cup T) = f(S) \cup f(T)$
b. $f(S \cap T) \subseteq f(S) \cap f(T)$.

Partial Functions

Definition: A partial function f from a set A to a set B is an assignment to each element a in a subset of A, called the domain of definition of f, of a unique element b in B. The sets A and B are called the domain and codomain of f, respectively. We say that f is undefined for elements in A that are not in the domain of definition of f. When the domain of definition of f equals A, we say that f is a total function.

Example: The function $f : Z \to R$ where $f(n) = \sqrt{n}$ is a partial function from Z to R where the domain of definition is the set of nonnegative integers. Note that f is undefined for negative integers.

$$f(n) = \begin{cases} \sqrt{n} & if \ \ n \geq 0 \\ \uparrow & if \ \ n < 0. \end{cases}$$