GEOPHYSICS, VOL. 46, NO. 6 (JUNE 1981); P. 932–933, 1 FIG.

Short Note
The Cole-Cole model in time domain induced polarization

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Recently Pelton et al (1978) used a Cole-Cole relaxation model to simulate the transient voltages that are observed during an induced-polarization survey. These authors took the impedance of the equivalent circuit $Z(\omega)$ to be

$$Z(\omega) = R_0 \left\{ 1 - m \left[1 - \frac{1}{1 + (i\,\omega\,\tau)^c} \right] \right\}. \tag{1}$$

They then gave the expression for the transient voltage $V_1(t)$ as

$$V_1(t) = m\,R_0 I_0 \left[\sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(n\,c + 1)} \left(\frac{t}{\tau} \right)^{nc} \right]. \tag{2}$$

In equation (2), I_0 was misprinted as $1/I_0$. In these equations, $m = 1/(1 + R_1/R_0)$ and R_1, R_0 and τ are constants to be determined for the given model. I_0 is the height of the step current that will flow in the transmitter. A disadvantage of equation (2) is that it is only slowly convergent for large t/τ. Pelton et al (1978) used a τ which ranged from 10^{-4} to 10^{-2}. The purpose of this note is to provide an alternative expression for $V_1(t)$ that is valid only at the later stages but which does not have this disadvantage. The trivial case of $c = 1.0$ is ignored.

From equation (1), the voltage $V(t)$ is

$$V(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{I_0 Z(\omega)}{i\,\omega} e^{i\omega t}\, d\omega. \tag{3}$$

The integral in equation (3) may be evaluated by contour integration about the branch cut along the imaginary ω-axis (see Figure 1). There are no poles in the upper half of the complex ω-plane. One notices that for $t > 0$, the contribution from the quarter circles is negligible because of Jordan's theorem.

The integral about the origin 0, I_1 is defined by the path P_0. In fact,

$$I_1 = \frac{1}{2\pi} \int_{P_0} \frac{e^{i\omega t} I_0 Z(\omega)}{i\,\omega}\, d\omega. \tag{4}$$

Writing $\omega = r\,e^{i\theta}$ in this integral and allowing r to become small, one finds that

$$I_1 = -R_0 I_0. \tag{5}$$

The contribution from the paths $P+$ and $P-$ may be obtained by writing $\omega = e^{i\pi/2}R$ on $P+$ and $\omega = e^{-3\pi/2\,i}R$ on $P-$. When this is done, we find that

$$\frac{1}{2\pi} \int_{P+} \frac{I_0 Z(\omega) e^{i\omega t}}{i\,\omega}\, d\omega + \frac{1}{2\pi} \int_{P-} \frac{I_0 Z(\omega) e^{i\omega t}}{i\,\omega}\, d\omega = \frac{1}{2\pi} \int_0^{\infty} \frac{I_0 m\,R_0}{R} \cdot e^{-Rt} \left[\frac{\tau^c R^c\, 2 \sin \pi\,c}{(1 + \cos \pi\,c\,\tau^c R^c)^2 + R^{2c}\tau^{2c} \sin^2 \pi\,c} \right] dR. \tag{6}$$

By Cauchy's theorem and equations (3), (5), and (6) one now finds that

$$V(t) = I_0 R_0 - \frac{I_0}{\pi} \int_0^{\infty} \frac{m\,R_0 e^{-Rt} \tau^c R^c \sin \pi\,c\, dR}{[(1 + \cos \pi\,c\,\tau^c R^c)^2 + R^{2c}\tau^{2c} \sin^2 \pi\,c]R}. \tag{7}$$

The interpretation of equation (7) is that the time-dependent term is the amount by which the voltage is reduced from the steady-state value of $I_0 R_0$.

Thus the transient voltage $V_1(t)$ of Pelton et al (1978) is

$$V_1(t) = \frac{I_0}{\pi} \int_0^{\infty} \frac{m\,R_0 e^{-xt/\tau} x^c \sin \pi\,c}{x[(1 + x^c \cos \pi\,c)^2 + x^{2c} \sin^2 \pi\,c]}\, dx. \tag{8}$$

To see how $V_1(t)$ behaves for large t/τ, one only needs to derive an asymptotic expansion for the integral in equation (8). This expansion is found by expanding the function $x^c/[(1 + x^{2c} + 2x^c \cos \pi\,c)]$ as a power series in x^c and then integrating term by term. Since

$$x^c/(1 + x^{2c} + 2x^c \cos \pi\,c) \approx x^c[1 - (2 \cos \pi\,c)x^c + (4 \cos^2 \pi\,c - 1)x^{2c}], \tag{9}$$

$$V(t) \approx \frac{I_0 m\,R_0}{\pi} \sin \pi\,c \left[\Gamma(c) \left(\frac{\tau}{t} \right)^c - 2\,\Gamma(2c) \cos \pi\,c \left(\frac{\tau}{t} \right)^{2c} + (4 \cos^2 \pi\,c - 1)\Gamma(3c) \left(\frac{\tau}{t} \right)^{3c} \right]. \tag{10}$$

Equation (10) clearly shows that the decay (growth) is much

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slower, at the later stages, than an exponential. It also provides a means of estimating that decay.

Pelton et al (1978) also used a modified form of $Z(\omega)$. Specifically, they took

$$Z(\omega) = R_0 \left[1 - \frac{m_1}{1 + (i\,\omega\,\tau_1)^{c_1}} \right] \frac{1}{[1 + (i\,\omega\,\tau_2)^{c_2}]}. \quad (11)$$

For this model we concur with the remark made by Pelton et al that the Cole-Cole model is only a very idealized representation of the mineralized rock. Nevertheless, the model has been useful, and for this reason it is discussed here. Once again one finds that the integral about the origin yields

$$-R_0 I_0 = \frac{I_0}{2\pi} \int_{P_0} \frac{Z(\omega)}{i\,\omega}\, d\omega. \quad (12)$$

The integral about the branch cut is given in equation (13).

$$\frac{1}{2\pi} \int_{P_+} \frac{I_0 Z(\omega)}{i\,\omega} e^{i\omega t} d\omega + \frac{1}{2\pi} \int_{P_-} \frac{I_0 Z(\omega)}{i\,\omega} d\omega$$

$$= -\frac{I_0 R_0}{2\pi} \int_0^{\infty} \frac{e^{-Rt}}{Ri} \left\{ \left[1 - \frac{m_1}{1 + (R\,e^{i\pi}\tau_1)^{c_1}} \right] \left[\frac{1}{1 + (R\,e^{i\pi}\tau_2)^{c_2}} \right] - \left[1 - \frac{m_1}{1 + (R\,e^{-i\pi}\tau_1)^{c_1}} \right] \left[\frac{1}{1 + (R\,e^{-i\pi}\tau_2)^{c_2}} \right] \right\} dR$$

$$= \frac{I_0 R_0}{\pi} \int_0^{\infty} \frac{e^{-Rt}}{R} \left\{ \frac{\sin \pi\, c_2 (R\tau_2)^{c_2}}{[1 + (\tau_2 R)^{2c_2} + 2(\tau_2 R)^{c_2} \cos \pi\, c_2]} - \frac{m_1 [(R\tau_1)^{c_1} \sin \pi\, c_1 + (R\tau_2)^{c_2} \sin \pi\, c_2]}{[1 + (R\tau_1)^{2c_1} + 2(R\tau_1)^{c_1} \cos \pi\, c_1][1 + (R\tau_2)^{2c_2} + 2(R\tau_2)^{c_2} \cos \pi\, c_2]} \right\} dR. \quad (13)$$

In this case, when both t/τ_1 and t/τ_2 are large,

$$V_1(t) \approx \frac{I_0 R_0}{\pi} \left\{ \Gamma(c_2) \sin \pi\, c_2 \left(\frac{\tau_2}{t} \right)^{c_2} - m_1 \left[\left(\frac{\tau_1}{t} \right)^{c_1} \sin \pi\, c_1 \Gamma(c_1) + \left(\frac{\tau_2}{t} \right)^{c_2} \sin \pi\, c_2 \Gamma(c_2) \right] \right\}. \quad (14)$$

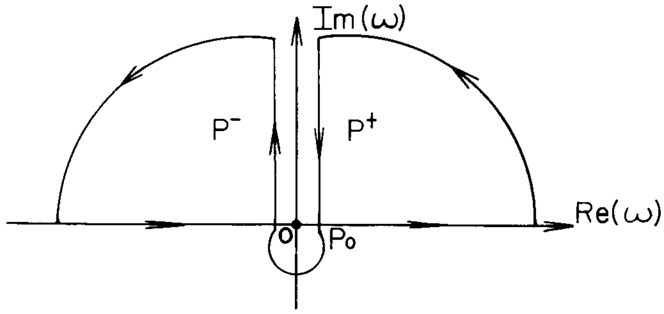

Fig. 1. Path for contour integration.

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