Q1

Tuesday, December 1, 2020 10:02 AM

1. Raleigh-Jean's law gives the radiant energy density (energy per unit volume) as

$$u(\nu)d\nu = \frac{8\pi\nu^2}{c^3}kT\,d\nu$$

Show that in terms of wavelength interval, Raleigh-Jean's law can be expressed as

$$u(\lambda)d\lambda = \frac{8\pi}{\lambda^4}kT\,d\lambda$$

$u(\nu)$: Energy / Vol / freq.

$u(\lambda)$: Energy / Vol / λ

$\therefore \quad u'(\lambda)d\lambda = u(\nu)d\nu$ = Energy / Vol

$$\nu = \frac{c}{\lambda}$$

$$\frac{d\nu}{d\lambda} = -\frac{c}{\lambda^2} \Rightarrow \boxed{d\nu = -\frac{c}{\lambda^2}d\lambda}$$

$$\underline{u'(\lambda)}\,d\lambda = \underline{u}\left(\frac{c}{\lambda}\right)\frac{c}{\lambda^2}d\lambda$$ (where is (-)ve sign?)

$$= \frac{8\pi}{c^3}\left(\frac{c}{\lambda}\right)^2 kT\,\frac{c}{\lambda^2}d\lambda$$

$$= \frac{8\pi kT}{\lambda^4}d\lambda$$

Q2

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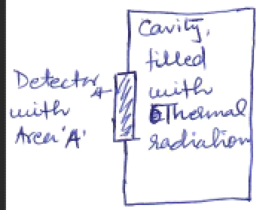
2. Use Planck's equation and show that the expression for radiant intensity (in terms of λ) is given by

$$I(\lambda)\, d\lambda = \frac{2\pi hc^2}{\lambda^5} \frac{1}{\exp(hc/\lambda kT) - 1}$$

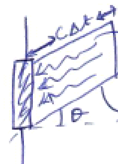
$$\mathbf{u}(\nu) = \frac{8\pi\nu^2}{c^3} \frac{h\nu}{e^{h\nu/k_B T} - 1}$$



Let ΔE is the Energy (radiation) that hits the detector in time Δt

Assume radiation has frequency 'f' and have momentum direction $|p|$

In time 'Δt', all the radiation in this cavity that is moving in this direction will hit the detector



Integrate over directions (θ & ϕ)

$$\Delta E = \int_0^{\pi/2} d\theta \int_0^{2\pi} d\phi \,(\text{energy in cylinder})\, \underbrace{P(\theta,\phi)}_{\text{probability that radiation is going in the right direction}}$$

if $\theta > \pi/2$, we are outside of cavity

$$= \int_0^{\pi/2} d\theta \int_0^{2\pi} d\phi \left(u \cdot \underbrace{A\cos\theta}_{\text{Area of slice } \perp^{lar} \text{ to } \vec{p}} \cdot c\Delta t \right) \underbrace{\frac{\sin\theta}{4\pi}}_{\text{isotropic radiation}}$$

$$= \frac{c u A \Delta t}{4\pi} \underbrace{\int_0^{\pi/2} \sin\theta \cos\theta \, d\theta}_{1/2} \; \underbrace{\int_0^{2\pi} d\phi}_{2\pi}$$

$$\Delta E = \frac{c u A \Delta t}{4}$$

(1)

Now power incident on detector $\equiv$ Intensity (Energy flux)

$$= \frac{\Delta E}{A \Delta t}$$

So Intensity $= \frac{\Delta E}{A\Delta t} = \frac{cu}{4} \longrightarrow$ (1)

Here $u(f,T)$ is spectral Energy density

$$u(f,T)\,df = \frac{\text{Energy of radiation with frequency in } f \,\&\, f+df}{\text{unit volume}}$$

Now recall Kirchoff's Law for Blackbody

$$e_f = J(f,T)(A_f = 1)$$

e_f: emitted power per unit Area per freq^n f

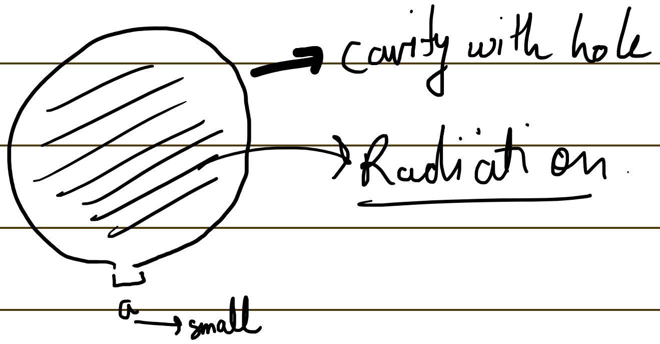
(2) $\leftarrow$ So $\boxed{J(f,T) \text{ is Intensity per freq}^n\, f}$

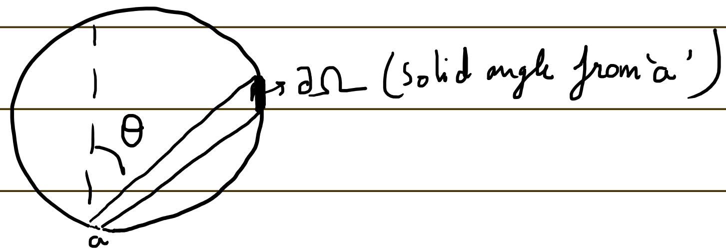
as (power per unit Area is Intensity)

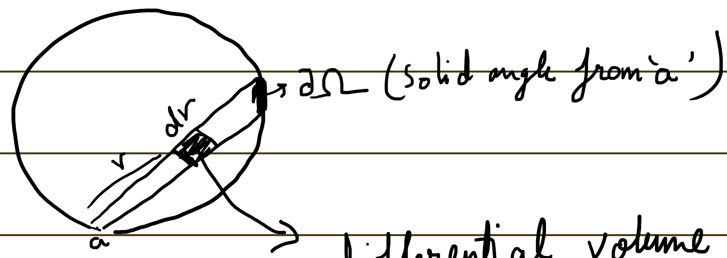
$\Rightarrow$ from (1) & (2)

$$\boxed{J(f,T) = \frac{c\,u(f,T)}{4}}$$

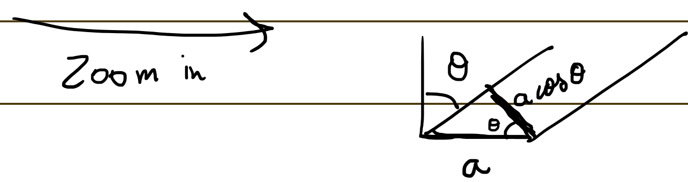
(2)







differential volume $dV = d\Omega\, r^2 dr$



Solid angle made by a on dV $= \dfrac{\text{Area of 'a' projected on } dV}{r^2} = \dfrac{a\cos\theta}{r^2}$

Probability of radiation from dV falling on a $= \dfrac{\frac{a\cos\theta}{r^2}}{4\pi} = \dfrac{a\cos\theta}{4\pi r^2}$

$\Delta E =$ Amount of radiation $\quad$ Total $\quad a\cos\theta$

ΔE = Amount of radiation coming from dV to a $=$ $\underbrace{\text{Total radiation from } dV}$ $\times \dfrac{a\cos\theta}{4\pi r^2}$

$d\Omega\, r^2 dr \longleftarrow dV \times u(\Omega)d\Omega \rightarrow$ Let 'u'

$$= \frac{a\cos\theta}{4\pi r^2} r^2 dr\, d\Omega\, u$$

$$= \frac{a\cos\theta}{4\pi} u \underbrace{dr}_{c\,dt} d\Omega$$

$$\frac{\Delta E}{a\,dt} = \frac{\text{Power}}{\text{Area}} = \frac{uc}{4\pi} \cos\theta \underbrace{d\Omega}_{\sin\theta\, d\theta\, d\phi}$$

$$= \frac{uc}{4\pi} \cos\theta \sin\theta\, d\theta\, d\phi$$

Integrate as in the images above.

Q3

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3. According to Planck, the spectral energy density $u(\lambda)$ of a black body maintained at temperature T is given by

A

$$u(\lambda, T) = \frac{8\pi hc}{\lambda^5} \frac{1}{\exp\left(\frac{hc}{\lambda k_B T}\right) - 1}$$

α

where λ denotes the wavelength of radiation emitted by the black body.

(a) Find an expression for $\lambda_{\max}$ at which $u(\lambda, T)$ attains its maximum value (at a fixed temperature T). $\lambda_{\max}$ should be in terms of T and fundamental constants h, c and k_B.

Maximize $u(\lambda, T)$ by varying λ (for a particular T)

$$\frac{d\,u(\lambda,T)}{d\lambda} = 0$$

$$A\left[\frac{-5}{\lambda^6}\frac{1}{\left(\exp\left(\frac{\alpha}{\lambda}\right)-1\right)} - \frac{\exp\left(\frac{\alpha}{\lambda}\right)}{\left(\exp\left(\frac{\alpha}{\lambda}\right)-1\right)^2}\left(-\frac{\alpha}{\lambda^2}\right)\cdot\frac{1}{\lambda^5}\right] = 0$$

$$-5 + \frac{\alpha}{\lambda}\frac{\exp\left(\frac{\alpha}{\lambda}\right)}{\left(\exp\left(\frac{\alpha}{\lambda}\right)-1\right)} = 0$$

Let $\frac{\alpha}{\lambda} = z$

$$\frac{z}{(1-\exp(-z))} = 5$$

$5(1-e^{-z}) = z \Rightarrow$ Use scientific Calci.

$z \approx 4.965$

$$\frac{hc}{\lambda KT} = 4.965$$

$$\boxed{\lambda_m T = \frac{hc}{4.965\, KT}}$$

(b) Expressing $\lambda_{\max}$ as $\frac{\alpha}{T}$, obtain an expression for $u_{\max}(T)$ in terms of α, T and the fundamental constants.

$$\lambda_{max} = \frac{\alpha}{T}$$

$$\boxed{u_{max}(T) = \frac{8\pi hc T^5}{\alpha^5} \frac{1}{\left(e^{\frac{hc}{\alpha K}} - 1\right)}}$$

Q4

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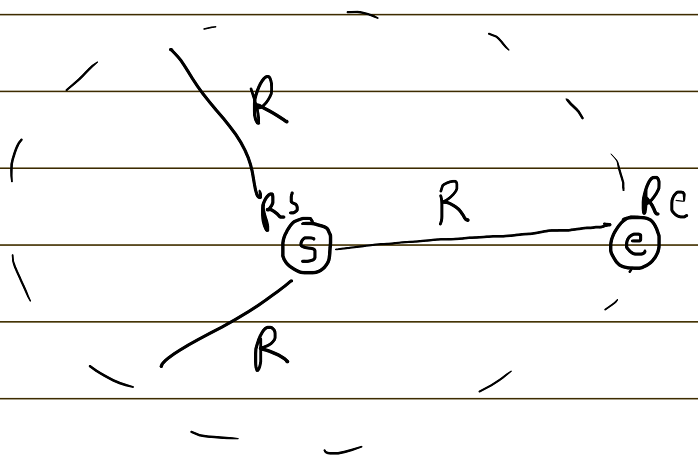
4. The earth rotates in a circular orbit about the sun. The radius of the orbit is 140×10^6 km. The radius of the earth is 6000 km and the radius of the sun is 700,000 km. The surface temperature of the sun is 6000 K. Assuming that the sun and the earth are perfect black bodies, calculate the equilibrium temperature of the earth.



$R = 140 \times 10^6$ Km

$R_e = 6000$ km

$R_S = 700{,}000$ km

$T_S = 6000$ K

$\Rightarrow \quad I = \sigma T^4 \quad \& \quad I \cdot A = \text{Power}$

$\therefore \quad P = \sigma A T^4$ (for black body)

Total Power emitted by Sun $= \sigma T_S^4 (4\pi R_S^2)$

$I_R =$ 'I' at 'R' (in outward direcn) $\Rightarrow \dfrac{\sigma T_S^4 (4\pi R_S^2)}{4\pi R^2}$

$= \sigma T_S^4 \dfrac{R_S^2}{R^2}$

$\therefore$ Power absorbed by Earth $= I_R \times \underline{\pi R_e^2}$?

In Eq^{m}

Power absorbed by Earth = Power emitted by Earth

$$\sigma\, 4\pi R_e^2 T_e^4 = \sigma T_s^4 \frac{R_s^2}{R^2} \times \pi R_e^2$$

$$T_e = \left(\frac{1}{4} \times \left(\frac{R_s}{R}\right)^2\right)^{1/4} T_s$$

$$T_e = T_s \sqrt{\frac{R_s}{2R}}$$

$$= T_s \sqrt{\frac{7 \times 10^5}{10^7 \times 14 \times 2}}$$

$$= \frac{T_s}{20}$$

$$= 300\,K.$$

Q5

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5. (a) Given Planck's formula for the energy density, obtain an expression for the Rayleigh Jeans formula for $U(\nu, T)$.

$$\mathbf{u}(\nu) = \frac{8\pi\nu^2}{c^3}\frac{h\nu}{e^{h\nu/k_BT}-1}$$

R.J. Law

Valid for Low freq.

$a = \frac{h\nu}{K_BT}$ is small (close to zero)

$\therefore$ $e^a = 1 + a + \frac{a^2}{2!} + \dots$

$e^a \simeq 1 + a$

$\therefore$ Plancks' law.

$$u(\nu, T)\,d\nu \simeq \frac{8\pi\nu^2}{c^3}\frac{h\nu}{1 + \frac{h\nu}{k_BT} - 1}\,d\nu$$

$$U(\nu,T)d\nu = \frac{8\pi\nu^2}{c^3} \times kT\,d\nu$$

R·J law

(b) For a black body at temperature T, $U(\nu, T)$ was measured at $\nu = \nu_0$. This value is found to be one tenth of the value estimated using Rayleigh Jeans formula. Obtain an implicit equation in terms of $h\nu/k_BT$

$$\frac{8\pi\nu_0^2}{c^3} \frac{h\nu_0}{\left(\exp\left(\frac{h\nu_0}{kT}\right) - 1\right)} = \frac{1}{10}\left[\frac{8\pi\nu_0^2}{c^3} kT\right]$$

$$\frac{h\nu_0}{\exp\left(\frac{h\nu_0}{kT}\right) - 1} = \frac{kT}{10}$$

$$\alpha = \frac{h\nu}{kT} \Rightarrow \boxed{10\alpha = \exp(\alpha) - 1}$$

(c) Solve the above equation to obtain the value of $h\nu/k_BT$, up to the first decimal place.

$$\frac{h\nu}{kT} = 3.165$$

$$\boxed{\frac{h\nu}{kT} = 3.2}$$

KT

$$\boxed{\frac{h\nu}{KT} = 3.2}$$

Q6

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6. Using appropriate approximations, derive Wiens' displacement law from Planck's formula for energy density of black body radiation.

$$u(\nu) = \frac{8\pi\nu^2}{c^3}\frac{h\nu}{e^{h\nu/k_BT} - 1}$$

Q7

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7. Derive the Stefan-Boltzmann law from the expression for $I(\lambda)$ given in problem 2

$$I(\lambda)\,d\lambda = \frac{2\pi hc^2}{\lambda^5}\frac{1}{\exp(hc/\lambda kT) - 1}\,d\lambda$$

$$\frac{c}{4}\underbrace{u(\lambda)\,d\lambda}_{\text{Energy/Vol}} = \underbrace{I(\lambda)\,d\lambda}_{\text{Energy/Area/time}}$$

We need Power/Area.

$$\frac{P}{A} = \int_0^\infty I(\lambda)\,d\lambda$$

$$= \int_0^\infty I(\nu)\,d\nu$$

$$= \int_0^\infty \frac{c}{4}\,u(\nu)\,d\nu$$

$$= \frac{c}{4}\int_0^\infty \frac{8\pi h}{c^3}\frac{\nu^3}{\left(\exp\left(\frac{h\nu}{kT}\right) - 1\right)}\,d\nu$$

$$= \frac{2\pi h}{c^2} \int_0 \frac{\quad}{\left(\exp\left(\frac{h\nu}{kT}\right) - 1\right)}$$

put $\frac{h\nu}{kT} = \alpha$, limit remains same.

$$\nu = \frac{kT\alpha}{h} \Rightarrow d\nu = \frac{kT}{h} d\alpha$$

$$\Rightarrow \frac{2\pi h}{c^2} \int_0^\infty \left(\frac{kT}{h}\right)^3 \frac{\alpha^3}{e^\alpha - 1} \frac{kT}{h} d\alpha$$

$$= \frac{2\pi h}{c^2} \left(\frac{kT}{h}\right)^4 \underbrace{\int_0^\infty \frac{\alpha^3}{e^\alpha - 1} d\alpha}_{\frac{\pi^4}{15}}$$

$$= \frac{2\pi k^4 T^4}{c^2 h^3} \frac{\pi^4}{15}$$

$$J(T) = \underbrace{\frac{2\pi^5}{15} \frac{k^4}{c^2 h^3}}_{\sigma} T^4$$