TRIGONOMETRY

- $\sin(A+B) = \sin A\cdot\cos B + \cos A\cdot\sin B$
- $\sin(A-B) = \sin A\cdot\cos B - \cos A\cdot\sin B$
- $\cos(A+B) = \cos A\cdot\cos B - \sin A\sin B$
- $\cos(A-B) = \cos A\cdot\cos B + \sin A\cdot\sin B$
- $\tan(A+B) = \dfrac{\tan A + \tan B}{1-\tan A\tan B}$
- $\tan(A-B) = \dfrac{\tan A - \tan B}{1+\tan A\cdot\tan B}$
- $\cot(A+B) = \dfrac{\cot B\cot A - 1}{\cot B + \cot A}$
- $\cot(A-B) = \dfrac{\cot B\cdot\cot A + 1}{\cot B - \cot A}$

- odd $F(x) \Rightarrow f(-x) = -f(x)$
- Even $f(x) \Rightarrow F(-x) = +F(x)$

$\cos(-x) = \cos x$	$\sec(-x) = \sec x$
$\sin(-x) = -\sin x$	$\operatorname{cosec}(-x) = -\operatorname{cosec} x$
$\tan(-x) = -\tan x$	$\cot(-x) = -\cot x$

- $2\sin A\cos B = \sin(A+B) + \sin(A-B)$
- $2\cos A\cdot\sin B = \sin(A+B) - \sin(A-B)$
- $2\cos A\cdot\cos B = \cos(A+B) + \cos(A-B)$
- $-2\sin A\cdot\sin B = \cos(A+B) - \cos(A-B)$
- $\sin(A+B)\cdot\sin(A-B) = \sin^2 A - \sin^2 B$
- $\cos(A+B)\cdot\cos(A-B) = \cos^2 A - \sin^2 B$

- $\sin C + \sin D = 2\cdot\sin\left(\frac{C+D}{2}\right)\cdot\cos\left(\frac{C-D}{2}\right)$
- $\sin C - \sin D = 2\cos\left(\frac{C+D}{2}\right)\cdot\sin\left(\frac{C-D}{2}\right)$
- $\cos C + \cos D = 2\cos\left(\frac{C+D}{2}\right)\cdot\cos\left(\frac{C-D}{2}\right)$
- $\cos C - \cos D = -2\sin\left(\frac{C+D}{2}\right)\cdot\sin\left(\frac{C-D}{2}\right)$
  $= 2\sin\left(\frac{C+D}{2}\right)\cdot\sin\left(\frac{D-C}{2}\right)$
- $\sin 2x = 2\sin x\cdot\cos x = \dfrac{2\tan x}{1+\tan^2 x}$
- $\cos 2x = \cos^2 x - \sin^2 x = 2\cos^2 x - 1$
  $= 1 - 2\sin^2 x = \dfrac{1-\tan^2 x}{1+\tan^2 x}$

$\sin^2\theta + \cos^2\theta = 1$	$1+\cos 2x = 2\cos^2 x$
$\sec^2\theta - \tan^2\theta = 1$	$1-\cos 2x = 2\sin^2 x$
$\operatorname{cosec}^2\theta - \cot^2\theta = 1$	$\tan x = \dfrac{1-\cos 2x}{\sin 2x}$

*General Solution of trigonometry

- $\sin x = 0 \Rightarrow x = n\pi$
- $\cos x = 0 \Rightarrow x = (2n+1)\pi/2$
- $\tan x = 0 \Rightarrow x = n\pi$
- $\sin x = \sin y \Rightarrow x = n\pi + (-1)^n\cdot y$
- $\cos x = \cos y \Rightarrow x = 2n\pi \pm y$

Trigonometric Values

- $\sin(90^\circ-\theta) = \cos\theta$
- $\cos(90^\circ-\theta) = \sin\theta$
- $\tan(90^\circ-\theta) = \cot\theta$
- $\cot(90^\circ-\theta) = \tan\theta$
- $\sec(90^\circ-\theta) = \csc\theta$
- $\mathrm{cosec}(90^\circ-\theta) = \sec\theta$

- $\sin(90^\circ+\theta) = \cos\theta$
- $\cos(90^\circ+\theta) = -\sin\theta$
- $\tan(90^\circ+\theta) = -\cot\theta$
- $\cot(90^\circ+\theta) = -\tan\theta$
- $\sec(90^\circ+\theta) = -\mathrm{cosec}\,\theta$
- $\mathrm{cosec}(90^\circ+\theta) = \sec\theta$

- $\sin(180^\circ-\theta) = \sin\theta$
- $\cos(180^\circ-\theta) = -\cos\theta$
- $\tan(180^\circ-\theta) = -\tan\theta$
- $\cot(180^\circ-\theta) = -\cot\theta$
- $\sec(180^\circ-\theta) = -\sec\theta$
- $\mathrm{cosec}(180^\circ-\theta) = \mathrm{cosec}\,\theta$

- $\sin(180^\circ+\theta) = -\sin\theta$
- $\cos(180^\circ+\theta) = -\cos\theta$
- $\tan(180^\circ+\theta) = \tan\theta$
- $\cot(180^\circ+\theta) = \cot\theta$
- $\sec(180^\circ+\theta) = -\sec\theta$
- $\mathrm{cosec}(180^\circ+\theta) = -\mathrm{cosec}\,\theta$

- $\sin(360^\circ-\theta) = -\sin\theta$
- $\cos(360^\circ-\theta) = \cos\theta$
- $\tan(360^\circ-\theta) = -\tan\theta$
- $\cot(360^\circ-\theta) = -\cot\theta$
- $\sec(360^\circ-\theta) = \sec\theta$
- $\mathrm{cosec}(360^\circ-\theta) = -\mathrm{cosec}\,\theta$

- $\sin(360^\circ+\theta) = \sin\theta$
- $\cos(360^\circ+\theta) = \cos\theta$
- $\tan(360^\circ+\theta) = \tan\theta$
- $\cot(360^\circ+\theta) = \cot\theta$
- $\sec(360^\circ+\theta) = \sec\theta$
- $\mathrm{cosec}(360^\circ+\theta) = \mathrm{cosec}\,\theta$

4. TRIGONOMETRIC FUNCTIONS OF ALLIED ANGLES :

(a) $\sin(2n\pi + \theta) = \sin\theta$, $\cos(2n\pi + \theta) = \cos\theta$, where $n \in I$

(b) $\boxed{\sin(-\theta) = -\sin\theta}$ $\boxed{\cos(-\theta) = \cos\theta}$

$\sin(90° - \theta) = \cos\theta$ $\cos(90° - \theta) = \sin\theta$

$\sin(90° + \theta) = \cos\theta$ $\cos(90° + \theta) = -\sin\theta$

$\sin(180° - \theta) = \sin\theta$ $\cos(180° - \theta) = -\cos\theta$

$\sin(180° + \theta) = -\sin\theta$ $\cos(180° + \theta) = -\cos\theta$

$\sin(270° - \theta) = -\cos\theta$ $\cos(270° - \theta) = -\sin\theta$

$\sin(270° + \theta) = -\cos\theta$ $\cos(270° + \theta) = \sin\theta$

Note :

(i) $\sin n\pi = 0$; $\cos n\pi = (-1)^n$; $\tan n\pi = 0$ where $n \in I$

(ii) $\sin(2n+1)\dfrac{\pi}{2} = (-1)^n$; $\cos(2n+1)\dfrac{\pi}{2} = 0$ where $n \in I$

5. IMPORTANT TRIGONOMETRIC FORMULAE :

(i) $\sin(A + B) = \sin A\cos B + \cos A\sin B$.

(ii) $\sin(A - B) = \sin A\cos B - \cos A\sin B$.

(iii) $\cos(A + B) = \cos A\cos B - \sin A\sin B$

(iv) $\cos(A - B) = \cos A\cos B + \sin A\sin B$

(v) $\tan(A + B) = \dfrac{\tan A + \tan B}{1 - \tan A\tan B}$

(vi) $\tan(A - B) = \dfrac{\tan A - \tan B}{1 + \tan A\tan B}$

(vii) $\cot(A + B) = \dfrac{\cot B\cot A - 1}{\cot B + \cot A}$

(viii) $\cot(A - B) = \dfrac{\cot B\cot A + 1}{\cot B - \cot A}$

(ix) $2\sin A\cos B = \sin(A + B) + \sin(A - B)$.

(x) $2\cos A\sin B = \sin(A + B) - \sin(A - B)$.

(xi) $2\cos A\cos B = \cos(A + B) + \cos(A - B)$

(xii) $2\sin A\sin B = \cos(A - B) - \cos(A + B)$

(xiii) $\sin C + \sin D = 2\sin\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right)$

(xiv) $\sin C - \sin D = 2\cos\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right)$

(xv) $\cos C + \cos D = 2\cos\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right)$

(xvi) $\cos C - \cos D = 2\sin\left(\frac{C+D}{2}\right)\sin\left(\frac{D-C}{2}\right)$

(xvii) $\sin 2\theta = 2\sin\theta\cos\theta = \frac{2\tan\theta}{1+\tan^2\theta}$

(xviii) $\cos 2\theta = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta = \frac{1-\tan^2\theta}{1+\tan^2\theta}$

(xix) $1 + \cos 2\theta = 2\cos^2\theta$ or $\cos\theta = \pm\sqrt{\frac{1+\cos 2\theta}{2}}$

(xx) $1 - \cos 2\theta = 2\sin^2\theta$ or $\sin\theta = \pm\sqrt{\frac{1-\cos 2\theta}{2}}$

(xxi) $\tan\theta = \frac{1-\cos 2\theta}{\sin 2\theta} = \frac{\sin 2\theta}{1+\cos 2\theta} = \pm\sqrt{\frac{1-\cos 2\theta}{1+\cos 2\theta}}$

(xxii) $\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}$

(xxiii) $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$.

(xxiv) $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$.

(xxv) $\tan 3\theta = \frac{3\tan\theta - \tan^3\theta}{1-3\tan^2\theta}$

(xxvi) $\sin^2 A - \sin^2 B = \sin(A+B).\sin(A-B) = \cos^2 B - \cos^2 A$.

(xxvii) $\cos^2 A - \sin^2 B = \cos(A+B).\cos(A-B)$.

Sunderdeep college of management and technology

Bca 4th sem

Subject:- Mathematics 3rd

Assignment :- 1

Q1. Define the following with examples

a) Modulus of complex no.
b) Argument of complex no.
c) limit of a sequence.
d) Bounded sequence & least upper bound & greatest lower bound.

Q2. Express 1+7i/(2-i)^2 in the modulus argument form.

Q3. The modulus of the sum of two complex no.can never exceed the sum of their moduli.

Q4.if Sin^-1(x/a)+Sin^-1(y/b)= p then prove that
x^2/a^2 + 2xy/ab cosp
+y^2/b^2=Sin^2p

Q5.Find the locus of the following complex no.

a) mod(z-i/z+i) = 5
b)mod(z-1/z+1)=pi/4
c)arg(z-1/z+1)= pi/3
d)mod(z-1)+mod(z+1) is lessthan equal to 4

Q6 Find the convergence of the following

a) (1 + 1/n+1)^n
b) (1 - 1/n)^n
c)(1 + 1/n)^ -n
d)(1 - 2/n+2)^n

SUBMISSION DATE: before 22nd march 2024

10.4 Exact Differential Equations of First Order

A differential equation of the form $M(x,y)dx + N(x,y)dy = 0$ is said to be exact if it can be directly obtained from its primitive by differentiation.

Theorem: The necessary and sufficient condition for the equation $M(x,y)dx + N(x,y)dy = 0$ to be exact is $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$.

Working rule to solve an exact differential equation

1. For the equation $M(x,y)dx + N(x,y)dy = 0$, check the condition for exactness i.e. $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$
2. Solution of the given differential equation is given by

$$\int M(taking\ y\ as\ constant)\ dx + \int N(terms\ not\ containg\ x)dy = C$$

Example 10 Solve the differential equation:

$(e^y + 1)\cos x\, dx + e^y \sin x\, dy = 0$ .....①

Solution: $M = (e^y + 1)\cos x$, $N = e^y \sin x$

$\frac{\partial M}{\partial y} = e^y \cos x$, $\frac{\partial N}{\partial x} = e^y \cos x$

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, ∴ given differential equation is exact.

Solution of ① is given by:

$$\underset{y\ constant}{\int (e^y + 1)\cos x\ \ dx} + \int 0\, dy = C$$

$\Rightarrow (e^y + 1)\sin x = C$

Example 11 Solve the differential equation:

$(\sec x \tan x\ \tan y - e^x)dx + (\sec x\, sec^2 y)dy = 0$ .....①

Solution: $M = \sec x \tan x\ \tan y - e^x$, $N = \sec x\, sec^2 y$

$\frac{\partial M}{\partial y} = \sec x \tan x\, sec^2 y$, $\frac{\partial N}{\partial x} = \sec x \tan x\, sec^2 y$

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, ∴ given differential equation is exact.

Solution of ① is given by:

$$\underset{y\ constant}{\int (\sec x \tan x\ \tan y - e^x)\, dx} + \int 0\, dy = C$$

$\Rightarrow \sec x \tan y - e^x = C$

Example 12 Solve the differential equation:

$\left[y\left(1 + \frac{1}{x}\right) + \cos y\right]dx + [x + \log x - x\sin y]dy = 0$ .....①

Solution: $M = y\left(1 + \frac{1}{x}\right) + \cos y$, $N = x + \log x - x\sin y$

$\frac{\partial M}{\partial y} = \left(1 + \frac{1}{x}\right) - \sin y$, $\frac{\partial N}{\partial x} = \left(1 + \frac{1}{x}\right) - \sin y$

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, ∴ given differential equation is exact.

Solution of ① is given by:

$$\int \left[y\left(1 + \frac{1}{x}\right) + \cos y\right]\ dx + \int 0\, dy = C$$

Solution of ① is given by:

$$\int_{y\ constant} \left[y\left(1+\frac{1}{x}\right)+\cos y\right] dx + \int 0\, dy = C$$

$\Rightarrow y\,(x+\log x) + x\cos y = C$

Example 13 Solve the differential equation:

$$x\,dx + y\,dy = \frac{a^2(x\,dy - y\,dx)}{x^2+y^2} \quad .....①$$

Solution: $\Rightarrow \left(x+\frac{a^2\,y}{x^2+y^2}\right)dx + \left(y-\frac{a^2\,x}{x^2+y^2}\right)dy = 0$

$$M = x+\frac{a^2\,y}{x^2+y^2},\ N = y-\frac{a^2\,x}{x^2+y^2}$$

$$\frac{\partial M}{\partial y} = \frac{a^2\,(x^2-y^2)}{(x^2+y^2)^2},\ \frac{\partial N}{\partial x} = \frac{a^2\,(x^2-y^2)}{(x^2+y^2)^2}$$

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, ∴ given differential equation is exact.

Solution of ① is given by:

$$\int_{y\ constant} \left(x+\frac{a^2\,y}{x^2+y^2}\right) dx + \int y\,dy = C$$

$\Rightarrow \frac{x^2}{2} + a^2\tan^{-1}\frac{x}{y} + \frac{y^2}{2} = \mathrm{C}$

$\Rightarrow x^2 + 2a^2\tan^{-1}\frac{x}{y} + y^2 = \mathrm{D}\ ,\ D = 2C$

10.5 Equations Reducible to Exact Differential Equations

Sometimes a differential equation of the form $M(x,y)dx + N(x,y)dy = 0$ is not exact i.e. $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$. It can be made exact by multiplying the equation by some function of x and y known as integrating factor (IF).

10.5.1 Integrating Factor (IF) Found By Inspection

Some non-exact differential equations can be grouped or rearranged and solved directly by integration, after multiplying by an integrating factor (IF) which can be found just by inspection as shown below:

Term	IF	Result
$xdy + ydx$	1. $\frac{1}{\mathrm{xy}}$	$\frac{xdy+ydx}{xy} = \frac{1}{\mathrm{y}}\mathrm{dy} + \frac{1}{x}\,\mathrm{dx} = d[\log(xy)]$
	2. $\frac{1}{(\mathrm{xy})^n}, n \neq 1$	$\frac{xdy+ydx}{(\mathrm{xy})^n} = \frac{\mathrm{d(xy)}}{(\mathrm{xy})^n} = -d\left[\frac{1}{(n-1)(\mathrm{xy})^{n-1}}\right]$
$xdy - ydx$	1. $\frac{1}{x^2}$	$\frac{xdy - ydx}{x^2} = d\left[\frac{y}{x}\right]$
	2. $\frac{1}{y^2}$	$\frac{xdy - ydx}{y^2} = -d\left[\frac{x}{y}\right]$
	3. $\frac{1}{\mathrm{xy}}$	$\frac{xdy - ydx}{\mathrm{xy}} = d\left[\log\frac{y}{x}\right]$

Term	IF	Result
$xdy + ydx$	1. $\frac{1}{xy}$	$\frac{xdy+ydx}{xy} = \frac{1}{y}dy + \frac{1}{x}dx = d[\log(xy)]$
	2. $\frac{1}{(xy)^n}, n \neq 1$	$\frac{xdy+ydx}{(xy)^n} = \frac{d(xy)}{(xy)^n} = -d\left[\frac{1}{(n-1)(xy)^{n-1}}\right]$
$xdy - ydx$	1. $\frac{1}{x^2}$	$\frac{xdy - ydx}{x^2} = d\left[\frac{y}{x}\right]$
	2. $\frac{1}{y^2}$	$\frac{xdy - ydx}{y^2} = -d\left[\frac{x}{y}\right]$
	3. $\frac{1}{xy}$	$\frac{xdy - ydx}{xy} = d\left[log\frac{y}{x}\right]$
	4. $\frac{1}{x^2+y^2}$	

		$\frac{xdy - ydx}{x^2 + y^2} = d\left[tan^{-1}\frac{y}{x}\right]$
	5. $\frac{1}{x\sqrt{x^2-y^2}}$	$\frac{xdy - ydx}{x\sqrt{x^2 - y^2}} = d\left[sin^{-1}\frac{y}{x}\right]$
$xdx + ydy$	1. $\frac{1}{x^2+y^2}$	$\frac{xdx + ydy}{x^2 + y^2} = \frac{1}{2}d[log(x^2 + y^2)]$
	2. $\frac{1}{(x^2+y^2)^n}, n \neq 1$	$\frac{xdx + ydy}{(x^2 + y^2)^n} = \frac{1}{2}d\left[\frac{(x^2 + y^2)^{-n+1}}{-n+1}\right]$

Example 14 Solve the differential equation:

$x\,dy - y\,dx + 2x^3dx = 0$ .....①

Solution: $\Rightarrow (-y + 2x^3)dx + xdy = 0$

$M = -y + 2x^3$, $N = x$

$\frac{\partial M}{\partial y} = -1$, $\frac{\partial N}{\partial x} = 1$

$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, ∴ given differential equation is not exact.

Taking $\frac{1}{x^2}$ as integrating factor due to presence of the term $(x\,dy - y\,dx)$

① may be rewritten as : $\frac{xdy - ydx}{x^2} + 2x\,dx = 0$

$\Rightarrow d\left[\frac{y}{x}\right] + 2x\,dx = 0$ ......②

Integrating ②, solution is given by : $\frac{y}{x} + x^2 = C$

$\Rightarrow y + x^3 = Cx$

Example 14 Solve the differential equation:

$x\,dy - y\,dx + 2x^3 dx = 0$ .....①

Solution: $\Rightarrow (-y + 2x^3)dx + xdy = 0$

$M = -y + 2x^3$, $N = x$

$\frac{\partial M}{\partial y} = -1$, $\frac{\partial N}{\partial x} = 1$

$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, ∴ given differential equation is not exact.

Taking $\frac{1}{x^2}$ as integrating factor due to presence of the term $(x\,dy - y\,dx)$

① may be rewritten as : $\frac{xdy - ydx}{x^2} + 2x\,dx = 0$

$\Rightarrow d\left[\frac{y}{x}\right] + 2x\,dx = 0$ ......②

Integrating ②, solution is given by : $\frac{y}{x} + x^2 = C$

$\Rightarrow y + x^3 = Cx$

Example 15 Solve the differential equation:

$y\,dx - x\,dy + (1 + x^2)dx + x^2 \cos y\,dy = 0$ .....①

Solution: $\Rightarrow (y + 1 + x^2)dx + (x^2 \cos y - x)dy = 0$

$M = y + 1 + x^2$, $N = x^2 \cos y - x$

$\frac{\partial M}{\partial y} = 1$, $\frac{\partial N}{\partial x} = 2x \cos y - 1$

$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, ∴ given differential equation is not exact.

Taking $\frac{1}{x^2}$ as integrating factor due to presence of the term $(y\,dx - x\,dy)$

① may be rewritten as : $\frac{ydy - xdx}{x^2} + \left(\frac{1}{x^2} + 1\right)dx + \cos y\,dy = 0$

$\Rightarrow -d\left[\frac{y}{x}\right] + \left(\frac{1}{x^2} + 1\right)dx + \cos y\,dy = 0$ ......②

Integrating ②, solution is given by : $-\frac{y}{x} + \left(-\frac{1}{x} + x\right) + \sin y = C$

$\Rightarrow x^2 - y - 1 + x \sin y = Cx$

Example 16 Solve the differential equation:

$x\,dx + y\,dy = a(x^2 + y^2)dy$ .....①

Solution: $\Rightarrow xdx + (y - a(x^2 + y^2))dy = 0$

$M = x$, $N = y - a(x^2 + y^2)$

$\frac{\partial M}{\partial y} = 0$, $\frac{\partial N}{\partial x} = -2ax$

$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, ∴ given differential equation is not exact.

Taking $\frac{1}{x^2 + y^2}$ as integrating factor due to presence of the term $(x\,dx + y\,dy)$

① may be rewritten as : $\frac{xdx + ydy}{x^2 + y^2} - a\,dy = 0$

$\Rightarrow \frac{1}{2} d[\log(x^2 + y^2)] - a\,dy = 0$

$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, ∴ given differential equation is not exact.

Taking $\frac{1}{x^2+y^2}$ as integrating factor due to presence of the term $(x\,dx + y\,dy)$

① may be rewritten as : $\frac{xdx+ydy}{x^2+y^2} - a\,dy = 0$

$\Rightarrow \frac{1}{2}d[log(x^2 + y^2)] - a\,dy = 0$

$\Rightarrow d[log(x^2 + y^2)] - 2\,a\,dy = 0$ ……②

Integrating ②, solution is given by: $(x^2 + y^2) - 2ay = C$, C is an arbitrary constant

Example 17 Solve the differential equation:

$a(x\,dy + 2y\,dx) = xy\,dy$ …..①

Solution: $\Rightarrow 2aydx + (ax - xy)dy = 0$

$M = 2ay\,,\ N = ax - xy$

$\frac{\partial M}{\partial y} = 2a\,,\ \frac{\partial N}{\partial x} = a - y$

$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, ∴ given differential equation is not exact.

Rewriting ① as $a(x\,dy + y\,dx) + ay\,dx = xy\,dy$ ……②

Taking $\frac{1}{xy}$ as integrating factor due to presence of the term $(x\,dy + y\,dx)$

② may be rewritten as : $a\frac{xdy+ydx}{xy} + \frac{a}{x}\,dx - dy = 0$

$\Rightarrow ad[\log(xy)] + \frac{a}{x}\,dx - dy = 0$ ……③

Integrating ③ solution is given by: $a\log(xy) + a\log x - y = C$

$\Rightarrow a\log(x^2y) - y = C$, C is an arbitrary constant

Example 18 Solve the differential equation:

$$x^4\frac{dy}{dx} + x^3y + \sec(xy) = 0 \quad .....①$$

Solution: $\Rightarrow (x^3y + \sec(xy))dx + x^4dy = 0$

$M = x^3y + \sec(xy)\,,\ N = x^4$

$\frac{\partial M}{\partial y} = x^3 + x\sec(xy)\tan(xy)\,,\ \frac{\partial N}{\partial x} = 4x^3$

$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, ∴ given differential equation is not exact.

Rewriting ① as: $x^3(x\,dy + y\,dx) + \sec(xy)\,dx = 0$

$\Rightarrow \frac{(x\,dy+y\,dx)}{\sec(xy)} - x^{-3}dx = 0$

$\Rightarrow \cos(xy)\,(x\,dy + y\,dx) - x^{-3}dx = 0$

$\Rightarrow d\,[\sin(xy)] - \frac{1}{2}d(x^{-2})dx = 0$ ……②

Integrating ②, we get the required solution as:

$\sin(xy) - \frac{x^{-2}}{2} = C$

$\Rightarrow 2\,x^2\sin(xy) - 1 = Cx^2$

10.5.2 Integrating Factor (IF) of a Non-Exact Homogeneous Equation

10.5.2 Integrating Factor (IF) of a Non-Exact Homogeneous Equation

If the equation $Mdx + Ndy = 0$ is a homogeneous equation, then the integrating factor (IF) will be $\frac{1}{Mx+Ny}$, provided $Mx + Ny \neq 0$

Example 19 Solve the differential equation:

$(x^3 + y^3)dx - xy^2\, dy = 0$ .....①

Solution: $M = x^3 + y^3$, $N = -xy^2$

$\frac{\partial M}{\partial y} = 3y^2$, $\frac{\partial N}{\partial x} = -y^2$

$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, $\therefore$ given differential equation is not exact.

As ① is a homogeneous equation , $\therefore$ IF $= \frac{1}{Mx+Ny} = \frac{1}{x^4 + xy^3 - xy^3} = \frac{1}{x^4}$

$\therefore$ ① may be rewritten as : $\left(\frac{1}{x} + \frac{y^3}{x^4}\right)dx - \frac{y^2}{x^3}dy = 0$ ......②

New $M = \frac{1}{x} + \frac{y^3}{x^4}$, New $N = -\frac{y^2}{x^3}$

$\frac{\partial M}{\partial y} = \frac{3y^2}{x^4}$, $\frac{\partial N}{\partial x} = \frac{3y^2}{x^4}$

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, $\therefore$ ② is an exact differential equation.

Solution of ② is given by:

$$\int_{y\ constant} \left(\frac{1}{x} + \frac{y^3}{x^4}\right) dx + \int 0\, dy$$

$\Rightarrow \log x - \frac{1}{3}\left(\frac{y}{x}\right)^3 = C$

Example 20 Solve the differential equation:

$(3y^4 + 3x^2y^2)dx + (x^3y - 3xy^3)\, dy = 0$ .....①

Solution: $M = 3y^4 + 3x^2y^2$, $N = x^3y - 3xy^3$

$\frac{\partial M}{\partial y} = 12y^3 + 6x^2y$, $\frac{\partial N}{\partial x} = 3x^2y - 3y^3$

$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, $\therefore$ given differential equation is not exact.

As ① is a homogeneous equation

$\therefore$ IF $= \frac{1}{Mx+Ny} = \frac{1}{3xy^4 + 3x^3y^2 + x^3y^2 - 3xy^4} = \frac{1}{4x^3y^2}$

$\therefore$ ① may be rewritten after multiplying by IF as:

$\left(\frac{3y^2}{4x^3} + \frac{3}{4x}\right)dx + \left(\frac{1}{4y} - \frac{3y}{4x^2}\right)dy = 0$ ......②

New $M = \frac{3y^2}{4x^3} + \frac{3}{4x}$, New $N = \frac{1}{4y} - \frac{3y}{4x^2}$

$\frac{\partial M}{\partial y} = \frac{6y}{4x^3} = \frac{3y}{2x^3}$, $\frac{\partial N}{\partial x} = \frac{3y}{2x^3}$

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, $\therefore$ ② is an exact differential equation.

Solution of ② is given by:

$$\int_{y\ constant} \left(\frac{3y^2}{4x^3} + \frac{3}{4x}\right) dx + \int \frac{1}{4y}\, dy$$

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, ∴ ② is an exact differential equation.

Solution of ② is given by:

$$\underset{y\ constant}{\int}\left(\frac{3y^2}{4x^3}+\frac{3}{4x}\right)dx + \int \frac{1}{4y}\,dy$$

$\Rightarrow \frac{-3y^2}{8x^2}+\frac{3}{4}\log x+\frac{1}{4}\log y = \mathrm{C}$

$\Rightarrow \log x^3 y - \frac{3y^2}{2x^2} = \mathrm{D}$, $\mathrm{D} = 4\mathrm{C}$

10.5.3 Integrating Factor of a Non-Exact Differential Equation of the Form

$\boldsymbol{yf_1(xy)dx + xf_2(xy)\,dy = 0}$: If the equation $Mdx + Ndy = 0$ is of the given form, then the integrating factor (IF) will be $\frac{1}{Mx-Ny}$ provided $Mx - Ny \neq 0$

Example 21 Solve the differential equation:

$y(1+xy)dx + x(1-xy)\,dy = 0$ .....①

Solution: $M = y + xy^2,\ N = x - x^2y$

$\frac{\partial M}{\partial y} = 1 + 2xy$, $\frac{\partial N}{\partial x} = 1 - 2xy$

$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, ∴ given differential equation is not exact.

As ① is of the form $yf_1(xy)dx + xf_2(xy)dy = 0$,

$\therefore \mathrm{IF} = \frac{1}{Mx-Ny} = \frac{1}{xy+x^2y^2-xy+x^2y^2} = \frac{1}{2x^2y^2}$

∴ ① may be rewritten after multiplying by IF as:

$\left(\frac{1}{2x^2y}+\frac{1}{2x}\right)dx + \left(\frac{1}{2xy^2}-\frac{1}{2y}\right)dy = 0$ ......②

New $M = \frac{1}{2x^2y}+\frac{1}{2x}$, New $N = \frac{1}{2xy^2}-\frac{1}{2y}$

$\frac{\partial M}{\partial y} = \frac{-1}{2x^2y^2}$, $\frac{\partial N}{\partial x} = \frac{-1}{2x^2y^2}$

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, ∴ ② is an exact differential equation.

Solution of ② is given by:

$$\underset{y\ constant}{\int}\left(\frac{1}{2x^2y}+\frac{1}{2x}\right)dx + \int -\frac{1}{2y}dy$$

$\Rightarrow \frac{-1}{2xy}+\frac{1}{2}\log x - \frac{1}{2}\log y = \mathrm{C}$

$\Rightarrow \log\frac{x}{y} - \frac{1}{xy} = \mathrm{D},\ \mathrm{D} = 2\mathrm{C}$

Example 22 Solve the differential equation:

$y(xy + 2x^2y^2)dx + x(xy - x^2y^2)\,dy = 0$ .....①

Solution: $M = xy^2 + 2x^2y^3,\ N = x^2y - x^3y^2$

Example 22 Solve the differential equation:

$$y(xy + 2x^2y^2)dx + x(xy - x^2y^2)\,dy = 0 \;.....①$$

Solution: $M = xy^2 + 2x^2y^3,\; N = x^2y - x^3y^2$

$$\frac{\partial M}{\partial y} = 2xy + 6x^2y^2\,,\; \frac{\partial N}{\partial x} = 2xy - 3x^2y^2$$

$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, ∴ given differential equation is not exact.

As ① is of the form $yf_1(xy)dx + xf_2(xy)dy = 0$,

$$\therefore \text{IF} = \frac{1}{Mx-Ny} = \frac{1}{x^2y^2+2x^3y^3-x^2y^2+x^3y^3} = \frac{1}{3x^3y^3}$$

∴ ① may be rewritten after multiplying by IF as:

$$\left(\frac{1}{x^2y} + \frac{2}{x}\right)dx + \left(\frac{1}{xy^2} - \frac{1}{y}\right)dy = 0 \;......②$$

New $M = \frac{1}{x^2y} + \frac{2}{x}$, New $N = \frac{1}{xy^2} - \frac{1}{y}$

$$\frac{\partial M}{\partial y} = \frac{-1}{x^2y^2}\,,\; \frac{\partial N}{\partial x} = \frac{-1}{x^2y^2}$$

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, ∴ ② is an exact differential equation.

Solution of ② is given by:

$$\int_{y\ constant}\left(\frac{1}{x^2y} + \frac{2}{x}\right)dx + \int -\frac{1}{y}dy$$

$$\Rightarrow \frac{-1}{xy} + 2\log x - \log y = C$$

$$\Rightarrow \log\frac{x^2}{y} - \frac{1}{xy} = C$$

10.5.4 Integrating Factor (IF) of a Non-Exact Differential Equation

$Mdx + Ndx = 0$ in which $\frac{\partial M}{\partial y}$ and $\frac{\partial N}{\partial x}$ are connected in a specific way as shown:

i. If $\dfrac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = f(x)$, a function of x alone, then IF $= e^{\int f(x)dx}$

ii. If $\dfrac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{M} = g(y)$, a function of y alone, then IF $= e^{\int -g(y)dy}$

Example 23 Solve the differential equation:

$$(x^3 + y^2 + x)dx + xy\,dy = 0 \;.....①$$

Solution: $M = x^3 + y^2 + x,\; N = xy$

$$\frac{\partial M}{\partial y} = 2y\,,\; \frac{\partial N}{\partial x} = y$$

$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, ∴ given differential equation is not exact.

As ① is neither homogeneous nor of the form $yf_1(xy)dx + xf_2(xy)dy = 0$,

$\therefore$ Computing $\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = y$

Clearly $\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{y}{xy} = \frac{1}{x} = f(x)$ say

$\therefore$ IF $= e^{\int f(x)dx} = e^{\int \frac{1}{x}dx} = e^{\log x} = x$

$\therefore$ ① may be rewritten after multiplying by IF as:

$(x^4 + xy^2 + x^2)dx + x^2y\,dy = 0$ ....②

New $M = x^4 + xy^2 + x^2$, New $N = x^2y$

$\frac{\partial M}{\partial y} = 2xy$, $\frac{\partial N}{\partial x} = 2xy$

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, $\therefore$ ② is an exact differential equation.

Solution of ② is given by:

$$\underset{y\ constant}{\int}(x^4 + xy^2 + x^2)\,dx + \int 0\,dy$$

$\Rightarrow \frac{x^5}{5} + \frac{x^2y^2}{2} + \frac{x^3}{3} = C$

Example 24 Solve the differential equation:

$(y^4 + 2y)dx + (xy^3 + 2y^4 - 4x)\,dy = 0$ .....①

Solution: $M = y^4 + 2y$, $N = xy^3 + 2y^4 - 4x$

$\frac{\partial M}{\partial y} = 4y^3 + 2$, $\frac{\partial N}{\partial x} = y^3 - 4$

$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, $\therefore$ given differential equation is not exact.

As ① is neither homogeneous nor of the form $yf_1(xy)dx + xf_2(xy)dy = 0$,

$\therefore$ Computing $\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = 3y^3 + 6$

Clearly $\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{M} = \frac{3y^3+6}{y^4+2y} = \frac{3}{y} = g(y)$ say

$\therefore$ IF $= e^{\int -g(y)dy} = e^{\int -\frac{3}{y}dy} = e^{-3\log y} = \frac{1}{y^3}$

$\therefore$ ① may be rewritten after multiplying by IF as:

$\left(y + \frac{2}{y^2}\right)dx + \left(x + 2y - \frac{4x}{y^3}\right)dy = 0$ ......②

New $M = y + \frac{2}{y^2}$, New $N = x + 2y - \frac{4x}{y^3}$

$\frac{\partial M}{\partial y} = 1 - \frac{4}{y^3}$, $\frac{\partial N}{\partial x} = 1 - \frac{4}{y^3}$

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$,∴ ② is an exact differential equation.

Solution of ② is given by:

$$\underset{y\ constant}{\int} \left(y + \frac{2}{y^2}\right) dx + \int 2y\, dy$$

$$\Rightarrow \left(y + \frac{2}{y^2}\right)x + y^2 = C$$

Example 25 Solve the differential equation:

$(x^2 - y^2 + 2x)dx - 2y\,dy = 0$ .....①

Solution: $M = x^2 - y^2 + 2x,\ N = -2y$

$\frac{\partial M}{\partial y} = -2y$, $\frac{\partial N}{\partial x} = 0$

$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, ∴ given differential equation is not exact.

As ① is neither homogeneous nor of the form

$y f_1(xy)dx + x f_2(xy)dy = 0$,

∴ Computing $\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = -2y$

Clearly $\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{-2y}{-2y} = 1 = f(x)$ say

∴ IF $= e^{\int f(x)dx} = e^{\int 1 dx} = e^x$

∴ ① may be rewritten after multiplying by IF as:

$e^x(x^2 - y^2 + 2x)dx - 2e^x y\,dy = 0$.......②

New $M = e^x(x^2 - y^2 + 2x)$, New $N = -2e^x y$

$\frac{\partial M}{\partial y} = -2e^x y$, $\frac{\partial N}{\partial x} = -2e^x y$

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, ∴ ② is an exact differential equation.

Solution of ② is given by:

$$\underset{y\ constant}{\int} e^x(x^2 - y^2 + 2x)\, dx + \int 0\, dy$$

$$\Rightarrow (x^2 - y^2 + 2x)e^x - (2x + 2)e^x + (2)e^x = C$$

$$\Rightarrow (x^2 - y^2)e^x = C$$, C is an arbitrary constant

$yf_1(xy)dx + xf_2(xy)dy = 0$,

$\therefore$ Computing $\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = -2y$

Clearly $\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{-2y}{-2y} = 1 = f(x)$ say

$\therefore$ IF $= e^{\int f(x)dx} = e^{\int 1 dx} = e^x$

$\therefore$ ① may be rewritten after multiplying by IF as:

$e^x(x^2 - y^2 + 2x)dx - 2e^x y\, dy = 0$.......②

New $M = e^x(x^2 - y^2 + 2x)$, New $N = -2e^x y$

$\frac{\partial M}{\partial y} = -2e^x y$, $\frac{\partial N}{\partial x} = -2e^x y$

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, $\therefore$ ② is an exact differential equation.

Solution of ② is given by:

$$\int_{y\ constant} e^x(x^2 - y^2 + 2x)\, dx + \int 0\, dy$$

$\Rightarrow (x^2 - y^2 + 2x)e^x - (2x + 2)e^x + (2)e^x = C$
$\Rightarrow (x^2 - y^2)e^x = C$, C is an arbitrary constant

Example 26 Solve the differential equation:

$2ydx + (2x \log x - xy)\, dy = 0$ .....①

Solution: $M = 2y$, $N = 2x \log x - xy$

$\frac{\partial M}{\partial y} = 2$, $\frac{\partial N}{\partial x} = 2(1 + \log x) - y$

$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, $\therefore$ given differential equation is not exact.

As ① is neither homogeneous nor of the form $yf_1(xy)dx + xf_2(xy)dy = 0$,

$\therefore$ Computing $\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = -2\log x + y$

Clearly $\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{-2\log x + y}{x(2\log x - y)} = -\frac{1}{x} = f(x)$ say

$\therefore$ IF $= e^{\int f(x)dx} = e^{\int \frac{-1}{x} dx} = e^{\log x^{-1}} = \frac{1}{x}$

$\therefore$ ① may be rewritten after multiplying by IF as:

$\frac{2y}{x} dx + (2\log x - y)\, dy = 0$ ....②

New $M = \frac{2y}{x}$, New $N = 2\log x - y$

$\frac{\partial M}{\partial y} = \frac{2}{x} = \frac{\partial N}{\partial x} = \frac{2}{x}$

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$,$\therefore$ ② is an exact differential equation.

Solution of ② is given by:

$$\int_{y\ constant} \frac{2y}{x}\, dx + \int -y\, dy$$

$\Rightarrow 2y\log x - \frac{y^2}{2} = C$

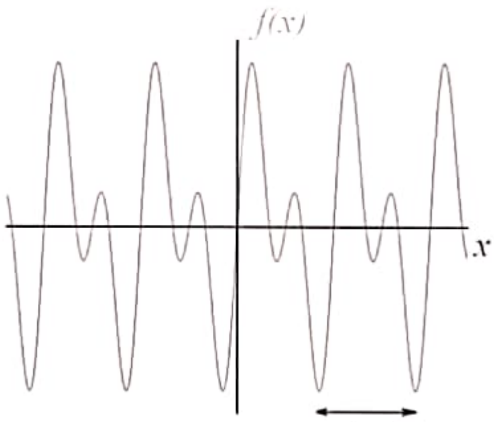
I. Introduction

Contents - Fourier series – Euler's formula – Dirichlet's conditions – Fourier series for a periodic function – Parseval's identity (without proof) – Half range cosine series and sine series – simple problems – Harmonic Analysis.

Periodic Functions

A function $f(x)$ is said to be periodic, if and only if $f(x + L) = f(x)$ is true for some value of L and for all values of x. The smallest value of L for which this equation is true for every value of x will be called the period of the function.

A graph of periodic function $f(x)$ that has period L exhibits the same pattern every L units along the $x -$ axis, so that $f(x + L) = f(x)$ for every value of x. If we know what the function looks like over one complete period, we can thus sketch a graph of the function over a wider interval of x (that may contain many periods). For example, $sinx$ and $cosx$ are periodic with period 2π and $tanx$ has period π.



Dirichlet's Conditions

(i) $f(x)$ is single valued and finite in $(c, c + 2\pi)$

(ii) $f(x)$ is continuous or piecewise continuous with finite number of finite discontinuities in $(c, c + 2\pi)$

(iii) $f(x)$ has a finite number of maxima and minima in $(c, c + 2\pi)$

Note 1: These conditions are not necessary but only sufficient for the existence of Fourier series.

Note 2: If $f(x)$ satisfies Dirichlet's conditions and $f(x)$ is defined in $(-\infty, \infty)$, then $f(x)$ has to be periodic of periodicity 2π for the existence of Fourier series of period 2π.

Note 3: If $f(x)$ satisfies Dirichlet's conditions and $f(x)$ is defined in $(c, c + 2\pi)$, then $f(x)$ need not be periodic for the existence of Fourier series of period 2π.

Note 4: If $x = a$ is a point of continuity of $f(x)$, then the value of Fourier series at $x = a$ is $f(a)$. If $x = a$ is a point of discontinuity of $f(x)$, then the value of Fourier series at $x = a$ is $\frac{1}{2}[f(a+) + f(a-)]$. In other words, specifying a particular value of $x = a$ in a Fourier series, gives a series of constants that should equal $f(a)$. However, if $f(x)$ is discontinuous at this value of x, then the series converges to a value that is half-way between the two possible function values.

Fourier Series

Periodic functions occur frequently in engineering problems. Such periodic functions are often complicated. Therefore, it is desirable to represent these in terms of the simple periodic functions of sine and cosine. A development of a given periodic function into a series of sines and cosines was studied by the French physicist and mathematician Joseph Fourier (1768-1830). The series of sines and cosines was named after him.

If $f(x)$ is a periodic function with period 2π defined in $(c, c + 2\pi)$ and the Dirichlet's conditions are satisfied, then $f(x)$ can be expanded as a **Fourier series** of the form

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

where the Fourier coefficients a_0, a_n *and* b_n are calculate using Euler's formula.

Euler's Formula

(1) $a_0 = \frac{1}{\pi}\int_c^{c+2\pi} f(x)dx$

(2) $a_n = \frac{1}{\pi}\int_c^{c+2\pi} f(x) \cos nx dx$

(3) $b_n = \frac{1}{\pi}\int_c^{c+2\pi} f(x) \sin nx dx$

Standard Integrals

1. $\int e^{ax} \sin bx \, dx = \frac{e^{ax}}{a^2+b^2}[a \sin bx - b \cos bx]$
2. $\int e^{ax} \cos bx \, dx = \frac{e^{ax}}{a^2+b^2}[a \cos bx + b \sin bx]$
3. Bernoulli's generalized formula of integration by parts

$\int uv dx = uv_1 - u'v_2 + u''v_3 - u'''v_4 + \cdots$

Trigonometric results

1. $\sin n\pi = 0,$ if n is an integer
2. $\cos n\pi = (-1)^n,$ if n is an integer

Example 1

Obtain the Fourier series of the following function defined in $(0, 2\pi)$.

$$f(x) = \begin{cases} x, & 0 < x < \pi \\ \pi, & \pi < x < 2\pi, \end{cases} \quad \text{and has period } 2\pi$$

Solution.

STEP ONE

$$\begin{aligned} a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\mathrm{d}x &= \frac{1}{\pi}\int_0^{\pi} f(x)\mathrm{d}x + \frac{1}{\pi}\int_{\pi}^{2\pi} f(x)\mathrm{d}x \\ &= \frac{1}{\pi}\int_0^{\pi} x\mathrm{d}x + \frac{1}{\pi}\int_{\pi}^{2\pi} \pi\cdot\mathrm{d}x \\ &= \frac{1}{\pi}\left[\frac{x^2}{2}\right]_0^{\pi} + \frac{\pi}{\pi}\Big[x\Big]_{\pi}^{2\pi} \\ &= \frac{1}{\pi}\left(\frac{\pi^2}{2} - 0\right) + \left(2\pi - \pi\right) \\ &= \frac{\pi}{2} + \pi \\ \text{i.e. } a_0 &= \frac{3\pi}{2}. \end{aligned}$$

STEP TWO

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,\mathrm{d}x \\ &= \frac{1}{\pi}\int_0^{\pi} x\cos nx\,\mathrm{d}x + \frac{1}{\pi}\int_{\pi}^{2\pi} \pi\cdot\cos nx\,\mathrm{d}x \\ &= \frac{1}{\pi}\left[\frac{1}{n}\left(\pi\sin n\pi - 0\cdot\sin n0\right) - \left[\frac{-\cos nx}{n^2}\right]_0^{\pi}\right] \\ &\qquad + \frac{1}{n}(\sin n2\pi - \sin n\pi) \end{aligned}$$

$$a_n = \frac{1}{\pi}\left[\frac{1}{n}\Big(0-0\Big) + \left(\frac{\cos n\pi}{n^2} - \frac{\cos 0}{n^2}\right)\right] + \frac{1}{n}\Big(0-0\Big)$$

$$= \frac{1}{n^2\pi}(\cos n\pi - 1),$$

$$a_n = \begin{cases} -\frac{2}{n^2\pi} & , n \text{ odd} \\ 0 & , n \text{ even.} \end{cases}$$

STEP THREE

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,\mathrm{d}x$$

$$= \frac{1}{\pi}\int_0^{\pi} x\sin nx\,\mathrm{d}x + \frac{1}{\pi}\int_{\pi}^{2\pi} \pi\cdot\sin nx\,\mathrm{d}x$$

$$= \frac{1}{\pi}\left[\left(\frac{-\pi\cos n\pi}{n} + 0\right) + \left[\frac{\sin nx}{n^2}\right]_0^{\pi}\right] - \frac{1}{n}(\cos 2n\pi - \cos n\pi)$$

$$= \frac{1}{\pi}\left[\frac{-\pi(-1)^n}{n} + \left(\frac{\sin n\pi - \sin 0}{n^2}\right)\right] - \frac{1}{n}(1-(-1)^n)$$

$$= -\frac{1}{n}(-1)^n + \qquad 0 \qquad -\frac{1}{n}(1-(-1)^n)$$

We now have

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}[a_n\cos nx + b_n\sin nx]$$

where $a_0 = \frac{3\pi}{2}$, $\quad a_n = \begin{cases} 0 & , n \text{ even} \\ -\frac{2}{n^2\pi} & , n \text{ odd} \end{cases}$, $\quad b_n = -\frac{1}{n}$

Example 2

Expand in Fourier series of periodicity 2π $f(x) = x\sin x$, for $0 < x < 2\pi$

Solution.

STEP ONE

$$a_0 = \frac{1}{\pi}\int_c^{c+2\pi} f(x)dx$$

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} xsinx\,dx$$

$$= \frac{1}{\pi}[x(-cosx) - 1.(-sinx)]_0^{2\pi}$$

$$= \frac{1}{\pi}[-2\pi cos2\pi + sin2\pi]$$

$$= \frac{1}{\pi}[-2\pi.1 + 0]$$

$$= \frac{1}{\pi}[-2\pi]$$

$$a_0 = -2$$

STEP TWO

$$a_n = \frac{1}{\pi}\int_c^{c+2\pi} f(x)cosnxdx$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} xsinxcosnx\,dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} x[sin(n+1)x - \sin(n-1)x]\,dx$$

$$= \frac{1}{2\pi}\left[x\left(\frac{-\cos(n+1)x}{n+1}\right) - 1.\left(\frac{-\sin(n+1)x}{(n+1)^2}\right) - [x\left(\frac{-\cos(n-1)x}{n-1}\right) - 1.\left(\frac{-\sin(n-1)x}{(n-1)^2}\right)]\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\left[2\pi\left(\frac{-\cos(n+1)2\pi}{n+1}\right) - 1.\left(\frac{-\sin(n+1)2\pi}{(n+1)^2}\right) - [2\pi\left(\frac{-\cos(n-1)2\pi}{n-1}\right) - 1.\left(\frac{-\sin(n-1)2\pi}{(n-1)^2}\right)]\right]$$

$$= \frac{1}{2\pi}\left[\left(\frac{-2\pi}{n+1}\right) - 1.(0) - [\left(\frac{-2\pi}{n-1}\right) - 1.(0)]\right]$$

$$= \left(\frac{-1}{n+1}\right) + \left(\frac{1}{n-1}\right)$$

$a_n = \frac{1}{n^2-1}$ provided $n \neq 1$.

$$a_1 = \frac{1}{\pi}\int_0^{2\pi} xsinxcosx\,dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} xsin2x\,dx$$

$$= \frac{1}{2\pi}\left[x\left(\frac{-cos2x}{2}\right) - 1.\left(\frac{-\sin 2x}{4}\right)\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\left[2\pi\left(\frac{-cos2(2\pi)}{2}\right) - 1.\left(\frac{-\sin 2(2\pi)}{4}\right)\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}[2\pi\left(\frac{-1}{2}\right) - 1.(0)]$$

$$a_1 = \frac{-1}{2}$$

STEP THREE

$$b_n = \frac{1}{\pi}\int_0^{2\pi} xsinxsinnx\, dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} x[cos(n-1)x - \cos(n+1)\, x]\, dx$$

$$= \frac{1}{2\pi}\left[x\left(\frac{\sin(n-1)x}{n-1}\right) - 1.\left(\frac{-\cos(n-1)x}{(n-1)^2}\right) - [x\left(\frac{\sin(n+1)x}{n+1}\right) - 1.\left(\frac{-\cos(n+1)x}{(n+1)^2}\right)]\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\left[2\pi\left(\frac{\sin(n-1)2\pi}{n-1}\right) - 1.\left(\frac{-\cos(n-1)2\pi}{(n-1)^2}\right) - [2\pi\left(\frac{\sin(n+1)2\pi}{n+1}\right) - 1.\left(\frac{-\cos(n+1)2\pi}{(n+1)^2}\right)]\right]$$

$$= \frac{1}{2\pi}\left[\left(\frac{1}{(n-1)^2}\right) - \left[\left(\frac{-1}{(n+1)^2}\right)\right] - \left(\frac{1}{(n-1)^2}\right) + \left[\left(\frac{-1}{(n+1)^2}\right)\right]\right]$$

$b_n = 0$ provided $n \neq 1$.

$$b_1 = \frac{1}{\pi}\int_0^{2\pi} xsinxsinx\, dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} xsin^2x\, dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} x\left(\frac{1 - cos2x}{2}\right) dx$$

$$= \frac{1}{2\pi}\left[x\left(x - \frac{sin2x}{2}\right) - 1.\left(\frac{x^2}{2} + \frac{cos2x}{4}\right)\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[2\pi\left(2\pi - \frac{sin2(2\pi)}{2}\right) - 1.\left(\frac{(2\pi)^2}{2} + \frac{cos2(2\pi)}{4}\right)\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[4\pi^2 - 2\pi^2 + \frac{1}{4} - \frac{1}{4}\right]$$

$$= \frac{1}{\pi}[2\pi^2]$$

$$b_1 = \pi$$

Therefore, the Fourier series expansion of the function $xsinx$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n\, cosnx + b_n\, sinnx)$$

$$xsinx = 1 - \frac{1}{2}cosx + 2\sum_{2}^{\infty}\frac{cosnx}{n^2-1} + \pi sinx$$

Example 3

Obtain all the Fourier coefficients of $f(x) = k$ where k is a constant, the periodicity being 2π.

Solution.

STEP ONE

$$a_0 = \frac{1}{\pi}\int_{c}^{c+2\pi} f(x)dx$$

$$a_0 = \frac{1}{\pi}\int_{0}^{2\pi} k\, dx$$

$$= \frac{k}{\pi}\int_{0}^{2\pi} dx$$

$$= \frac{k}{\pi}\,[x]_0^{2\pi}$$

$$= \frac{k}{\pi}\,[2\pi]$$

$$a_0 = 2k$$

STEP TWO

$$a_n = \frac{1}{\pi}\int_{c}^{c+2\pi} f(x)\, cosnxdx$$

$$a_n = \frac{1}{\pi}\int_{0}^{2\pi} k\, cosnxdx$$

$$= \frac{k}{\pi}\int_{0}^{2\pi} cosnxdx$$

$$= \frac{k}{\pi}\left[\frac{sinnx}{n}\right]_0^{2\pi}$$

$$= \frac{k}{\pi}\left[\frac{sin2n\pi - sin0}{n}\right]$$

$$a_n = 0$$

STEP THREE

[redacted]

$$b_n = \frac{1}{\pi}\int_{c}^{c+2\pi} f(x)\, sinnxdx$$

$$b_n = \frac{1}{\pi}\int_{0}^{2\pi} k\ sinnxdx$$

$$= \frac{k}{\pi}\int_0^{2\pi} sinnxdx$$

$$= \frac{k}{\pi}\left[\frac{-cosnx}{n}\right]_0^{2\pi}$$

$$= \frac{k}{\pi}\left[\frac{cos2n\pi - cos0}{n}\right]$$

$$= \frac{k}{\pi}\left[\frac{1-1}{n}\right]$$

$$b_n = 0$$

Even and Odd Functions

The function f(x) is said to be even, if f(-x) = f(x).

The function f(x) is said to be odd, if f(-x) = -f(x).

If $f(x)$ is an even function with period 2π defined in $(-\pi, \pi)$, then $f(x)$ can be expanded as a Fourier cosine series:

$$\boldsymbol{f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\, cosnx}$$

where the Fourier coefficients $\boldsymbol{a_0}$ and $\boldsymbol{a_n}$ are calculated by

(1) $a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)dx$

(2) $a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\ cosnxdx$

If $f(x)$ is an odd function with period 2π defined in $(-\pi, \pi)$, then $f(x)$ can be expanded as a Fourier sine series:

$$\boldsymbol{f(x) = \sum_{n=1}^{\infty} b_n\, sinnx}$$

where the Fourier coefficient $\boldsymbol{b_n}$ is calculated by $b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\ sinnxdx$

Example 4

Find the Fourier series for $f(x) = |cosx|$ in $(-\pi, \pi)$ of periodicity 2π.

Solution.

Since $f(x) = |cosx|$ is an even function, $f(x)$ will contain only cosine terms.

Therefore, $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \, cosnx$

STEP ONE

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)dx$$

$$= \frac{2}{\pi}\int_0^{\pi} |cosx|dx$$

$$= \frac{2}{\pi}\left[\int_0^{\frac{\pi}{2}} cosx \, dx + \int_{\frac{\pi}{2}}^{\pi} (-cosx)dx\right]$$

(Since in $(0, \frac{\pi}{2})$, $cosx$ is positive and in $(\frac{\pi}{2}, \pi)$ $cosx$ is negative)

$$= \frac{2}{\pi}\left[(sinx)_0^{\frac{\pi}{2}} - (sinx)_{\frac{\pi}{2}}^{\pi}\right]$$

$$= \frac{2}{\pi}\left[sin\frac{\pi}{2} - sin0 - sin\pi + sin\frac{\pi}{2}\right]$$

$$= \frac{2}{\pi}[1 - 0 - 0 + 1]$$

$$a_0 = \frac{4}{\pi}$$

STEP TWO

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)cosnxdx$$

$$= \frac{2}{\pi}\int_0^{\pi} |cosx|cosnxdx$$

$$= \frac{2}{\pi}\left[\int_0^{\frac{\pi}{2}} cosxcosnx \, dx + \int_{\frac{\pi}{2}}^{\pi} (-cosxcosnx)dx\right]$$

$$= \frac{1}{\pi}\left[\int_0^{\frac{\pi}{2}} \cos(n+1)\,x + \cos(n-1)x \, dx - \int_{\frac{\pi}{2}}^{\pi} \cos(n+1)\,x + \cos\,(n-1)xdx\right]$$

$$= \frac{1}{\pi}\left[\left\{ \frac{\sin(n+1)x}{n+1} + \frac{\sin(n-1)x}{n-1} \right\}_0^{\pi/2} \right.$$

$$\left. - \left\{ \frac{\sin(n+1)x}{n+1} + \frac{\sin(n-1)x}{n-1} \right\}_{\pi/2}^{\pi} \right]$$

$$= \frac{1}{\pi}\left[\frac{\sin(n+1)\pi/2}{n+1} + \frac{\sin(n-1)\pi/2}{n-1} + \frac{\sin(n+1)\pi/2}{n+1} \right.$$

$$\left. + \frac{\sin(n-1)\pi/2}{n-1} \right] \text{ if } n \neq 1$$

$$= \frac{2}{\pi}\left[\frac{1}{n+1}\left\{ \sin\frac{n\pi}{2}\cos\frac{\pi}{2} + \cos\frac{n\pi}{2}\sin\frac{\pi}{2} \right\} + \frac{1}{n-1} \right.$$

$$\left. \times \left\{ \sin\frac{n\pi}{2}\cos\frac{\pi}{2} - \cos\frac{n\pi}{2}\sin\frac{\pi}{2} \right\} \right] \text{ if } n \neq 1$$

$$= \frac{2}{\pi}\left[\frac{1}{n+1} - \frac{1}{n-1} \right]\cos\frac{n\pi}{2}$$

$$= -\frac{4}{\pi(n^2-1)}\cos\frac{n\pi}{2} \text{ if } n \neq 1$$

$$a_1 = \frac{1}{\pi}\int_{-\pi}^{\pi} |\cos x| \cos x \, dx$$

$$= \frac{2}{\pi}\left[\int_0^{\pi} |\cos x| \cos x \right]$$

$$= \frac{2}{\pi}\left[\int_0^{\pi/2} \cos^2 x\,dx - \int_{\pi/2}^{\pi} \cos^2 x\,dx\right]$$

$$= \frac{2}{\pi}\left[\frac{1}{2}\cdot\frac{\pi}{2} - \int_{\pi/2}^{\pi} \frac{1+\cos 2x}{2}\,dx\right]$$

$$= \frac{2}{\pi}\left[\frac{\pi}{4} - \frac{1}{2}\left(x + \frac{\sin 2x}{2}\right)_{\pi/2}^{\pi}\right]$$

$$= \frac{2}{\pi}\left[\frac{\pi}{4} - \frac{\pi}{4}\right]$$

$$= 0.$$

$$\therefore \quad |\cos x| = \frac{2}{\pi} - \frac{4}{\pi}\sum_{n=2}^{\infty} \frac{1}{n^2-1}\cos\frac{n\pi}{2}\cos nx$$

Example 5.

Find the Fourier series of $f(x) = e^x$ in $(-\pi, \pi)$ of periodicity 2π.

Solution. Let $\quad f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$

where $\quad a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} e^x\,dx$

$$= \frac{1}{\pi}(e^x)_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}(e^{\pi} - e^{-\pi})$$

$$= \frac{2}{\pi}\sinh \pi$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} e^x \cos nx\,dx$$

$$= \frac{1}{\pi}\left[\frac{e^x}{1+n^2}\,(\cos nx + n\sin nx)\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi\,(1+n^2)}\,[e^{\pi}\,(-1)^n + e^{-\pi}\,(-1)^n\,]$$

$$= \frac{2\,(-1)^n}{\pi\,(1+n^2)}\,\sinh\pi$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} e^x \sin nx\,dx$$

$$= \frac{1}{\pi}\left[\frac{e^x}{1+n^2}\,(\sin nx - n\cos nx)\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi\,(1+n^2)}\,[e^{\pi}\,(-n)\,(-1)^n + e^{-\pi}\,n\,(-1)^n\,]$$

$$= \frac{-2\,(-1)^n\cdot n}{\pi\,(1+n^2)}\,\sinh\pi$$

$$e^x = \frac{\sinh\pi}{\pi}\left[1 + \sum_{n=1}^{\infty}\frac{2\,(-1)^n}{1+n^2}\,(\cos nx - n\sin nx)\right]$$

Example 6

Derive the Fourier series of $f(x) = x + x^2$ in $(-\pi, \pi)$ of periodicity 2π and hence deduce $\Sigma \frac{1}{n^2} = \frac{\pi^2}{6}$.

Solution.

STEP ONE

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} (x + x^2)\, dx$$

$$= \frac{1}{\pi}\left[\frac{x^2}{2} + \frac{x^3}{3}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{\pi^2}{2} + \frac{\pi^3}{3} - \left(\frac{(-\pi)^2}{2} + \frac{(-\pi)^3}{3}\right)\right]$$

$$= \frac{1}{\pi}\left[\frac{\pi^2}{2} + \frac{\pi^3}{3} - \frac{\pi^2}{2} + \frac{\pi^3}{3}\right]$$

$$a_0 = \frac{2\,\pi^2}{3}$$

STEP TWO

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)cosnxdx$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} (x + x^2)cosnx\, dx$$

$$= \frac{1}{\pi}\left[(x + x^2)\left(\frac{sinnx}{n}\right) - (1 + 2x)\left(\frac{-cosnx}{n^2}\right) + (2)\left(\frac{-sinnx}{n^3}\right)\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[(1 + 2\pi)\left(\frac{cosn\pi}{n^2}\right) - (1 - 2\pi)\left(\frac{cosn\pi}{n^2}\right)\right]$$

$$= \frac{1}{\pi}\left[2\pi\left(\frac{(-1)^n}{n^2}\right) + 2\pi\left(\frac{(-1)^n}{n^2}\right)\right]$$

$$= \frac{4}{n^2}(-1)^n$$

STEP THREE

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)sinnxdx$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} (x + x^2)sinnx\, dx$$

$$= \frac{1}{\pi}\left[(x + x^2)\left(\frac{-cosnx}{n}\right) - (1 + 2x)\left(\frac{-sinnx}{n^2}\right) + (2)\left(\frac{cosnx}{n^3}\right)\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[(\pi + \pi^2)\left(\frac{-cosn\pi}{n}\right) + 2\left(\frac{cosn\pi}{n^3}\right) - (-\pi + \pi^2)\left(\frac{-cosn\pi}{n}\right) - 2\left(\frac{cosn\pi}{n^3}\right)\right]$$

$$= \frac{1}{\pi}\left[2\pi\left(\frac{-(-1)^n}{n}\right)\right]$$

2

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n\, cosnx + b_n\, sinnx)$$

$$f(x) = \frac{\pi^3}{3} + \sum_{n=1}^{\infty}\left(\frac{4}{n^2}(-1)^n\, cosnx + \frac{2}{n}(-1)^{n+1}\, sinnx\right) \text{ \_\_\_\_\_\_\_\_\_\_\_\_ } (1)$$

STEP FOUR

Deduction:

The end points of the range are $x = \pi$ and $x = -\pi$. Therefore, the value of Fourier series at $x = \pi$ is the average value of $f(x)$ at the points $x = \pi$ and $x = -\pi$. Hence put $x = \pi$ in (1),

$$\Rightarrow \frac{f(-\pi)}{2} = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{1}{n^2}(-1)^n\, cosn\pi$$

$$\Rightarrow \frac{(\pi+\pi^2)+(-\pi+\pi^2)}{2} = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{1}{n^2}(-1)^{2n}$$

$$\Rightarrow \frac{2\pi^2}{3} = 4\sum_{n=1}^{\infty}\frac{1}{n^2}$$

$$\Rightarrow \sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}$$

Example 7.

Expand f(x)=x², when $-\pi < x < \pi$ in a Fourier series of periodicity 2π. Hence deduce that

(i) $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots \text{to } \infty = \frac{\pi^2}{6}$

(ii) $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \cdots \text{to } \infty = \frac{\pi^2}{12}$

(iii) $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots \text{to } \infty = \frac{\pi^2}{8}$

$f(x)$ is an even function of x in $-\pi < x < \pi$. Hence bn = 0 and only cosine terms will be present. Therefore,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \quad ....(i)$$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} x^2 dx = \frac{2\pi^2}{3}$$

For $n = 1, 2, 3, \ldots$

$$a_n = \frac{2}{\pi}\int_0^{\pi} x^2 \cos nx \, dx$$

$$= \frac{2}{\pi}\left[(x^2)\left(\frac{\sin nx}{n}\right) - (2x)\left(\frac{-\cos nx}{n^2}\right) + (2)\left(\frac{-\sin nx}{n^3}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\frac{2\pi}{n^2}\cos n\pi\right] = \frac{4(-1)^n}{n^2}.$$

Substituting these values in (i),

$$f(x) = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos nx \quad ....(ii)$$

i.e., $x^2 = \frac{\pi^2}{3} - 4\left[\frac{\cos x}{1^2} - \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} - \cdots\right]$, in $-\pi < x < \pi$.

The function $f(x) = x^2$ is continuous at $x = 0$. Hence the sum of the Fourier series equals the value of the function at $x = 0$. Putting $x = 0$, in (ii),

$$0 = \frac{\pi^2}{3} - 4\left[\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \cdots\right]$$

$$\therefore \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \cdots = \frac{\pi^2}{12} \quad ....(iii)$$

$x = \pi$ is an end point. Hence the sum of the Fourier series at $x = \pi$ equals $\frac{1}{2}\{f(-\pi+0) + f(\pi-0)\}$

Putting $x = \pi$ in the series of (ii),

$$\frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n \cos n\pi}{n^2} = \frac{1}{2}[f(-\pi+0) + f(\pi-0)]$$

$$= \frac{1}{2}[\pi^2 + \pi^2] = \pi^2$$

$$\therefore 4\sum_{n=1}^{\infty}\frac{1}{n^2} = \pi^2 - \frac{\pi^2}{3} = \frac{2}{3}\pi^2$$

$$\therefore \quad 4\sum_{n=1}^{\infty}\frac{1}{n^2}=\pi^2-\frac{\pi^2}{3}=\frac{2}{3}\pi^2$$

$$\text{i.e.,}\quad \sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\cdots=\frac{\pi^2}{6} \qquad \text{....}(iv)$$

Adding (*iii*) and (*iv*),

$$2\left(\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\cdots\right)=\frac{\pi^2}{4}$$

$$\text{i.e.,}\quad \frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\cdots \text{ to } \infty=\frac{\pi^2}{8}.$$

Example 8 *Find the Fourier series of periodicity* 2π

$$\textit{for}\quad f(x)=\begin{cases} x \text{ when } -\pi<x<0 \\ 0 \text{ when } 0<x<\dfrac{\pi}{2} \\ x-\dfrac{\pi}{2} \text{ when } \dfrac{\pi}{2}<x<\pi \end{cases}$$

Solution. Let $f(x)=\dfrac{a_0}{2}+\displaystyle\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx)$ ...(1)

where $a_0=\dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi}f(x)\,dx$

Taking $c=-\pi$ in the Euler formulas we have

$$a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\,dx=\frac{1}{\pi}\left\{\int_{-\pi}^{0}f(x))\,dx+\int_{0}^{\pi}f(x)\,dx\right\}.$$

Now using the hypothesis for the value of $f(x)$, we get

$$a_0=\frac{1}{\pi}\left\{\int_{-\pi}^{0}(-k)\,dx+\int_{0}^{\pi}k\,dx\right\}=\frac{1}{\pi}\left\{\left(-kx\right)_{-\pi}^{0}+\left(kx\right)_{0}^{\pi}\right\}$$

$$=\frac{1}{\pi}\{(0-k\pi)+(k\pi-0)\}$$

Thus $a_0=0$. Again for $n=1,2,3,\ldots.$

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos nx\,dx$$

$$=\frac{1}{\pi}\left\{\int_{-\pi}^{0}f(x)\cos nx\,dx+\int_{0}^{\pi}f(x)\cos nx\,dx\right\}.$$

Substituting the values supplied for $f(x)$, we have

$$a_n=\frac{1}{\pi}\left\{\int_{-\pi}^{0}(-k)\cos nx\,dx+\int_{0}^{\pi}k\cos nx\,dx\right\}$$

$$= \frac{1}{\pi}\left\{\left(-k\frac{\sin nx}{n}\right)_{-\pi}^{0} + \left(k\frac{\sin nx}{n}\right)_{0}^{\pi}\right\}$$

Since sin 0, sin $(-n\pi)$ and sin $n\pi$ are all zero, we get $a_n = 0$.

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

$$= \frac{1}{\pi}\left\{\int_{-\pi}^{0} f(x)\sin nx\,dx + \int_{0}^{\pi} f(x)\sin nx\,dx\right\}$$

$$= \frac{1}{\pi}\left\{\int_{-\pi}^{0} (-k)\sin nx\,dx + \int_{0}^{\pi} k\sin nx\,dx\right\}$$

$$= \frac{1}{\pi}\left\{\left[k\frac{\cos nx}{n}\right]_{-\pi}^{0} + \left[-k\frac{\cos nx}{n}\right]_{0}^{\pi}\right\}$$

$$= \frac{1}{\pi}\left[\left\{\frac{k}{n}\cos 0 - \frac{k}{n}\cos(-n\pi)\right\} + \left\{-\frac{k}{n}\cos n\pi + \frac{k}{n}\cos 0\right\}\right]$$

But $\cos(-\alpha) = \cos\alpha$, giving $\cos(-n\pi) = \cos n\pi$; further, $\cos 0 = 1$.

Hence $b_n = \frac{k}{n\pi}[\{1 - \cos n\pi\} + \{-\cos n\pi + 1\}] = \frac{k}{n\pi}(2 - 2\cos n\pi)$

$\therefore\ b_n = \frac{2k}{n\pi}(1 - \cos n\pi)$. Now $\cos n\pi = \begin{cases} -1, & \text{for odd } n \\ +1, & \text{for even } n \\ (-1)^n, & \text{for any integer n} \end{cases}$

Hence $b_1 = \frac{4k}{\pi}$; $b_2 = 0$; $b_3 = \frac{4k}{3\pi}$; $b_4 = 0$; $b_5 = \frac{4k}{5\pi}$; $b_6 = 0$;

$$b_7 = \frac{4k}{7\pi} \cdots\cdots$$

Using the values of a_n and b_n in (i) we obtain

$$f(x) = \frac{4k}{\pi}\left\{\sin x + \tfrac{1}{3}\sin 3x + \tfrac{1}{5}\sin 5x + \tfrac{1}{7}\sin 7x + \cdots \text{ to } \infty\right\}$$

In the above equation putting $x = \pi/2$, we get

$$f\left(\frac{\pi}{2}\right) = \frac{4k}{\pi}\left\{1 - \tfrac{1}{3} + \tfrac{1}{5} - \tfrac{1}{7} + \cdots \text{ to } \infty\right\}$$

But, by hypothesis, $f\left(\frac{\pi}{2}\right) = k$.

Hence $k = \frac{4k}{\pi}\left\{1 - \tfrac{1}{3} + \tfrac{1}{5} - \tfrac{1}{7} + \cdots \text{ to } \infty\right\}$

Multiplying both the sides by $\frac{\pi}{4k}$, we have

$$\frac{\pi}{4} = 1 - \tfrac{1}{3} + \tfrac{1}{5} - \tfrac{1}{7} + \cdots \text{ to } \infty.$$

Note. Functions of the type given in this example occur as external force acting on mechanical systems, electromotive forces in electric circuits etc.

Root Mean Square (RMS)Value

The root-mean-square value of a function $y = f(x)$ over a given (a, b) is defined as

$$\bar{y} = \sqrt{\left\{\frac{\int_a^b y^2\,dx}{b-a}\right\}}$$

If the interval is taken as $(c, c + 2\pi)$, then

$$\bar{y}^2 = \frac{1}{2\pi}\int_c^{c+2\pi} y^2\,dx$$

Suppose that $y = f(x)$ is expressed as a Fourier-series of periodicity 2π in $(c, c+2\pi)$. then,

$$y = f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \qquad \text{...(ii)}$$

$$\left.\begin{aligned} \text{where} \quad a_0 &= \frac{1}{\pi}\int_c^{c+2\pi} f(x)\,dx \\ a_n &= \frac{1}{\pi}\int_c^{c+2\pi} f(x)\cos nx\,dx \\ \text{and} \quad b_n &= \frac{1}{\pi}\int_c^{c+2\pi} f(x)\sin nx\,dx. \end{aligned}\right\} \qquad \text{...(iii)}$$

Multiply (ii) by $f(x)$ and integrate term by term with respect to x over the given range. Thus,

$$\int_c^{c+2\pi} [f(x)]^2\,dx = \frac{a_0}{2}\int_c^{c+2\pi} f(x)\,dx + \sum_{n=1}^{\infty}\left[a_n \int_c^{c+2\pi} f(x)\cos nx\,dx + b_n \int_c^{c+2\pi} f(x)\sin nx\,dx\right]$$

$$= \frac{a_0}{2}(\pi a_0) + \sum_{n=1}^{\infty} [a_n(\pi a_n) + b_n(\pi b_n)] \text{ using } (iii)$$

$$\int_c^{c+2\pi} [f(x)]^2\,dx = 2\pi\left[\frac{a_0^2}{4} + \frac{1}{2}\sum_{n=1}^{\infty}(a_n^2 + b_n^2)\right]$$

$$= (\text{Range})\left[\frac{a_0^2}{4} + \frac{1}{2}\sum_{n=1}^{\infty}(a_n^2 + b_n^2)\right]$$

$$\bar{y}^2 = \frac{a_0^2}{4} + \frac{1}{2}\sum_{n=1}^{\infty}(a_n^2 + b_n^2).$$

Ex 9. *Find the Fourier series of periodicity 2π for $f(x) = x^2$, in $-\pi < x < \pi$. Hence show that*

$$\frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \cdots + \text{to } \infty = \frac{\pi^4}{90}.$$

In example 7, we have proved

$$f(x) = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos nx,$$ which is the first part of this problem. The coefficients a_0, a_n, b_n were seen to be

$$a_0 = \frac{2\pi^2}{3},\ a_n = \frac{4(-1)^n}{n^2},\ b_n = 0.$$

Hence using the root-mean-square value in series,

$$2\pi\left[\frac{a_0^2}{4} + \frac{1}{2}\sum_{n=1}^{\infty}(a_n^2 + b_n^2)\right] = \int_{-\pi}^{\pi} [f(x)^2]\, dx = \int_{-\pi}^{\pi} x^4 dx$$

$$2\pi\left[\frac{\pi^4}{9} + \frac{1}{2}\sum_{n=1}^{\infty}\frac{16}{n^4}\right] = \frac{2}{5}\pi^5$$

$$8\sum_{n=1}^{\infty}\frac{1}{n^4} = \frac{\pi^4}{5} - \frac{\pi^4}{9} = \frac{4\pi^4}{45}$$

$$\therefore \sum_{n=1}^{\infty}\frac{1}{n^4} = \frac{\pi^4}{90}.$$

Change of Interval

Example 10 *Find the Fourier series of periodicity 3 for $f(x) = 2x - x^2$ in $0 < x < 3$*

Here the range and the period are same (equal to 3)

It is a full range series.

$\therefore\ 2l = 3\ ;\ l = \frac{3}{2}$

$$\text{Let } f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{2n\pi x}{3} + b_n \sin\frac{2n\pi x}{3}\right)$$

where $$a_0 = \frac{1}{\frac{3}{2}}\int_0^3 (2x - x^2)\,dx = \frac{2}{3}\left[x^2 - \frac{x^3}{3}\right]_0^3 = 0$$

$$a_n = \frac{1}{\frac{3}{2}}\int_0^3 (2x - x^2)\cos\frac{2n\pi x}{3}\,dx$$

$$= \frac{2}{3}\left[(2x - x^2)\left(\frac{\sin\frac{2n\pi x}{3}}{\frac{2n\pi}{3}}\right) - (2 - 2x)\left(-\frac{\cos\frac{2n\pi x}{3}}{\frac{4n^2\pi^2}{9}}\right) + (-2)\left(-\frac{\sin\frac{2n\pi x}{3}}{\frac{8n^3\pi^3}{27}}\right)\right]_0^3$$

$$= \frac{2}{3}\left[\frac{-9}{n^2\pi^2} - \frac{9}{2n^2\pi^2}\right]$$

$$= -\frac{9}{n^2\pi^2}$$

$$b_n = \frac{1}{\frac{3}{2}}\int_0^3 (2x - x^2)\sin\frac{2n\pi x}{3}\,dx$$

$$=\frac{2}{3}\left[(2x-x^2)\left(-\frac{\cos\frac{2n\pi x}{3}}{\frac{2n\pi}{3}}\right)-(2-2x)\left(-\frac{\sin\frac{2n\pi x}{3}}{\frac{4n^2\pi^2}{9}}\right)+(-2)\left(\frac{\cos\frac{2n\pi x}{3}}{\frac{8n^3\pi^3}{27}}\right)\right]_0^3$$

$$=\frac{3}{n\pi}$$

$$\therefore\quad f(x)=-\frac{9}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\cos\left(\frac{2n\pi x}{3}\right)+\frac{3}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\left(\frac{2n\pi x}{3}\right)$$

Half-Range Fourier Series

Example 11

Express $f(x)=x(\pi-x), 0<x<\pi$ as a Fourier series of periodicity 2π containing (i) sine terms only and (ii) cosine terms only. Hence deduce, $1-\frac{1}{3^3}+\frac{1}{5^3}-\frac{1}{7^3}+\cdots=\frac{\pi^3}{32}$ and $\frac{1}{1^2}-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\cdots=\frac{\pi^2}{12}$.

Solution.

(i) sine series:

$$\text{Let } f(x)=\sum_{1}^{\infty}b_n\sin nx$$

$$\text{where}\qquad b_n=\frac{2}{\pi}\int_0^{\pi}x(\pi-x)\sin nx\,dx$$

$$=\frac{2}{\pi}\left[\{\pi x-x^2\}\left(-\frac{\cos nx}{n}\right)-(\pi-2x)\left(\frac{\sin nx}{n^2}\right)+(-2)\left(\frac{\cos nx}{n^3}\right)\right]_0^{\pi}$$

$$=\frac{2}{\pi}\left[-\frac{2}{n^3}\{(-1)^n-1\}\right]$$

$$= \frac{2}{\pi}\left[\{\pi x - x^2\}\left(-\frac{\cos nx}{n}\right) - (\pi - 2x)\left(\frac{\sin nx}{n^2}\right) + (-2)\left(\frac{\cos nx}{n^3}\right)\right]_0^\pi$$

$$= \frac{2}{\pi}\left[-\frac{2}{n^3}\{ (-1)^n - 1 \}\right]$$

$$= \frac{4}{\pi n^3}\left[1 - (-1)^n \right]$$

$= 0$ if n is even

$= \frac{8}{\pi n^3}$ if n is odd

$$\therefore \; f(x) = \frac{8}{\pi} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^3} \sin (2n-1)\, x.$$

Setting $x = \pi/2$ which is a point of continuity we get first deduction .

(ii) cosine series:

$$\text{Let } f(x) = \frac{a_0}{2} + \sum_{1}^{\infty} a_n \cos nx$$

where $$a_0 = \frac{2}{\pi} \int_0^\pi x(\pi - x)\, dx$$

$$= \frac{2}{\pi}\left[\frac{\pi x^2}{2} - \frac{x^3}{3}\right]_0^\pi = \frac{\pi^2}{3}$$

$$a_n = \frac{2}{\pi} \int_0^\pi (\pi x - x^2) \cos nx \, dx$$

$$= \frac{2}{\pi}\left[(\pi x - x^2)\left(\frac{\sin nx}{n}\right) - (\pi - 2x)\left(-\frac{\cos nx}{n^2}\right) + (-2)\left(-\frac{\sin nx}{n^3}\right)\right]_0^\pi$$

$$= \frac{2}{\pi}\left[-\frac{\pi}{n^2}(-1)^n - \frac{\pi}{n^2}\right] = -\frac{2}{n^2}[\,1 + (-1)^n\,]$$

$= 0$ for n odd

$= -\frac{4}{n^2}$ for n even

$$x(\pi - x) = \frac{\pi^2}{6} - 4 \sum_{n=2,4,6\ldots}^{\infty} \frac{1}{n^2} \cos nx$$

$$= \frac{4}{\pi n^3}\left[1-(-1)^n \right]$$

$= 0$ if n is even

$= \dfrac{8}{\pi n^3}$ if n is odd

$$\therefore \quad f(x) = \frac{8}{\pi} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^3} \sin (2n-1)\, x.$$

Setting $x = \pi/2$ which is a point of continuity we get first deduction .

(ii) **cosine series:**

$$\text{Let } f(x) = \frac{a_0}{2} + \sum_{1}^{\infty} a_n \cos nx$$

where $\quad a_0 = \dfrac{2}{\pi} \displaystyle\int_0^{\pi} x(\pi - x)\, dx$

$$= \frac{2}{\pi}\left[\frac{\pi x^2}{2} - \frac{x^3}{3} \right]_0^{\pi} = \frac{\pi^2}{3}$$

$$a_n = \frac{2}{\pi} \int_0^{\pi} (\pi x - x^2) \cos nx\, dx$$

$$= \frac{2}{\pi}\left[(\pi x - x^2)\left(\frac{\sin nx}{n} \right) - (\pi - 2x)\left(-\frac{\cos nx}{n^2} \right) + (-2)\left(-\frac{\sin nx}{n^3} \right) \right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[-\frac{\pi}{n^2}(-1)^n - \frac{\pi}{n^2} \right] = -\frac{2}{n^2}[\, 1 + (-1)^n\,]$$

$= 0$ for n odd

$= -\dfrac{4}{n^2}$ for n even

$$x(\pi - x) = \frac{\pi^2}{6} - 4 \sum_{n=2,4,6\ldots}^{\infty} \frac{1}{n^2} \cos nx$$

Order and degree of a differential equation

The highest order derivative present in the differential equation is the order of the differential equation.

Degree is the highest power of the highest order derivative in the differential equation, after the equation has been cleared from fractions and the radicals as for as the derivatives are concerned.

For example, consider the differential equation

$$\frac{d^3y}{dx^3}+\left(\frac{d^2y}{dx^2}\right)^3+\left(\frac{dy}{dx}\right)^5+y=7$$

Here the highest order derivatives is $\frac{d^3y}{dx^3}$ (i.e 3rd order derivative). So the order of the differential equation is 3.

Now the power of highest order derivative $\frac{d^3y}{dx^3}$ is 1.

$\therefore$ The degree of the differential equation is 1.

Example 4.1

Find the order and degree of the following differential equations.

(i) $\frac{d^2y}{dx^2}+3\left(\frac{dy}{dx}\right)^2+4y=0$

(ii) $\frac{d^2y}{dx^2}-2\frac{dy}{dx}+3y=0$

(iii) $\frac{d^3y}{dx^3}-3\left(\frac{dy}{dx}\right)^6+2y=x^2$

(iv) $\left[1+\frac{d^2y}{dx^2}\right]^{\frac{3}{2}}=a\frac{d^2y}{dx^2}$

(v) $y'+(y'')^2=(x+y'')^2$

(vi) $\frac{d^3y}{dx^3}-\left(\frac{dy}{dx}\right)^{\frac{1}{2}}=0$

(vii) $y=2\left(\frac{dy}{dx}\right)^2+4x\frac{dx}{dy}$

Solution

(i) $\frac{d^2y}{dx^2}+3\left(\frac{dy}{dx}\right)^2+4y=0$

Highest order derivative is $\frac{d^2y}{dx^2}$

$\therefore$ order = 2

Power of the highest order derivative $\frac{d^2y}{dx^2}$ is 1.

$\therefore$ Degree =1

(ii) $\frac{d^2y}{dx^2}-2\frac{dy}{dx}+3y=0$

Highest order derivative is $\frac{d^2y}{dx^2}$

$\therefore$ order =2

Power of the highest order derivative $\frac{d^2y}{dx^2}$ is 1.

$\therefore$ degree = 1

(iii) $\frac{d^3y}{dx^3}-3\left(\frac{dy}{dx}\right)^6+2y=x^2$

$\therefore$ order =3, Degree =1

(iv) $\left[1+\frac{d^2y}{dx^2}\right]^{\frac{3}{2}}=a\frac{d^2y}{dx^2}$

Here we eliminate the radical sign.

Squaring both sides, we get

$$\left[1+\frac{d^2y}{dx^2}\right]^3=a^2\left(\frac{d^2y}{dx^2}\right)^2$$

$\therefore$ Order=2, degree =3

(v) $y'+(y'')^2=(x+y'')^2$

$$y'+(y'')^2=x^2+2xy''+(y'')^2$$

$$y'=x^2+2xy''\Rightarrow\frac{dy}{dx}=x^2+2x\frac{d^2y}{dx^2}$$

$\therefore$ Order=2, degree=1

$\therefore$ Order=2, degree=1

(vi) $$\frac{d^3y}{dx^3}-\left(\frac{dy}{dx}\right)^{\frac{1}{2}}=0$$

Here we eliminate the radical sign.

For this write the equation as

$$\frac{d^3y}{dx^3}=\left(\frac{dy}{dx}\right)^{\frac{1}{2}}$$

Squaring both sides, we get

$$\left(\frac{d^3y}{dx^3}\right)^2=\frac{dy}{dx}$$

$\therefore$ Order=3, degree=2

(vii) $$y=2\left(\frac{dy}{dx}\right)^2+4x\frac{dx}{dy}$$

$$y=2\left(\frac{dy}{dx}\right)^2+4x\frac{1}{\left(\frac{dy}{dx}\right)}$$

$$y\frac{dy}{dx}=2\left(\frac{dy}{dx}\right)^3+4x$$

$\therefore$ order=1, degree=3

Family of Curves

Sometimes a family of curves can be represented by a single equation with one or more arbitrary constants. By assigning different values for constants, we get a family of curves. The arbitrary constants are called the parameters of the family.

For example,

(i) $y^2 = 4ax$ represents the equation of a family of parabolas having the origin as vertex where '*a*' is the parameter.

(ii) $x^2 + y^2 = a^2$ represents the equation of family of circles having the origin as centre, where '*a*' is the parameter.

(iii) $y = mx + c$ represents the equation of a family of straight lines in a plane, where m and c are parameters.

Linear differential equations of first order

A differential equation is said to be linear when the dependent variable and its derivatives occur only in the first degree and no product of these occur.

Linear differential equations of first order:

A differential equation is said to be linear when the dependent variable and its derivatives occur only in the first degree and no product of these occur.

The most general form of a linear equation of the first order is $dy/dx + Py = Q$ (1)

P and Q are functions of x alone.

Equation (1) is linear in y. The solution is given by $ye^{\int pdx} = \int Qe^{\int pdx}dx + c$. Here $e^{\int pdx}$ is known as an integrating factor and is denoted by I.F.

Note

For the differential equation $\frac{dx}{dy} + Px = Q$ (linear in x) where P and Q are functions of y alone, the solution is $xe^{\int pdy} = \int Qe^{\int pdy}dy + c$

Example

Example 4.20

Solve $\frac{dy}{dx} + \frac{y}{x} = x^3$

Solution:

Given $\frac{dy}{dx} + \frac{1}{x}y = x^3$

It is of the form $\frac{dy}{dx} + Py = Q$

Example

Example 4.20

Solve $\frac{dy}{dx}+\frac{y}{x}=x^3$

Solution:

Given $\frac{dy}{dx}+\frac{1}{x}y=x^3$

It is of the form $\frac{dy}{dx}+Py = Q$

Here $P=\frac{1}{x}$, $Q=x^3$

$$\int Pdx = \int\frac{1}{x}dx=\log x$$

$$I.F = e^{\int pdx}=e^{\log x}=x$$

The required solution is $y(I.F) = \int Q(I.F)dx+c$

$$yx = \int x^3.x\ dx+c$$

$$= \int x^4dx+c$$

$$= \frac{x^5}{5}+c$$

$$\therefore \quad yx = \frac{x^5}{5}+c$$

Example 4.21

Solve $\cos^2 x\frac{dy}{dx}+y=\tan x$

Solution:

The given equation can be written as $\frac{dy}{dx}+\frac{1}{\cos^2 x}y=\frac{\tan x}{\cos^2 x}$

$$\frac{dy}{dx}+y\sec^2 x = \tan x\sec^2 x$$

It is of the form $\frac{dy}{dx}+Py = \text{Q}$

Here $\text{P}=\sec^2 x, \text{Q} = \tan x\sec^2 x$

$$\int Pdx = \int\sec^2 x\ dx = \tan x$$

$$I.F = e^{\int pdx}=e^{\tan x}$$

The required solution is $y(I.F) = \int Q(I.F)dx+c$

$$\therefore \qquad yx = \frac{x^5}{5} + c$$

Example 4.21

Solve $\cos^2 x \frac{dy}{dx} + y = \tan x$

Solution:

The given equation can be written as $\frac{dy}{dx} + \frac{1}{\cos^2 x} y = \frac{\tan x}{\cos^2 x}$

$$\frac{dy}{dx} + y\sec^2 x = \tan x \sec^2 x$$

It is of the form $\frac{dy}{dx} + Py = Q$

Here $\quad P = \sec^2 x, Q = \tan x \sec^2 x$

$$\int P dx = \int \sec^2 x \, dx = \tan x$$

$$I.F = e^{\int p dx} = e^{\tan x}$$

The required solution is $y(I.F) = \int Q(I.F) dx + c$

$$y e^{\tan x} = \int \tan x \sec^2 x \, e^{\tan x} \, dx + c$$

Put $\quad \tan x = t$

Then $\quad \sec^2 x dx = dt$

$$\therefore y e^{\tan x} = \int t e^t \, dt + c$$

$$= \int t d(e^t) + c$$

$$= t e^t - e^t + c$$

$$= \tan x \, e^{\tan x} - e^{\tan x} + c$$

$$y e^{\tan x} = e^{\tan x} (\tan x - 1) + c$$

Example 4.22

Solve $(x^2+1)\frac{dy}{dx} + 2xy = 4x^2$

Solution:

The given equation can be reduced to

$$\frac{dy}{dx} + \frac{2x}{x^2+1} y = \frac{4x^2}{x^2+1}$$

Example 4.22

Solve $\left(x^2+1\right)\dfrac{dy}{dx}+2xy=4x^2$

Solution:

The given equation can be reduced to

$$\frac{dy}{dx}+\frac{2x}{x^2+1}y = \frac{4x^2}{x^2+1}$$

It is of the form $\dfrac{dy}{dx}+Py = Q$

Here $P=\dfrac{2x}{x^2+1}, Q=\dfrac{4x^2}{x^2+1}$

$$\int Pdx = \int\frac{2x}{x^2+1}dx = \log(x^2+1)$$

$$I.F = e^{\int pdx} = e^{\log\left(x^2+1\right)} = x^2+1$$

The required solution is $y(IF) = \int Q(I.F)dx+c$

$$y(x^2+1) = \int\frac{4x^2}{x^2+1}\left(x^2+1\right)dx+c$$

$$y(x^2+1) = \frac{4x^3}{3}+c$$

Example 4.23

Solve $\dfrac{dy}{dx}-3y\cot x=\sin 2x$ given that $y=2$ when $x=\dfrac{\pi}{2}$

Solution:

Given $\dfrac{dy}{dx}-\left(3\cot x\right).y=\sin 2x$

It is of the form $\dfrac{dy}{dx}+Py = Q$

Here $P=-3\cot x$, $Q=\sin 2x$

$$\int Pdx = \int -3\cot x dx = -3\log\sin x = -\log\sin^3 x = \log\frac{1}{\sin^3 x}$$

$$\text{I.F.} = e^{\log\frac{1}{\sin^3 x}} = \frac{1}{\sin^3 x}$$

The required solution is $y\,(I.F) = \int Q(I.F)dx+c$

$$y\frac{1}{\sin^3 x} = \int \sin 2x\,\frac{1}{\sin^3 x}\,dx+c$$

$$y(x^2+1) = \int \frac{4x^2}{x^2+1}(x^2+1)dx + c$$
$$y(x^2+1) = \frac{4x^3}{3} + c$$

Example 4.23

Solve $\frac{dy}{dx} - 3y\cot x = \sin 2x$ given that $y = 2$ when $x = \frac{\pi}{2}$

Solution:

Given $\frac{dy}{dx} - (3\cot x).y = \sin 2x$

It is of the form $\frac{dy}{dx} + Py = Q$

Here $P = -3\cot x$, $Q = \sin 2x$

$$\int Pdx = \int -3\cot x dx = -3\log\sin x = -\log\sin^3 x = \log\frac{1}{\sin^3 x}$$

$$\text{I.F.} = e^{\log\frac{1}{\sin^3 x}} = \frac{1}{\sin^3 x}$$

The required solution is $y\ (I.F) = \int Q(I.F)dx + c$

$$y\frac{1}{\sin^3 x} = \int \sin 2x \frac{1}{\sin^3 x}\ dx + c$$
$$y\frac{1}{\sin^3 x} = \int 2\sin x\cos x \times \frac{1}{\sin^3 x}dx + c$$
$$= 2\int \frac{1}{\sin x} \times \frac{\cos x}{\sin x}\ dx + c$$
$$= 2\int \operatorname{cosec} x \cot x dx + c$$

$$y\frac{1}{\sin^3 x} = -2\operatorname{cosec} x + c \qquad (1)$$

Now $\qquad y = 2$ when $x = \frac{\pi}{2}$

(1) $\Rightarrow$ $\qquad 2\left(\frac{1}{1}\right) = -2\times 1 + c \Rightarrow c = 4$

$\therefore$ (1) $\Rightarrow$ $\qquad y\frac{1}{\sin^3 x} = -2\operatorname{cosec} x + 4$

Example 4.24

Method of solving first order Homogeneous differential equation

Check $f(x, y)$ and $g(x, y)$ are homogeneous functions of same degree.

i.e. $\frac{dy}{dx} = F\left(\frac{y}{x}\right)$

Put $y = vx$ and $\frac{dy}{dx} = v + x\frac{dv}{dx}$

The given differential equation becomes v x dv/dx =F(v)

Separating the variables, we get

$$x\frac{dv}{dx} = F(v) - v \Rightarrow \frac{dv}{F(v)-v} = \frac{dx}{x}$$

By integrating we get the solution in terms of v and x.

Replacing v by y/x we get the solution.

Note

Sometimes it becomes easier by taking the Homogeneous differential equation as $\frac{dx}{dy} = F\left(\frac{x}{y}\right)$ (1)

In this method we have to substitute $x = vy$ and $\frac{dx}{dy} = v + x\frac{dv}{dy}$ then (1) reduces to variable separable type. By integrating, we get the solution in terms of v and y. The solution is deduced by replacing $v = \frac{x}{y}$.

Example 4.15

Solve the differential equation $y^2\,dx + (xy + x^2)dy = 0$

Solution

$$y^2dx + (xy + x^2)\,dy = 0$$

$$(xy + x^2)\,dy = -y^2dx$$

$$\frac{dy}{dx} = \frac{-y^2}{xy + x^2} \qquad (1)$$

Example 4.15

Solve the differential equation $y^2\, dx + (xy + x^2)dy = 0$

Solution

$$y^2 dx + (xy + x^2)\, dy = 0$$

$$(xy + x^2)\, dy = -y^2 dx$$

$$\frac{dy}{dx} = \frac{-y^2}{xy + x^2} \qquad (1)$$

It is a homogeneous differential equation, same degree in x and y

Put $y = vx$ and $\dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$

$\therefore$ (1) becomes

$$v + x\frac{dv}{dx} = \frac{-v^2x^2}{x\ vx + x^2}$$

$$= \frac{-v^2}{v+1}$$

$$x\frac{dv}{dx} = \frac{-v^2}{v+1} - v$$

$$= \frac{-v^2 - v^2 - v}{v+1}$$

$$x\frac{dv}{dx} = \frac{-\left(v + 2v^2\right)}{1+v}$$

Now, separating the variables

$$\frac{1+v}{v(1+2v)}\, dv = \frac{-dx}{x}$$

$$\frac{(1+2v) - v}{v(1+2v)}\, dv = \frac{-dx}{x} \qquad (\because\ 1+v = 1+2v-v)$$

$$\frac{1}{v} - \frac{1}{1+2v}\, dv = \frac{-dx}{x}$$

On Integration we have

$$\int\left(\frac{1}{v} - \frac{1}{1+2v}\right) dv = -\int\frac{dx}{x}$$

$$\log v - \frac{1}{2}\log(1+2v) = -\log x + \log c$$

Now, separating the variables

$$\frac{1+v}{v(1+2v)}\,dv = \frac{-dx}{x}$$

$$\frac{(1+2v)-v}{v(1+2v)}\,dv = \frac{-dx}{x} \qquad (\because\ 1+v = 1+2v-v)$$

$$\frac{1}{v}-\frac{1}{1+2v}\,dv = \frac{-dx}{x}$$

On Integration we have

$$\int\left(\frac{1}{v}-\frac{1}{1+2v}\right)dv = -\int\frac{dx}{x}$$

$$\log v-\frac{1}{2}\log(1+2v) = -\log x+\log c$$

$$\log\left(\frac{v}{\sqrt{1+2v}}\right) = \log\left(\frac{c}{x}\right)$$

$$\frac{v}{\sqrt{1+2v}} = \frac{c}{x}$$

Replace $v = \dfrac{y}{x}$ we get

$$\frac{\frac{y}{x}}{\sqrt{1+\frac{2y}{x}}} = \frac{c}{x}$$

$$\frac{y\sqrt{x}}{\sqrt{x+2y}} = c$$

$$\frac{y^2x}{x+2y} = k$$

where $k = c^2$

Note

$\int\frac{1+v}{v^2+2v}\,dv$ can be done by the method of partial fraction also.

fraction also.

Example 4.16

Solve the differential equation $\frac{dy}{dx}=\frac{x-y}{x+y}$.

Solution:

$$\frac{dy}{dx}=\frac{x-y}{x+y} \qquad (1)$$

This is a homogeneous differential equation.

Now put $y=vx$ and $\frac{dy}{dx}=v+x\frac{dv}{dx}$

$$\therefore \quad (1) \Rightarrow \quad v+x\frac{dv}{dx} = \frac{x-vx}{x+vx}$$

$$= \frac{1-v}{1+v}$$

$$x\frac{dv}{dx} = \frac{1-v}{1+v}-v$$

$$= \frac{1-2v-v^2}{1+v}$$

$$\frac{1+v}{v^2+2v-1}dv = \frac{-dx}{x}$$

Multiply 2 on both sides

$$\frac{2+2v}{v^2+2v-1}dv = -2\frac{dx}{x}$$

On Integration

$$\int\frac{2+2v}{v^2+2v-1}dv = -2\int\frac{dx}{x}$$

$$\log(v^2+2v-1) = -2\log x + \log c$$

$$v^2+2v-1 = \frac{c}{x^2}$$

$$x^2\,(v^2+2v-1) = c$$

Now, Replace $\quad v = \frac{y}{x}$

$$x^2\left[\frac{y^2}{x^2}+\frac{2y}{x}-1\right] = c$$

$y^2+2xy-x^2 = c$ is the solution.

Exercise 4.2: First order and first degree differential

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Exercise 4.2

1. Solve: (i) $\frac{dy}{dx}=ae^y$ (ii) $\frac{1+x^2}{1+y}=xy\frac{dy}{dx}$

2. Solve : $y(1-x)-x\frac{dy}{dx}=o$

3. Solve: (i) $ydx-xdy=0$ (ii) $\frac{dy}{dx}+e^x+ye^x=0$

4. Solve: $\cos x(1+\cos y)dx-\sin y(1+\sin x)dy=0$

5. Solve: $(1-x)dy-(1+y)dx=0$

6. Solve: (i) $\frac{dy}{dx}=y\sin 2x$ (ii) $\log\left(\frac{dy}{dx}\right)=ax+by$

7. Find the curve whose gradient at any point P (x, y) on it is x−a / y-b and which passes through the origin.

1. Solve (i) $\frac{dy}{dx}=ae^y$

Sol. Given $\frac{dy}{dx}=ae^y$

Separating the variables we get,

$\frac{dy}{e^y}=adx \Rightarrow e^{-y}\,dy=adx$

Integrating both sides we get,

$$\int e^{-y}dy = a\int dx$$
$$-e^{-y} = ax+c$$
$$ax+e^{-y}+c = 0$$

(ii) $\frac{1+x^2}{\ldots} = \ldots\frac{dy}{\ldots}$

(ii) $\frac{1+x^2}{1+y} = xy\frac{dy}{dx}$

Sol. Given $\frac{1+x^2}{1+y} = xy\frac{dy}{dx}$

Separating the variables we get,

$$\frac{(1+x^2)dx}{x} = y(1+y)dy$$

$$\Rightarrow \left(\frac{1}{x}+\frac{x^2}{x}\right)dx = (y+y^2)dy$$

$$\Rightarrow \left(\frac{1}{x}+x\right)dx = (y+y^2)dy$$

Integrating both sides we get,

$$\int\left(\frac{1}{x}+x\right)dx = \int(y+y^2)dy$$

$$\Rightarrow \log x + \frac{x^2}{2} = \frac{y^2}{2}+\frac{y^3}{3}+c$$

2. Solve: $y(1-x) - x\frac{dy}{dx} = 0$

Sol.

Given $y(1-x) - x\frac{dy}{dx} = 0$

$$\Rightarrow y(1-x) = x\frac{dy}{dx}$$

Separating the variables we get,

$$\Rightarrow \frac{(1-x)}{x}dx = \frac{dy}{y}$$

$$\Rightarrow \left(\frac{1}{x}-1\right)dx = \frac{dy}{y}$$

Integrating both sides we get,

$$\int\left(\frac{1}{x}-1\right)dx = \int\frac{dy}{y}$$

$$\Rightarrow \log x - x = \log y + c$$

3. Solve (i) $ydx - xdy = 0$

$$\Rightarrow \quad \frac{(1-x)}{x}dx = \frac{dy}{y}$$

$$\Rightarrow \quad \left(\frac{1}{x}-1\right)dx = \frac{dy}{y}$$

Integrating both sides we get,

$$\int\left(\frac{1}{x}-1\right)dx = \int\frac{dy}{y}$$

$$\Rightarrow \quad \log x - x = \log y + c$$

3. **Solve (i)** $ydx - xdy = 0$

Sol.

Given $ydx - xdy = 0$

$$\Rightarrow \quad ydx = xdy$$

Separating the variables we get,

$$\frac{dx}{x} = \frac{dy}{y}$$

Integrating both sides we get,

$$\int\frac{dx}{x} = \int\frac{dy}{y}$$

$$\log x = \log y + \log c$$

$$\log x = \log cy$$

$$[\because \log m + \log n = \log mn]$$

$$x = cy$$

(ii) $\dfrac{dy}{dx} + e^x + ye^x = 0$

$$\Rightarrow \quad \frac{dy}{dx} = -e^x(1+y)$$

Separating the variables we get,

$$\Rightarrow \quad \frac{dy}{1+y} = -e^x\,dx$$

$$\Rightarrow \log(1+y) = -e^x + c$$

4. **Solve: $\cos x\,(1+\cos y)\,dx - \sin y\,(1+\sin x)\,dy = 0$.**

Sol.

$$\cos x\,(1+\cos y)dx = \sin y\,(1+\sin x)dy$$

(ii) $\frac{dy}{dx} + e^x + ye^x = 0$

$$\Rightarrow \quad \frac{dy}{dx} = -e^x(1+y)$$

Separating the variables we get,

$$\Rightarrow \quad \frac{dy}{1+y} = -e^x\,dx$$

$$\Rightarrow \log(1+y) = -e^x + c$$

4. Solve: $\cos x\,(1+\cos y)\,dx - \sin y\,(1+\sin x)\,dy = 0$.

Sol.

$$\cos x\,(1+\cos y)dx = \sin y\,(1+\sin x)dy$$

Separating the variables we get

$$\frac{\cos x}{1+\sin x}dx = \frac{\sin y}{1+\cos y}dy$$

Integrating both sides we get

$$\int \frac{\cos x}{1+\sin x}dx = \int \frac{\sin y}{1+\cos y}dy$$

put $1+\sin x = t \Rightarrow \cos x\,dx = dt$

Also $1+\cos y = s \Rightarrow -\sin y\,dy = ds$

$$\Rightarrow \quad \sin y\,dy = -ds$$

$$\Rightarrow \quad \int \frac{dt}{t} = -\int \frac{ds}{s}$$

$$\Rightarrow \quad \log t = \log s + \log c$$

$$\Rightarrow \quad \log t = \log\left(\frac{c}{s}\right)$$

$\left[\because \log m - \log n = \log \frac{m}{n}\right]$

$$\Rightarrow \quad t = \frac{c}{s}$$

$$\Rightarrow \quad 1+\sin x = \frac{c}{1+\cos y}$$

$[\because t = 1+\sin x \;\&\; s = 1+\cos y]$

$$\Rightarrow (1+\sin x)(1+\cos y) = c$$

5. Solve: $(1 - x)\, dy - (1 + y)\, dx = 0$

Sol.

$$(1 - x)dy = (1 + y)dx$$

Separating the variables we get,

$$\frac{dy}{1+y} = \frac{dx}{1-x}$$

Integrating both sides we get,

$$\int \frac{dy}{1+y} = \int \frac{dx}{1-x}$$

$$\log(1+y) = \frac{\log(1-x)}{-1} + \log c$$

$$\log(1+y) = -\log(1-x) + \log c$$

$$\Rightarrow \log(1+y) + \log(1-x) = \log c$$

$$\Rightarrow \log(1+y)(1-x) = \log c$$

$$\Rightarrow (1+y)(1-x) = c$$

Multiplying by a negative sign we get,

$(x - 1)(y + 1) = -c = C$ where $C = -c$

6. Solve (i) $\dfrac{dy}{dx} = y \sin 2x$

Sol.

Separating the variables we get,

$$\frac{dy}{y} = \sin 2x\, dx$$

Integrating both sides we get,

$$\int \frac{dy}{y} = \int \sin 2x\, dx$$

$$\Rightarrow \log y = \frac{-\cos 2x}{2} + c$$

(ii) Solve $\log\left(\dfrac{dy}{dx}\right) = ax + by$

Sol.

Given $\log\left(\dfrac{dy}{dx}\right) = ax + by$

$$\Rightarrow \frac{dy}{dx} = e^{ax+by}$$

$$\Rightarrow \quad \log y = \frac{-\cos 2x}{2} + c$$

(ii) Solve $\log\left(\frac{dy}{dx}\right) = ax + by$

Sol.

Given $\log\left(\frac{dy}{dx}\right) = ax + by$

$$\Rightarrow \quad \frac{dy}{dx} = e^{ax+by}$$

[Since logarithmic & exponential are reversible functions]

$$\Rightarrow \quad \frac{dy}{dx} = e^{ax} \times e^{by} \quad [\because a^m \times a^n = a^{m+n}]$$

Separating the variables we get,

$$\frac{dy}{e^{by}} = e^{ax}\,dx$$

$$\Rightarrow \quad e^{-by}\,dy = e^{ax}\,dx$$

Integrating both sides we get,

$$\int e^{-by}dy = \int e^{ax}dx$$

$$\Rightarrow \quad \frac{e^{-by}}{-b} = \frac{e^{ax}}{a} + c$$

$$\Rightarrow \quad \frac{-e^{-by}}{b} = \frac{e^{ax}}{a} + c$$

$$\Rightarrow \quad \frac{e^{ax}}{a} = \frac{e^{-by}}{b} + c$$

7. Find the curve whose gradient at any point 0 on it is $\frac{x-a}{y-b}$ and which passes through the origin.

Sol.

Given Gradient $= \frac{x-a}{y-b}$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{x-a}{y-b}$$

Differential Equation in which variables are separable

If in an equation it is possible to collect all the terms of x and dx on one side and all the terms of y and dy on the other side, then the variables are said to be separable. Thus the general form of such an equation is

$f(x)dx = g(y)dy$ (or) $f(x)dx + g(y)dy = 0$

By direct integration we get the solution.

Example 4.6

Solve: $(x^2 + x + 1)\,dx + (y^2 - y + 3)dy = 0$

Solution:

Given $\quad (x^2+x+1)dx+(y^2-y+3)dy = 0$

It is of the form $\quad f(x)dx+g(y)dy = 0$

Integrating , we get

$$\int(x^2+x+1)dx+\int(y^2-y+3)\,dy = c$$

$$\left(\frac{x^3}{3}+\frac{x^2}{2}+x\right)+\left(\frac{y^3}{3}-\frac{y^2}{2}+3y\right) = c$$

Example 4.7

Solve $\dfrac{dy}{dx}=e^{x-y}+x^2e^{-y}$

Solution :

Given $\dfrac{dy}{dx}=e^{x-y}+x^2e^{-y} = e^{-y}e^x + e^{-y}x^2$

$$= e^{-y}\left(e^x+x^2\right)$$

Separating the variables, we get $e^y dy=\left(e^x+x^2\right)dx$

Integrating, we get $\quad \int e^y dy=\int\left(e^x+x^2\right)dx$

$$e^y = e^x+\frac{x^3}{3}+c$$

Example 4.7

Solve $\frac{dy}{dx}=e^{x-y}+x^2e^{-y}$

Solution :

Given $\frac{dy}{dx}=e^{x-y}+x^2e^{-y} = e^{-y}e^x + e^{-y}x^2$

$$=e^{-y}\left(e^x+x^2\right)$$

Separating the variables, we get $e^y dy=\left(e^x+x^2\right)dx$

Integrating, we get $\int e^y dy=\int\left(e^x+x^2\right)dx$

$$e^y=e^x+\frac{x^3}{3}+c$$

Example 4.8

Solve $3e^x\tan y dx+(1+e^x)\sec^2 y dy=0$ given $y(0)=\frac{\pi}{4}$

Solution:

Given $3e^x\tan y dx+(1+e^x)\sec^2 y\, dy = 0$

$$3e^x\tan y\ dx = -(1+e^x)\sec^2 y\ dy$$

$$\frac{3e^x}{1+e^x}dx = -\frac{\sec^2 y}{\tan y}dy$$

Integrating, we get $3\int\frac{e^x}{1+e^x}dx = -\int\frac{\sec^2 y}{\tan y}dy+c$

$$3\log(1+e^x) = -\log\tan y+\log c \quad \left[\because \int\frac{f'(x)}{f(x)}dx=\log f(x)\right]$$

$$\log(1+e^x)^3+\log\tan y = \log c$$

$$\log\left[\left(1+e^x\right)^3\tan y\right] = \log c$$

$$(1+e^x)^3\tan y = c \qquad (1)$$

Given $y(0)=\frac{\pi}{4}(i.e)\quad y=\frac{\pi}{4}\ at\ x=0$

$(1)\Rightarrow \quad (1+e^0)^3\tan\frac{\pi}{4} = c$

$$2^3(1) = c$$

$\Rightarrow \quad c = 8$

Hence the required solution is $(1+e^x)^3\tan y = 8$

$$(1) \Rightarrow \qquad (1+e^0)^3 \tan\frac{\pi}{4} = c$$

$$2^3(1) = c$$

$$\Rightarrow \qquad c = 8$$

Hence the required solution is $(1+e^x)^3 \tan y = 8$

Example 4.9

Solve $\sec^2 x \tan y dx + \sec^2 y \tan x dy = 0$

Solution:

Separating the variables, we get

$$\frac{\sec^2 x}{\tan x}dx + \frac{\sec^2 y}{\tan y}dy = 0$$

Integrating, we get

$$\int\frac{\sec^2 x}{\tan x}dx + \int\frac{\sec^2 y}{\tan y}dy = c$$

$$\log\tan x + \log\tan y = \log c$$

$$\log(\tan x \tan y) = \log c$$

$$\tan x \tan y = c$$

Example 4.10

Solve $ydx - xdy - 3x^2 y^2 e^{x^3} dx = 0$

Solution:

Given equation can be written as $\dfrac{ydx - xdy}{y^2} - 3x^2 e^{x^3} dx = 0$

Integrating, $\displaystyle\int\frac{ydx - xdy}{y^2} - \int 3x^2 e^{x^3} dx = c$

$$\int d\left(\frac{x}{y}\right) - \int e^t dt = c \qquad (\text{where } t = x^3 \text{ and } dt = 3x^2 dx)$$

$$\frac{x}{y} - e^t = c$$

$$\frac{x}{y} - e^{x^3} = c$$

Solve $ydx - xdy - 3x^2y^2e^{x^3}dx = 0$

Solution:

Given equation can be written as $\dfrac{ydx - xdy}{y^2} - 3x^2e^{x^3}dx = 0$

Integrating, $\displaystyle\int \frac{ydx - xdy}{y^2} - \int 3x^2e^{x^3}dx = c$

$$\int d\left(\frac{x}{y}\right) - \int e^t dt = c \qquad \text{(where } t = x^3 \text{ and } dt = 3x^2dx \text{)}$$

$$\frac{x}{y} - e^t = c$$

$$\frac{x}{y} - e^{x^3} = c$$

Example 4.11

Solve : $x - y\dfrac{dx}{dy} = a\left(x^2 + \dfrac{dx}{dy}\right)$

Solution:

Given $x - y\dfrac{dx}{dy} = a\left(x^2 + \dfrac{dx}{dy}\right)$

$$x - y\frac{dx}{dy} = ax^2 + a\frac{dx}{dy}$$

$$x - ax^2 = a\frac{dx}{dy} + y\frac{dx}{dy}$$

$$x(1 - ax) = (a + y)\frac{dx}{dy}$$

By separating the variables, we get

$$\frac{dx}{x(1-ax)} = \frac{dy}{a+y}$$

$$\left(\frac{a}{1-ax} + \frac{1}{x}\right)dx = \frac{dy}{a+y}$$

Integrating, $\displaystyle\int\left(\frac{a}{1-ax} + \frac{1}{x}\right)dx = \frac{dy}{a+y}$

$$-\log(1-ax) + \log x = \log(a+y) + \log c$$

$$\log\left(\frac{x}{1-ax}\right) = \log\left(c(a+y)\right)$$

$$\left(\frac{x}{1-ax}\right) = c\,(a+y)$$

$x = (1-ax)(a+y)c$ which is the required solution

VECTOR CALCULUS

INTRODUCTION

In this chapter we study the basics of vector calculus with the help of a standard vector differential operator. Also we introduce concepts like gradient of a scalar valued function, divergence and curl of a vector valued function, discuss briefly the properties of these concepts and study the applications of the results to the evaluation of line and surface integrals in terms of multiple integrals.

2.1 GRADIENT – DIRECTIONAL DERIVATIVE

Vector differential operator

The vector differential operator ∇ (read as Del) is denoted by $\nabla = \vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}$ where $\vec{i}, \vec{j}, \vec{k}$ are unit vectors along the three rectangular axes OX, OY and OZ.

It is also called Hamiltonian operator and it is neither a vector nor a scalar, but it behaves like a vector.

The gradient of a scalar function

If $\varphi(x, y, z)$ is a scalar point function continuously differentiable in a given region of space, then the gradient of φ is defined as $\nabla\varphi = \vec{i}\frac{\partial\varphi}{\partial x} + \vec{j}\frac{\partial\varphi}{\partial y} + \vec{k}\frac{\partial\varphi}{\partial z}$

It is also denoted as Grad φ.

Note

(i) $\nabla\varphi$ is a vector quantity.

(ii) $\nabla\varphi = 0$ if φ is constant.

(iii) $\nabla(\varphi_1\varphi_2) = \varphi_1\nabla\varphi_2 + \varphi_2\nabla\varphi_1$

(iv) $\nabla\left(\frac{\varphi_1}{\varphi_2}\right) = \frac{\varphi_2\nabla\varphi_1 - \varphi_1\nabla\varphi_2}{{\varphi^2}_2}$ if $\varphi_2 \neq 0$

(v) $\nabla(\varphi \pm \chi) = \nabla\varphi \pm \nabla\chi$

Problems based on Gradient

Example: 2.1 Find the gradient of φ where φ is $3x^2y - y^3z^2$ at $(1, -2, 1)$.

Solution:

Given $\varphi = 3x^2y - y^3z^2$

Grad $\varphi = \vec{i}\frac{\partial\varphi}{\partial x} + \vec{j}\frac{\partial\varphi}{\partial y} + \vec{k}\frac{\partial\varphi}{\partial z}$

Now $\frac{\partial\varphi}{\partial x} = 6xy$, $\frac{\partial\varphi}{\partial y} = 3x^2 - 3y^2z^2$, $\frac{\partial\varphi}{\partial z} = -2y^3z$

$\therefore$ grad $\varphi = \vec{i}\,6xy + \vec{j}(3x^2 - 3y^2z^2) - \vec{k}2y^3z$

$\therefore (grad\ \varphi)_{(1,-2,\ 1)} = -12\vec{i} - 9\vec{j} + 16\vec{k}$

Example: 2.2 If $\varphi = \log(x^2 + y^2 + z^2)$ then find $\nabla\varphi$.

Solution:

Given $\varphi = \log(x^2 + y^2 + z^2)$

$$\nabla\varphi = \vec{i}\frac{\partial\varphi}{\partial x} + \vec{j}\frac{\partial\varphi}{\partial y} + \vec{k}\frac{\partial\varphi}{\partial z}$$

$$= \vec{i}\left(\frac{2x}{x^2+y^2+z^2}\right) + \vec{j}\left(\frac{2y}{x^2+y^2+z^2}\right) + \vec{k}\left(\frac{2z}{x^2+y^2+z^2}\right)$$

$$= \frac{2}{x^2+y^2+z^2}(x\vec{i} + y\vec{j} + z\vec{k}) = \frac{2}{r^2}\vec{r}$$

Example: 2.3 Find $\nabla(r), \nabla\left(\frac{1}{r}\right), \nabla(\log r)$ where $r = |\vec{r}|$ and $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$.

Solution:

Given $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$

$$\Rightarrow |\vec{r}| = r = \sqrt{x^2 + y^2 + z^2}$$

$$\Rightarrow r^2 = x^2 + y^2 + z^2$$

$$2r\frac{\partial r}{\partial x} = 2x, \quad 2r\frac{\partial r}{\partial y} = 2y, \quad 2r\frac{\partial r}{\partial z} = 2z$$

$$\Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r}, \quad \frac{\partial r}{\partial y} = \frac{y}{r}, \quad \frac{\partial r}{\partial z} = \frac{z}{r}$$

(i) $\nabla(r) = \vec{i}\frac{\partial r}{\partial x} + \vec{j}\frac{\partial r}{\partial y} + \vec{k}\frac{\partial r}{\partial z}$

$$= \vec{i}\frac{x}{r} + \vec{j}\frac{y}{r} + \vec{k}\frac{z}{r}$$

$$= \frac{1}{r}(x\vec{i} + y\vec{j} + z\vec{k}) = \frac{1}{r}\vec{r}$$

(ii) $\nabla\left(\frac{1}{r}\right) = \vec{i}\frac{\partial\left(\frac{1}{r}\right)}{\partial x} + \vec{j}\frac{\partial\left(\frac{1}{r}\right)}{\partial y} + \vec{k}\frac{\partial\left(\frac{1}{r}\right)}{\partial z}$

$$= \vec{i}\left(\frac{-1}{r^2}\right)\frac{\partial r}{\partial x} + \vec{j}\left(\frac{-1}{r^2}\right)\frac{\partial r}{\partial y} + \vec{k}\left(\frac{-1}{r^2}\right)\frac{\partial r}{\partial z}$$

$$= \left(-\frac{1}{r^2}\right)\left[\vec{i}\frac{x}{r} + \vec{j}\frac{y}{r} + \vec{k}\frac{z}{r}\right]$$

$$= -\frac{1}{r^3}(x\vec{i} + y\vec{j} + z\vec{k}) = -\frac{1}{r^3}\vec{r}$$

(iii) $\nabla(\log r) = \sum\vec{i}\frac{\partial(\log r)}{\partial x}$

$$= \sum\vec{i}\frac{1}{r}\frac{\partial r}{\partial x}$$

$$= \sum\vec{i}\frac{1}{r}\frac{x}{r}$$

$$= \sum\vec{i}\frac{x}{r^2}$$

$$= \frac{1}{r^2}(x\vec{i} + y\vec{j} + z\vec{k}) = \frac{1}{r^2}\vec{r}$$

Example: 2.4 Prove that $\nabla(r^n) = nr^{n-2}\,\vec{r}$

Solution:

Given $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$

$$\nabla(r^n) = \vec{i}\frac{\partial r^n}{\partial x} + \vec{j}\frac{\partial r^n}{\partial y} + \vec{k}\frac{\partial r^n}{\partial z}$$

$$= \vec{i}\, nr^{n-1}\frac{\partial r}{\partial x} + \vec{j}\, nr^{n-1}\frac{\partial r}{\partial y} + \vec{k}\, nr^{n-1}\frac{\partial r}{\partial z}$$

$$= nr^{n-1}\left[\vec{i}\left(\frac{x}{r}\right) + \vec{j}\left(\frac{y}{r}\right) + \vec{k}\left(\frac{z}{r}\right)\right]$$

$$= \frac{nr^{n-1}}{r}(x\vec{i} + y\vec{j} + z\vec{k}) = nr^{n-2}\vec{r}$$

Example: 2.5 Find $|\nabla\varphi|$ if $\varphi = 2xz^4 - x^2y$ at $(2,-2,-1)$

Solution:

Given $\varphi = 2xz^4 - x^2y$

$$\nabla\varphi = \vec{i}\frac{\partial\varphi}{\partial x} + \vec{j}\frac{\partial\varphi}{\partial y} + \vec{k}\frac{\partial\varphi}{\partial z}$$

Now $\frac{\partial\varphi}{\partial x} = 2z^4 - 2xy,\ \frac{\partial\varphi}{\partial y} = -x^2,\ \frac{\partial\varphi}{\partial z} = 8xz^3$

$$\therefore \nabla\varphi = \vec{i}(2z^4 - 2xy) + \vec{j}(-x^2) + \vec{k}(8xz^3)$$

$$\therefore (\nabla\varphi)_{(2,-2,-1)} = 10\vec{i} - 4\vec{j} - 16\vec{k}$$

$$|\nabla\varphi| = \sqrt{100 + 16 + 256} = \sqrt{372}$$

Directional Derivative (D.D) of a scalar point function

The derivative of a point function (scalar or vector) in a particular direction is called its directional derivative along the direction.

The directional derivative of a scalar function φ in a given direction $\vec{a}$ is the rate of change of φ in that direction. It is given by the component of $\nabla\varphi$ in the direction of $\vec{a}$.

The directional derivative of a scalar point function in the direction of $\vec{a}$ is given by

$$\text{D.D} = \frac{\nabla\varphi \cdot \vec{a}}{|\vec{a}|}$$

The maximum directional derivative is $|\nabla\varphi|$ or $|\text{grad}\ \varphi|$.

Problems based on Directional Derivative

Example: 2.6 Find the directional derivative of $\varphi = 4xz^2 + x^2yz$ at $(1,-2,1)$ in the direction of $2\vec{i} - \vec{j} - 2\vec{k}$.

Solution:

Given $\varphi = 4xz^2 + x^2yz$

$$\nabla\varphi = \vec{i}\frac{\partial\varphi}{\partial x} + \vec{j}\frac{\partial\varphi}{\partial y} + \vec{k}\frac{\partial\varphi}{\partial z}$$

$$= \vec{i}(2xyz + 4z^2) + \vec{j}(x^2z) + \vec{k}(x^2y + 8xz)$$

$$\therefore (\nabla\varphi)_{(1,-2,-1)} = 8\vec{i} - \vec{j} - 10\vec{k}$$

Given $\vec{a} = 2\vec{i} - \vec{j} - 2\vec{k}$

$$|\vec{a}| = \sqrt{4+1+4} = 3$$

$$\text{D.D} = \frac{\nabla\varphi \cdot \vec{a}}{|\vec{a}|}$$

$$= (8\vec{i} - \vec{j} - 10\vec{k}) \cdot \frac{(2\vec{i} - \vec{j} - 2\vec{k})}{3}$$

$$= \frac{1}{3}(16 + 1 + 20) = \frac{37}{3}$$

Example: 2.7 Find the directional derivative of $\varphi(x, y, z) = xy^2 + yz^3$ at the point $P(2, -1, 1)$ in the direction of PQ where Q is the point $(3, 1, 3)$

Solution:

Given $\varphi = xy^2 + yz^3$

$$\nabla\varphi = \vec{i}\frac{\partial\varphi}{\partial x} + \vec{j}\frac{\partial\varphi}{\partial y} + \vec{k}\frac{\partial\varphi}{\partial z}$$

$$= \vec{i}\,(y^2) + \vec{j}\,(2xy + z^3) + \vec{k}\,(3yz^2)$$

$$\therefore (\nabla\varphi)_{(2,-1,\ 1)} = \vec{i} - 3\vec{j} - 3\vec{k}$$

Given $\vec{a} = \overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP}$

$$= \left(3\vec{i} + \vec{j} + 3\vec{k}\right) - \left(2i - \vec{j} + \vec{k}\right)$$

$$= \vec{i} + 2\vec{j} + 2\vec{k}$$

$$|\vec{a}| = \sqrt{1 + 4 + 4} = 3$$

$$\text{D. D} = \frac{\nabla\varphi \cdot \vec{a}}{|\vec{a}|}$$

$$= \frac{(\vec{i} - 3\vec{j} - 3\vec{k}) \cdot (\vec{i} + 2\vec{j} + 2\vec{k})}{3}$$

$$= \frac{1}{3}(1 - 6 - 6) = -\frac{11}{3}$$

Example: 2.8 In what direction from $(-1, 1, 2)$ is the directional derivative of $\varphi = xy^2 z^3$ a maximum? Find also the magnitude of this maximum.

Solution:

Given $\varphi = xy^2 z^3$

$$\nabla\varphi = \vec{i}\frac{\partial\varphi}{\partial x} + \vec{j}\frac{\partial\varphi}{\partial y} + \vec{k}\frac{\partial\varphi}{\partial z}$$

$$= \vec{i}\,(y^2z^3) + \vec{j}\,(2xy\,z^3) + \vec{k}\,(3xy^2z^2)$$

$$\therefore (\nabla\varphi)_{(-1,\ 1,\ 2)} = 8\vec{i} - 16\vec{j} - 12\vec{k}$$

The maximum directional derivative occurs in the direction of $\nabla\varphi = 8\vec{i} - 16\vec{j} - 12\vec{k}$.

∴ The magnitude of this maximum directional derivative

$$|\nabla\varphi| = \sqrt{64 + 256 + 144} = \sqrt{464}$$

Example: 2.9 Find the directional derivative of the scalar function $\varphi = xyz$ in the direction of the outer normal to the surface $z = xy$ at the point$(3, 1, 3)$.

Solution:

Given $\varphi = xyz$

$$\nabla\varphi = \vec{i}\frac{\partial\varphi}{\partial x} + \vec{j}\frac{\partial\varphi}{\partial y} + \vec{k}\frac{\partial\varphi}{\partial z}$$

$$= \vec{i}\,(yz) + \vec{j}\,(xz) + \vec{k}\,(xy)$$

$\therefore (\nabla\varphi)_{(3,\ 1,\ 3)} = 3\vec{i} + 9\vec{j} + 3\vec{k}$

Given surface is $z = xy \Rightarrow z - xy = 0$

$$\nabla\chi = \vec{i}\frac{\partial\chi}{\partial x} + \vec{j}\frac{\partial\chi}{\partial y} + \vec{k}\frac{\partial\chi}{\partial z}$$

$$= \vec{i}\,(-y) + \vec{j}\,(-x) + \vec{k}\,(1)$$

Let $\vec{a} = \nabla\chi_{(3,1,3)} = -\vec{i} - 3\vec{j} + \vec{k}$

$$\Rightarrow |\vec{a}| = \sqrt{1+9+1} = \sqrt{11}$$

$$\text{D. D} = \frac{\nabla\varphi\cdot\vec{a}}{|\vec{a}|}$$

$$= \frac{(3\vec{i}+9\vec{j}+3\vec{k})\cdot(-\vec{i}-3\vec{j}+\vec{k})}{\sqrt{11}}$$

$$= \frac{1}{\sqrt{11}}(-3-27+3) = -\frac{27}{\sqrt{11}}$$

Example: 2.10 Find the directional derivative of $\varphi = xy + yz + zx$ at $(1, 2, 0)$ in the direction of $\vec{i} + 2\vec{j} + 2\vec{k}$. Find also its maximum value.

Solution:

Given $\varphi = xy + yz + zx$

$$\nabla\varphi = \vec{i}\frac{\partial\varphi}{\partial x} + \vec{j}\frac{\partial\varphi}{\partial y} + \vec{k}\frac{\partial\varphi}{\partial z}$$

$$= \vec{i}\,(y+z) + \vec{j}\,(x+z) + \vec{k}\,(y+x)$$

$\therefore (\nabla\varphi)_{(1,\ 2,\ 0)} = 2\vec{i} + \vec{j} + 3\vec{k}$

Given $\vec{a} = \vec{i} + 2\vec{j} + 2\vec{k}$

$$|\vec{a}| = \sqrt{1+4+4} = 3$$

$$\text{D. D} = \frac{\nabla\varphi\cdot\vec{a}}{|\vec{a}|}$$

$$= \frac{(2\vec{i}+\vec{j}+3\vec{k})\cdot(\vec{i}+2\vec{j}+2\vec{k})}{3}$$

$$= \frac{1}{3}(2+2+6) = \frac{10}{3}$$

Maximum value is $|\nabla\varphi| = \sqrt{4+1+9} = \sqrt{14}$

Unit normal vector to the surface

If $\varphi(x, y, z)$ be a scalar function, then $\varphi(x, y, z) = c$ represents a surface and the unit normal vector to the surface φ is given by $\hat{n} = \frac{\nabla\varphi}{|\nabla\varphi|}$

Normal Derivative $= |\nabla\varphi|$

Problems based on unit normal vector

Example: 2.11 Find the unit normal to the surface $x^2 + y^2 = z$ at the point $(1, -2, 5)$.

Solution:

Given $\varphi = x^2 + y^2 - z$

$$\nabla\varphi = \vec{i}\frac{\partial\varphi}{\partial x} + \vec{j}\frac{\partial\varphi}{\partial y} + \vec{k}\frac{\partial\varphi}{\partial z}$$

$$= \vec{i}\,(2x) + \vec{j}\,(2y) + \vec{k}\,(-1)$$

$$\therefore (\nabla\varphi)_{(1,-2,\ 5)} = 2\vec{i} - 4\vec{j} - \vec{k}$$

$$|\nabla\varphi| = \sqrt{4+16+1} = \sqrt{21}$$

Unit normal $\hat{n} = \frac{\nabla\varphi}{|\nabla\varphi|} = \frac{2\vec{i}-4\vec{j}-\vec{k}}{\sqrt{21}}$

Example: 2.12 Find the unit normal to the surface $x^2 + xy + y^2 + xyz$ at the point $(1, -2, 1)$.

Solution:

Given $\varphi = x^2 + xy + y^2 + xyz$

$$\nabla\varphi = \vec{i}\frac{\partial\varphi}{\partial x} + \vec{j}\frac{\partial\varphi}{\partial y} + \vec{k}\frac{\partial\varphi}{\partial z}$$

$$= \vec{i}\,(2x + y + yz) + \vec{j}\,(x + 2y + xz) + \vec{k}\,(xy)$$

$$\therefore (\nabla\varphi)_{(1,-2,\ 1)} = -2\vec{i} - 2\vec{j} - 2\vec{k}$$

$$|\nabla\varphi| = \sqrt{4+4+4} = \sqrt{12} = 2\sqrt{3}$$

Unit normal $\hat{n} = \frac{\nabla\varphi}{|\nabla\varphi|} = \frac{-2\vec{i}-2\vec{j}-2\vec{k}}{2\sqrt{3}}$

$$= \frac{-1}{\sqrt{3}}\left(\vec{i} + \vec{j} + \vec{k}\right)$$

Example: 2.13 Find the normal derivative to the surface $x^2y + xz^2$ at the point $(-1, 1, 1)$.

Solution:

Given $\varphi = x^2y + xz^2$

$$\nabla\varphi = \vec{i}\frac{\partial\varphi}{\partial x} + \vec{j}\frac{\partial\varphi}{\partial y} + \vec{k}\frac{\partial\varphi}{\partial z}$$

$$= \vec{i}\,(2xy + z^2) + \vec{j}\,(x^2) + \vec{k}\,(2xz)$$

$$\therefore (\nabla\varphi)_{(-1,\ 1,\ 1)} = -\vec{i} + \vec{j} - 2\vec{k}$$

Normal derivative $|\nabla\varphi| = \sqrt{1+1+4} = \sqrt{6}$

Example: 2.14 What is the greatest rate of increase of $\varphi = xyz^2$ at the point $(1, 0, 3)$.

Solution:

Given $\varphi = xyz^2$

$$\nabla\varphi = \vec{i}\frac{\partial\varphi}{\partial x} + \vec{j}\frac{\partial\varphi}{\partial y} + \vec{k}\frac{\partial\varphi}{\partial z}$$

$$= \vec{i}\,(y\,z^2) + \vec{j}\,(xz^2) + \vec{k}\,(2xyz)$$

$$\therefore (\nabla\varphi)_{(1,\ 0,\ 3)} = 0\vec{i} + 9\vec{j} + 0\vec{k}$$

$\therefore$ Greatest rate of increase $|\nabla\varphi| = \sqrt{9^2} = 9$

Angle between the surfaces

$$\cos\theta = \frac{\nabla\varphi_1 \cdot \nabla\varphi_2}{|\nabla\varphi_1||\nabla\varphi_2|}$$

$$\Rightarrow \theta = \cos^{-1}\left[\frac{\nabla\varphi_1 \cdot \nabla\varphi_2}{|\nabla\varphi_1|\,|\nabla\varphi_2|}\right]$$

Problems based on angle between two surfaces

Example: 2.15 Find the angle between the surfaces $z = x^2 + y^2 - 3$ and $x^2 + y^2 + z^2 = 9$ at tl

point $(2, -1, 2)$.

Solution:

Given $\varphi = x^2 + y^2 - z - 3$

$$\nabla\varphi_1 = \vec{i}\frac{\partial\varphi_1}{\partial x} + \vec{j}\frac{\partial\varphi_1}{\partial y} + \vec{k}\frac{\partial\varphi_1}{\partial z}$$

$$= \vec{i}\,(2x) + \vec{j}\,(2y) + \vec{k}\,(-1)$$

$$\therefore (\nabla\varphi_1)_{(2,-1,\ 2)} = 4\vec{i} - 2\vec{j} - \vec{k}$$

$$|\nabla\varphi_1| = \sqrt{16+4+1} = \sqrt{21}$$

$$\nabla\varphi_2 = \vec{i}\frac{\partial\varphi_2}{\partial x} + \vec{j}\frac{\partial\varphi_2}{\partial y} + \vec{k}\frac{\partial\varphi_2}{\partial z}$$

$$= \vec{i}\,(2x) + \vec{j}\,(2y) + \vec{k}\,(2z)$$

$$\therefore (\nabla\varphi_2)_{(2,-1,\ 2)} = 4\vec{i} - 2\vec{j} + 4\vec{k}$$

$$|\nabla\varphi_2| = \sqrt{16+4+16} = \sqrt{36} = 6$$

The angle between the surfaces is $\cos\theta = \frac{\nabla\varphi_1 \cdot \nabla\varphi_2}{|\nabla\varphi_1|\,|\nabla\varphi_2|}$

$$= \frac{(4\vec{i}-2\vec{j}-\vec{k})\,(4\vec{i}-2\vec{j}+4\vec{k})}{\sqrt{21}(6)}$$

$$= \frac{16+4-4}{\sqrt{21}(6)}$$

$$= \frac{16}{\sqrt{21}(6)} = \frac{8}{3\sqrt{21}}$$

$$\Rightarrow \theta = \cos^{-1}\left[\frac{8}{3\sqrt{21}}\right]$$

Example: 2.16 Find the angle between the normals to the surfaces $x^2 = yz$ at the point $(1,1,1)$ *and* $(2,4,1)$.

Solution:

Given $\varphi = x^2 - yz$

$$\nabla\varphi = \vec{i}\frac{\partial\varphi}{\partial x} + \vec{j}\frac{\partial\varphi}{\partial y} + \vec{k}\frac{\partial\varphi}{\partial z}$$

$$= \vec{i}\,(2x) + \vec{j}\,(-z) + \vec{k}\,(-y)$$

$$\therefore (\nabla\varphi_1)_{(1,\ 1,\ 1)} = 2\vec{i} - \vec{j} - \vec{k}$$

$$|\nabla\varphi_1| = \sqrt{4+1+1} = \sqrt{6}$$

$$\therefore (\nabla\varphi_2)_{(2,\ 4,\ 1)} = 4\vec{i} - \vec{j} - 4\vec{k}$$

$$|\nabla\varphi_2| = \sqrt{16+1+16} = \sqrt{33}$$

The angle between the surfaces is $\cos\theta = \frac{\nabla\varphi_1 \cdot \nabla\varphi_2}{|\nabla\varphi_1|\,|\nabla\varphi_2|}$

$$= \frac{(2\vec{i}-\vec{j}-\vec{k})\,(4\vec{i}-\vec{j}-4\vec{k})}{\sqrt{6}\sqrt{33}}$$

$$= \frac{8+1+4}{\sqrt{6}\sqrt{33}}$$

$$= \frac{13}{\sqrt{2(3)}\sqrt{11(3)}} = \frac{13}{3\sqrt{22}}$$

$$\Rightarrow \theta = cos^{-1}\left[\frac{13}{3\sqrt{22}}\right]$$

Example: 2.17 Find the angle between the surfaces $x\,log\,z = y^2 - 1$ and $x^2y = 2 - z$ at the point $(1,1,1)$.

Solution:

Given $\varphi_1 = y^2 - x\,log\,z - 1$

$$\nabla\varphi_1 = \vec{i}\frac{\partial\varphi_1}{\partial x} + \vec{j}\frac{\partial\varphi_1}{\partial y} + \vec{k}\frac{\partial\varphi_1}{\partial z}$$

$$= \vec{i}\,(-\log z) + \vec{j}\,(2y) + \vec{k}\left(-\frac{x}{z}\right)$$

$$\therefore (\nabla\varphi_1)_{(1,\ 1,\ 1)} = 0\vec{i} + 2\vec{j} - \vec{k}$$

$$|\nabla\varphi_1| = \sqrt{0+4+1} = \sqrt{5}$$

$$\nabla\varphi_2 = \vec{i}\frac{\partial\varphi_2}{\partial x} + \vec{j}\frac{\partial\varphi_2}{\partial y} + \vec{k}\frac{\partial\varphi_2}{\partial z}$$

$$= \vec{i}\,(2xy) + \vec{j}\,(x^2) + \vec{k}\,(1)$$

$$\therefore (\nabla\varphi_2)_{(1,\ 1,\ 1)} = 2\vec{i} + \vec{j} + \vec{k}$$

$$|\nabla\varphi_2| = \sqrt{4+1+1} = \sqrt{6}$$

The angle between the surfaces is $\cos\theta = \frac{\nabla\varphi_1 \cdot \nabla\varphi_2}{|\nabla\varphi_1|\,|\nabla\varphi_2|}$

$$= \frac{(0\vec{i}+2\vec{j}-\vec{k})\cdot(2\vec{i}+\vec{j}+\vec{k})}{\sqrt{5}\sqrt{6}}$$

$$= \frac{0+2-1}{\sqrt{30}}$$

$$= \frac{1}{\sqrt{30}}$$

$$\Rightarrow \theta = cos^{-1}\left[\frac{1}{\sqrt{30}}\right]$$

Problems based on orthogonal surfaces

Two surfaces are orthogonal if $\nabla\varphi_1 \cdot \nabla\varphi_2 = 0$

Example: 2.18 Find a and b such that the surfaces $ax^2 - byz = (a+2)x$ and $4x^2y + z^3 = 4$ cut orthogonally at $(1,-1,2)$.

Solution:

Given $ax^2 - byz = (a+2)x$

Let $\varphi_1 = ax^2 - byz - (a+2)x$

$$\nabla\varphi_1 = \vec{i}\frac{\partial\varphi_1}{\partial x} + \vec{j}\frac{\partial\varphi_1}{\partial y} + \vec{k}\frac{\partial\varphi_1}{\partial z}$$

$$= \vec{i}\left(2ax - (a+2)\right) + \vec{j}\,(-bz) + \vec{k}\,(-by)$$

$$\therefore (\nabla \varphi_1)_{(1,-1,\ 2)} = \vec{i}(a-2) + \vec{j}\,(-2b) + \vec{k}(b)$$

Let $\varphi_2 = 4x^2y + z^3 - 4$

$$\nabla\varphi_2 = \vec{i}\frac{\partial \varphi_2}{\partial x} + \vec{j}\frac{\partial \varphi_2}{\partial y} + \vec{k}\frac{\partial \varphi_2}{\partial z}$$

$$= \vec{i}\,(8xy) + \vec{j}\,(4x^2) + \vec{k}\,(3z^2)$$

$$\therefore (\nabla \varphi_2)_{(1,-1,\ 2)} = -8\vec{i} + 4\vec{j} + 12\vec{k}$$

Since the two surfaces are orthogonal if $\nabla\varphi_1 \cdot \nabla\varphi_2 = 0$

$$\Rightarrow \left(\vec{i}(a-2) + \vec{j}\,(-2b) + \vec{k}(b)\right) \cdot \left(-8\vec{i} + 4\vec{j} + 12\vec{k}\right) = 0$$

$$\Rightarrow -8\,(a-2) - 8b + 12b = 0$$

$$\Rightarrow -8a + 16 - 8b + 12b = 0$$

$$\Rightarrow -8a + 16 + 4b = 0$$

$\div$ by 4 $\Rightarrow -2a + 4 + b = 0$

$$\Rightarrow 2a - b - 4 = 0 \ \ldots\ (1)$$

To find a and b we need another equation in a and b.

The point $(1,-1,2)$ lies in $ax^2 - byz - (a+2)x = 0$

$$\therefore a - b\,(-1)\,(2) - (a+2)\,(1) = 0$$

$$\Rightarrow a + 2b - a - 2 = 0$$

$$\Rightarrow 2b - 2 = 0$$

$$\Rightarrow b = 1$$

Substitute $b = 1$ in (1) we get

$$\Rightarrow 2a - 1 - 4 = 0$$

$$\Rightarrow 2a - 5 = 0$$

$$\Rightarrow a = \frac{5}{2}$$

Example: 2.19 Find the values of a and b so that the surfaces $ax^3 - by^2z = (a+3)x^2$ and $4x^2y - z^3 = 11$ may cut orthogonally at $(2,-1,-3)$.

Solution:

Given $ax^3 - by^2z = (a+3)x^2$

Let $\varphi_1 = ax^3 - by^2z - (a+3)x^2$

$$\nabla\varphi_1 = \vec{i}\frac{\partial \varphi_1}{\partial x} + \vec{j}\frac{\partial \varphi_1}{\partial y} + \vec{k}\frac{\partial \varphi_1}{\partial z}$$

$$= \vec{i}\left(3ax^2 - 2x(a+3)\right) + \vec{j}\,(-2byz) + \vec{k}\,(-by^2)$$

$$\therefore (\nabla \varphi_1)_{(2,-1,-3)} = \vec{i}(8a - 12) + \vec{j}\,(-6b) + \vec{k}(-b)$$

Let $\varphi_2 = 4x^2y - z^3 - 11$

$$\nabla\varphi_2 = \vec{i}\frac{\partial \varphi_2}{\partial x} + \vec{j}\frac{\partial \varphi_2}{\partial y} + \vec{k}\frac{\partial \varphi_2}{\partial z}$$

$$= \vec{i}\,(8xy) + \vec{j}\,(4x^2) + \vec{k}\,(-3z^2)$$

$$\therefore (\nabla \varphi_2)_{(2,-1,-3)} = -16\vec{i} + 16\vec{j} - 27\vec{k}$$

Given the two surfaces cut orthogonally if $\nabla\varphi_1 \cdot \nabla \varphi_2 = 0$

$$\Rightarrow \left(\vec{i}(8a-12) + \vec{j}\,(-6b) - \vec{k}(b)\right) \cdot \left(-16\vec{i} + 16\vec{j} - 27\vec{k}\,\right) = 0$$

$$\Rightarrow -16\,(8a-12) - 16(6b) + 27b = 0$$

$$\Rightarrow -128a + 192 - 69b = 0$$

$$\Rightarrow 128a + 69b - 192 = 0 \ \ldots (1)$$

To find a and b we need another equation in a and b.

The point $(2,-1,-3)$ lies in $ax^3 - by^2z - (a+3)x^2 = 0$

$$\therefore 8a - b\,(1)\,(-3) - (a+3)\,(4) = 0$$

$$\Rightarrow 4a + 3b - 12 = 0 \ \ldots (2)$$

Solving (1) and (2) we get, $a = -\frac{7}{3}$ & $b = \frac{64}{9}$

Equation of the tangent plane and normal to the surface

Equation of the tangent plane is $(\vec{r} - \vec{a}) \cdot \nabla\varphi = 0$

Equation of the normal line is $(\vec{r} - \vec{a}) \times \nabla\varphi = \vec{0}$

Problems based on tangent plane

Example: 2.20 Find the equation of the tangent plane and normal line to the surface $xyz = 4$ at the point $\vec{i} + 2\vec{j} + 2\vec{k}$.

Solution:

Given $\varphi = xyz - 4$

$$\nabla\varphi = \vec{i}\frac{\partial\varphi}{\partial x} + \vec{j}\frac{\partial\varphi}{\partial y} + \vec{k}\frac{\partial\varphi}{\partial z}$$

$$= \vec{i}\,(yz) + \vec{j}\,(xz) + \vec{k}\,(xy)$$

$$\therefore (\nabla \varphi)_{(1,\ 2,\ 2)} = 4\vec{i} + 2\vec{j} + 2\vec{k}$$

Equation of the tangent plane at the point $\vec{a} = \vec{i} + 2\vec{j} + 2\vec{k}$ is $(\vec{r} - \vec{a}) \cdot \nabla\varphi = 0$

$$\Rightarrow \left[(x\vec{i} + y\vec{j} + z\vec{k}) - \vec{i} + 2\vec{j} + 2\vec{k}\right] \cdot (4\vec{i} + 2\vec{j} + 2\vec{k}) = 0$$

$$\Rightarrow \left[(x-1)\,\vec{i} + (y-2)\,\vec{j} + (z-2)\,\vec{k}\right] \cdot (4\vec{i} + 2\vec{j} + 2\vec{k}) = 0$$

$$\Rightarrow 4(x-1) + 2(y-2) + 2(z-2) = 0$$

$$\Rightarrow 4x - 4 + 2y - 4 + 2z - 4 = 0$$

$$\Rightarrow 4x + 2y + 2z = 12$$

$$\Rightarrow 2x + y + z = 6$$

Equation of the normal line $(\vec{r} - \vec{a}) \times \nabla\varphi = \vec{0}$

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ x-1 & y-2 & z-2 \\ 4 & 2 & 2 \end{vmatrix} = \vec{0}$$

$$\Rightarrow \vec{i}\,[2\,(y-2) - 2\,(z-2)] - \vec{j}\,[2\,(x-1) - 4\,(z-2)] + \vec{k}\,[2\,(x-1) - 4\,(y-2)]$$

$$= 0\,\vec{i} + 0\,\vec{j} + 0\vec{k}$$

Equating the coefficients of $\vec{i}, \vec{j}, \vec{k}$ we get

$$\Rightarrow 2\,(y-2) - 2\,(z-2) = 0$$

$$\Rightarrow (y-2) = (z-2) \quad \dots (1)$$

$$\Rightarrow 2\,(x-1) - 4\,(z-2) = 0$$

$$\Rightarrow (x-1) = 2\,(z-2)$$

$$\Rightarrow \frac{x-1}{2} = (z-2) \quad \dots (2)$$

$$\Rightarrow 2\,(x-1) - 4\,(y-2) = 0$$

$$\Rightarrow (x-1) = 2\,(y-2)$$

$$\Rightarrow \frac{x-1}{2} = (y-2) \quad \dots (3)$$

From $(1), (2)$ *and* (3) we get $\frac{x-1}{2} = \frac{y-2}{1} = \frac{z-2}{1}$

Which is the required equation of the normal line.

Exercise: 2.1

1. Find $\nabla\varphi$ if $\varphi = \frac{1}{2}\log(x^2 + y^2 + z^2)$ **Ans:** $\frac{\vec{r}}{r^2}$

2. Find the directional derivative of

   (i) $\varphi = 2xy + z^2$ at the point $(1, -1, 3)$ in the direction $\vec{i} + 2\vec{j} + 2\vec{k}$. **Ans:** $\frac{14}{3}$

   (ii) $\varphi = xy^2 + yz^3$ at the point $(2, -1, 1)$ in the direction of PQ where Q is the point $(3, 1, 3)$. **Ans:** $\frac{-11}{3}$

3. Prove that the directional derivative of $\varphi = x^3y^2z$ at $(1, 2, 3)$ is maximum along the direction $9\vec{i} + 3\vec{j} + \vec{k}$. Also, find the maximum directional derivative. **Ans:** $4\sqrt{91}$

4. Find the unit tangent vector to the curve $\vec{r} = (t^2+1)\vec{i} + (4t-3)\vec{j} + (2t^2 - 65)\vec{k}$ at $t = 1$. **Ans:** $\frac{\vec{i}+2\vec{j}-\vec{k}}{\sqrt{6}}$

5. Find a unit normal to the following surfaces at the specified points.

   (i) $x^2y + 2xz = 4$ at $(2, -2, 3)$ **Ans:** $\pm\frac{1}{3}(\vec{i} - 2\vec{j} - 2\vec{k})$

   (ii) $x^2 + y^2 = z$ at $(1, -2, 5)$ **Ans:** $\frac{1}{\sqrt{21}}(2\vec{i} - 4\vec{j} - \vec{k})$

   (ii) $xy^3z^2 = 4$ at $(-1, -1, 2)$ **Ans:** $\frac{1}{\sqrt{11}}(-\vec{i} - 3\vec{j} + \vec{k})$

   (iv) $x^2 + y^2 = z$ at $(1, 1, 2)$ **Ans:** $\frac{1}{3}(2\vec{i} + 2\vec{j} - \vec{k})$

6. Find the angle between the surfaces $x^2 - y^2 - z^2 = z$ and $xy + yz - zx - 18 = 0$ at the

point $(6,4,3)$. **Ans:** $cos^{-1}\left[\frac{-24}{\sqrt{86}\sqrt{61}}\right]$

7. Find the angle between the surfaces $xy^2z = 3x + z^2$ and $3x^2 - y^2 + 2z = 1$ at the point $(1,-2,1)$. **Ans:** $cos^{-1}\left[\frac{-3}{7\sqrt{6}}\right]$

8. Find the equation of the tangent plane to the surfaces $2xz^2 - 3xy - 4x = 7$ at the point $(1,-1,2)$. **Ans:** $7x - 3y + 8z - 26 = 0$

9. Find the equation of the tangent plane to the surfaces $xz^2 + x^2y = z - 1$ at the point $(1,-3,2)$. **Ans:** $2x - y - 3z + 1 = 0$

10. Find the angle between the surfaces $x\log z = y^2 - 1$ and $x^2y = 2 - z$ at the point $(1,1,1)$. **Ans:** $cos^{-1}\left[\frac{1}{\sqrt{30}}\right]$

2.2 DIVERGENCE, CURL – IRROTATIONAL AND SOLENOIDAL VECTORS

Divergence of a vector function

If $\vec{F}(x,y,z)$ is a continuously differentiable vector point function in a given region of space, then the divergence of $\vec{F}$ is defined by

$$\nabla.\vec{F} = \text{div}\,\vec{F} = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right).(F_1\vec{i} + F_2\vec{j} + F_3\vec{k})$$

$$\text{div}\vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} \quad \text{where}\vec{F} = F_1\vec{i} + F_2\vec{j} + F_3\vec{k}$$

Note: $\nabla.\vec{F}$ Is a scalar point function.

Solenoidal vector

A vector $\vec{F}$ is said to be solenoidal if $div\,\vec{F} = 0$ (i.e)$\nabla.\vec{F} = 0$

Curl of a vector function

If $\vec{F}(x,y,z)$ is a differentiable vector point function defines at each point (x,y,z) in some region of space, then the curl of $\vec{F}$ is defined by

$$\text{Curl}\,\vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

$$= \vec{i}\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) - \vec{j}\left(\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z}\right) + \vec{k}\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)$$

Where $\vec{F} = F_1\vec{i} + F_2\vec{j} + F_3\vec{k}$

Note: $\nabla \times \vec{F}$ Is a vector point function.

Irrotational vector

A vector is said to be irrotational if Curl $\vec{F} = 0$ $(i.e)$ $\nabla \times \vec{F} = 0$

Scalar potential

If $\vec{F}$ is an irrotational vector, then there exists a scalar function φ such that $\vec{F} = \nabla\varphi$. Such a scalar function is called scalar potential of $\vec{F}$.

Problems based on Divergence and Curl of a vector

Example: 2.21 If $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$ then find div $\vec{r}$ and curl$\vec{r}$

Solution:

Given $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$

Now div $\vec{r} = \nabla \cdot \vec{r}$

$$= \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z)$$

$$= 1 + 1 + 1 = 3$$

And curl $\vec{r} = \nabla \times \vec{r}$

$$\nabla \times \vec{r} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix}$$

$$= \vec{i}\left(\frac{\partial}{\partial y}(z) - \frac{\partial}{\partial z}(y)\right) - \vec{j}\left(\frac{\partial}{\partial x}(z) - \frac{\partial}{\partial z}(x)\right) + \vec{k}\left(\frac{\partial}{\partial x}(y) - \frac{\partial}{\partial y}(x)\right)$$

$$= \vec{i}(0) + \vec{j}(0) + \vec{k}(0) = \vec{0}.$$

Example: 2.22 If $\vec{F} = xy^2\vec{i} + 2x^2yz\vec{j} - 3yz^2\vec{k}$ find $\nabla.\vec{F}$ and $\nabla \times \vec{F}$ at the point (1,-1, 1).

Solution:

Given $\vec{F} = xy^2\vec{i} + 2x^2yz\vec{j} - 3yz^2\vec{k}$

(i) $\nabla.\vec{F} = \frac{\partial}{\partial x}(xy^2) + \frac{\partial}{\partial y}(2x^2yz) + \frac{\partial}{\partial z}(-3yz^2)$

$$= y^2 + 2x^2z - 6yz$$

$$\nabla.\vec{F}_{(1,-1,1)} = 1 + 2 + 6 = 9$$

(ii) $$\nabla \times \vec{r} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy^2 & 2x^2yz & 3yz^2 \end{vmatrix}$$

$$= \vec{i}\left[\frac{\partial(-3yz^2)}{\partial y} - \frac{\partial(2x^2yz)}{\partial z}\right] - \vec{j}\left[\frac{\partial(-3yz^2)}{\partial x} - \frac{\partial(xy^2)}{\partial z}\right] + \vec{k}\left[\frac{\partial(2x^2yz)}{\partial x} - \frac{\partial(xy^2)}{\partial y}\right]$$

$$= \vec{i}(-3z^2 - 2x^2y) - \vec{j}(0) + \vec{k}(4xyz - 2xy)$$

$$\nabla \times \vec{F}_{(1,-1,1)} = \vec{i}(-3 + 2) + \vec{k}(-4 + 2)$$

$$= -\vec{i} - 2\vec{k}$$

Example: 2.23 If $\vec{F} = (x^2 - y^2 + 2xz)\vec{i} + (xz - xy + yz)\vec{j} + (z^2 + x^2)\vec{k}$, then find $\nabla \cdot \vec{F}$, $\nabla(\nabla \cdot \vec{F})$, $\nabla \times \vec{F}$, $\nabla \cdot (\nabla \times \vec{F})$, and $\nabla \times (\nabla \times \vec{F})$ at the point (1,1,1).

Solution:

Given $\vec{F} = (x^2 - y^2 + 2xz)\vec{i} + (xz - xy + yz)\vec{j} + (z^2 + x^2)\vec{k}$

(i) $\nabla \cdot \vec{F} = \frac{\partial}{\partial x}(x^2 - y^2 + 2xz) + \frac{\partial}{\partial y}(xz - xy + yz) + \frac{\partial}{\partial z}(z^2 + x^2)$

$$= (2x + 2z) + (-x + z) + 2z$$

$$= x + 5z$$

$\therefore \nabla \cdot \vec{F}_{(1,1,1)} = 6$

(ii) $\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 - y^2 + 2xz & xz - xy + yz & z^2 + x^2 \end{vmatrix}$

$$= \vec{i}\left[\frac{\partial(z^2+x^2)}{\partial y} - \frac{\partial(xz-xy+yz)}{\partial z}\right] - \vec{j}\left[\frac{\partial(z^2+x^2)}{\partial x} - \frac{\partial(x^2-y^2+2xz)}{\partial z}\right] + \vec{k}\left[\frac{\partial(xz-xy+yz)}{\partial x} - \frac{\partial(x^2-y^2+2xz)}{\partial y}\right]$$

$$= -(x+y)\vec{i} - (2x - 2x)\vec{j} + (y + z)\vec{k}$$

$\therefore \nabla \times \vec{F}_{(1,1,1)} = -2\vec{i} + 2\vec{k}$

(iii) $\nabla(\nabla \cdot \vec{F}) = \vec{i}\frac{\partial}{\partial x}(x + 5z) + \vec{j}\frac{\partial}{\partial y}(x + 5z) + \vec{k}\frac{\partial}{\partial z}(x + 5z)$

$$= \vec{i} + 5\vec{k}$$

$\therefore \nabla(\nabla \cdot \vec{F})_{(1,1,1)} = \vec{i} + 5\vec{k}$

(iv) $\nabla \cdot (\nabla \times \vec{F}) = \frac{\partial}{\partial x}(-(x + y)) + \frac{\partial}{\partial y}(0) + \frac{\partial}{\partial z}(y + z)$

$$= -1 + 0 + 1$$

$\nabla \cdot (\nabla \times \vec{F})_{(1,1,1)} = 0$

(v) $\nabla \times (\nabla \times \vec{F}) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -(x + y) & 0 & y + z \end{vmatrix}$

$\therefore \nabla \times (\nabla \times \vec{F})_{(1,1,1)} = \vec{i} + \vec{k}$

Example: 2.24 Find div $\vec{F}$ and curl $\vec{F}$, where $\vec{F} = \text{grad}(x^3+y^3+z^3-3xyz)$

Solution:

Given $\vec{F} = \text{grad}(x^3+y^3+z^3-3xyz)$

$$= \vec{i}\frac{\partial}{\partial x}(x^3 + y^3 + z^3 - 3xyz) + \vec{j}\frac{\partial}{\partial y}(x^3 + y^3 + z^3 - 3xyz) + \vec{k}\frac{\partial}{\partial z}(x^3 + y^3 + z^3 - 3xyz)$$

$\vec{F} = \vec{i}(3x^2 - 3yz) + \vec{j}(3y^2 - 3xz) + \vec{k}(3z^2 - 3xy)$

Now div $\vec{F} = \nabla \cdot \vec{F} = \frac{\partial}{\partial x}(3x^2 - 3yz) + \frac{\partial}{\partial y}(3y^2 - 3xz) + \frac{\partial}{\partial z}(3z^2 - 3xy)$

$$= 6x + 6y + 6z$$

$$= 6(x + y + z)$$

Curl $\vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3x^2 - 3yz & 3y^2 - 3xz & 3z^2 - 3xy \end{vmatrix}$

$$= \vec{i}[-3x + 3x] - \vec{j}[-3y + 3y] + \vec{k}[-3z + 3z]$$

$$= \vec{0}$$

Example: 2.25 Find div(grad φ) and curl(grad φ) at (1,1,1) for $\varphi = x^2y^3z^4$

Solution:

Given $\varphi = x^2y^3z^4$

$$\text{grad}\,\varphi = \nabla\varphi = \vec{i}\frac{\partial\varphi}{\partial x} + \vec{j}\frac{\partial\varphi}{\partial y} + \vec{k}\frac{\partial\varphi}{\partial z}$$

$$= \vec{i}(2xy^3z^4) + \vec{j}(x^2 3y^2z^4) + \vec{k}(x^2y^3 4z^3)$$

$$\text{Div}(\text{grad}\,\varphi) = \nabla\cdot(\text{grad}\,\varphi)$$

$$= \frac{\partial}{\partial x}(2xy^3z^4) + \frac{\partial}{\partial y}(x^2 3y^2z^4) + \frac{\partial}{\partial z}(x^2y^3 4z^3)$$

$$= 2y^3z^4 + 6x^2yz^4 + 12x^2y^3z^4$$

$$\therefore \text{Div}(\text{grad}\,\varphi)_{(1,1,1)} = 2 + 6 + 12 = 20$$

$$\text{Curl}(\text{grad}\,\varphi) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xy^3z^4 & x^2 3y^2z^4 & x^2y^3 4z^3 \end{vmatrix}$$

$$= \vec{i}(12x^2y^2z^3 - 12x^2y^2z^3) - \vec{j}(8xy^3z^3 - 8xy^3z^3) + \vec{k}(6xy^2z^4 - 6xy^2z^4)$$

$$= \vec{0}$$

$$\therefore \text{Curl grad}\varphi_{(1,1,1)} = \vec{0}$$

Vector Identities

1) $\nabla\cdot(\varphi\vec{F}) = \varphi(\nabla\cdot\vec{F}) + \vec{F}\cdot\nabla\varphi$
2) $\nabla\times(\varphi\vec{F}) = \varphi(\nabla\times\vec{F}) + (\nabla\varphi)\times\vec{F}$
3) $\nabla\cdot(\vec{A}\times\vec{B}) = \vec{B}\cdot(\nabla\times\vec{A}) - \vec{A}\cdot(\nabla\times\vec{B})$
4) $\nabla\times(\vec{A}\times\vec{B}) = \vec{A}(\nabla\cdot\vec{B}) - \vec{B}(\nabla\cdot\vec{A}) + (\vec{B}\cdot\nabla)\vec{A} - (\vec{A}\cdot\nabla)\vec{B}$
5) $\nabla(\vec{A}\cdot\vec{B}) = \vec{A}\times(\nabla\times\vec{B}) - (\vec{A}\cdot\nabla)\vec{B} + \vec{B}\times(\nabla\times\vec{A}) - (\vec{B}\cdot\nabla)\vec{A}$
6) $\nabla\cdot(\nabla\varphi) = \vec{0}$
7) $\nabla\cdot(\nabla\times\vec{F}) = 0$
8) $\nabla\times(\nabla\times\vec{F}) = \nabla(\nabla\cdot\vec{F}) - \nabla^2\vec{F}$
9) $\nabla\cdot\nabla\varphi = (\nabla\cdot\nabla)\varphi = \nabla^2\varphi$ where $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ is a laplacian operator

1) If φ is a scalar point function, $\vec{F}$ is a vector point function, then $\nabla.(\varphi\vec{F}) = \varphi(\nabla\cdot\vec{F}) + \vec{F}\cdot\nabla\varphi$

Proof:

$$\nabla\cdot(\varphi\vec{F}) = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)\cdot(\varphi\vec{F})$$

$$= \sum\vec{i}\cdot\frac{\partial}{\partial x}(\varphi\vec{F})$$

$$= \sum\vec{i}\cdot\left(\varphi\frac{\partial\vec{F}}{\partial x} + \vec{F}\frac{\partial\varphi}{\partial x}\right)$$

$$= \varphi\left(\sum \vec{i} \cdot \frac{\partial \vec{F}}{\partial x} + \vec{F}\frac{\partial \varphi}{\partial x}\right) + \vec{F} \cdot \left(\sum \vec{i}\frac{\partial \varphi}{\partial x}\right)$$

$$\therefore \nabla \cdot (\varphi\,\vec{F}) = \varphi(\nabla \cdot \vec{F}) + \vec{F} \cdot \nabla\varphi$$

2) If $\boldsymbol{\varphi}$ is a scalar point fuction, $\vec{\mathbf{F}}$ is a vector point function, then $\boldsymbol{\nabla} \times (\boldsymbol{\varphi}\,\vec{\mathbf{F}}) = \boldsymbol{\varphi}(\boldsymbol{\nabla} \times \vec{\mathbf{F}}) + (\boldsymbol{\nabla\varphi}) \times \vec{\mathbf{F}}$

Proof:

$$\nabla \times (\varphi\,\vec{F}) = \sum \vec{i} \times \frac{\partial}{\partial x}(\varphi\,\vec{F})$$

$$= \sum \vec{i} \times \left[\varphi\frac{\partial \vec{F}}{\partial x} + \vec{F}\frac{\partial \varphi}{\partial x}\right]$$

$$= \sum \vec{i} \times \left(\frac{\partial \varphi}{\partial x}\vec{F} + \varphi\frac{\partial \vec{F}}{\partial x}\right)$$

$$= \left(\sum \vec{i}\frac{\partial \varphi}{\partial x}\right) \times \vec{F} + \varphi\left[\sum \vec{i} \times \frac{\partial \vec{F}}{\partial x}\right]$$

$$\therefore \nabla \times (\varphi\,\vec{F}) = \nabla\varphi \times \vec{F} + \varphi(\nabla \times \vec{F})$$

3) If $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$ are vector point functions, then $\boldsymbol{\nabla} \cdot (\vec{\mathbf{A}} \times \vec{\mathbf{B}}) = \vec{\mathbf{B}} \cdot (\boldsymbol{\nabla} \times \vec{\mathbf{A}}) - \vec{\mathbf{A}} \cdot (\boldsymbol{\nabla} \times \vec{\mathbf{B}})$

Proof:

$$\nabla.(\vec{A} \times \vec{B}) = \sum \vec{i} \cdot \frac{\partial}{\partial x}(\vec{A} \times \vec{B})$$

$$= \sum \vec{i} \cdot \left(\vec{A} \times \frac{\partial \vec{B}}{\partial x} + \frac{\partial \vec{A}}{\partial x} \times \vec{B}\right)$$

$$= \sum \vec{i} \cdot \left(\vec{A} \times \frac{\partial \vec{B}}{\partial x}\right) + \sum \vec{i} \cdot \left(\frac{\partial \vec{A}}{\partial x} \times \vec{B}\right)$$

$$= -\left(\sum \vec{i} \times \frac{\partial \vec{B}}{\partial x}\right) \cdot \vec{A} + \left(\sum \vec{i} \times \frac{\partial \vec{A}}{\partial x}\right).\vec{B}$$

$$= -(\nabla \times \vec{B}).\vec{A} + (\nabla \times \vec{A}) \cdot \vec{B}$$

$$\therefore \nabla \cdot (\vec{A} \times \vec{B}) = \vec{B} \cdot (\nabla \times \vec{A}) - \vec{A} \cdot (\nabla \times \vec{B}) \qquad [\because (\nabla \times \vec{A}) \cdot \vec{B} = \vec{B} \cdot (\nabla \times \vec{A})]$$

(4) If $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$ are vector point functions, then

$\boldsymbol{\nabla} \times (\vec{\mathbf{A}} \times \vec{\mathbf{B}}) = \vec{\mathbf{A}}(\boldsymbol{\nabla} \cdot \vec{\mathbf{B}}) - \vec{\mathbf{B}}(\boldsymbol{\nabla} \cdot \vec{\mathbf{A}}) + (\vec{\mathbf{B}} \cdot \boldsymbol{\nabla})\vec{\mathbf{A}} - (\vec{\mathbf{A}} \cdot \boldsymbol{\nabla})\vec{\mathbf{B}}$

Proof:

$$\nabla \times (\vec{A} \times \vec{B}) = \sum \vec{i} \times \frac{\partial}{\partial x}(\vec{A} \times \vec{B})$$

$$= \sum \vec{i} \times \left(\frac{\partial \vec{A}}{\partial x} \times \vec{B} + \vec{A} \times \frac{\partial \vec{B}}{\partial x}\right)$$

$$= \sum \vec{i} \times \left(\frac{\partial \vec{A}}{\partial x} \times \vec{B}\right) + \sum \vec{i} \times \left(\vec{A} \times \frac{\partial \vec{B}}{\partial x}\right)$$

We know that $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\,\vec{b} - (\vec{a} \cdot \vec{b})\,\vec{c}$

$$\nabla \times (\vec{A} \times \vec{B}) = \sum\left[(\vec{i} \cdot \vec{B})\frac{\partial \vec{A}}{\partial x} - \left(\vec{i} \cdot \frac{\partial \vec{A}}{\partial x}\right)\vec{B}\right] + \sum\left[\left(\vec{i} \cdot \frac{\partial \vec{B}}{\partial x}\right)\vec{A} - (\vec{i} \cdot \vec{A})\frac{\partial \vec{B}}{\partial x}\right]$$

$$= \left(\sum \vec{i} \cdot \frac{\partial \vec{B}}{\partial x}\right)\vec{A} - \left(\sum \vec{i} \cdot \frac{\partial \vec{A}}{\partial x}\right)\vec{B} + \sum(\vec{B} \cdot \vec{i})\frac{\partial \vec{A}}{\partial x} - \sum(\vec{A} \cdot \vec{i})\frac{\partial \vec{B}}{\partial x}$$

$$= \left(\sum \vec{i} \cdot \frac{\partial \vec{B}}{\partial x}\right)\vec{A} - \left(\sum \vec{i} \cdot \frac{\partial \vec{A}}{\partial x}\right)\vec{B} + \left(\vec{B} \cdot \sum \vec{i}\frac{\partial}{\partial x}\right)\vec{A} - \left(\vec{A} \cdot \sum \vec{i}\frac{\partial}{\partial x}\right)\vec{B}$$

$$\therefore \nabla \times (\vec{A} \times \vec{B}) = \vec{A}(\nabla \cdot \vec{B}) - \vec{B}(\nabla \cdot \vec{A}) + (\vec{B} \cdot \nabla)\vec{A} - (\vec{A} \cdot \nabla)\vec{B}$$

(5) If $\vec{A}$ and $\vec{B}$ are vector point functions, then

$$\nabla(\vec{A}\cdot\vec{B}) = \vec{A}\times(\nabla\times\vec{B}) + (\vec{A}\cdot\nabla)\vec{B} + \vec{B}\times(\nabla\times\vec{A}) + (\vec{B}\cdot\nabla)\vec{A}$$

Proof:

$$\nabla(\vec{A}\cdot\vec{B}) = \sum \vec{i}\,\frac{\partial}{\partial x}(\vec{A}\cdot\vec{B})$$

$$= \sum \vec{i}\left(\frac{\partial\vec{A}}{\partial x}\cdot\vec{B} + \vec{A}\cdot\frac{\partial\vec{B}}{\partial x}\right)$$

$$= \sum \vec{i}\left(\vec{B}\cdot\frac{\partial\vec{A}}{\partial x}\right) + \sum \vec{i}\left(\vec{A}\cdot\frac{\partial\vec{B}}{\partial x}\right)$$

$$= \sum\left(\vec{B}\cdot\frac{\partial\vec{A}}{\partial x}\right)\vec{i} + \sum\left(\vec{A}\cdot\frac{\partial\vec{B}}{\partial x}\right)\vec{i} \quad \dots (1)$$

We know that $\vec{a}\times(\vec{b}\times\vec{c}) = (\vec{a}\cdot\vec{c})\,\vec{b} - (\vec{a}\cdot\vec{b})\,\vec{c}$

$\therefore (\vec{a}\cdot\vec{b})\,\vec{c} = (\vec{a}\cdot\vec{c})\,\vec{b} - \vec{a}\times(\vec{b}\times\vec{c})$

Consider $\sum\left(\vec{B}\cdot\frac{\partial\vec{A}}{\partial x}\right)\vec{i} = \sum\left[(\vec{B}\cdot\vec{i})\frac{\partial\vec{A}}{\partial x} - \vec{B}\times\left(\frac{\partial\vec{A}}{\partial x}\times\vec{i}\right)\right]$

$$= \sum\left(\vec{B}\cdot\vec{i}\,\frac{\partial}{\partial x}\right)\vec{A} + \sum\left[\vec{B}\times\left(\vec{i}\times\frac{\partial\vec{A}}{\partial x}\right)\right]$$

$$= (\vec{B}\cdot\nabla)\vec{A} + \sum\left[\vec{B}\times\left(\vec{i}\,\frac{\partial}{\partial x}\times\vec{A}\right)\right]$$

$$= (\vec{B}\cdot\nabla)\vec{A} + \vec{B}\times(\nabla\times\vec{A}) \quad \dots (2)$$

In (2) interchanging $\vec{A}$ and $\vec{B}$ we get,

$$\sum\left(\vec{A}\cdot\frac{\partial\vec{B}}{\partial x}\right)\vec{i} = (\vec{A}\cdot\nabla)\vec{B} + \vec{A}\times(\nabla\times\vec{B}) \quad \dots (3)$$

Substitute in equation (1)

$$(1) \Rightarrow \nabla(\vec{A}\cdot\vec{B}) = (\vec{B}\cdot\nabla)\vec{A} + \vec{B}\times(\nabla\times\vec{A}) + (\vec{A}\cdot\nabla)\vec{B} + \vec{A}\times(\nabla\times\vec{B})$$

(6) If φ is a scalar point function, then $\nabla\times(\nabla\varphi) = \vec{0}$.

(or)

Prove that $curl(grad\,\varphi) = 0$.

Solution:

$$\nabla\varphi = \vec{i}\frac{\partial\varphi}{\partial x} + \vec{j}\frac{\partial\varphi}{\partial y} + \vec{k}\frac{\partial\varphi}{\partial z}$$

$$\nabla\times\nabla\varphi = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial\varphi}{\partial x} & \frac{\partial\varphi}{\partial y} & \frac{\partial\varphi}{\partial z} \end{vmatrix}$$

$$= \sum\vec{i}\left[\frac{\partial^2\varphi}{\partial y\partial z} - \frac{\partial^2\varphi}{\partial z\partial y}\right]$$

$$= \sum\vec{i}\,(\vec{0}) = \vec{0}$$

(7) If $\vec{F}$ is a vector point function, then $\nabla\cdot(\nabla\times\vec{F}) = 0$.

(or)

Prove that $div(curl\,\vec{F}) = 0$.

Solution:

Let $\vec{F} = F_1\vec{i} + F_2\vec{j} + F_3\vec{k}$

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

$$= \vec{i}\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) - \vec{j}\left(\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z}\right) + \vec{k}\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)$$

$$\nabla \cdot (\nabla \times \vec{F}) = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right) \cdot$$

$$\left[\vec{i}\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) - \vec{j}\left(\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z}\right) + \vec{k}\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\right]$$

$$= \frac{\partial^2 F_3}{\partial x \partial y} - \frac{\partial^2 F_2}{\partial x \partial z} - \frac{\partial^2 F_3}{\partial y \partial x} + \frac{\partial^2 F_1}{\partial y \partial z} + \frac{\partial^2 F_2}{\partial z \partial x} - \frac{\partial^2 F_1}{\partial z \partial y}$$

$$= 0$$

(8) If is a vector point function, then $\nabla \times (\nabla \times \vec{F}) = \nabla(\nabla \cdot \vec{F}) - \nabla^2\vec{F}$

(or)

Prove that $curl\,(curl\,\vec{F}) = grad\,(div\,\vec{F}) - \nabla^2\vec{F}$

Solution:

Let $\vec{F} = F_1\vec{i} + F_2\vec{j} + F_3\vec{k}$

$$\nabla \times (\nabla \times \vec{F}) = \vec{i}\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) - \vec{j}\left(\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z}\right) + \vec{k}\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)$$

And $\nabla \cdot \vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$

L.H.S $\nabla \times (\nabla \times \vec{F}) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} & -\frac{\partial F_3}{\partial x} + \frac{\partial F_1}{\partial z} & \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \end{vmatrix}$

$$= \vec{i}\left[\frac{\partial^2 F_2}{\partial y \partial x} - \frac{\partial^2 F_1}{\partial y^2} - \frac{\partial^2 F_3}{\partial z \partial x} + \frac{\partial^2 F_1}{\partial z^2}\right] - \vec{j}\left[\frac{\partial^2 F_2}{\partial x^2} - \frac{\partial^2 F_1}{\partial x \partial y} - \frac{\partial^2 F_3}{\partial z \partial y} + \frac{\partial^2 F_2}{\partial z^2}\right]$$

$$+\vec{k}\left[-\frac{\partial^2 F_3}{\partial x^2} + \frac{\partial^2 F_1}{\partial x \partial z} - \frac{\partial^2 F_3}{\partial y^2} + \frac{\partial^2 F_2}{\partial y \partial z}\right]$$

R.H.S $\nabla(\nabla \cdot \vec{F}) - \nabla^2\vec{F}$

$$= \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)\left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right) - \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)(F_1\vec{i} + F_2\vec{j} + F_3\vec{k})$$

$$= \vec{i}\left[\frac{\partial^2 F_1}{\partial x^2} + \frac{\partial^2 F_2}{\partial x \partial y} + \frac{\partial^2 F_3}{\partial x \partial z}\right] + \vec{j}\left[\frac{\partial^2 F_1}{\partial y \partial x} + \frac{\partial^2 F_2}{\partial y^2} + \frac{\partial^2 F_3}{\partial y \partial z}\right] + \vec{k}\left[\frac{\partial^2 F_1}{\partial z \partial x} + \frac{\partial^2 F_2}{\partial z \partial y} + \frac{\partial^2 F_3}{\partial z^2}\right]$$

$$- \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)(F_1\vec{i} + F_2\vec{j} + F_3\vec{k})$$

$$= \vec{i}\left[\frac{\partial^2 F_2}{\partial x \partial y} + \frac{\partial^2 F_3}{\partial x \partial z} - \frac{\partial^2 F_1}{\partial y^2} - \frac{\partial^2 F_1}{\partial z^2}\right] - \vec{j}\left[\frac{\partial^2 F_2}{\partial x^2} - \frac{\partial^2 F_1}{\partial x \partial y} - \frac{\partial^2 F_3}{\partial z \partial y} + \frac{\partial^2 F_2}{\partial z^2}\right] +$$

$$\vec{k}\left[-\frac{\partial^2 F_3}{\partial x^2} + \frac{\partial^2 F_1}{\partial x \partial z} - \frac{\partial^2 F_3}{\partial y^2} + \frac{\partial^2 F_2}{\partial y \partial z}\right]$$

$$\text{L.H.S} = \text{R.H.S}$$

$$\therefore \nabla \times (\nabla \times \vec{F}) = \nabla(\nabla \cdot \vec{F}) - \nabla^2 \vec{F}$$

(9) $\boldsymbol{\nabla \cdot (\nabla\varphi) = (\nabla \cdot \nabla)\, \varphi = \nabla^2 \varphi}$

Proof:

$$\nabla\varphi = \vec{\imath}\frac{\partial\varphi}{\partial x} + \vec{\jmath}\frac{\partial\varphi}{\partial y} + \vec{k}\frac{\partial\varphi}{\partial z}$$

$$\therefore \nabla \cdot (\nabla\varphi) = \frac{\partial}{\partial x}\left(\frac{\partial\varphi}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial\varphi}{\partial y}\right) + \frac{\partial}{\partial z}\left(\frac{\partial\varphi}{\partial z}\right)$$

$$= \frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} + \frac{\partial^2\varphi}{\partial z^2}$$

$$\nabla \cdot \nabla = \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

$$\nabla \cdot (\nabla\varphi) = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\varphi = \nabla^2\varphi$$

Example: 2.26 Find (i) $\boldsymbol{\nabla \cdot \vec{r}}$ **(ii)** $\boldsymbol{\nabla \times \vec{r}}$

Solution:

$$\text{Let } \vec{r} = x\,\vec{\imath} + y\,\vec{\jmath} + z\,\vec{k}$$

(i) $$\nabla \cdot \vec{r} = \left(\vec{\imath}\,\frac{\partial}{\partial x} + \vec{\jmath}\,\frac{\partial}{\partial y} + \vec{k}\,\frac{\partial}{\partial z}\right) \cdot \left(x\,\vec{\imath} + y\,\vec{\jmath} + z\,\vec{k}\right)$$

$$= \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z)$$

$$= 1 + 1 + 1 = 3$$

(ii) $$\nabla \times \vec{r} = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix}$$

$$= \vec{\imath}(0) + \vec{\jmath}\,(0) + \vec{k}\,(0) = \vec{0}$$

Example: 2.27 Find $\boldsymbol{\nabla \cdot \left(\frac{1}{r}\,\vec{r}\right)}$ **where** $\boldsymbol{\vec{r} = x\,\vec{\imath} + y\,\vec{\jmath} + z\,\vec{k}}$

Solution:

$$\nabla \cdot \left(\frac{1}{r}\,\vec{r}\right) = \nabla \cdot \left[\frac{1}{r}\left(x\,\vec{\imath} + y\,\vec{\jmath} + z\,\vec{k}\right)\right]$$

$$= \left(\vec{\imath}\,\frac{\partial}{\partial x} + \vec{\jmath}\frac{\partial}{\partial y} + \vec{k}\,\frac{\partial}{\partial z}\right) \cdot \left(\frac{x}{r}\,\vec{\imath} + \frac{y}{r}\,\vec{\jmath} + \frac{z}{r}\,\vec{k}\right)$$

$$= \sum \frac{\partial}{\partial x}\left(\frac{x}{r}\right)$$

$$= \sum\left[\frac{1}{r}(1) + x\left(-\frac{1}{r^2}\right)\frac{\partial r}{\partial x}\right]$$

$$= \sum\left[\frac{1}{r} - \frac{x}{r^2}\left(\frac{x}{r}\right)\right] \qquad \left(\because \frac{\partial r}{\partial x} = \frac{x}{r}\right)$$

$$= \sum\left[\frac{1}{r} - \frac{x^2}{r^3}\right]$$

$$= \frac{3}{r} - \frac{1}{r^3}\left(x^2 + y^2 + z^2\right)$$

$$= \frac{3}{r} - \frac{r^2}{r^3} \qquad \because r^2 = (x^2 + y^2 + z^2)$$

$$= \frac{3}{r} - \frac{1}{r} = \frac{2}{r}$$

Example: 2.28 If is a constant vector and is the position vector of any point, prove that

(i) $\nabla \cdot (\vec{a} \times \vec{r}) = 0$ (ii) $\nabla \times (\vec{a} \times \vec{r}) = 2\vec{a}$

Solution:

Let $\vec{r} = x\,\vec{\imath} + y\,\vec{\jmath} + z\,\vec{k}$

$$\vec{a} = a_1\,\vec{\imath} + a_2\,\vec{\jmath} + a_3\,\vec{k}$$

$$\vec{a} \times \vec{r} = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ a_1 & a_2 & a_3 \\ x & y & z \end{vmatrix}$$

$$= \vec{\imath}(a_2 z - a_3 y) - \vec{\jmath}(a_1 z - a_3 x) + \vec{k}(a_1 y - a_2 x)$$

(i) $\nabla \cdot (\vec{a} \times \vec{r}) = \frac{\partial}{\partial x}(a_2 z - a_3 y) + \frac{\partial}{\partial y}(-a_1 z + a_3 x) + \frac{\partial}{\partial z}(a_1 y - a_2 x)$

$$= 0 + 0 + 0 = 0$$

(ii) $\nabla \times (\vec{a} \times \vec{r}) = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ a_2 z - a_3 y & -a_1 z + a_3 x & a_1 y - a_3 x \end{vmatrix}$

`$= \vec{\imath}(a_1 + a_1) - \vec{\jmath}(-a_2 - a_2) + \vec{k}(a_3 + a_3)$

$$= 2a_1\vec{\imath} + 2a_2\vec{\jmath} + 2a_3\vec{k}$$

$$= 2\left(a_1\vec{\imath} + a_2\vec{\jmath} + a_3\vec{k}\right) = 2\vec{a}$$

Example: 2.29 Prove that $curl(f(r)\vec{r}) = \vec{0}$

Solution:

Let $f(r)\vec{r} = f(r)\left[x\,\vec{\imath} + y\,\vec{\jmath} + z\,\vec{k}\right]$

$$= xf(r)\vec{\imath} + yf(r)\vec{\jmath} + zf(r)\vec{k}$$

$$\nabla \times (f(r)\vec{r}) = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xf(r) & yf(r) & zf(r) \end{vmatrix}$$

$$= \sum \vec{\imath}\left[zf'(r)\frac{\partial r}{\partial y} - yf'(r)\frac{\partial r}{\partial z}\right]$$

$$= \sum \vec{\imath}\left[zf'(r)\left(\frac{y}{r}\right) - yf'(r)\left(\frac{z}{r}\right)\right]$$

$$= \sum \vec{\imath}\left[\frac{zy}{r}f'(r) - \frac{zy}{r}f'(r)\right]$$

$$= \sum \vec{\imath}\,(0)$$

$$= 0\,\vec{\imath} + 0\,\vec{\jmath} + 0\,\vec{k} = \vec{0}$$

Example: 2.30 Prove that $curl[\varphi\,\nabla\varphi] = \vec{0}$

(or)

Prove that $\nabla \times [\varphi\,\nabla\varphi] = \vec{0}$

Solution:

$$\varphi\nabla\varphi = \varphi\left[\vec{i}\frac{\partial\varphi}{\partial x} + \vec{j}\frac{\partial\varphi}{\partial y} + \vec{k}\frac{\partial\varphi}{\partial z}\right]$$

$$= \vec{i}\left(\varphi\frac{\partial\varphi}{\partial x}\right) + \vec{j}\left(\varphi\frac{\partial\varphi}{\partial y}\right) + \vec{k}\left(\varphi\frac{\partial\varphi}{\partial z}\right)$$

$$\nabla \times (\varphi\nabla\varphi) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \varphi\frac{\partial\varphi}{\partial x} & \varphi\frac{\partial\varphi}{\partial y} & \varphi\frac{\partial\varphi}{\partial z} \end{vmatrix}$$

$$= \sum\vec{i}\left[\frac{\partial}{\partial y}\left(\varphi\frac{\partial\varphi}{\partial z}\right) - \frac{\partial}{\partial z}\left(\varphi\frac{\partial\varphi}{\partial y}\right)\right]$$

$$= \sum\vec{i}\left[\varphi\frac{\partial^2\varphi}{\partial y\partial z} + \frac{\partial\varphi}{\partial y}\cdot\frac{\partial\varphi}{\partial z} - \varphi\frac{\partial^2\varphi}{\partial z\partial y} - \frac{\partial\varphi}{\partial y}\cdot\frac{\partial\varphi}{\partial z}\right]$$

$$= \sum\vec{i}\,(0)$$

$$= 0\,\vec{i} + 0\,\vec{j} + 0\,\vec{k} = \vec{0}$$

Example: 2.31 If $\vec{\omega}$ is a constant vector and $\vec{v} = \vec{\omega}\times\vec{r}$, then prove that $\vec{\omega} = \frac{1}{2}(\nabla\times\vec{v})$.

Solution:

Let $\vec{r} = x\,\vec{i} + y\,\vec{j} + z\,\vec{k}$

$\vec{\omega} = \omega_1\,\vec{i} + \omega_2\,\vec{j} + \omega_3\,\vec{k}$

$$\vec{\omega}\times\vec{r} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \omega_1 & \omega_2 & \omega_3 \\ x & y & z \end{vmatrix}$$

$$= \vec{i}(\omega_2 z - \omega_3 y) - \vec{j}(\omega_1 z - \omega_3 x) + \vec{k}(\omega_1 y - \omega_2 x)$$

$$\nabla\times\vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \omega_2 z - \omega_3 y & -\omega_1 z + \omega_3 x & \omega_1 y - \omega_2 x \end{vmatrix}$$

$$= \vec{i}(\omega_1 + \omega_1) - \vec{j}(-\omega_2 - \omega_2) + \vec{k}(\omega_3 + \omega_3)$$

$$= 2\omega_1\vec{i} + 2\omega_2\vec{j} + 2\omega_3\vec{k}$$

$$= 2\left(\omega_1\vec{i} + \omega_2\vec{j} + \omega_3\vec{k}\right) = 2\vec{\omega}$$

$$\vec{\omega} = \frac{1}{2}(\nabla\times\vec{v})$$

Problems based on solenoidal vector and irrotational vector and scalar potential

Example: 2.32 Prove that the vector $\vec{F} = z\,\vec{i} + x\,\vec{j} + y\,\vec{k}$ is solenoidal.

Solution:

Given $\vec{F} = z\,\vec{i} + x\,\vec{j} + y\,\vec{k}$

To prove $\nabla\cdot\vec{F} = 0$

$$\nabla\cdot\vec{F} = \frac{\partial}{\partial x}(z) + \frac{\partial}{\partial y}(x) + \frac{\partial}{\partial z}(y)$$

$$= 0$$

$\therefore\ \vec{F}$ is solenoidal.

Example: 2.33 Show that the vector $\vec{F} = 3y^4z^2\vec{\imath} + 4x^3z^2\vec{\jmath} - 3x^2y^2\vec{k}$ is solenoidal.

Solution:

Given $\vec{F} = 3y^4z^2\vec{\imath} + 4x^3z^2\vec{\jmath} - 3x^2y^2\vec{k}$

To prove $\nabla \cdot \vec{F} = 0$

$$\nabla \cdot \vec{F} = \frac{\partial}{\partial x}(3y^4z^2) + \frac{\partial}{\partial y}(4x^3z^2) + \frac{\partial}{\partial z}(3x^2y^2)$$

$$= 0 + 0 + 0 = 0$$

$\therefore\ \vec{F}$ is solenoidal.

Example: 2.34 If $\vec{F} = (x + 3y)\vec{\imath} + (y - 2z)\vec{\jmath} + (x + \lambda z)\vec{k}$ is solenoidal, then find the value of λ.

Solution:

Given $\vec{F}$ is solenoidal.

$$(ie)\nabla \cdot \vec{F} = 0$$

$$\Rightarrow \frac{\partial}{\partial x}(x + 3y) + \frac{\partial}{\partial y}(y - 2z) + \frac{\partial}{\partial z}(x + \lambda z) = 0$$

$$\Rightarrow 1 + 1 + \lambda = 0$$

$$\therefore \lambda = -2$$

Example: 2.35 Find a such that $(3x - 2y + z)\vec{\imath} + (4x + ay - z)\vec{\jmath} + (x - y + 2z)\vec{k}$ is solenoidal.

Solution:

Given $(3x - 2y + z)\vec{\imath} + (4x + ay - z)\vec{\jmath} + (x - y + 2z)\vec{k}$ is solenoidal.

$$(ie)\nabla \cdot \vec{F} = 0$$

$$\Rightarrow \frac{\partial}{\partial x}(3x - 2y + z) + \frac{\partial}{\partial y}(4x + ay - z) + \frac{\partial}{\partial z}(x - y + 2z) = 0$$

$$\Rightarrow 3 + a + 2 = 0$$

$$\therefore a = -5$$

Example: 2.36 Show that the vector $\vec{F} = (6xy + z^3)\vec{\imath} + (3x^2 - z)\vec{\jmath} + (3xz^2 - y)\vec{k}$ is irrotational.

Solution:

Given $\vec{F} = (6xy + z^3)\vec{\imath} + (3x^2 - z)\vec{\jmath} + (3xz^2 - y)\vec{k}$

To prove $curl\,\vec{F} = 0$

$$(i.e) To\ prove\ \nabla \times \vec{F} = 0$$

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 6xy + z^3 & 3x^2 - 3z^2 & 3xz^2 - y \end{vmatrix}$$

$$= \vec{\imath}(-1 + 1) - \vec{\jmath}(3z^2 - 3z^2) + \vec{k}(6x - 6x) = \vec{0}$$

$\therefore\ \vec{F}$ is irrotational.

Example: 2.37 Find the constants a, b, c so that the vectors is irrotational

$\vec{F} = (x+2y+az)\vec{i} + (bx+3y-z)\vec{j} + (4x+cy+2z)\vec{k}$.

Solution:

Given $\vec{F} = (x+2y+az)\vec{i} + (bx+3y-z)\vec{j} + (4x+cy+2z)\vec{k}$ **is irrotational.**

$$(ie)\nabla\times\vec{F} = 0$$

$$\nabla\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x+2y+az & bx+3y-z & 4x+cy+2z \end{vmatrix} = \vec{0}$$

$$\Rightarrow \vec{i}(c+1) - \vec{j}(4-a) + \vec{k}(b-2) = \vec{0}$$

$$\Rightarrow c+1 = 0\ ; \qquad 4-a = 0\ ; \qquad b-2 = 0$$

$$\Rightarrow c = -1\ ; \qquad 4 = a\ ; \qquad b = 2$$

Example: 2.38 Prove that $\vec{F} = (6xy+z^3)\vec{i} + (3x^2-z)\vec{j} + (3xz^2-y)\vec{k}$ **is irrotational and find** φ **such that** $\vec{F} = \nabla\varphi$**.**

Solution:

$$\text{Given } \vec{F} = (6xy+z^3)\vec{i} + (3x^2-z)\vec{j} + (3xz^2-y)\vec{k}$$

To prove $\nabla\times\vec{F} = 0$

$$\nabla\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 6xy+z^3 & 3x^2-z & 3xz^2-y \end{vmatrix}$$

$$= \vec{i}(-1+1) - \vec{j}(3z^2-3z^2) + \vec{k}(6x-6x)$$

$$= \vec{0}$$

$\therefore\ \vec{F}$ is irrotational.

To find φ such that $\vec{F} = \nabla\varphi$.

$$\nabla\varphi = \vec{i}\frac{\partial\varphi}{\partial x} + \vec{j}\frac{\partial\varphi}{\partial y} + \vec{k}\frac{\partial\varphi}{\partial z}$$

Equating the coefficients of $\vec{i},\vec{j}$ and $\vec{k}$ we get,

$$\frac{\partial\varphi}{\partial x} = 6xy+z^3; \qquad \frac{\partial\varphi}{\partial y} = 3x^2-z; \qquad \frac{\partial\varphi}{\partial z} = 3xz^2-y$$

Integrating the above equations partially with respect to x, y, z respectively

$$\varphi = 3x^2y + xz^3 + f_1(y,z)$$

$$\varphi = 3x^2y - yz + f_2(x,z)$$

$$\varphi = xz^3 - yz + f_3(x,y)$$

$\therefore \varphi = 3x^2y + xz^3 - yz + c$ where c is constant.

Example: 2.39 Prove that $\vec{F} = (y^2\cos x + z^3)\vec{i} + (2y\sin z - 4)\vec{j} + (3xz^2)\vec{k}$ **is irrotational and find its scalar potential.**

Solution:

Given $\vec{F} = (y^2 cos\, x + z^3)\vec{\imath} + (2y\, sinz - 4)\vec{\jmath} + (3xz^2)\vec{k}$

To prove $\nabla \times \vec{F} = 0$

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 cos\, x + z^3 & 2y\, sinz - 4 & 3xz^2 \end{vmatrix}$$

$$= \vec{\imath}(0-0) - \vec{\jmath}(3z^2 - 3z^2) + \vec{k}(2y\cos x - 2ycos\, x)$$

$$= \vec{0}$$

$\therefore$ $\vec{F}$ is irrotational.

To find φ such that $\vec{F} = \nabla\varphi$.

$$\nabla\varphi = \vec{\imath}\frac{\partial\varphi}{\partial x} + \vec{\jmath}\frac{\partial\varphi}{\partial y} + \vec{k}\frac{\partial\varphi}{\partial z}$$

Equating the coefficients of $\vec{\imath}, \vec{\jmath}$ and $\vec{k}$ we get,

$\frac{\partial\varphi}{\partial x} = y^2\cos x + z^3$; $\quad \frac{\partial\varphi}{\partial y} = 2ysin\, x - 4$; $\quad \frac{\partial\varphi}{\partial z} = 3xz^2$

Integrating the above equations partially with respect to x, y, z respectively

$\varphi = y^2 \sin x + z^3 x + f_1(y,z)$

$\varphi = y^2 \sin x - 4y + f_2(x,z)$

$\varphi = xz^3 + f_3(x,y)$

$\therefore \varphi = y^2\sin x + z^3 x - 4y + c$ is scalar potential.

Example: 2.40 Prove that $\vec{F} = (2x + yz)\vec{\imath} + (4y + zx)\vec{\jmath} + (6z - xy)\vec{k}$ is solenoidal as well as irrotational also find the scalar potential of $\vec{F}$.

Solution:

Given $\vec{F} = (2x + yz)\vec{\imath} + (4y + zx)\vec{\jmath} + (6z - xy)\vec{k}$

(i) To prove $\vec{F}$ is solenoidal.

$(ie)\, To\, prove\, \nabla \cdot \vec{F} = 0$

$$\nabla \cdot \vec{F} = \frac{\partial}{\partial x}(2x + yz) + \frac{\partial}{\partial y}(4y + zx) + \frac{\partial}{\partial z}(-6z + xy)$$

$$= 2 + 4 - 6 = 0$$

$\therefore$ $\vec{F}$ is solenoidal.

(ii) To prove $\vec{F}$ is irrotational.

$(ie)\, To\, prove\, \nabla \times \vec{F} = 0$

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x + yz & 4y + zx & -6z + xy \end{vmatrix}$$

$$= \vec{\imath}(x - x) - \vec{\jmath}(y - y) + \vec{k}(z - z)$$

$= \vec{0}$

$\therefore$ $\vec{F}$ is irrotational.

(iii) To find φ such that $\vec{F} = \nabla\varphi$.

$$(2x + yz)\vec{i} + (4y + zx)\vec{j} + (6z - xy)\vec{k} = \vec{i}\frac{\partial\varphi}{\partial x} + \vec{j}\frac{\partial\varphi}{\partial y} + \vec{k}\frac{\partial\varphi}{\partial z}$$

Equating the coefficients of $\vec{i}, \vec{j}$ and $\vec{k}$ we get,

$\frac{\partial\varphi}{\partial x} = 2x + yz;$ $\quad \frac{\partial\varphi}{\partial y} = 4y + zx;$ $\quad \frac{\partial\varphi}{\partial z} = -6z + xy$

Integrating the above equations partially with respect to x, y, z respectively

$\varphi = x^2 + xyz + f_1(y,z)$

$\varphi = 2y^2 + xyz + f_2(x,z)$

$\varphi = -3z^2 + xyz + f_3(x,y)$

$\therefore \varphi = x^2 + 2y^2 - 3z^2 + xyz + c$ where c is a constant.

$\therefore$ φ is a scalar potential of $\vec{F}$.

Example: 2.41 If $\nabla\varphi = 2xyz^3\vec{i} + x^2z^3\vec{j} + 3x^2yz^2\vec{k}$ find φ if $\varphi(-1,2,2) = 4$

Solution:

Given $\nabla\varphi = 2xyz^3\vec{i} + x^2z^3\vec{j} + 3x^2yz^2\vec{k}$ ... (1)

We know that $\nabla\varphi = \vec{i}\frac{\partial\varphi}{\partial x} + \vec{j}\frac{\partial\varphi}{\partial y} + \vec{k}\frac{\partial\varphi}{\partial z}$ ... (2)

Comparing (1) and (2)

$\frac{\partial\varphi}{\partial x} = 2xyz^3;$ $\quad \frac{\partial\varphi}{\partial y} = x^2z^3;$ $\quad \frac{\partial\varphi}{\partial z} = 3x^2yz^2$

Integrating the above equations partially with respect to x, y, z respectively

$\varphi = x^2yz^3 + f_1(y,z)$

$\varphi = x^2yz^3 + f_2(x,z)$

$\varphi = x^2yz^3 + f_3(x,y)$

$\therefore \varphi = x^2yz^3 + c$ where c is a constant.

Given $\varphi(-1,2,2) = 4$

$\Rightarrow 16 + c = 4$

$\Rightarrow c = -12$

$\therefore \varphi = x^2yz^3 - 12$

Example: 2.42 If $\vec{A}$ and $\vec{B}$ are irrotational, then prove that $\vec{A} \times \vec{B}$ is solenoidal.

Solution:

Given $\vec{A}$ and $\vec{B}$ are irrotational.

$(ie)\nabla \times \vec{A} = 0$ and $\nabla \times \vec{B} = 0$

We know that $\nabla \cdot (\vec{A} \times \vec{B}) = (\nabla \times \vec{A}) \cdot \vec{B} - (\nabla \times \vec{B}) \cdot \vec{A}$

$= 0 \cdot \vec{A} - 0 \cdot \vec{B}$

$$= 0$$

Hence $\vec{A} \times \vec{B}$ is solenoidal.

Example: 2.43 if $\vec{A}$ is a constant vector, then prove that (i) $div\,\vec{A} = 0$ and (ii) $curl\,\vec{A} = 0$

Solution:

Let $\vec{A} = A_1\vec{\imath} + A_2\vec{\jmath} + A_3\vec{k}$

$\frac{\partial A_1}{\partial x} = 0; \quad \frac{\partial A_2}{\partial y} = 0; \quad \frac{\partial A_3}{\partial z} = 0$

(i) $\nabla \cdot \vec{A} = \frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z}$

$= 0 + 0 + 0 = 0$

Hence $div\,\vec{A} = 0$.

(ii) $\nabla \times \vec{A} = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_1 & A_2 & A_3 \end{vmatrix}$

$= \vec{\imath}(0-0) - \vec{\jmath}(0-0) + \vec{k}(0-0)$

$= \vec{0}$

$\therefore\ curl\,\vec{F} = \vec{0}$

Example: 2.44 If φ and χ are differentiable scalar fields, prove $\nabla\varphi \times \nabla\chi$ is solenoidal.

Solution:

Consider $\nabla \cdot (\nabla\varphi \times \nabla\chi)$

$= \nabla\chi \cdot \nabla \times (\nabla\varphi) - \nabla\varphi \cdot [\nabla \times (\nabla\chi)] \qquad [\because \nabla \cdot (\vec{A} \times \vec{B}) = \vec{B} \cdot (\nabla \times \vec{A}) - \vec{A} \cdot (\nabla \times \vec{B})]$

$= \nabla\chi \cdot 0 - \nabla\varphi \cdot 0$

$= 0$

$\therefore \nabla\varphi \times \nabla\chi$ is solenoidal.

Example: 2.45 Find $f(r)$ if the vector $f(r)\vec{r}$ is both solenoidal and irrotational.

Solution:

(i) Given $f(r)\vec{r}$ is solenoidal.

$\therefore\ \nabla \cdot (f(r)\vec{r}) = 0$

We know that $\vec{r} = x\vec{\imath} + y\vec{\jmath} + z\vec{k}$

$\therefore\ f(r)\vec{r} = f(r)\,x\vec{\imath} + f(r)\,y\vec{\jmath} + f(r)\,z\vec{k}$

Now $\nabla \cdot (f(r)\vec{r}) = 0$

$\Rightarrow \left(\vec{\imath}\frac{\partial}{\partial x} + \vec{\jmath}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right) \cdot \left(f(r)\,x\,\vec{\imath} + f(r)\,y\,\vec{\jmath} + f(r)\,z\,\vec{k}\right) = 0$

$\Rightarrow \frac{\partial}{\partial x}(f(r)\,x) + \frac{\partial}{\partial y}(f(r)\,y) + \frac{\partial}{\partial z}(f(r)\,z) = 0$

$\Rightarrow \sum \frac{\partial}{\partial x}(f(r)\,x) = 0$

$$\Rightarrow \sum\left[f(r)\cdot 1 + x\, f'(r)\frac{\partial r}{\partial x}\right] = 0$$

$$\Rightarrow \sum\left[f(r) + x\, f'(r)\frac{x}{r}\right] = 0$$

$$\Rightarrow \sum\left[f(r) + \frac{x^2}{r}\, f'(r)\right] = 0$$

$$\Rightarrow 3f(r) + f'(r)\left[\frac{x^2}{r} + \frac{y^2}{r} + \frac{z^2}{r}\right] = 0$$

$$\Rightarrow 3f(r) + \frac{f'(r)}{r}[r^2] = 0 \quad [\because\ x^2 + y^2 + z^2 = r^2]$$

$$\Rightarrow 3f(r) + f'(r)\, r = 0$$

$$\Rightarrow f'(r)\, r = -3f(r)$$

$$\Rightarrow \frac{f'(r)}{f(r)} = \frac{-3}{r}$$

Integrating with respect to r, we get

$$\Rightarrow \int \frac{f'(r)}{f(r)}\, dr = \int \frac{-3}{r}\, dr$$

$$\Rightarrow \log f(r) = -3\log r + \log c$$

$$= \log r^{-3} + \log c$$

$$= \log\left(\frac{1}{r^3}\right) + \log c$$

$$= \log\left(\frac{c}{r^3}\right)$$

$$\therefore f(r) = \frac{c}{r^3}$$

(ii) Given $f(r)\vec{r}$ is irrotational.

$$\nabla \times f(r)\vec{r} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xf(r) & yf(r) & zf(r) \end{vmatrix}$$

$$= \sum \vec{i}\left[z\frac{\partial}{\partial y} f(r) - y\frac{\partial}{\partial z} f(r)\right]$$

$$= \sum \vec{i}\left[zf'(r)\frac{\partial r}{\partial y} - y\, f'(r)\frac{\partial r}{\partial z}\right]$$

$$= \sum \vec{i}\left[zf'(r)\frac{y}{r} - y\, f'(r)\frac{z}{r}\right]$$

$$= \sum \vec{i} f'(r)\left[\frac{zy}{r} - \frac{zy}{r}\right]$$

$$= \vec{0} \text{ for all } f(r)$$

Example: 2.46 Prove that $r^n\vec{r}$ is irrotational for every n and solenoidal only for $n = -3$.

Solution:

We know that $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$

$$\therefore r^n\vec{r} = r^n x\vec{i} + r^n y\vec{j} + r^n z\vec{k}$$

(i) To prove $r^n\vec{r}$ is irrotational.

$$\nabla \times (r^n \vec{r}) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ r^n\, x & r^n\, y & r^n\, z \end{vmatrix}$$

$$= \sum \vec{i}\left[z\, nr^{n-1}\frac{\partial r}{\partial y} - y\, nr^{n-1}\frac{\partial r}{\partial z}\right]$$

$$= \sum \vec{i}\left[z\, nr^{n-1}\frac{y}{r} - y\, nr^{n-1}\frac{z}{r}\right]$$

$$= \sum \vec{i}\left[\, nr^{n-1}\frac{zy}{r} - nr^{n-1}\frac{zy}{r}\right]$$

$$= \sum \vec{i}\,(0)$$

$$= 0\,\vec{i} + 0\,\vec{j} + 0\,\vec{k} = \vec{0}$$

$\therefore$ $r^n\vec{r}$ is irrotational for every n.

(ii) To prove $r^n\vec{r}$ is solenoidal.

$$\nabla \cdot (r^n\vec{r}) = \nabla \cdot \left(r^n\, x\vec{i} + r^n\, y\,\vec{j} + r^n\, z\,\vec{k}\right)$$

$$= \sum \frac{\partial}{\partial x}(r^n\, x)$$

$$= \sum \left[r^n\,(1) + xnr^{n-1}\,\frac{\partial r}{\partial x}\right]$$

$$= \sum \left[r^n \;+\; xnr^{n-1}\,\frac{x}{r}\right]$$

$$= \sum [r^n \;+\; x^2nr^{n-2}\,]$$

$$= 3r^n \;+ nr^{n-2}(x^2 + \; y^2 + \; z^2)$$

$$= 3r^n \;+ nr^{n-2}(r^2)$$

$$= 3r^n \;+ nr^n$$

$$= r^n(3 \;+ n)$$

When $n = -3$, we get $\nabla \cdot (r^n\vec{r}) = 0$

$\therefore$ $r^n\vec{r}$ is solenoidal only if $n = -3$.

Problems based on Laplace operator

Example: 2.47 Find $\nabla^2(\log r)$

Solution:

$$\nabla^2(\log r) = \sum \frac{\partial^2}{\partial x^2}(\log r)$$

$$= \sum \frac{\partial}{\partial x}\left(\frac{1}{r}\frac{\partial r}{\partial x}\right)$$

$$= \sum \frac{\partial}{\partial x}\left(\frac{1}{r^2}\, x\right)$$

$$= \sum \left[\frac{1}{r^2}(1) + \; x\left(-\frac{2}{r^3}\right)\frac{\partial r}{\partial x}\right]$$

$$= \sum \left[\frac{1}{r^2} - \; x\left(\frac{2}{r^3}\right)\frac{x}{r}\right]$$

$$= \sum \left[\frac{1}{r^2} - \; \frac{2x^2}{r^4}\right]$$

$$= \frac{3}{r^2} - \frac{2}{r^4}(x^2 + \; y^2 + \; z^2)$$

$$= \frac{3}{r^2} - \frac{2}{r^4}(r^2)$$

$$= \frac{3}{r^2} - \frac{2}{r^2} = \frac{1}{r^2}$$

Example: 2.48 Prove that $\nabla^2(r^n) = n(n+1)r^{n-2}$, where $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$ and $r = |\vec{r}|$ and hence deduce $\nabla^2\left(\frac{1}{r}\right)$.

(or)

Prove that $div\,(grad\, r^n) = n(n+1)\, r^{n-2}$

Solution:

Let $r = |\vec{r}| = \sqrt{x^2 + y^2 + z^2}$

Hence $\frac{\partial r}{\partial x} = \frac{x}{r}$; $\quad \frac{\partial r}{\partial y} = \frac{y}{r}$; $\quad \frac{\partial r}{\partial z} = \frac{z}{r}$

$$\nabla^2(r^n) = \sum \frac{\partial^2}{\partial x^2}(r^n)$$

$$= \sum \frac{\partial}{\partial x}\left[n r^{n-1} \frac{\partial r}{\partial x}\right]$$

$$= \sum \frac{\partial}{\partial x}\left[n r^{n-1} \frac{x}{r}\right]$$

$$= \sum \frac{\partial}{\partial x}[n\, x r^{n-2}]$$

$$= \sum n\left[x(n-2)r^{n-3}\frac{\partial r}{\partial x} + r^{n-2}(1)\right]$$

$$= \sum n\left[x(n-2)r^{n-3}\frac{x}{r} + r^{n-2}\right]$$

$$= \sum[n[(n-2)r^{n-4}x^2 + r^{n-2}]]$$

$$= \sum[n(n-2)r^{n-4}x^2 + n\, r^{n-2}]$$

$$= n(n-2)r^{n-4}(x^2 + y^2 + z^2) + 3\, n\, r^{n-2}$$

$$= n(n-2)r^{n-4}\, r^2 + 3\, n\, r^{n-2}$$

$$= n(n-2)r^{n-2} + 3\, n\, r^{n-2}$$

$$= n\, r^{n-2}(n-2+3)$$

$$= n\, r^{n-2}(n+1) \quad \dots (1)$$

(ii) $\nabla^2\left(\frac{1}{r}\right) = \nabla^2(r^{-1})$

$$= (-1)(-1+1)\, r^{-1-2} \quad \text{by (1)}$$

$$= (-1)(0)\, r^{-3} = 0$$

Example: 2.49 Prove that $\nabla^2(r^n\vec{r}) = n(n+3)r^{n-2}\vec{r}$

Solution:

We have $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$

Hence $\frac{\partial \vec{r}}{\partial x} = \vec{i}$; $\quad \frac{\partial \vec{r}}{\partial y} = \vec{j}$; $\quad \frac{\partial \vec{r}}{\partial z} = \vec{k}$

Also $r = |\vec{r}| = \sqrt{x^2 + y^2 + z^2}$

Hence $\frac{\partial r}{\partial x} = \frac{x}{r}$; $\frac{\partial r}{\partial y} = \frac{y}{r}$; $\frac{\partial r}{\partial z} = \frac{z}{r}$

$$\nabla^2(r^n\vec{r}) = \sum \frac{\partial^2}{\partial x^2}(r^n\vec{r})$$

$$= \sum \frac{\partial}{\partial x}\left[r^n \frac{\partial \vec{r}}{\partial x} + n\, r^{n-1} \frac{\partial r}{\partial x}\vec{r}\right]$$

$$= \sum \frac{\partial}{\partial x}\left[r^n\, \vec{\imath} + n\, r^{n-1} \frac{x}{r}\vec{r}\right]$$

$$= \sum \frac{\partial}{\partial x}\left[r^n\, \vec{\imath} + n\, r^{n-2}\, x\, \vec{r}\right]$$

$$= \sum\left[n\, r^{n-1} \frac{\partial r}{\partial x}\vec{\imath} + n\left[r^{n-2}x\left(\frac{\partial \vec{r}}{\partial x}\right) + r^{n-2}(1)\vec{r} + \left[(n-2)\, r^{n-3} \frac{\partial r}{\partial x}\right] x\vec{r}\right]\right]$$

$$= \sum\left[n\, r^{n-1} \frac{x}{r}\vec{\imath} + n\, r^{n-2}\, x\vec{\imath} + n\, r^{n-2}\vec{r} + n(n-2)\, r^{n-3} \frac{x}{r}\, x\vec{r}\right]$$

$$= \sum[n\, r^{n-2}x\vec{\imath} + n\, r^{n-2}\, x\vec{\imath} + n\, r^{n-2}\vec{r} + n(n-2)\, r^{n-4}\, x^2\vec{r}]$$

$$= n\, r^{n-2}(x\vec{\imath} + y\vec{\jmath} + z\vec{k}) + n\, r^{n-2}(x\vec{\imath} + y\vec{\jmath} + z\vec{k}) + 3n\, r^{n-2}\vec{r}$$

$$n(n-2)\, r^{n-4}\vec{r}(x^2 + y^2 + z^2)$$

$$= n\, r^{n-2}\vec{r} + n\, r^{n-2}\vec{r} + 3n\, r^{n-2}\vec{r} + n(n-2)r^{n-4}\vec{r}r^2$$

$$= 5n\, r^{n-2}\vec{r} + n(n-2)r^{n-2}\vec{r}$$

$$= n\, r^{n-2}\vec{r}(5 + n - 2)$$

$$= n\, r^{n-2}\vec{r}(n+3)$$

$$= n(n+3)r^{n-2}\vec{r}$$

Example: 2.50 Prove that $\nabla^2 f(r) = f''(r) + \left(\frac{2}{r}\right) f'(r)$

Solution:

$$\nabla^2 f(r) = \sum \frac{\partial^2}{\partial x^2} f(r)$$

$$= \sum \frac{\partial}{\partial x}\left[f'(r)\, \frac{\partial r}{\partial x}\right]$$

$$= \sum \frac{\partial}{\partial x}\left[f'(r)\, \frac{x}{r}\right]$$

$$= \sum \frac{\partial}{\partial x}\left[f'(r)x\, \frac{1}{r}\right]$$

$$= \sum\left[f'(r)x\left[\frac{-1}{r^2}\frac{\partial r}{\partial x}\right] + f'(r)\,(1)\,\frac{1}{r} + f''(r)\, \frac{\partial r}{\partial x}\, x\frac{1}{r}\right]$$

$$= \sum\left[f'(r)x\, \frac{-1}{r^2}\frac{x}{r} + f'(r)\, \frac{1}{r} + f''(r)\, \frac{x}{r}\, x\frac{1}{r}\right]$$

$$= \sum\left[f'(r)\, \frac{-1}{r^3}x^2 + f'(r)\, \frac{1}{r} + f''(r)\, \frac{1}{r^2}\, x^2\right]$$

$$= f'(r)\, \frac{-1}{r^3}(x^2 + y^2 + z^2) + \frac{3}{r}f'(r) + f''(r)\, \frac{1}{r^2}\,(x^2 + y^2 + z^2)$$

$$= -f'(r)\, \frac{1}{r^3}(r^2) + \frac{3}{r}f'(r) + f''(r)\, \frac{1}{r^2}\,(r^2)$$

$$= -f'(r)\, \frac{1}{r} + \frac{3}{r}f'(r) + f''(r)$$

$$= f''(r) + \frac{2}{r}f'(r)$$

Exercise: 2.2