Reinforcement Learning Assignment-1

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January 18, 2022

1. Histogram obtained after sampling $N = 100$ samples:

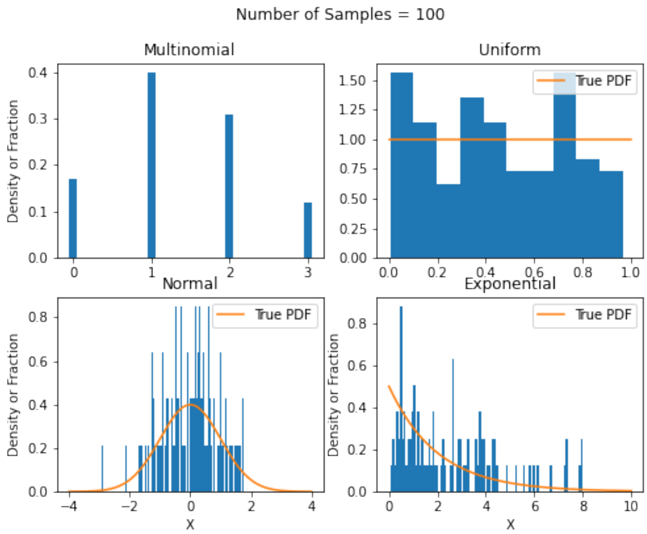


Figure 1: Histogram obtained after sampling $N = 100$ samples.

Here, since the number of samples is less hence we see a lot deviation between the plotted histogram and the true PDF.

Histogram obtained after sampling $N = 1000$ samples:

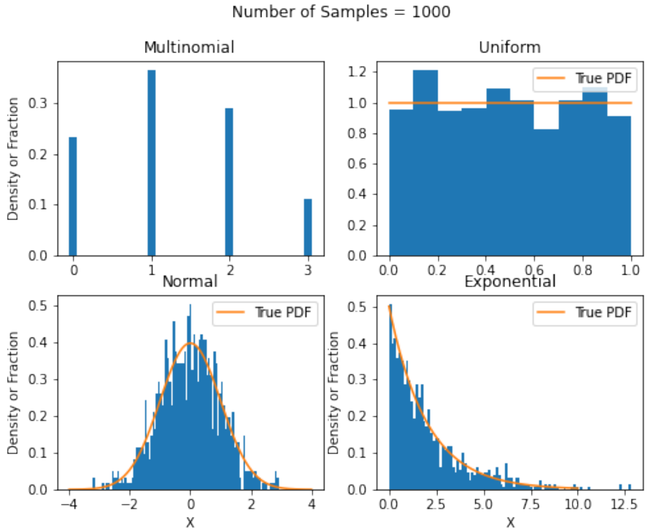


Figure 2: Histogram obtained after sampling $N = 1000$ samples.

Histogram obtained after sampling $N = 10000$ samples:

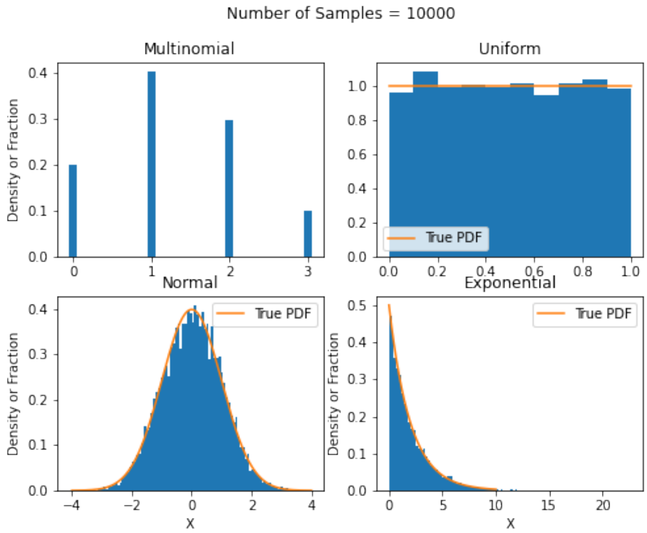


Figure 3: Histogram obtained after sampling $N = 10000$ samples.

We observe that as we increase the number of samples the plotted histogram and the true PDF tend to match. This way we are guaranteed that the samples were drawn from the same distribution.

2. Let $U \sim Unif(0,1)$ and $\Phi(x)$ denote the standard normal CDF. Then, we can show that if $X = \Phi^{-1}(U)$, then $X \sim \mathcal{N}(0,1)$. This is because $P(X \leq t) = P(\Phi^{-1}(U) \leq t) = P(U \leq \Phi(t)) = \Phi(t)$. Now, since the CDF X is $\Phi(x)$, hence, $X \sim \mathcal{N}(0,1)$.

   Let u be a sample from $Unif(0,1)$. Then using the above fact, we generate samples standard normal random samples as $x = \Phi^{-1}(u)$. Then, to generate normal samples with mean μ

and variance σ^2 we can simply transform as $y = \sigma x + \mu$, where y is the sample from normal random variable with mean μ and variance σ^2.

The plotted histogram are the samples drawn using the method described above using $\mu = 1$ and $\sigma^2 = 1$. The graph also shows the true PDF for the standard normal distribution. Since, the two match, we are guaranteed that the samples were drawn from the standard normal distribution.

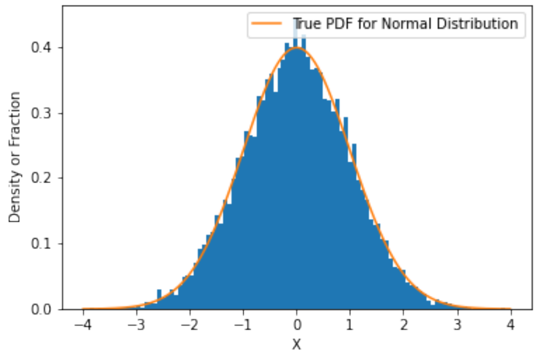


Figure 4: Histogram obtained after sampling $N = 10000$ samples.

3. In order to compute the integral, we can represent the integration as an expectation of a random variable. Then using Law of Large Numbers, we can compute the expectation, by sampling from the distribution of that random variable.

   Let's $X \sim Unif(0, \pi)$ and $f(x)$ be it's PDF i.e.,

$$f(x) = \begin{cases} \frac{1}{\pi} & if 0 \leq x \leq \pi \\ 0 & otherwise \end{cases}$$

   (a) The graph for $\sqrt(sin(x))$ is plotted as below:

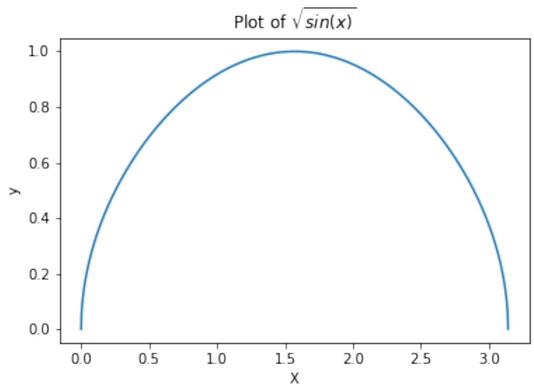


Figure 5: Graph for $\sqrt(sin(x))$

Let $g(X) = \sqrt(sin(X))$, where X is as defined above. Then

$$\mathbb{E}[g(X)] = \int_{-\infty}^{\infty} g(x) f(x)\, dx = \frac{1}{\pi} \int_{0}^{\pi} g(x)\, dx \tag{1}$$

Let $X_1, X_2, ..., X_N \overset{i.i.d.}{\sim}$ Unif$(0, \pi)$ and $Y_i = g(X_i)$ for $i \in 1, 2, ...N$. Then according to Law of Large Numbers, we have

$$\lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} Y_i = \mathbb{E}[g(X)]$$

Now,

$$\int_0^\pi \sqrt{(sin(x))}\, dx = \pi \left(\lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} Y_i \right) \tag{2}$$

The value of the integral obtained by this method is 2.4061907079036167.

(b) The graph for $\sqrt{(sin(x))} exp(-x^2)$ is plotted as below:

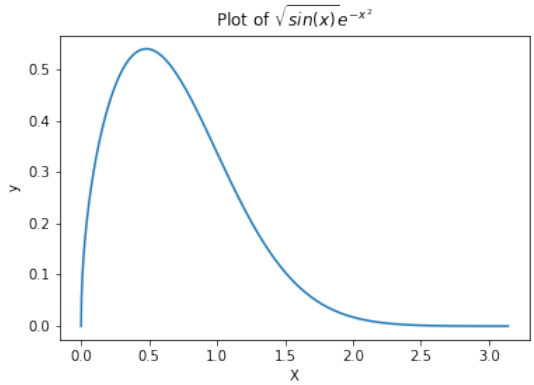
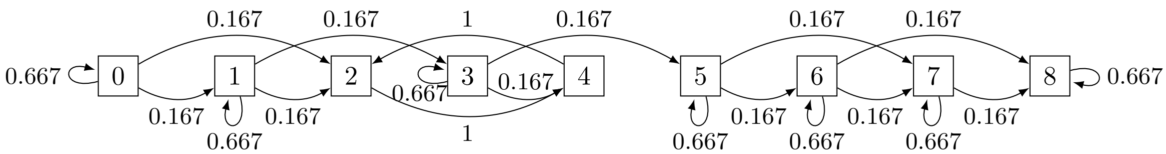


Figure 6: Graph for $\sqrt{(sin(x))} exp(-x^2)$

We can follow the same thing as above but by defining $g(X) = \sqrt{(sin(X))} exp(-X^2)$. The value of the integral obtained by this method is 0.5737398658239544.

4. The Markov Chain for the game can be drawn as follows:



The transition probability matrix can be represented as follows:

$$P = \begin{bmatrix} 2/3 & 1/6 & 1/6 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2/3 & 1/6 & 1/6 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2/3 & 1/6 & 1/6 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2/3 & 1/6 & 1/6 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2/3 & 1/6 & 1/6 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5/6 & 1/6 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

The initial distribution of states can be represented as follows:

$$\pi_0 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Simulation

We can consider an indicator random variable I such that it takes value of 1 when the player reaches the end state else it takes the value of 0. Now, the expectation of this random variable is the probability of the player ever reaching the end state. Using, Law of Large Numbers we can calculate this probability by simulating this Markov Chain (i.e. playing this game) multiple times and calculating the average number of times the player reaches the end state.

The question now that arises is that how long should we simulate each game? For this we calculate the value of $\pi_0 P^T$ for large values of T. We observe that the values of $\pi_0 P^T$ for large values of T alternate between $\begin{bmatrix} 0 & 0 & 0.482 & 0 & 0.393 & 0 & 0 & 0 & 0.125 \end{bmatrix}$ and $\begin{bmatrix} 0 & 0 & 0.393 & 0 & 0.482 & 0 & 0 & 0 & 0.125 \end{bmatrix}$. We know from this that in the long run following things happen:

- The probability of player being in states 0, 1, 3, 5, 6 and 7 is almost 0.
- The probability of player being in states 2 and 4 alternates between 0.393 and 0.482.
- The probability of player being in state 8 is 0.125.

In other words, in the long run the player can only be in states 2, 4 or 8. Therefore, while simulating the Chain, we need to only simulate to the point where the player either reaches the state 2 or 4 (then the player cannot reach end state) or the point where the player reaches the state 5 (because then the player is guaranteed in the long run will reach the end state).

On simulating the Chain for 10,000 times the probability of ever reaching the end state is found to be 0.12469.

Analytic Solution

Let us consider a random variable X_n denoting the state of the player at timestamp n. Let I be an indicator random variable such that it takes value of 1 when the player reaches the end state else it takes the value of 0. Now, the probability of the player ever reaching the end state is sum of probabilities of it reaching the end state at timestamp $T = 5, 6, ...$, where the probability of a player reaching the end state at time T is probability that the player is in state 8 at time T but was not in state 8 at $T-1$. We start the sum from 5 because the earliest the player can reach the end state is at timestamp 5. This can be written down as follows,

$$P(I = 1) = \sum_{t=5}^{\infty} P(X_t = 8, X_{t-1} \neq 8) = \sum_{t=5}^{\infty} P\left(X_t = 8, \bigcup_{i=0}^{7} (X_{t-1} = i)\right)$$

The last equality holds because if the player is not in state 8 at time $T-1$ implies that the player was in one of the states from 0 to 7. We can further simplify the above equation as

$$\begin{aligned} P(I = 1) &= \sum_{t=5}^{\infty} \sum_{i=0}^{7} P(X_t = 8, X_{t-1} = i) \\ &= \sum_{t=5}^{\infty} \sum_{i=0}^{7} P(X_t = 8 | X_{t-1} = i) P(X_{t-1} = i) \\ &= \sum_{t=5}^{\infty} (P(X_t = 8 | X_{t-1} = 6) P(X_{t-1} = 6) + (P(X_t = 8 | X_{t-1} = 7) P(X_{t-1} = 7) \end{aligned}$$

The equality holds because the transition probabilities $P(X_t = 8 | X_{t-1} = i) = 0 \forall i \in 0, 1, ..., 5$.