University of Vienna SS 2024
250082 VO Linear Algebra 2
Mock examination

**X**

**Student's full name:** ______________________________

**Student's matriculation number:** ______________________________

(1) This sheet, provided ahead of the examination, collects *some* examination rules to help you understand and follow the examination procedure. The procedure itself is aimed at allowing all the students taking the examination to demonstrate the knowledge they acquired through the course and to receive fair grading on the basis of that knowledge. Some rules are essential primarily to the organized character of the examination and not immediately to its fairness, even if the former serves to ensure the latter. Other rules are immediately crucial for the fairness of the examination. By understanding the rules and making an earnest effort to comply with them, you support the organized character of the examination, its fairness and, eventually, the success of our joint educational endeavor. Substantial violations of examination rules not only undermine those values but also may be formally and individually penalized.

(2) The examination is conducted by examiners. The instructor of the course is the sole examiner until he explicitly appoints other persons for them to assist as additional examiners.

(3) All students taking the examination should carry their own student identification cards valid on the day of examination. In exceptional circumstances, a government-issued identification document with a photograph or other means of identification may be used instead of a student identification card.

(4) The following rules apply to the whole duration of the examination, from the moment you enter the examination room until (i) the examiners finish collecting examination booklets or (ii) you definitely leave the examination room, whichever occurs first. Within that period, the students taking the examination have **105 minutes** for solving the examination problems and recording their solutions in their examination booklets. That period starts with an announcement of an examiner (to be issued when everyone is ready to start); outside that period, you should keep your examination booklet closed, so that all students have the same amount of time for solving the examination problems and recording their solutions.

(5) Once you have been instructed to open the examination booklet, please check that it contains exactly 22 *numbered* pages and is correctly stapled. Do not unstaple the booklet at any time. **All drafts and examination booklets must remain in the examination room at all times**. Leaving the examination room during the exam for *any* reason before definitely submitting your examination booklet to an examiner, you are required to leave your drafts and examination booklet on your desk, collected and closed.

(6) You should record your answers **in the examination booklet** in the spaces provided. You may continue your answers on the extra sheets at the end of the booklet if necessary. The loose paper provided is intended for rough work and will not be collected and considered for grading. Ask the examiners if you need more sheets to be attached to your examination booklet. In any case, reference the extra sheets by page numbers in the spaces provided in the booklet (for example: "see also page 12").

(7) **The examination is written, individual and closed book**.
   - Graded is the work you record in written form in the examination booklet.
   - All the work toward solving the examination problems (whether recorded or not recorded in your examination booklet) should be your own.
   - During the examination, the use of any materials (printed, handwritten or electronic) or electronic devices that can potentially serve for communication, calculation or reference purposes with respect to this examination is strictly prohibited, both inside and outside the examination

**see overleaf!**

room. All such materials and devices (switched to a fully silent mode) that you bring into the examination room should be deposited in the area designated by the instructor upon your arrival in the examination room. This provision covers but is not limited to cellphones, smartphones, smartwatches, headphones, electronic notepads, computers and calculators, the use of which is strictly prohibited (even for tracking time). The examiners will be regularly informing you of the current time. It is permitted to use analog and digital watches that cannot be used for communication or for accessing reference sources. A consequence of this provision is that **electronic devices cannot be used for emergency communication during the examination**. If you need to be able to receive an emergency message in any form during the examination, an arrangement should be made with the instructor ahead of the examination. If during the examination you need to keep using medical wearable devices and associated communicating electronic devices, you should make an arrangement with the instructor ahead of the examination.

– You are not permitted to receive assistance in solving the examination problems from another person or information system or to give such assistance to another student taking the examination. If a curious situation occurs in which another person attempts to assist you in solving the examination problems, you should immediately report that to one of the examiners.

These provisions are crucial for the fairness of the examination. Any violations of these provisions should be reported to the examiners as soon as possible.

(8) If you decide to finish the examination early, please submit your examination booklet to one of the examiners before leaving the examination room.

(9) The examination consists of problems, some of which consist of parts. Complete as much as you can. Problems and problem parts may be solved in any order. Insofar as it makes sense in context, you may answer problem parts (for full credit) without having correctly answered preceding problem parts and use as true any claims given as true in those preceding problem parts.

(10) The number of points assigned to each problem is indicated in a circle. The maximum total number of points is **58**. To maximize your grade, show all the essential details of your work and properly justify your answers. The tentative number of points necessary for a positive grade is **27**. You can receive the highest grade ("sehr gut") without perfectly solving all problems.

(11) In solving any problem or problem part (P), you may, with the exceptions explained below, use any definition or **correct** result (AUX) from any one (SRC) of the following sources:

– lectures and lecture notes;
– homework assignment (given as a problem).

In any case, you need to make sure that (AUX) is valid in the specific setting of (P) and that you clearly indicate (SRC) and clearly refer to (AUX) before applying it. At this examination, please use verbal references, not numbered. The following are examples of appropriate referencing: "the definition of matrix inversion", "the reflexivity of matrix transposition", "the inversion of permutation matrices", "the lemma on iterated Gaussian elimination", "the result on cross approximation and the pivoted LU decomposition", "the definition of a vector space", "the definition of a basis".

The exceptions when you **cannot** use (AUX) from (SRC) in solving (P) are as follows:

– you are specifically instructed not to use (AUX);
– (AUX) renders (P) entirely trivial, e.g., because (P) asks you to prove (AUX);
– using (AUX) creates a circular argument: you are trying to use (AUX) to solve (P) while (P) precedes (AUX) in the course and is used to obtain the result or conclusion of (P) in the course.
– (AUX) chronologically follows (P) in (SRC). This item is a measure against circular argument.

(12) Clarification requests should be made directly to one of the examiners.

(13) Under all circumstances, keep calm and carry on.

## Before Submitting Your Work

(1) Check that your copy of the exam booklet contains **all the 22 numbered pages** provided to you, any **additional sheets** provided by the examiners that you wish to submit (**also numbered**) and nothing else. Do not submit your draft paper!

(2) Check that your **full name** and **matriculation number** are clearly written on the front side of the **cover sheet** (at the top).

(3) Check that each of the additional pages at the end of the booklet that you would like to be graded clearly indicates the **number** of the corresponding problem and, where applicable, the **letter** of the problem part.

(4) After the last modifications in your exam booklet and before submitting it, indicate the total number of pages (should be **even**) and sign your sumbmission below.

**Total number of pages in this submission:** _______

**Signature:** ____________________

**1.** **(12) Decomposition of a polynomial space**

Let $V$ be a vector space over a field $\mathbb{F}$.

**(a)** (2) Define the sum of subspaces $U$ and $W$ of $V$.

**(b)** (2) Let $U$ and $W$ be subspaces of $V$. Define that $U$ and $W$ are linearly independent (i.e., form a direct sum).

**(c)** (4) Let $U$ and $W$ be subspaces of $V$. Show that $U$ and $W$ are linearly independent (i.e., form a direct sum) if and only if $U \cap W = \{0\}$.

For $n \in \mathbb{N}$, consider the vector space

$$\mathcal{P}_n = \left\{p\colon \mathbb{R} \to \mathbb{R}\colon p(t) = a_0 + \sum_{k=1}^{n} a_k t^k \text{ for all } t \in \mathbb{R} \text{ with } a_0, \ldots, a_n \in \mathbb{R}\right\}.$$

of real algebraic polynomials of degree at most $n$ and its subspaces

$$\mathcal{P}_n' = \{p \in \mathcal{P}_n \colon p(-t) = p(t) \text{ for every } t \in \mathbb{R}\}$$

and

$$\mathcal{P}_n'' = \{p \in \mathcal{P}_n \colon p(-t) = -p(t) \text{ for every } t \in \mathbb{R}\}.$$

**(d)** 2 Show that $\mathcal{P}_n = \mathcal{P}_n' + \mathcal{P}_n''$.

**(e)** 2 Show that $\mathcal{P}'_n$ and $\mathcal{P}''_n$ are linearly independent (i.e., that they form a direct sum).

**2.** 9 **Matrix similarity**

Let $\mathbb{F}$ be a field and $n \in \mathbb{N}$.

**(a)** 1 Define that matrices $A, B \in \mathbb{F}^{n\times n}$ are similar.

**(b)** 2 Give the definition of an eigenvalue of a matrix $A$ over a field $\mathbb{F}$.

**(c)** 3 Assume that matrices $A, B \in \mathbb{F}^{n\times n}$ are similar. State how the eigenvalues of $B$ are related to those of $A$. Prove your claim.

**(d)** 3 Establish whether the following two matrices are similar:

$$A = \begin{bmatrix} 1 & 0 & 3 \\ 2 & -5 & 0 \\ 1 & 0 & -1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 5 & 2 \\ 1 & -1 & 0 \\ 0 & 2 & 1 \end{bmatrix}.$$

**3.** (6) **Matrix diagonalizability**

Consider the matrix

**(a)** (1) Let $V$ be a vector space of dimension $n \in \mathbb{N}$ over a field $\mathbb{F}$. Define that a linear transformation $\varphi \in \mathcal{L}(V, V)$ is diagonalizable.

**(b)** (1) Let $\mathbb{F}$ be a field and $n \in \mathbb{N}$. Define that a matrix $A \in \mathbb{F}^{n \times n}$ is diagonalizable over $\mathbb{F}$.

**(c)** 1 Let $V$ be a vector space of dimension $n \in \mathbb{N}$ over a field $\mathbb{F}$ with a basis $v_1, \ldots, v_n$. Consider a linear transformation $\varphi \in \mathcal{L}(V, V)$ and its matrix $A$ with respect to the bases $v_1, \ldots, v_n$ and $v_1, \ldots, v_n$.
Argue why the diagonalization of $A$ yields the diagonalization of $\varphi$.

**(d)** 3 Establish whether the following matrix is diagonalizable (i) over $\mathbb{R}$ (as a real matrix) and (ii) over $\mathbb{C}$ (as a complex matrix):

$$A = \begin{bmatrix} 1 & -1 & 1 \\ -1 & 3 & -1 \\ 1 & -1 & 2 \end{bmatrix}.$$

**4.** (12) **Eigenvalues and the structure of a linear transformation**

Consider the real matrix

$$A = \begin{bmatrix} 4 & 1 & 2 \\ 1 & 4 & 2 \\ -1 & -1 & 1 \end{bmatrix}.$$

**(a)** (1) Show that $\lambda = 3$ is an eigenvalue of $A$.

**(b)** (5) Establish whether the matrix $A$ is diagonalizable over $\mathbb{R}$ (as a real matrix). Establish whether the matrix $A$ is diagonalizable over $\mathbb{C}$ (as a complex matrix).

**(c)** 5 Obtain an entrywise expression for the matrix exponential $\exp(tA)$ with $t \in \mathbb{R}$ in terms of a finite number of field operations of $\mathbb{R}$ and in terms of elementary real-valued functions of real argument.

**(d)** 1 Solve the following initial-value problem on $(0, +\infty)$ for the initial value $x_0 = (1, -1, 2) \in \mathbb{R}^3$:

$$\begin{cases} \dot{x} = Ax, \\ x(0) = x_0 \,. \end{cases}$$

**5.** (10) **Operator norm**

Let $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$ and $m, n \in \mathbb{N}$.

**(a)** (2) Define the matrix (operator) norm $\|\cdot\|$ on $\mathbb{F}^{m \times n}$ induced by norms $\|\cdot\|_*$ and $\|\cdot\|_\star$ on $\mathbb{F}^m$ and $\mathbb{F}^n$.

**(b)** (8) For every $p \in [0, +\infty]$, consider the matrix (operator) norm $\|\cdot\|_p$ on $\mathbb{F}^{m \times n}$ induced by the Hölder $p$-norms $\|\cdot\|_*$ and $\|\cdot\|_\star$ on $\mathbb{F}^m$ and $\mathbb{F}^n$.
Find the sharp equivalence constants for such operator norms $\|\cdot\|_2$ and $\|\cdot\|_\infty$ on $\mathbb{F}^{m \times n}$ (and prove that the constants are indeed sharp).

**6.** (9) **Hermitian positive-definite matrices and inner products**

Assume $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$. Consider a vector space $V$ of dimension $n \in \mathbb{N}$ over the field $\mathbb{F}$ with a basis $v_1, \ldots, v_n$ and the associated analysis operator $\mathbf{\Psi} \in \mathcal{L}(V, \mathbb{F}^n)$.

**(a)** (2) Give the definition of an inner product on $V$.

**(b)** (3) Consider an Hermitian positive-definite matrix $G \in \mathbb{F}^{n \times n}$. Show that the function $\langle \cdot, \cdot \rangle \colon V \times V \to \mathbb{F}$ given by

$$\langle u, v \rangle = \big(\mathbf{\Psi}(u)\big)^{\mathsf{H}} \cdot G \cdot \mathbf{\Psi}(v) \quad \text{for all} \quad u, v \in V$$

is an inner product on $V$.

**(c)** 4 Consider an inner product $\langle \cdot, \cdot \rangle$ on $V$. Define the Gram matrix $G$ of $v_1, \dots, v_n$ with respect to the inner product $\langle \cdot, \cdot \rangle$. Show that the matrix $G$ is Hermitian and positive definite.