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Distributed containment control of multi-agent systems with general linear dynamics in the presence of multiple leaders

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SUMMARY

This paper considers the containment control problems for both continuous-time and discrete-time multi-agent systems with general linear dynamics under directed communication topologies. Distributed dynamic containment controllers based on the relative outputs of neighboring agents are constructed for both continuous-time and discrete-time cases, under which the states of the followers will asymptotically converge to the convex hull formed by those of the leaders if, for each follower, there exists at least one leader that has a directed path to that follower. Sufficient conditions on the existence of these dynamic controllers are given. Static containment controllers relying on the relative states of neighboring agents are also discussed as special cases. Copyright © 2011 John Wiley & Sons, Ltd.

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1. INTRODUCTION

Cooperative control of a group of agents has received compelling attentions from various scientific communities for its potential applications in spacecraft formation flying, sensor networks, cooperative surveillance, and so forth [1, 2]. In the area of cooperative control of multi-agent systems, consensus is an important and fundamental problem, which is closely related to formation control [3] and flocking problems [4, 5]. The main idea of consensus is to develop distributed control policies that enable the agents to reach an agreement on a state of interest. Consensus algorithms are studied in [6–11] for a group of single, double, and high-order integrators with fixed/switching network topologies. Different static and dynamic consensus protocols are designed in [12–17] to reach consensus for multi-agent systems with general linear dynamics. The relatively complete coverage of the literature on consensus can be found in [1, 2].

The earlier mentioned references mainly focus on consensus for a group of agents without any leader. However, in some practical applications, there might exist one or even multiple leaders in the agent network. Tracking control for multi-agent consensus with an active leader is considered in [18, 19] by using a neighbor-based state-estimation rule. Distributed tracking algorithms are proposed, respectively, in [20] and [21] for a network of continuous-time and discrete-time agents to track a time-varying leader. In the presence of multiple leaders, the containment control problem arises, where the followers are to be driven into a given geometric space spanned by the leaders [22]. Containment control has several potential applications. For instance, a group of autonomous vehicles (designated as leaders) equipped with necessary sensors to detect the hazardous obstacles

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can be used to safely maneuver another group of vehicles (designated as followers) from one target to another, by ensuring that the followers are contained within the moving safety area formed by the leaders [22, 23]. In [22], a hybrid containment control law is proposed to drive the followers into the convex hull spanned by the leaders. Distributed containment control problems are studied in [23, 24] for a group of autonomous agents with multiple stationary/dynamic leaders under fixed and switching directed communication topologies. The authors in [25, 26] further study the containment control problem for a collection of rigid bodies. In particular, [25] discusses the case with multiple stationary leaders whereas [26] studies the case of dynamic leaders with finite-time convergence.

This paper considers the containment control problems for both continuous-time and discrete-time multi-agent systems with general linear dynamics under directed communication topologies. Distributed dynamic containment controllers based on the relative outputs of neighboring agents are proposed for both the continuous-time and discrete-time cases. In the continuous-time case, a multi-step algorithm is presented to construct a dynamic containment controller, under which the states of the followers will asymptotically converge to the convex hull formed by those of the leaders, if for each follower there exists at least one leader that has a directed path to that follower. It is shown that a sufficient condition on the existence of such a controller is that each agent is stabilizable and detectable. In the discrete-time case, in light of the modified algebraic Riccati equation, an algorithm is given to design a dynamic containment controller that solves the containment control problem. For the case where the state matrix A of each agent has no eigenvalues with magnitude larger than 1, a sufficient condition on the existence of such a discrete-time controller is that each agent is stabilizable and detectable. For the case where A has at least eigenvalue outside the unit circle, the communication topology has to further satisfy a constraint related to the unstable eigenvalues of A. Furthermore, as special cases of the dynamic controllers, static containment controllers relying on the relative states of neighboring agents are also discussed for both the continuous-time and discrete-time cases. In contrast to [22–24] where the agent dynamics are restricted to be single or double integrators and to [25, 26], which considers second-order Euler–Lagrange systems, the main contribution of the current paper is that the results obtained here are applicable to multi-agent systems with general linear dynamics.

The rest of this paper is organized as follows. Some basic notation and useful results of the graph theory are reviewed in Section 2. The containment control problems for continuous-time and discrete-time multi-agent systems are considered, respectively, in Sections 3 and 4. Simulation examples are presented in Section 5 to illustrate the analytical results. Conclusions are drawn in Section 6.

2. NOTATIONS AND GRAPH THEORY

Let $\mathbf{R}^{n\times n}$ and $\mathbf{C}^{n\times n}$ be the set of $n \times n$ real matrices and complex matrices, respectively. The superscript T means transpose for real matrices and H means conjugate transpose for complex matrices. I_N represents the identity matrix of dimension N. Matrices, if not explicitly stated, are assumed to have compatible dimensions. Denoted by $\mathbf{1}$ is the column vector with all entries equal to 1. $\mathrm{Re}(\zeta)$ denotes the real part of $\zeta \in \mathbf{C}$. $A \otimes B$ denotes the Kronecker product of the matrices A and B. For real symmetric matrices X and Y, $X > (\geqslant) Y$ means that the matrix $X - Y$ is positive (semi-) definite. A matrix is Hurwitz (in the continuous-time case) if all of its eigenvalues have negative real parts, whereas it is Schur stable (in the discrete-time case) if all of its eigenvalues have magnitude less than 1. For a set $X = \{x_1, \cdots, x_n\}$ in $V \subseteq \mathbf{R}^p$, its convex hull $\mathrm{co}(X)$ is defined as $\mathrm{co}(X) = \left\{\sum_{i=1}^{n} \alpha_i x_i | x_i \in V, \alpha_i \geqslant 0, \sum_{i=1}^{n} \alpha_i = 1\right\}$.

A directed graph $\mathcal{G}$ is a pair $(\mathcal{V}, \mathcal{E})$, where $\mathcal{V} = \{v_1, \cdots, v_N\}$ is a nonempty finite set of nodes, and $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ is a set of edges, in which an edge is represented by an ordered pair of distinct nodes. For an edge (v_i, v_j), node v_i is called the parent node, node v_j is the child node, and v_i is a neighbor of v_j. A graph with the property that $(v_i, v_j) \in \mathcal{E}$ implies $(v_j, v_i) \in \mathcal{E}$ is said to be undirected. A path from node v_{i_1} to node v_{i_l} is a sequence of ordered edges of the form $(v_{i_k}, v_{i_{k+1}})$, $k = 1, \cdots, l-1$. A directed graph contains a directed spanning tree if there exists a node called the root, which has no parent node, such that the node has a directed path to every other node in the graph.

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The adjacency matrix $\mathcal{A} = [a_{ij}] \in \mathbf{R}^{N\times N}$ associated with the directed graph $\mathcal{G}$ is defined by $a_{ii} = 0$, $a_{ij} = 1$ if $(v_j, v_i) \in \mathcal{E}$ and $a_{ij} = 0$ otherwise. The Laplacian matrix $\mathcal{L} = [\mathcal{L}_{ij}] \in \mathbf{R}^{N\times N}$ is defined as $\mathcal{L}_{ii} = \sum_{j\neq i} a_{ij}$ and $\mathcal{L}_{ij} = -a_{ij}$, $i \neq j$. Let $\mathcal{D} = [d_{ij}] \in \mathbf{R}^{N\times N}$ be a row-stochastic matrix associated with $\mathcal{G}$ with the additional assumption that $d_{ii} > 0$, $d_{ij} > 0$ if $(v_j, v_i) \in \mathcal{E}$ and $d_{ij} = 0$ otherwise. It is straightforward to verify that zero is an eigenvalue of $\mathcal{L}$ with $\mathbf{1}$ as the corresponding eigenvector, and all nonzero eigenvalues have positive real parts. One is an eigenvalue of $\mathcal{D}$ with $\mathbf{1}$ as the corresponding eigenvector, and all other eigenvalues of $\mathcal{D}$ are in the open unit disk.

3. CONTAINMENT CONTROL OF MULTI-AGENT SYSTEMS WITH GENERAL CONTINUOUS-TIME LINEAR DYNAMICS

Consider a group of N identical agents with general continuous-time linear dynamics, described by

$$\begin{aligned}\dot{x}_i &= Ax_i + Bu_i,\\ y_i &= Cx_i, \quad i = 1, \cdots, N,\end{aligned} \tag{1}$$

where $x_i \in \mathbf{R}^n$, $u_i \in \mathbf{R}^p$, and $y_i \in \mathbf{R}^q$ are, respectively, the state, the control input, and the measured output of the i-th agent, and A, B, and C are constant matrices with compatible dimensions.

A variety of static and dynamic consensus protocols were proposed for the agents with dynamics given by (1), for example, in [12–16]. Algorithms were given in [12–16] to construct protocols such that the agents reach consensus. However, these references only considered the case with at most one leader in the group.

Next, we consider the case with multiple leaders. Suppose that there are M $(M < N)$ followers and $N - M$ leaders. An agent is called a leader if the agent has no neighbor. An agent is called a follower if the agent has at least one neighbor. Without loss of generality, we assume that the agents indexed by $1, \cdots, M$, are followers, whereas the agents indexed by $M+1, \cdots, N$ are leaders whose control inputs are set to be zero. We use $\mathcal{R} \triangleq \{M+1, \cdots, N\}$ and $\mathcal{F} \triangleq \{1, \cdots, M\}$ to denote, respectively, the leader set and the follower set. The communication topology among the N agents is represented by a directed graph $\mathcal{G}$. Note that, here, the leaders do not receive any information.

Assumption 1
Suppose that, for each follower, there exists at least one leader that has a directed path to that follower.

The objective is to design distributed controllers to solve the containment control problem, defined as follows.

Definition 1
The containment control problem is solved for the agents in (1) if the states of the followers, under a certain distributed controller, asymptotically converge to the convex hull formed by those of the leaders.

3.1. Dynamic containment controller

In this subsection, it is assumed that each agent has access to the relative output measurements with respect to its neighbors. We let $u_i = 0$, $i \in \mathcal{R}$. Motivated by the consensus protocols in [12, 15], we propose the following distributed dynamic containment controller for each follower as

$$\begin{aligned}\dot{v}_i &= Av_i + Bu_i + cL \sum_{j\in\mathcal{F}\cup\mathcal{R}} a_{ij}\left[C(v_i - v_j) - (y_i - y_j)\right],\\ u_i &= cK \sum_{j\in\mathcal{F}\cup\mathcal{R}} a_{ij}(v_i - v_j), \quad i \in \mathcal{F},\end{aligned} \tag{2}$$

where $v_i \in \mathbf{R}^n$ is the state of the controller corresponding to the i-th follower, $v_j \in \mathbf{R}^n$ are the state variables of $\dot{v}_j = Av_j$, $j \in \mathcal{R}$, a_{ij} is the (i, j)-th entry of the adjacency matrix $\mathcal{A}$ associated with $\mathcal{G}$,

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$c > 0 \in \mathbf{R}$ denotes the coupling strength, and $L \in \mathbf{R}^{n\times q}$ and $K \in \mathbf{R}^{p\times n}$ are feedback gain matrices. In (2), the term $\sum_{j\in\mathcal{F}\cup\mathcal{R}} a_{ij}(v_i - v_j)$ means that the agents need to transmit the internal states of their corresponding controllers to their neighbors via the communication topology $\mathcal{G}$

Denoted by $\mathcal{L}$ is the Laplacian matrix associated with $\mathcal{G}$. Because the leaders have no neighbors, $\mathcal{L}$ can be partitioned as

$$\mathcal{L} = \begin{bmatrix} \mathcal{L}_1 & \mathcal{L}_2 \\ 0_{(N-M)\times M} & 0_{(N-M)\times(N-M)} \end{bmatrix}, \tag{3}$$

where $\mathcal{L}_1 \in \mathbf{R}^{M\times M}$ and $\mathcal{L}_2 \in \mathbf{R}^{M\times(N-M)}$.

Lemma 1 ([26]*)*
Under Assumption 1, all the eigenvalues of $\mathcal{L}_1$ have positive real parts, each entry of $-\mathcal{L}_1^{-1}\mathcal{L}_2$ is nonnegative, and each row of $-\mathcal{L}_1^{-1}\mathcal{L}_2$ has a sum equal to 1.

Next, an algorithm is presented to select the control parameters in (2).

Algorithm 1
Under Assumption 1, the containment controller (2) can be constructed as follows:

(i) Solve the following LMI:

$$AP + PA^T - 2BB^T < 0, \tag{4}$$

to get one solution $P > 0$. Then, choose the matrix $K = -B^T P^{-1}$.

(ii) Take $L = -Q^{-1}C^T$, where $Q > 0$ is a solution to the following LMI:

$$A^T Q + QA - 2C^T C < 0. \tag{5}$$

(iii) Select the coupling strength $c \geqslant c_{th}$, with

$$c_{th} = \frac{1}{\min\limits_{i=1,\cdots,M}\{\mathrm{Re}(\lambda_i)\}}, \tag{6}$$

where λ_i, $i = 1, \cdots, M$, are the eigenvalues of $\mathcal{L}_1$.

Remark 1
As shown in [12], a necessary and sufficient condition on the existence of a positive-definite solution to the LMI (4) is that (A, B) is stabilizable. Clearly, the LMI (5) is dual to (4). Therefore, a necessary and sufficient condition on the feasibility of Algorithm 1 is that (A, B, C) is stabilizable and detectable. Note that the feedback gain matrices K and L in (2) are designed by solving the LMIs (4) and (5). This can also be done by solving continuous-time algebraic Ricatti equations, as in [15].

Theorem 1
Assume that (A, B, C) is stabilizable and detectable. For a directed communication graph $\mathcal{G}$ satisfying Assumption 1, the controller (2) constructed by Algorithm 1 solves the containment control problem for the agents described by (1). Specifically, $\lim_{t\to\infty}(x_f(t) - \varpi_x(t)) = 0$ and $\lim_{t\to\infty}(v_f(t) - \varpi_v(t)) = 0$, where $x_f = [x_1^T, \cdots, x_M^T]^T$, $v_f = [v_1^T, \cdots, v_M^T]^T$,

$$\varpi_x(t) \triangleq (-\mathcal{L}_1^{-1}\mathcal{L}_2 \otimes e^{At})\begin{bmatrix} x_{M+1}(0) \\ \vdots \\ x_N(0) \end{bmatrix}, \quad \varpi_v(t) \triangleq (-\mathcal{L}_1^{-1}\mathcal{L}_2 \otimes e^{At})\begin{bmatrix} v_{M+1}(0) \\ \vdots \\ v_N(0) \end{bmatrix}. \tag{7}$$

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Proof

Let $z_i = \left[x_i^T, v_i^T\right]^T$, $z_f = \left[z_1^T, \cdots, z_M^T\right]^T$, and $z_l = \left[z_{M+1}^T, \cdots, z_N^T\right]^T$. Then, the closed-loop network dynamics resulting from (1) and (2) can be written as

$$\begin{aligned} \dot{z}_f &= (I_M \otimes \mathcal{S} + c\mathcal{L}_1 \otimes \mathcal{H})\, z_f + c(\mathcal{L}_2 \otimes \mathcal{H}) z_l, \\ \dot{z}_l &= (I_{N-M} \otimes \mathcal{S})\, z_l, \end{aligned} \tag{8}$$

where

$$\mathcal{S} = \begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix}, \quad \mathcal{H} = \begin{bmatrix} 0 & BK \\ -LC & LC + BK \end{bmatrix}.$$

Let $\xi_i = \sum_{j \in \mathcal{F} \cup \mathcal{R}} a_{ij}(z_i - z_j)$, $i \in \mathcal{F}$, and $\xi = [\xi_1^T, \cdots, \xi_M^T]^T$. Then, we have

$$\xi = (\mathcal{L}_1 \otimes I_{2n}) z_f + (\mathcal{L}_2 \otimes I_{2n}) z_l. \tag{9}$$

Considering the special structure of $\mathcal{L}$ as in (3), we can obtain from (8) and (9) that ξ satisfies the following dynamics:

$$\begin{aligned} \dot{\xi} &= (\mathcal{L}_1 \otimes I_{2n}) \dot{z}_f + (\mathcal{L}_2 \otimes I_{2n}) \dot{z}_l \\ &= (\mathcal{L}_1 \otimes I_{2n}) \left[(I_M \otimes \mathcal{S}) z_f + c(\mathcal{L}_1 \otimes \mathcal{H}) z_f + c(\mathcal{L}_2 \otimes \mathcal{H}) z_l\right] \\ &\quad + (\mathcal{L}_2 \otimes I_{2n})(I_{N-M} \otimes \mathcal{S}) z_l \\ &= (\mathcal{L}_1 \otimes \mathcal{S} + c\mathcal{L}_1^2 \otimes \mathcal{H}) \left[(\mathcal{L}_1^{-1} \otimes I_{2n})\xi - (\mathcal{L}_1^{-1}\mathcal{L}_2 \otimes I_{2n}) z_l\right] \\ &\quad + (c\mathcal{L}_1\mathcal{L}_2 \otimes \mathcal{H} + \mathcal{L}_2 \otimes \mathcal{S}) z_l \\ &= (I_M \otimes \mathcal{S} + c\mathcal{L}_1 \otimes \mathcal{H})\xi. \end{aligned} \tag{10}$$

Under Assumption 1, it follows from Lemma 1 that all the eigenvalues of $\mathcal{L}_1$ have positive real parts. Let $U \in \mathbf{C}^{M \times M}$ be such a unitary matrix that $U^H \mathcal{L}_1 U = \Lambda$, where Λ is an upper-triangular matrix with λ_i, $i = 1, \cdots, M$, as its diagonal entries. Let $\tilde{\xi} \triangleq \left[\tilde{\xi}_1^T, \cdots, \tilde{\xi}_N^T\right]^T = U^H \otimes I_{2n})\xi$. Then, it follows from (10) that

$$\dot{\tilde{\xi}} = (I_M \otimes \mathcal{S} + c\Lambda \otimes \mathcal{H})\tilde{\xi}. \tag{11}$$

By noting that Λ is upper-triangular, it is clear that (11) is asymptotically stable if and only if the following M systems

$$\dot{\tilde{\xi}}_i = (\mathcal{S} + c\lambda_i \mathcal{H})\tilde{\xi}_i, \quad i = 1, \cdots, M, \tag{12}$$

are simultaneously asymptotically stable. Multiplying the left and right sides of $\mathcal{S} + c\lambda_i\mathcal{H}$ by $T = \begin{bmatrix} I & -I \\ 0 & I \end{bmatrix}$ and T^{-1}, respectively, we get

$$T(\mathcal{S} + c\lambda_i \mathcal{H})T^{-1} = \begin{bmatrix} A + c\lambda_i LC & 0 \\ -c\lambda_i LC & A + c\lambda_i BK \end{bmatrix}. \tag{13}$$

By steps (i) and (iii) in Algorithm 1, it follows that there exists a $P > 0$ satisfying

$$\begin{aligned} (A + c\lambda_i BK)P + P(A + c\lambda_i BK)^H &= AP + PA^T - 2c\mathrm{Re}(\lambda_i) BB^T \\ &\leqslant AP + PA^T - 2BB^T < 0, \quad i = 1, \cdots, M. \end{aligned}$$

That is, $A + c\lambda_i BK$, $i = 1, \cdots, M$, are Hurwitz. Similarly, by steps (ii) and (iii) in Algorithm 1, we know that $A + c\lambda_i LC$, $i = 1, \cdots, M$, are also Hurwitz. Therefore, using (11), (12), and (13), it follows that (10) is asymptotically stable. Then, it follows from (9) that $\|z_f(t) + \left(\mathcal{L}_1^{-1}\mathcal{L}_2 \otimes I_{2n}\right) z_l(t)\| \to 0$, as $t \to \infty$, from which it is easy to get that $x_f(t) \to \left(-\mathcal{L}_1^{-1}\mathcal{L}_2 \otimes I_n\right) x_l(t)$ and $v_f(t) \to \left(-\mathcal{L}_1^{-1}\mathcal{L}_2 \otimes I_n\right) v_l(t)$, as $t \to \infty$,

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where $x_l = \left[x_{M+1}^T, \cdots, x_N^T\right]^T$ and $v_l = \left[v_{M+1}^T, \cdots, v_N^T\right]^T$. By noting that $x_j(t) = \mathrm{e}^{At} x_j(0)$, $v_j(t) = \mathrm{e}^{At} v_j(0)$, $j \in \mathcal{R}$, it then follows that $\lim_{t\to\infty}\left(x_f(t) - \varpi_x(t)\right) = 0$ and $\lim_{t\to\infty}(v_f(t) - \varpi_v(t)) = 0$. By Lemma 1, we know that the states of the followers asymptotically converge to the convex hull formed by those of the leaders, that is, the containment problem is solved. □

Remark 2
Containment control of multi-agent systems was previously studied in [22–26], where the agent dynamics are restricted to be single or double integrators in [22–24] and to be second-order Euler–Lagrange systems in [25, 26]. In contrast, Theorem 1 is applicable to multi-agent systems with general linear dynamics. For the special case with only one leader, Theorem 1 implies that the states of the followers will asymptotically approach the state of the leader.

Remark 3
Algorithm 1 has a favorable decoupling feature. Specifically, the first two steps deal with only the agent dynamics and the feedback gain matrices of (2), whereas the last step tackles the communication topology. Therefore, the containment controller (2) constructed via Algorithm 1 for a given communication graph satisfying Assumption 1 can be directly used for any other communication graph satisfying Assumption 1, with the only additional task of appropriately adjusting the coupling strength c.

3.2. *Static containment controller*

In this subsection, a special case where the relative states between neighboring agents are available is considered. For this case, a distributed static containment controller is proposed as

$$u_i = cF \sum_{j \in \mathcal{F} \cup \mathcal{R}} a_{ij}(x_i - x_j), \quad i \in \mathcal{F}, \tag{14}$$

where c and a_{ij} are defined as in (2), and $F \in \mathbf{R}^{p\times n}$ is the feedback gain matrix to be designed. Let $x_f = \left[x_1^T, \cdots, x_M^T\right]^T$ and $x_l = \left[x_{M+1}^T, \cdots, x_N^T\right]^T$. Using (14) for (1) gives the closed-loop network dynamics as

$$\begin{aligned}\dot{x}_f &= (I_M \otimes A + c\mathcal{L}_1 \otimes BF)x_f + c(\mathcal{L}_2 \otimes BF)x_l,\\ \dot{x}_l &= (I_{N-M} \otimes A)x_l,\end{aligned}$$

where $\mathcal{L}_1$ and $\mathcal{L}_2$ are defined in (3).

The following algorithm is given for designing the control parameters in (14).

Algorithm 2
Under Assumption 1, the controller (14) can be constructed as follows:

(i) Choose the feedback gain matrix $F = -B^T P^{-1}$, where $P > 0$ is a solution to the LMI (4).
(ii) Select the coupling strength $c \geqslant c_{th}$, with c_{th} given by (6).

Corollary 1
Assume that (A, B) is stabilizable and the communication graph $\mathcal{G}$ satisfies Assumption 1. Then, the static controller (14) constructed by Algorithm 2 solves the containment control problem for the agents in (1). Specifically, $\lim_{t\to\infty}(x_f(t) - \varpi_x(t)) = 0$, where $x_f(t)$ and $\varpi_x(t)$ are defined in (7).

Proof
The proof follows similar steps to those in the proof of Theorem 1, and is thus omitted here. □

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4. CONTAINMENT CONTROL OF MULTI-AGENT SYSTEMS WITH GENERAL DISCRETE-TIME LINEAR DYNAMICS

This section focuses on the discrete-time counterpart of the last section. Consider a group of N identical agents with general discrete-time linear dynamics, described by

$$\begin{aligned} x_i^+ &= Ax_i + Bu_i, \\ y_i &= Cx_i, \quad i = 1, \cdots, N, \end{aligned} \tag{15}$$

where $x_i = x_i(k) \in \mathbf{R}^n$, $x_i^+ = x_i(k+1)$, $u_i \in \mathbf{R}^p$, and $y_i \in \mathbf{R}^q$ are, respectively, the state at the current time instant, the state at the next time instant, the control input, the measured output of the i-th agent. Without loss of generality, it is assumed throughout this section that B is of full column rank and C has full row rank.

Similar to the last section, we assume that the agents indexed by $1, \cdots, M$, are followers, whereas the agents indexed by $M+1, \cdots, N$, are leaders. The leader set and the follower set are denoted, respectively, by $\mathcal{R} = \{M+1, \cdots, N\}$ and $\mathcal{F} = \{1, \cdots, M\}$. The communication graph $\mathcal{G}$ among the N agents is directed and satisfies Assumption 1.

4.1. Dynamic containment controller

We again let $u_i = 0$, $i \in \mathcal{R}$. Similar to the continuous-time case, we propose the following distributed dynamic containment controller for each follower as

$$\begin{aligned} \hat{v}_i^+ &= Av_i + Bu_i + L \sum_{j \in \mathcal{F} \cup \mathcal{R}} d_{ij} \left[C(\hat{v}_i - \hat{v}_j) - (y_i - y_j) \right], \\ u_i &= K \sum_{j \in \mathcal{F} \cup \mathcal{R}} d_{ij} (\hat{v}_i - \hat{v}_j), \quad i \in \mathcal{F}, \end{aligned} \tag{16}$$

where $\hat{v}_i \in \mathbf{R}^n$ is the state of the controller corresponding to the i-th follower, $\hat{v}_j \in \mathbf{R}^n$ are the state variables of $\hat{v}_j^+ = A\hat{v}_j$, $j \in \mathcal{R}$, d_{ij} is the (i,j)-th entry of the row-stochastic matrix $\mathcal{D}$ associated with $\mathcal{G}$, and $L \in \mathbf{R}^{n \times q}$ and $K \in \mathbf{R}^{p \times n}$ are feedback gain matrices.

Because the last $N-M$ agents are leaders that have no neighbors, $\mathcal{D}$ can be partitioned as

$$\mathcal{D} = \begin{bmatrix} \mathcal{D}_1 & \mathcal{D}_2 \\ 0_{(N-M) \times M} & I_{N-M} \end{bmatrix}, \tag{17}$$

where $\mathcal{D}_1 \in \mathbf{R}^{M \times M}$ and $\mathcal{D}_2 \in \mathbf{R}^{M \times (N-M)}$.

Lemma 2
Under Assumption 1, all the eigenvalues of $\mathcal{D}_1$ lie in the open unit disk, each entry of $(I_M - \mathcal{D}_1)^{-1}\mathcal{D}_2$ is nonnegative, and each row of $(I_M - \mathcal{D}_1)^{-1}\mathcal{D}_2$ has a sum equal to 1.

Proof
Consider the following new row-stochastic matrix:

$$\overline{\mathcal{D}} = \begin{bmatrix} \mathcal{D}_1 & \mathcal{D}_2 \mathbf{1}_{N-M} \\ 0_{1 \times M} & 1 \end{bmatrix}.$$

According to the definition of the directed spanning tree in Section 2, the graph associated with $\overline{\mathcal{D}}$ has a directed spanning tree if $\mathcal{G}$ satisfies Assumption 1. Therefore, by Lemma 3.4 in [7], if Assumption 1 holds, then 1 is a simple eigenvalue of $\overline{\mathcal{D}}$, and all the other eigenvalues of $\overline{\mathcal{D}}$ are in the open unit disk, which further implies that all the eigenvalues of $\mathcal{D}_1$ lie in the open unit disk.

Because every entry of $\mathcal{D}_1$ is nonnegative, and the spectral radius of $\mathcal{D}_1$ is less than 1, $I_M - \mathcal{D}_1$ is a nonsingular M-matrix [27]. Therefore, each entry of $(I_M - \mathcal{D}_1)^{-1}$ is nonnegative [27]. Clearly, each entry of $(I_M - \mathcal{D}_1)^{-1}\mathcal{D}_2$ is also nonnegative. Note that $(I_N - \mathcal{D})\mathbf{1}_N = 0$, implying that

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$(I_M - \mathcal{D}_1)\mathbf{1}_M - \mathcal{D}_2\mathbf{1}_{N-M} = 0$. That is, $(I_M - \mathcal{D}_1)^{-1}\mathcal{D}_2\mathbf{1}_{N-M} = \mathbf{1}_M$, which shows that each row of $(I_M - \mathcal{D}_1)^{-1}\mathcal{D}_2$ has a sum equal to 1. □

Before moving forward, we introduce the following modified algebraic Riccati equation (MARE) [28, 29]:

$$P = A^T PA - \delta A^T PB(B^T PB)^{-1} B^T PA + R, \tag{18}$$

where $P > 0$, $R > 0$, and $\delta > 0 \in \mathbf{R}$.

The following lemma shows the existence of solutions for the MARE.

Lemma 3 ([28, 29]*)*
Let (A, B) be stabilizable. Then, the following hold.

(i) There exists a critical value $\delta_c \in [0, 1)$ such that the MARE (18) has a unique positive-definite solution P for any $\delta > \delta_c$. Moreover, $\delta_c = 0$ if A has no eigenvalues with magnitude larger than 1. For the case where A has at least one eigenvalue with magnitude larger than 1 and B is of rank one, $\delta_c = 1 - \frac{1}{\Pi_i |\lambda_i^u(A)|^2}$, where $\lambda_i^u(A)$ are the unstable eigenvalues of A. For the case where A has at least one eigenvalue with magnitude larger than 1 and B is invertible, $\delta_c = 1 - \frac{1}{\max_i |\lambda_i^u(A)|^2}$.

(ii) If the MARE (18) has a unique positive-definite solution P, then $P = \lim_{k\to\infty} P_k$ for any initial condition $P_0 > 0$, where P_k satisfies

$$P(k+1) = A^T P(k)A - \delta A^T P(k)B(B^T P(k)B)^{-1} B^T P(k)A + R.$$

Next, an algorithm for determining the control parameters in (16) is presented.

Algorithm 3
Under Assumption 1, the containment controller (16) can be constructed as follows:

(i) Choose $K = -(B^T PB)^{-1} B^T PA$, where $P > 0$ is the unique solution to the following MARE:

$$P = A^T PA - \left(1 - \max_{i=1,\cdots,M} |\hat{\lambda}_i|^2\right) A^T PB(B^T PB)^{-1} B^T PA + R, \tag{19}$$

with $R > 0$ and $\hat{\lambda}_i$, $i = 1, \cdots, M$, being the eigenvalues of $\mathcal{D}_1$.

(ii) Select $L = -AQC^T(CQC^T)^{-1}$, where $Q > 0$ is the unique solution to the following MARE:

$$Q = AQA^T - \left(1 - \max_{i=1,\cdots,M} |\hat{\lambda}_i|^2\right) AQC^T(CQC^T)^{-1}CQA^T + R. \tag{20}$$

Remark 4
According to Lemma 3, for the case where A has at least an eigenvalue outside the unit circle, a sufficient condition for the existence of the controller (16) by using Algorithm 3 is that (A, B, C) is stabilizable and detectable, and $\max_{i=1,\cdots,M} |\hat{\lambda}_i|^2 < 1 - \delta_c$, where δ_c is defined in Lemma 3. For the case where A has no eigenvalues with magnitude larger than 1, the sufficient condition is reduced to that (A, B, C) is stabilizable and detectable.

Remark 5
When applying Algorithms 1, 2, and 3 to construct the containment controllers, we do not have to know the exact eigenvalues of $\mathcal{L}_1$ in (3) or $\mathcal{D}_1$ in (17). Instead, we can use a lower bound for the real parts of the eigenvalues of $\mathcal{L}_1$ or an upper bound for the magnitudes of the eigenvalues of $\mathcal{D}_1$. By choosing the lower bound reasonably small in the continuous-time case and the upper bound close to 1 in the discrete-time case, the containment controllers given by these algorithms will maintain

certain robustness margin with respect to modifications of the communication topology, like adding or deleting some communication links.

Theorem 2

Assume that the directed communication graph $\mathcal{G}$ satisfies Assumption 1. Let (A, B, C) be stabilizable and detectable. Then, the controller given by Algorithm 3 solves the containment control problem for the agents described by (16). Specifically, $\lim_{k\to\infty}\left(x_f(k) - \psi_x(k)\right) = 0$ and $\lim_{k\to\infty}\left(\hat{v}_f(k) - \psi_v(k)\right) = 0$, where $x_f = \left[x_1^T, \cdots, x_M^T\right]^T$, $\hat{v}_f = \left[\hat{v}_1^T, \cdots, \hat{v}_M^T\right]^T$,

$$\begin{aligned} \psi_x(k) &\triangleq \left[(I_M - \mathcal{D}_1)^{-1}\mathcal{D}_2 \otimes A^k\right]\begin{bmatrix} x_{M+1}(0) \\ \vdots \\ x_N(0)\end{bmatrix}, \\ \psi_v(k) &\triangleq \left[(I_M - \mathcal{D}_1)^{-1}\mathcal{D}_2 \otimes A^k\right]\begin{bmatrix} \hat{v}_{M+1}(0) \\ \vdots \\ \hat{v}_N(0)\end{bmatrix}. \end{aligned} \tag{21}$$

Proof

Let $\hat{z}_i = \left[x_i^T, \hat{v}_i^T\right]^T$, $\hat{z}_f = \left[\hat{z}_1^T, \cdots, \hat{z}_M^T\right]^T$, and $\hat{z}_l = \left[\hat{z}_{M+1}^T, \cdots, \hat{z}_N^T\right]^T$. Then, we can obtain from (15) and (16) that the collective network dynamics can be written as

$$\begin{aligned} \hat{z}_f^+ &= [I_M \otimes \mathcal{S} + (I_M - \mathcal{D}_1) \otimes \mathcal{H}]\,\hat{z}_f - (\mathcal{D}_2 \otimes \mathcal{H})\hat{z}_l, \\ \hat{z}_l^+ &= (I_{N-M} \otimes \mathcal{S})\hat{z}_l, \end{aligned} \tag{22}$$

where $\mathcal{S}$ and $\mathcal{H}$ are defined in (8). Let $\zeta_i = \sum_{j\in\mathcal{F}\cup\mathcal{R}} d_{ij}\left(\hat{z}_i - \hat{z}_j\right)$, $i \in \mathcal{F}$, and $\zeta = \left[\zeta_1^T, \cdots, \zeta_M^T\right]^T$. Then, we have

$$\zeta = [(I_M - \mathcal{D}_1) \otimes I_{2n}]\,\hat{z}_f - (\mathcal{D}_2 \otimes I_{2n})\,\hat{z}_l. \tag{23}$$

By following similar steps to those in the proof of Theorem 1, we can obtain from (22) and (23) that ζ satisfies the following dynamics:

$$\zeta^+ = [I_M \otimes \mathcal{S} + (I_M - \mathcal{D}_1) \otimes \mathcal{H}]\,\zeta. \tag{24}$$

Under Assumption 1, it follows from Lemma 2 that all the eigenvalues of $I_M - \mathcal{D}_1$ have positive real parts. Let $\hat{U} \in \mathbf{C}^{M\times M}$ be such a unitary matrix that $\hat{U}^H(I_M - \mathcal{D}_1)\hat{U} = \hat{\Lambda}$, where $\hat{\Lambda}$ is an upper-triangular matrix with $1 - \hat{\lambda}_i$, $i = 1, \cdots, M$, on the diagonal. Let $\tilde{\zeta} \triangleq \left[\tilde{\zeta}_1^T, \cdots, \tilde{\zeta}_N^T\right]^T = \left(\hat{U}^H \otimes I_{2n}\right)\zeta$. Then, (24) can be rewritten as

$$\tilde{\zeta}^+ = \left(I_M \otimes \mathcal{S} + \hat{\Lambda} \otimes \mathcal{H}\right)\tilde{\zeta}. \tag{25}$$

Clearly, (11) is asymptotically stable if and if the following M systems

$$\tilde{\zeta}_i^+ = \left[\mathcal{S} + (1 - \hat{\lambda}_i)\mathcal{H}\right]\tilde{\zeta}_i, \quad i = 1, \cdots, M, \tag{26}$$

are simultaneously asymptotically stable. As shown in the proof of Theorem 1, $\mathcal{S}+(1-\hat{\lambda}_i)\mathcal{H}$ are similar to $\begin{bmatrix} A + (1-\hat{\lambda}_i)LC & 0 \\ -(1-\hat{\lambda}_i)LC & A + (1-\hat{\lambda}_i)BK \end{bmatrix}$, $i = 1, \cdots, M$. In light of step (i) in Algorithm 3,

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we can obtain

$$
\begin{aligned}
&\left[A+(1-\hat{\lambda}_i)BK\right]^H P\left[A+(1-\hat{\lambda}_i)BK\right]-P \\
&\quad= A^TPA-P+\left[-2\mathbf{Re}(1-\hat{\lambda}_i)+|1-\hat{\lambda}_i|^2\right]A^TPB(B^TPB)^{-1}B^TPA \\
&\quad= A^TPA-P+\left(|\hat{\lambda}_i|^2-1\right)A^TPB(B^TPB)^{-1}B^TPA \\
&\quad\leqslant A^TPA-P-\left(1-\max_{i=1,\cdots,M}|\hat{\lambda}_i|^2\right)A^TPB(B^TPB)^{-1}B^TPA \\
&\quad= -R<0.
\end{aligned}
\tag{27}
$$

Then, (27) implies that $A+(1-\hat{\lambda}_i)BK$, $i=1,\cdots,N$, are Schur stable. Similarly, by step (ii) in Algorithm 3, we can show that $A+(1-\hat{\lambda}_i)LC$, $i=1,\cdots,M$, are also Schur stable. Therefore, considering (25) and (26), we obtain that (24) is asymptotically stable, which, by (23), implies that $\|z_f(k)-\left[(I_M-\mathcal{D}_1)^{-1}\mathcal{D}_2\otimes I_{2n}\right]z_l(k)\|\to 0$, as $k\to\infty$. Then, we can obtain that $\lim_{k\to\infty}\left(x_f(k)-\psi_x(k)\right)=0$ and $\lim_{k\to\infty}\left(v_f(k)-\psi_v(k)\right)=0$. In virtue of Lemma 2, the states of the followers asymptotically converge to the convex hull formed by those of the leaders, that is, the containment control problem is solved. □

Remark 6
Theorem 2 gives the discrete-time counterpart of the results in Theorem 1. In contrast to the continuous-time case where the Laplacian matrix $\mathcal{L}$ is used to represent the communication graph, the row-stochastic matrix $\mathcal{D}$ is utilized here in the discrete-time case. Different from Theorem 1 in the last section, Algorithm 3 for constructing discrete-time containment controllers and the proof of Theorem 2 rely on the MARE. Furthermore, the eigenvalues of $\mathcal{D}$ in the discrete-time case have to further satisfy a constraint related to the unstable eigenvalues of the state matrix A, when A has eigenvalues outside the unit circle.

4.2. *Static containment controller*

In this subsection, a special case where the relative states between neighboring agents are available is considered. For this case, a distributed static containment controller is proposed as

$$
u_i = F\sum_{j\in\mathcal{F}\cup\mathcal{R}} d_{ij}(x_i-x_j), \quad i\in\mathcal{F}, \tag{28}
$$

where d_{ij} is defined as in (16) and $F\in\mathbf{R}^{p\times n}$ is the feedback gain matrix to be designed.

Let $x_f=\left[x_1^T,\cdots,x_M^T\right]^T$ and $x_l=\left[x_{M+1}^T,\cdots,x_N^T\right]^T$. Using (14) for (1) gives the closed-loop network dynamics as

$$
\begin{aligned}
x_f^+ &= [I_N\otimes A+(I_M-\mathcal{D}_1)\otimes BF]x_f-(\mathcal{D}_2\otimes BF)x_l, \\
x_l^+ &= (I_{N-M}\otimes A)x_l,
\end{aligned}
$$

where $\mathcal{D}_1$ and $\mathcal{D}_2$ are defined in (17).

Corollary 2
Assume that (A,B) is stabilizable and the communication graph $\mathcal{G}$ satisfies Assumption 1. Then, the controller (28) with $F=-(B^TPB)^{-1}B^TPA$ solves the containment control problem, where $P>0$ is the unique solution to (19). Specifically, $\lim_{k\to\infty}(x_f(k)-\psi_x(k))=0$, where $x_f(k)$ and $\psi_x(k)$ are defined in (21).

5. SIMULATION EXAMPLES

In this section, two simulation examples are provided to validate the effectiveness of the theoretical results.

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Example 1

In this example, we take the agents in (1) as the Caltech multi-vehicle wireless testbed vehicles, whose linearized dynamics can be described by (1) [30], with

$$x_i = \begin{bmatrix} x_{i1} & x_{i2} & x_{i3} & \dot{x}_{i1} & \dot{x}_{i2} & \dot{x}_{i3} \end{bmatrix}^T,$$
$$A = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -0.2003 & -0.2003 & 0 & 0 \\ 0 & 0 & 0.2003 & 0 & -0.2003 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1.6129 \end{bmatrix},$$
$$B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0.9441 & 0.9441 \\ 0.9441 & 0.9441 \\ -28.7097 & 28.7097 \end{bmatrix},$$

where x_{i1} and x_{i2} are, respectively, the positions of the i-th vehicle along the x and y coordinates, x_{i3} is the orientation of the i-th vehicle. The objective is to design a static containment controller in the form of (14) by using local state information of neighboring vehicles.

Solving the LMI (4) by using the LMI toolbox of MATLAB gives the feedback gain matrix of (14) as

$$F = \begin{bmatrix} -0.0089 & 0.0068 & 0.0389 & -0.0329 & 0.0180 & 0.0538 \\ 0.0068 & -0.0089 & -0.0389 & 0.0180 & -0.0329 & -0.0538 \end{bmatrix}.$$

For illustration, let the communication graph $\mathcal{G}$ be given by Figure 1, where nodes $7, 8, 9$ are three leaders and the others are followers. It can be verified that $\mathcal{G}$ satisfies Assumption 1. Correspondingly, the matrix $\mathcal{L}_1$ in (3) is

$$\mathcal{L}_1 = \begin{bmatrix} 3 & 0 & 0 & -1 & -1 & -1 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ -1 & -1 & 2 & 0 & 0 & 0 \\ -1 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 \end{bmatrix},$$

whose eigenvalues are 0.8213, 1, 2, $2.3329 \pm 0.6708i$, and 3.5129. By Algorithm 1, we choose the coupling strength $c \geqslant 1.2176$. The positions and orientations of the nine vehicles under the controller (14) with F as above and $c = 2$ are depicted in Figure 2, from which it can be observed that the containment control problem is indeed solved.

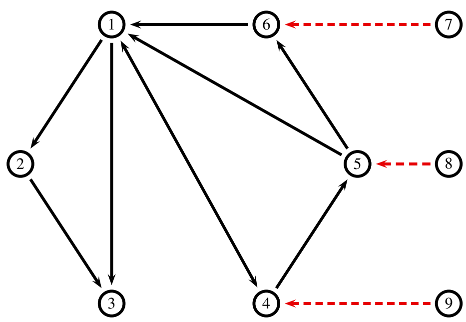

Figure 1. The communication topology.

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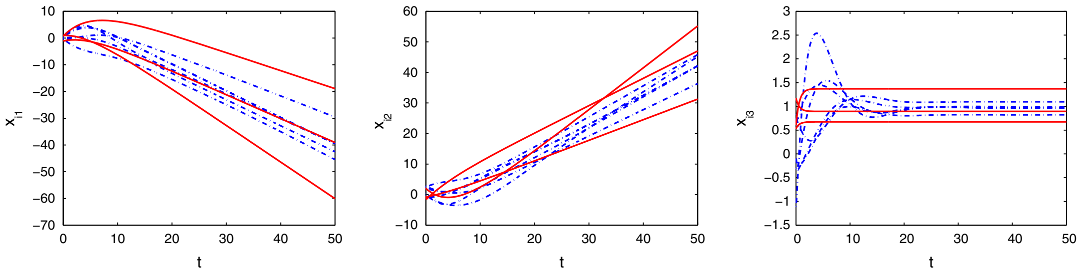

Figure 2. The positions and orientations of the nine vehicles under (14). The solid and dashdotted lines denote, respectively, the trajectories of the leaders and the followers.

Example 2

Consider a network of discrete-time double integrators described by (15), with

$$x_i = \begin{bmatrix} x_{i1} \\ x_{i2} \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix}.$$

To illustrate Theorem 2, assume that the communication topology $\mathcal{G}$ is given as in Figure 1. The corresponding $\mathcal{D}_1$ in (17) is

$$\mathcal{D}_1 = \begin{bmatrix} 0.4 & 0 & 0 & 0.1 & 0.3 & 0.2 \\ 0.5 & 0.5 & 0 & 0 & 0 & 0 \\ 0.3 & 0.2 & 0.5 & 0 & 0 & 0 \\ 0.5 & 0 & 0 & 0.5 & 0 & 0 \\ 0 & 0 & 0 & 0.4 & 0.4 & 0.2 \\ 0 & 0 & 0 & 0 & 0.3 & 0.7 \end{bmatrix},$$

whose eigenvalues are $\hat{\lambda}_i = 0.1809, 0.2361 \pm 0.3094i, 0.5$, and 0.8469. Clearly, $\max_{i=1,\cdots,6} |\hat{\lambda}_i| = 0.8469$ in (19) and (20). By solving MAREs (19) and (20) with $\max_{i=1,\cdots,6} |\hat{\lambda}_i| = 0.9$, we can obtain the feedback gain matrices of (16) as $K = -\begin{bmatrix} 0.1038 & 1.1038 \end{bmatrix}$ and $L = -\begin{bmatrix} 1.1038 \\ 0.1038 \end{bmatrix}$. The state trajectories of the agents are depicted in Figure 3, which shows that the containment problem is solved in this case.

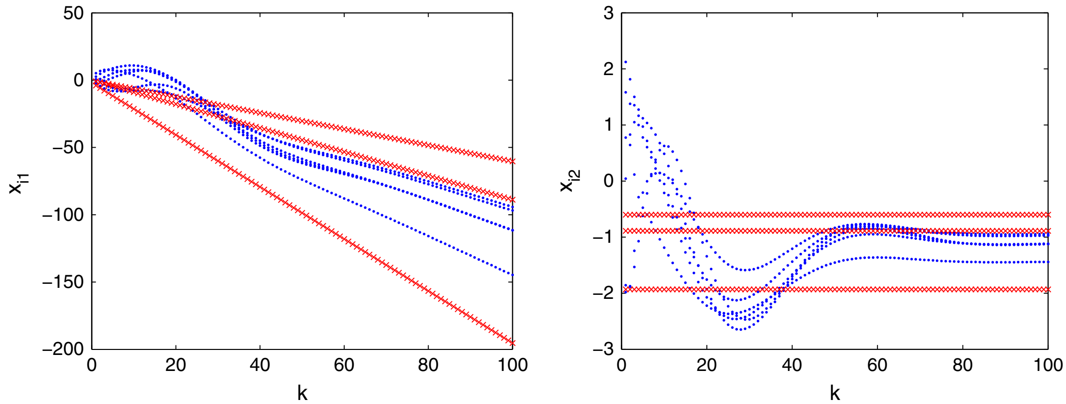

Figure 3. The state trajectories of the agents (15) under (16). The x-shaped and dotted lines denote, respectively, the trajectories of the leaders and the followers.

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6. CONCLUSIONS

In this paper, the containment control problems have been considered for both continuous-time and discrete-time multi-agent systems with general linear dynamics under directed communication topologies. Distributed dynamic containment controllers based on only the relative outputs of neighboring agents have been constructed for both continuous-time and discrete-time cases, under which the states of the followers can asymptotically converge to the convex hull formed by those of the leaders if, for each follower, there exists at least one leader that has a directed path to that follower. The current paper has extended the existing results on containment control to multi-agent systems with general continuous-time and discrete-time linear dynamics.

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