

0x02 - Digital Arithmetic

ENGR 3410: Computer Architecture

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Fall 2020

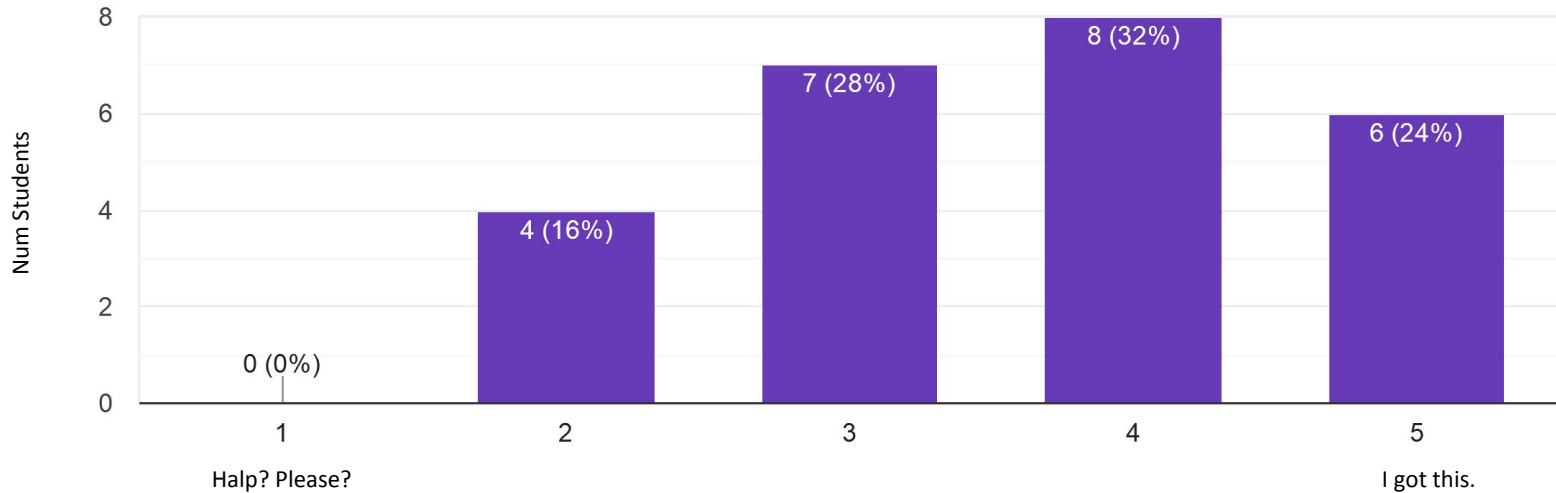
Housekeeping

- HW1 was due last night, solution on Canvas.
- HW2 live now, due next Monday
- Lab 1, Part 0 due last night
- Lab 1, Parts 1 & 2 due next Monday (9/21)

Feedback - Understanding

How are you feeling about your current level of understanding of the topics we've seen so far?

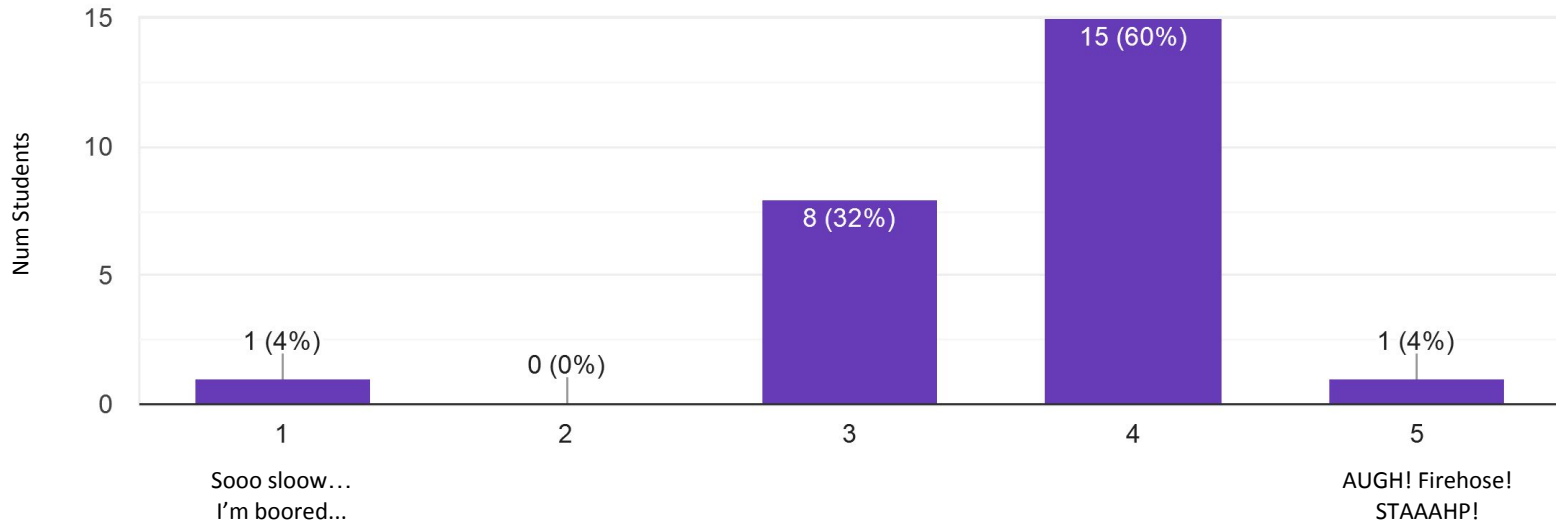
25 responses



Feedback - Pacing

How's the lecture pace?

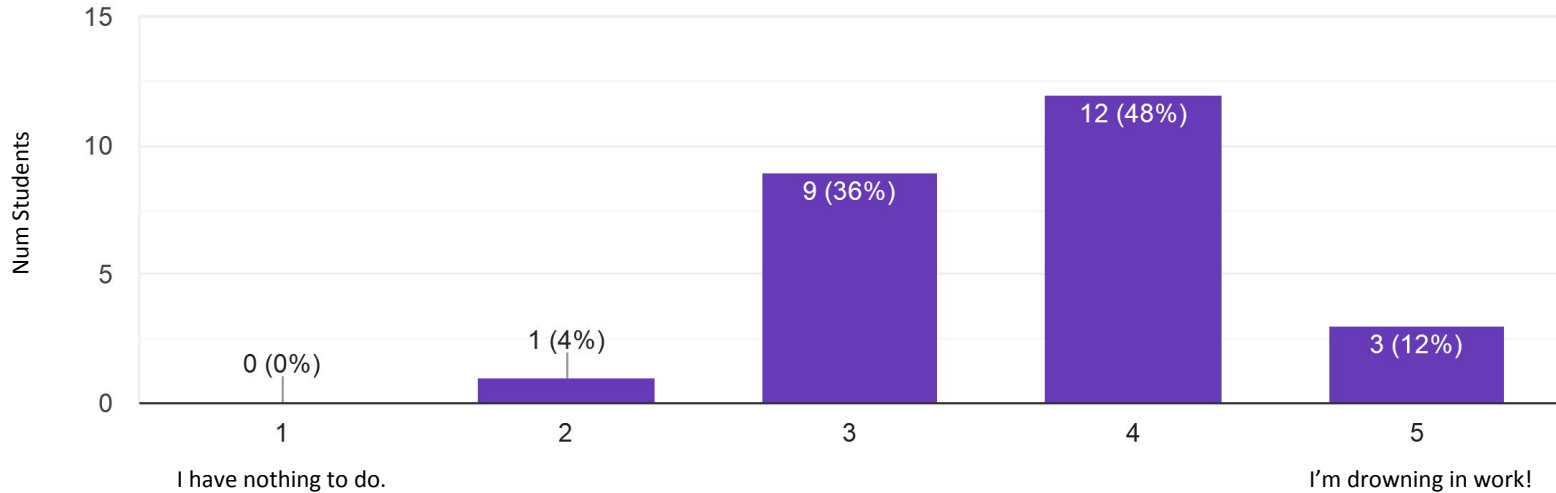
25 responses



Feedback - Workload

What's your workload like this week?

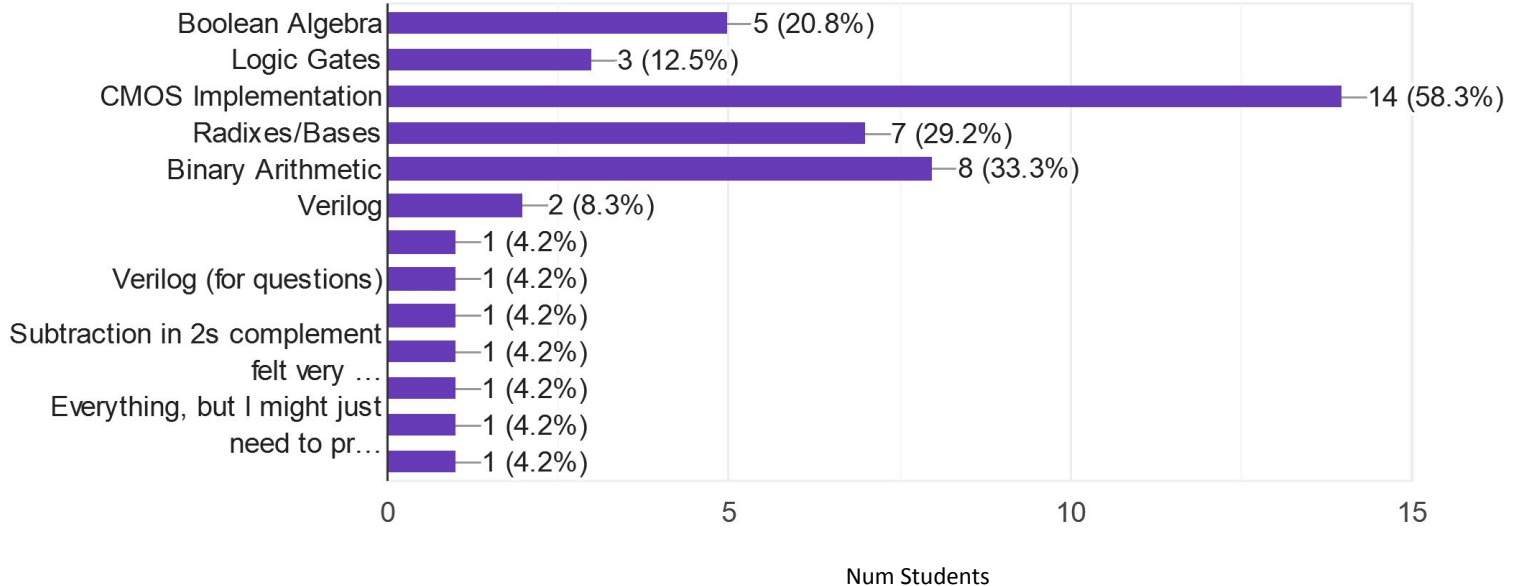
25 responses



Feedback - More, Plz

I wish we spent more time on...

24 responses



~~14~~ in radix r

Review

Radix to Decimal

$x_r \rightarrow x_{10}$

$$x = \sum_{i=0}^{n-1} r^i d_i$$

$d_2 d_1 d_0$

$r = 2$

$010 = 2$

$$2^2(0) + 2^1(1) + 2^0(0)$$

Decimal to Radix

$x_{10} \rightarrow x_r$

```
i = 0;
while (x > 0):
    d[i] = x % r
    x = x / r
    i++
```

$x_{10} = 100$

$r = 16$

$x_{16} = 0x64$

$d[5] = 100 \% 16$
 $= 4$

$x = 100 // 16$

$= 6$
 $d[1] = 6 \% 16$
 $= 6$

complement
 $(r-1)x + 1 = (r-1)$

Review

$r-1$'s complement

0	1	2	3	4	5	6	7	8	9
9	8	7	6	5	4	3	2	1	0

Subtraction via R's Complement

Want to solve:

$$x - y = ?$$

$$\begin{array}{r} \cancel{2}^1 \cancel{0}^1 3 \\ - \quad 2 \quad 3 \\ \hline 180 \end{array} \quad \begin{array}{r} \cancel{x}^1 2 \cancel{0}^1 3 \\ + 977 \\ \hline \cancel{x} 180 \end{array}$$

Example Inputs, $R = 10$:

$$x = 203, y = 23$$

Recall that $x - y = x + y'$

$$x + (-y)$$

$y = 2023$
 $y' = 976 + 1$ \swarrow R 's complement
 $= 977$

Review

Two's Complement (R=2)

$$x' = -x = \sim x + 1$$

2's complement

negative

bitwise NOT

Allows us to use Adders for everything

$$\begin{array}{r} 0011 (+3) \\ - 0100 (+4) \\ \hline \end{array}$$

$$A - B = A + B'$$

$$\begin{array}{r} 0011 (+3) \\ + 1100 (-4) \\ \hline \end{array}$$

$$1111 (-1)$$

?

$$x = 0100$$

$$\sim x = 1011$$

$$\sim x + 1 = 1100$$

$$\begin{array}{r} \sim 1111 \\ 0000 \\ + \\ \hline 1 \end{array}$$

2⁴

2³ positive
negative 1

$$x' = -x = \sim x + 1$$

$$-(-4)$$

$$\begin{array}{r} 1100 \\ 0011 + 1 \\ \downarrow \\ 0100 \end{array}$$

Today

Wrap up Subtraction

Multiply and Divide

Intro to Fixed- and Floating-point numbers

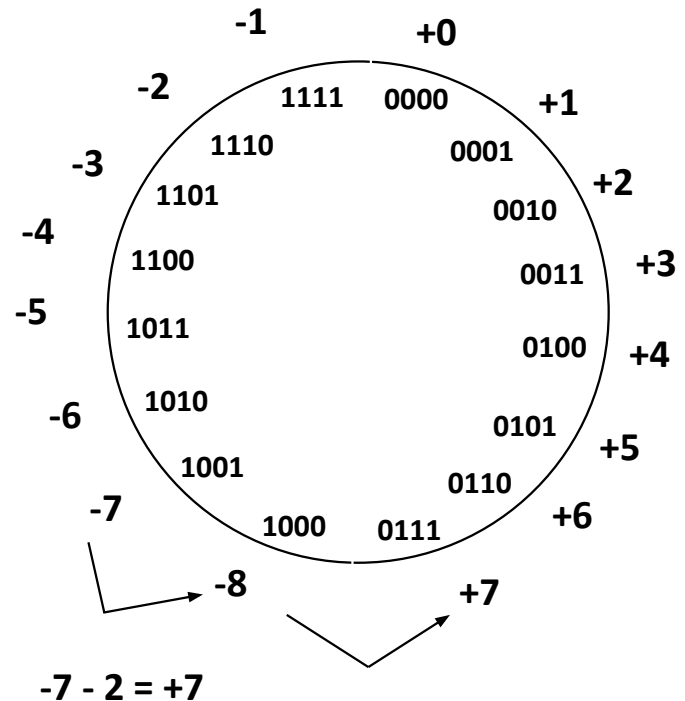
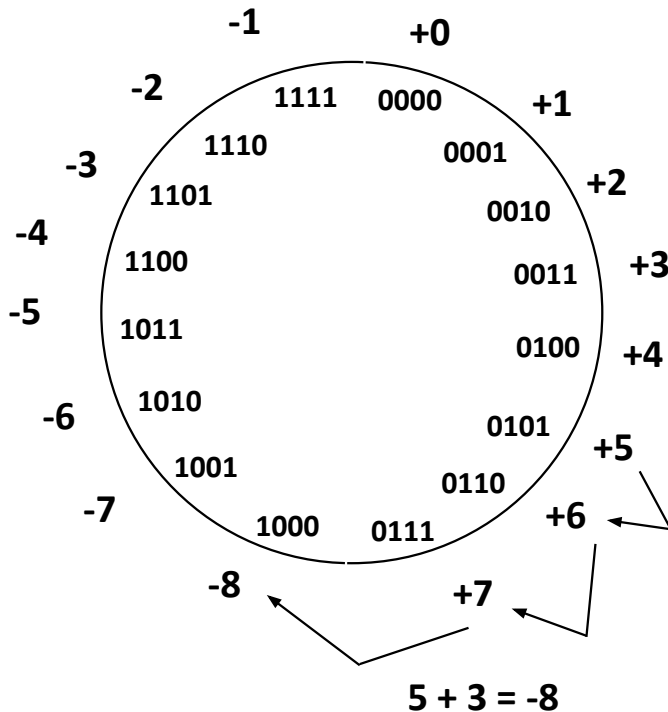
Big Takeaway

2's Complement turns stuff into addition
(mostly)

Overflows in Two's Complement

Add two positive numbers to get a negative number

or two negative numbers to get a positive number



Overflow Detection in 2's Complement

overflow

$$\begin{array}{r}
 \text{5} \quad 0101 \\
 \underline{\text{3} \quad 0011} \\
 \text{-8} \quad 1000
 \end{array}$$

no

$$\begin{array}{r}
 \text{5} \quad 0101 \\
 \underline{\text{2} \quad 0010} \\
 \text{7} \quad 0111
 \end{array}$$

overflow

$$\begin{array}{r}
 \text{-7} \quad 1001 \\
 \underline{\text{-2} \quad 1110} \\
 \text{7} \quad 1011
 \end{array}$$

no

$$\begin{array}{r}
 \text{-3} \quad 1101 \\
 \underline{\text{-5} \quad 1011} \\
 \text{-8} \quad 1000
 \end{array}$$

unsigned

$$\begin{array}{r}
 1000 \\
 + 1000 \\
 \hline
 10000
 \end{array}$$

next state discussion

correct

8-bit

Detecting Overflow

C _{in}	C _{out}	Overflow
0	0	0
0	1	1
1	0	1
1	1	0

A	B	C _{in}	Sum	C _{out}	Overflow
0	0	0	0	0	0
0	0	1	1	0	1
0	1	0	1	0	0
0	1	1	0	1	0
1	0	0	1	0	0
1	0	1	0	1	0
1	1	0	0	1	1
1	1	1	1	1	0

sign of output
MSB

C_{in} C_{out}
||
AND
Overflow

(Unsigned) Multiplication

M digit x N digit
= M+N digit

Like grade school, right?

How many bits of output?

How to handle negative?

- Sign-Magnitude
- Sign-Extend
- Booth's Algorithm
- Many more...

$$\begin{array}{r} 0010 = 2 \\ \times 1011 = 11 \\ \hline \end{array}$$

$$\begin{array}{r} 0010 \\ 0010 \\ 0000 \\ + 0010 \\ \hline 0010110 \end{array}$$

6 d₇ d₀

$$2^6 0 + 2^5 0 + 2^4 1 + 2^3 0 + 2^2 1 + 2^1 1 + 2^0 0$$
$$16 + 4 + 2 = 22$$

1011 \rightarrow 2D \rightarrow 0100 + 1 0101 5

Two's Complement Multiplication

$$\begin{array}{r} 0010 = 2 \\ \times 1011 = -5 \\ \hline 0010 \end{array}$$

$$0010$$

$$0000$$

$$00010$$

$$00010110$$

↑
this isn't
1, we were
expecting 10

$$\begin{array}{r} 1011 \\ \times 0010 \\ \hline 0000 \end{array}$$

$$+ 1011$$

$$00010110$$

something's
wrong!

Two's Complement Multiplication

$$\begin{array}{r}
 0010 \\
 x 1011 \\
 \hline
 00000010 \\
 00000100 \\
 00000000 \\
 00001000 \\
 00010000 \\
 01000000 \\
 10000000 \\
 + 00000000 \\
 \hline
 11110110
 \end{array}$$

Handwritten notes: "also 0's" (pointing to leading zeros), "same" (pointing to the multiplier).

$$\begin{array}{r}
 1011 = -5 \\
 x 0010 = 2 \\
 \hline
 00000000 \\
 + 11110111 \\
 \hline
 11110110 = -10? \\
 \\
 00001001 \\
 + 00001001 \\
 \hline
 00001010 \\
 8 + 2 = 10
 \end{array}$$

Handwritten notes: "same" (pointing to the multiplier), "8 + 2 = 10" (at the bottom).

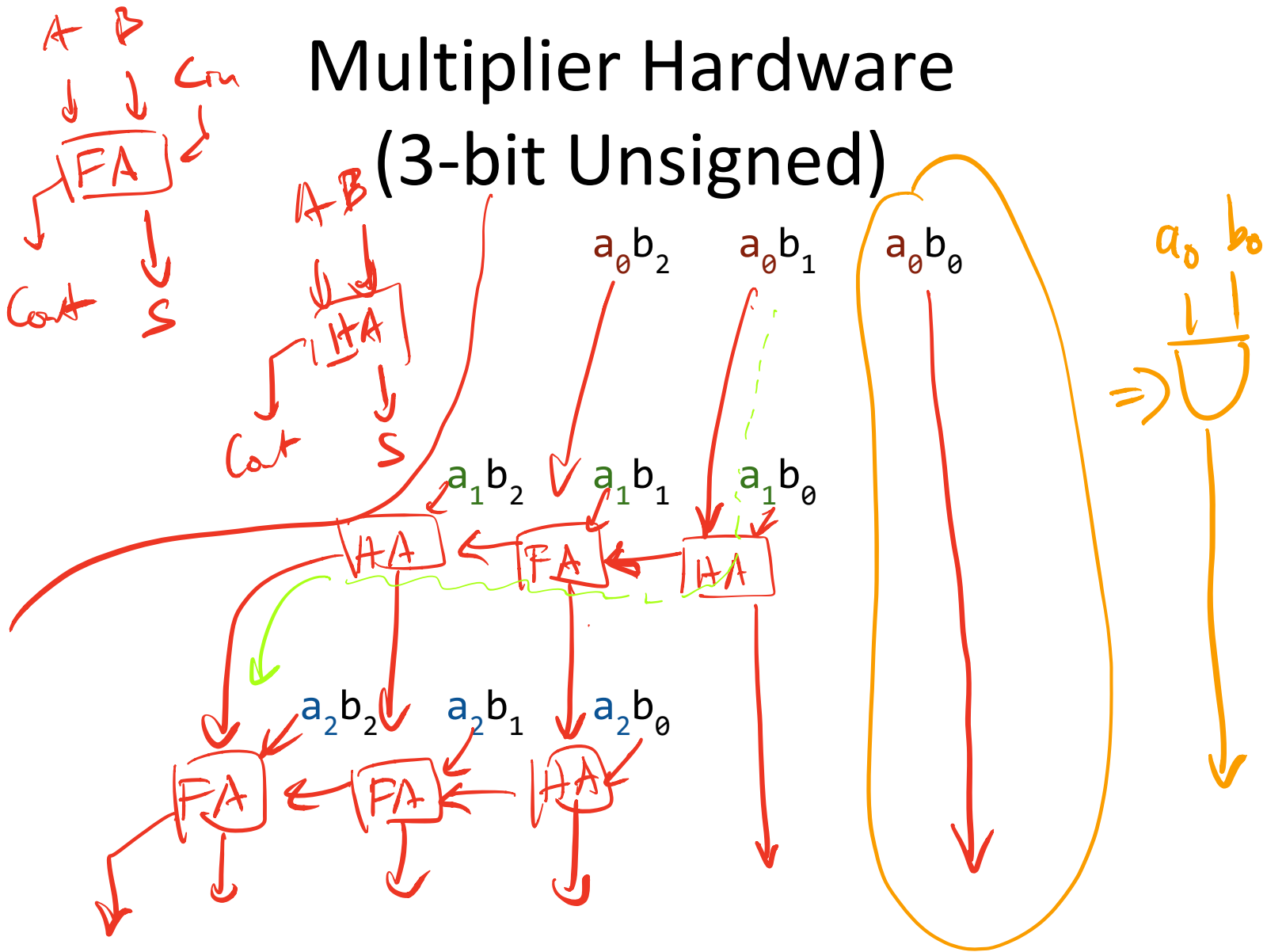
Multiplier Hardware (Unsigned)

$$a_0 = D - a_0 b_0$$

		b_2	b_1	b_0
x		a_2	a_1	a_0
		$a_0 b_2$	$a_0 b_1$	$a_0 b_0$
	$a_1 b_2$	$a_1 b_1$	$a_1 b_0$	
	$a_2 b_2$	$a_2 b_1$	$a_2 b_0$	

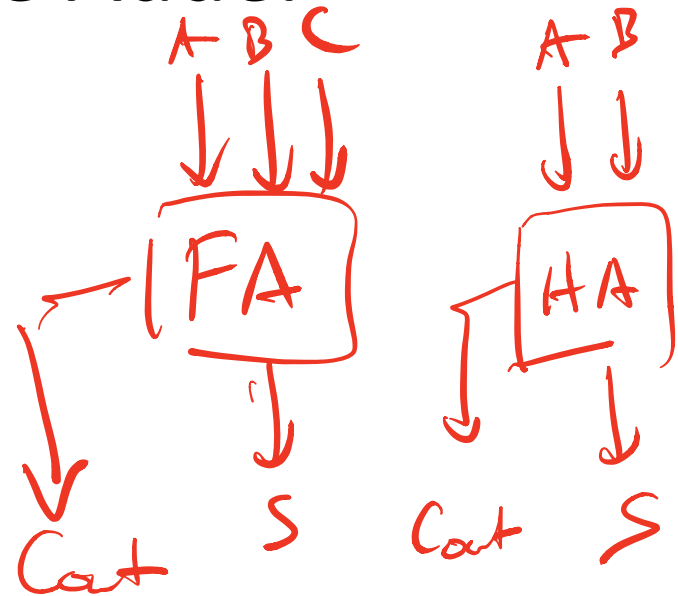
$$\begin{array}{r}
 011 \\
 \times 010 \\
 \hline
 000 \\
 011 \\
 000
 \end{array}$$

Multiplier Hardware (3-bit Unsigned)



Carry Save Adder

A	B	C _{in}	Carry	S
0	0	0	0	0
0	0	1	0	1
0	1	0	0	1
0	1	1	1	0
1	0	0	0	1
1	0	1	1	0
1	1	0	1	0
1	1	1	1	1



$$\begin{array}{r}
 1 \\
 111 \\
 + 010 \\
 \hline
 1001
 \end{array}$$

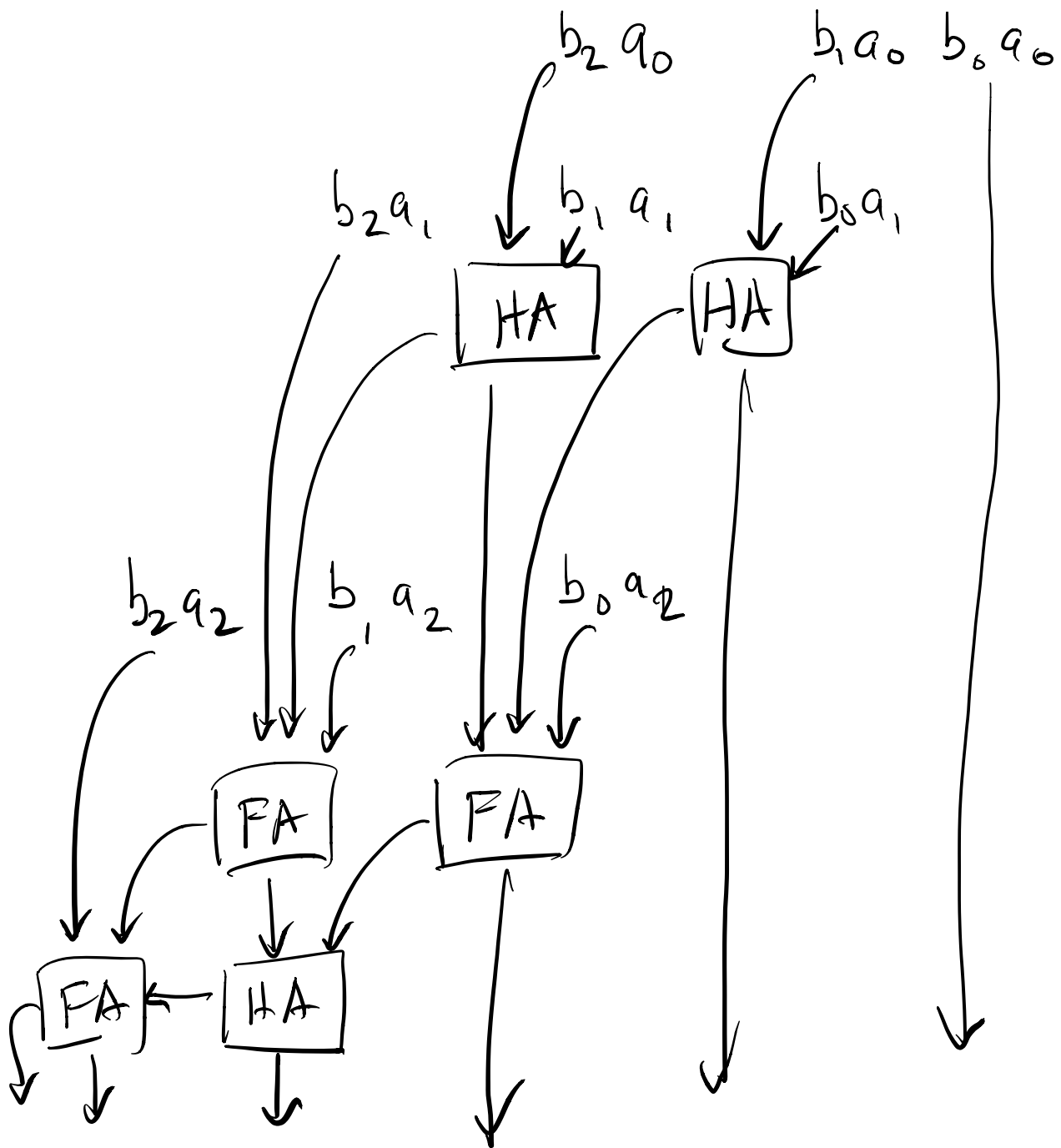
Carry Save Array Multiplier (3-bit Unsigned)

$$a_0 b_2 \quad a_0 b_1 \quad a_0 b_0$$

$$a_1 b_2 \quad a_1 b_1 \quad a_1 b_0$$

$$a_2 b_2 \quad a_2 b_1 \quad a_2 b_0$$

Carry Save Multiplier (3-bit Unsigned)



Division

$$\begin{array}{r} 1001 \overline{) 11011} \\ \underline{1001} \\ 1001 \\ \underline{1001} \\ 0 \end{array}$$

Fixed Point Numbers

Integer scaled by a **fixed** factor.

Fixed # of digits before/after the radix point

$$x = \sum_i r^i d_i$$

allow $i < 0$

^{3 2 1 0 -1 -2 -3 -4}
0101.1100

$$2^3 \cdot 1 + 2^2 \cdot 0 + 2^1 \cdot 1 + 2^0 \cdot 1 + 2^{-1} \cdot 1 + 2^{-2} \cdot 1 + 2^{-3} \cdot 0 + 2^{-4} \cdot 0$$
$$4 + 1 + 0.5 + 0.25$$
$$5.75$$

Fixed Point Numbers

IQ Notation

I4Q2 Signed, 4 bits before, 2 bits after

XXXX.XX

U2Q5 Unsigned, 2 before, 5 after

XX.XXXXX