Utsav Mehta - 20CS10069

Linear Algebra in AI and ML

Q1. Prove that M remains a column stochastic matrix.
Prove that M has only positive entries.

⇒ The representation of matrix M in the random surfer model can be considered as:

(i) A considerably ~~proportion~~ of time will ~~be~~ spent to abandon the current page and teleport to another page
(ii) from any page (say x), the surfer follows one of the outbounds links and traverse to one of the neighbours of x.
(iii) The dumping factor (denoted by p) represents the chances/probability that the surfer leaves current page and teleports to another.
(iv) The probability of choosing such a page is clearly $1/n$. This fact is used in construction of B's structure.

∴ The page-rank matrix is as follows:

$M = (1-p)A + p.B$, where $p \in (0,1)$

transition matrix

$B = \frac{1}{n}\begin{bmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1 \end{bmatrix} = \begin{bmatrix} 1/n & \cdots & 1/n \\ 1/n & \ddots & \vdots \\ 1/n & \cdots & 1/n \end{bmatrix}$

Stochastic ~~column~~ column matrix.

∴ $A_{ij} \geq 0$ and $\sum_{i=1}^{n} (A_i)_j = 1$ (as Σ probability $= 1$)

∴ for any column j of M, the following holds:

$M_j = (1-p)A_j + pB_j = (1-p)A_j + p \cdot \frac{1}{n} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$

∴ for every row i, the following holds:

$(M_i)_j = \{(1-p)(A_i)_j + \frac{p}{n}\}$

$$\sum_{i=1}^{n} (M_i)_j = \sum_{i=1}^{n} \{(1-p)(A_i)_j + p/n\}$$

$$= (1-p)\cdot 1 + \sum_{i=1}^{n} p/n = 1-p+p = 1$$

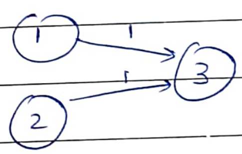
$\left[\sum_{i=1}^{n} (A_i)_j = 1\right.$ as shown previously$\left.\right]$

Hence M is a column stochastic [if $p<1$, $M_{ij} > 0$].

Q2. Redo the computations for Page Rank with the transition matrix A replaced with the matrix M, for the graphs representing the Dangling Nodes, respectively Disconnected components. Do the problems mentioned in there still occur?

$\Rightarrow$ Following problems are encountered in Page Rank:

- Nodes with edges: (outgoing)



for such graphs,

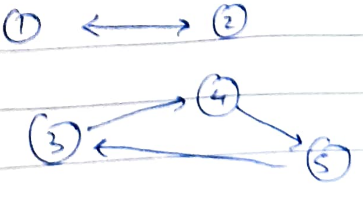
$$M = (1-p)A + pB = (1-p)\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix} + p\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} p/3 & p/3 & p/3 \\ p/3 & p/3 & p/3 \\ 1-2p/3 & 1-2p/3 & p/3 \end{bmatrix}$$

$$v_0 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \Rightarrow Mv_0 = \begin{bmatrix} p \\ p \\ 2-p \end{bmatrix} \; [\text{for } p \in (0,1)] = v_1$$

$$v_2 = Mv_1 = \begin{bmatrix} p(p+2)/3 \\ p(p+2)/3 \\ p(8-5p)/3 \end{bmatrix}$$

- Nodes with edges: (Disconnected components)



For the above graph,

$$A = \left[\begin{array}{cc|ccc} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

$$\therefore M = (1-p)A + pB = \begin{bmatrix} p/5 & 1-4p/5 & p/5 & p/5 & p/5 \\ 1-4p/5 & p/5 & p/5 & p/5 & p/5 \\ p/5 & p/5 & p/5 & p/5 & 1-4p/5 \\ p/5 & p/5 & 1-4p/5 & p/5 & p/5 \\ p/5 & p/5 & p/5 & 1-4p/5 & p/5 \end{bmatrix}$$

Considering no loss generality ($p = 0.15$),

$$M = \begin{bmatrix} 0.03 & 0.88 & 0.03 & 0.03 & 0.03 \\ 0.88 & 0.03 & 0.03 & 0.03 & 0.03 \\ 0.03 & 0.03 & 0.03 & 0.03 & 0.88 \\ 0.03 & 0.03 & 0.88 & 0.03 & 0.03 \\ 0.03 & 0.03 & 0.03 & 0.88 & 0.03 \end{bmatrix}$$

Again, for $v_0 = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$, $v_1 = Mv_0 = \cancel{\begin{bmatrix} 0.97 \\ 0.97 \\ 0.97 \\ 0.97 \\ 0.97 \end{bmatrix}} \begin{bmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} = v_0$

And similarly, $v_k = M^k v_0 = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$ is clearly convergent.

→ Thus the matrix M connects the network & eliminates the "dangling nodes".

→ Hence, a node with no incoming edge poses no reason to be reallocated.

→ Hence, by power-method convergence, a theorem as M is a +ve, column stochastic $n \times n$ matrix.

→ The probabilistic eigen vectors corresponding to eigen value $=1$ be z^+ (say). If z is a column vector with all entries $= 1/n$

→ Thus, $z, Mz, \ldots M^r z$ converges to z^+. This clearly eradicates the issues by using M.