*I*nformation theory notes

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Contents.

- 1 - Basics of information theory.

Information theory, alongside data compression, have a key role in modern CS. The former defines a set of theoretical tools, usefull to understand the limitations of what is computable; the latter allows to reduce the amount of space required (in terms of bits) without losing information.

- 1.1 - Shannon's Entropy.

When talking about information we all have a general notion of what it represents, but can we define it formally? And also, is there a way to measure information?

The first to answer these questions was *Claude Shannon*, by many condsidered the father of modern information theory. In [2] Shannon analizes various comunication systems: from the discrete noiceless one to the continous noisy one. A general structure of these systems is shown in *Figure 1*.

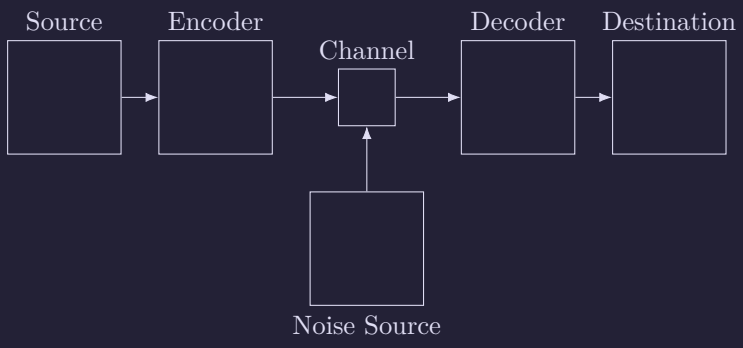


Figure 1: Diagram of a general communication system.

- The *Source* or, more precisely, the *Information Source* referes to some entity (a human, a computer, etc) that produces messages.
- The *Encoder* encodes the messages coming from the source, and transmits them trough the channel. .
- The *channel* is the physical mean trough which the messages are transmitted.
- The *Decoder* has the opposite role of the *Encoder*.
- The *Destination* is the entity to which the messages are meant for.

Remark. What follows in this section, and the those following, refers just to the discrete noiceless case. We also assume the source to be memoryless, meaning that each symbol produced is indipendent from the previous one.

Let S be a source of information. Let Σ be the alphabet of symbols used by the source, and for each symbols let $p_i = \Pr(S = \sigma_i)$ at any given time. What we are looking for is some function H, if it exists, that measures the uncertainty we have about S.

As we will show in *Section 1.1.1*, the only form H can assume is the following:

$$H = -K \sum_{i=1}^{n=|\Sigma|} p_i \log_b p_i. \tag{1}$$

where K is a positive constant and b is usually 2.

We define $H(S)$ to be the *entropy* of S. Formally, the entropy measures the average amount of information stored in each symbol of the source output. Thus, we can define the information as the quantity of uncertainty lost, once the outcome of a given event is known.

Note. In this context, by "event" we refer to the symbol produced by the source at a given time.

- 1.1.1 - Entropy: an axiomatic definition.

In *Section 1.1* we stated that the only form H can assume is the one of *Equation* (1). The reason behind this, though one may use different approaches (eg. the connection with the entropy in quantum mechanics), is the way entropy is defined axiomatically.

Axiom. Let H be a measure of uncertainty of a source S, then H must satisfy the following properties.

1. H must be a continuos function of the probabilities. Meaning that if $H(S) = H(p_1, \dots, p_n)$, then H is continuos in [0, 1].
2. $H_2\left(\frac{1}{2}, \frac{1}{2}\right) = 1$.
3. If τ is a permutation of the probabilities, then $H(\tau) = H_n(p_1, \dots, p_n)$. Meaning that, H must not care of the order of the probabilities.
4. If $A(n) = H_n\left(\frac{1}{n}, \dots, \frac{1}{n}\right)$, then H is monotonically increasing. Equally probable events imply greater uncertainty.
5. If a choice can be divided into k consecutive ones, then the original value of H must be the weighted sum of the new H. Meaning that

$$H_n(p_1, \dots, p_n) = H_{n-k+1}(p_1 + \dots + p_k, p_{k+1}, \dots, p_n) + (p_1 + \dots + p_k) H_k\left(\frac{p_1}{p_1 + \dots + p_k}, \dots, \frac{p_k}{p_1 + \dots + p_k}\right).$$

Theorem (Theorem 2, [2]) *The only function H that satisfies the above axioms, has the form of* Equation (1).

Proof Let's show how *Equation* (1) satisfies all the axioms.

1. From axiom 5 we have $A(nm) = A(n) + A(m)$. Let's note in fact that:

$$\begin{aligned} A(nm) &= H_{nm}\left(\frac{1}{nm}, \dots, \frac{1}{nm}\right) \\ &= H_{nm-n+1}\left(\frac{1}{m}, \frac{1}{nm}, \dots, \frac{1}{nm}\right) + \frac{1}{m} H_n\left(\frac{1}{n}, \dots, \frac{1}{n}\right) \\ &\underbrace{= \dots =}_{m \text{ times}} H_{nm-nm+m}\left(\frac{1}{m}, \dots, \frac{1}{m}\right) + H_n\left(\frac{1}{n}, \dots, \frac{1}{n}\right) \\ &= A(m) + A(n). \end{aligned}$$

2. From the previous point, we also have that $A(n^k) = kA(n)$, for any n, k.

3. Let us consider some r big enough such that $\exists k : 2^k \leq n^r < 2^{k+1}$. It follows that

$$k \log_b 2 \leq r \log_b n < (k+1) \log_b 2.$$

   Let's divide everything by $r \log_b 2$, we now have

$$\frac{k}{r} \leq \log_2 n < \frac{k+1}{r} \implies \left| \log_2 n - \frac{k}{r} \right| < \frac{1}{r}.$$

   From axiom 4 follows $A(2^k) \leq A(n^r) < A(n^{k+1})$, which means that $kA(2) \leq rA(n) < (k+1)A(2)$. From a similar reasoning as before, we get

$$\left| \frac{A(n)}{A(2)} - \frac{k}{r} \right| < \frac{1}{r} \implies A(n) \approx \frac{A(2)}{\log_b 2} \log_b n$$

   since ${}^{A(n)}/_{A(2)} \approx \log_b n$. Thus, $A(n) = c \log_b n$.

4. Suppose $p \in \mathbb{Q} \iff p = \frac{r}{s}, r \leq s$. Then we have

$$\begin{aligned} A(s) &= H_s\left(\frac{1}{s}, \dots, \frac{1}{s}\right) \\ &= H_{s-r+1}\left(\frac{r}{s}, \frac{1}{s}, \dots, \frac{1}{s}\right) + \frac{r}{s} H_r\left(\frac{1}{r}, \dots, \frac{1}{r}\right) \\ &= \dots = H_2\left(\frac{r}{s}, 1 - \frac{r}{s}\right) + \frac{s-r}{s} A(s-r) + \frac{r}{s} A(s). \end{aligned}$$

From the previous point, we can conclude that

$$A(n) = -c\left[(1-p)\log_b(1-p) + p\log_b p\right].$$

Also, since $\mathbb{Q}$ is dense in $\mathbb{R}$ and H is continuos, we can extend what above to any $p \in \mathbb{R}$.

5. Lastly, *Equation* (1) can be easily proved to hold, by induction on n.

■

The proof we provide is berely a sketch. See [2] (*Appendix 2*) for a detailed proof.

- 1.2 - Encoding of a source and Sardinas-Patterson algorithm.

Let's now focus our attention onto the source itself. In general, the alphabet these use may not be suitable for transmission, for some reason or another. For this particular reason, an *encoder* is used to convert the source alphabet, to a new one, which is more suitable for transmission. On the recieving side, a *decoder* is used to convert back to the original alphabet. This process is called *source encoding/decoding*.

Formally, given a source S defined on some alphabet, and X a new alphabet, called *input alphabet*, we define a function $C : S \to X$ that maps sequences of symbols of S to sequences of symbols in X. A sequence of symbols in X is called a *codeword*.

Before we consider any specific type of *code*, let us consider the general case. Let's C be a generic mapping from a source S to some new alphabet X. Since we have not imposed any condition on C, one may define it in such a way that two, or more, source symbols share the same codeword. It easy to observe that, in doing so, the decoded message may not be correct or even unique.

Example. Let's consider the code shown below. How should we de-

Source	Code
s_1	0
s_2	11
s_3	01
s_4	11

code the text 000111? We have two possibilities: either as $s_1s_1s_3s_2$ or as $s_1s_1s_3s_4$. As said before, the decoded message might not be unique. Also, unless the context is given, there's no way to know which one is correct.

From the above example, we can conclude that a "good" code must encode each source symbols with a unique codeword. We call such codes *non-singular codes.*

- 1.2.1 - Uniquely decodable codes (UD).

One may think that non-singular codes are enough, but in most cases they are not. In fact, it might happen that a codeword is prefix of another, making the decode process tedious.

Example. Let's consider the following codes. How should we decode

Source	Code
s_1	0
s_2	1
s_3	01
s_4	11

the string 000111? Once again, we have many possibilities: for example $s_1s_1s_3s_2s_2$ or $s_1s_1s_3s_4$. Additionally, unless contex is given, we don't know which one is correct.

We say that a code is UD if and only each sequence of codewords corresponds to at most one sequence of source symbols.

From this definition, two questions follow:

- How we construct such codes?
- How can we check whether a given code is UD or not?

The next section focus on the second question, we'll then answer the first.

- 1.2.2 - Sardinas-Patterson algorithm.

The *Sardinas-Patterson* algorithm provides a way to check whether a code C, is UD or not. Conceptually the algorithm, and the theorem from which it derives, are based on the following remark: consider a string that is the concatenation of codewords. If we try to construct two distinct factorization, each word in of the factorization is either part of a word in the other factorization, or it starts with a prefix that is suffix of a word in the other factorization. Hence, a code is non-UD if it happens that a suffix is itself a codeword.

As stated before the algorithm is based on a theorem, given below.

Theorem 1.1 *Given C a code on an alphabet Σ, consider the sets $S_0, S_1, \dots$ such that:*

- $S_0 = C$

- $S_i = \{\omega \in \Sigma^* | \exists \alpha \in S_0, \exists \beta \in S_{i-1} : \alpha = \beta\omega \vee \beta = \alpha\omega\}$

then necessary and sufficient condition for C *to be UD is that,* $\forall n > 0, S_0 \cap S_n = \varnothing$.

Therefore, the algorithm has three halt conditions:

- $\forall i > 0, S_0 \cap S_i \neq \varnothing$ the the code is not UD.
- $\forall i > 0, S_i = \varnothing$ then the code is UD.
- $\forall i, j > 0, S_i = S_j$ then the code is UD.

Example. Let's consider the following code: $C = \{a, c, ad, abb, bad, deb, bbcde\}$. Is it UD? Applying Sardinas-Patterson, step by step we get the following:

Iter 1. Let $\alpha_0 = ab, \alpha_1 = abb$ and $\beta_0 = \beta_1 = a$ then, $S_1 = \{d, bb\}$.

Iter 2. Let $\alpha_0 = deb, \alpha_1 = bbcde$ and $\beta_0 = d, \beta_1 = bb$ then, $S_2 = \{eb, cde\}$.

Iter 3. Let $\alpha_0 = c$ and $\beta = cde$ then, $S_3 = \{de\}$.

Iter 4. Let $\alpha_0 = deb$ and $\beta = de$ then, $S_4 = \{b\}$.

Iter 5. Let $\alpha_0 = bad$ and $\beta = b$ then, $S_5 = \{ad\}$.

From the above steps (summarised in *Table 1*), follows that the code is not UD since $S_0 \cap S_5 = \{ab\} \neq \varnothing$.

S_0	S_1	S_2	S_3	S_4	S_5
a	d	eb			
c					
ad					
abb	bb	cde	de	b	ab
bad					
deb					
bbcde					

Table 1: Steps of Sardinas-Patterson for the code C.

Exercise 1.1 Apply the Sardinas-Patterson algorithm to the following codes:

- $C_0 = \{a, b, cd, abb, abcd\}$
- $C_1 = \{0, 01, 100, 11001, 01011\}$
- $C_2 = \{010, 0001, 0110, 11000, 00011, 00110, 11110, 101011\}$

- 1.2.3 - Prefix codes.

Let's look back at the example of *Section 1.2.1.* Given the codewords $\{0, 1, 01, 11\}$ for the source symbol s_1, s_2, s_3, s_4 respectively, how should we decode the string 000111? As said before, unless some context is given, we can't be sure.

Then what's the issue? Essentially, even though to each source symbol is assigned a distinct codeword, these are not prefix-free, meaning that some of the codewords are prefix of others (**eg.** the codeword for s_1 is prefix of the codeword for s_3). From what just stated, a natural solution to the problem is to define codes that are prefix-free. One can also prove that, prefix-free (or simply prefix) codes are also instataneous (each source symbol can be decoded without reading further in the string).

Theorem. *Let C be a prefix-free code, then C is UD.*

Proof It follows from *Theorem 1.1.* ■

- 1.3 - Average code length and Kraft inequality.

In the previous section, we've introduced prefix codes. The question now is, given two distinct prefix-free codes, which one is more convenient? A good choice would be the one that, on average, has the shortest length.

Definition (Average code length (ACL)) Let C be a code with a source alphabet $S = \{s_1, \dots, s_n\}$ and a code alphabet $X = \{x_1, \dots, x_m\}$. Let $\{c_1, \dots, c_n\}$ be the codewords with lengths $l_1, \dots, l_n$ respectively. And let $\{p_1, \dots, p_n\}$ be the probabilities for the source symbols. Then, we define the quantity

$$L_S(C) = \sum_{i=1}^{n} p_i l_i$$

as the average code length of the code C.

From what above, it makes sense to look for the UD code with the lowest average code length. That is, among all the UD codes for the same source and with the same code alphabet, the one with the lowest ACL. But how do we find such codes? A good starting point is the entropy of the source. Once again, thanks to Shannon, we have the following result.

Theorem (Shannon) *Let C be a UD code for a memoryless source S, whose probabilities are $\{p_1, \dots, p_n\}$, and alphabet code X with a size d. Then*

$$L_S(C) \geq \frac{H(S)}{\log_b d}$$

- 1.3.1 - The Kraft-McMillan inequality.

We have disscussed what prefix-free codes are, and what the average code length represents. We now provide a necessary and sufficient condition for the existence of a prefix code: the *Kraft-McMillan inequality*.

Theorem 1.2 (Kraft-McMillan) *Let us consider a source alphabet* $S = \{s_1, \dots, s_n\}$ *and a code alphabet* $X = \{x_1, \dots, x_d\}$. *Let* $l_1, \dots, l_n$ *be a set of lengths. Then necessary and sufficient condition for the existence of a prefix-free code* C *over the alphabet* X *with codewords lenghts* $l_1, \dots, l_n$ *is that*

$$\sum_{i=1}^{n} d^{-l_i} \leq 1$$

with d *size of the code alphabet.*

In other terms, if a set of lengths satisfies the inequality then, there exist at least a way to arrange the codewords into a prefix code. A proof of this theorem is beyond the scope of this *notes*.

- 1.3.2 - Optimal codes and the Shannon-Fano encoding.

The concept of average code length allows us to define a intresting class of codes: the *compact* (or optimal) codes. These are UD codes that have the lowest ACL.

A first attempt to achive such codes was firstly proposed by Shannon and *Robert Mario Fano*, who developed the so called Shannon-Fano encoding (1949).

Shannon-Fano encoding.

We will be considering the binary case. The encoding itself is very simple: we order the symbols in decreasing order, divide the symbols in two sets such that the sum of probabilities in each set is almost equal, encode the symbols in the first set with 0 and the other with 1. Lastly, repeat the procedure for each set recursively.

Example. Let's assume the alphabet $\{a, b, c, d, e\}$ whose probabilities are $\{\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{16}\}$. Applying the steps above we get the following.

Symbol	Encoding
a	0
b	10
c	110
d	1110
e	1111

One can prove that Shannon-Fano encoding is not optimal.

- 2 - Huffman encoding.

The first to solve the problem of creating optimal code was *D.A. Huffman*, who in [1] provides a method to create compact codes.

Before we actually analyze such method, it's of intrest to point out some properties that optimal codes must satisfy. Let S be a source with some probability distribution, let's also assume that $p_1 \geq p_2 \geq \ldots \geq p_n$. Let C be a compact prefix code for S such that c_i is the codeword associated to p_i. Then:

1. To reduce the expected code length, the shortes codewords are associated to the symbols with the higtest probability. This means that
$$p_i \geq p_j \implies |c_i| \leq |c_j|$$
If this was not the case, swapping the codewords would result in a code with lower ACL.
2. The least probable symbols have codewords with the same length.
3. The longest codewords differ only by the last symbol.

- 2.1 - The algorithm.

We treat the binary case; the extension to the general case is immediate. Let
$$S = \begin{pmatrix} s_1 & s_2 & \cdots & s_n \\ p_1 & p_2 & \cdots & p_n \end{pmatrix}$$
be a source, such that the probabilities are sorted non-increasingly. Denote by $R(S)$ the *reduced soource*, obtained by replacing the two least common symbols from S, with one whose probabily is the sum of those of the replaced symbols. This means
$$R(S) = \begin{pmatrix} s_1 & s_2 & \cdots & (s_{n-1}, s_n) \\ p_1 & p_2 & \cdots & p_{n-1} + p_n \end{pmatrix}$$
Assume C_R to be a binary prefix code for $R(S)$, and let z be the codeword associated to the merged symbols (s_{n-1}, s_n). Then, a prefix code C for S can be obtained by C_R by assigning to the i-th symbol of S the i-th codeword in C_R, for $i \leq n-2$. The codeword for s_{n-1} and s_n are simply $z0$ and $z1$ respectively.

- 2.1.1 - The encoding.

Let ω be a string to encode, produced by some source S. To encode such string we have to:

1. Scan ω to get the probabilities of each symbol. Unless these are known a priori.
2. Sort the symbols non-increasingly according to their probabilities.
3. Build a (binary) tree where the leaves are the source symbols, and each node is the sum of the (two) lowest probabilities. Repeat the process until a root is formed.
4. Build the code by traversing the tree from the root to a leaf, and assigning a 0 to the left paths, a 1 to the right one.

Example. To ease things out, the example skips the scanning step.

Let $S = \{a, b, c, d, e\}$ whose probabilities are $\left\{\frac{1}{3}, \frac{1}{6}, \frac{1}{3}, \frac{1}{12}, \frac{1}{12}\right\}$, respectively. Applying the steps above we have the following.

Step 1. Sorting the symbols according to the probabilities, we get the following order: a, c, b, d, e.

Step 2. To build the tree let's consider the two least probable symbols, in this case d and e. The sum of their probabilities will define a new node x in the tree (the ■ one in the figures below). The next node y (the ■ one) is given by x and the third least probable symbol b. At this point we can make two choices to define the node z: we can either merge a and c or merge c and y. Both will build a optimal code.

For the sake of this example, let's merge c and y, the other choice is shown in *Figure 2.b*. Lastly, we merge the last two symbols, leading to the tree in *Figure 2.a*.

(a) **(b)**

Figure 2: Huffman trees for the source S.

Step 3. Traversing the tree we get $C = \{0, 10, 110, 1110, 1111\}$ for tree in *Figure 2.a* or $C = \{00, 01, 10, 110, 111\}$ for the one in *Figure 2.b*.

From the steps above, it's obvious that the slowest part of the encoding is given by the neccessity to scan the text twice.

- 2.1.2 - The decoding.

Let's assume we recieve a text encoded using Huffman, how do we decode it? We remark that to decode the text we need to know either the Huffman tree, or the source probabilities, which represent in both cases an overhead to the bare encoding. Assuming that such overhead is know, the decoding is immediate. In fact, we read the recieved text and traverse the tree accordingly until reaching a leaf. Repeating the process for the whole text.

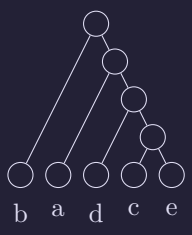
Example. Let $\omega = 00101111100001$ be a text encoded using Huffman. Let's assume that the tree used is the one in *Figure 2.b*. Starting from the root, we read a zero as first symbol of the encoded text, thus we consider the left subtree. Since we are not at a leaf we keep reading the encoded text. We read another zero, so once again we consider the left subtree. Since we reached a leaf we stop and decode 00 as a. We move back to the root and repeat the process. After few steps we decode $\omega = abedac$.

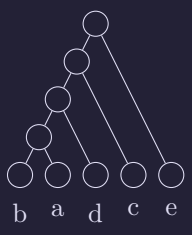
- 2.1.3 - Exercises.

Exercise 2.1 Apply Huffman encoding to the following strings.

- $\omega_0 = abbabccadeabf$
- $\omega_1 = \mathbf{0100100010010 10}$
- $\omega_2 = baabbbacabbdeffea$

Exercise 2.2 Determine which of the following is the Huffman tree for the source $S = \{a, b, c, d, e\}$ whose probabilities are, respectively, the following: $\{\frac{1}{4}, \frac{1}{2}, \frac{1}{16}, \frac{1}{8}, \frac{1}{16}\}$.





Exercise 2.3 Using the tree from *Exercise 2.2*, decode the following strings:

- $\omega_0 = 111011000100010$
- $\omega_1 = 111100110110110011110$

- 2.1.4 - Adaptive Huffman.

As it was pointed out previously, the main issue with the classic Huffman approach, is the necessity to scan the text twice, which slows down the encoding. Also, to work properly we need to know the symbols probabilities in advace, which is not always the case. To solve both these issues, a new approach has been developed known as *Adaptive Huffman*.

The key point of this new approach is the fact that the tree is built and updated dynamically. Essentially we:

- Start with an initially empty tree, or one which only contains a special **Not Yet Transfered** node.
- For each symbol in the text:
  - If the symbol was already encoded, endode it with the existing code.
  - If it's a new symbol: encode the **NYT** node first, then encode the symbol itself and add the symbol to the tree as a leaf.
  - After encoding a symbol, update the tree: increment the frequency of the symbol and its ancestors. Reorganize the tree to mantain the Huffman properties.

One can prove that the adaptive approach has a better locality the the classic one.

- 2.2 - Canonical Huffman.

Let us consider some limitations of the Huffman encoding discussed so far. One of the main problems, is the fact that the encoder has to transmit alongside the text, the tree used for it: a not so trivial overhead. Additionally, the decoding phase is slow due to the necessity to traverse the tree for each symbol.

Let us observe that there exist codes, that are compact, that are not produced by the Huffman algorithm. We now introduce the so called *Canonical Huffman encoding*, which is a variant of Huffman, that solves the issues above by requiring the encoder to just transmit the lengtht of the codewords.

Such new encoding, comes in handy when the source alphabet is large, and a fast decoding is mandatory.

- 2.2.1 - The encoding.

Let C be the usual Huffman encoding for some source. To obtain the canonical version from it, we need to:

1. Compute the length of the codewords in C, with respect to each symbol in the source alphabet.
2. Construct the num array which stores at each entry num[l], how many symbol have length l.
3. Construct the symb array which stores at each entry symb[l], the symbols having length l.
4. Construct the fc array that stores at each entry fc[l], the first codeword off all symbols having codeword length l. Such construction is done via the following snippet.

```
fc[MAX] = 0 // MAX = 3 in the example
for(l = MAX - 1; l >= 1; l--){
    fc[l] = (fc[l + 1] + num[l + 1]) >> 1;
}
```

Note. The snippet shown uses a reverse approach, which usually produces codewords that differ from those built via a forward approach.

5. Assign consecutive codewords to the symbols in symb[l] starting from symb[l].

Example. Let C be the Huffman code obtained by the tree in *Figure 2.b*.

1. Let l_i be the length of the i-th codeword, we have: $l_0 = l_1 = l_2 = 2, l_3 = l_4 = 3$.
2. Computing num we have: num[1] = 0, num[2] = 3 and num[3] = 2.
3. Computing symb we get: symb[1] = null, symb[2, 0] = a, symb[2, 1] = c, symb[2, 2] = b, symb[3, 0] = d and symb[3, 1] = e. To ease things out, we use a second index to move within a list when needed.
4. Computing fc we get: fc[1] = 2 (this value will be used in the decoding process), fc[2] = 1, fc[3] = 0.
5. Let's observe that symb[1] = null, meaning there're no codewords to assign. Considering symb[2] we assign 01 to a, 10 to c and 11 to b. For what regards symb[3] we have that 000 is assigned to d and 001 to e.

We assume a 1-based indexing for the arrays and a 0-based one for the lists.

- 2.2.2 - The decoding.

Let's assume ω to be a a encoded text via the Canonical Huffman encoding. To decode such text, we simply apply the following procedure.

```
char v = next_bit();
// Assume next_bit() as a
     primitive
int l = 1;
while(v < fc[l]){
    v = 2*v + next_bit();
    l++;
}
return symb[l, v - fc[l]];
```

Example. Let $\omega = 01100000011011$ be a text encoded using the Canonical Huffman encoding, and let symb and fc be those computed in the previous example. Applying the snippet above, one can verify that $\omega = acdecb$.

- 2.2.3 - Exercises.

Exercise 2.4 Apply the canonical Huffman encoding to the codes obtained in *Exercise 2.1.*

Exercise 2.5 Rewrite the procedure to construct the fc array using a forward approach.

- 3 - Aritmethic coding.

Let us recall that, given C a code, $L_S(C)$ represents the average code length of the codewords in C.

From it, we can define two other important quantities: the efficiency and the redundancy of the code. Formally, we define the efficiency as

$$\vartheta(C) = \frac{H_d(S)}{L_S(C)}$$

which, from Shannon's theorem, is a value in the range $[0, 1]$. We define the redundancy of the code as

$$\rho(S) = 1 - \vartheta(C).$$

When considering Huffman, one can prove that its redundancy approaches 1 when the most frequent symbol probability approaches 1.

The question then is: can we do better than Huffman? The answer is yes, but we have to use a completely different approach: instead of assigning a codeword to each symbol, we assign one to the whole text. The method we are about to present is known as *aritmethic coding (AC).*

- 3.1 - The encoding.

The AC encoding is pretty straightforward, as summarized in the steps below. Let ω be the text to encode and let's assume the alphabet, alongside the probabities, is know. Then:

1. We start by initializing the range $[0, 1]$.
2. For each symbol in ω:
   (a) Divide the current range into n intervals (n the size of the alphabet), each proportional to the probabities of the symbols.
   (b) Select as current subinteval the one corresponding to the symbol that is begin analyzed.
3. Output the binary representation[1] of the lowerbound of the last subinteval.

Example. Let $\omega = abac$ and let $S = \{a, b, c\}$ with probabilities $\{\frac{1}{2}, \frac{1}{4}, \frac{1}{4}\}$. By applying the above steps, we get:

Step 1. We initialize the range $[0, 1]$.

Step 2. Let us consider each iteraction, each step can be observed in *Figure 3*. We read "a" anddthus, we select the range $[0, \frac{1}{2}]$. We proceed to updated the ranges, and repeat the process until the end of the text.

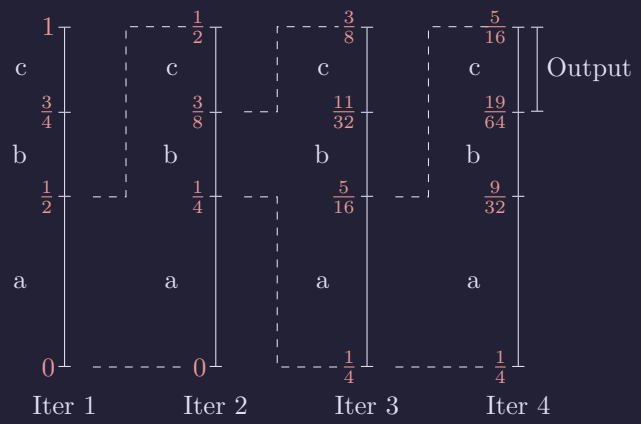


Figure 3: Step by step encoding of $\omega = abac$ via the AC.

Step 3. We return the mid value of the last range.

Algorithmically speaking, we simply apply the following code.

```
double AC_encode(char* S, double* cumFun, double* Prob){
    double high = 1;
    double low = 0;
    int i = 1;
    double range;
    while (i <= n)  {
        range = high - low;
        low = low + range * cumFun[S[i]];
        high = low + range * (cumFun[S[i]] + Prob[S[i]]);
    }
    return high - low;
}
```

Figure 4: C implementation of the AC encoding.

How many bits do we need to represent the encoding? From the code in *Figure 4*, we can deduce that the size of the final range is given by

$$range_n = \prod_{i=1}^{n} Prob[S[i]]$$

from which we can derive that the outpus size is indipendent of the string symbols permutation. Then by the above remark, and the following results, we can deduce that we need $l_n + \frac{s_n}{2}$ bits, where l_n and s_n are, respectively, the last two values of low and high in the code above.

Lemma. *Let $x = 0.b_1b_2\ldots$ be the binary representation of a real number. Then, it's truncation to the first d bits is also a real number,*

Proof Let's observe that

$$x - trunc_d(x) = \sum_{i=1}^{\infty} b_{d+i}2^{-(d+i)} \leq \sum_{i=1}^{\infty} 1 * 2^{-(d+i)} = 2^{-d}\sum_{i=1}^{\infty} 2^{-i} = 2^{-d}$$

Therefore,

$$x - trunc_d(x) \leq 2^{-d} \iff x - d^{-d} \leq trunc_d(x)$$

■

Corollary. Given $x \in [low, high]$ its truncation to the first $\left\lceil \log \frac{2}{high-low} \right\rceil$ falls within the range.

Proof It follows by the above *Lemma* putting $d = \left\lceil \log \frac{2}{high-low} \right\rceil$. ■

[1] We assume the reader knows how fractional numbers are represented in binary.

- 3.2 - The decoding.

Deconding a text compressed via the AC encoding, is straightforward as its encoding. In fact, the decoding proceeds similarly to the encoding. We just

1. Initialize the range $[0, 1]$.
2. We divide the subrange according to the probabilities, as done for the encoding.
3. At each step we select as current interval the one where the encoded text fall in to.

Algorithmically speaking we proceed applying the procedure in *Figure 5*.

```
char* AC_decode (int encodedText, int textLength, double*
      Prob, double* cumFun){
    double high = 1;
    double low = 0;
    int i = 1;
    double range;
    char* decoded = malloc(sizeof(char * textLength));
    while (i <= n) {
        // divide the current range into subranges
            according to the probabilities
        // return the symbol x associated to the current
            range
        range = high - low;
        decoded[i -1] = x;
        low = low + range * cumFun[x];
        high = low + range * (cumFun[x] + Prob[x]);
    }
    return decoded;
}
```

Figure 5: C like implementation of the AC decoding.

- 3.3 - The adaptive version.

It's easy to observe from *Figure 4*, that the necessity to know the probabilities a priori, slows down the encoding. The question then is: can we improve the AC encoding? The answer is yes, by assuming an initial equality in the probabilities. Briefly, before any symbol is read, we assume that each of the alphabet symbol has the same probability to appear within the text. Then, at each step, we simply apply the same procedure in *Figure 4* and then update the probabilities.

What about the decoding? We can apply the same logic, with a little differce: instead of updating the probabilities after a symbols is read, we

update them, each time a symbol is decoded by applying the procedure in *Figure 5*.

Example. Let $\omega = abcb$ be the text to encode. Before any character is read, we have the following probabilities: $p_a = p_b = p_c = \frac{1}{3}$. Once the first character (a) is read we select as current range the $[0, \frac{1}{3}]$ then we update the probabilities as $p_a = \frac{1}{2}, p_b = p_c = \frac{1}{4}$. We repeat the same process until the end of the text, and encode ω as $[\frac{43}{180}, \frac{11}{45}]$.

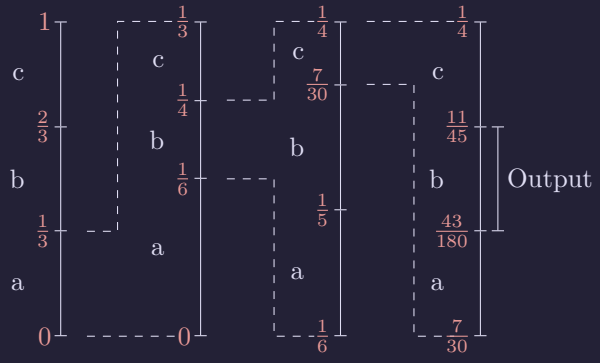


Figure 6: Step by step encoding of $\omega = abcb$ via the adaptive AC.

Let's now decode 0.2416, with $n = 4$. Again we assume equality in the probabilities, and thus, we decode (a) and update the probabilities. Repeating the process, we decode the text $abcb$.

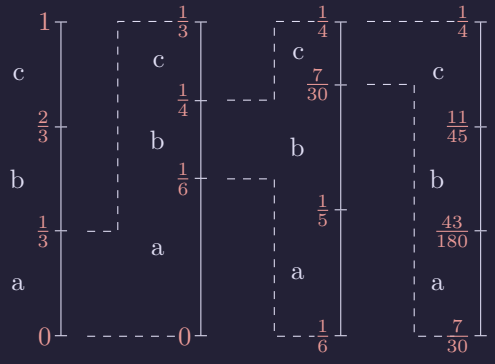


Figure 7: Step by step decoding of 0.2416 via the adaptive AC.

References.

[1] David A. Huffman. "A method for the construction of minimum-redundancy codes". In: *Proceeding of I.R.E.* (1952), pp. 1092–1102.

[2] Claude Shannon. "A mathematical theory of communication". In: *Bell System Technical Journal* 27 (1948), pp. 397–423, 623–656.

Acronyms.

AC aritmethic coding. 14–18

ACL average code length. 7–9

UD uniquely decodable codes. 5–8