University of Toronto Scarborough
Department of Computer & Mathematical Sciences

FINAL EXAMINATION

MATB24H – Linear Algebra II

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Date: December 19, 2015
Start Time: 2:00 PM
Duration: 3 hours

FAMILY NAME: _____

GIVEN NAMES: _____

STUDENT NUMBER: _____

SIGNATURE: _____

DO NOT OPEN THIS BOOKLET UNTIL INSTRUCTED TO DO SO.

NOTES:

- Your signature above indicates that you have abided by the UofT Code of Behaviour while writing this exam.

- NO AIDS.

- No electronic devices of any kind (e.g. calculators, smart phones, smart watches, tablets, etc.) allowed.

- There are 11 numbered pages in the exam. It is your responsibility to ensure that, at the start of the exam, this booklet has all its pages.

- Answer all questions. EXPLAIN and JUSTIFY your answers. Proofs must be given when appropriate.

- **Show all your work**. Credit will not be given for numerical answers if the work is not shown. If you need more space use the back of the page or the last page and indicate clearly the location of your continuing work.

question	1	2	3	4	5	6	7	8	9	10	11	total
marks	15	6	7	10	15	15	12	15	5	5	5	110

1. [**15 points**]   Multiple Choice
   Each question has exactly one correct answer.

   (1) Let $\boldsymbol{u}$, $\boldsymbol{v}$ and $\boldsymbol{w}$ be three vectors in an inner product space V. The inner products among the three vectors are given by

$\langle\,,\,\rangle$	$\boldsymbol{u}$	$\boldsymbol{v}$	$\boldsymbol{w}$
$\boldsymbol{u}$	3	1	0
$\boldsymbol{v}$	1	5	-3
$\boldsymbol{w}$	0	-3	4

   for example, $\langle \boldsymbol{u},\,\boldsymbol{v}\rangle = 1$, $\langle \boldsymbol{v},\,\boldsymbol{w}\rangle = -3$.

   Then $\|\boldsymbol{u} - \boldsymbol{v} + \boldsymbol{w}\|$ is

   (a) 16   (b) 10   (c) 4   (d) 2

   Your answer: _____

   (2) Let $A \in M_n(\mathbb{C})$, Schur's Theorem states that

   (a) If A is a Hermitian matrix, its eigenvectors corresponding to distinct eigenvalues are orthogonal.

   (b) If A is a Hermitian matrix, A is unitarily diagonalizable.

   (c) A is a normal matrix if and only if A is unitarily diagonalizable.

   (d) If the characteristic polynomial of A splits over $\mathbb{C}$ then A is unitarily equivalent to a complex upper triangular matrix.

   Your answer: _____

   (3) The magnitude of vector $\boldsymbol{v} = [i,\, 2 - i,\, 1 + 3i]$ is

   (a) $-6 + 2i$ (b) $\sqrt{16 + 2i}$ (c) $\sqrt{15}$ (d) 4

   Your answer: _____

   (4) Which of the following statements is true?

   (a) The vector space $P_7(\mathbb{R})$ is isomorphic to $\mathbb{R}^8$.

   (b) The set of vectors $\{[0, 1, 1, 1],\ [1, 0, 0, 1],\ [1, 1, 1, 0]\}$ is linearly independent over $\mathbb{Z}_2$.

   (c) Every basis of $M_{2,2}$ contains a noninvertible matrix.

   (d) If T is a linear transformation from V to V, $\ker(T)$ must be included in $\mathrm{range}(T)$.

   Your answer: _____

   (5) If $W = \mathrm{sp}\left(\begin{bmatrix} i \\ 2 - i \\ 1 + 3i \end{bmatrix}\right)$, $W^\perp$ is

   (a) $\mathrm{sp}\left(\begin{bmatrix} 1 + 2i \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 + i \\ 0 \\ 1 \end{bmatrix}\right)$ (c) $\mathrm{sp}\left(\begin{bmatrix} -2 + 3i \\ 1 \\ 1 \end{bmatrix}\right)$

   (b) $\mathrm{sp}\left(\begin{bmatrix} 1 - 2i \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 - i \\ 0 \\ 1 \end{bmatrix}\right)$ (d) Two (none or all) of them.

   Your answer: _____

2. [**6 points**]   Let $V = (\mathbb{Z}_7)^4$, and let $W = \big\{ (a,\ 2a,\ a + 2b,\ b + c) \ \big| \ a,\ b,\ c \in \mathbb{Z}_7 \big\}$.

(a) Show that W is a subspace of V.

(b) Find a basis for W.

3. **[7 points]**

   (a) Find the volume of the 3-box in $\mathbb{R}^4$ determined by the vectors $[1, 1, 0, 1]$, $[0, 1, 2, 2]$ and $[-1, -1, 1, 0]$.

   (b) Let $T : \mathbb{R}^3 \to \mathbb{R}^4$ be given by $T([x, y, z]) = [2x - y,\ 2x + z,\ 2y + z,\ x + y + z]$. Find the volume of the image under T of a 3-box in $\mathbb{R}^3$ of volume 2.

4. [**10 points**]

   (a) Let $W = \text{sp}(\boldsymbol{a}_1, \boldsymbol{a}_2, \cdots, \boldsymbol{a}_k)$ be a k–dimensional subspace of $\mathbb{C}^n$. Give the formula for the projection matrix that gives the orthogonal projection from $\mathbb{C}^n$ to W.

   (b) Let $W = \text{sp}\left(\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \right)$ in $\mathbb{R}^3$.

      i. Find the projection matrix that gives the orthogonal projection from $\mathbb{R}^3$ to W.

      ii. Find the orthogonal projection of $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ on W.

5. **[15 marks]** Let $\beta = \left(x^2 + x, \; x^2 + 1, \; x + 1 \right)$
   and $\gamma = \left(x^2, \; 2x^2 + x + 1, \; 2x + 1 \right)$ be ordered bases of $P_2(\mathbb{R})$.

   (a) Find $\left[I \right]_\beta^\gamma$, the change of coordinate matrix from β to γ.

   (b) Let $f(x) = 3 + 2x + x^2$. Find the coordinate vector $\left[f(x) \right]_\beta$.

   (c) Define $T : P_2(\mathbb{R}) \to P_2(\mathbb{R})$ by $T\big(g(x)\big) = x\, g'(x)$. Compute $\left[T \right]_\gamma$.

6. [**15 points**]

    (a) Transform the ordered basis $\big([1, 0, 0], [0, 4, 1], [3, 7, -2]\big)$ for $\mathbb{R}^3$ into an orthonormal basis for $\mathbb{R}^3$ by using the Gram-Schmidt process.

    (b) Find a QR factorization for the matrix $A = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 4 & 7 \\ 0 & 1 & -2 \end{bmatrix}$.

7. [**12 points**] Let $A = \begin{bmatrix} 2 & 3 & 4 \\ -2 & -2 & -2 \\ 1 & 1 & 1 \end{bmatrix}$.

(a) Find the eigenvalues of A.

(b) Find a Jordan basis for A.

(c) Find a Jordan canonical form for A which corresponds to the Jordan basis found in part (b).

8. **[15 points]** Let $A = \begin{bmatrix} 3 & 0 & -2 \\ 0 & 3 & 0 \\ -2 & 0 & 3 \end{bmatrix}$.

(a) Find the eigenvalues of A.

(b) Find a matrix C such that $D = C^{-1} A C$ is an orthogonal diagonalization of A.

(c) Find an orthogonal substitution that diagonalizes the quadratic form $f(x, y, z) = 3x^2 + 3y^2 - 4xz + 3z^2$ and find the diagonalized form.

9. [**5 points**]   Let A be an orthogonal $n \times n$ matrix and let $\boldsymbol{x}$ and $\boldsymbol{y}$ be column vectors in $\mathbb{R}^n$. Prove that
$$(A\,\boldsymbol{x}) \cdot (A\,\boldsymbol{y}) = \boldsymbol{x} \cdot \boldsymbol{y}$$

10. [**5 points**]   Let A be an $n \times n$ normal matrix. Show that A is Hermitian if and only if the eigenvalues of A are real.

11. [**5 points**]   Prove that, for vectors $\boldsymbol{v}_1, \boldsymbol{v}_2,\ \cdots, \boldsymbol{v}_n \in \mathbb{C}^n$, $\left\{\boldsymbol{v}_1, \boldsymbol{v}_2,\ \cdots\ \boldsymbol{v}_n\right\}$ is a basis for $\mathbb{C}^n$ if and only if $\left\{\overline{\boldsymbol{v}_1}, \overline{\boldsymbol{v}_2},\ \cdots\ \overline{\boldsymbol{v}_n}\right\}$ is a basis for $\mathbb{C}^n$.

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