University of Toronto at Scarborough
Department of Computer & Mathematical Sciences

Final Test

MATB24
Linear Algebra II

Examiner: L. de Thanhoffer de Volcsey

Date: Thursday, August 10th, 2017
Duration: 120 minutes

FAMILY NAME: ____________________

GIVEN NAMES: ____________________

STUDENT NUMBER: ____________________

TUTORIAL NUMBER: ____________________

SIGNATURE: ____________________

DO NOT OPEN THIS BOOKLET UNTIL INSTRUCTED TO DO SO.

NOTES:

- No calculators, or any electronic aid is permitted. If you have a cell phone in the room, make sure it is away from you and **turned off**.
- No books, notebooks or scrap paper are permitted.
- There are 10 numbered pages and 5 questions on the test. It is your responsibility to ensure at the start of the test that this booklet has all its pages.
- Please leave all the pages of this booklet stapled. Do not remove any pages.
- Answer all questions in the space provided. Show your work and **justify your answers** for full credit.

FOR MARKERS ONLY	
Question	Marks
1	/20
2	/20
3	/20
4	/20
5	/20
TOTAL	**/100**

1. **[20 points]** Consider the vector space $\mathcal{C}\big([-\pi,\pi]\big) = \{f : [-\pi,\pi] \longrightarrow \mathbb{R} \,|\, f \text{ is continuous}\}$ together with the inner product

$$\langle f, g\rangle = \int_{-\pi}^{\pi} f \cdot g$$

   (a) Argue why this is indeed an inner product.

   (b) let $f(x) = e^x$. Compute $||f||$.

   (c) Let $f(x) = \sin(x)$ and $g(x) = \cos(x)$. Show that f and g are orthogonal.

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2. **[20 points]**

   (a) Consider the complex numbers $z_1 = 1+i$, $z_2 = 2-i$, $z_3 = 1+i\sqrt{3}$ and $z_4 = \sqrt{3}+i$

      i. compute $\frac{\overline{z_1}}{z_2}$.

      ii. Compute the polar form of $\frac{z_3}{z_4}$.

   (b) Prove that the following matrix is Hermitian:

$$\begin{bmatrix} 1 & 1+i & 2i \\ 1-i & 5 & -3 \\ -2i & -3 & 0 \end{bmatrix}$$

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3. **[20 points]** Let T denote the linear transformation

$$T : \mathcal{P}_2(\mathbb{R}) \longrightarrow \mathcal{P}_2(\mathbb{R}) : a + bX + cX^2 \longrightarrow: (a + 2c) + (2a + 3b - 4c)X + (2c)X^2$$

(a) Find the matrix A_T corresponding to this linear transformation after the choice of basis $(1, X, X^2) \subset \mathcal{P}_2(\mathbb{R})$.

(b) Compute the eigenvalues and eigenspaces.

(c) Exhibit an invertible matrix C and a diagonal matrix D such that

$$A_T = C \cdot D \cdot C^{-1}$$

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4. **[20 points]** Consider the set $\{(1,-1,1),(1,0,1),(1,1,2)\}$.

   (a) Show that this set forms a basis for the vector space $\mathbb{R}^3$.

   (b) Find an orthonormal basis for $\mathbb{R}^3$ by using the Gram-Schmidt method starting with the given basis.

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5. **[20 points]** Argue why the statements below are True/False in each case (ie prove or give a counterexample):

   (a) Every square matrix is diagonalizable.

   (b) Every change of coordinates matrix is invertible.

   (c) If an $n \times n$ matrix has n different eigenvalues, it is diagonalizable.

   (d) Every matrix has a Jordan canonical form.