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18.02 Multivariable Calculus
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18.02 Lecture 18. – Tue, Oct 23, 2007

Change of variables.

Example 1: area of ellipse with semiaxes a and b : setting $u = x/a$, $v = y/b$,

$$\iint_{(x/a)^2 + (y/b)^2 < 1} dx dy = \iint_{u^2 + v^2 < 1} ab du dv = ab \iint_{u^2 + v^2 < 1} du dv = \pi ab.$$

(substitution works here as in 1-variable calculus: $du = \frac{1}{a} dx$, $dv = \frac{1}{b} dy$, so $du dv = \frac{1}{ab} dx dy$.

In general, must find out the scale factor (ratio between $du dv$ and $dx dy$)?

Example 2: say we set $u = 3x - 2y$, $v = x + y$ to simplify either integrand or bounds of integration. What is the relation between $dA = dx dy$ and $dA' = du dv$? (area elements in xy - and uv -planes).

Answer: consider a small rectangle of area $\Delta A = \Delta x \Delta y$, it becomes in uv -coordinates a parallelogram of area $\Delta A'$. Here the answer is independent of which rectangle we take, so we can take e.g. the unit square in xy -coordinates.

In the uv -plane, $\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$, so this becomes a parallelogram with sides given by vectors $\langle 3, 1 \rangle$ and $\langle -2, 1 \rangle$ (picture drawn), and area = $\det = \begin{vmatrix} 3 & 1 \\ -2 & 1 \end{vmatrix} = 5$ ($= \begin{vmatrix} 3 & -2 \\ 1 & 1 \end{vmatrix}$).

For any rectangle $\Delta A' = 5\Delta A$, in the limit $dA' = 5dA$, i.e. $du dv = 5dx dy$. So $\iint \dots dx dy = \iint \dots \frac{1}{5} du dv$.

General case: approximation formula $\Delta u \approx u_x \Delta x + u_y \Delta y$, $\Delta v \approx v_x \Delta x + v_y \Delta y$, i.e.

$$\begin{bmatrix} \Delta u \\ \Delta v \end{bmatrix} \approx \begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix}.$$

A small xy -rectangle is approx. a parallelogram in uv -coords, but scale factor depends on x and y now. By the same argument as before, the scale factor is the determinant.

Definition: the Jacobian is $J = \frac{\partial(u, v)}{\partial(x, y)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix}$. Then $du dv = |J| dx dy$.

(absolute value because area is the absolute value of the determinant).

Example 1: polar coordinates $x = r \cos \theta$, $y = r \sin \theta$:

$$\frac{\partial(x, y)}{\partial(r, \theta)} = \begin{vmatrix} x_r & x_\theta \\ y_r & y_\theta \end{vmatrix} = \begin{vmatrix} \cos \theta & -r \sin \theta \\ \sin \theta & r \cos \theta \end{vmatrix} = r \cos^2 \theta + r \sin^2 \theta = r.$$

So $dx dy = r dr d\theta$, as seen before.

Example 2: compute $\int_0^1 \int_0^1 x^2 y dx dy$ by changing to $u = x$, $v = xy$ (usually motivation is to simplify either integrand or region; here neither happens, but we just illustrate the general method).

1) Area element: Jacobian is $\frac{\partial(u, v)}{\partial(x, y)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ y & x \end{vmatrix} = x$, so $du dv = x dx dy$, i.e. $dx dy = \frac{1}{x} du dv$.

2) Express integrand in terms of u, v : $x^2 y dx dy = x^2 y \frac{1}{x} du dv = xy du dv = v du dv$.

3) Find bounds (picture drawn): if we integrate $du dv$, then first we keep $v = xy$ constant, slice looks like portion of hyperbola (picture shown), parametrized by $u = x$. The bounds are: at the top boundary $y = 1$, so $v/u = 1$, i.e. $u = v$; at the right boundary, $x = 1$, so $u = 1$. So the inner

integral is \int_v^1 . The first slice is $v = 0$, the last is $v = 1$; so we get

$$\int_0^1 \int_v^1 v \, du \, dv.$$

Besides the picture in xy coordinates (a square sliced by hyperbolas), I also drew a picture in uv coordinates (a triangle), which some students may find is an easier way of getting the bounds for u and v .

18.02 Lecture 19. – Thu, Oct 25, 2007

Handouts: PS7 solutions; PS8.

Vector fields.

$\vec{F} = M\hat{i} + N\hat{j}$, where $M = M(x, y)$, $N = N(x, y)$: at each point in the plane we have a vector \vec{F} which depends on x, y .

Examples: velocity fields, e.g. wind flow (shown: chart of winds over Pacific ocean); force fields, e.g. gravitational field.

Examples drawn on blackboard: (1) $\vec{F} = 2\hat{i} + \hat{j}$ (constant vector field); (2) $\vec{F} = x\hat{i}$; (3) $\vec{F} = x\hat{i} + y\hat{j}$ (radially outwards); (4) $\vec{F} = -y\hat{i} + x\hat{j}$ (explained using that $\langle -y, x \rangle$ is $\langle x, y \rangle$ rotated 90° counterclockwise).

Work and line integrals.

$W = (\text{force}).(\text{distance}) = \vec{F} \cdot \Delta\vec{r}$ for a small motion $\Delta\vec{r}$. Total work is obtained by summing these along a trajectory C : get a “line integral”

$$W = \int_C \vec{F} \cdot d\vec{r} \left(= \lim_{\Delta\vec{r} \rightarrow 0} \sum_i \vec{F} \cdot \Delta\vec{r}_i \right).$$

To evaluate the line integral, we observe C is parametrized by time, and give meaning to the notation $\int_C \vec{F} \cdot d\vec{r}$ by

$$\int_C \vec{F} \cdot d\vec{r} = \int_{t_1}^{t_2} \left(\vec{F} \cdot \frac{d\vec{r}}{dt} \right) dt.$$

Example: $\vec{F} = -y\hat{i} + x\hat{j}$, C is given by $x = t$, $y = t^2$, $0 \leq t \leq 1$ (portion of parabola $y = x^2$ from (0,0) to (1,1)). Then we substitute expressions in terms of t everywhere:

$$\vec{F} = \langle -y, x \rangle = \langle -t^2, t \rangle, \quad \frac{d\vec{r}}{dt} = \left\langle \frac{dx}{dt}, \frac{dy}{dt} \right\rangle = \langle 1, 2t \rangle,$$

so $\int_C \vec{F} \cdot d\vec{r} = \int_0^1 \vec{F} \cdot \frac{d\vec{r}}{dt} dt = \int_0^1 \langle -t^2, t \rangle \cdot \langle 1, 2t \rangle dt = \int_0^1 t^2 dt = \frac{1}{3}$. (in the end things always reduce to a one-variable integral.)

In fact, the definition of the line integral does not involve the parametrization: so the result is the same no matter which parametrization we choose. For example we could choose to parametrize the parabola by $x = \sin \theta$, $y = \sin^2 \theta$, $0 \leq \theta \leq \pi/2$. Then we'd get $\int_C \vec{F} \cdot d\vec{r} = \int_0^{\pi/2} \dots d\theta$, which would be equivalent to the previous one under the substitution $t = \sin \theta$ and would again be equal to $\frac{1}{3}$. In practice we always choose the simplest parametrization!

New notation for line integral: $\vec{F} = \langle M, N \rangle$, and $d\vec{r} = \langle dx, dy \rangle$ (this is in fact a differential: if we divide both sides by dt we get the component formula for the velocity $d\vec{r}/dt$). So the line integral

becomes

$$\int_C \vec{F} \cdot d\vec{r} = \int_C M dx + N dy.$$

The notation is dangerous: this is not a sum of integrals w.r.t. x and y , but really a line integral along C . To evaluate one must express everything in terms of the chosen parameter.

In the above example, we have $x = t$, $y = t^2$, so $dx = dt$, $dy = 2t dt$ by implicit differentiation; then

$$\int_C -y dx + x dy = \int_0^1 -t^2 dt + t(2t) dt = \int_0^1 t^2 dt = \frac{1}{3}$$

(same calculation as before, using different notation).

Geometric approach.

Recall velocity is $\frac{d\vec{r}}{dt} = \frac{ds}{dt} \hat{\mathbf{T}}$ (where s = arclength, $\hat{\mathbf{T}}$ = unit tangent vector to trajectory).

So $d\vec{r} = \hat{\mathbf{T}} ds$, and $\int_C \vec{F} \cdot d\vec{r} = \int_C \vec{F} \cdot \hat{\mathbf{T}} ds$. Sometimes the calculation is easier this way!

Example: C = circle of radius a centered at origin, $\vec{F} = x\hat{i} + y\hat{j}$, then $\vec{F} \cdot \hat{\mathbf{T}} = 0$ (picture drawn), so $\int_C \vec{F} \cdot \hat{\mathbf{T}} ds = \int 0 ds = 0$.

Example: same C , $\vec{F} = -y\hat{i} + x\hat{j}$, then $\vec{F} \cdot \hat{\mathbf{T}} = |\vec{F}| = a$, so $\int_C \vec{F} \cdot \hat{\mathbf{T}} ds = \int a ds = a(2\pi a) = 2\pi a^2$; checked that we get the same answer if we compute using parametrization $x = a \cos \theta$, $y = a \sin \theta$.

18.02 Lecture 20. – Fri, Oct 26, 2007

Line integrals continued.

Recall: line integral of $\vec{F} = M\hat{i} + N\hat{j}$ along a curve C : $\int_C \vec{F} \cdot d\vec{r} = \int_C M dx + N dy = \int_C \vec{F} \cdot \hat{\mathbf{T}} ds$.

Example: $\vec{F} = y\hat{i} + x\hat{j}$, $\int_C \vec{F} \cdot d\vec{r}$ for $C = C_1 + C_2 + C_3$ enclosing sector of unit disk from 0 to $\pi/4$. (picture shown). Need to compute $\int_{C_i} y dx + x dy$ for each portion:

1) x -axis: $x = t$, $y = 0$, $dx = dt$, $dy = 0$, $0 \leq t \leq 1$, so $\int_{C_1} y dx + x dy = \int_0^1 0 dt = 0$. Equivalently, geometrically: along x -axis, $y = 0$ so $\vec{F} = x\hat{j}$ while $\hat{\mathbf{T}} = \hat{i}$ so $\int_{C_1} \vec{F} \cdot \hat{\mathbf{T}} ds = 0$.

2) C_2 : $x = \cos \theta$, $y = \sin \theta$, $dx = -\sin \theta d\theta$, $dy = \cos \theta d\theta$, $0 \leq \theta \leq \frac{\pi}{4}$. So

$$\int_{C_2} y dx + x dy = \int_0^{\pi/4} \sin \theta (-\sin \theta) d\theta + \cos \theta \cos \theta d\theta = \int_0^{\pi/4} \cos(2\theta) d\theta = \left[\frac{1}{2} \sin(2\theta) \right]_0^{\pi/4} = \frac{1}{2}.$$

3) C_3 : line segment from $(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$ to $(0, 0)$: could take $x = \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}t$, $y = \text{same}$, $0 \leq t \leq 1$, ... but easier: C_3 backwards (“ $-C_3$ ”) is $y = x = t$, $0 \leq t \leq \frac{1}{\sqrt{2}}$. Work along $-C_3$ is opposite of work along C_3 .

$$\int_{C_3} y dx + x dy = \int_{1/\sqrt{2}}^0 t dt + t dt = - \int_0^{1/\sqrt{2}} 2t dt = -[t^2]_0^{1/\sqrt{2}} = -\frac{1}{2}.$$

If \vec{F} is a gradient field, $\vec{F} = \nabla f = f_x \hat{i} + f_y \hat{j}$ (f is called “potential function”), then we can simplify evaluation of line integrals by using the fundamental theorem of calculus.

Fundamental theorem of calculus for line integrals:

$$\int_C \nabla f \cdot d\vec{r} = f(P_1) - f(P_0) \text{ when } C \text{ runs from } P_0 \text{ to } P_1.$$

Equivalently with differentials: $\int_C f_x dx + f_y dy = \int_C df = f(P_1) - f(P_0)$. Proof:

$$\int_C \nabla f \cdot d\vec{r} = \int_{t_0}^{t_1} (f_x \frac{dx}{dt} + f_y \frac{dy}{dt}) dt = \int_{t_0}^{t_1} \frac{d}{dt} (f(x(t), y(t))) dt = [f(x(t), y(t))]_{t_0}^{t_1} = f(P_1) - f(P_0).$$

E.g., in the above example, if we set $f(x, y) = xy$ then $\nabla f = \langle y, x \rangle = \vec{F}$. So \int_{C_i} can be calculated just by evaluating $f = xy$ at end points. Picture shown of C , vector field, and level curves.

Consequences: for a gradient field, we have:

- Path independence: if C_1, C_2 have same endpoints then $\int_{C_1} \nabla f \cdot d\vec{r} = \int_{C_2} \nabla f \cdot d\vec{r}$ (both equal to $f(P_1) - f(P_0)$ by the theorem). So the line integral $\int_C \nabla f \cdot d\vec{r}$ depends only on the end points, not on the actual trajectory.
- Conservativeness: if C is a closed loop then $\int_C \nabla f \cdot d\vec{r} = 0$ ($= f(P) - f(P)$).
(e.g. in above example, $\int_C = 0 + \frac{1}{2} - \frac{1}{2} = 0$.)

WARNING: this is only for gradient fields!

Example: $\vec{F} = -y\hat{i} + x\hat{j}$ is not a gradient field: as seen Thursday, along C = circle of radius a counterclockwise ($\vec{F} \parallel \hat{T}$), $\int_C \vec{F} \cdot d\vec{r} = 2\pi a^2$. Hence \vec{F} is not conservative, and not a gradient field.

Physical interpretation.

If the force field \vec{F} is the gradient of a potential f , then work of \vec{F} = change in value of potential.

E.g.: 1) \vec{F} = gravitational field, f = gravitational potential; 2) \vec{F} = electrical field; f = electrical potential (voltage). (Actually physicists use the opposite sign convention, $\vec{F} = -\nabla f$).

Conservativeness means that energy comes from change in potential f , so no energy can be extracted from motion along a closed trajectory (conservativeness = conservation of energy: the change in kinetic energy equals the work of the force equals the change in potential energy).

We have four equivalent properties:

- (1) \vec{F} is conservative ($\int_C \vec{F} \cdot d\vec{r} = 0$ for any closed curve C)
- (2) $\int F \cdot d\vec{r}$ is path independent (same work if same end points)
- (3) \vec{F} is a gradient field: $\vec{F} = \nabla f = f_x \hat{i} + f_y \hat{j}$.
- (4) $M dx + N dy$ is an exact differential ($= f_x dx + f_y dy = df$.)

((1) is equivalent to (2) by considering C_1, C_2 with same endpoints, $C = C_1 - C_2$ is a closed loop.
(3) \Rightarrow (2) is the FTC, \Leftarrow will be key to finding potential function: if we have path independence then we can get $f(x, y)$ by computing $\int_{(0,0)}^{(x,y)} \vec{F} \cdot d\vec{r}$. (3) and (4) are reformulations of the same property).