18.02 Multivariable Calculus
Fall 2007

18.02 Lecture 29. – **Tue, Nov 20, 2007**

Recall statement of divergence theorem: $\iint_S \mathbf{F} \cdot d\vec{S} = \iiint_D \operatorname{div} \mathbf{F}\, dV$.

Del operator. $\nabla = \langle \partial/\partial x, \partial/\partial y, \partial/\partial z \rangle$ (symbolic notation!)

$\nabla f = \langle \partial f/\partial x, \partial f/\partial y, \partial f/\partial z \rangle$ = gradient.

$\nabla \cdot \mathbf{F} = \langle \partial/\partial x, \partial/\partial y, \partial/\partial z \rangle \cdot \langle P, Q, R \rangle = P_x + Q_y + R_z$ = divergence.

Physical interpretation. $\operatorname{div} \mathbf{F}$ = source rate = flux generated per unit volume. Imagine an incompressible fluid flow (i.e. a given mass occupies a fixed volume) with velocity $\mathbf{F}$, then $\iiint_D \operatorname{div} \mathbf{F}\, dV = \iint_S \mathbf{F} \cdot \hat{\mathbf{n}}\, dS$ = flux through S is the net amount leaving D per unit time = total amount of sources (minus sinks) in D.

Proof of divergence theorem. To show $\iint_S \langle P, Q, R \rangle \cdot d\vec{S} = \iiint_D (P_x + Q_y + R_z)\, dV$, we can separate into sum over components and just show $\iint_S R\hat{\boldsymbol{k}} \cdot d\vec{S} = \iiint_D R_z\, dV$ & same for P and Q.

If the region D is vertically simple, i.e. top and bottom surfaces are graphs, $z_1(x,y) \leq z \leq z_2(x,y)$, with (x,y) in some region U of xy-plane: r.h.s. is

$$\iiint_D R_z\, dV = \iint_U \Big(\int_{z_1(x,y)}^{z_2(x,y)} R_z\, dz\Big)\, dx\, dy = \iint_U (R(x,y,z_2(x,y)) - R(x,y,z_1(x,y))\, dx\, dy.$$

Flux through top: $d\vec{S} = \langle -\partial z_2/\partial x, -\partial z_2/\partial y, 1 \rangle dx\, dy$, so $\iint_{\text{top}} R\hat{\boldsymbol{k}} \cdot d\vec{S} = \iint R(x,y,z_2(x,y))\, dx\, dy$.

Bottom: $d\vec{S} = -\langle -\partial z_1/\partial x, -\partial z_1/\partial y, 1 \rangle dx\, dy$, so $\iint_{\text{bottom}} R\hat{\boldsymbol{k}} \cdot d\vec{S} = \iint -R(x,y,z_1(x,y))\, dx\, dy$.

Sides: sides are vertical, $\hat{\mathbf{n}}$ is horizontal, $\mathbf{F}$ is vertical so flux = 0.

Given any region D, decompose it into vertically simple pieces (illustrated for a donut). Then $\iiint_D$ = sum of pieces (clear), and $\iint_S$ = sum of pieces since the internal boundaries cancel each other.

Diffusion equation: governs motion of smoke in (immobile) air (dye in solution, ...)

The equation is: $\dfrac{\partial u}{\partial t} = k\nabla^2 u = k\left(\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial^2 u}{\partial z^2}\right)$,
where $u(x,y,z,t)$ = concentration of smoke; we'll also introduce $\mathbf{F}$ = flow of the smoke. It's also the heat equation (u = temperature).

Equation uses two inputs:

1) Physics: $\mathbf{F} = -k\nabla u$ (flow goes from highest to lowest concentration, faster if concentration changes more abruptly).

2) Flux and quantity of smoke are related: if D bounded by closed surface S, then $\iint_S \mathbf{F} \cdot \hat{\mathbf{n}}\, dS = -\frac{d}{dt}\iiint_D u\, dV$. (flux out of D = - variation of total amount of smoke inside D)

By differentiation under integral sign, the r.h.s. is $-\iiint_D \frac{\partial}{\partial t} u\, dV$ (This can be explained in terms of integral as a sum of $u(x_i, y_i, z_i, t)\Delta V_i$ and derivative of sum is sum of derivatives) and by divergence theorem the l.h.s. is $\iiint_D \operatorname{div} \mathbf{F}\, dV$. Dividing by volume of D, we get

$$-\frac{1}{vol(D)} \iiint_D \frac{\partial u}{\partial t}\, dV = \frac{1}{vol(D)} \iiint_D \operatorname{div} \mathbf{F}\, dV.$$

Same average values over any region; taking limit as D shrinks to a point, get $\partial u/\partial t = -\operatorname{div} \mathbf{F}$.

Combining, we get the diffusion equation: $\partial u/\partial t = -\operatorname{div} \mathbf{F} = +k\operatorname{div}(\nabla u) = k\nabla^2 u$.