18.02 Multivariable Calculus
Fall 2007

18.02 Lecture 33. – **Thu, Dec 6, 2007**

Handouts: PS12 solutions; exam 4 solutions; review sheet and practice final.

Applications of div and curl to physics.

Recall: curl of velocity field = 2· angular velocity vector (of the rotation component of motion).

E.g., for uniform rotation about z-axis, $\boldsymbol{v} = \omega(-y\hat{\boldsymbol{\imath}} + x\hat{\boldsymbol{\jmath}})$, and $\nabla \times \boldsymbol{v} = 2\omega\hat{\boldsymbol{k}}$.

Curl singles out the rotation component of motion (while div singles out the stretching component).

Interpretation of curl for force fields.

If we have a solid in a force field (or rather an acceleration field!) $\mathbf{F}$ such that the force exerted on Δm at (x, y, z) is $\mathbf{F}(x, y, z)\,\Delta m$: recall the *torque* of the force about the origin is defined as $\tau = \vec{r} \times \mathbf{F}$ (for the entire solid, take $\iiint \ldots\, \delta\, dV$), and measures how $\mathbf{F}$ imparts rotation motion.

For translation motion: $\dfrac{\text{Force}}{\text{Mass}} = \text{acceleration} = \dfrac{d}{dt}(\text{velocity})$.

For rotation effects: $\dfrac{\text{Torque}}{\text{Moment of inertia}} = \text{angular acceleration} = \dfrac{d}{dt}(\text{angular velocity})$.

Hence: $\text{curl}(\dfrac{\text{Force}}{\text{Mass}}) = 2\,\dfrac{\text{Torque}}{\text{Moment of inertia}}$.

Consequence: if $\mathbf{F}$ derives from a potential, then $\nabla \times \mathbf{F} = \nabla \times (\nabla f) = 0$, so $\mathbf{F}$ does not induce any rotation motion. E.g., gravitational attraction by itself does not affect Earth's rotation. (not strictly true: actually Earth is deformable; similarly, friction and tidal effects due to Earth's gravitational attraction explain why the Moon's rotation and revolution around Earth are synchronous).

Div and curl of electrical field. – part of Maxwell's equations for electromagnetic fields.

1) Gauss-Coulomb law: $\nabla \cdot \vec{E} = \dfrac{\rho}{\epsilon_0}$ (ρ = charge density, ϵ_0 = physical constant).

By divergence theorem, can reformulate as: $\displaystyle\iint_S \vec{E} \cdot \hat{\mathbf{n}}\, dS = \iiint_D \nabla \cdot \vec{E}\, dV = \frac{Q}{\epsilon_0}$, where Q = total charge inside the closed surface S.

This formula tells how charges influence the electric field; e.g., it governs the relation between voltage between the two plates of a capacitor and its electric charge.

2) Faraday's law: $\nabla \times \vec{E} = -\dfrac{\partial \vec{B}}{\partial t}$ ($\vec{B}$= magnetic field).

So in presence of a varying magnetic field, $\vec{E}$ is no longer conservative: if we have a closed loop of wire, we get a non-zero voltage ("induction" effect). By Stokes, $\displaystyle\oint_C \vec{E} \cdot d\vec{r} = -\frac{d}{dt} \iint_S \vec{B} \cdot \hat{\mathbf{n}}\, dS$.

This principle is used e.g. in transformers in power adapters: AC runs through a wire looped around a cylinder, which creates an alternating magnetic field; the flux of this magnetic field through another output wire loop creates an output voltage between its ends.

There are two more Maxwell equations, governing div and curl of $\vec{B}$: $\nabla \cdot \vec{B} = 0$, and $\nabla \times \vec{B} = \mu_0 \vec{J} + \epsilon_0 \mu_0 \dfrac{\partial \vec{E}}{\partial t}$ (where $\vec{J}$ = current density).