University of Toronto at Scarborough
Department of Computer & Mathematical Sciences

MIDTERM

MATC34 – Complex Variables

Examiner: Lisa Jeffrey

Date: November 3, 2018
Time: 17:00PM–19:00PM

FAMILY NAME: ______________________________

GIVEN NAME(S): ______________________________

STUDENT NUMBER: ______________________________

SIGNATURE: ______________________________

DO NOT OPEN THIS BOOKLET UNTIL INSTRUCTED TO DO SO

NOTES:

- There are 6 numbered pages in the test including the blank pages. It is your responsibility to ensure that, at the start of the exam, this booklet has all its pages and that all pages are handed in.
- No calculators allowed.

  No books or calculators may be used

You may use any theorems stated in class, as long as you state them clearly and correctly.

FOR MARKERS ONLY

Question	Marks
1	/ 20
2	/ 20
3	/ 20
4	/ 20
5	/ 20
TOTAL	**/100**

1. **(20 points)**

   (a) (7 points) Find the limit of the function $f(z) = \bar{z}$ as $z \to 0$. Here z is a complex number.

   Solution: The limit is

   $$\lim_{z \to 0} \bar{z} = \lim_{r \to 0} re^{-i\theta} = 0$$

   (where (r, θ) are polar coordinates for z).

   (b) (7 points) Is the function $f(z)$ defined by

   $$f(z) = \bar{z}$$

   differentiable at $z = 0$? If you think so, give a proof and compute $\frac{df}{dz}$ at this value. If you think not, show why the complex derivative at 0 does not exist.

   Solution: The derivative would be

   $$\lim_{h \to 0} \bar{h}/h = \lim_{r \to 0} e^{-2i\theta}$$

   (in polar coordinates as in the first part). This limit does not exist.

   (c) (6 points) Is the function $f(z)$ defined by

   $$f(z) = z$$

   differentiable at $z = 0$? If you think so, give a proof and compute $\frac{df}{dz}$ at this value. If you think not, show why the complex derivative at 0 does not exist.

   Solution: This derivative is

   $$\lim_{h \to 0} h/h = 1$$

   (where h is a complex number). So the derivative does exist, and its value is 1.

2. **(20 points)** Let γ denote the contour around the boundary of the square with corners $1+i$, $1-i$, $-1+i$, $-1-i$ oriented counterclockwise. Evaluate the following integrals:

   (a) (10 points) $\int_\gamma \frac{1}{(z-2)(z-4)} dz$
   Solution: Use Cauchy's theorem and the deformation theorem. The only points where this not holomorphic are $z = 2$ and $z = 4$, Neither of these points is inside the square so the answer is 0

   (b) (10 points) $\int_\gamma (z + \bar{z}) dz$
   Solution: This is

$$\int 2x dx + \int_1^{-1} 2x dx + \int_{-1}^{1} 1 dy + \int_1^{-1} (-1) dy$$

$$= 0 + 0 + 2 + 2 = 4.$$

3. **(20 points)** Let f be the function

$$f(z) = \frac{1}{4+z^2}$$

(a) (7 points) Compute the integral of f around the circle $|z| = 1$.
Solution: This function is holomorphic except at $z = \pm 2i$. These points are outside the unit circle so the integral is 0 by Cauchy.

(b) Compute the integral of f around the circle $|z| = 4$.
Solution: This function is holomorphic except at $z = \pm 2i$. we factor

$$\frac{1}{z^2+4} = \frac{1}{(z+2i)(z-2i)}$$

By partial fractions this equals

$$\frac{1}{z+2i} - \frac{1}{z-2i}$$

The integral $\int_\gamma \frac{1}{z+2i} = 2\pi i$ by the deformation theorem. .
Both points $\pm 2i$ are inside the circle γ. Each contributes $2\pi i$ but with opposite signs, so the integral is 0.

(c) Compute the integral of f around the semicircle which is the union of $\{4e^{i\theta} | 0 \le \theta \le \pi\}$ and the line segment $\{x| -4 \le x \le 4\}$ which is a subset of the real axis.
Solution: Only the point $z = 2i$ is inside the semicircle. we factor

$$\frac{1}{z^2+4} = \frac{1}{(z+2i)(z-2i)}$$

By partial fractions this equals

$$\frac{1}{z+2i} - \frac{1}{z-2i}$$

The integral $\int_\gamma \frac{1}{z+2i} = 0$ by Cauchy.
The integral $\int_\gamma \frac{1}{z-2i} = 2\pi i$ (by the Deformation Theorem).

4. **(20 points)** Find a power series $\sum_{n=0}^{\infty} c_n(z-1)^n$ so that

$$\frac{1}{z} = \sum_{n=0}^{\infty} c_n(z-1)^n.$$

What is its radius of convergence?

Solution:

$$z = (z-1)+1$$

so

$$\frac{1}{z} = \frac{1}{(z-1)+1} = \sum_{n=0}^{\infty}(-1)^n(z-1)^n$$

The radius of convergence is 1 (because the radius of convergence of $\sum_{n=0}^{\infty} z^n$ is 1).

5. **(20 points)** Use the Cauchy integral formula to compute the integral

$$\int_\gamma \frac{e^z dz}{(z^2+4)(z-1)}$$

where

$$\gamma(t) = 3e^{it}/2$$

is a closed contour (it is the circle of radius 3/2 and center 0) traversed counterclockwise.

Solution:

The Cauchy integral formula states that

$$2\pi i f(a) = \int_\gamma \frac{f(z)}{z-a}.$$

In this case choose

$$a = 1$$

and

$$f(z) = \frac{e^z dz}{(z^2+4)}.$$

Hence the integral is

$$2\pi i e/5.$$

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