Lectures on complex analysis

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Abstract

These are the collected lecture notes for Math 113.

Contents

Chapter 1

Complex numbers and holomorphic functions

In this first chapter I will give you a taste of complex analysis, and recall some basic facts about the complex numbers. We define holomorphic functions, the subject of this course. These functions turn out to be much more well-behaved than the functions you have encountered in real analysis. We will mention the most striking such properties that we will prove during the courses. We also give many examples of holomorphic functions, ending with the most exciting example: the Riemann zeta-function.

We will go over this material in more depth in the coming lectures. However, you can take a look at [SS03, Section 1.1], [MH87, Section 1.1 and 1.2] for more details about the complex numbers.

1.1 Complex numbers

Complex numbers have their origin in attempts to understand the roots of polynomials. By the sixteenth century, mathematicians had found formulas for the solutions of the equation

$$p(x) = 0,$$

when p is a quadratic or cubic polynomial. For example, when $p(x) = ax^2 + bx + c$ such solutions x_0 are given by the well-known *quadratic formula*

$$x_0 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

When $b^2 - 4ac \geq 0$ the graph of p intersects the x-axis, and the above formula can be interpreted as giving the x-coordinate of these intersections. When $b^2 - 4ac < 0$ there are no such intersections, which is reflected by the fact that at first sight $\sqrt{b^2 - 4ac}$ is meaningless when the expression $b^2 - 4ac$, called the *discriminant* is negative. This suggests one way to interpret the quadratic formula: if the discriminant is positive it gives two roots, if the discriminant is zero it gives a single root, and if it is negative it gives no roots.

However, it was observed that if one introduced a new number i satisfying $i^2 = -1$ to the number system, and let square roots of negative numbers be multiples of i, they not only appeared to give roots for the polynomial p when the discriminant is negative, but could be fruitfully manipulated to solve various other algebraic problems. That is, the transition from real to complex numbers gives the quadratic formula a useful interpretation even when the discriminant is negative. Throughout the seventeenth and eighteenth centuries these novel complex numbers took on increasingly important roles in mathematics, until eventually in the nineteenth they were considered as being equal in status to the real numbers.

1.1.1 The algebra of complex numbers

Let us introduce some notation which we will use throughout the course. When a is a real number, we use the notation $a \in \mathbb{R}$. A number of the form bi with $b \in \mathbb{R}$ is called an *imaginary* number, and a number of the form $a + bi$ is called an *complex number*. In particular, through the course, real numbers and imaginary numbers are subsets of the complex numbers. Given a complex number $a + bi$, a is its *real part* and b its *imaginary part*. Observe we can record $a + bi$ as a pair (a, b) of real numbers. In fact, we shall take this as the starting point of our formal definition.

Definition 1.1.1. The set $\mathbb{C}$ of complex numbers is the set of all pairs $(a, b) \in \mathbb{R}^2$. We will write a pair (a, b) as $a + bi$.

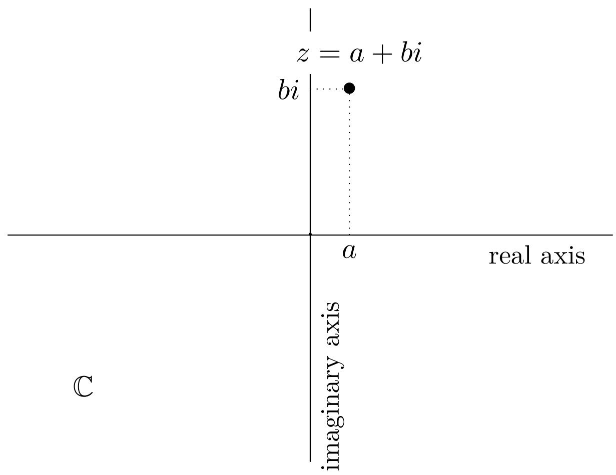


Figure 1.1 A point in the complex plane, its real part giving its x-coordinate and its imaginary part its y-coordinate.

What makes numbers useful is that one can perform algebraic operations with them. Knowing how to define such operations for the real numbers, we can define addition and multiplication of complex numbers:

$$(a + bi) + (c + di) := (a + c) + (b + d)i$$

$$(a + bi) \cdot (c + di) := (ac - bd) + (ad + bc)i.$$

The latter rule is easy to reconstruct if you recall that multiplication of complex numbers should be an extension of that of real numbers, and that we required i to have the property that $i^2 = -1$. We will often drop the $\cdot$ from the notation of multiplication.

These addition and multiplication rules have the same formal properties as the real numbers: the only non-obvious one is that non-zero complex numbers have multiplicative inverses. To see this is the case, let's try to solve

$$(a + bi) \cdot (c + di) = (ac - bd) + (ad + bc)i = 1.$$

If we attempt to solve the linear equations

$$ac - bd = 1 \qquad \text{and} \qquad ad + bc = 0$$

for c, d in terms of a, b, we get

$$(a + bi)^{-1} = c + di = \frac{a - bi}{a^2 + b^2}. \tag{1.1}$$

In other words, the complex numbers are a *field*. This means that they satisfy the following axioms (this spells out the "obvious" formal properties I didn't spell out above):

Proposition 1.1.2. $(\mathbb{C}, +, \cdot, 0, 1)$ *is a field, that is, has the following properties:*

(i) $a + b = b + a$,

(ii) $(a + b) + c = a + (b + c)$,

(iii) $a + 0 = a = 0 + a$,

(iv) for all $a \in \mathbb{C}$, *there exists* $b \in \mathbb{C}$ *such that* $a + b = 0$,

(v) $ab = ba$,

(vi) $(ab)c = a(bc)$,

(vii) $1a = a = a1$,

(viii) for all $a \in \mathbb{C} \setminus \{0\}$, *there exists* $b \in \mathbb{C}$ *such that* $ab = 1$,

(ix) $(a + b)c = ab + bc$ *and* $a(b + c) = ab + ac$.

As a exercise, you should identify which of these properties are referred to as "commutativity," "associativity," and "distributivity."

In fact, the complex numbers are best type of field; they are *algebraically closed.* This means that every non-constant polynomial p with coefficients in $\mathbb{C}$ has a root in $\mathbb{C}$, i.e. for every polynomial p which is not constant there exists a $z_0 \in \mathbb{C}$ such that $p(z_0) = 0$. This is a fact that we will be able to prove quite early in this course using the techniques of complex analysis:

Theorem 1.1.3 (Fundamental theorem of algebra)**.** *If* $p(z) = a_n z^n + \cdots + a_1 z + a_0$ *is a polynomial with complex coefficients which is not constant (i.e.* $n > 0$*), then there is a complex number* z_0 *such that* $p(z_0) = 0$.

This should be surprising. It is not clear at all that adding the solutions $\pm i$ of the single equation $x^2 + 1 = 0$ means you can now also solve other quadratic polynomials, or polynomials of higher degree, let alone polynomials with complex coefficients.

1.1.2 The geometry of complex numbers

The real numbers have two more important properties: a notion of absolute value, and an order. Complex numbers inherit the former, a notion of absolute value, but not the latter (as you'll see on the homework).

The norm

To define the distance between complex numbers, we use a geometric interpretation of complex numbers. Indeed, using our definition of complex numbers as pairs of real numbers we can draw the complex numbers $\mathbb{C}$ as a plane: the *complex plane*. The real numbers lie on the x-axis and the imaginary numbers lie on the y-axis. Under this identification the complex $a+bi$ corresponds to the vector $(a,b) \in \mathbb{R}^2$. and addition of complex numbers is vector-addition.

The absolute value of a real number is its distance from the origin. This makes it clear how we should define the *absolute value* of a complex number, which we will occasionally also refer to as the *norm*. It should defined as the distance to the origin:

$$|a+bi| := \sqrt{a^2+b^2}.$$

Observe that $|z| = 0$ if and only if $z = 0$, as behooves a norm. It interacts with addition and multiplication as follows:

Proposition 1.1.4. $|-|\colon \mathbb{C} \to \mathbb{R}_{\geq 0}$ *is a norm, that is, has the following properties:*

(i) $|z| = 0$ *if and only if* $z = 0$,

(ii) $|zw| = |z||w|$,

(iii) $|z+z'| \leq |z| + |z'|$.

As usual, one can define distances in terms of absolute values. Just like the distance between two real numbers x, x' is given by $|x - x'|$, we should think of $|z - z'|$ as the distance between the two complex numbers z, z'. It is just the Euclidean distance in the complex plane.

The complex conjugation

It is helpful to express the norm in terms of another operation on complex numbers, *complex conjugation*. This is given by sending i to $-i$, or more precisely by

$$\overline{a+bi} := a - bi.$$

That such an operation exists and is useful, is not surprising: when formally adding the solution i of $x^2+1=0$ to the real numbers, you can't distinguish between i and $-i$, and from this should result a symmetry of the complex numbers.[1]

[1] In fact, these are all the symmetries of the complex numbers as a field extension of the real numbers. That is, the *Galois group* $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$ is $\mathbb{Z}/2$ with non-identity element given by complex conjugation.

We have that $|z|^2 = z\bar{z}$. From this we get the following expression for the multiplicative inverse:

$$z^{-1} = \frac{\bar{z}}{|z|^2},$$

which you should compare to (1.1).

The polar representation of complex numbers

When we were thinking of complex numbers as elements in the plane, we were using cartesian coordinates. We can also use polar coordinates: recording the distance to the origin and the angle to the positive x-axis. A non-zero complex number $a + bi$ has polar coordinates

$$(\text{distance}, \text{angle}) = (r, \theta) =: (|z|, \arg(z)).$$

That is, the first is the norm $|z|$ of a complex number z, the second is the so-called *argument* $\arg(z)$. In other words, I am defining these by demanding that

$$z = |z|(\cos(\arg(z)) + i \sin(\arg(z))).$$

The argument is a bit more subtle than the norm: if we can take $\arg(z) = \theta$ in the above formula, we can also take it to be $\theta + k2\pi$ for any $k \in \mathbb{Z}$. The way we resolve this ambiguity is by declaring the argument to be an element of the quotient $\mathbb{R}/2\pi\mathbb{Z}$, i.e. identifying two real numbers when they differs by a multiple of 2π. This will continue to be a source of headaches, especially when trying to define logarithms.

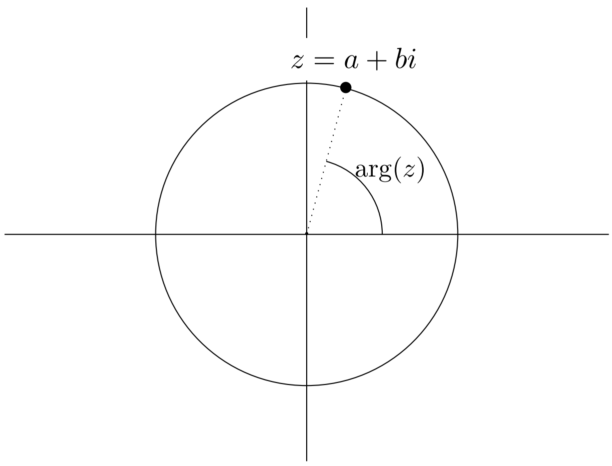


Figure 1.2 The circle in the complex plane around the origin and through z has radius $|z|$.

Addition of complex numbers was easy to represent in cartesian coordinates, but multiplication not so much so. For polar coordinates the situation is reversed: multiplication is easy to represent, but addition not so much. This is encoded by the following lemma, the second part of which is proven by the addition formulas for sines and cosines:

Lemma 1.1.5. *We have that* $|zw| = |z||w|$, $\arg(zw) \equiv \arg(z) + \arg(w) \pmod{2\pi\mathbb{Z}}$.

This gives us some insight into the geometry of multiplication by a complex number z: it scales the complex plane by $|z|$ and rotates it by angle $\arg(z)$.

1.2 Holomorphic functions

The main objects of interest in this course are complex-valued functions defined on a some subset of the complex plane:

$$f\colon \mathbb{C} \supset \Omega \longrightarrow \mathbb{C}.$$

We do not care about arbitrary functions: we care about those that are complex-differentiable. These are called *holomorphic*. This definition concerns the behavior of $f(z_i)$ as $z_i \to z_0 \in \Omega$ and for this to be reasonable we should be able to approach z_0 from all directions. That is, Ω should be open, i.e. for each point in Ω we can find a little disk around it which is also contained in Ω.

Definition 1.2.1. Let Ω be open, then $f\colon \mathbb{C} \supset \Omega \longrightarrow \mathbb{C}$ is *complex-differentiable* at $z_0 \in \Omega$ if there exists an $a \in \mathbb{C}$ such that

$$\lim_{h \to 0} \frac{|f(z_0 + h) - f(z_0) - ah|}{|h|} = 0.$$

We then write $a = f'(z_0)$, and say a is the derivative of f at z_0.

Definition 1.2.2. We say that f is *holomorphic* if it is complex-differentiable at all $z_0 \in \Omega$.

This implies f is continuous, but at first sight does not imply that f' is continuous. However, holomorphic functions have many miraculous properties, the first of which is the following:

(1) If f is holomorphic then not only f' continuous, but it is again holomorphic.

By induction, if f is once complex-differentiable then f is infinitely many times complex-differentiable!

Let me try to dispel some of the mystery of this astounding rigidity property of holomorphic functions. Using cartesian coordinates, the subset $\Omega \subset \mathbb{C}$ can be thought of as a subset of $\mathbb{R}^2$ and the holomorphic function $f\colon \mathbb{C} \supset \Omega \to \mathbb{C}$ gives a function $\mathbb{R}^2 \supset \Omega \to \mathbb{R}^2$. Explicitly, this is the function

$$\begin{aligned} \mathbb{R}^2 \supset \Omega &\longrightarrow \mathbb{R}^2 \\ (x, y) &\longmapsto (\mathrm{Re}(f(x + yi)), \mathrm{Im}(f(x + yi))). \end{aligned}$$

We will see that if f is holomorphic, this is a differentiable function of two variables. That is, at each $z_0 = x_0 + y_0 i$ it has a total derivative at $z_0 = x_0 + y_0 i$, a (2×2)-matrix with real entries, which gives the best linear approximation to the function. For an arbitrary differentiable function of two variables, this matrix can be anything. For a holomorphic function, this matrix comes from multiplication by the complex number $f'(z_0)$ and is given by a *composition of rotation and scaling.* Shearing or scaling by differing amounts in different directions are *not* allowed. That the total derivative is so restricted at each point is what gives holomorphic functions their rigidity.

1.2.1 Examples of holomorphic functions

From the definition of a holomorphic function, one can directly check that functions

$$\begin{aligned} 1\colon \mathbb{C} &\longrightarrow \mathbb{C} \\ z &\longmapsto 1, \end{aligned}$$

$$\begin{aligned} z\colon \mathbb{C} &\longrightarrow \mathbb{C} \\ z &\longmapsto z, \end{aligned}$$

are holomorphic. Indeed, we will see that the rules of complex differentiation of holomorphic functions are the same as those for ordinary differentiation of differentiable functions. The derivatives of the above functions are just $1' = 0$ and $z' = 1$.

Similarly, a sum of holomorphic functions is holomorphic with derivative the sum of the derivatives, and a product of holomorphic functions is holomorphic with derivative given by the product rule. Using finite linear combinations of products of these basic functions, we can form any polynomial:

Example 1.2.3. Every polynomial with complex coefficients is holomorphic.

We can form more complicated functions using infinite sums. Within its radius of convergence, the power series $\sum_{n=0}^{\infty} a_n z^n$ define a holomorphic function.

Example 1.2.4. The exponential function can be extended from real numbers to the cmoplex numbers by the familiar everywhere-convergent power series:

$$e^z = \sum_{n=0}^{\infty} \frac{1}{n!} z^n.$$

This will our definition. From this we can define the cosine and sine:

$$\cos(z) = \frac{e^z + e^{-z}}{2}, \qquad \sin(z) = \frac{e^{iz} - e^{-iz}}{2i},$$

as well as other trigonometric functions. Using these definitions, we can write the polar representation of a non-zero complex number z more conveniently as:

$$z = |z| e^{i \arg(z)}.$$

In fact, another special property of holomorphic functions which we shall prove in this course, is that locally holomorphic functions can be defined using power series:

(2) If f is holomorphic, then for each $z_0 \in \Omega$ there is a disk of radius $\epsilon > 0$ around z_0 such that f can be described on this disk by a convergent power series

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n.$$

That is, all holomorphic functions are *analytic*. This is in contract with differentiable functions, e.g. the function

$$\begin{aligned} g\colon \mathbb{R} &\longrightarrow \mathbb{R} \\ x &\longmapsto \begin{cases} e^{-1/x^2} & \text{if } x > 0, \\ 0 & \text{otherwise,} \end{cases} \end{aligned}$$

is not given by a power series near $x = 0$.

1.2.2 Integration of holomorphic functions

Since holomorphic functions are continuous functions, we can try to integrate them. The complex plane gives us more freedom to define such integrals than the real line: it is unambiguous what integrating $g\colon \mathbb{R} \to \mathbb{R}$ from a real number a to a larger real number b means, but for a holomorphic function $f\colon \mathbb{C} \to \mathbb{C}$ we need to give a path γ from a complex number $a \in \mathbb{C}$ to a complex number $b \in \mathbb{C}$ to make sense of the integral. That is, we can write

$$\int_a^b g(x)dx$$

but should write

$$\int_\gamma f(z)dz.$$

One can then ask how this depends on the path γ from a to b. Here we meet the most important property of holomorphic functions for physics and engineering:

(3) If γ_0 can be deformed into γ_1 through a family of paths γ_t with the same endpoints and depending continuously on t, then

$$\int_{\gamma_0} f(z)dz = \int_{\gamma_1} f(z)dz.$$

This gives a very powerful method to compute integrals, since you can pick γ to make your life as easy as possible.

Example 1.2.5. After the first couple of weeks you will be able to use this to compute that

$$\int_0^1 \log(\sin(\pi z))dz = -\log(2).$$

Example 1.2.6 (Contour integration). If $\gamma(0) = \gamma(1)$ and γ can be deformed in Ω to a constant map while keeping its endpoints fixed, then

$$\int_\gamma f(z)dz = 0.$$

This is surprising, as γ could be some large circle.

1.3 The Riemann zeta-function

A much more interesting example of a holomorphic function is the *Riemann zeta-function*. For $\mathrm{Re}(s) > 1$ this is defined as[2]

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

To see that difficulties occur with this definition when $\mathrm{Re}(s) \leq 1$, observe that for $s = 1$ we get the divergent harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$.

Remark 1.3.1. It is in fact an instance of a general "point-counting function" construction in algebraic geometry, applied to the scheme $\mathrm{Spec}(\mathbb{Z})$, the algebro-geometric counterpart of the integers.

The Riemann zeta-function actually extends to a holomorphic function $\Omega := \mathbb{C}\setminus\{1\} \to \mathbb{C}$. At $z = 1$, it blows up in the sense that $|\zeta(s)| \to \infty$ as $s \to 1$. That such an extension exists is not a special property of Riemann zeta function, but a general property of holomorphic functions. We will give precise formulations during the course, but for now let us phrase it as:

(4) Holomorphic functions defined on small subsets $\Omega' \subset \mathbb{C}$ admit unique extensions to much larger subsets $\Omega \subset \mathbb{C}$.

The behavior of the zeroes of Riemann ζ-function in Ω is closely related to the distribution of the prime numbers. This is made more plausible from the following equivalent definition $\zeta(s)$ for $\mathrm{Re}(s) > 1$: using that $\frac{1}{1-p^s} = \sum_{k=0}^{\infty} p^{sk}$, unique prime factorization gives (we are skipping some justifications about the manipulations of infinite products and sums)

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \text{ prime}} \left(\sum_{k=0}^{\infty} p^{-sk} \right) = \prod_{p \text{ prime}} \frac{1}{1 - p^{-s}}.$$

For example, let us deduce the following from what we've asserted about the ζ-function so far:

Proposition 1.3.2. *There are infinitely many primes.*

Proof. We use that $|\zeta(s)| \to \infty$ as $s \to 1$, and give a proof by contradiction. Indeed, if there were only finitely many primes $p_1, \ldots, p_r$, then our product expansion for $\mathrm{Re}(s) \to 1$ is finite

$$\zeta(s) = \frac{1}{1 - p_1^{-s}} \cdots \frac{1}{1 - p_r^{-s}}$$

which converges to a large but finite value as $s \to 1$. □

A much more subtle statement about the behavior of ζ-function is:

Theorem 1.3.3. *$\zeta(s)$ has no zeroes satisfying* $\mathrm{Re}(s) \geq 1$.

[2] We switched from z to s as notation for our complex variable here. Using s for the Riemann zeta-function has been the convention since Riemann's paper.

This has implications for the prime-counting function

$$\begin{aligned}\pi\colon \mathbb{R}_{\geq 0} &\longrightarrow \mathbb{R}\\ x &\longmapsto \#\{\text{primes less than or equal to } x\}.\end{aligned}$$

A somewhat complicated argument, which relies a lot on the special properties of integrals of holomorphic functions, deduces from this statement about the Riemann zeta function the following statement about the prime-counting function:

Corollary 1.3.4 (Prime number theorem)**.** *We have that*

$$\pi(x) \sim \frac{x}{\log(x)},$$

where $f(x) \sim g(x)$ *means* $f(x)/g(x) \to 1$ *as* $x \to \infty$*.*

A much stronger error term on the prime number theorem would follow if we knew that the only zeroes of $\zeta(s)$ lie on the line $\mathrm{Re}(s) = 1/2$. This is the famous *Riemann hypothesis*. It is one of the Millennium Prize problems, and you get a million dollars if you solve it.[3] You can find a plot of the values of $\zeta(s)$ on the line $\mathrm{Re}(s) = 1/2$ at http://mathworld.wolfram.com/pdf/posters/Zeta.pdf.

Conjecture 1.3.5 (Riemann hypothesis)**.** *All zeroes of* $\zeta(s)$ *which are not* $-2k$ *for* $k > 0$ *lie on the line* $\mathrm{Re}(s) = 1/2$*.*

This has consequences for the error term in the prime number theorem (we will not prove this in the course):

Corollary 1.3.6. *If the Riemann hypothesis is true, then*

$$\left|\pi(x) - \frac{x}{\log(x)}\right| < \frac{1}{8\pi}\sqrt{x}\log(x) \qquad \textit{for } x \geq 2657.$$

[3]See https://www.claymath.org/millennium-problems/riemann-hypothesis. I would recommend against trying to do so until you have learned more mathematics, as this problem has withstood more than a century of attacks by some of the best mathematicians.

Chapter 2

Functions on the complex plane

Last lecture we outlined some of the topics of the course. Today we recall the complex plane and holomorphic functions, and establish rigorously their basic properties. This is [SS03, Sections 1,1, 1.2].

2.1 The complex plane

As we discussed last time, the objects of interest in this course are in particular complex-valued functions defined on subsets of $\mathbb{C}$. Let's discuss the complex plane $\mathbb{C}$ in more detail.

2.1.1 Recollection: the algebra of complex numbers

Recall that the complex numbers are of the form $a + bi$ with $a, b \in \mathbb{R}$ and $i^2 = -1$. Here $a = \mathrm{Re}(a + bi)$ is the *real part* and $b = \mathrm{Im}(a + bi)$ is the *imaginary part*. As a set we can identify them with $\mathbb{R}^2$, $a + bi \in \mathbb{C}$ corresponding to $(a, b) \in \mathbb{R}^2$. The addition and multiplication on $\mathbb{C}$ follow by demanding that (commutativity, associativity, distributivity, and unitality), and that the equation $i^2 = -1$ holds.

With these operations, $\mathbb{C}$ is a field: every non-zero $z \in \mathbb{C}$ has a multiplicative inverse given by

$$z^{-1} = \frac{\bar{z}}{|z|^2},$$

with $\overline{a + bi} = a - bi$ the *complex conjugate*, and $|a + bi| = \sqrt{a^2 + b^2}$ the *absolute value* (or *norm*).

In terms of the complex conjugate, we have

$$\mathrm{Re}(z) = \frac{z + \bar{z}}{2}, \quad \mathrm{Im}(z) = \frac{z - \bar{z}}{2i}.$$

2.1.2 The topology of complex numbers

Using the absolute value (or norm), we will make $\mathbb{C}$ into a metric space. Using this, we make sense of convergence, as well as open, closed, and compact subsets.

Distance

The absolute value $|-|$ gives a notion of distance on $\mathbb{C}$. In fact, it provides a *metric* on $\mathbb{C}$: we define the distance between two points $z, w \in \mathbb{C}$ to be

$$|z - w|,$$

and this satisfies (i) $|z - w| \geq 0$ with equality if and only if $z = w$, (ii) the triangle inequality $|z - w| \leq |z - y| + |y - w|$. Some useful additional properties are

$$|\mathrm{Re}(z)| \leq |z|, \quad |\mathrm{Im}(z)| \leq |z|, \quad \text{and} \quad ||z| - |w|| \leq |z - w|.$$

The latter follows from the triangle inequality, as we have $|z| = |(z - w) - (-w)| \leq |z - w| + |w|$ and similarly $|w| = |(z - w) - z| \leq |z - w| + |z|$.

Remark 2.1.1. Since the distance on $\mathbb{C}$ defined by absolute value coincides with the Euclidean distance upon identifying it with $\mathbb{R}^2$, the properties of $\mathbb{C}$ and $\mathbb{R}^2$ as metric spaces coincide.

Convergence

Convergence of sequences is defined as for the reals:

Definition 2.1.2. A sequence $\{z_n\}_{n\in\mathbb{N}}$ of complex numbers *converges* to $z \in \mathbb{C}$ when $|z_n - z| \to 0$ as $n \to \infty$.

Recall this means that for all $\epsilon > 0$ there is an $N \in \mathbb{N}$ such that $|z_n - z| < \epsilon$ for all $n > N$. We say that z is the *limit* of the sequence $\{z_n\}_{n\in\mathbb{N}}$, which is unique. Using the inequalities for the absolute given above, one easily deduces that

Lemma 2.1.3. *A sequence $\{z_n\}_{n\in\mathbb{N}}$ of complex numbers converges to z if and only if the sequences of real numbers $\{\mathrm{Re}(z_n)\}_{n\in\mathbb{N}}$, $\{\mathrm{Im}(z_n)\}_{n\in\mathbb{N}}$ converge to $\mathrm{Re}(z)$ and $\mathrm{Im}(z)$ respectively.*

Sometimes you don't know the limit z yet, but you still want to check a sequence converges. The notion of a Cauchy sequence takes care of that:

Definition 2.1.4. A sequence $\{z_n\}_{n\in\mathbb{N}}$ of complex numbers is a *Cauchy sequence* if $|z_n - z_m| \to 0$ as $n, m \to \infty$.

Recall this means that for all $\epsilon > 0$ there is an $N \in \mathbb{N}$ such that $|z_n - z_m| < \epsilon$ for all $n, m > N$. As in the previous lemma, $\{z_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence if and only if the sequences of real numbers $\{\mathrm{Re}(z_n)\}_{n\in\mathbb{N}}$, $\{\mathrm{Im}(z_n)\}_{n\in\mathbb{N}}$ are.

The real numbers are *complete*: every Cauchy sequence converges. By the previous observation, this implies the same for the complex numbers (this is the first instance of Remark 2.1.1):

Theorem 2.1.5. $\mathbb{C}$ *is complete, i.e. every Cauchy sequence converges.*

Open and closed subsets

The open disk of radius r around z_0 is given by

$$D_r(z_0) \coloneqq \{z \in \mathbb{C} \mid |z - z_0| < r\}.$$

This is the prototypical example of an open subset:

Definition 2.1.6. A subset $\Omega \subset \mathbb{C}$ is *open* if for each $z_0 \in \mathbb{C}$ there exists a $r > 0$ such that $D_r(z_0) \subset \Omega$.

Example 2.1.7. $\mathbb{C}$ and $\varnothing$ are both open, as is $\{z \in \mathbb{C} \mid \mathrm{Re}(z) > 0\}$. A good intution is that a subset defined by a strict inequality tends to be open.

Lemma 2.1.8. *Open subsets have the following properties:*

- *If $\{\Omega_i\}_{i \in I}$ is a collection of open subsets, then $\bigcup_i \Omega_i$ is open.*
- *If $\{\Omega_1, \ldots, \Omega_k\}$ is a finite collection of open subsets, then $\bigcap_{i=1}^k \Omega_i$ is open.*

Definition 2.1.9. A subset $\Omega \subset \mathbb{C}$ is *closed* if it is the complement of an open subset.

By taking complement in the above examples and lemma, we get:

Example 2.1.10. $\mathbb{C}$ and $\varnothing$ are both closed, as is $\{z \in \mathbb{C} \mid \mathrm{Re}(z) \leq 0\}$. A good intution is that a subset defined by a non-strict inequality tends to be closed.

Lemma 2.1.11. *Closed subsets have the following properties:*

- *If $\{\Omega_i\}_{i \in I}$ is a collection of closed subsets, then $\bigcap_i \Omega_i$ is xlosed.*
- *If $\{\Omega_1, \ldots, \Omega_k\}$ is a finite collection of closed subsets, then $\bigcup_{i=1}^k \Omega_i$ is closed.*

Closed subsets have a useful characterization in terms of convergent sequences, which maybe explains the terminology "closed": you can't create a larger subset by adding limit points.

Lemma 2.1.12. *A subset $\Omega \subset \mathbb{C}$ is closed if and only if each convergent sequence $\{z_n\}_{n \in \mathbb{N}}$ with $z_n \in \Omega$ has its limit in Ω.*

Compact subsets

Particularly nice subsets are those are closed and bounded, in the sense that the diameter

$$\mathrm{diam}(\Omega) \coloneqq \sup_{z,w \in \Omega} |z - w|$$

is bounded.

The Bolzano–Weierstrass theorem says a subset of $\mathbb{R}^2$ is closed and bounded if and only if it is compact, in the sense that every sequence has a convergent subsequence. By Remark 2.1.1, the same holds for $\mathbb{C}$:

Theorem 2.1.13. *A subset $\Omega \subset \mathbb{C}$ is closed and bounded if and only if every sequence $\{z_n\}_{n \in \mathbb{N}}$ has a convergent subsequence.*

Remark 2.1.14. There is an equivalent formulation of compactness in terms of open covers, see [SS03, Theorem 1.3]. This is the Heine–Borel theorem.

Next lecture we will use the following fact, which we prove as an exercise with the definitions:

Definition 2.1.15. Suppose that $\Omega_1 \supset \Omega_2 \supset \Omega_3 \supset \cdots$ is a collection of non-empty compact subsets whose diameter goes to 0. Then $\bigcap_{n=1}^{\infty} \Omega_n$ consists of a single point.

Proof. We first show the intersection is non-empty. Pick a point z_n in each Ω_n. Then $\{z_n\}_{n \in \mathbb{N}}$ is a Cauchy sequence because the diameters of the Ω_n go to 0: for $n, m > N$, $|z_n - z_m| \leq \operatorname{diam}(\Omega_N)$ and this goes to 0 as $N \to \infty$. Thus it has a limit $z \in \mathbb{C}$. Since $z_n \in \Omega_N$ for $n \geq N$ and Ω_N is closed, $z \in \Omega_N$. Because N was arbitrary here, $z \in \bigcap_{n=1}^{\infty} \Omega_n$.

We next show by contradiction there is only a single point in the intersection. Suppose that $z, z' \in \bigcap_{n=1}^{\infty} \Omega_n$ and $z \neq z'$. Then $|z - z'|$ is non-zero, and hence bigger than some $\epsilon > 0$. Let N be such that $\operatorname{diam}(\Omega_N) < \epsilon$. Then z and z' can't both be in Ω_N, which gives the contradiction. □

2.2 Holomorphic functions

Now that we have discussed more carefully some of the properties of the complex plane, we can revisit the definition of holomorphic functions.

2.2.1 Continuity

Having a metric on $\mathbb{C}$, we can define continuity for complex-valued functions defined on subsets of $\mathbb{C}$. It is the straight-forward generalization of continuity for real-valued functions defined on subsets of $\mathbb{R}$.

Definition 2.2.1. A function $f\colon \mathbb{C} \supset \Omega \to \mathbb{C}$ is *continuous at* $z \in \Omega$ if

$$\lim_{h \to 0} f(z+h) = f(z).$$

We say that f is *continuous* if it is continuous at all $z \in \Omega$.

In other words, for all $\epsilon > 0$ there exists a $\delta > 0$ such that $|h| < \delta$ implies $|f(z+h) - f(z)| < \epsilon$. This has a reformulation in terms of sequences: it is continuous at $z \in \mathbb{C}$ if and only if for all sequence $\{z_n\}_{n \in \mathbb{N}}$ in Ω converging to z we have $\lim_{n \to \infty} f(z_n) = f(z)$.

By Remark 2.1.1, a function $f\colon \mathbb{C} \supset \Omega \to \mathbb{C}$ is continuous if and only if the corresponding function $F\colon \mathbb{R}^2 \supset \Omega \to \mathbb{R}^2$ of two variables, given by

$$(x, y) \longmapsto (\operatorname{Re}(f(x+yi)), \operatorname{Im}(f(x+yi))),$$

is continuous. This makes clear that if f is continuous, so is the real-valued function $z \mapsto |f(z)|$. We can then use the fact that every continuous real-valued function attains a minimum and maximum on each compact subset of $\mathbb{R}^n$ to deduce that:

Proposition 2.2.2. *If $f\colon \mathbb{C} \supset \Omega \to \mathbb{C}$ is continuous, then $z \mapsto |f(z)|$ is bounded on each compact subset $\Omega' \subset \Omega$ and moreover attains a minimum and maximum on Ω'.*

2.2.2 Complex differentiation

Like we generalized continuity from $\mathbb{R}$ to $\mathbb{C}$, we generalize differentiability from $\mathbb{R}$ to $\mathbb{C}$.

Definition 2.2.3. A function $f\colon \mathbb{C} \supset \Omega \to \mathbb{C}$ is *complex-differentiable at* $z \in \Omega$ if there is an $a \in \mathbb{C}$ such that

$$\lim_{h \to 0} \frac{f(z+h) - f(z)}{h} = a.$$

We then say that a is the derivative of f at z, and set $f'(z) := a$.

If Ω is open, we say that f is *holomorphic* if it is complex-differentiable at all $z \in \Omega$. If Ω is not open, we say that f is *holomorphic* if it extends to a holomorphic function on a open subset Ω' containing Ω.

In other words, for all $\epsilon > 0$ there exists a $\delta > 0$ such that $0 < |h| < \delta$ implies $|f(z+h) - f(z) - ah|/|h| < \epsilon$. Observe that h is a complex number, and so approaches z from all directions.

Example 2.2.4. For $c \in \mathbb{C}$, the constant function f given by $z \mapsto c$ defined on all of $\mathbb{C}$ is holomorphic: if we take $a = 0$, then we have

$$|f(z+h) - f(z)|/|h| = 0 \quad \text{for all } h > 0.$$

The identity function g given by $z \mapsto z$ defined on all of $\mathbb{C}$ is also holomorphic: if we take $a = 1$, then we have

$$|g(z+h) - g(z) - h|/|h| = |z + h - z - h|/|h| = 0 \quad \text{for all } h > 0.$$

Similarly, we can explicit check that each polynomial $a_n z^n + \cdots + a_1 z + a_0$ defines a holomorphic function $\mathbb{C} \to \mathbb{C}$ with derivative given by $n a_n z^{n-1} + \cdots + a_1$. This also follows from the previous two examples and the forthcoming Lemma 2.2.7.

Non-example 2.2.5. The function f given by $z \mapsto \bar{z}$ is *not* holomorphic. For example, moving over the real axis we get

$$\lim_{\mathbb{R} \ni a \to 0} \frac{f(a) - f(0)}{a} = \lim_{\mathbb{R} \ni a \to 0} \frac{a}{a} = 1,$$

but moving over the imaginary axis we get

$$\lim_{\mathbb{R} \ni b \to 0} \frac{f(bi) - f(0)}{bi} = \lim_{\mathbb{R} \ni b \to 0} \frac{-bi}{bi} = -1.$$

A helpful reformulation says that f is linear up to an error term:

Lemma 2.2.6. *f is complex-differentiable at z with derivative $f'(z)$ if and only if*

$$f(z+h) = f(z) + f'(z)h + \phi(z+h)h$$

and $|\phi(z+h)| \to 0$ as $h \to 0$

Proof. The direction $\Leftarrow$ is easy, as the difference quotient is just

$$\frac{|f(z+h)-f(z)-f'(z)h|}{|h|}=\frac{|\phi(z+h)h|}{|h|}=|\phi(z+h)|,$$

which indeed goes to 0 as $h\to 0$.

For the converse, we define $\phi(z+h)$ by

$$\phi(z+h)=\begin{cases}\frac{f(z+h)-f(z)-f'(z)h}{h} & \text{if } h\neq 0,\\ 0 & \text{if } h=0.\end{cases}$$

That its absolute value goes to 0 as $h\to 0$ is exactly saying that f is complex-differentiable at z with derivative $f'(z)$. □

The complex derivative has the same properties as the real derivative:

Lemma 2.2.7. *We have that:*

- *If f is complex-differentiable at z, then f is continuous at z.*
- *If f,g are holomorphic on Ω, so is $f+g$. Furthermore $(f+g)'=f'+g'$.*
- *If f,g are holomorphic on Ω, so is fg. Furthermore $(fg)'=f'g+fg'$.*
- *If $f\colon\Omega\to U\subset\mathbb{C}$ and $g\colon U\to\mathbb{C}$ are holomorphic, then $g\circ f$ is holomorphic on Ω. Furthermore*
$$(g\circ f)'(z)=g'(f(z))f'(z).$$

2.2.3 The Cauchy–Riemann equations

Even though $f\colon\mathbb{C}\supset\Omega\to\mathbb{C}$ is continuous if and only if the corresponding function $F\colon\mathbb{R}^2\supset\Omega\to\mathbb{R}^2$, Non-example 2.2.5 shows that it is *not* the case that f is holomorphic if and only if F is a differentiable function of two variables. Indeed, for $z\mapsto\bar{z}$ the corresponding function of two variables is $(x,y)\mapsto(x,-y)$ and this is certainly a differentiable function of two variables with total derivative everywhere given by

$$\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}.$$

So what is the relationship between holomorphic functions and differentiable functions of two variables? We mentioned one such relationship during last lecture: for F to be differentiable at $\vec{z}=(a,b)$ there needs to exists a (2×2)-matrix A such that

$$\lim_{\vec{h}\to\vec{0}}\frac{||(F(\vec{z}+\vec{h})-F(\vec{z})-A\vec{h}||}{||\vec{h}||}=0.$$

If F comes from a complex-valued function which is complex-differentiable at the point $a+bi$, then A needs be a product of a scaling and a rotation. These are given by

$$\begin{bmatrix}\lambda & 0\\ 0 & \lambda\end{bmatrix}\qquad\begin{bmatrix}\cos(\theta) & \sin(\theta)\\ -\sin(\theta) & \cos(\theta),\end{bmatrix}$$

and thus satisfies $a = d$ and $b = -c$ in terms of its entries

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

In other words, if we write

$$F(x, y) = (u(x, y), v(x, y)),$$

we get the *Cauchy–Riemann equations*

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

Thus we have shown one direction of the following result:

Theorem 2.2.8. *A differentiable function $F\colon \mathbb{R}^2 \supset \Omega \to \mathbb{R}^2$ comes from a holomorphic function $f\colon \mathbb{C} \supset \Omega \to \mathbb{C}$ if and only if F satisfies the Cauchy–Riemann equations.*

I will not prove the other direction here, instead look at pages 10–13 of [SS03].

Chapter 3

Integration and Goursat's theorem

Last time we defined holomorphic function. Today we study their integrals along piecewise-smooth curves. In particular, we will establish Goursat's theorem on the integral of a holomorphic function along a triangle. These are [SS03, Section 1.3, 2.1].

3.1 Integration along curves

Given a holomorphic function $f\colon \mathbb{C} \supset \Omega \to \mathbb{C}$, we can attempt to define its integral along curves in Ω. We first need a definition of "curve" suitable to our purposes:

Definition 3.1.1. A *parametrized curve* is a function $\gamma\colon [a, b] \to \Omega$. It is *smooth* if $\gamma\colon [a, b] \to \Omega$ is differentiable with continuous derivative. It is *piecewise-smooth* if it is continuous and there exist $a = a_0 < a_1 < \cdots < a_n = b$ such that $z(t)$ is smooth on the intervals $[a_i, a_{i+1}]$,

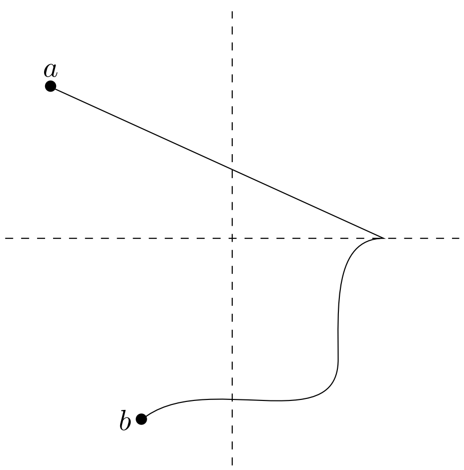


Figure 3.1 A piecewise-smooth curve in $\mathbb{C}$.

We want to allow different parametrizations, i.e. consider two piecewise-smooth parametrized curves

$$\gamma\colon [a,b] \to \Omega, \quad \tilde{\gamma}\colon [c,d] \to \Omega$$

as giving rise to the same piecewise-smooth curve if there exists a piecewise-smooth increasing homeomorphism $\tau\colon [a,b] \to [c,d]$ such that $\tilde{\gamma} = \gamma \circ \tau$.

Remark 3.1.2. We want it be increasing, instead of just any piecewise-smooth homeomorphism, because we want to preserve the orientation of the interval. We will soon see that if one uses a reparametrization that changes the orientation, the integral changes by a sign. This is easily verified by hand, but the more subtle reason is given in Math 132: if you reverse the orientation on the interval, the induced orientation on the boundary is also reversed, and hence you ought to reverse the limits a and b in order to get the same answer out.

Example 3.1.3. The line segment

$$\begin{aligned} \gamma\colon [0,1] &\longrightarrow \mathbb{C} \\ t &\longmapsto ta, \end{aligned}$$

is a smooth parametrized curve from 0 to a. We could have run through the same line segment at twice the speed, obtaining a different parametrization

$$\begin{aligned} \tilde{\gamma}\colon [0,1/2] &\longrightarrow \mathbb{C} \\ t &\longmapsto 2ta. \end{aligned}$$

Definition 3.1.4. If $\gamma\colon [a,b] \to \Omega$ is a smooth parametrized curve and $f\colon \Omega \to \mathbb{C}$ is continuous, we define

$$\int_\gamma f(z)dz := \int_a^b f(\gamma(t))\gamma'(t)dt,$$

where the integral of a complex-valued function is given by integrating the real and imaginary parts separately.

If $\gamma\colon [a,b] \to \Omega$ is instead piecewise-smooth, we define this as

$$\int_\gamma f(z)dz := \sum_{i=1}^{n} \int_{a_{i-1}}^{a_i} f(\gamma(t))\gamma'(t)dt.$$

The term $\gamma'(t)$ should remind you of the Jacobian in multivariable calculus, and serves to make the value of the integral independent of the choice of parametrization. Indeed, the change-of-variables theorem and chain rule imply that if $\tilde{\gamma} = \gamma \circ \tau$ then we have (we are assuming everything is smooth for the sake convenience, to work with piecewise-smooth functions just insert some sums)

$$\begin{aligned} \int_a^b f(\gamma(t))\gamma'(t)dt &= \int_c^d f(\gamma(\tau(s)))\gamma(\tau(s))\tau'(s)ds \\ &= \int_c^d f(\gamma(\tau(s)))(\gamma \circ \tau)'(s)ds \\ &= \int_c^d f(\tilde{\gamma}(s))\tilde{\gamma}'(s)ds. \end{aligned}$$

Example 3.1.5. In particular, the following is well-defined:

$$\text{length}(\gamma) = \int_\gamma |\gamma'(t)| dt,$$

intuitively just the *length* of γ in $\mathbb{R}^2$.

Now that we've shown it to be well-defined, it makes sense to state:

Lemma 3.1.6. *Integration has the following properties:*

(i) It is linear, i.e. $\int_\gamma af(z) + bg(z)dz = a\int_\gamma f(z)dz + b\int_\gamma g(z)dz$.

(ii) Reversing orietation gives a sign: if γ^- *is* γ *run backwards, then* $\int_{\gamma^-} f(z)dz = -\int_\gamma f(z)dz$.

(iii) There is an inequality

$$\left|\int_\gamma f(z)dz\right| \le \sup_{t\in[a,b]} |f(\gamma(t))| \cdot \text{length}(\gamma).$$

Proof. Part (i) follows from the corresponding property of the integral of real-valued function, and (ii) follows because once we fix an parametrization, changing the orientation amounts to precomposing γ by $t \mapsto -t$ contributing a -1 to the derivative. For part we use the integral version of triangle inequality (obtained from the usual triangle inequality by some manipulation of Riemann sums)

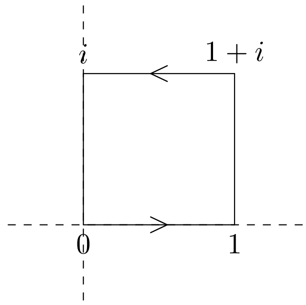
$$\begin{aligned}
\left|\int_a^b f(\gamma(t))\gamma'(t)dt\right| &\le \int_a^b |f(\gamma(t))||\gamma'(t)|dt \\
&\le \int_a^b \left(\sup_{t\in[a,b]} |f(\gamma(t))|\right) |\gamma'(t)|dt \\
&= \sup_{t\in[a,b]} |f(\gamma(t))| \int_a^b |\gamma'(t)|dt \\
&= \sup_{t\in[a,b]} |f(\gamma(t))| \cdot \text{length}(\gamma).
\end{aligned}$$

□

3.1.1 Examples of integrals

This setup makes everything sound more complicated than it actually is. Let us do some examples:

Example 3.1.7. Let us integrate e^z along the boundary of the unit square (positively oriented, i.e. ran through counter-clockwise)



It is the sum of four integrals, obtained by parametrizing each of the four line segments in the obvious manner:

$$\text{bottom edge:} \qquad \int_0^1 e^z dz = e^1 - e^0,$$

$$\text{right edge:} \qquad \int_0^1 e^{1+iz} i dz = e^{1+i} - e^1,$$

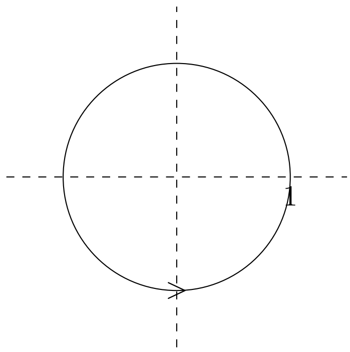
$$\text{top edge:} \qquad \int_0^1 e^{1+i-z}(-1) dz = e^i - e^{1+i},$$

$$\text{left edge:} \qquad \int_0^1 e^{i-iz}(-i) dz = e^0 - e^i.$$

Adding these up gives 0.

This is as expected: e^z is holomorphic on all of $\mathbb{C}$ and since the square can be contracted to a point in $\mathbb{C}$, one of the special properties of holomorphic functions discussed in the first lecture tells us that the answer should be the same as integration over the constant path. This vanishes because $\gamma'_{\text{const}}(t) = 0$ for all t.

Example 3.1.8. Let's continue with some easy integrals over the unit circle C_1 (positively oriented, i.e. ran through counter-clockwise):



We shall use the convenient parametrization of this circle in terms of polar representations of complex number as

$$\gamma(\theta) = e^{i\theta} \quad \theta \in [0, 2\pi].$$

Firstly, we have that

$$\int_{C_1} 1dz = \int_0^{2\pi} ie^{i\theta} d\theta = \left[e^{2\pi i} - e^0\right] = 0.$$

Secondly, we have that

$$\int_{C_1} zdz = \int_0^{2\pi} e^{i\theta} ie^{i\theta} d\theta = \int_0^{2\pi} ie^{2i\theta} d\theta = \frac{1}{2}\left[e^{4\pi i} - e^0\right] = 0.$$

I'll leave it to you to verify that the same happens for all powers z^n, $n \geq 0$: their integrals over C_1 vanish.

What the previous two examples have in common is the following: if f has a *primitive* F on Ω, i.e. $F'(z) = f(z)$, then $\int_\gamma f(z)dz = 0$ when γ is closed. This is because whenever f has a primitive and γ is a piecewise-smooth curve from a to b, we have

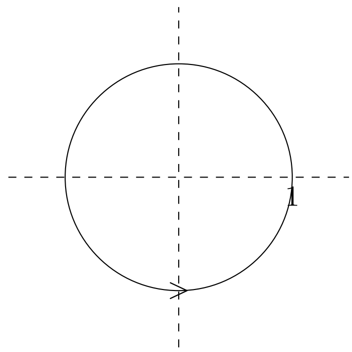
$$\begin{aligned}
\int_\gamma f(z)dz &= \int_{t_0}^{t_1} f(\gamma(t))\gamma'(t)dt \\
&= \int_{t_0}^{t_1} F'(\gamma(t))\gamma'(t)dt \\
&= \int_{t_0}^{t_1} (F \circ \gamma)'(t)dt \\
&= F(\gamma(t_1)) - F(\gamma(t_0)) \\
&= F(b) - F(a).
\end{aligned}$$

Here a just happens to equal b.

Example 3.1.9. Let us next consider *the most important integral in complex analysis.* It concerns the holomorphic function

$$\begin{aligned}
1/z\colon \mathbb{C} \setminus \{0\} &\longrightarrow \mathbb{C} \\
z &\longmapsto 1/z,
\end{aligned}$$

and is along the positively-oriented unit circle we saw before:



Parametrizing as before, we get:

$$\begin{aligned}\int_{C_1} \frac{1}{z} dz &= \int_0^{2\pi} \frac{1}{e^{i\theta}} i e^{i\theta} d\theta \\ &= \int_0^{2\pi} i d\theta \\ &= 2\pi i.\end{aligned}$$

This doesn't contradict the miraculous property discussed in the first lecture, which says that integrals along homotopic paths are equal: the circle can't be contracted within the domain Ω, which has a hole at the origin. Observe that this computation implies that f does not have a primitive on $\mathbb{C} \setminus \{0\}$!

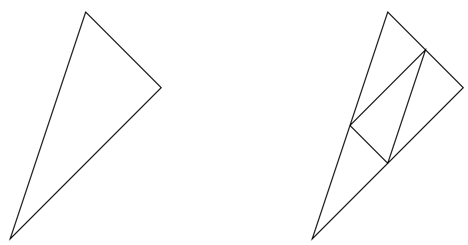
3.2 Goursat's theorem

3.2.1 Goursat's theorem for triangles

Theorem 3.2.1 (Goursat)**.** *Suppose that* $f : \mathbb{C} \supset \Omega \to \mathbb{C}$ *is holomorphic. Let* $T \subset \Omega \subset \mathbb{C}$ *be a solid triangle. Let* ∂T *be the piecewise-smooth curve obtained by positively orientation the boundary of* T *(i.e. moving over it in a counterclockwise fashion). Then we have that*

$$\int_{\partial T} f(z) dz = 0.$$

Proof. We can subdivide our triangle T into four congruent triangles, each half the size:



If we orient the boundaries of the four triangles $T_1^1, T_1^2, T_1^3, T_1^4$ positively as well, then the integrals over paired edges cancel out, and we get

$$\int_{\partial T} f(z) dz = \sum_{i=1}^{4} \int_{\partial T_1^i} f(z) dz.$$

This implies

$$\left| \int_{\partial T} f(z) dz \right| \leq \sum_{i=1}^{4} \left| \int_{\partial T_1^i} f(z) dz \right| \leq 4 \left| \int_{\partial T_1^{i_1}} f(z) dz \right|,$$

for some i_1 (of course the one whose integral has largest absolute value). If we perform the subdivision procedure n times, we get

$$\left| \int_{\partial T} f(z) dz \right| \leq 4^n \left| \int_{\partial T_n^{i_n}} f(z) dz \right| \tag{3.1}$$

for some triangle $T_n^{i_n}$ in the nth subdivision.

Observe that $T_1^{i_1} \supset T_2^{i_2} \supset T_3^{i_3} \supset \cdots$ is a nested collection of non-empty compact subsets, so its intersection contains a unique point z_0. Near z_0 we can write f as

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + \phi(z)(z - z_0),$$

with $|\phi(z - z_0)| \to 0$ as $z \to z_0$. Since integrals are linear

$$\int_{\partial T_n^{i_n}} f(z)dz = \int_{\partial T_n^{i_n}} f(z_0) + f'(z_0)(z - z_0)dz + \int_{\partial T_n^{i_n}} \phi(z)(z - z_0)dz.$$

The first of these vanishes by direct calculation, so we get that it's equal to

$$\int_{\partial T_n^{i_n}} \phi(z)(z - z_0)dz,$$

whose absolute value can be estimated as

$$\sup_{z \in \partial T_n^{i_n}} |\phi(z)(z - z_0)| \cdot \mathrm{length}(\partial T_n^{i_n}).$$

Now, $\sup_{z \in \partial T_n^{i_n}} |\phi(z)(z - z_0)| \leq \sup_{z \in \partial T_n^{i_n}} |\phi(z)| \cdot \sup_{z \in \partial T_n^{i_n}} |(z - z_0)|$, and the latter is $\leq \mathrm{diam}(\partial T_n^{i_n}) = 2^{-n}\mathrm{diam}(T)$. On the other hand, $\mathrm{length}(\partial T_n^{i_n}) = 2^{-n}\mathrm{length}(\partial T)$. We conclude that

$$\left| \int_{\partial T_n^{i_n}} \phi(z)(z - z_0)dz \right| \leq \sup_{z \in \partial T_n^{i_n}} |\phi(z)| \frac{\mathrm{diam}(T)\mathrm{length}(\partial T)}{4^n}.$$

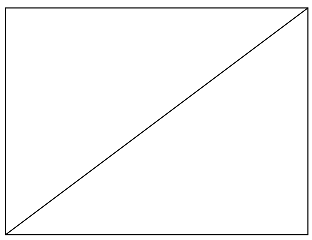
Combining this with (3.1), we get

$$\left| \int_{\partial T} f(z)dz \right| \leq \sup_{z \in \partial T_n^{i_n}} |\phi(z)| \mathrm{diam}(T)\mathrm{length}(\partial T),$$

and since $|\phi(z)| \to 0$ as $z \to z_0$, $\sup_{z \in \partial T_n^{i_n}} |\phi(z)| \to 0$ as $z \to z_0$ as well. □

Remark 3.2.2. Observe it was crucially important that f was defined on all of T, not just near ∂T, as the subdivided triangles have boundaries which eventually become dense in T.

A triangle is a slightly unusual shape, but other shapes are easily decomposed into them or approximated by them. For example, we can decompose a rectangle into two triangles:



If we oriented the boundaries of the triangles positively, the integrals along the diagonal cancel out and what remains is the integral over the positively-oriented boundary of the rectangle. Thus by applying Goursat's theorem to both triangles we get:

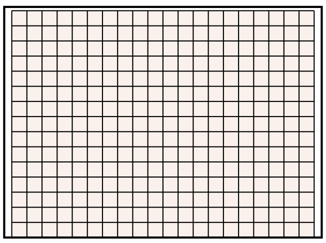
Corollary 3.2.3. *Suppose that* $f\colon \mathbb{C} \supset \Omega \to \mathbb{C}$ *is holomorphic. Let* $R \subset \Omega \subset \mathbb{C}$ *be a solid rectangle. Let* ∂R *be the piecewise-smooth curve obtained by positively orientation the boundary of* R. *Then we have that*

$$\int_{\partial R} f(z)dz = 0.$$

3.2.2 A down-to-earth argument in support of Goursat's theorem

Let us give a argument that, though not rigorous, gives some intuition for why Goursat's theorem for rectangles, here Corollary 3.2.3, follows from the Cauchy–Riemann equations. These equations we deduced by observing that multiplication by a complex numbers geometrically is given by a composition of scaling and rotation

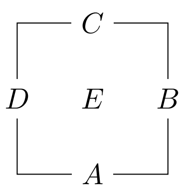
We start similarly to the prove of Goursat's theorem, with the observation that we can approximately fill out our rectangle with squares whose sides have some small length $\epsilon > 0$:



As before, If we integrate over the positively-oriented boundaries of each of the squares then all of the internal edges cancel out, and we get something that will approximate well the integral over the boundary of the rectangle as $\epsilon \to 0$:

$$\int_{\partial(\text{rectangle})} f(z)dz \approx \sum_{\text{squares}} \int_{\partial(\text{square})} f(z)dz.$$

The number of squares grows as $1/\epsilon^2$, so what we need to prove that is the integral over the boundary of each of the squares grows as a larger power of ϵ than ϵ^2 as $\epsilon \to 0$. For example, if it grew as ϵ^3, then our integral (which is independent of ϵ) would grow as $\epsilon^3/\epsilon^{=}\epsilon$ as $\epsilon \to 0$ and hence need to be 0. Make this estimate for the integral over the boundary of each of the squares, we label the midpoints of the edges and the center of the square as



and use that the integral over a square can be approximated well by:

$$f(A)\epsilon + f(B)i\epsilon - f(C)\epsilon - f(D)i\epsilon = \epsilon(f(A) - f(C)) + i\epsilon(f(B) - f(D)).$$

For general differentiable functions $f(x, y) = u(x, y) + iv(x, y))$, $f(A) - f(C)$ and $f(B) - f(D)$ are of order ϵ; they are approximately

$$-\frac{\partial u}{\partial y}(E) \cdot \epsilon - i\frac{\partial v}{\partial y}(E) \cdot \epsilon \quad \text{and} \quad \frac{\partial u}{\partial x}(E) \cdot \epsilon + i\frac{\partial v}{\partial x}(E) \cdot \epsilon$$

respectively. The results is that the integral is of order ϵ^2; combined with the $1/\epsilon^2$ squares we see that we expect a finite but non-zero value to result.

Now we finally use that f is holomorphic. This means that u and v satisfy the Cauchy–Riemann equations:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

Multiplying out our approximation to the integral we get

$$\begin{aligned} &-\frac{\partial u}{\partial y}(E) \cdot \epsilon^2 - i\frac{\partial v}{\partial y}(E)\epsilon^2 + i\frac{\partial u}{\partial x}(E) \cdot \epsilon^2 - \frac{\partial v}{\partial x}(E) \cdot \epsilon^2 \\ &= \left(-\frac{\partial u}{\partial y}(E) - \frac{\partial v}{\partial x}(E)\right)\epsilon^2 + i\left(-\frac{\partial v}{\partial y}(E)\epsilon^2 + \frac{\partial u}{\partial x}(E)\right)\epsilon^2, \end{aligned}$$

separating the real and imaginary parts. But the Cauchy–Riemann equations tell us these cancel!

Chapter 4

Cauchy's integral formula

Last time we proved that the integral of a holomorphic function over the boundary of a triangle or rectangle vanishes. Today, we use this to prove that holomorphic functions on disks (and several other shapes) have primitives, and use this to compute some integrals.

4.1 Holomorphic functions have primitives on disks

Let time we proved that if $f\colon \mathbb{C} \supset \Omega \to \mathbb{C}$ is a holomorphic function and $T, R \subset \Omega$ are a triangle or rectangle, then we have that

$$\int_{\partial T} f(z)dz = 0 \quad \text{and} \quad \int_{\partial R} f(z)dz = 0.$$

We will now use this to prove that if f is holomorphic on the open unit disk $D_1(0)$, then F has a primitive on $D_1(0)$. That is, there exists a holomorphic function $F\colon \mathbb{C} \supset D_1(0) \to \mathbb{C}$ such that $F'(z) = f(z)$.

Our strategy is to mimic the fundamental theorem of analysis for real-valued functions of a single variable: a primitive of a continuous f is given by $F(x) = \int_0^x f(x)dx$. In our case there is a similar prefered linear path from the origin to $z \in D_1(0)$, and Goursat's theorem will be used to show that integrating along this gives a primitive.

Theorem 4.1.1. *If f is holomorphic on $D_1(0)$, then there exists a holomorphic function $F\colon \mathbb{C} \supset D_1(0) \to \mathbb{C}$ such that $F'(z) = f(z)$.*

Proof. For $z \in D_1(0)$, let γ_z be the smooth curve from 0 to z given by

$$\begin{aligned} \gamma_z\colon [0,1] &\longrightarrow D_1(0) \\ t &\longmapsto tz. \end{aligned}$$

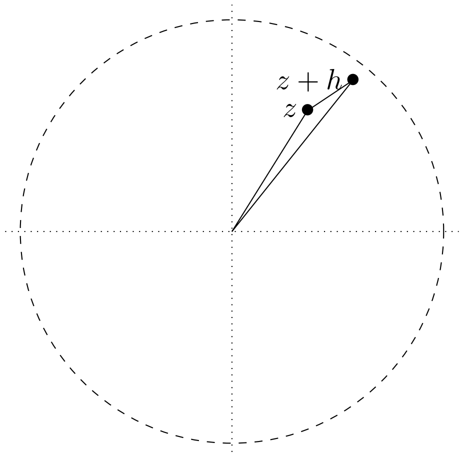
Then we define $F\colon D_1(0) \to \mathbb{C}$ by

$$F(z) := \int_{\gamma_z} f(w)dw.$$

What we need to show is that F is holomorphic and that $F'(z) = f(z)$. Do so, we study the differrence quotient

$$F(z+h) - F(z) = \int_{\gamma_{z+h}} f(w)dw - \int_{\gamma_z} f(w)dw.$$

From the picture



we see that there is a triangle with side γ_z and γ_{z+h} with opposite orientations, as well as new side $\gamma_{z+h,z}$ given the line segment from z to $z+h$. By Goursat's theorem the integral over its boundary vanishes, so we get that

$$\left|\int_{\gamma_{z+h}} f(w)dw - \int_{\gamma_z} f(w)dw\right| = \left|\int_{\gamma_{z+h,z}} f(w)dw.\right|$$

We know that we can write $f(z+u) = f(z)+f'(z)u+\psi(z+u)u$ with $\lim_{u\to 0}|\psi(z+u)| = 0$. Thus we can make the following estimate (note that the changes of variables from z to u means we replace $\gamma_{z+h,z}$ by γ_h)

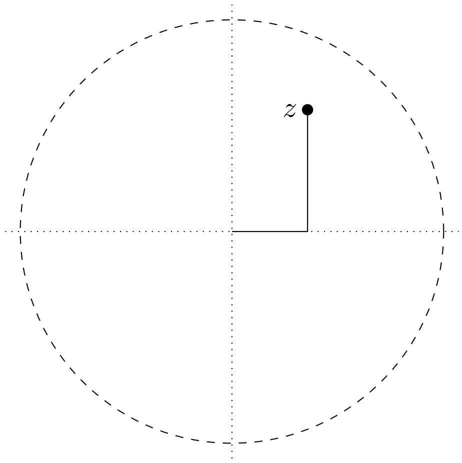
$$\begin{aligned}\left|\int_{\gamma_h} f(w)dw - f(z)h\right| &\leq \left|\int_{\gamma_h} f'(z)wdw\right| + \left|\int_{\gamma_h} \psi(z+w)wdw\right| \\ &\leq \frac{1}{2}|f'(z)||h|^2 + \sup_{w\in\gamma_h}|\psi(z+w)w| \cdot \mathrm{length}(\gamma_h) \\ &= |h|\left(\frac{1}{2}|f'(z)||h| + \sup_{w\in\gamma_h}|\psi(z+w)w|\right)\end{aligned}$$

Since $\lim_{u\to 0}|\psi(z+u)| = 0$, it is bounded. This means we can estimate $\sup_{w\in\gamma_h}|\psi(z+w)w|$ as being $\leq C|h|$ as $h\to 0$. We conclude that

$$\lim_{h\to 0}\frac{\left|\int_{\gamma_{z+h,z}} f(w)dw - f(z)h\right|}{|h|} \leq \lim_{h\to 0}\left(\frac{1}{2}|f'(z)||h| + C|h|\right) = 0.$$

We conclude that $F(z)$ is holomorphic with derivative $f(z)$. $\square$

Remark 4.1.2. There is another reasonable choice of piecewise-smooth path from 0 to z:



By Goursat's theorem, if we define F using this path instead, we get the same function.

Since holomorphic functions with a primitive have the property that integrals over closed curves vanish, we conclude that:

Corollary 4.1.3 (Cauchy's theorem)**.** *Suppose $C \subset \Omega$ is a circle whose interior is also contained in Ω and $f\colon \mathbb{C} \supset \Omega \to \mathbb{C}$ is holomorphic, then*

$$\int_C f(z)dz = 0.$$

Proof. Since Ω is open, there is a slightly large disk $D_r(z_0)$ containing C. By scaling and translation, we may assume that $D_r(z_0) = D_1(0)$. Now apply the previous theorem. $\square$

This argument is not specific to disks. In essence, we only use one property of Ω: in each path component of Ω there is a piecewise-linear path from some basepoint z_0 to z, which is unique up to replacing one or two edges of a triangle by the remaining two or one edges. This is not a great definition, and we will give a much better one later, but it clearly applies to a collection of useful shapes (see p. 42 of [SS03]):

- sectors,
- semi-circles,
- indented semi-circles,
- "keyholes",
- squares,
- rectangles,
- parallelograms,
- regular polygons.

As a consequence, if such a shape is contained in Ω the integral over its boundary vanishes.

4.2 Computing some integrals

We can use this to compute some interesting integrals:

Example 4.2.1. We will prove that for $\xi \in \mathbb{R}$

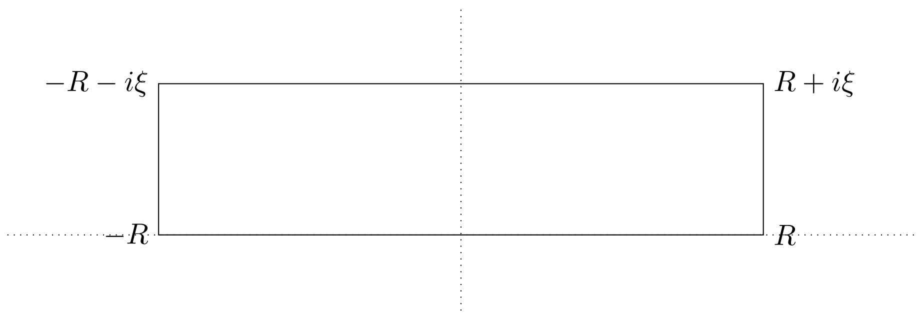
$$e^{-\pi\xi^2} = \int_{-\infty}^{\infty} e^{-\pi x^2 - 2\pi i x\xi} dx.$$

In other words, we prove that $e^{-\pi x^2}$ is its own Fourier transform (up to some constants, depending on your definition of the Fourier transform). For $\xi = 0$, this gives the famous formula

$$1 = \int_{-\infty}^{\infty} e^{-\pi x^2} dx.$$

We will *assume* this is already known.

For the proof we integrate $f(z) = e^{-\pi z^2}$ over the positively-oriented boundary of the rectangle with corners $R, R + i\xi, -R - i\xi, -R$:



Our intention is to eventually let $R \to \infty$.

Since f is holomorphic, by the version of Cauchy's theorem for rectangles this integral vanishes. The integral over the right sides is given by

$$\int_0^{\xi} e^{-\pi R^2 + -2\pi i R y + \pi y^2} i dy,$$

and thus its absolute value can be estimated as $\leq Ce^{-\pi R^2}$ with C independent of R. Thus by letting $R \to \infty$ we can make this arbitrarily small. Thus we have that

$$0 = \int_{-\infty}^{\infty} e^{-\pi x^2} dx - \int_{-\infty}^{\infty} e^{-\pi x^2 - 2\pi i x\xi + \pi\xi^2} dx.$$

Using that we know the first integral, this gives

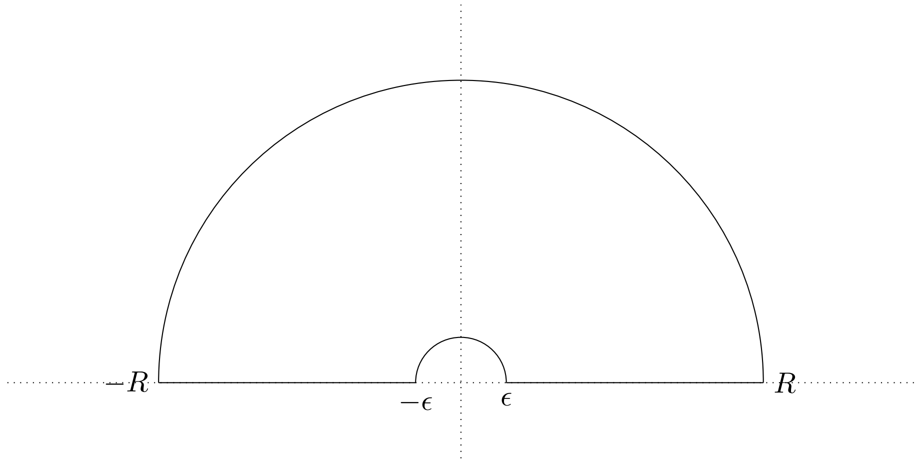
$$e^{\pi\xi^2} \int_{-\infty}^{\infty} e^{-\pi x^2 - 2\pi i x\xi} dx = 1,$$

and the desired formula follows.

Example 4.2.2. We will prove that

$$\int_0^\infty \frac{\sin(x)}{x} dx = \frac{\pi}{2}.$$

Here we will integrate the holomorphic function $f(z) = \frac{e^{iz}}{z}$ over the positively-oriented boundary of the following indented semi-circle:



This vanishes by the version of Cauchy's theorem for indented semicircles. The point is now that the integrals over the two line segments can be combined as the integral over the single line segment $[\epsilon, R]$ of the function

$$\frac{e^{ix}}{x} - \frac{e^{-ix}}{x} = 2i\frac{\sin(x)}{x}.$$

The integral over the large semicircle can be estimated by observing $|e^{iz}| = e^{-\mathrm{Im}(z)}$ so that $|(e^{iz})/z| \leq (e^{-\mathrm{Im}(z)})/|z|$, and hence we get

$$\left| \int_{\text{large semi-circle}} \frac{e^{iz}}{z} dz \right| \leq \int_0^\pi e^{-R\sin(x)} dx.$$

For $\epsilon > 0$ there is a $S > 0$ such that if $R > S$, then $e^{-R\sin(x)} < \epsilon$ except on the intervals $[0, \epsilon]$ and $[1-\epsilon, 1]$. Since $e^{-R\sin(x)} \leq 1$, we can thus estimate under these conditions the integral by 3ϵ. In particular, it vanishes as $R \to \infty$.

We can write $e^{iz}/z = 1/z + E(z)$ where $E(z)$ is bounded near 0, using the Taylor expansion $e^z = \sum_{n=0}^\infty z^n/n!$. Thus the integral over the little semi-circle can be written as

$$\int_{\text{small semi-circle}} 1/z + E(z) dz = \int_{\text{small semi-circle}} \frac{1}{z} dz + \int_{\text{small semi-circle}} E(z) dz.$$

Since $E(z)$ is bounded, the absolute value of the right integral can be estimate as $\leq C\pi\epsilon$, so goes to 0 as $\epsilon \to 0$. On the other hand, the right integral can be computed as

$$\int_{\text{small semi-circle}} \frac{1}{z} dz = -\int_0^\pi \frac{1}{e^{i\theta}} ie^{i\theta} d\theta = -\pi i.$$

Letting $\epsilon \to 0$ and $R \to \infty$, we thus get that

$$-\pi i + \int_0^\infty 2i \frac{\sin(x)}{x} = 0,$$

and the result follows.

Example 4.2.3. You should read Example 2 on p. 44 of [SS03] to see a proof of

$$\int_0^\infty \frac{1-\cos(x)}{x^2} dx = \frac{\pi}{2}.$$

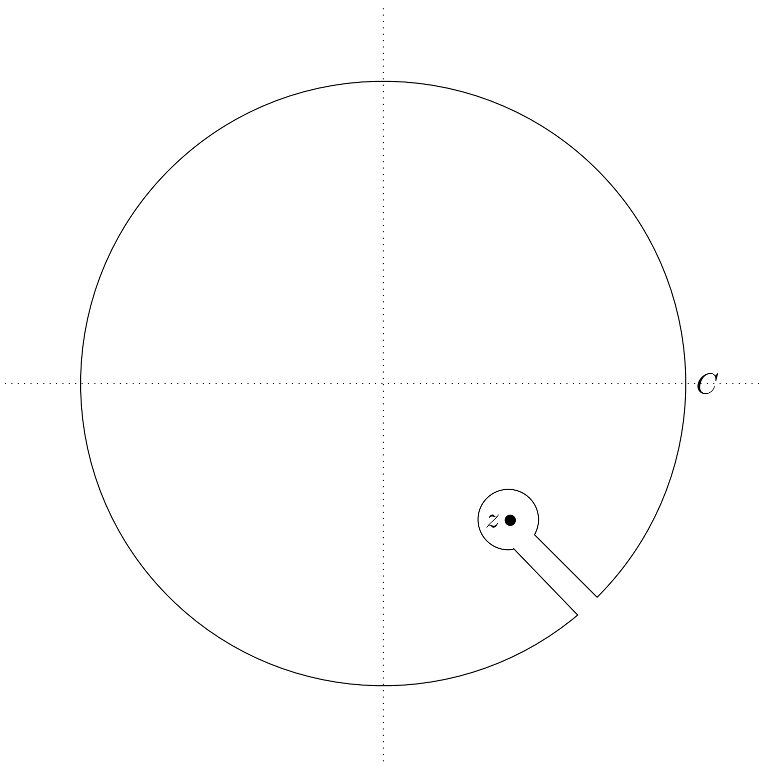
4.3 Cauchy integral formula

We will give one of the most important formula's in complex analysis, from which all the miraculous facts mentioned in the first lecture are deduced:

Theorem 4.3.1 (Cauchy's integral formula)**.** *Suppose that $f\colon \mathbb{C} \supset \Omega \to \mathbb{C}$ is holomorphic and $C \subset \Omega$ is a circle whose interior is also contained in Ω. Then for any z in the interior of C, we have that*

$$f(z) = \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{\zeta - z} d\zeta.$$

Proof. Without loss of generality C is the unit circle. We will integrate along the following path:



Since $g(\zeta) = \frac{f(\zeta)}{\zeta - z}$ is holomorphic on a neighborhood of the interior of this path, the integral over it vanishes. Letting the little corrider close up, the integrals over both sides cancel out, and we get

$$0 = \int_{\text{big circle}} g(\zeta)d\zeta - \int_{\text{little circle}} g(\zeta)d\zeta = \int_C \frac{f(\zeta)}{\zeta - z}d\zeta - \int_{\text{little circle}} g(\zeta)d\zeta,$$

where the sign comes from the fact that in our path the little circle appears negatively-oriented. So it suffices to prove that

$$\int_{\text{little circle}} g(\zeta)d\zeta = 2\pi i f(z).$$

This follows by writing g near z as

$$g(w) = \frac{f(\zeta)}{\zeta - z} = \frac{f(\zeta) - f(z)}{\zeta - z} + \frac{f(z)}{\zeta - z}.$$

Since f is holomorphic, the first one is bounded when ζ is near z (as it converges to $f'(z)$). Thus as the radius of the little circle goes to 0, the integral goes to

$$\int_{\text{little circle}} \frac{f(z)}{\zeta - z}d\zeta$$

and we can explicitly evaluate this as

$$\int_0^{2\pi} \frac{f(z)}{e^{i\theta}} ie^{i\theta} d\theta = 2\pi i f(z). \qquad \square$$

Observe that what we are integrating is a function of z which is infinitely many differentiable. This leads to the following corollary, the only subtlety being that we need to move the derivative inside the integral.

Corollary 4.3.2. *If $f\colon \mathbb{C} \supset \Omega \to \mathbb{C}$ is holomorphic, then it is infinitely many times complex differentable. Moreover, if $C \subset \Omega$ is a circle whose interior is also contained in Ω and contains z, then*

$$f^{(n)}(z) = \frac{n!}{2\pi i}\int_C \frac{f(\zeta)}{(\zeta - z)^{n+1}}dz$$

Proof. The proof is by induction on n, the case $n = 0$ being Cauchy's theorem. We start by using the induction hypothesis to write the difference quotient as

$$\frac{f^{(n-1)}(z+h) - f^{(n-1)}(z)}{h} = \frac{(n-1)}{2\pi i}\int_C f(\zeta)\frac{1}{h}\left[\frac{1}{(\zeta - z - h)} - \frac{1}{(\zeta - z)^n}\right]dz.$$

Writing $A = 1/(\zeta - z - h)$ and $B = 1/(\zeta - z)$, we see that we need to understand $A^n - B^n$. This can be written as $(A - B)(A^{n-1} + A^{n-2}B + \cdots + B^{n-1})$, with $A - B = h/((\zeta - z - h)(\zeta - z))$, so we get

$$\frac{(n-1)}{2\pi i}\int_C f(\zeta)\frac{1}{h}\frac{h}{(\zeta - z - h)(\zeta - z)}\left[A^{n-1} + A^{n-2}B + \cdots + B^{n-1}\right]dz.$$

As $h \to z$ the integrand converges absolutely uniformly, and thus we can take the limit inside. Moreover, both A and B converge to $1/(\zeta - z)$. The conclusion is that the different quotient converges as $h \to 0$, and that it converges to the following value:

$$\frac{(n-1)}{2\pi i}\int_C f(\zeta)\frac{1}{h}\frac{h}{(\zeta-z)(\zeta-z)}\frac{n}{(\zeta-z)^{n-1}}dz = \frac{n!}{2\pi i}\int_C \frac{f(\zeta)}{(\zeta-z)^{n+1}}dz.$$

This completes the induction step. □

This implies that we can estimate $f(z)$ and the derivatives $f^{(n)}(z)$ in terms of the values of f on a circle:

Corollary 4.3.3 (Cauchy's inequalities)**.** *Given f, C of radius R, and z as above, we have that*

$$|f^{(n)}(z)| \leq \frac{n!}{R^n}\sup_{\zeta \in C}|f(\zeta)|.$$

Proof. We use the estimate $|\int_\gamma g(z)dz| \leq \text{length}(\gamma)\cdot\sup_{\zeta\in\gamma}|g(\zeta)|$. Here $\text{length}(C_1) = 2\pi R$ and

$$\sup_{\zeta\in C}\left|\frac{f(\zeta)}{(\zeta-z)^{n+1}}\right| = \frac{1}{R^{n+1}}\sup_{\zeta\in\gamma}|g(\zeta)|.$$ □

Chapter 5

Applications of Cauchy's integral formula

Last time we proved Cauchy's integral formula. Today we deduce several consequences from this, which requires a discussion of power series.

5.1 Cauchy integral formula

Let us start with some recollections:

Theorem 5.1.1 (Cauchy's integral formula). *Suppose that $f\colon \mathbb{C} \supset \Omega \to \mathbb{C}$ is holomorphic and $C \subset \Omega$ is a circle whose interior is also contained in Ω. Then for any z in the interior of C, we have that*

$$f(z) = \frac{1}{2\pi i}\int_C \frac{f(\zeta)}{\zeta - z}d\zeta.$$

Observe that what we are integrating is a function of z which is infinitely many times differentiable, which implied:

Corollary 5.1.2. *If $f\colon \mathbb{C} \supset \Omega \to \mathbb{C}$ is holomorphic, then it is infinitely many times complex differentable. Moreover, if $C \subset \Omega$ is a circle whose interior is also contained in Ω and contains z, then*

$$f^{(n)}(z) = \frac{n!}{2\pi i}\int_C \frac{f(\zeta)}{(\zeta - z)^{n+1}}dz$$

This implies that we can estimate $f(z)$ and the derivatives $f^{(n)}(z)$ in terms of the values of f on a circle:

Corollary 5.1.3 (Cauchy's inequalities). *Given f, C of radius R, and z as above, then if z is the center of C we have that*

$$|f^{(n)}(z)| \leq \frac{n!}{R^n}\mathrm{sup}_{\zeta \in C}|f(\zeta)|.$$

5.1.1 Liouville's theorem and the fundamental theorem of algebra

Let us give a first application of this:

Theorem 5.1.4 (Liouville)**.** *If $f\colon \mathbb{C} \to \mathbb{C}$ is analytic and bounded, then it is constant.*

Proof. Suppose the bound is $|f(\zeta)| \leq C$. Then the Cauchy inequality for a circle of radius R centered as z and $n = 1$ gives

$$|f'(z)| \leq \frac{B}{R}.$$

Letting $R \to \infty$ we see that $f'(z) = 0$. Since $\mathbb{C}$ is connected, this implies that f is constant. ☐

We now prove something promised in the first lecture: $\mathbb{C}$ is algebraically closed.

Corollary 5.1.5 (Fundamental theorem of algebra)**.** *If $p(z) = a_n z^n + \cdots + a_1 z + a_0$ is not constant, then it has a root in $\mathbb{C}$.*

Proof. If p is not constant we may assume $a_n \neq 0$. Then we can estimate

$$|p(z)| \geq |a_n||z|^n - (|a_{n-1}||z|^{n-1} + \cdots + |a_0|),$$

which goes to ∞ as $|z| \to \infty$. Thus if p has no root, it stays a bounded distance from 0 and hence $1/p(z)$ is a bounded holomorphic function. By Liouville's theorem it is constant and hence so is p. This is of course not true, by the estimate we just gave. ☐

Corollary 5.1.6. *Every polynomial $p(z) = a_n z^n + \cdots + a_1 z + a_0$ with $a_n \neq 0$ can be written as*

$$a_n(z - w_1) \cdots (z - w_n)$$

for some $w_i \in \mathbb{C}$.

Proof sketch. This is an induction over n, the case $n = 0$ being obvious. By the fundamental theorem of algebra we can find a root w_1 of $p(z)$. This implies that $p(z) = (z - w_1)q(z)$ for q of degree $n - 1$, and we can apply the induction hypothesis. For more details, see [SS03, Corollary 4.7]. ☐

5.2 Holomorphic functions are analytic

5.2.1 Power series

We would like to have more examples of holomorphic functions, generalizing exponentials and trigoniometric functions. These are defined using power series, the infinite sum variation of polynomials.

For example, the exponential function is defined by

$$e^z := \sum_{n=0}^{\infty} \frac{z^n}{n!}.$$

This sum makes sense, and in fact converges absolutely: $|z^n/n!| = |z|^n/n!$, and from the usual real-valued exponential we know that $\sum_{n=0}^{\infty} \frac{|z|^n}{n!} = e^{|z|} < \infty$. Thus the above formula defines a function $\mathbb{C} \to \mathbb{C}$, and we will soon see it is holomorphic with derivative computed term by term. The result is that it's equal to its own deritative:

$$(e^z)' = \sum_{n=0}^{\infty} n\frac{z^{n-1}}{n!} = \sum_{n=1}^{\infty} \frac{z^{n-1}}{(n-1)!} = e^z.$$

Power series don't need to converge everywhere. The classical example is the geometric series

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n,$$

which converges absolutely for $|z| < 1$. In general, there is a critical radius inside of which a power series converges and outside of which it diverges (on the circle itself the behavior is very complicated).

Theorem 5.2.1. *Let*

$$\sum_{n=0}^{\infty} a_n z^n$$

be a power series, and define $R \in [0, \infty]$ *by*[1]

$$1/R = \limsup_{n\to\infty} |a_n|^{1/n} = \lim_{n\to\infty} \left(\sup\{|a_m|^{1/m} \mid m \geq n\}\right).$$

Then the power series converges absolutely for z *with* $|z| < R$ *and diverges for some* z *with* $|z| > R$*.*

The number R is called the *radius of convergence.*

Example 5.2.2. For $\frac{1}{1-z}$, the radius of convergence is given by $1/R = \limsup_{n\to\infty} 1^{1/n} = 1$, as expected.

Proof of Theorem 5.2.1. We will only prove the statement about convergence, which is the only one we shall be using. We shall also assume that $R \neq 0, \infty$, and set $L = 1/R$.

For $|z|$ satisfying $|z| < R$, choose $\epsilon > 0$ such that $(L + \epsilon)|z| = r < 1$. By definition $|a_n|^{1/n} < L + \epsilon$ for n sufficiently large, so

$$|a_n||z|^n \leq (L + \epsilon)^n |z|^n < r^n.$$

By comparison to the geometric series $\sum_{n=0}^{\infty} r^n$, the series $\sum_{n=0}^{\infty} a_n z^n$ converges absolutely. $\square$

Observe that the candidate derivative of $\sum_{n=0}^{\infty} a_n z^n$ is $\sum_{n=0}^{\infty} n a_n z^{n-1}$, and its radius of convergence is the same. A great trick is that the radius of convergence is not affected by multiplication with z, which makes na_n the coefficient of z^n instead of $(n+1)a_{n+1}$, and so it suffices to prove that:

Lemma 5.2.3. *We have that* $\limsup_{n\to\infty} |na_n|^{1/n} = \limsup_{n\to\infty} |a_n|^{1/n}$.

[1] If the right hand side is 0, take $R = \infty$. If the right hand side is ∞, take $R = 0$.

Proof. We claim that if $\lim_{n\to\infty} b_n = b > 0$ and $\limsup_{n\to\infty} c_n = c$, then

$$\limsup_{n\to\infty} b_n c_n = bc.$$

We assume $c \neq \infty$, and we may without loss of generality assume that all c_n are positive (this is certainly the case when we want to apply the result). For $\epsilon > 0$ we have that $(b-\epsilon)c_n \leq b_n c_n \leq (b+\epsilon)c_n$ for n large enough, taking lim sup we get

$$(b-\epsilon)c \leq \limsup_{n\to\infty} b_n c_n \leq (b+\epsilon)c.$$

Since ϵ was arbitrary, we can let it go to 0 and the claim follows.

It is easy to see that $\limsup_{n\to\infty} |n|^{1/n} = 1$,[2] so we get that $\limsup_{n\to\infty} |na_n|^{1/n} = \limsup_{n\to\infty} |a_n|^{1/n}$. □

In particular, $\sum_{n=0}^{\infty} na_n z^{n-1}$ has the same radius of convergence as $\sum_{n=0}^{\infty} a_n z^n$. There is a general fact that you can exchange differentation and infinite sums, as long as the infinite sums which appear converge absolutely. A proof can be found in [SS03, Theorem 2.7]:

Theorem 5.2.4. *If $\sum_{n=0}^{\infty} a_n z^n$ has radius of convergence R, then it is holomorphic on the open disk $D_R(0)$ with derivative $\sum_{n=0}^{\infty} na_n z^{n-1}$.*

Since the radius of convergence of $\sum_{n=0}^{\infty} na_n z^{n-1}$ is the same as that of $\sum_{n=0}^{\infty} a_n z^n$, iterating this theorem we see that we can differentiate as many times as we want:

Corollary 5.2.5. *Within its radius of convergence, $\sum_{n=0}^{\infty} a_n z^n$ is infinitely many times complex differentiable.*

So far we discussed power series expansions about the origin. You can expand around $z_0 \in \mathbb{C}$ by studying $\sum_{n=0}^{\infty} a_n (z-z_0)^n$ instead.

5.2.2 Analytic functions

Definition 5.2.6. A function $f\colon \mathbb{C} \supset \Omega \to \mathbb{C}$ is *analytic* if for each $z_0 \in \Omega$ we can a disk $D_R(z_0) \subset \Omega$ and a power series

$$\sum_{n=0}^{\infty} a_n (z-z_0)^n$$

with radius of convergence $> R$, so that

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n \qquad \text{for } z \in D_R(z_0).$$

This sum is called a power series expansion, or Taylor series, for f at z_0.

Example 5.2.7. $\exp(z)$, $\sin(z)$, and $\cos(z)$ are analytic on all of $\mathbb{C}$. So is any polynomial.

[2] For example, take log to get $1/n \log(n)$ and observe logarithms grow slower than linear functions.

Since being complex-differentiable is a local property, Theorem 5.2.1 implies that any analytic function is holomorphic. It follows from Cauchy's formula that the converse is also true:

Theorem 5.2.8. *Suppose $f\colon \mathbb{C} \supset \Omega \to \mathbb{C}$ is holomorphic and there is a closed disk $\overline{D}_R(z_0) \subset \Omega$, then f has a power series expansion*

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \qquad \text{for } z \in \overline{D}_R(z_0),$$

with $a_n = \frac{f^{(n)}(z_0)}{n!}$. In particular this power series has radius of convergence $> R$.

Proof. Our starting point is

$$f(z) = \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{\zeta - z} d\zeta,$$

with C a circle of radius $R + \epsilon$ around z_0. We write

$$\frac{1}{\zeta - z} = \frac{1}{\zeta - z_0} \frac{1}{1 - \frac{z - z_0}{\zeta - z_0}},$$

and use the geometric series to write this is as

$$\frac{1}{\zeta - z_0} \sum_{n=0}^{\infty} \left(\frac{z - z_0}{\zeta - z_0} \right)^n .$$

This converges absolutely uniformly when $\frac{z-z_0}{\zeta - z_0} < 1$, so in particular when $|z - z_0| \le R$. Hence we interchange the sum and integral in Cauchy's integral formula to obtain

$$\begin{aligned} f(z) &= \sum_{n=0}^{\infty} \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{\zeta - z_0} \left(\frac{z - z_0}{\zeta - z_0} \right)^n d\zeta \\ &= \sum_{n=0}^{\infty} \frac{1}{2\pi i} \left(\int_C \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} d\zeta \right) (z - z_0)^n \\ &= \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} (z - z_0)^n. \end{aligned}$$

The last equation used the integral expression for the nth derivative. □

In this theorem is hidden a rather strong statement: if the radius of convergence of a power series

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n$$

representing f is R, then it is impossible to extend f to some open neighborhood of $\overline{D}_R(z_0)$: if you could, then the radius of convergence of the power series would have to be larger.

5.2.3 Analytic continuation

Having established two of the miraculous properties of holomorphic functions (being infinitely many times differentiable, and being analytic), we give a precise statement of the claim that "many holomorphic functions extend uniquely to larger domains."

This will be a consequence of the following result, which says that the global behavior of a holomorphic function is determined locally, as long as the domain is connected:

Theorem 5.2.9. *Suppose that Ω is connected and there is a sequence of distinct points $\{z_n\}_{n\in\mathbb{N}}$ in Ω with a limit point $z \in \Omega$. Suppose that $f\colon \mathbb{C} \supset \Omega \to \mathbb{C}$ is holomorphic, then if $f(z_n) = 0$ for all $n \in \mathbb{N}$, then $f(z) = 0$ for all $z \in \Omega$.*

Proof. Consider the power series expansion $f(w) = \sum_{n=0}^{\infty} a_n(w-z)^n$ around z. We will prove that all a_n vanish. This follows by observing that if n is smallest such that $a_n \neq 0$, then $f(w) = a_n(w-z)^n(1+g(w))$ with $g(w) \to 0$ as $z \to 0$. This is contradicts z being a limit point of zeroes.

Now consider the subset $\mathrm{int}(f^{-1}(0))$. This is by definition open. It is also closed, because if a sequence $z'_1, z'_2, \ldots \in \mathrm{int}(f^{-1}(0))$ converges to $z' \in \Omega$, then by the previous argument f vanishes on an open disk $D_{R'}(z')$. But a non-empty subset of a connected Ω which is both open and closed must be all of Ω. $\square$

Example 5.2.10. If f vanishes on a non-empty open subset of a connected Ω, it has to be identically zero. This means bump functions can never be holomorphic.

Corollary 5.2.11. *Let Ω and $\{z_n\}_{n\in\mathbb{N}}$ be as above. Suppose that $f, g\colon \mathbb{C} \supset \Omega \to \mathbb{C}$ are holomorphic, then if $f(z_n) = g(z_n)$ for all $n \in \mathbb{N}$, then $f(z) = 0$ for all $z \in \Omega$.*

Proof. Apply the previous theorem to $f - g$. $\square$

This gives a way of extending holomorphic functions *uniquely*; a procedure called analytic continuation. For example, if $f\colon \Omega\colon \mathbb{C}$ and $g\colon \Omega' \to \mathbb{C}$ are holomorphic functions on connected open subsets, $\Omega \cap \Omega' \neq \varnothing$, and $f = g$ on $\Omega \cap \Omega'$, then

$$\begin{aligned} \Omega \cup \Omega' &\longrightarrow \mathbb{C} \\ z &\longmapsto \begin{cases} f(z) & \text{if } z \in \Omega, \\ g(z) & \text{if } z \in \Omega' \end{cases} \end{aligned}$$

is holomorphic. This is not the interesting part: by the corollary this is the *only* possible extension of f to $\Omega \cup \Omega'$. In fact, in this argument only one of Ω or Ω' needs to be path-connected.

It is this technique that we will use to extend $\zeta(s)$ defined as a sum for $\mathrm{Re}(s) > 1$, to $\mathbb{C} \setminus \{1\}$.

Example 5.2.12. Suppose that $f\colon \mathbb{C} \supset \Omega \to \mathbb{C}$ has as Tayler series at $z_0 \in \Omega$ with convergence radius R. Then $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ is a holomorphic function on $D_R(z_0)$. This disk may very well be larger than Ω. Since $D_R(z_0)$ is path-connected, we can thus extend f uniquely to $\Omega \cup D_R(z_0)$ by

$$f(z) = \begin{cases} f(z) & \text{if } z \in \Omega, \\ \sum_{n=0}^{\infty} a_n(z-z_0)^n & \text{if } z \in D_R(z_0). \end{cases}$$

We can keep iterating this construction of extensions using Taylor series. Does this end? The answer is subtle: a holomorphic function may not have a “maximal” domain of definition. Eventually, we will see that the logarithm provides an example of this.

Chapter 6

Morera's theorem and its applications

Last time we gave two applications of Cauchy's integral formula: we proved that holomorphic functions are analytic, and we used this to construct unique extensions of holomorphic functions ("analytic extension"). This is Sections 5.1–5.4 of Chapter 2 of [SS03].

6.1 Morera's theorem

Two lectures ago we proved that every holomorphic function $f\colon \mathbb{C} \supset \Omega \to \mathbb{C}$ on an open disk Ω has a primitive F, i.e. a holomorphic F satisfying $F' = f$. This was proven by constructing F using Goursat's theorem, which says that for every triangle $T \subset \Omega$ we have $\int_{\partial T} f(z)dz = 0$. In fact, we only used that f is continuous and that $\int_{\partial T} f(z)dz = 0$ for each triangle T. This leads to the following result.

Theorem 6.1.1 (Morera's theorem)**.** *Suppose we are given a continuous function $f\colon \mathbb{C} \supset \Omega \to \mathbb{C}$, such that for every triangle $T \subset \Omega$ we have $\int_{\partial T} f(z)dz = 0$. Then f is holomorphic.*

Proof. Since being holomorphic is a local property, we may assume that Ω is a disk. By the preceding remark, there is a holomorphic $F\colon \mathbb{C} \supset \Omega \to \mathbb{C}$ such that $F' = f$. Since every holomorphic function is infinitely many times complex-differentiable, F' is holomorphic and hence so is f. $\square$

We shall use this to recognize that a continuous function is holomorphic.

6.2 Constructing holomorphic functions by limits

6.2.1 Sequences

Let's start by constructing holomorphic functions by as limits of sequences:

Question 6.2.1. Suppose that $f_0, f_1, f_2, \cdots \colon \mathbb{C} \supset \Omega \to \mathbb{C}$ are holomorphic and the limit $\lim_{n\to\infty} f_n(z)$ exists for all $z \in \Omega$. When is the function $\lim_{n\to\infty} f_n$ holomorphic?

Example 6.2.2. A special case of this appeared in our discussion of power series: the power series $\sum_{n=0}^{\infty} a_n z^n$ is the limit of the partial sums $s_m = \sum_{n=0}^{m} a_n z^n$ when it coverges. This happens only within its radius of convergence R, given by $1/R = \liminf |a_n|^{1/n}$. That is, where the partial sums converges we get a holomorphic function.

Definition 6.2.3. A sequence $f_0, f_1, f_2, \ldots : \mathbb{C} \supset \Omega \to \mathbb{C}$ of functions *converges uniformly to* f *on a subset* $K \subset \Omega$ if for all $\epsilon > 0$ there exists an N such that for all $n \geq N$ we have $|f_n(z) - f(z)| < \epsilon$ for all $z \in K$.

We don't need to know f beforehand, and may replace convergent sequences by Cauchy sequences. That is, $f_0, f_1, f_2, \ldots : \mathbb{C} \supset \Omega \to \mathbb{C}$ of functions *converges uniformly on a subset* $K \subset \Omega$ if for all $\epsilon > 0$ there exists an N such that for all $n, m \geq N$ we have $|f_n(z) - f_m(z)| < \epsilon$ for all $z \in K$.

Theorem 6.2.4. *Suppose that* $f_0, f_1, f_2, \cdots : \mathbb{C} \supset \Omega \to \mathbb{C}$ *are holomorphic and converge uniformly on all compact subsets* $K \subset \Omega$*. Then* $f = \lim_{n \to \infty} f_n$ *is holomorphic.*

Proof. By Morera's theorem, it suffices to prove that $\int_{\partial T} f(z)dz = 0$ for all triangles $T \subset \Omega$. Now we use a fact from real analysis (whose proof you do not need to know), which says that uniform convergence is a sufficient condition for interchanging limits and integrals:

$$\int_{\partial T} f(z)dz = \int_{\partial T} \lim_{n\to\infty} f_n(z)dz = \lim_{n\to\infty} \int_{\partial T} f_n(z)dz.$$

But each of the terms in the sequence on the right hand side vanish: f_n is holomorphic so Goursat's theorem applies. The limit is then also 0, and the result follows. □

Remark 6.2.5. This should be very surprising: a limit of differentiable real-valued functions need not be differentiable again. For example, by the Weierstrass approximation theorem any continuous function can be approximated by polynomials uniformly on compacts.

We got the complex-differentiability of $f = \lim_n f_n$ through a rather complicated chain of arguments. Thus it is not clear what its derivative is. It turns out to be what you expect it to be, by the following result:

Proposition 6.2.6. *Suppose that* $f_0, f_1, f_2, \cdots : \mathbb{C} \supset \Omega \to \mathbb{C}$ *are holomorphic and converge uniformly on all compact subsets* $K \subset \Omega$*. Then the derivatives* $f_0', f_1', f_2', \cdots$ *converge uniformly on compacts to* f'*.*

Proof. Let's start with some reductions: by replacing f_n with $f_n - f$ we may assume that $f = 0$ and hence $f' = 0$. Since every compact subset can be covered by finitely many closed balls and we can translate, it suffices to take $K = \overline{D}_r(0)$.

Our goal is now to prove that $\sup_{z \in \overline{D}_r(0)} |f_n'(z)| \to 0$ as $n \to \infty$. To get this estimate, we observe that since Ω is open, there is a $\delta > 0$ such that $\overline{D}_{r+\delta}(z_0) \subset \Omega$. As f_n converge uniformly to 0 on $\overline{D}_{r+\epsilon}(0)$, this follows once we prove the estimate

$$\sup_{z \in \overline{D}_r(0)} |f_n'(z)| \leq C \sup_{z \in \overline{D}_{r+\delta}(0)} |f_n(z)|$$

for some constant $C > 0$.

But this follows from the Cauchy inequality applied to a circle C of radius $r > \delta$ around z:

$$|f_n'(z)| \leq \frac{1}{r} \sup_{z \in C} |f_n(z)| \leq \frac{1}{\delta} \sup_{z \in \overline{D}_{r+\epsilon}(0)} |f_n(z)|$$

when we take $C = \frac{1}{\delta}$. ☐

By induction the same holds for the higher derivatives: the sequence of holomorphic functions $f_0^{(k)}, f_1^{(k)}, f_2^{(k)}, \cdots$ converges uniformly on compact subsets to $f^{(k)}$.

6.2.2 Sums

Taking the functions to be the partial sums $s_m = \sum_{n=0}^m f_n$, we obtain:

Corollary 6.2.7. *Suppose that $f_0, f_1, f_2, \cdots : \mathbb{C} \supset \Omega \to \mathbb{C}$ are holomorphic and the sum $\sum_{n=0}^\infty f_n$ converges uniformly on all compact subsets $K \subset \Omega$. Then $f = \sum_{n=0}^\infty f_n$ is holomorphic.*

Proof. By definition, convergence of the sum means that the partial sums s_m converge as $m \to \infty$. Now apply Theorem 6.2.4. ☐

As a consequence of the propositions, we see that we can compute the derivative of a sum as in the corollary, by taking the sum of the derivatives of the terms. The higher derivatives are similar.

Example 6.2.8. For $\mathrm{Re}(s) > 1$, the Riemann ζ-function

$$\zeta(s) = \sum_{n=1}^\infty \frac{1}{n^s}$$

is defined by a sum. We claim this converges uniformly on all half-planes $\{s \in \mathbb{C} \mid \mathrm{Re}(s) \geq 1 + \epsilon\}$ for $\epsilon > 0$. Indeed, we have

$$\left| \frac{1}{n^s} \right| = \frac{1}{n^{\mathrm{Re}(s)}} \leq \frac{1}{n^{1+\epsilon}},$$

which is a convergent sum as $\epsilon > 0$. For example, you can estimate from above by

$$1 + \int_1^\infty \frac{1}{t^{1+\epsilon}} dt = 1 + \left[-(1+\epsilon) \frac{1}{t^\epsilon} \right]_1^\infty = 2 + \epsilon.$$

Since every compact subset of $\{s \in \mathbb{C} \mid \mathrm{Re}(s) > 1\}$ is contained in one of these half-planes, the sum defining the Riemann ζ-function is uniformly convergent on compacts. By the Corollary $\zeta(s)$ is holomorphic on $\{s \in \mathbb{C} \mid \mathrm{Re}(s) > 1\}$.

6.2.3 Integrals

Since integrals are similarly limits of finite sums, now Riemann sums instead of partial sums, we get a similar result for holomorphic functions defined by integrals:

Corollary 6.2.9. *Suppose that $F\colon \mathbb{C} \times [0,1] \supset \Omega \times [0,1] \to \mathbb{C}$ has the property that (i) for each $s \in [0,1]$, $F(-,t)\colon \mathbb{C} \supset \Omega \to \mathbb{C}$ is holomorphic, (ii) F is continuous. Then $\int_0^1 F(z,t)dt$ defines a holomorphic function $\Omega \to \mathbb{C}$.*

Proof sketch. Apply Theorem 6.2.4 to the Riemann sums $\frac{1}{N}\sum_{k=1}^N f(z,k/N)$. See pages 56-57 of [SS03] for the details. □

Of course we may replace $[0,1]$ by any other closed interval $[a,b]$. As before, the derivative F' of F will be as expected, $\int_0^1 \frac{\partial F(z,t)}{\partial z}dt$. The higher derivatives are similar.

Example 6.2.10. For $\mathrm{Re}(s) > 0$, the Gamma function

$$\Gamma(z) = \int_0^\infty e^{-t}t^{z-1}ds$$

is defined by an integral. Let us sketch why it is holomorphic, details will appear later in this course. Firstly, we claim that uniformly on compacts in $\{s \in \mathbb{C} \mid \mathrm{Re}(s) > 0\}$, we have that

$$\lim_{\epsilon \to 0} \int_\epsilon^{1/\epsilon} e^{-t}t^{z-1}dt.$$

If so, we can use Theorem 6.2.4 to prove that Γ is holomorphic if we can prove that the function $(z,t) \longmapsto e^{-t}t^{z-1}$ satisfies the condition of the previous Corollary. But its clear that (i) it's holomorphic for fixed $t \in [\epsilon, 1/\epsilon]$, and (ii) continuous.

The Gamma function will play a role when studying the Riemann ζ-function. It also appears in the formula for the volume of a ball B_1 of radius 1 in $\mathbb{R}^n$:

$$\mathrm{vol}(B_1) = \frac{\pi^{n/2}}{\Gamma(n/2+1)}.$$

6.2.4 Pointwise convergence

Suppose we drop the condition that $f_n \to f$ uniformly on compacts, and only demand that $f_n(z) \to f(z)$. This was our original question, and we can answer it with some sledgehammers from real analysis and complex analysis: the Baire category theorem and Montel's theorem. We will prove the latter in a later lecture.

Theorem 6.2.11 (Corollary of the Baire category theorem)**.** *If $K \subset \mathbb{C}$ is compact and we can write $K = \bigcup_{j=1}^\infty K_j$ as a countable union of compact subsets, then some K_j is somewhere dense in K (i.e. its closure contains an open disk).*

Theorem 6.2.12 (Corollary of Montel's theorem)**.** *If a sequence $f_0, f_1, f_2, \ldots \colon \mathbb{C} \supset \Omega \to \mathbb{C}$ is a sequence of functions such that for all compacts $K \subset \Omega$ there is a $B > 0$ such that $|f_n(z)| < B$ for all $z \in K$. Then the sequence has a subsequence which converges uniformly on compacts.*

Theorem 6.2.13. *If $f_0, f_1, f_2, \ldots : \mathbb{C} \supset \Omega \to \mathbb{C}$ is a sequence of holomorphic functions that converges pointwise to a function f. Then f is holomorphic on a dense open subset of Ω.*

Proof. Let $\overline{D}_r(z)$ be a closed disk in Ω. Since z and r are arbitrary, it suffices to prove that f is holomorphic on an open subset of $\overline{D}_r(z)$.

Take $K = \overline{D}_r(z)$ and define

$$K_j = \{z \in \overline{D}_r(z) \mid |f_n(z)| \leq j \text{ for all } n \geq 0.\}.$$

Since the $f_n(z)$ converges to $f(z)$, they are bounded and $\bigcup_{j=1}^{\infty} K_j = K$. Furthermore, each K_j is an intersection $\bigcap_n \{z \in \overline{D}_r(z) \mid |f_n(z)| \leq j\}$ of closed subsets. Hence K_j is closed, and since it also bounded it is compact. By the Baire category theorem some K_j has the property that $\overline{K_j}$ contains an open disk $D_{r'}(z')$. Thus the restrictions of $f_0, f_1, f_2, \ldots$ to any compact are in absolute value bounded by j.

By Montel's theorem, there is a subsequence which converges on compacts, necessarily to f. By Theorem 6.2.4, f is holomorphic on $D_{r'}(z')$. □

6.3 Schwarz reflection

Since being holomorphic is a local condition, if $f\colon \mathbb{C} \supset \Omega \to \mathbb{C}$ and $\tilde{f}\colon \tilde{\Omega} \to \mathbb{C}$ are holomorphic functions and $f = \tilde{f}$ on $\Omega \cap \tilde{\Omega}$, then

$$z \longmapsto \begin{cases} f(z) & \text{if } z \in \Omega, \\ \tilde{f}(z) & \text{if } z \in \tilde{\Omega}, \end{cases}$$

is holomorphic. We can think of this as an extension of f from Ω to $\Omega \cup \tilde{\Omega}$. This extension is unique if either Ω or $\tilde{\Omega}$ is connected.

Today we give a different result for producing extensions of holomorphic functions: the *Schwarz reflection principle.* Let $\Omega \subset \mathbb{C}$ be an open subset symmetric with respect to reflection in the real axis, i.e. $\overline{z} \in \Omega$ if and only $z \in \Omega$. Then the subsets $\Omega^+ \subset \{z \in \mathbb{C} \mid \mathrm{Im}(z) \geq 0\}$ and $\Omega^- \subset \{z \in \mathbb{C} \mid \mathrm{Im}(z) \leq 0\}$ are mapped to each other by complex conjugation, and they intersect in $\Omega^+ \cap \Omega^- \subset \mathbb{R}$.

Let $f^+\colon \mathbb{C} \supset \Omega^+ \to \mathbb{C}$ and $f^-\colon \mathbb{C} \supset \Omega^- \to \mathbb{C}$ be continuous functions which coincide on $\Omega^+ \cap \Omega^- = \Omega \cap \mathbb{R}$. Then the formula

$$f(z) := \begin{cases} f^+(z) & \text{if } z \in \Omega^+, \\ f^-(z) & \text{if } z \in \Omega^-, \end{cases}$$

defines a continuous function on Ω.

Lemma 6.3.1. *With notation as above, further suppose that f^+ is holomorphic on $\Omega^+ \cap \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$ and f^- is holomorphic on $\Omega^+ \cap \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$. Then f is also holomorphic.*

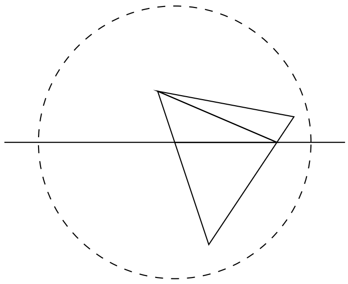
Proof. It is clear that f is complex-differentiable at $z \in \Omega \setminus \mathbb{R}$. To prove it is complex-differentiable at $z_0 \in \Omega \cap \mathbb{R}$, we may restrict our attention to an open ball $D_1(z_0) \subset \Omega$.

Let us take a triangle $T \subset \Omega$. If T lies in the interior of Ω^+, then $\int_{\partial T} f(z)dz = \int_{\partial T} f^+(z)dz$, which vanishes by Goursat. If T lies in Ω^+, $\int_{\partial T} f(z)dz$ is the limit of $\int_{\partial T_i} f^+(z)dz$ with T_i in the interior of Ω^+ and also vanishes. The same is true for Ω^-, so it suffices to restrict our attention to T which intersect both the interior of Ω^+ and Ω^-.

In that case, we can write the integral

$$\int_{\partial T} f(z)dz$$

as a sum of the integral over the boundaries of two or three triangles, each of which vanishes. The following is a typical example:



Thus we have proven that the integral over the boundaries of all triangles vanishes, and conclude by Morera's theorem that f is holomorphic on $D_1(z_0)$ as well. □

Since there is no condition on the real axis apart from continuity, we can use this to easily construct examples:

Theorem 6.3.2 (Schwarz reflection principle)**.** *Let Ω be as above, and suppose that $f^+ \colon \Omega^+ \to \mathbb{C}$ is continuous satisfying (i) its restriction to $\Omega \cap \mathbb{R}$ is real-valued, and (ii) its restriction to $\Omega^+ \cap \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$ is holomorphic. Then the function*

$$f(z) := \begin{cases} f^+(z) & \text{if } z \in \Omega^+, \\ \overline{f^+(\overline{z})} & \text{if } z \in \Omega^-, \end{cases}$$

is also holomorphic.

Proof. Since f is real-valued on the real axis, $f^+(z) = \overline{f^+(\overline{z})}$ for $z \in \mathbb{R}$. Furthermore, to verify $\overline{f^+(\overline{z})}$ is holomorphic, we use that if near z_0 the function f has a Taylor series $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ then $\overline{f^+(\overline{z})}$ is given near $\overline{z_0}$ by

$$\sum_{n=0}^{\infty} \overline{a_n}(z - \overline{z_0})^n.$$

This has the same radius of convergence as the original Taylor series, and hence exhibits $\overline{f^+(\overline{z})}$ as a holomorphic function near z_0. □

Observe that the extension f of f^+ to Ω is unique. If $\tilde{f}$ is another extension, it must necessary coincide with f on $\Omega^+ \cap \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$. Since this is open and meets every connected component of Ω, this means that $\tilde{f} = f$ on each connected component of Ω and hence they are equal.

Remark 6.3.3. There are straightforward generalizations of this result: we can replace the condition that f takes real values on the real axis by the condition that it takes imaginary values: then $\overline{f^+(\overline{z})}$ needs to be replaced by $-\overline{f^+(\overline{z})}$. We can similarly replace the real axis by the imaginary axis, or even the circle of radius 1 around the origin. The latter will be an exercise.

Example 6.3.4. Suppose that $f\colon \mathbb{C} \to \mathbb{C}$ takes real values on the real axis, and imaginary values on the imaginary axis. Then we can apply the Schwarz reflection principle twice to the restriction of f to $\{z \in \mathbb{C} \mid \mathrm{Re}(z) \geq 0 \text{ and } \mathrm{Im}(z) \geq 0\}$ and get a holomorphic function:

$$g(z) = \begin{cases} f(z) & \text{if } \mathrm{Re}(z) \geq 0 \text{ and } \mathrm{Im}(z) \geq 0, \\ \overline{f(\overline{z})} & \text{if } \mathrm{Re}(z) \geq 0 \text{ and } \mathrm{Im}(z) \leq 0, \\ -\overline{f(-\overline{z})} & \text{if } \mathrm{Re}(z) \leq 0 \text{ and } \mathrm{Im}(z) \geq 0, \\ -f(-z) & \text{if } \mathrm{Re}(z) \leq 0 \text{ and } \mathrm{Im}(z) \leq 0. \end{cases}$$

Indeed, this is arranged to be continuous and holomorphic on each of the open quadrants. Observe that g satisfies $g(-z) = -g(z)$, i.e. g is odd.

Since f and g coincide on $\{z \in \mathbb{C} \mid \mathrm{Re}(z) \geq 0 \text{ and } \mathrm{Im}(z) \geq 0\}$, they must be equal and hence f must be odd. Thus we have proven that if $f\colon \mathbb{C} \to \mathbb{C}$ takes real values on the real axis, and imaginary values on the imaginary axis, it must be odd!

Chapter 7

Poles

We give now discuss holomorphic functions with singularities, i.e. there are some points where they are not defined. We start the classification of these singularities, and discuss the case of poles. Then we explain how to evaluate integrals $\int_\gamma f(z)dz$ along loops γ that contain a pole in their interior. This is Section 1–2 of Chapter 3 of [SS03].

7.1 Poles and zeroes

7.1.1 Isolated singularities

Definition 7.1.1. A point $z_0 \in \mathbb{C}$ is an *isolated singularity* of a holomorphic function $f\colon \Omega \to \mathbb{C}$ if Ω contains a punctured disk $D_r(z_0) \setminus \{z_0\}$.

There are three types of isolated singularities:

- *removable singularities*, where $|f(z)|$ is bounded as $z \to z_0$,
- *poles*, where $|f(z)|$ goes to ∞ as $z \to z_0$,
- *essential singularities*, all other cases.

Today we will talk about the second case, which we will approach by studying $1/f(z)$. We will talk about removable and essential singularities next lecture.

7.1.2 Zeroes and poles

Definition 7.1.2. A point $z_0 \in \Omega$ is a *zero* of a holomorphic function $f\colon \Omega \to \mathbb{C}$ if $f(z_0) = 0$.

Suppose Ω is connected and z_0 is a zero of a holomorphic function $f\colon \Omega \to \mathbb{C}$. Unless it is 0 everywhere, it can't vanish in a neighborhood of z_0. In fact, something stronger is true: there is an open neighborhood U of z_0 in Ω, $n \geq 1$, and a non-vanishing holomorphic function $g\colon U \to \mathbb{C}$ such that

$$f(z) = (z - z_0)^n g(z) \text{ for } z \in U.$$

This is proven by observing that on a neighborhood of z_0, we have

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n.$$

Letting n be the lowest integer such that $a_n \neq 0$ (which exists since f is not 0 everywhere), we can take

$$g(z) = 1 + \sum_{m=1}^{\infty} a_{n+m}(z - z_0)^m.$$

Furthermore, fixing U the integer n and function g are unique.

Definition 7.1.3. We call n the *multiplicity* of the zero z_0 (or say z_0 is a zero *of order* n).

Example 7.1.4. Recall that

$$\cos(\pi z) = \frac{e^{\pi i z} + e^{-\pi i z}}{2}.$$

This means $\cos(\pi z) = 0$ is equivalent to $e^{\pi i z} = -e^{-\pi i z}$. Using the fact that the exponential function has no zeroes, this happens when $e^{2\pi i z} = -1$. This is the case when $z = k + 1/2$ for $k \in \mathbb{Z}$. Thus $\cos(\pi z)$ has infinitely many zeroes, exactly at all half-integers on the real line.

We claim they all have multiplicity 1. Let's prove this at $z = 1/2$: as $\cos(\pi(z-1/2)) = -\sin(\pi z)$, we may equivalently consider the multiplicitive of $\sin(\pi z)$ at the origin. But the Taylor series $\sin(\pi z) = \pi z +$ higher order terms starts with a linear term.

Let us apply this to poles, and for that purpose we consider a holomorphic function $f \colon \Omega \to \mathbb{C}$ such that $D_r(z_0) \setminus \{z_0\} \subset \Omega$ for some $r > 0$.

Definition 7.1.5. We say f as above has a *pole* at z_0 if there is an $r > 0$ such that f is non-zero on $D_r(z_0) \setminus \{z_0\}$, and the function

$$z \longmapsto \begin{cases} 1/f(z) & \text{if } z \in D_r(z_0) \setminus \{z_0\}, \\ 0 & \text{if } z = z_0, \end{cases}$$

is holomorphic.

Writing $1/f$ for the above function, it has a 0 of some order n at z_0 and hence we can write

$$1/f(z) = (z - z_0)^n g(z)$$

with g non-vanishing. The integer n is unique, and we say z_0 is a pole of *order* n. A pole of order 1 is said to be *simple*.

Example 7.1.6. The function $\frac{1}{\cos(\pi z)}$ has poles of order 1 at each half-integer. The function $\frac{z-1/2}{\cos(\pi z)}$ has poles at each half-integer except 1/2 (there it will turn out to have a removable singularity).

We then get that:

Proposition 7.1.7. *If f has a pole of order n at z_0, we can write*

$$f(z) = \frac{1}{(z - z_0)^n} h(z)$$

on $D_r(z_0) \setminus \{z_0\}$ with h a non-vanishing holomorphic function $D_r(z_0) \to \mathbb{C}$.

Writing $h(z) = \sum_{n=0}^{\infty} b_n (z - z_0)^n$, we get

$$f(z) = \frac{a_{-n}}{(z - z_0)^n} + \cdots + \frac{a_{-1}}{z - z_0} + H(z),$$

where $a_{-n+i} = b_i$ and $H\colon D_r(z_0) \to \Omega$ is holomorphic.

The terms

$$\frac{-a_{-n}}{(z - z_0)^n} + \cdots + \frac{a_{-1}}{z - z_0}$$

are the *principal part*, and a_{-1} is the *residue* of f at z_0. Let us record this more prominently:

$$\operatorname{res}_{z_0} f \coloneqq a_{-1}.$$

Example 7.1.8. We can compute the residue as

$$\operatorname{res}_{z_0} f = \lim_{z \to z_0} \frac{1}{(n-1)!} \frac{d^{n-1}}{dz^{n-1}} (z - z_0)^n f(z).$$

7.1.3 Laurent series

If we were to expand H by a power series $H(z) = \sum_{j=0}^{\infty} a_j z^j$, we would get

$$f(z) = \sum_{j=-n}^{\infty} a_j z^j,$$

the *Laurent series* of f around the pole z_0.

More generally, a Laurent series can have infinitely many negative powers. These exist as series expansions for holomorphic functions on annuli. For $0 < r < R$, let $\mathbb{A}_{r,R} = \{z \in \mathbb{C} \mid r < |z| < R\}$ be an annulus around the origin.

Theorem 7.1.9. *Suppose that $f\colon \Omega \to \mathbb{C}$ is holomorphic and $\overline{\mathbb{A}_{r,R}} \subset \Omega$. Then we have that*

$$f(z) = \sum_{n=-\infty}^{\infty} a_n z^n$$

on $\overline{\mathbb{A}_{r,R}}$, where the series converges absolutely.

Proof sketch. Using a "slit annulus contour" we see that for some small $\delta > 0$ we can write

$$f(z) = \frac{1}{2\pi i} \int_{C_{R+\delta}} \frac{f(\zeta)}{z - \zeta} d\zeta - \frac{1}{2\pi i} \int_{C_{r-\delta}} \frac{f(\zeta)}{z - \zeta} d\zeta.$$

We can expand the term $\frac{1}{z-\zeta} = -\frac{1}{\zeta} \frac{1}{1 - z/\zeta}$ in the first integral as a geometric series, and the term $\frac{1}{z-\zeta} = \frac{1}{z} \frac{1}{1-\zeta/z}$ in the second integral as a geometric series. The first gives the positive powers of z, the second the negative powers. $\square$

7.2 The residue formula

Recall that if $f\colon \Omega \to \mathbb{C}$ is holomorphic and $C \subset \Omega$ is a circle whose interior is also contained in Ω, then we have

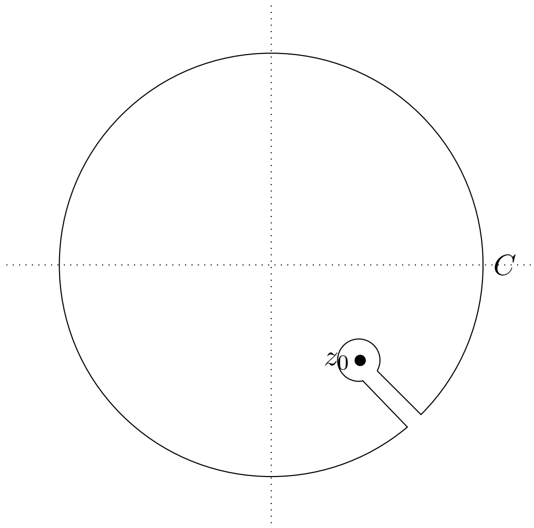
$$\int_C f(z)dz = 0.$$

Next suppose that it is the interior of C is also contained in Ω *with the exception of a pole* z_0. Then we have the following:

Theorem 7.2.1. *Let f, C be as above, then*

$$\int_C f(z)dz = 2\pi i \operatorname{res}_{z_0} f.$$

Proof. Consider a "keyhole contour" as follows:



By Cauchy's theorem, the integral over this vanishes. Letting the distance δ between the parallel line segments go to 0, we get that

$$\int_C f(z)dz - \int_{C_\epsilon} f(z)dz = 0,$$

where C_ϵ is the tiny circle of radius ϵ around z_0 that appears when $\delta \to 0$.

We conclude that

$$\int_C f(z)dz = \int_{C_\epsilon} \frac{a_{-n}}{(z-z_0)^n} + \cdots + \frac{a_{-1}}{z-z_0} + H(z)dz,$$

which is the equal to the sums of the integrals of each of the terms on the right hand side.

The integral $\int_{C_\epsilon} H(z)dz$ vanishes because H is holomorphic on the interior of C_ϵ. The other integrals we can compute explicitly using the parametrization $\theta \mapsto z_0 + \epsilon e^{i\theta}$ of the circle C_ϵ:

$$\int_{C_\epsilon} \frac{a_{-j}}{(z-z_0)^j} dz = \int_0^{2\pi} a_{-j} i e^{i(j-1)\theta} d\theta = \begin{cases} 2\pi i a_{-1} & \text{if } j = 1, \\ 0 & \text{otherwise.} \end{cases}$$

This proves the theorem, once we recall that $\mathrm{res}_{z_0} f$ is by definition a_{-1}. □

More generally, if there are k poles $z_1, \ldots, z_k$ in the interior of C we can use a "multiple keyhole contour" to prove:

Corollary 7.2.2. *Let f, C be as above, then*

$$\int_C f(z)dz = 2\pi i \sum_{j=1}^{k} \mathrm{res}_{z_j} f.$$

More generally, we can replace C by your favorite toy contour; rectangles, keyholes, indented semicircles, etc.

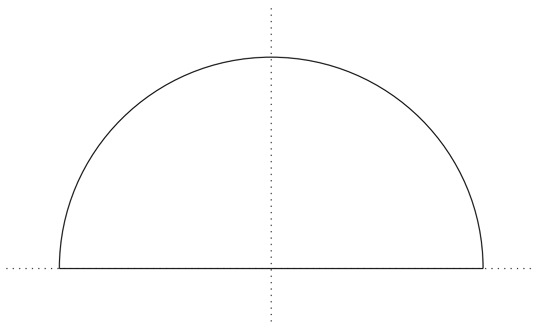
7.3 Integrals using the residue formula

The residue formula is one of the most powerful methods to evaluate integrals.

Example 7.3.1. Let us compute

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2} dx.$$

We shall take a contour given by a semi-circle S_R of radius $R > 1$ in the upper half-plane:



Since $f(z) = \frac{1}{1+z^2}$ has two poles, at $\pm i$, this contains a single pole at $z_0 = i$. The residue is given by writing

$$\frac{1}{1+z^2} = \frac{1}{z+i} \frac{1}{z-i},$$

which by expanding the first term around i gives us the Laurent series

$$f(z) = \frac{-i}{2} \frac{1}{z-i} + \text{higher order terms}$$

around i. Thus the residue at i at $-i/2$, and we get

$$\int_{S_R} \frac{1}{1+z^2} dz = 2\pi i \cdot -i/2 = \pi.$$

We can write the integral over S_R as the sum of the integral

$$\int_{-R}^{R} \frac{1}{1+x^2} dx$$

and the integral over the upper semicircle S_R^+, which we can estimate as being at most

$$\text{length}(S_R^+) \cdot \sup_{z \in S_R^+} \left| \frac{1}{1+z^2} \right| . = \pi R \frac{1}{\sqrt{1+R^4}},$$

which goes to 0 as $R \to \infty$. Thus letting $R \to \infty$ we get that

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2} dx = \lim_{R \to \infty} \int_{S_R} \frac{1}{1+z^2} dz = \pi.$$

Example 7.3.2. Let's come up with a crazy integral you can do to impress your friends. We take $f(z) = \frac{e^z}{z}$ and integrate over the circle C_1 of radius 1 around the origin. On the one hand, we have a pole with residue 1 at the origin so we get

$$\int_C f(z) dz = 2\pi i.$$

On the other hand, we can evaluate it by parametrizing the circle by $\theta \mapsto e^{i\theta}$ to get

$$\begin{aligned} \int_C f(z) dz &= \int_0^{2\pi} \frac{e^{e^{i\theta}}}{e^{i\theta}} i e^{i\theta} d\theta \\ &= \int_0^{2\pi} i e^{\cos(\theta) + i \sin(\theta)} d\theta \\ &= \int_0^{2\pi} i e^{\cos(\theta)} \left(\cos(\sin(\theta) + i \sin(\sin(\theta)) \right) d\theta. \end{aligned}$$

Taking imaginary parts of both expressions we get

$$2\pi = \int_0^{2\pi} e^{\cos(\theta)} \cos(\sin(\theta)) d\theta.$$

It's unlikely you'd be able to evaluate the right hand side without knowing its origin.

Example 7.3.3. Let us prove compute the average of the squared absolute value of a polynomial over the unit circle: for a polynomial

$$p(z) = a_n z^n + \cdots + a_1 z + a_0,$$

we will compute

$$\frac{1}{2\pi} \int_0^{2\pi} |p(e^{i\theta})|^2 d\theta.$$

Since we are on the unit circle C_1, $\bar{z} = z^{-1}$, so we have

$$|p(z)|^2 = p(z)\overline{p(z)} = p(z)\overline{p}(z^{-1})$$

with

$$\overline{p}(z) = \overline{a_n} z^n + \cdots + \overline{a_1} z + \overline{a_0}.$$

That is, we have

$$\frac{1}{2\pi} \int_0^{2\pi} |p(e^{i\theta})|^2 d\theta = \frac{1}{2\pi i} \int_{C_1} p(z)\overline{p}(z^{-1}) \frac{1}{z} dz.$$

We now observe that $p(z)\overline{p}(z^{-1})\frac{1}{z}$ has pole at 0, with residue given by $\sum_{j=0}^{n} a_j\overline{a_j}$. Using the residue theorem, we conclude that

$$\frac{1}{2\pi}\int_0^{2\pi} |p(e^{i\theta})|^2 d\theta = |a_n|^2 + \cdots + |a_1|^2 + |a_0|^2.$$

Here is a fun consequence by estimate the integral: the coefficients of a polynomial satisfy:

$$|a_i|^2 \leq \frac{1}{2\pi}\mathrm{length}(C_1) \cdot \sup_{|z|=1} \left| p(z)\overline{p}(z^{-1})\frac{1}{z}\right|^2 = \sup_{|z|=1} |p(z)|^2.$$

There are three types of isolated singularities:

- *removable singularities*, where $|f(z)|$ is bounded as $z \to z_0$,
- *poles*, where $|f(z)|$ goes to ∞ as $z \to z_0$,
- *essential singularities*, all other cases.

Today we will talk about the second case, which we will approach by studying $1/f(z)$. We will talk about removable and essential singularities next lecture.

7.3.1 Zeroes and poles

Definition 7.3.4. A point $z_0 \in \Omega$ is a *zero* of a holomorphic function $f\colon \Omega \to \mathbb{C}$ if $f(z_0) = 0$.

Suppose Ω is connected and z_0 is a zero of a holomorphic function $f\colon \Omega \to \mathbb{C}$. Unless it is 0 everywhere, it can't vanish in a neighborhood of z_0. In fact, something stronger is true: there is an open neighborhood U of z_0 in Ω, $n \geq 1$, and a non-vanishing holomorphic function $g\colon U \to \mathbb{C}$ such that

$$f(z) = (z - z_0)^n g(z) \text{ for } z \in U.$$

This is proven by observing that on a neighborhood of z_0, we have

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n.$$

Letting n be the lowest integer such that $a_n \neq 0$ (which exists since f is not 0 everywhere), we can take

$$g(z) = 1 + \sum_{m=1}^{\infty} a_{n+m}(z - z_0)^m.$$

Furthermore, fixing U the integer n and function g are unique.

Definition 7.3.5. We call n the *multiplicity* of the zero z_0 (or say z_0 is a zero *of order* n).

Let us apply this to poles, and for that purpose we consider a holomorphic function $f\colon \Omega \to \mathbb{C}$ such that $D_r(z_0) \setminus \{z_0\} \subset \Omega$ for some $r > 0$.

Definition 7.3.6. We say f as above has a *pole* at z_0 if there is an $r > 0$ such that f is non-zero on $D_r(z_0) \setminus \{z_0\}$, and the function

$$z \longmapsto \begin{cases} 1/f(z) & \text{if } z \in D_r(z_0) \setminus \{z_0\}, \\ 0 & \text{if } z = z_0, \end{cases}$$

is holomorphic.

Writing $1/f$ for the above function, it has a 0 of some order n at z_0 and hence we can write

$$1/f(z) = (z - z_0)^n g(z)$$

with g non-vanishing. The integer n is unique, and we say z_0 is a pole of *order* n. A pole of order 1 is said to be *simple*. We then get that:

Proposition 7.3.7. *If f has a pole of order n at z_0, we can write*

$$f(z) = \frac{1}{(z - z_0)^n} h(z)$$

on $D_r(z_0) \setminus \{z_0\}$ with h a non-vanishing holomorphic function $D_r(z_0) \to \mathbb{C}$.

Writing $h(z) = \sum_{n=0}^{\infty} b_n (z - z_0)^n$, we get

$$f(z) = \frac{-a_{-n}}{(z - z_0)^n} + \cdots + \frac{a_{-1}}{z - z_0} + H(z),$$

where $a_{-n+i} = b_i$ and $H\colon D_r(z_0) \to \Omega$ is holomorphic.

The terms

$$\frac{-a_{-n}}{(z - z_0)^n} + \cdots + \frac{a_{-1}}{z - z_0}$$

are the *principal part*, and a_{-1} is the *residue* of f at z_0. Let us record this more prominently:

$$\operatorname{res}_{z_0} f := a_{-1}.$$

Example 7.3.8. We can compute the residue as

$$\operatorname{res}_{z_0} f = \lim_{z \to z_0} \frac{1}{(n-1)!} \frac{d^{n-1}}{dz^{n-1}} (z - z_0)^n f(z).$$

7.4 The residue formula

Recall that if $f\colon \Omega \to \mathbb{C}$ is holomorphic and $C \subset \Omega$ is a circle whose interior is also contained in Ω, then we have

$$\int_C f(z) dz = 0.$$

Next suppose that it is the interior of C is also contained in Ω *with the exception of a pole z_0*. Then we have the following:

Theorem 7.4.1. *Let f, C be as above, then*

$$\int_C f(z)dz = 2\pi i \operatorname{res}_{z_0} f.$$

More generally, if there are k poles $z_1, \ldots, z_k$ in the interior of C we have (which includes the case $k = 0$):

Corollary 7.4.2. *Let f, C be as above, then*

$$\int_C f(z)dz = 2\pi i \sum_{j=1}^{k} \operatorname{res}_{z_i} f.$$

More generally, we can replace C by your favorite toy contour; rectangles, keyholes, indented semicircles, etc.

Chapter 8

Singularities

We finish the classification of singularities, discussing removable and essential singularities. This is Section 3 of Chapter 3 of [SS03].

8.1 Removable singularities

Suppose that $f\colon \Omega \to \mathbb{C}$ has an isolated singularity at z_0, i.e. $D_r(z_0) \setminus \{z_0\} \subset \Omega$.

Definition 8.1.1. We say a singularity z_0 of f is a *removable singularity* if we can extend f to a holomorphic function $\Omega \cup \{z_0\}$.

Such an extension is unique if it is exists, since $f(z_0)$ is determined by $f(z)$ as $z \to z_0$ by continuity.

Theorem 8.1.2. *$|f(z)|$ is bounded as $z \to z_0$ if and only if z_0 is a removable singularity.*

Proof. The direction $\Leftarrow$ follows from continuity. For the direction $\Rightarrow$, we take a circle $C \subset \Omega \cup \{z_0\}$ centered at z_0. It suffices to extend f to the interior of C. We will prove that

$$\mathrm{int}(C) \ni z \longmapsto \frac{1}{2\pi i}\int_C \frac{f(\zeta)}{z-\zeta}d\zeta$$

is holomorphic and agrees with f on $\mathrm{int}(C) \setminus \{z_0\}$.

That it is holomorphic follows from a previous theorem we had about defining holomorphic functions by integrals, applied to the integral

$$z \mapsto \frac{1}{2\pi i}\int_0^{2\pi} \frac{f(z_0+\epsilon e^{i\theta})}{z-z_0-\epsilon e^{i\theta}} i\epsilon e^{i\theta} d\theta$$

obtained by parametrizing the circles.

To prove this integral agrees with f at $z \neq z_0$ in the interior of C, we use a "double keyhole contour" to get that

$$\frac{1}{2\pi i}\int_C \frac{f(\zeta)}{z-\zeta}d\zeta = \frac{1}{2\pi i}\int_{C_\delta} \frac{f(\zeta)}{z-\zeta}d\zeta + \frac{1}{2\pi i}\int_{C'_\delta} \frac{f(\zeta)}{z-\zeta}d\zeta$$

where C_δ is a tiny circle of radius $\delta > 0$ around z and C'_δ is a tiny circle of radius $\delta > 0$ around z_0. On the one hand, by the Cauchy integral formula, the first term is $f(z)$. On the other hand, we can estimate the second term as

$$\leq \operatorname{length}(C'_\delta) \sup_{z \in C'_\delta} 2\pi \left| \frac{f(\zeta)}{z - \zeta} \right|.$$

Since f is bounded by $z \to z_0$ and ζ stays away from z_0, the term within $| - |$ is bounded as $z \to z_0$. Hence we can estimate the right hand side as $\leq 2\pi\delta C$, which goes to 0 as $\delta \to 0$. □

We can use this to characterize poles:

Corollary 8.1.3. *$|f(z)| \to \infty$ as $z \to z_0$ if and only if z_0 is a pole.*

Proof. For direction $\Leftarrow$, recall that if f has a pole as z_0, then $1/f$ has a zero at z_0. Thus $1/|f(z)| \to 0$ as $z \to z_0$ which is equivalent to $|f(z)| \to \infty$ as $z \to z_0$.

For the direction $\Rightarrow$, observe that $1/f(z)$ will be bounded near z_0. Hence the singularity is removable, and by continuity the function f extends to have a zero at z_0. □

8.2 Essential singularities

It remains to discuss the "garbage" case of essential singularities.

Definition 8.2.1. We say a singularity z_0 of f is an *essential singularity* if it is neither removable nor a pole.

The surprise is that we can still say anything about this case:

Theorem 8.2.2 (Casorati–Weierstrass)**.** *If $f\colon D_r(z_0) \setminus \{z_0\} \to \mathbb{C}$ is holomorphic with an essential singularity at z_0, then $f(D_r(z_0) \setminus \{z_0\})$ is dense in $\mathbb{C}$.*

Proof. We give a proof by contradiction. If the image is not dense, then there exists a $w \in \mathbb{C}$ and $\delta > 0$ such that $|f(z) - w| > \delta$ for all $z \in D_r(z_0) \setminus z_0$. Then the function

$$z \longmapsto \frac{1}{f(z) - w}$$

is holomorphic and bounded by $1/\delta$ as $z \to z_0$. Hence its singularity at z_0 is removable. If it extends to have non-zero value at z_0, then z_0 was a removable singularity of f, which is ruled out by the hypothesis. If it extends to have a zero, then z_0 was a pole of f, which is ruled out by the hypothesis. □

8.2.1 Application: the automorphisms of $\mathbb{C}$

Theorem 8.2.3. *If $f : \mathbb{C} \to \mathbb{C}$ is a bijective holomorphic map with holomorphic inverse, then $f(z) = az + b$ for some $a, b \in \mathbb{C}$ with $a \neq 0$.*

This will follow from the following stronger result:

Proposition 8.2.4. *f $f : \mathbb{C} \to \mathbb{C}$ is an injective holomorphic map, then $f(z) = az + b$ for some $a, b \in \mathbb{C}$ with $a \neq 0$.*

The proof of this proposition in turn will rely on a result which we will prove in the next lecture:

Theorem 8.2.5 (Open mapping theorem)**.** *If Ω is connected and $f : \Omega \to \mathbb{C}$ is holomorphic and not constant, then f maps open sets to open sets.*

Proof of Proposition 8.2.4. The function $f(1/z)$ has a removable singularity, pole, or essential singularity at 0. We will prove it has a pole, and then deduce from this that f is a polynomial of degree 1.

First we rule out that it is a removable singularity. Since f is injective, it can't be bounded and hence $f(1/z)$ is not bounded as $z \to 0$.

Next we rule out that that it is an essential singularity. Write $g(z) = f(1/z)$, which is also injective. If g has an essential singularity at 0, consider the value $u \coloneqq g(1)$. By the open mapping theorem, $g(D_{1/2}(1))$ is open in $\mathbb{C}$, so contains $D_\delta(u)$ for some $\delta > 0$. Since g is injective, this implies that $g(D_{1/2}(0) \setminus \{0\})$ avoids $D_\delta(u)$. But this contradicts the Casorati–Weierstrass theorem applied to $D_{1/2}(0)$.

Hence $f(1/z)$ has a pole at 0. This means we can write it as

$$f(1/z) = \frac{a_{-n}}{z^n} + \cdots + \frac{a_{-1}}{z} + H(z)$$

near 0. This implies that

$$f(z) - (a_{-n} z^n + \cdots + a_{-1} z)$$

is bounded, and hence constant by Liouville's theorem. Thus $f(z)$ is a polynomial. We claim that the only injective polynomials are those of degree 1. For suppose that f is injective of degree $n > 1$. Then it must have a single zero of multiplicity n. Thus $f(z) = (z - z_0)^n$ and this takes the same value all points $z_0 + e^{2\pi i \cdot k/n}$ for $k = 0, 1, \ldots, n-1$. $\square$

8.3 Meromorphic functions and the complex projective plane

Removable singularities are uniquely removable, so we may safely assume our holomorphic functions don't have them. Essential singularities are hard, so it is reasonable to restrict our attention to holomorphic function that only have poles:

Definition 8.3.1. A holomorphic function $f : \mathbb{C} \setminus \{z_1, z_2, \ldots\} \to \mathbb{C}$ is said to be *meromorphic* if the set $\{z_1, z_2, \ldots\}$ is discrete and each singularity z_i of f is a pole.

By considering the singularity of $g(1/z)$ at the origin, we see that g either has a removable singularity, pole, or essential singurality *at infinity.* Again, we focus our attention on the case without essential singularities:

Definition 8.3.2. A meromorphic function is said to be *meromorphic in the extended complex plane* if it has a removable singurality or pole at infinity.

We can think of such a function in terms of complex manifolds. An f which is meromorphic in the extended complex plane extends to a function $\hat{f}\colon \mathbb{C}P^1 \to \mathbb{C}P^1$ on the *extended complex plane*

$$\mathbb{C}P^1 = \mathbb{C} \cup \{\infty\}$$

as follows:

- $\hat{f}(z_i) = \infty$,
- if f has a removable singularity at ∞, $\hat{f}(\infty) = \lim_{z\to\infty} f(z)$.
- if f has a pole at ∞, $\hat{f}(\infty) = \infty$.

The notation reflects that $\mathbb{C}P^1$ can also be though of as the complex projective plane, whose points complex lines through the origin in $\mathbb{C}^2$. It is homeomorphic to S^2 and can be covered by two copies of $\mathbb{C}$:

$$z \longmapsto z \qquad \text{and} \qquad z \longmapsto 1/z,$$

the latter with the understanding that $1/0 = \infty$. These are the charts making $\mathbb{C}P^1$ into a complex 1-dimensional manifold. A function $\mathbb{C}P^1 \to \mathbb{C}P^1$ is holomorphic if it is with respect to these coordinates:

Definition 8.3.3. A function $\hat{f}\colon \mathbb{C}P^1 \to \mathbb{C}P^1$ is *holomorphic* if its restrictions

- $z \mapsto \hat{f}(z)$ to $\hat{f}^{-1}(\mathbb{C}P^1 \setminus \{\infty\}) \setminus \{\infty\}$,
- $z \mapsto \hat{f}(1/z)$ to $\hat{f}^{-1}(\mathbb{C}P^1 \setminus \{\infty\}) \setminus \{0\}$,
- $z \mapsto 1/\hat{f}(z)$ to $\hat{f}^{-1}(\mathbb{C}P^1 \setminus \{0\}) \setminus \{\infty\}$,
- $z \mapsto 1/\hat{f}(1/z)$ to $\hat{f}^{-1}(\mathbb{C}P^1 \setminus \{0\}) \setminus \{0\}$,

are holomorphic.

Lemma 8.3.4. *If a function f is meromorphic on the extended complex plane, its extension $\hat{f}\colon \mathbb{C}P^1 \to \mathbb{C}P^1$ is holomorphic. Conversely, the restriction of a holomorphic map $\hat{f}\colon \mathbb{C}P^1 \to \mathbb{C}P^1$ to $\hat{f}^{-1}(\mathbb{C}) \setminus \{\infty\}$ is meromorphic on the extended complex plane.*

Since $\mathbb{C}P^1$ is compact and holomorphic functions are quite rigid, you might imagine there are not many functions which are meromorphic in the extended complex plane:

Theorem 8.3.5. *If f is a meromorphic in the extended complex plane then it is a rational function, i.e. a quotient of two polynomials.*

Proof. Near each pole $z_i \in \mathbb{C}$, we can write f as

$$f(z) = f_k(z) + g_k(z)$$

where $f_k(z)$ is the principal part, a polynomial in $1/(z - z_i)$, and g_k is holomorphic near z_i. Similarly, around ∞ we can write

$$f(1/z) = \tilde{f}_\infty(z) + \tilde{g}_\infty(z)$$

where $\tilde{f}_\infty(z)$ is the principal part, a polynomial in $1/z$, and $\tilde{g}_\infty$ is holomorphic near the origin. Now set $f_\infty(z) = \tilde{f}_\infty(1/z)$, a polynomial in z, and $g_\infty(z) = \tilde{g}_\infty(1/z)$

Observe that since f is holomorphic near ∞, it can only have finitely many poles in $\mathbb{C}$. Thus the function

$$F = f - f_\infty - \sum_{k=1}^{n} f_k$$

is holomorphic. By constructing it only has removable singularities, both at the z_i's and ∞. Thus we can extend it to a holomorphic $\mathbb{C} \to \mathbb{C}$ which is bounded. By Liouville it is constant and we conclude that

$$f = \text{constant} + f_\infty + \sum_{k=1}^{n} f_k.$$

Thus f is a sum of a polynomial and finitely many polynomials in $1/(z-z_i)$. By combining denominators, we see it is rational. ☐

Corollary 8.3.6. *Up to a multiplicative constant, functions which are meromorphic in the extended complex plane are determined by their zeroes and poles (with multiplicity).*

8.3.1 Application: the automorphisms of $\mathbb{C}P^1$

We saw above that the automorphisms of $\mathbb{C}$ are given by $z \mapsto az + b$ for $a, b \in \mathbb{C}$ and $a \neq 0$. The extended complex plane has more automorphisms, because we can move ∞.

Theorem 8.3.7. *If $\hat{f} \colon \mathbb{C}P^1 \to \mathbb{C}P^1$ is a bijective holomorphic map with holomorphic inverse, then $f(z) = \frac{az+b}{cz+d}$ for some $a, b, c, d \in \mathbb{C}$ with $ad - bc \neq 0$.*

The maps $z \mapsto \frac{az+b}{cz+d}$ are called *Möbius transformations.* If only depends on a, b, c, d up to rescaling by a non-zero complex numbers.

If $ad = bc$, then it is constant as

$$\frac{az + b}{cz + d} = \frac{a(cz + d)}{c(cz + d)} - \frac{ad - bc}{c(cz - d)} = \frac{a}{c}.$$

If we record the coefficients as a (2×2)-matrix

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

then the composition of $\frac{az+b}{cz+d}$ with $\frac{a'z+b'}{c'z+d'}$ has coefficients given by the matrix multiplication

$$\begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

The conclusion is that the Möbius transformation form a group under composition, isomorphic to the group $\mathrm{PGL}_2(\mathbb{C})$ of invertible (2×2)-matrices with complex entries up to rescaling.

Proof of Theorem 8.3.7. Suppose $\hat{f}(\infty) \neq \infty$. Then $\hat{f}(\infty) = x$ for some $x \in \mathbb{C}$. The Möbius transformation g given by $z \mapsto \frac{1}{z-x}$ sends x to ∞, so $g \circ \hat{f}$ sends ∞ to ∞. Thus it restricts to an automorphism of $\mathbb{C}$ and hence must be a Möbius transformation h given by $z \mapsto az + b$. This means that $\hat{f}$ must have been the Möbius transformation $g^{-1} \circ h$. $\square$

Remark 8.3.8. This theorem says that the automorphisms of $\mathbb{C}P^1$ as a 1-dimensional complex manifold are given by a Lie group. This is in contrast with the automorphisms of it as a 2-dimensional smooth manifold; the diffeomorphisms of S^2 form a infinite-dimensional Lie group.

Chapter 9

The argument principle

The argument principle is a technique to count zeroes and poles of a meromorphic function within a circle. We then give a number of applications. This is Section 4 of Chapter 3 of [SS03].

9.1 The argument principle

Suppose we want to recover whether f has a zero at z_0, and what multiplicity it has, using an integral. If the multiplicity is n, then we can write

$$f(z) = (z - z_0)^n g(z)$$

with g non-vanishing near z_0. Then we have

$$\frac{f'(z)}{f(z)} = \frac{n}{z - z_0} + \frac{g'(z)}{g(z)},$$

where the term $g'(z)/g(z)$ is holomorphic near z_0. In other words, it has a pole at z_0 with residue n.

Remark 9.1.1. This is an example of a general phenomena:

$$\frac{(f_1 f_2)'}{f_1 f_2} = \frac{f_1' f_2 + f_1 f_2'}{f_1 f_2} = \frac{f_1'}{f_1} + \frac{f_2'}{f_2}.$$

Similarly, if f has a pole of order n near z_0 then we can write

$$f(z) = (z - z_0)^{-n} h(z)$$

and thus

$$\frac{f'(z)}{f(z)} = \frac{-n}{z - z_0} + \frac{h'(z)}{h(z)}.$$

In other words, it has a pole at z_0 with residue $-n$.

Theorem 9.1.2 (Argument principle). *Suppose that f is meromorphic on Ω containing a circle C and its interior. If on C, f has neither zeroes nor poles, then*

$$\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}dz = \#\{\textit{zeroes of } f \textit{ in } C\} - \#\{\textit{zeroes of } f \textit{ in } C\},$$

where the zeroes and poles are counted with multiplicity.

Proof. Apply the residue theorem and the above observation that $f'(z)/f(z)$ has a pole with residue n at each zero of f of multiplicity n, and a pole with residue $-n$ at each pole of f with order n. □

A similar result holds for toy contours. The argument principle says that contour integrals of f'/f can be used to count zeroes and poles. We will three applications of this idea.

9.2 Rouché's theorem

The first concerns the relationship of the zeroes and poles of f to those of small pertubations of f.

Theorem 9.2.1 (Rouché). *Suppose that f and g are holomorphic on Ω containing a circle and its interior. Then if*

$$|f(z)| > |g(z)| \textit{ for all } z \in C,$$

both f and $f+g$ have the same number of zeroes inside C.

Proof. Consider the family of holomorphic functions given by

$$f_t(z) = f(z) + tg(z) \text{ with } t \in [0,1].$$

Let n_t denote the number of zeroes of f_t within C; the hypothesis tells us f_t has no zeroes on C, so this is given by

$$n_t = \frac{1}{2\pi i}\int_C \frac{f_t'(z)}{f_t(z)}dz.$$

This is an integer-valued function depending continuously on t, so must be constant. We conclude that $n_0 = n_1$. □

The same is true for toy contours.

Example 9.2.2. We will prove that the function $f(z) = z + 2 - e^z$ has exactly one zero in the half-plane $\{z \in \mathbb{C} \mid \mathrm{Im}(z) < 0\}$. To do so, it suffices to prove that it has one zero in a sufficiently large semicircle S_R in this half plane, of radius R centered at the origin 0.

Since $z+2$ has a unique zero in the interior of this semicircle when $R > 1$, we can apply Rouché to this contour with $f(z) = z+2$ and $g(z) = -e^z$. It then suffices to observe that

$$|z+2| \geq 2 > 1 \geq |-e^z| = e^{-\mathrm{Re}(z)}.$$

9.3 The open mapping theorem

Recall that a function $f\colon \Omega \to \mathbb{C}$ is *open* if it sends open subsets to open subsets.

Theorem 9.3.1. *If Ω is connected and $f\colon \Omega \to \mathbb{C}$ is holomorphic and non-constant, then f is open.*

If we didn't assume Ω was connected, we just need f to be non-constant on all connected components.

Proof. By restricting f to an open subset of interest, it suffices to prove that $f(\Omega)$ is open. Suppose $w_0 = f(z_0)$, then we must prove that all points w near w_0 must also be in the image of f.

Consider the functoin $f(z) - w$. We want to use Rouché to prove that this has a zero. To do so, we write

$$f(z) - w = (f(z) - w_0) + (w_0 - w) = F(z) + G(z).$$

Since f is not constant, we know that $f(z) = w_0 + a_n(z - z_0)^n(1 + h(z))$ for some n with $a_n \neq 0$ and $h(z_0) = 0$s. That is, $F(z) = a_n(z - z_0)^n(1 + h(z))$. In particular, $F(z) \neq 0$ on the circle C_δ given by $|z - z_0| = \delta$ for $\delta > 0$ sufficiently small. Take $\delta > 0$ small enough so that C_δ and its interior are in Ω. Since C_δ is compact, there is an $\epsilon > 0$ such that $|F(z)| > \epsilon$ on C_δ.

Then the hypothesis for Rouché's theorem apply whenever $|G(z)| = |w_0 - w| < \epsilon$. Thus for $|w_0 - w| < \epsilon$ we have that $F(z) = f(z_0) - w_0$ and $F(z) + G(z) = f(z) - w$ have the same number of zeroes within C_δ. As $F(z)$ has at least one, so does $F(z) + G(z)$. We conclude from this that w is in the image of f. □

Last lecture we saw an application of the open mapping theorem: the classification of injective holomorphic functions $f\colon \mathbb{C} \to \mathbb{C}$; these are polynomials of degree 1.

9.4 The maximum modulus principle

The final application asks whether $|f|$ can have local maxima:

Theorem 9.4.1 (Maximum modulus principle)**.** *If Ω is connected and $f\colon \Omega \to \mathbb{C}$ is holomorphic and non-constant, then $|f|$ can't attain a maximum in Ω.*

Proof. If it did, this would violate the open mapping theorem. □

Corollary 9.4.2. *If Ω is connected with compact closure $\overline{\Omega}$ and $f\colon \overline{\Omega} \to \mathbb{C}$ is continuous and holomorphic in the interior Ω, then*

$$\max_{z \in \overline{\Omega}} |f(z)| \leq \max_{z \in \partial\Omega} |f(z)|.$$

Proof. Since $\overline{\Omega}$ is compact, the continuous function $|f|$ attains a maximum somewhere in $\overline{\Omega}$. The maximum modulus principle rules out that this occurs in the interior. □

Example 9.4.3. Suppose Ω contains the closed unit disk, and that $|f(z)| = 1$ when $|z| = 1$. By the maximum modulus principle $|f(z)| < 1$ when $|z| < 1$.

We claim that f must have a zero in the unit disk. If it didn't, then $1/f$ would be holomorphic on a neighborhood of the unit disk and since $|f(z)| < 1$ when $|z| < 1$ would violate the maximum modulus principle.

Now recall for $|w| < 1$ the Blaschke factor

$$\beta_w \colon z \longmapsto \frac{w - z}{1 - \bar{w}z}.$$

It is holomorphic near the closed unit disk, preserves the unit circle, interchanges 0 and w, and satisfies $\beta_w \circ \beta_w = \mathrm{id}$. By applying the above argument to $\beta_w \circ f$, we see that $\beta_w \circ f$ must have a zero, and hence $\beta_w \circ \beta_w \circ f = f$ must take the value w on the unit disk.

By letting w vary, we conclude the following result: each f as above has the entire closed unit disk in its image.

Chapter 10

Homotopy invariance and the logarithm

We finally finish the proof that integrals of holomorphic functions along paths are homotopy-invariant. This implies that holomorphic functions have primitives on simply-connected subsets. We then study the logarithm and its primitives. This is Sections 5 and 6 of Chapter 3 of [SS03].

10.1 Integrals along paths are homotopy-invariant

We now give the most general form of Cauchy's theorem, which dispenses with the notions of toy contours. Recall that Cauchy's theorem says that under certain conditions

$$\int_C f(z)dz = 0,$$

and that this is certainly the case if f admits a primitive on a neighborhood of C. Thus finding the most general form of Cauchy's theorem is closely related to the finding the most general statement about the existence of primitives.

When we defined primitives in previous proofs, we did so by integrating along piecewise-linear paths. By Goursat's theorem this was independent of the choice of piecewise-linear path. To define primitives more generally, we need a more general condition under which integrals are independent of the chosen path.

Definition 10.1.1. Two (piecewise smooth) curves $\gamma_0, \gamma_1 \colon [a,b] \to \Omega$ with $\gamma_0(a) = \alpha = \gamma_1(a)$ and $\gamma_0(b) = \beta = \gamma_0(b)$ are said to be *homotopic rel endpoints* if there exists a continuous map $G\colon [a,b] \times [0,1] \to \Omega$ such that each $\gamma_s \coloneqq G|_{[a,b]\times\{s\}}$ is piecewise smooth.

It is not hard to see that among paths parametrized by $[a,b]$ with the same endpoints, homotopy is an equivalence relation.

Theorem 10.1.2. *If $f\colon \Omega \to \mathbb{C}$ is holomorphic and γ_0, γ_1 are homotopic rel endpoints, then*

$$\int_{\gamma_0} f(z)dz = \int_{\gamma_1} f(z)dz.$$

Proof. The idea is to show that the integrals $\int_{\gamma_t} f(z)dz$ are all equal. Thus we only need to show that the values don't change if paths are close to each other. We will do so by writing the difference between them as a sum of integrals of holomorphic functions with a primitive along closed paths.

Since $[a,b] \times [0,1]$ is compact, so is its image K under G. Thus we can find an $\epsilon > 0$ such that for all $z \in K$, the disk $D_{3\epsilon}(z) \subset \Omega$. (For details of this statement in point-set topology, see [SS03, p. 94].) Again using that $[a,b] \times [0,1]$ is compact, we see that G is uniformly continuous. Thus there is a $\delta > 0$ such that $|\gamma_{s_1}(t) - \gamma_{s-2}(t)| < \epsilon$ for all $t \in [a,b]$ whenever $|s_1 - s_2| < \delta$. We will now prove that $\int_{\gamma_{s_1}} f(z)dz = \int_{\gamma_{s_2}} f(z)dz$. By covering $[0,1]$ with finitely intervals of size δ, this proves the theorem.

To do so, we choose disks $D_0, \ldots, D_n$ of radius ϵ, points $z_0, \ldots, z_{n+1}$ on γ_{s_1}, and points $w_0, \ldots, w_{n+1}$ on γ_{s_2}, so that the disks cover both curves and $z_i, z_{i+1}, w_i, w_{i+1} \in D_i$. We may assume $z_0 = \alpha = w_0$ and $z_{n+1} = \beta = w_{n+1}$. The conditions on the radii of the D_i imply that $D_i \subset \Omega$.

Now we recall that f has a primitive F_i on D_i. Since primitives differ by a constant, $F_{i+1}(z_i) - F_i(z_i) = F_{i+1}(w_i) - F_i(w_i)$. This implies

$$F_{i+1}(z_{i+1}) - F_{i+1}(w_{i+1}) = F_i(z_i) - F_i(w_i). \tag{10.1}$$

Now we have that

$$\begin{aligned}
\int_{\gamma_{s_1}} f(z)dz - \int_{\gamma_{s_2}} f(z)dz &= \sum_{i=0}^{n} (F_i(z_{i+1}) - F_i(z_i)) - \sum_{i=0}^{n} (F_i(w_{i+1}) - F_i(w_i)) \\
&= \sum_{i=0}^{n} (F_{i+1}(z_{i+1}) - F_{i+1}(w_{i+1})) - (F_i(z_i) - F_i(w_i)) \\
&= (F_n(z_{n+1}) - F_n(w_{n+1})) - (F_0(z_0) - F_0(w_0)) \\
&= (\beta - \beta) - (\alpha - \alpha) = 0.
\end{aligned}$$

The third equality used a telescoping of most of the sum, using (10.1). $\square$

Corollary 10.1.3. *If a closed path γ is homotopic to a constant path, then*

$$\int_\gamma f(z)dz = 0.$$

Definition 10.1.4. $\Omega \subset \mathbb{C}$ is *simply-connected* is if it is connected and any pair of paths with the same endpoints is homotopic.

Remark 10.1.5. Taking the endpoints to be equal, we see that on simply-connected Ω every closed curve is homotopic to a constant path. Conversely, if Ω is connected and has the property that every closed curve is homotopic through closed paths to a constant path, then Ω is simply-connected.

Example 10.1.6. An open disk is simply-connected by linear interpolation. More generally, every open convex set is.

Example 10.1.7. The split plane $\mathbb{C} \setminus \{(-\infty, 0]\}$ is simply-connected. Here one can linearly interpolate the radius and argument independently.

Example 10.1.8. The punctured plane $\mathbb{C} \setminus \{0\}$ is not simply-connected. For example, going from 1 to -1 across the upper-half plane is not homotopic to going from 1 to -1 across the bottom half-plane. One can prove this by integrating $1/z$ along semicircle and obtaining a different answer.

The use of this definition is the following:

Theorem 10.1.9. *A holomorphic function with simply-connected domain Ω has a primitive.*

Proof. We follow the proof of the existence of a primitive on a disk. It starts by defining the primitive: we fix a point $z_0 \in \Omega$ and we define $F \colon \Omega \to \mathbb{C}$ by

$$F(z) := \int_{\gamma_{z_0,w}} f(w)dz,$$

where $\gamma_{z_0,w}$ is a path from z_0 to w. Such a path exists since being connected implies path-connected, and the value is independent of the path by Theorem 10.1.2.

To check this is a primitive, we need to verify it is holomorphic with derivative f. However, if we take h such that $z+h$ is in a disk around $z \in \Omega$ which is also contained in Ω, then the concatenation of $\gamma_{z_0,z}$, the straight line segment η from z to $z+h$ and $\gamma_{z_0,z+h}$ run in opposite direction, is a closed curve. This is homotopic to a constant path, hence the integral over it vanishes. We conclude that

$$F(z+h) - F(z) = \int_\eta f(w)dw.$$

The proof now continues as in the case of a disk to show that

$$\lim_{h \to 0} \frac{F(z+h) - F(z)}{h} = f(z),$$

proving that F is holomorphic and $F' = f$. $\square$

Corollary 10.1.10. *If f is holomorphic with simply-connected domain Ω, then for any closed path γ we have*

$$\int_\gamma f(z)dz = 0.$$

Remark 10.1.11. This also follows by combining Corollary 10.1.3 with the definition of simply-connectedness.

10.2 The logarithm

We have long ignored the logarithm, as its usual definition as the primitive of $1/z$ does not make sense: its domain $\mathbb{C} \setminus \{0\}$ is not simply-connected, and as $\int_{C_1} 1/z dz = 2\pi i$ with C_1 the circle of radius 1 around the origin, $1/z$ can't have a primitive on $\mathbb{C} \setminus \{0\}$. Our solution will be to restrict the domain.

More down-to-earth, giving $z = re^{i\theta}$ it seems reasonable to define

$$\log(z) = \log(r) + i\theta.$$

But then one realizes that θ is only well-defined up to addition of 2π. If we try to fix a choice, we realize as one passes around the origin counterclockwise 2π is added to θ. As we don't want to lose continuity by restricting the imaginary part to $[0, 2\pi)$, the solution is again to restrict the domain so we can't go around the origin.

What domain Ω should we restrict to? The previous section tells us it should suffice to take Ω simply-connected.

Theorem 10.2.1. *If Ω is simply-connected and satisfies $1 \in \Omega$ and $0 \notin \Omega$, then there is a holomorphic function $\log_\Omega \colon \Omega \to \mathbb{C}$ such that (i) $e^{\log_\Omega(z)} = z$, and (ii) for r real near 1, $\log_\Omega(r)$ is the usual real-valued logarithm $\log(r)$.*

Proof. The function $\log_\Omega$ is the primitive of $1/z$ with the property that $\log_\Omega(1) = 1$. This exists by the previous section.

To check (i) holds, we should prove that $ze^{-\log_\Omega(z)} = 1$. By constructing this is true at $z = 1$, and since Ω is connected it suffices to prove the derivative of $ze^{-\log_\Omega(z)}$ vanishes. But this is

$$\frac{d}{dz}\left(ze^{-\log_\Omega(z)}\right) = e^{-\log_\Omega(z)} + z\frac{1}{z}e^{-\log_\Omega(z)} = 0.$$

To check (ii) holds, we use that the primitive of $1/z$ is constructed by integrals along paths. Near 1 on the real axis, we can take a path that moves over the real axis and get the ordinary logarithm as a result. $\square$

The argument also works when we drop the condition that $1 \in \Omega$, though we can't state (ii) anymore. However, we can instead decide to fix the value of the logarithm near any other point $z_0 \in \Omega$.

Example 10.2.2. Taking $\Omega = \mathbb{C} \setminus \{z \mid \mathrm{Re}(z) \geq 0\}$, we get the *principal branch* of the logarithm and we drop Ω from the notation. We claim that for $z = re^{i\theta}$ with $-\pi < \theta < \pi$ this is given by

$$\log(z) = \log(r) + i\theta.$$

This is proven by integrating for 1 along the path in Figure 8 of [SS03]. It has a Taylor series expansion given as follows:

$$\log(1 + z) = \sum_{n=1}^{\infty} (-1)^n \frac{z^n}{n} \qquad \text{for } |z| < 1.$$

To prove this, observe it agrees with the principal branch on the positive real axis and has the same derivative.

Note that $\log(z_1 z_2) \neq \log(z_1) + \log(z_2)$; this is impossible when demanding the imaginary parts lies between $-\pi$ and π. Indeed, if the value of the addition has imaginary part exceeding πi, we ought to subtract $2\pi i$.

In keeping with the previous example, we refer to the $\log_\Omega$ as other *branches* of the logarithm.

Example 10.2.3. We can now define z^α when $\alpha \in \mathbb{C}$ and $z \notin \{z \in \mathbb{C} \mid \mathrm{Re}(z) \leq 0\}$ by taking $z^\alpha = e^{\alpha \log(z)}$. This uses the principal branch. If we want to define this for z with $\mathrm{Re}(z) \leq 0$ we need to pick a logarithm on a different simply-connected Ω avoiding 0.

10.2.1 Logarithms of holomorphic functions

Using Theorem 10.2.1 we can also take logarithms of functions.

Theorem 10.2.4. *If f is a non-vanishing holomorphic function with simply-connected domain Ω, then there exists a holomorphic function $g\colon \Omega \to \mathbb{C}$ such that $f(z) = e^{g(z)}$. The function g is unique up to addition of $2\pi i$.*

Proof. Take g to be a primitive of $\frac{f'}{f}$ with constant chosen such that $f(z)e^{-g(z)} = 1$. Then we prove as before that $\frac{d}{dz}(f(z)e^{-g(z)}) = 0$ to conclude to that $f(z) = e^{g(z)}$. $\square$

Chapter 11

Pre-midterm recap

Today we do a recap of the material covered by the midterm exam. It covers Chapters 1-3 of [SS03] except Sections 2.5.5 and 3.5, 3.6, and 3.7.

11.1 Holomorphic functions and their properties

The objects of interest to this course are holomorphic functions, which are maps $f\colon \mathbb{C} \supset \Omega \to \mathbb{C}$ with $\Omega \subset \mathbb{C}$ open, which are everywhere complex-differentiable. This means that for all $z \in \Omega$ there exists a $f'(z) \in \mathbb{C}$ such that

$$\lim_{h \to 0} \frac{f(z+h) - f(z)}{h} = f'(z).$$

Using the real and imaginary parts to consider f as a function $\mathbb{R}^2 \supset \Omega \to \mathbb{R}^2$, it is not just differentiable but its derivatives satisfy the Cauchy–Riemann equations.

This condition is much stronger than you might suspect at first, as we proved it has the following consequences:

- Every holomorphic function is infinitely many times complex-differentiable.
- Every holomorphic function is analytic, that is, for z near $z_0 \in \Omega$ a holomorphic function f is equal to its Taylor expansion

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \qquad \text{with } a_n = \frac{1}{n!} \frac{d^n f}{dz^n}(z_0).$$

- The integral of a holomorphic function along a path only depends on its homotopy class rel endpoints. That is, if we have a continuous map $G\colon [a,b] \times [0,1] \to \Omega$ such that $G|_{[a,b]\times\{t\}}$ is piecewise-smooth for all $t \in [0,1]$ then

$$\int_{\gamma_0} f(z)dz = \int_{\gamma_1} f(z)dz.$$

  As a consequence, integrals along closed paths that can be contracted to a point vanish. For example, if $C \subset \Omega$ is a circle whose interior is also contained in Ω, then $\int_C f(z)dz = 0$ (*Cauchy's theorem*).

- Holomorphic functions often extend uniquely to larger domains. A concrete instance of this is the result that if $\Omega \subset \Omega'$ are path-connected and $g\colon \Omega' \to \mathbb{C}$ extends $f\colon \Omega \to \mathbb{C}$, then g is unique.

  Another instance of this is the *Schwarz reflection principle*, which says that if Ω is symmetric with respect to complex conjugation, and $f\colon \Omega \cap \{z \mid \mathrm{Im}(z) \geq 0\} \to \mathbb{C}$ is continuous and holomorphic on $\Omega \cap \{z \mid \mathrm{Im}(z) > 0\}$, then f extends uniquely to Ω.

11.2 Constructing holomorphic functions

How do get holomorphic functions? We saw several ways of constructing them:

- We have verified some elementary functions are holomorphic on all of $\mathbb{C}$: polynomials, e^z, $\cos(z)$, $\sin(z)$. Some other elementary functions are holomorphic on subsets of $\mathbb{C}$: $1/z$ on $\mathbb{C} \setminus \{0\}$, and the principal branch $\log(z)$ of the logarithm on $\mathbb{C} \setminus \{z \mid \mathrm{Re}(z) \leq 0\}$.
- Holomorphic functions are closed under the usual constructions: addition, scaling, multiplication, dividing, composition.
- If a sum of holomorphic functions converge uniformly on compacts, the result is again holomorphic. Taylor series are a special case of this, and the construction of e^z is a particular case of this.

  This is a special case of a limit of holomorphic functions which converges uniformly on compacts having a holomorphic limit. That result not only applies the above result about sums, but also to integrals: if $F\colon \Omega \times [a,b] \to \mathbb{C}$ is continuous and each $F|_{\Omega \times \{s\}}$ is holomorphic, then $z \mapsto \int_a^b F(z,s)ds$ is holomorphic.

11.3 The classification of singularities

Maybe the most important example an complex analysis is the function $1/z\colon \mathbb{C}\setminus\{0\} \to \mathbb{C}$. It is the prototypical example of a function with a pol, the most interesting type of singularity that a holomorphic function can have. We say that $f\colon \Omega \to \mathbb{C}$ has a *singularity at* z_0 if $D_\epsilon(z_0) \subset \Omega$ but $z_0 \notin \Omega$. That is, f is not defined at z_0 but is defined at all nearby points. We proved there are three types of singularities:

(i) *removable singularities*: this is the case where you can extend f holomorphically over z_0, and is characterized by f being bounded near z_0. The prototypical example is the singularity at 0 of the constant function $1\colon \mathbb{C} \setminus \{0\} \to \mathbb{C}$.

(ii) *poles*: this is the case where f is given by

$$f(z) = \frac{a_{-n}}{(z-z_0)^n} + \cdots + \frac{a_{-1}}{z-z_0} + g(z) \qquad \text{with } a_n \neq 0,$$

with $g\colon \Omega \cup \{z_0\} \to \mathbb{C}$ holomorphic. Here n is the *order* of the pole, and a_{-1} is the *residue* of f at z_0.

(iii) *essential singularities*: these are the singularities which are not removable or poles. The prototypical example is the singularity at 0 of $e^{1/z}\colon \mathbb{C}\setminus\{0\} \to \mathbb{C}$. The Casorati–Weierstrass theorem says that the image under f of a small disk around an essential singularity is dense.

11.4 Evaluation of integrals

The classical application of complex analysis is the evaluation of integrals. This is done through the residue theorem, which has Cauchy's theorem as a special case: if $f\colon \Omega \to \mathbb{C}$ and γ is a toy contour in Ω such that f is defined on its interior except at some poles $z_1, \ldots, z_n$, then we have

$$\int_\gamma f(z)dz = 2\pi i \sum_{i=1}^n \mathrm{res}_{z_i}(f).$$

Usually, γ will depend on some parameter R that we will want to take infinity, in which case some parts of the paths won't contribute. It's best to do an example.

Example 11.4.1. Let us evaluate

$$\int_{-\infty}^{\infty} \frac{x^2}{1+x^4}dx.$$

We will apply the residue theorem to the function $f(z) = \frac{z^2}{1+z^4}$, integrated over the toy contour given by the path $[-R, R] \subset \mathbb{R}$ closed up by a semi-circle in the upper half plane. We will assume $R > 1$ and denote this S_R.

The poles of $f(z)$ are at the roots of unity

$$z_1 = e^{\pi/4i}, \quad z_2 = e^{3\pi/4i}, \quad z_3 = e^{5\pi/4i}, \quad z_4 = e^{7\pi/4i}$$

and S_R encircles the first two of these. The residue at one of these points is given by writing

$$\frac{z^2}{1+z^4} = \frac{(z-z_i)^2 + 2z_i(z-z_i) + z_i^2}{(z-z_i)\prod_{j\neq i}(z-z_j)}.$$

We conclude that the residue at z_i is

$$\frac{z_i^2}{\prod_{j\neq i}(z_i - z_j).}$$

To evaluate this, we observe that $\prod_{j\neq i}(z_i - z_j)$ is what you get when you evaluate the derivative of $1+z^4$ at z_i, so it is $4z_i^3$. Thus the residue is $\frac{1}{4z_i}$.

The conclusion is that

$$\int_{S_R} \frac{z^2}{1+z^4}dz = \frac{2\pi i}{4}(e^{-\pi/4i} + e^{-3\pi/4i}) = \frac{\pi}{\sqrt{2}}$$

for any $R > 1$.

We want to let $R \to \infty$, and verify that the integral over the semicircle then vanishes. Indeed, its absolute value can be estimate by

$$\sup_{z\in\text{semicircle}} \left|\frac{z^2}{1+z^4}\right| \cdot \text{length(semicircle)} \leq \frac{R^2}{1+R^4}\pi R \to 0 \qquad \text{as } R \to \infty.$$

We conclude that

$$\int_{-\infty}^{\infty} \frac{x^2}{1+x^4}dx = \frac{\pi}{\sqrt{2}}.$$

This shows the importance of being able to find poles and residues. This is not discussed systematically in [SS03], instead you can look at [MH87, Section 4.1].

11.5 Integral formula's and growth behavior of holomorphic functions

The important properties of holomorphic functions given above were proven by integral formula's. The most important of this is the Cauchy integral formula:

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_C \frac{f(\zeta)}{(\zeta - z)^{n+1}} d\zeta,$$

with $C \subset \Omega$ a circle whose interiors is also contained in Ω and contains z. (You should try to deduce this from the Taylor expansion and the residue theorem.)

This is the basis of an important estimate: if C is a circle as above, of radius R centered at z, then

$$|f^{(n)}(z)| \leq \frac{n!}{R^n} \cdot \sup_{\zeta \in C} |f(\zeta)|.$$

Applied to a bounded function defined on all of $\mathbb{C}$, this says that $|f'(z)| \leq C/R$. Letting $R \to \infty$ we deduce that $f'(z) = 0$ so f is constant. This is Liouville's theorem.

Theorems with a similar flavor are the argument principle and maximum modules principle:

- The *argument principle* says that if γ is a toy contour in Ω such that f is defined on its interior except at some poles $z_1, \ldots, z_n$, then if f has no zeroes on γ we have

  $$\frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)} dz = \#\text{zeroes of } f \text{ in } \gamma - \#\text{poles of } f \text{ in } \gamma,$$

  where zeroes and poles are too be counted with multiplicity, resp. order.

  This implies *Rouché's theorem*, which says that a sufficiently small perturbation of a holomorphic function doesn't change the number of zeroes it has within a toy contour.

- The *maximum modulus principle* says that if f is holomorphic, then $|f|$ can't have local maxima unless it is locally constant.

  This implies the *open mapping theorem*, which says that f takes open subsets to open subsets unless it is locally constant.

Chapter 12

Growth of zeroes

Today we start the next part of the course: the study of holomorphic functions whose domain is all of the complex plane $\mathbb{C}$. We will answer three questions about such functions:

(1) What do the sets of zeroes of such functions look like?

(2) How do these functions grow as $|z| \to \infty$?

(3) To what extent are such functions determined by their zeroes?

This is Sections 5.1 and 5.2 of [SS03]. The application to polynomials is from [Mah60].

12.1 The mean value property

For later use, we give another expression for the Taylor series coefficients a_n in

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n.$$

Theorem 12.1.1. *If the Taylor series above converges on $D_R(z_0)$, we have*

$$a_n = \frac{1}{2\pi r^n} \int_0^{2\pi} f(z_0 + re^{i\theta}) e^{-in\theta} d\theta$$

for $n \geq 0$ and $0 < r < R$. If we have $n < 0$, the right hand side vanishes.

Proof. Recalling that $a_n = \frac{f^{(n)}(z_0)}{n!}$, we can use the Cauchy integral formula for the nth derivative to get

$$a_n = \frac{1}{2\pi i} \int_{C_r(z_0)} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} d\zeta.$$

Let us parametrize the circle as $\theta \mapsto z_0 + re^{i\theta}$. Then we get

$$\begin{aligned} a_n &= \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(z_0 + re^{i\theta})}{(z_0 + re^{i\theta} - z_0)^{n+1}} ire^{i\theta} d\theta \\ &= \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{i\theta}) r^{-(n+1)} e^{-i(n+1)\theta} re^{i\theta} d\theta \\ &= \frac{1}{2\pi r^n} \int_0^{2\pi} f(z_0 + re^{i\theta}) e^{-in\theta} d\theta. \end{aligned}$$

For $n < 0$, we get that the expression in the theorem is given by

$$\frac{1}{2\pi r^n}\int_0^{2\pi} f(z_0 + re^{i\theta})e^{-in\theta}d\theta = rac12\pi i\int_{C_r(z_0)} f(\zeta)(\zeta - z_0)^{-n-1}d\zeta$$

which vanishes by Cauchy's theorem when $n < 0$ or equivalently $-n-1 \geq 0$. □

Taking $n = 0$, we get:

Corollary 12.1.2 (Mean-value property)**.** *If $f\colon D_R(z_0) \to \mathbb{C}$ is holomorphic, then*

$$f(z_0) = \frac{1}{2\pi}\int_0^{2\pi} f(z_0 + re^{i\theta})d\theta$$

for $0 < r < R$.

We can deduce from this a mean-value property for harmonic functions by taking the real part, as every harmonic function is the real part of a holomorphic function.

12.2 Jensen's formula

As usual, D_R and C_R are the open disk and circle of radius $R > 0$ around the origin. Let $\Omega \subset \mathbb{C}$ contain $\overline{D_R}$ and $f\colon \Omega \to \mathbb{C}$ be holomorphic. If f is not constant and f has no zeroes on C_R, it can have only finite many zeroes in D_R: otherwise the set of zeroes would have an accumulation point in $\overline{D_R}$ and f would have to vanish on the connected components of Ω containing D_R.

Theorem 12.2.1 (Jensen's formula)**.** *Let $\Omega \subset \mathbb{C}$ contain $\overline{D_R}$ and $f\colon \Omega \to \mathbb{C}$ be holomorphic. Suppose that f is not constant, $f(0) \neq 0$, and f has no zeroes on C_R. If $z_1, \ldots, z_r$ are the zeroes of f inside C_R counted with multiplicity, then*

$$\log|f(0)| = \sum_{k=1}^{r} \log\left(\frac{|z_k|}{R}\right) + \frac{1}{2\pi}\int_0^{2\pi} \log|f(Re^{i\theta})|d\theta. \tag{12.1}$$

You should think of this as a function that constricts the location of zeroes of f in terms of the values of f at the origin and the circle C_R.

Proof. There are four steps:

Step 1 The first observation is that $f_1, f_2\colon \Omega \to \mathbb{C}$ satisfy the hypothesis, then so does their product $f_1 f_2$. Furthermore, we claim that if f_1 and f_2 satisfy the conclusions of the theorem, then so does $f_1 f_2$. For the left hand side of (12.1) this is easy: $\log|f_1(0)f_2(0)| = \log|f_1(0)| + \log|f_2(0)|$. The same additivity holds for the integral in the right hand side of (12.1):

$$\frac{1}{2\pi}\int_0^{2\pi} \log|f_1(Re^{i\theta})f_2(Re^{i\theta})|d\theta = \frac{1}{2\pi}\int_0^{2\pi} \log|f_1(Re^{i\theta})|d\theta + \frac{1}{2\pi}\int_0^{2\pi} \log|f_2(Re^{i\theta})|d\theta.$$

Finally, the set of zeroes of $f_1 f_2$ is the union of the sets of zeroes of f_1 and f_2: when $z_1, \dots, z_r$ are the zeroes of f_1 and $z'_1, \dots, z'_s$ are the zeroes of f_2, $z_1, \dots, z_r, z'_1, \dots, z'_s$ are the zeroes of $f_1 f_2$. Thus we have that the sum

$$\sum_{k=1}^{r} \left(\frac{|z_k|}{R} \right) + \sum_{j=1}^{s} \left(\frac{|z'_j|}{R} \right)$$

is the sum that appears in the right hand side of (12.1) for $f_1 f_2$.

Step 2 Next note that

$$g(z) = \frac{f(z)}{(z - z_1) \cdots (z - z_r)}$$

is bounded near $z_1, \dots, z_r$. Hence these singularities are removable and g gives a holomorphic function $\Omega \to \mathbb{C}$ without zeroes in C_R. Writing

$$f(z) = (z - z_1) \cdots (z - z_r) g(z),$$

we see it suffices to prove (12.1) for linear functions, i.e. $z - z_i$, or functions that have no zeroes within C_R, i.e. g.

Step 3 We first prove (12.1) for functions g that have no zeroes within C_R. In this case, we have to prove that

$$\log |g(0)| = \frac{1}{2\pi} \int_0^{2\pi} \log |g(Re^{i\theta})| d\theta.$$

To do so, recall that since g is non-vanishing and holomorphic on the simply-connected set $D_{R+\epsilon}$, we can write $g(z) = e^{h(z)}$ with $h\colon D_{R+\epsilon} \to \mathbb{C}$ holomorphic. Rephrased in this property, we have to prove that

$$\operatorname{Re}(h(0)) = \frac{1}{2\pi} \int_0^{2\pi} \operatorname{Re}(h(Re^{i\theta})) d\theta.$$

But this is obtained by taking the real part of the mean-value property for h with $z_0 = 0$ and $r = R$:

$$h(0) = \frac{1}{2\pi} \int_0^{2\pi} h(Re^{i\theta}) d\theta.$$

Step 4 We finally prove (12.1) for a linear function $z - z_0$ for $z_0 \in D_R$. We must show that

$$\log |z_0| = \log \left(\frac{|z_0|}{R} \right) + \frac{1}{2\pi} \int_0^{2\pi} \log |Re^{i\theta} - z_0| d\theta.$$

Rewriting, this is equivalent to

$$0 = \int_0^{2\pi} \log |e^{i\theta} - z_0/R| d\theta.$$

Writing $a = z_0/R$ and substituting $-\theta$ for θ, this is equivalent to

$$\int_0^{2\pi} \log |1 - ae^{i\theta}| d\theta = 0.$$

This follows by applying the mean-value property to the harmonic function $\log |1 - az|$, which vanishes at the origin. $\square$

We can reformulate Jensen's formula in terms of the function

$$\begin{aligned}\mathfrak{n}(r)\colon \mathbb{R}_{\geq 0} &\longrightarrow \mathbb{Z}_{\geq 0}\\ r &\longmapsto \#\{\text{zeroes of } f \text{ in } D_r \text{ counted with multiplicity}\},\end{aligned}$$

which counts the number of zeroes by radius.

Proposition 12.2.2. *Let $\Omega \subset \mathbb{C}$ contain $\overline{D_R}$ and $f\colon \Omega \to \mathbb{C}$ be holomorphic such that $f(0) \neq 0$. Then for $0 < r < R$ we have*

$$\int_0^R \frac{\mathfrak{n}(r)}{r} dr = \frac{1}{2\pi} \int_0^{2\pi} \log |f(Re^{i\theta})| d\theta - \log |f(0)|.$$

Proof. By Jensen's formula, it suffices to prove that

$$\int_0^R \frac{\mathfrak{n}(r)}{r} dr = \sum_{k=1}^{r} \log\left(\frac{R}{|z_k|}\right),$$

where as before $z_1, \dots, z_r$ are the zeroes of f counted with multiplicity.

Since $\mathfrak{n}(r) = \sum_{k=1}^r \eta_k(r)$ with $\eta_k(r) = 1$ if $r > |z_k|$ and 0 otherwise, it suffices to prove that

$$\int_0^R \frac{\eta_k(r)}{r} dr = \log\left(\frac{R}{|z_k|}\right).$$

This follows since $\int_0^R \frac{\eta_k(r)}{r} dr = \int_{|z_k|}^R \frac{1}{r} dr = \log(R) - \log(|z_k|)$. ☐

12.3 Functions of finite order

Using the version of Jensen's formula given in Proposition 12.2.2, we will prove a stronger result about the locations of zeroes for the following restricted class of holomorphic functions:

Definition 12.3.1. For $\rho > 0$, a holomorphic $f\colon \mathbb{C} \to \mathbb{C}$ has *order of growth* $\leq \rho$ if there exist A, B such that $|f(z)| \leq Ae^{B|z|^\rho}$.

Definition 12.3.2. A holomorphic $f\colon \mathbb{C} \to \mathbb{C}$ is *of finite order* if it has order of growth $\leq \rho$ for some $\rho > 0$. Its *order of growth* is given by the infimum over all ρ such that f has order of growth $\leq \rho$.

Example 12.3.3. The function e^z has order of growth 1, the function e^{z^3} has order of growth 3, and polynomials have order of growth 0 (the infimum over all positive numbers).

Theorem 12.3.4. *If $f\colon \mathbb{C} \to \mathbb{C}$ is holomorphic with order of growth $\leq \rho$, then*

(i) $\mathfrak{n}(r) \leq Cr^\rho$ for some $C > 0$ and r sufficiently large,

(ii) if $z_1, z_2, \dots$ denote the zeroes of f not equal to 0, then for all $s > \rho$ we have

$$\sum_{k=1}^{\infty} \frac{1}{|z_k|^s} < \infty.$$

Proof. We start with the first part. Without loss of generality $f(0) \neq 0$; otherwise we may replace $f(z)$ by $f(z)/z^\ell$ with ℓ the order of the zero at f at the origin. This only changes $\mathfrak{n}(r)$ by a constant. We have

$$\int_0^R \frac{\mathfrak{n}(x)}{x} dx = \frac{1}{2\pi} \int_0^{2\pi} \log|f(Re^{i\theta})| d\theta - \log|f(0)|.$$

We will estimate $\mathfrak{n}(r)$ by the left hand side, and use the growth condition to estimate the right hand side.

For the former, by integrating over a smaller interval, we get

$$\int_r^{2r} \frac{\mathfrak{n}(x)}{x} dx \leq \frac{1}{2\pi} \int_0^{2\pi} \log|f(Re^{i\theta})| d\theta - \log|f(0)|.$$

Since $\mathfrak{n}(x)$ is increasing, we have

$$\int_r^{2r} \frac{\mathfrak{n}(x)}{x} dx \geq \mathfrak{n}(x) \int_r^{2r} \frac{1}{x} dx = \mathfrak{n}(r)(\log(2r) - \log(r)) = \mathfrak{n}(r)\log(2).$$

For the latter, the growth estimate $|f(z)| \leq Ae^{B|z|^\rho}$ gives

$$\int_0^{2\pi} \log|f(Re^{i\theta})| d\theta \leq \int_0^{2\pi} \log(Ae^{B(2r)^\rho}) d\theta = \int_0^{2\pi} \log(A) + B(2r)^\rho d\theta \leq C' r^\rho$$

for some $C' > 0$ absorbing various constants and r sufficiently large.

Combined, we get

$$\mathfrak{n}(r)\log(2) \leq C' r^\rho - \log|f(0)|,$$

so that picking an appropriate constant $C > 0$, we get

$$\mathfrak{n}(r) \leq C r^\rho$$

for r sufficiently large.

For the second part, we use the first part to estimate the sum over all roots with absolute value ≥ 1 (there are only finitely many with absolute value < 1):

$$\begin{aligned}
\sum_{|z_k| \geq 1} |z_k|^{-s} &= \sum_{j=0}^{\infty} \left(\sum_{2^j \leq |z_k| < 2^{j+1}} |z_k|^{-s} \right) \\
&\leq \sum_{j=0}^{\infty} 2^{-js} \mathfrak{n}(2^{j+1}) \\
&\leq C' \sum_{j=0}^{\infty} 2^{-js} 2^{(j+1)\rho} \\
&\leq C' \sum_{j=0}^{\infty} 2^{(\rho - s)j} < \infty.
\end{aligned}$$

In the first inequality, we use that $|z_k|^{-s} \leq 2^{-js}$ for $2^j \leq |z_k| \leq 2^{j+1}$. In the second, we used the growth estimate (modifying the constant C to C' to absorb the fact that our estimate only holds for r sufficiently large). In the third, we used that $2^{-js}2^{(j+1)\rho} = 2^{(\rho-s)j+\rho}$ and absorbed the 2^ρ into C' to get C''. The final geometric series converges because $s > \rho$. □

Example 12.3.5. The condition $s > \rho$ in the last part can't be improved: take $f(z) = \sin(\pi z)$. It has order of growth ≤ 1, as $f(z) = \frac{1}{2i}(e^{i\pi z} - e^{-i\pi z})$ implies $|f(z) \leq Ce^{\pi|z|}$. (In fact, it is equal to 1 since it grows exactly as e^x on the imaginary axis.) However, the zeroes are at $n \in \mathbb{Z}$ with order 1 and we indeed have that

$$\sum_{k=1}^{\infty} \frac{1}{|z_k|^s} = 2\sum_{n=1}^{\infty} \frac{1}{n^s}$$

and this converges if and only if $s > 1$.

12.4 An application of Jensen's formula to polynomials

What happens when we apply Jensen's formula to a polynomial $p(z) = a_n z^n + \cdots + a_1 z + a_0$? We might as well assume $a_n = 1$ for simplicity and need $a_0 \neq 0$ to get $p(0) \neq 0$. Let $z_1, \ldots, z_n$ be the zeroes of p, counted with multiplicity, and assume that $|z_i| \neq 1$ for all i. That is, p has no zeroes on the circle C_1. Let $z_1, \ldots, z_r$ denote the zeroes of p with absolute value < 1, i.e. those in D_1. Then Jensen's formula says

$$\log|a_0| = \sum_{k=1}^{r} \log|z_k| + \frac{1}{2\pi}\int_0^{2\pi} \log|f(e^{i\theta})| d\theta.$$

Proposition 12.4.1. *We have that*

$$\log(|a_0| + \cdots + |a_n|) \leq \frac{2^n}{2\pi}\int_0^{2\pi} \log|f(e^{i\theta})| d\theta.$$

Proof. Reorganizing Jensen's formula, we get

$$\log\left|\frac{a_0}{z_1 \cdots z_r}\right| = \frac{1}{2\pi}\int_0^{2\pi} \log|f(e^{i\theta})| d\theta.$$

Let us now recall that a_0 is the product of all zeroes, so $\frac{a_0}{z_1 \cdots z_r} = z_{r+1} \cdots z_n$, and we get

$$\sum_{k=r+1}^{n} \log|z_k| = \frac{1}{2\pi}\int_0^{2\pi} \log|f(e^{i\theta})| d\theta.$$

Since $\log|z_i| < 0$ if $|z_i| < 1$, we see that for any subset $I \subset \{1, \ldots, n\}$, we have

$$\sum_{i \in I} \log|z_i| \leq \sum_{k=r+1}^{n} \log|z_k|,$$

and hence

$$\sum_{i \in I} \log|z_i| \leq \frac{1}{2\pi}\int_0^{2\pi} \log|f(e^{i\theta})| d\theta. \tag{12.2}$$

Now observe that the coefficient a_s is equal to the sum $\sum_{|I|=n-s} \prod_{i \in I} z_i$ (as usual, the empty product is 1). Thus we have that (with $|a_n| = 1$)

$$|a_0| + \cdots + |a_n| \leq \sum_{I \subset \{1,\ldots,n\}} \prod_{i \in I} |z_i|.$$

Now the logarithm is concave, i.e. $\log(x+y) \leq \log(x) + \log(y)$, so we have

$$\log(|a_0| + \cdots + |a_n|) \leq \sum_{I \subset \{1,\ldots,n\}} \sum_{i \in I} \log |z_i|.$$

Each of the 2^n terms $\sum_{i\in I} \log |z_i|$ can be estimated by $\frac{1}{2\pi}\int_0^{2\pi} \log |f(e^{i\theta})| d\theta$, so we conclude that

$$\log(|a_0| + \cdots + |a_n|) \leq \frac{2^n}{2\pi} \int_0^{2\pi} \log |f(e^{i\theta})| d\theta. \qquad \square$$

From (12.2), we can deduce another consequence. By the triangle inequality, we have that $|f(e^{i\theta})| \leq |a_0| + \cdots + |a_n|$. Thus we can estimate the right hand side as $\leq \log(|a_0| + \cdots + |a_n|)$. This directly gives

Lemma 12.4.2 (Feldman's inequality)**.** *For every $I \subset \{1, \ldots, n\}$, we have*

$$\sum_{i \in I} \log |z_i| \leq |a_0| + \cdots + |a_n|.$$

Remark 12.4.3. We can also use this observation to bound $\frac{1}{2\pi}\int_0^{2\pi} \log |f(e^{i\theta})| d\theta$ from above and below as follows:

$$\frac{1}{2^n} \log(|a_0| + \cdots + |a_n|) \leq \frac{1}{2\pi} \int_0^{2\pi} \log |f(e^{i\theta})| d\theta \leq \log(|a_0| + \cdots + |a_n|).$$

Chapter 13

Weierstrass' construction

We continue our study of holomorphic functions whose domain is all of the complex plane $\mathbb{C}$. Our goal for this lecture and the next one is the Hadamard factorization theorem, which describes an entire function of finite order in terms of its zeroes. That is, it is a generalization of the factorization of polynomials to holomorphic functions which do not grow too fast.

Today we prepare by studying infinite products, and Weierstrass construction of an entire function with prescribed zeroes. This is Sections 5.3 and 5.4 of [SS03].

13.1 Infinite products

You may have seen the expression

$$\frac{\sin(\pi z)}{\pi} = z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right).$$

In this section we make sense of such infinite products, and explain how to deduce this as an example.

Definition 13.1.1. An infinite product $\prod_{n=1}^{\infty}(1+a_n)$ converges if the sequence of partial products $\prod_{n=1}^{N}(1+a_n)$ does.

Proposition 5.3.1 of [SS03] gives a sufficient condition, which is proven by taking the logarithm after discarding the finitely many terms with $|a_n| \geq 1/2$:

Proposition 13.1.2. *If $\sum_{n=1}^{\infty} |a_n| < \infty$, then the infinite product $\prod_{n=1}^{\infty}(1+a_n)$ converges. Furthermore, it is 0 if and only if at least one of the terms is 0.*

We will need one fact from its proof though: the rate of convergence for the product depends on that of the sum.

This has the following consequence, which is line with previous results about infinite sums and integrals:

Theorem 13.1.3. *Suppose that $\{F_n\}_{n\geq 1}$ is a sequence of holomorphic functions $\Omega \to \mathbb{C}$. Suppose that there exist $c_n > 0$ such that $\sum_{n=1}^{\infty} c_n < \infty$ and $|F_n(z) - 1| < c_n$ for all $z \in \Omega$.*

Then we have that the product $\prod_{n=1}^{\infty} F_n(z)$ *converges uniformly on* Ω *to a holomorphic function* $F(z)$*. For* $z \in \Omega$*,* $F(z) = 0$ *if and only if at least one the terms* $F_n(z)$ *is* 0*. If* $F(z) \neq 0$*, we have*

$$\frac{F'(z)}{F(z)} = \sum_{n=1}^{\infty} \frac{F_n'(z)}{F_n(z)}.$$

Proof. We may write $F_n(z) = 1 + a_n(z)$, where $\sum_{n=1}^{\infty} |a_n(z)| < \infty$ for all $z \in \Omega$. The first two parts that follows from the previous proposition; it is uniform because the rate of convergence of the sum is controlled by the rate of convergence of $\sum_{n=1}^{\infty} c_n$.

For the third part, we recall that if a sequence $G_N(z)$ of holomorphic functions converges uniformly on compacts to $G(z)$, then the derivatives $G_N'(z)$ converge uniformly on compacts to $G_N'(z)$. We shall apply this to $G_N(z) = \prod_{n=1}^{N} F_n(z)$, which are all non-zero on K and hence have absolute value which is uniformly bounded below. Thus $G_N'(z)/G_N(z)$ converges uniformly on compacts to $G'(z)/G(z)$. Now we just need the fact that

$$\frac{G_N'(z)}{G_N(z)} = \sum_{n=1}^{N} \frac{F_n'(z)}{F_n(z)}.$$

□

13.1.1 The product formula for the sine

Let us now prove that

$$\frac{\sin(\pi z)}{\pi} = z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right).$$

Let us denote the left hand side by $G(z)$ and the right hand side by $P(z)$. Our first observation is that since $\sum_{n=1}^{\infty} \left|\frac{z^2}{n^2}\right| = |z|^2 \sum_{n=1}^{\infty} n^2 < \infty$ for all $z \in \mathbb{C}$, the infinite product in $P(z)$ converges uniformly to a holomorphic function. Furthermore, $P(z)$ is non-zero if and only $z \notin \mathbb{Z}$, and we have

$$\frac{P'(z)}{P(z)} = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{2z}{z^2 - n^2}.$$

We will show that the same expression holds for $\frac{G'(z)}{G(z)}$. If so, we would have

$$\left(\frac{P(z)}{G(z)}\right)' = \frac{P(z)}{G(z)} \left(\frac{P'(z)}{P(z)} - \frac{G'(z)}{G(z)}\right) = 0.$$

This means that $P(z) = cG(z)$ on $\mathbb{C} \setminus \mathbb{Z}$ for some constant c, and by continuity this is true everywhere. By evaluating the derivative at 0, we get

$$P'(0) = 1, \qquad G'(0) = 1,$$

so we see that $c = 1$. The second is easy; for the first observe that $\tilde{P}(z) = \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right)$ holomorphic, so $P'(0) = \tilde{P}(0)$. It is easy to see that $\tilde{P}(0) = 1$.

Thus, it remains to prove that

$$\frac{G'(z)}{G(z)} = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{2z}{z^2 - n^2}.$$

Observe that $\frac{G'(z)}{G(z)} = \pi \cot(\pi z) = \pi \frac{\cos(\pi z)}{\sin(\pi z)}$. We claim that $F(z) = \pi \cot(\pi z)$ satisfies the following conditions:

(i) F is defined on $\mathbb{C} \setminus \mathbb{Z}$,

(ii) $F(z+1) = F(z)$ when $z \in \mathbb{C} \setminus \mathbb{Z}$,

(iii) $F(z) = \frac{1}{z} + F_0(z)$ with $F_0(z)$ holomorphic near 0,

(iv) $F(z)$ is odd,

(v) $F(z)$ is bounded on $|\mathrm{Re}(z)| \leq 1/2$, $|\mathrm{Im}(z)| \geq 1$.

In fact, (i), (ii), and (iii) imply that all singularities are simple poles with residue 1. Properties (i), (ii) and (iv) are clear for $\pi \cot(\pi z)$, and (iii) follows using the Taylor expansion $\sin(\pi z) = \pi z(1 + g(z))$ with $g(0) = 0$.

We claim that

$$H(z) = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{2z}{z^2 - n^2}$$

satisfies the same properties. Properties (i), (iii), and (iv) are clear. To see (ii) holds, we observe that $\frac{2z}{z^2-n^2} = \frac{1}{z+n} + \frac{1}{z-n}$, so the right hand side is $\lim_{N\to\infty} \sum_{n=-N}^{N} \frac{1}{z+n}$. Replacing z by $z+1$ just shifts the indices (up to a small error term from the boundary terms).

We claim that

$$\Delta(z) = F(z) - H(z)$$

is bounded. This means it extends to a constant function, which is zero by (iv). By (ii), it suffices to prove it is bounded on the strip $\{z \in \mathbb{C} \mid \mathrm{Re}(z) \leq 1/2\}$. By (iii), we have that Δ is bounded for z in the strip with $|\mathrm{Im}(z)| \leq 1$. By (v) it is bounded on $|\mathrm{Re}(z)| \leq 1/2$, $|\mathrm{Im}(z)| \geq 1$, as both $F(z)$ and $H(z)$ are bounded.

We will give the outline for (v) for $F(z)$ and $H(z)$ now, for details see pages 143–144 of [SS03]. For $F(z)$ this follows by writing for $z = x + iy$

$$\cot(\pi z) = i \frac{e^{-2\pi y} + e^{-2\pi i x}}{e^{-2\pi y} - e^{-2\pi i x}},$$

and observe that as $y \to \pm\infty$ this remains bounded. For $H(z)$ this follows by writing

$$H(z) = \frac{1}{x+iy} + \sum_{n=1}^{\infty} \frac{2(x+iy)}{x^2 - y^2 - n^2 + 2ixy},$$

and observe that if $|x| \leq 1/2$ and $y > 1$, we can estimate its absolute value by $A + B\sum_{n=1}^{\infty} \frac{y}{y^2+n^2}$. The sum can be estimated by the integral $\int_0^{\infty} \frac{y}{y^2+x^2} dx$ which can be shown to be independent of y by substituting yx for x. In conclusion, for $y > 1$ it is bounded. The argument for $y < -1$ is similar.

13.2 Weierstrass' construction

The goal of the Weierstrass construction is, given a sequence $\{a_n\}_{n\in\infty}$ with $|a_n| \to \infty$, to construct a holomorphic function which vanishes exactly at the a_n. Note that the a_n are allowed to be repeated, say k times, in which case we want our function to have a zero at a_n of multiplicity k.

This uses the so-called *canonical factors.*

Definition 13.2.1. For $k \geq 0$, we define

$$E_0(z) := 1 - z, \quad \text{and} \quad E_k(z) := (1-z)e^{z+z^2/2+\cdots+z^k/k} \text{ for } k \geq 1.$$

These are essentially $1 - z$, with more and more control over the growth of $1 - E_k(z)$ for $|z|$ small:

Lemma 13.2.2. *If $|z| \leq 1/2$, then $|1 - E_k(z)| \leq c|z|^{k+1}$ for some constant $c > 0$ independent of k.*

Proof. For $|z| \leq 1/2$, we have a convergent power series expression

$$\log(1-z) = -\sum_{n=1}^{\infty} z^n/n.$$

Thus we get

$$E_k(z) = e^{\log(1-z)+z+z^2/2+\cdots+z^k/k} = e^w$$

$$\text{with } w = -\sum_{n=k+1}^{\infty} z^n/n.$$

Since $|z| \leq 1/2$ we can estimate $|w|$ by

$$|w| \leq |z|^{k+1} \sum_{n=k+1}^{\infty} |z|^{n-k-1}/n.$$

Using the crude estimate $|z|^{n-k-1}/n \leq 1/2^{n-k-1}$, we can estimate this as

$$\leq |z|^{k+1} \sum_{j=0}^{\infty} 2^{-j} = 2|z|^{k+1}.$$

Thus in particular $|w| \leq 1$.

Since $e^w - 1 = w(1 + F(w))$ with $F(0) = 0$. Since $w \mapsto |1 + F(w)|$ is continuous, it attains a maximum on w with $|w| \leq 1$. Thus we have that

$$|1 - E_k(z)| = |1 - e^w| \leq c'|w| \leq c|z|^{k+1}$$

with $c = 2c'$. □

Proposition 13.2.3. *Given a sequence $\{a_n\}_{n\in\infty}$ with $|a_n| \to \infty$ as $n \to \infty$, there exists an entire function f such that the zeroes of f are exactly the a_n (counted with multiplicity).*

Proof. Suppose 0 occurs m times among the a_n, and henceforth suppose all a_n are non-zero. Then we take

$$f(z) = z^m \prod_{n=1}^{\infty} E_n(z/a_n).$$

If suffices to prove that $f(z)$ has the desired properties on the disk D_R. There are two types of canonical factors, (i) those with $|a_n| \leq 2R$, (ii) those with $|a_n| > 2R$. We can then write the product as

$$\left(z^m \prod_{|a_n| \leq 2R} E_n(z/a_n) \right) \cdot \left(\prod_{|a_n| > 2R} E_n(z/an) \right).$$

The first term provide the correct zeroes on the disk D_R: a zero of order m at the origin, and from each canonical factor $E_n(z/a_n)$ a simple zero at a_n (if a_n appears r times in the sequence, then we have r canonical factors provide a simple zero at a_n and hence we get a zero of order r at a_n).

If thus remains to show that second term is holomorphic and non-vanishing on D_R. If $|a_n| > 2R$ and $|z| < R$, then $|z/a_n| \leq 1/2$. Hence by the previous lemma we have

$$|1 - E_n(z/a_n)| \leq c|z/a_n|^{n+1} \leq c/2^{n+1}$$

with c independent on n. Thus the infinite product

$$\prod_{|a_n| > 2R} E_n(z/a_n)$$

converges to a holomorphic function on $|z| < R$. It does not vanish on D_R since none of the terms in the product does. ☐

Theorem 13.2.4. *Given a sequence $\{a_n\}_{n \in \infty}$ with $|a_n| \to \infty$ as $n \to \infty$, if g is an entire function such that the zeroes of g are exactly the a_n (counted with multiplicity) then $g(z) = f(z)e^{h(z)}$ with f as in the previous proposition and h entire.*

Proof. Consider $\frac{g(z)}{f(z)}$. This is holomorphic on $\mathbb{C} \setminus \{a_n \mid n \geq 1\}$. Since the zeroes of g cancel out of those of f, it has removable singularities only and hence extends to an entire function. This function vanishes nowhere, so is of the form $e^{h(z)}$. The conclusion follows. ☐

Chapter 14

Hadamard factorization theorem

We continue our study of holomorphic functions whose domain is all of the complex plane $\mathbb{C}$. We now finish the proof of the Hadamard factorization theorem, which describes an entire function of finite order in terms of its zeroes. This uses the Weierstrass construction of a holomorphic function with specified zeroes. This is Section 5.5 of [SS03].

14.1 Hadamard factorization theorem

Last lecture we proved that when f has a zero of order m at the origin and further zeroes exactly given by $\{a_n\}_{n\geq 1}$ with $a_n \to \infty$, then we have that

$$f(z) = e^{g(z)} z^k \prod_{n=1}^{\infty} E_n(z/a_n),$$

with E_n the canonical factor $(1-z)e^{z+z^2/2+\cdots+z^n/n}$.

The Hadamard factorization theorem improves on this when f is of finite order. Recall that we say f has order of growth $\leq \rho$ if $|f(z)| \leq Ae^{B|z|^\rho}$ for some constants $A, B > 0$, and the order of growth of f is the infimum over all such ρ's.

Theorem 14.1.1 (Hadamard factorization theorem)**.** *Suppose f is entire and has order of growth ρ satisfying $k \leq \rho \leq k+1$ with $k \geq 0$ an integer. If f has a zero of order m at the origin and further zeroes exactly given by $\{a_n\}_{n\geq 1}$ with $a_n \to \infty$, then we have that*

$$f(z) = e^{P(z)} z^m \prod_{n=1}^{\infty} E_k(z/a_n),$$

with P a polynomial of degree $\leq k$.

Example 14.1.2. Recall that $\frac{\sin(\pi z)}{\pi}$ has order of growth 1. The above theorem then tells us that

$$\frac{\sin(\pi z)}{\pi} = e^{az+b} \prod_{n\in\mathbb{Z}} (1 - z/n)$$

for some constant a, b. Collecting the terms $1 - z/n$ and $1 + z/n$, we get

$$e^{az+b} z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right).$$

In fact, from the product expression for the sine we know that in fact $a = 0 = b$. To prove this, one evaluates the first and third derivatives at 0.

Example 14.1.3. Suppose that f is entire and of order ρ *not* an integer. We claim it must have infinitely many zeroes. If not, then by Theorem 14.1.1 we have

$$f(z) = e^{P(z)} z^m \prod_{n=1}^{r} E_k(z/a_n)$$

with P a polynomial of degree k. Since $e^{P(z)}$ as well as $E_k(z) = (1-z)e^{z+z^2/2+\cdots+z^k/k}$ are of order $\leq k$, it is easy to estimate that $f(z)$ as above must be of order $\leq k$. This contradicts the fact that it is of order $\rho > k$.

14.1.1 Preparatory lemmas

The main technical input of Weierstrass' construction are estimate on the canonical factors. Since the Hadamard factorization theorem is an improvement of this, it is not surprising that we need to prove better estimates on the canonical factors.

Lemma 14.1.4. *We have that*

$$|E_k(z)| \geq e^{-c|z|^{k+1}} \quad \textit{for } |z| \leq 1/2, \textit{ some } c \geq 0,$$

$$|E_k(z)| \geq |1-z| e^{-c'|z|^k} \quad \textit{for } |z| \geq 1/2, \textit{ some } c \geq 0.$$

Proof. For the first part, recall that $|z| \leq 1/2$, we can use the power series expansion for $\log(1-z) = \sum_{n=1}^{\infty} -z^n/n$ to write

$$E_k(z) = e^{\log(1-z)+\sum_{n=1}^{k} z^n/n} = e^{-\sum_{n=k+1}^{\infty} z_n/n} = e^w,$$

for $w = -\sum_{n=k+1}^{\infty} z^n/n$. Now use that $|e^w| \geq e^{-|w|}$, as well as the expression $|w| = |z|^{k+1} \left|\sum_{n=k+1}^{\infty} z^{n-k-1}\right|$. Since $z \mapsto \left|\sum_{n=k+1}^{\infty} z^{n-k-1}\right|$ is a continuous function, it attains a maximum c when $|z| \leq 1/2$.

For the second part, we simply write

$$|E_k(z)| = |1-z| \left|e^{z+z^2/2+\cdots+z^k/k}\right|.$$

Now observe that $\left|e^{z+z^2/2+\cdots+z^k/k}\right| \geq e^{-|z+z^2/2+\cdots+z^k/k|}$ and that $|z+z^2/2+\cdots+z^k/k| \leq c'|z|^k$ for some constant $c' \geq 0$ by the type of estimates we used for the proof of the Liouville theorem. $\square$

We shall use this to estimate the infinite product of canonical factors away from some disks:

Lemma 14.1.5. *For s with $\rho < s < k+1$ we have*

$$\left|\prod_{n=1}^{\infty} E_k(z/a_n)\right| \geq e^{-C|z|^s}$$

on the complement of a union of disks centered at the a_n of radius $|a_n|^{-k-1}$.

Proof. As in the Weierstrass construction, we write

$$\prod_{n=1}^{\infty} E_k(z/a_n) = \left(\prod_{|a_n|\leq 2|z|} E_k(z/a_n)\right) \cdot \left(\prod_{|a_n|>2|z|} E_k(z/a_n)\right)$$

and estimate the two terms in the product separately.

For the second term, we have

$$\begin{aligned}\left|\prod_{|a_n|>2|z|} E_k(z/a_n)\right| &= \prod_{|a_n|>2|z|} |E_k(z/a_n)| \\ &\geq \prod_{|a_n|>2|z|} e^{-c|z/a_n|^{k+1}} \\ &\geq e^{-c|z|^{k+1}\sum_{|a_n|>2|z|}|a_n|^{-k-1}}\end{aligned}$$

Now observe that $|a_n|^{-k-1} = |a_n|^{-s}|a_n|^{s-k-1}$ and since $|a_n| > 2|z|$ (in fact $|a_n| > |z|$ would have sufficed) and $s < k+1$, this can be estimate as $\leq C_1|a_n|^{-s}|z|^{s-k-1}$. Since $\sum_n |a_n|^{-s}$ converges by the assumption on the order of growth of f, we can estimate $C' \sum_{|a_n|>2|z|} |a_n|^{-k-1}$ as $\leq C_2|z|^{s-k-1}$. The result is that

$$\left|\prod_{|a_n|>2|z|} E_k(z/a_n)\right| \geq e^{-C_2|z|^s}.$$

(This did not use that z was in the complement of the disks centered at a_n and radius $|a_n|^{-k-1}$.)

For the first term, we have the estimate

$$\left|\prod_{|a_n|\leq 2|z|} E_k(z/a_n)\right| \geq \left(\prod_{|a_n|\leq 2|z|} |1 - z/a_n|\right) \cdot \left(\prod_{|a_n|\leq 2|z|} e^{-c'|z/a_n|^k}\right),$$

and we will investigate both terms in the product separately. For the latter one, $\prod_{|a_n|\leq 2|z|} e^{-c'|z/a_n|^k} = e^{-c'|z|^k \sum_{|a_n \leq 2|z|} |a_n|^k}$ we can make an estimate as above to see it is $\geq e^{-C_3|z|^s}$.

The more subtle one is the former one, $\prod_{|a_n|\leq 2|z|} |1 - z/a_n|$. We will take σ such $\rho < \sigma < s$. When z is distance at least $|a_n|^{-k-1}$ from a_n, we have that

$$|1 - z/a_n| \geq |a_n|^{-k-2},$$

and thus the product is $\geq e^{-(k+2)\sum_{|a_n|\leq 2|z|}\log|a_n|}$. This is something we can estimate in terms of the function $\mathfrak{n}(r)$, counting the number of zeroes (with multiplicity) within a circle of radius r:

$$(k+2)\sum_{|a_n|\leq 2|z|}\log|a_n| \leq (k+2)\mathfrak{n}(2z)\log|2z| \leq C_4|z|^\sigma \log|2z| \leq C_5|z|^s.$$

In the end, we get $\prod_{|a_n|\leq 2|z|}|1-z/a_n| \geq e^{-C_5|z|^s}$.

Combing all these results, we get

$$\left|\prod_{n=1}^{\infty} E_k(z/a_n)\right| \geq e^{-C_2|z|^s}e^{-C_3|z|^s}e^{-C_5|z|^s} = e^{-c|z|^s}$$

with $c = C_2+C_3+C_5$, as long as z avoids the disks centered at a_n of radius $|a_n|^{-k-1}$. □

Since the radii $|a_n|^{-k-1}$ sum to a finite sum, they can't possibly exhaust each of the intervals $[R, R+1]$ of possible radii. Thus we see that:

Corollary 14.1.6. *There exist a sequence $r_1, r_2, \ldots$ of radii with $r_i \to \infty$, such that*

$$\left|\prod_{n=1}^{\infty} E_k(z/a_n)\right| \geq e^{-C|z|^s}$$

when $|z| = r_i$.

14.1.2 The proof

Proof of Theorem 14.1.1. That the infinite product

$$E(z) = z^m \prod_{n=1}^{\infty} E_k(z/a_n)$$

requires nothing but the estimates of the previous section: when n is large enough $|z/a_n| \leq 1/2$ and we have $|1-E_k(z/a_n)| \leq c|z/a_n|^k$. Since the sum $\sum_{n=1}^{\infty}|a_n|^{-k-1}$ converges by our previous results on the zeroes of functions of order $< k+1$, this infinite product converges to an entire function. Further, it exactly has the same zeroes as f (counted with multiplicity).

Instead, our fancy estimates come in when we try to understand the entire functon $f(z)/E(z)$, which is non-vanishing. Hence we can write it as $f(z)/E(z) = e^{g(z)}$ for some entire function g. We must soch that g is a polynomial P of degree $\leq k$. However, on each circle of radius r_i around the origin, we have

$$|e^{g(z)}| \leq \frac{|f(z)|}{|E(z)|} \leq e^{C'|z|^s} \quad \text{when } |z| = r_i,$$

since f is of order $\rho < s$ and we have $|E(z)| \geq e^{-C|z|^s}$. Since $|e^{g(z)}| = e^{\mathrm{Re}(g(z))}$, it suffices to prove that when $\mathrm{Re}(g(z)) \leq C'|z|^s$, then g is polynomial of degree $\leq k$ (recall $k \leq \rho < s < k+1$).

This shall use a consequence of the formula for a_n which we used to derive the mean value property: if $h(z) = \sum_{n=0}^{\infty} a_n z^n$, then for $n \geq 0$ we have that

$$a_n = \frac{1}{2\pi r^n} \int_0^{2\pi} h(re^{i\theta}) e^{-in\theta} d\theta.$$

When we take $n > 0$ to be negative in the right hand side, the integral vanishes. In particular, the complex conjugate of the case with $-n$ gives that $\frac{1}{2\pi r^n} \int_0^{2\pi} \overline{h(re^{i\theta})} e^{-in\theta} d\theta$ is 0. If $n > 0$, we add this to the expression in the case n, we get

$$\begin{aligned} a_n &= a_n + 0 \\ &= \frac{1}{2\pi r^n} \int_0^{2\pi} h(re^{i\theta}) e^{-in\theta} d\theta + \frac{1}{2\pi r^n} \int_0^{2\pi} \overline{h(re^{i\theta})} e^{-in\theta} d\theta \\ &= \frac{1}{2\pi r^n} \int_0^{\pi} \mathrm{Re}(h)(re^{i\theta}) e^{-in\theta} d\theta. \end{aligned}$$

We may as well subtract a term $\frac{1}{2\pi r^n} \int_0^{\pi} C' r^s e^{-in\theta} d\theta$, since this is 0 anyway: for $n > 0$ we get

$$a_n = \frac{1}{2\pi r^n} \int_0^{\pi} (\mathrm{Re}(h)(re^{i\theta}) - C' r^s e^{-in\theta} d\theta.$$

Taking r to be one of the r_i given above, we get

$$|a_n| \leq \frac{1}{\pi r^n} \int_0^{2\pi} |(\mathrm{Re}(h)(re^{i\theta}) - C' r^s e^{-in\theta}| d\theta \leq \frac{1}{\pi r^n} \int_0^{2\pi} C' r^s - u(re^{i\theta}) d\theta,$$

and we compute the right hand side is $\frac{C'}{\pi} r^{s-n} - \frac{2}{\pi} \mathrm{Re}(a_0) r^n$. If $n \geq k+1 > s$, this goes to 0 as $r_i \to \infty$. Thus we conclude that $g(z) = a_0 + a_1 z \cdots + a_k z^k$, i.e. it is a polynomial of degree $\leq k$. ☐

Remark 14.1.7. For $n = 0$ we get a different result in the above argument:

$$2\mathrm{Re}(a_0) = \frac{1}{2\pi r^n} \int_0^{\pi} \mathrm{Re}(h)(re^{i\theta}) d\theta.$$

14.2 Picard's little theorem for functions of finite order

Picard's little theorem says that if f is entire function which omits two values, then f is constant. We shall this under the assumption that f is of finite order.

Theorem 14.2.1. *If f is entire function of finite order which omits two values, then f is constant.*

Proof. If f omits a, then $f(z) - a$ is still of finite order and never vanishes. By the Hadamard factorization theorem we have $f(z) - a = e^{p(z)}$ for a polynomial $p(z)$. From the fundamental theorem, if p has degree ≥ 1 then it is surjective and hence $e^{p(z)}$ only omits the value 0. This would contradict the assumption that f omits two values, so p must have degree 0. We conclude that $f(z) - a$ is constant and hence so is f. ☐

Example 14.2.2. The function $e^{z^2} - \sin(z)$ has finite order (in fact 2), so all points $z \in \mathbb{C}$ except possibly two are in its image.

Chapter 15

The Gamma function

Having finished the study of entire functions, we now move on the Riemann zeta function and its applications to prime numbers. As preparation, we study the Γ-function; the "simplest" function with simple poles at $0, -1, -2, \ldots$, which appears in the functional equation for the Riemann zeta function. This is Section 6.1 of [SS03].

15.1 The Gamma function

When we discussed the construction of holomorphic functions by limits, such as sums or integrals, we proved that the formula

$$\Gamma(s) \coloneqq \int_0^\infty e^{-t} t^{s-1} dt$$

defines a holomorphic function on $\{s \in \mathbb{C} \mid \mathrm{Re}(s) > 0\}$: the Gamma function. I recommend your read [SS03, Proposition 6.1.1] to refresh your memory, but the strategy is to write

$$\Gamma(s) = \int_0^\epsilon e^{-t} t^{s-1} dt + \int_\epsilon^{1/\epsilon} e^{-t} t^{s-1} dt + \int_{1/\epsilon}^\infty e^{-t} t^{s-1} dt,$$

show that the middle term is holomorphic for all ϵ when $\mathrm{Re}(s) > 0$ and show that the other two terms go to 0 as $\epsilon \to 0$.

We will today extend this to $\mathbb{C} \setminus \{0, -1, -2, \ldots\}$, necessarily uniquely, and discuss its most important properties. In the next section we will describe how the Gamma function is a particular instance of a Mellin transform, applied to e^{-t}.

15.1.1 Analytic extension

To extend Γ, we make the following observation:

Proposition 15.1.1. *When* $\mathrm{Re}(s) > 0$, *we have* $\Gamma(s+1) = s\Gamma(s)$.

Proof. By integrating $\frac{d}{dt}(e^{-t} t^s)$ we get

$$e^{-1/\epsilon}(1/\epsilon)^{-s} - e^{-\epsilon}\epsilon^s = \int_\epsilon^{1/\epsilon} \frac{d}{dt}(e^{-t} t^s) dt = -\int_\epsilon^{1/\epsilon} e^{-t} t^s dt + s \int_\epsilon^{1/\epsilon} e^{-t} t^{s-1} dt.$$

As $\epsilon \to 0$, the left hand side goes to 0 and the right hand side to $-\Gamma(s+1) + s\Gamma(s)$. This proves the formula. $\square$

Example 15.1.2. In particular, we have

$$\Gamma(n+1) = n\,\Gamma(n) = n(n-1)\,\Gamma(n-1) = \cdots n!\,\Gamma(1),$$

and since $\Gamma(1) = \int_0^\infty e^{-t}dt = \left[-e^{-t}\right]_0^\infty = 1$, we see that $\Gamma(n+1) = n!$. In other words, the Gamma function is a holomorphic extension of the factorial function.

Theorem 15.1.3. *The function $\Gamma(s)$ has a unique extension to a meromorphic function on $\mathbb{C}$, with simple poles at $0, -1, -2, \ldots$ and $\mathrm{res}_{s=-n}\Gamma(s) = (-1)^n/n!$.*

Proof. Observe that for $m \geq 1$, we have that

$$\frac{\Gamma(s+m)}{(s+m-1)(s+m-2)\cdots s}$$

is meromorphic on $\{s \in \mathbb{C} \mid \mathrm{Re}(s) \geq -m\}$ with poles at $0, -1, \ldots, -(m-1)$, and coincides with $\Gamma(s)$ on $\{s \in \mathbb{C} \mid \mathrm{Re}(s) \geq -m\}$. Furthermore, for the residue at $n \in \{0, -1, \ldots, -(m-1)\}$ simply observe that it is given by factoring out the term $1/(s-n)$ and evaluating at $-n$:

$$\begin{aligned}\mathrm{res}_{s=-n}\Gamma(s) &= \frac{\Gamma(-n+m)}{(m-n-1)!(-1)(-2)\cdots(-n)} \\ &= \frac{(m-n-1)!}{(m-n-1)!(-1)(-2)\cdots(-n)} = \frac{(-1)^n}{n!}.\end{aligned}$$

All these extensions agree on the overlaps of their domains, as these are connected subsets of $\mathbb{C}$ and all of them coincide with $\Gamma(s)$ as defined by integral on $\mathrm{Re}(s) > 0$. We obtain a well-defined extension as

$$s \longmapsto \frac{\Gamma(s+m)}{(s+m-1)(s+m-2)\cdots s} \qquad \text{if } \mathrm{Re}(s) > -m \text{ and } s \neq 0, -1, -2, \ldots$$ $\square$

We shall denote this extension also by $\Gamma(s)$.

Remark 15.1.4. On page 163 of [SS03], a different explanation for the extension is given. We write

$$\Gamma(s) = \int_0^1 e^{-t}t^{s-1}dt + \int_1^\infty e^{-t}t^{s-1}dt.$$

Expanding e^{-t} as $\sum_{n=0}^\infty \frac{t^n}{n!}$ and integrating the terms we get that the first term is equal to $\sum_{n=0}^\infty \frac{(-1)^n}{n!(n+s)}$. This gives an expression

$$\Gamma(s) = \sum_{n=0}^\infty \frac{(-1)^n}{n!(n+s)} + \int_1^\infty e^{-t}t^{s-1}dt$$

at first valid for $\mathrm{Re}(s) > 0$. However, by doing some estimates one can show that this combination of an infinite sum and integral defines a meromorphic function on all of $\mathbb{C}$!

15.1.2 Symmetry

Recall that we studied the function $\frac{\sin(\pi s)}{\pi}$, which has a simple zero at all integers $n \in \mathbb{Z}$. In some sense $1/\Gamma(s)$ is "half" of the function:

Theorem 15.1.5. *For all $s \in \mathbb{C}$ we have*

$$\Gamma(s)\,\Gamma(1-s) = \frac{\pi}{\sin(\pi s)}.$$

Proof. It suffices to prove that this is true for s real satisfying $0 < s < 1$, as the uniqueness for extensions of holomorphic functions then implies it is true everywhere.

For $0 < s < 1$ we have that

$$\Gamma(1-s) = \int_0^\infty e^{-u}u^{-s}du = t \in_0^\infty e^{-vt}(vt)^{-s}dv$$

by substituting $v = tu$. Thus we get

$$\begin{aligned}
\Gamma(1-s)\Gamma(s) &= \int_0^\infty e^{-t}t^{s-1}\Gamma(1-s)ds \\
&= \int_0^\infty e^{-t}t^{s-1}\left(t\int_0^\infty e^{-vt}(vt)^{-s}dv\right)dt \\
&= \int_0^\infty \int_0^\infty e^{-t(1+v)}v^{-s}dv\,dt \\
&= \int_0^\infty \int_0^\infty e^{-t(1+v)}v^{-s}dt\,dv \\
&= \int_0^\infty \frac{v^{-s}}{1+v}dv.
\end{aligned}$$

Now we observe that by substituting $v = e^x$, we have $\int_0^\infty \frac{v^{-s}}{1+v}dv = \int_{-\infty}^\infty \frac{e^{(s-1)x}}{1+e^x}dx$. We evaluate this in an earlier lecture as an example of Cauchy's theorem, and the value is $\frac{\pi}{\sin(\pi(1-s))}$, which is equal to $\frac{\pi}{\sin(\pi s)}$ as the sine function is odd. $\square$

Example 15.1.6. Take $s = 1/2$ to get $\Gamma(1/2)^2 = \pi$. Hence $\Gamma(1/2) = \sqrt{2}$ as $\Gamma(s)$ is easily seen to be positive on the positive real line.

15.1.3 Growth

We shall next study the growth of $1/\Gamma(s)$:

Theorem 15.1.7. *The function $1/\Gamma(s)$ extends to an entire function with simple zeroes at $0, -1, -2, \ldots$ and no other zeroes. We have that*

$$\left|\frac{1}{\Gamma(s)}\right| \le C_1 e^{C_2|s|\log|s|}$$

for some constants $C_1, C_2 > 0$.

Proof. For the first part we write

$$\frac{1}{\Gamma(s)} = \Gamma(1-s)\frac{\sin(\pi s)}{\pi}$$

and observe that the zeroes of the sines cancel out the poles of $\Gamma(1-s)$ at $1, 2, \ldots$. The zeroes at $0, -1, -2, \ldots$ remain.

For the growth estimate, we shall use that

$$\frac{1}{\Gamma(s)} = \frac{\sin(\pi s)}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!(n+1-s)} + \frac{\sin(\pi s)}{\pi}\int_1^{\infty} e^{-t}t^{-s}dt.$$

We first estimate the second term: $\left|\frac{\sin(\pi s)}{\pi}\right| \leq Ce^{\pi|s|}$ and

$$\left|\int_1^{\infty} e^{-t}t^{-s}dt\right| \leq \int_1^{\infty} e^{-t}t^{|\mathrm{Re}(s)|}dt.$$

Now pick an integer n such that $|\mathrm{Re}(s)| \leq n \leq |\mathrm{Re}(s)|+1$ and observe

$$\begin{aligned}\int_1^{\infty} e^{-t}t^{|\mathrm{Re}(s)|}dt &= \int_1^{\infty} e^{-t}t^{n}dt \\ &= \Gamma(n) = n! \\ &\leq n^n = e^{n\log n} \\ &\leq e^{(|\mathrm{Re}(s)|+1)\log(|\mathrm{Re}(s)|+1)}.\end{aligned}$$

Now using $|\mathrm{Re}(s)| \leq |s|$, we get that the second term can be estimated by

$$\left|\frac{\sin(\pi s)}{\pi}\int_1^{\infty} e^{-t}t^{-s}dt\right| \leq Ce^{\pi|s|}e^{(|s|+1)\log(|s|+1)} \leq C_1'e^{C_2'|s|\log|s|}$$

for suitable $C_1, C_2 > 0$.

Thus it remains to prove that the first term to bounded by $C_1''e^{C_2''|s|}$. There are two cases: if $|\mathrm{Im}(s)| \geq 1$, then $|\frac{1}{n+1-s}| \leq 1$ and we can estimate it as

$$\left|\frac{\sin(\pi s)}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!(n+1-s)}\right| \leq \left|\frac{\sin(\pi s)}{\pi}\right| e$$

and we have a further estimate $|\sin(\pi s)| \leq Ce^{\pi|s|}$ so this has the desired bound.

The harder case is $|\mathrm{Im}(s)| \leq 1$. Fix an integer k such that $k - 1/2 \leq \mathrm{Re}(s) \leq k + 1/2$. If $k \leq 0$, then $|\frac{1}{n+1-s}| \leq 1$ again and we do the same estimate as above. If $k \geq 1$, then we split the sum as

$$\frac{\sin(\pi s)}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!(n+1-s)} = (-1)^{k-1}\frac{\sin(\pi s)}{(k-1)!(k-s)\pi} + \sum_{n\neq k-1}\frac{(-1)^n}{n!(n+1-s)}\frac{\sin(\pi s)}{\pi}.$$

Both of these are bounded by $C_1'''e^{C_2'''|s|}$; for the second term this is done similarly as before by estimate $\frac{1}{n+1-s}$ and for the first term we use that the zero of $\sin(\pi s)$ cancels the pole of $1/(k-s)$. □

Corollary 15.1.8. $1/\Gamma(s)$ *is of order* ≤ 1.

Proof. It suffices to observe that for any $\epsilon > 0$, $C|z|\log|z| \leq C'|z|^{1+\epsilon}$. Thus $1/\Gamma(s)$ is of order $\leq 1+\epsilon$ for any $\epsilon > 0$. □

15.1.4 Product formula

Since $1/\Gamma(s)$ is a entire function of finite order whose zeroes we know, the Hadamard factorization theorem gives a product expression for it. In particular, its order is ≤ 1 and it has simple zeroes at $0, -1, -2, \ldots$. Thus we get

$$1/\Gamma(s) = e^{as+b} s \prod_{n=1}^{\infty} (1 + s/n) e^{-s/n} \tag{15.1}$$

for constants $a, b \in \mathbb{C}$ to be determined.

Proposition 15.1.9. *We may take $b = 0$ and $a = \gamma \coloneqq \lim_{N\to\infty} \left(-\log(N) + \sum_{n=1}^{N} \frac{1}{n}\right)$.*

That the limit in this proposition exists is not completely obvious, see page 167 of [SS03]; γ is called the *Euler–Mascheroni constant*. It is a famous open conjecture that it is irrational.

Proof. Since the residue of $\Gamma(s)$ at $s = 0$ is 1, we have that $\lim_{s\to 0} s\Gamma(s) = 1$. On the other hand, the limit as $s \to 0$ of (15.1) is e^{-b} so b must be a multiple of $2\pi i$; we may take 0.

Since $\Gamma(1) = 1$, we may evaluate (15.1) at $s = 1$ to get

$$e^a \prod_{n=1}^{\infty} (1 + 1/n) e^{-1/n}.$$

We can write the product as a limit of $N \to \infty$ of the exponential of $\sum_{n=1}^{N} \log(1 + 1/n) - \sum_{n=1}^{N} 1/n$. Writing $\log(1 + 1/n)$ as $\log(1 + n) - \log(n)$, half of the terms cancel to give $\log(1+N) - \sum_{n=1}^{N} 1/n$. This is almost the correct description to give the Euler–Mascheroni constant as $N \to \infty$, except we have $\log(1 + N)$ instead of $\log(N)$. To see this doesn't matter, write $\log(1+N)$ as $\log(N) - \log(1+1/N)$ and that $\lim_{N\to\infty} \log(1+1/N) = 0$. □

Example 15.1.10. Recall the product expansion

$$\frac{\sin(\pi s)}{\pi} = z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right) = z \prod_{n=1}^{\infty} (1 - z/n)(1 + z/n).$$

In the expression

$$\frac{1}{\Gamma(s)\Gamma(1-s)} = \frac{\sin(\pi s)}{\pi},$$

the terms z and $1 + z/n$ for $n \geq 1$ come from $\frac{1}{\Gamma(s)}$ and the terms $1 + z/n$ for $n \geq 1$ from $\frac{1}{\Gamma(1-s)}$.

15.2 Mellin transforms

The *Mellin transform* of a suitable function (e.g. continuous and quickly decaying) is

$$\mathcal{M}(f)(s) = \int_0^{\infty} f(t) t^s \frac{dt}{t}.$$

Thus the Gamma function is nothing but the Mellin transform of e^{-x}. There is a precise sense in which the Mellin transform is a Fourier transform.

The ordinary Fourier transform of a periodic function $f\colon \mathbb{R} \to \mathbb{C}$

$$\mathcal{F}(f)(n) = \int_0^1 f(t)e^{i2\pi nt}dt$$

relates functions on $\mathbb{R}/\mathbb{Z}$ (i.e. periodic functions) to functions on $\mathbb{Z}$ (i.e. the Fourier coefficients). Closely related is the Fourier transform of non-periodic functions

$$\mathcal{F}(f)(s) = \int_{-\infty}^{\infty} f(t)e^{ist}dt,$$

relations functions on $\mathbb{R}$ to functions on $\mathbb{R}$.

To see how these are related and what role the Mellin transform plays, we give a more general perspective. In the first case we consider the group $(\mathbb{R}/\mathbb{Z}, +)$. The characters, i.e. homomorphism $\lambda\colon (\mathbb{R}/\mathbb{Z}, +) \to U(1) := \{z \in \mathbb{C} \mid |z| = 1\}$ are given by $t \mapsto e^{i2\pi nt}$ for some $n \in \mathbb{Z}$. It is a result of Pontryagin that because $G = (\mathbb{R}/\mathbb{Z}, +)$ is locally compact abelian group, so is the set of $\hat{G}$ (the Pontryagin dual of G). Similarly, in the second case we consider the group $(\mathbb{R}, +)$. Its characters, i.e. homomorphism $\lambda\colon (\mathbb{R}, +) \to \mathbb{C}$ are given by $t \mapsto e^{i2\pi st}$ for some $s \in \mathbb{R}$.

Thus in general a Fourier transform relates functions on a locally compact abelian group to functions on its Pontryagin dual. The transformation is given by sending $f\colon G \to \mathbb{C}$ to the function

$$\hat{G} \ni \lambda \longmapsto \int_G f(g)\lambda(g)d\mu,$$

where μ is a G-invariant measure (unique up to scaling).

The Mellin transform is then the case $G = (\mathbb{R}_{>0}, \cdot)$. Just like $(\mathbb{R}, +)$ it is its own Pontryagin dual, with the character corresponding to $s \in i\mathbb{R}$ given by $t \mapsto t^s = e^{is\log t} \in U(1)$ (there is no issue about branches of the logarithm since $t \in \mathbb{R}_+$). The invariant measure is $\frac{dt}{t}$. In other words, the slogan is:

> The Mellin transform is the Fourier transform for the multiplicative group of positive real numbers.

Chapter 16

The Riemann zeta function

Last lecture we extended the Gamma function to a meromorphic function on $\mathbb{C}$ and discussed its properties. Today we do the same for the Riemann zeta function.[1] This serves as preparation for the eventual proof of the prime number theorem. This is Section 6.2 of [SS03].

16.1 The Riemann zeta function

When we discussed the construction of holomorphic functions by infinite sums, we gave the Riemann zeta function as an example: for $\mathrm{Re}(s) > 1$ the following sum converges uniformly on compacts

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

and hence defines a holomorphic function. We shall extend it to a meromorphic function, with a simple pole at $s = 1$. (Recall from the first lecture that having a spole at $s = 1$ implied that there are infinitely many primes.) The argument will relate ζ to Γ and a new function θ.

16.1.1 The theta function

Let us consider the function

$$\begin{aligned} \theta \colon \mathbb{R}_{>0} &\longrightarrow \mathbb{R} \\ t &\longmapsto \sum_{n=-\infty}^{\infty} e^{-\pi n^2 t}. \end{aligned}$$

To verify it is well-defined, we estimate $|e^{-\pi n^2 t}| \leq e^{-\pi |n| t}$, so the sum can be estimated

$$\left| \sum_{n=-\infty}^{\infty} e^{-\pi n^2 t} \right| \leq 1 + 2 \sum_{n=1}^{\infty} (e^{-\pi t})^n = 1 + 2 \frac{1}{1 - e^{-\pi t}}.$$

[1] Since Γ is a capital Greek letter we capitalize "Gamma" when referring to the Gamma function, but since ζ is lowercase we do not capitalize "zeta" when referring to the Riemann zeta function.

This shows that the sum is uniformly convergent on the sets $t > \epsilon$, and hence defines a continuous function. A more refined estimate shows it is smooth, but we will not use this.

Proposition 16.1.1 (Functional equation for θ)**.** *For* $t > 0$ *we have*

$$\theta(t) = 1/\sqrt{t}\,\theta(1/t).$$

Proof. Consider the holomorphic function $f(z) = e^{-\pi t z^2}$. Within the rectangular contour R_N with corners $N + 1/2 - i, N + 1/2 + i, -N - 1/2 + i, -N - 1/2 - i$ the function $f(z)/(e^{2\pi i z} - 1)$ has poles at $-N, -(N-1), \cdots, N-1, N$ with residues $\frac{f(n)}{2\pi i}$. By the residue theorem and canceling a $\frac{1}{2\pi i}$, we get

$$\int_{R_N} \frac{f(z)}{e^{2\pi i z} - 1} dz = \sum_{|n| \leq N} f(n).$$

The integrals over the vertical line segments can be estimated as $\leq C e^{-\pi t N^2}$ so go to 0 as $N \to \infty$. We thus get that

$$\sum_{n \in \mathbb{Z}} f(n) = \int_{-\infty}^{\infty} \frac{f(x-i)}{e^{2\pi i (x-i)} - 1} dx - \int_{-\infty}^{\infty} \frac{f(x+i)}{e^{2\pi i(x+i)} - 1} dx.$$

Now observe that $|e^{2\pi i(x-i)}| > 1$ and $|e^{2\pi i(x+i)}| < 1$ so we can write

$$\frac{1}{e^{2\pi i(x-i)} - 1} = e^{-2\pi i(x-i)} \sum_{n=0}^{\infty} e^{-2\pi n i(x-i)} = \sum_{n=1}^{\infty} e^{-2\pi n i(x-i)},$$

$$\frac{1}{e^{2\pi i(x+i)} - 1} = -\sum_{n=0}^{\infty} e^{2\pi n i(x+i)}.$$

Substituting these and exchanging sums and integrals (justified by the absolute convergence of either), we get

$$\sum_{n \in \mathbb{Z}} f(n) = \sum_{n=0}^{\infty} \int_{-\infty}^{\infty} f(x+i) e^{2\pi n i(x+i)} dx + \sum_{n=1}^{\infty} \int_{-\infty}^{\infty} f(x-i) e^{-2\pi n i(x-i)} dx.$$

Now we recall an integral we did: $\int_{-\infty}^{\infty} e^{-\pi x^2} e^{-2\pi i x \xi} dx = e^{-\pi \xi^2}$. Investigating a term in the first sum, we get

$$\int_{-\infty}^{\infty} f(x+i) e^{2\pi n i(x+i)} dx = \int_{-\infty}^{\infty} e^{-\pi t (x+i)^2 + 2\pi n i (x+i)} dx.$$

Let us substitute $y = \sqrt{t}(x+i)$ to get this is equal to

$$\frac{1}{\sqrt{t}} \int_{-\infty}^{\infty} e^{-\pi y^2} e^{2\pi i n/\sqrt{t} y} dy = \frac{1}{\sqrt{t}} e^{-(n/\sqrt{t})^2} = \frac{1}{\sqrt{t}} e^{-n^2/t}.$$

Similarly, we obtain an expression

$$\int_{-\infty}^{\infty} f(x-i)e^{-2\pi n i(x-i)}dx = \frac{1}{\sqrt{t}}e^{-n^2/t}.$$

The conclusion is that

$$\theta(t) = \sum_{n\in\mathbb{Z}} e^{-\pi n^2 t} = \sum_{n\in\mathbb{Z}} \frac{1}{\sqrt{t}}e^{-n^2/t} = \frac{1}{\sqrt{t}}\theta(1/t). \qquad \square$$

Remark 16.1.2. If we have done Chapter 4 of [SS03], this would have been an easily consequence of the Poisson summation formula in Fourier theory.

We will also need two estimates:

(i) $|\theta(t)-1| \le Ce^{-\pi t}$ as $t\to\infty$ because

$$|\theta(t)-1| \le e^{-\pi t}2\sum_{n=1}^{\infty} e^{-\pi(n^2-1)t} \le e^{-\pi t}\sum_{n=0}^{\infty} e^{-\pi n t}$$

and the latter sum can be bounded by a constant for $t \ge 1$.

(ii) Combining this with the functional equation we get that as $t\to 0$,

$$|\theta(t)| \le C/\sqrt{t}.$$

16.1.2 Expressing ζ in terms of θ

Proposition 16.1.3. *for* $\mathrm{Re}(s) > 1$ *we have*

$$\pi^{-s/2}\Gamma(s/2)\zeta(s) = \frac{1}{2}\int_0^\infty u^{s/2-1}(\theta(u)-1)du.$$

Proof. We start on the right hand side and observe that

$$\frac{\theta(u)-1}{2} = \sum_{n=1}^{\infty} e^{-\pi n^2 u}.$$

The estimates given above justify that we can exchange the sum and the integral to get

$$= \frac{1}{2}\int_0^\infty u^{s/2-1}(\theta(u)-1)du = \sum_{n=1}^{\infty} = \frac{1}{2}\int_0^\infty u^{s/2-1}e^{-\pi n^2 u}du.$$

If we substitute $u = \frac{t}{\pi n^2}$ in the terms, we get

$$\frac{1}{(\pi n^2)^{-s/2}}\int_0^\infty t^{s/2}e^{-t}dt = \pi^{-s/2}\Gamma(s/2)\frac{1}{n^s}.$$

Summing this over $n \ge 1$, we get exactly $\pi^{-s/2}\Gamma(s/2)\zeta(s)$. $\square$

It is a good idea to consider $\pi^{-s/2}\Gamma(s/2)\zeta(s)$ instead of $\Gamma(s)$, and we give it a name

$$\xi(s) \coloneqq \pi^{-s/2}\Gamma(s/2)\zeta(s),$$

at the moment defined for $\mathrm{Re}(s) > 1$.[2] The following is not only the basic input for the extension of ζ, but expresses a symmetry property similar to that of the Gamma function:

Theorem 16.1.4. *The function ξ extends to a meromorphic function with poles at $s = 0, 1$ and satisfies*

$$\xi(s) = \xi(1-s)$$

for all $s \in \mathbb{C}$.

Proof. As the above proposition tells us, the function $\psi(u) \coloneqq (\theta(u) - 1)/2$ is closely related to the xi function. The functional equation $\theta(u) = u^{-1/2}\,\theta(1/u)$ gives one for ψ:

$$\psi(u) = u^{-1/2}\psi(1/u) + \frac{1}{2u^{1/2}} - \frac{1}{2}.$$

We use this to rewrite $\xi(s)$ for $\mathrm{Re}(s) > 1$, using the trick of splitting the integral into the sections $[0, 1]$ and $[1, \infty)$, familiar from the Gamma function:

$$\begin{aligned}\xi(s) &= \frac{1}{2}\int_0^\infty u^{s/2-1}(\theta(u) - 1)du \\ &= \int_0^\infty u^{s/2-1}\psi(u)du \\ &= \int_0^1 u^{s/2-1}\psi(u)du + \int_1^\infty u^{s/2-1}\psi(u)du.\end{aligned}$$

Now we use the functional equation in the first term, so it is:

$$\begin{aligned}&\int_0^1 u^{s/2-1}\left(u^{-1/2}\psi(1/u) + \frac{1}{2u^{1/2}} - \frac{1}{2}\right)du \\ &\quad= \int_0^1 u^{s/2-3/2}\psi(1/u) + \frac{u^{s/2-1/2}}{2} - \frac{u^{s/2-1}}{2}du \\ &\quad= \frac{1}{s-1} - \frac{1}{s} + \int_1^\infty v^{-s/2-1/2}\psi(v)dv\end{aligned}$$

by substituting $v = 1/u$.

The end result is that

$$\xi(s) = \frac{1}{s-1} - \frac{1}{1-s} + \int_1^\infty \left(u^{s/2-1} + u^{-s/2-1/2}\right)\psi(u)du. \tag{16.1}$$

We claim that the integral defines an entire function. To see this is case, we first observe that

$$\begin{aligned}[1, R] \times \mathbb{C} &\longrightarrow \mathbb{C} \\ (s, u) &\longmapsto (u^{s/2-1} + u^{-s/2-1/2})\psi(u)\end{aligned}$$

[2] See https://terrytao.wordpress.com/2008/07/27/tates-proof-of-the-functional-equation/ for a high-level explanation why ξ is better than ζ.

is continuous and holomorphic when we fix s. Thus $s \mapsto \int_1^R \left(u^{s/2-1} + u^{-s/2-1/2}\right)\psi(u)du$ defines a holomorphic function. Now we let $R \to \infty$ and need to check uniform convergence on compacts to see that the result is still holomorphic. But if $|s| \le D$ then

$$\begin{aligned} \left|\int_R^\infty \left(u^{s/2-1} + u^{-s/2-1/2}\right)\psi(u)du\right| &\le \int_R^\infty \left(u^{|s|/2+1} + u^{|s|/2+1/2}\right) Ce^{-\pi u}du \\ &\le \int_R^\infty C' u^{D+1} e^{-\pi u} du < \infty \end{aligned}$$

since $e^{-\pi u}$ outgrows any polynomial.

Therefore (16.1) gives a meromorphic extension of ξ to $\mathbb{C}$, with evident simple poles at $s = 0, 1$. The functional equation is easily read off from this, especially when we write $u^{s/2-1} + u^{-s/2-1/2} = u^{s/2-1} + u^{(1-s)/2-1}$. □

Corollary 16.1.5. *The function ζ extends to a meromorphic function with a simple pole at $s = 1$.*

Proof. Writing ξ for the extension given above, we can define an extension of ζ as

$$\zeta(s) = \frac{\pi^{s/2}}{\Gamma(s/2)}\xi(s),$$

recalling that $1/\Gamma$ is an entire function. Furthermore, $1/\Gamma(s/2)$ has simple zeroes at $0, -1, -2, \cdots$, so one of the two poles of $\xi(s)$ cancels out and only the one at $s = 1$ remains. □

16.1.3 Estimates on the line $\mathrm{Re}(s) = 1$

We now give a different argument which extends ζ to $\mathrm{Re}(\zeta) > 0$. Its purpose is to give an estimate on the line $\mathrm{Re}(s) = 1$, which will be used to prove some results about primes later.

Proposition 16.1.6. *There are entire function $\delta_n(s)$ such that (i) $|\delta_n(s) \le \frac{|s|}{n^{\mathrm{Re}(s)+1}}$, (ii) we have*

$$\sum_{n=1}^N \frac{1}{n^s} - \int_1^N \frac{1}{x^s}dx = \sum_{n=1}^N \delta_n(s)$$

for $\mathrm{Re}(s) > 0$.

Proof. To make (ii), we need to define

$$\delta_n(s) := \int_n^{n+1} \frac{1}{n^s} - \frac{1}{x^s}dx.$$

This is entire by our usual results about defining holomorphic functions by integrals. To get the estimate, we start with

$$\left|\frac{1}{n^s} - \frac{1}{x^s}\right| \le \left|\frac{1}{n^{\mathrm{Re}(s)}} - \frac{1}{x^{\mathrm{Re}(s)}}\right|$$

we note that by the mean-value theorem for $f(x) = x^{-\mathrm{Re}(s)}$, we have

$$x =^{-\mathrm{Re}(s)}= f(x) = f(n) + (x-n)f'(c) = n^{-\mathrm{Re}(s)} - (x-n)\mathrm{Re}(s)c^{-\mathrm{Re}(s)-1}.$$

for some $c \in [n, x]$. Reorganizing and estimating $|x-n| \leq 1$, $|\mathrm{Re}(s)| \leq |s|$ and $c^{-\mathrm{Re}(s)-1} \leq n^{-\mathrm{Re}(s)-1}$, we get

$$\left| \frac{1}{n^s} - \frac{1}{x^s} \right| \leq \frac{|s|}{n^{\mathrm{Re}(s)+1}}$$

for $n \leq x \leq n+1$. □

Corollary 16.1.7. *For* $\mathrm{Re}(s) > 0$ *we have*

$$\zeta(s) - \frac{1}{s-1} = \sum_{n=1}^{\infty} \delta_n(s)$$

with right hand side holomorphic.

Proof. The estimate on the $\delta_n(s)$ proves that the sums converges uniformly for $\mathrm{Re}(s) \leq \epsilon$ for any $\epsilon > 0$, so it remains to observe that the equation follows by letting $N \to \infty$ in (ii) and evaluating the integral. □

We can now give our estimate on $\zeta(s)$ near the line $\mathrm{Re}(s) = 1$.

Lemma 16.1.8. *Write* $s = \sigma + i\tau$ *with* $\sigma, \tau \in \mathbb{R}$. *Then for all* $\sigma_0 \in [0,1]$ *and* $\epsilon \in (0,1)$ *there exists constants* $c_\epsilon, c'_\epsilon > 0$ *such that*

- $|\zeta(s)| \leq c_\epsilon |\tau|^{1-\sigma_0+\epsilon}$ *for* $\sigma \geq \sigma_0$ *and* $|\tau| \geq 1-\epsilon$,
- $|\zeta'(s)| \leq c'_\epsilon |\tau|^\epsilon$ *if* $\sigma \geq 1$ *and* $|\tau| \geq 1$.

Proof. We start with the observation that $|\delta_n(s)| \leq \frac{2}{n^\sigma}$ in addition to $|\delta_n(s)| \leq \frac{|s|}{\leq n^{\sigma+1}}$. Thus we have for $\rho \geq 0$ to be fixed later that

$$|\delta_n(s)| = |\delta_n(s)|^\rho |\delta_n(s)|^{1-\rho} = \left(\frac{|s|}{n^{\sigma+1}}\right)^\rho \left(\frac{2}{n^\sigma}\right)^{1-\rho}.$$

Estimate from above by σ_0 replacing σ and simplifying, we get this is $\leq \frac{2|s|^\rho}{n^{\sigma_0+\rho}}$.

Now we take $\rho = 1 - \sigma_0 + \epsilon$ and use $\zeta(s) = \frac{1}{s-1} + \sum_{n=1}^\infty \delta_n(s)$ to get

$$|\zeta(s)| \leq \left|\frac{1}{1-s}\right| + 2|s|^{1-\sigma_0+\epsilon} \sum_{n=1}^{\infty} \frac{1}{n^{1+\epsilon}}.$$

Since the sum converges and $\left|\frac{1}{1-s}\right|$ is bounded when $|\tau| \geq 1-\epsilon$, part (i) follows.

By adjusting the constant to take care of the compact part where $|\tau| \leq 2$, we may assume that $|\tau| \geq 2$. Part (ii) follows from part (i) using the Cauchy integral formula

$$\zeta'(s) = \frac{1}{2\pi i} \int_0^{2\pi} \zeta(s + re^{i\theta}) d\theta$$

where r is a circle of radius $\epsilon/2$. This doesn't contain a pole as $\epsilon \in (0,1)$. We may apply (i) with $\sigma_0 = 1 - \epsilon/2$ to get

$$|\zeta'(s)| \leq \frac{1}{2\pi} \int_0^{2\pi} c_{\epsilon/2}(|\tau| - \epsilon/2)^\epsilon d\theta \leq c'_\epsilon |\tau|^\epsilon.$$

□

Chapter 17

The zeroes of the Riemann zeta function

Last lectures we extended the Gamma function and Riemann zeta function to meromorphic functions on $\mathbb{C}$ and we discussed their basic properties. Today we start using this to eventually prove the prime number theorem. This is Section 7.1 of [SS03].

17.1 Recollection on the prime number theorem

One of the highlights of this course is the application of complex analysis to the study of the prime-counting function

$$\begin{aligned} \pi\colon \mathbb{R}_{\geq 0} &\longrightarrow \mathbb{R} \\ x &\longmapsto \#\{\text{primes less than or equal to } x\}. \end{aligned}$$

In particular, we will leverage the fact that $\zeta(s)$ does not vanish on the line $\mathrm{Re}(s) = 1$ to prove the following result:

Theorem 17.1.1 (Prime number theorem)**.** *We have that*

$$\pi(x) \sim \frac{x}{\log(x)},$$

where $f(x) \sim g(x)$ *means that* $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 1$.

It'll take several lectures to prove this, and today we'll only prove the input on the zeroes of the Riemann zeta function on the line $\mathrm{Re}(s) = 1$. Let us start with making precise the relationship between the primes and the Riemann zeta function.

17.1.1 The Riemann zeta function as a product

We start with a fact that may seems obvious, but requires a bit of estimates to make precise.

Proposition 17.1.2. *For* $\mathrm{Re}(s) > 1$ *we have that*

$$\zeta(s) = \prod_{\text{primes } p} \frac{1}{1 - p^{-s}}.$$

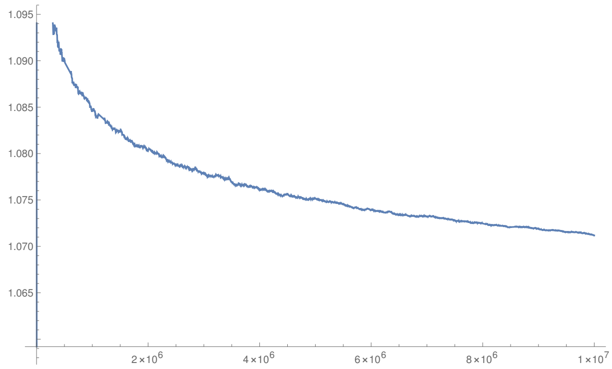


Figure 17.1 The quotient $\pi(x)/(x/\log(x))$ for $x \leq 10^7$. Note the convergence is not very fast.

Proof. We first observe that the right hand side indeed defines a holomorphic function. We will use the criterion that $\prod_{n=1}^{\infty} f_n(z)$ of holomorphic functions gives a holomorphic function if there exist constants $c_n > 0$ such that $\sum_{n=1}^{\infty} c_n < \infty$ and $|f_n(z) - 1| \leq c_n$. To check this estimate on $\mathrm{Re}(s) \geq 1 + \epsilon$, we start noting that in this case we use the geometric series expansion

$$\frac{1}{1-p^{-s}} = \sum_{n=1}^{\infty} \frac{1}{p^{ns}}.$$

So we can make the estimates

$$|f_p(z) - 1| \leq \sum_{n=2}^{\infty} p^{-n\mathrm{Re}(s)} \leq \sum_{n=2}^{\infty} p^{-(1+\epsilon)n} = \frac{p^{-(1+\epsilon)}}{1-p^{-(1+\epsilon)}} \leq Cp^{-(1+\epsilon)},$$

with $C = \frac{1}{1-2^{-(1+\epsilon)}}$. The sum over p can then be bounded by the sum over all n, which indeed converges.

Now that we know that both the left and right hand side are holomorphic, it suffices to prove the identity holds when s is real and $s > 1$.

Let us now deduce the equality from a pair of estimates. Since every number $\leq N$ can be written uniquely as a product of primes $\leq N$, we have

$$\begin{aligned}\sum_{n=1}^{N} \frac{1}{n^s} &\leq \prod_{p \leq N} \left(1 + \frac{1}{p^s} + \cdots + \frac{1}{p^{Ns}}\right) \\ &\leq \prod_{p \leq N} \frac{1}{1-p^{-s}} \\ &\leq \prod_{p} \frac{1}{1-p^{-s}}.\end{aligned}$$

Letting $N \to \infty$, we get that $\sum_{n=1}^{\infty} \frac{1}{n^s} \leq \prod_p \frac{1}{1-p^{-s}}$.

Hence it suffices to establish the reverse inequality. This follows by letting $N \to \infty$ in the similar estimate

$$\prod_{p \leq N} \left(1 + \frac{1}{p^s} + \cdots + \frac{1}{p^{Ns}}\right) \leq \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

□

Recall that our convergence criteria for infinite products also says it vanishes if and only if one of the terms vanish; thus $\zeta(s) \neq 0$ when $\mathrm{Re}(s) > 1$.

17.2 The trivial zeroes of Riemann zeta function

To get the first properties of $\zeta(s)$, and in particular finding its zeroes on the real line, we use the functional equation for $\xi(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s)$. This was $\xi(s) = \xi(1-s)$ and hence expands to

$$\pi^{-s/2}\Gamma(s/2)\zeta(s) = \pi^{-(1-s)/2}\Gamma(\tfrac{1-s}{2})\zeta(1-s),$$

so that

$$\zeta(s) = \pi^{s-1/2}\frac{\Gamma(\frac{1-s}{2})}{\Gamma(s/2)}\zeta(1-s).$$

From this we can deduce the following:

Theorem 17.2.1. *The only zeroes of* $\zeta(s)$ *outside the* critical strip $0 \leq \mathrm{Re}(s) \leq 1$ *are at* $-2, -4, -6, \cdots$.

Proof. It suffices to only consider the subset $\mathrm{Re}(s) \leq 0$. Then neither $\pi^{s-1/2}$ and $\zeta(1-s)$ vanishes. The enumerator $\Gamma(\frac{1-s}{2})$ has neither zeroes nor poles there, and the denominator $\Gamma(s/2)$ has poles at $s = -2, -4, -6, \ldots$. This implies the result. $\square$

The zeroes given in the above theorem are called the *trivial zeroes*. There are many zeroes known in the critical strip, but all of them lie on the line $\mathrm{Re}(s) = 1/2$. It is an open problem whether this is always true:

Conjecture 17.2.2 (Riemann hypothesis)**.** *All non-trivial zeroes of the Riemann zeta function lie on the line* $\mathrm{Re}(s) = 1/2$.

17.3 The Riemann zeta function has no zeroes on $\mathrm{Re}(s) = 1$

We shall now rule out the existence of zeroes on the right edge of the critical strip; by the functional equation this also rules out zeroes on the left edge.

Theorem 17.3.1. *The Riemann zeta function has no zeroes on the line* $\mathrm{Re}(s) = 1$.

Of course, since $\zeta(s)$ has a pole at $s = 1$ there are no zeroes near 1. This is thus really about $1 + i\tau$ with $|\tau|$ large.

Lemma 17.3.2. *If* $\mathrm{Re}(s) > 1$ *then*

$$\log(\zeta(s)) = \sum_{p,m} \frac{p^{-ms}}{m} = \sum_{n=1}^{\infty} c_n n^{-s}$$

with $c_n = 1/m$ *if* $n = p^m$ *and* 0 *otherwise.*

Proof. Since $\zeta(s)$ has no zeroes on the simply-connected domain $\mathrm{Re}(s) > 1$, $\log(\zeta(s))$ is a well-defined holomorphic function. We leave it to the reader to prove that the middle and right sums also define holomorphic functions on $\mathrm{Re}(s) > 1$. By analytic continuation, it then again suffices to prove the identities for s real.

Both are easy: for the first we write

$$\log(\zeta(s)) = \log\left(\prod_{\text{primes } p} \frac{1}{1 - p^{-s}}\right) = \sum_{\text{primes } p} -\log(1 - p^{-s}),$$

and now use that for $|x| \leq 1/2$ we have that $-\log(1-x) = \sum_{m=1}^{\infty} x^m/m$. The second is tautologically true, by the definition of the coefficients c_n. $\square$

Lemma 17.3.3. *If $\sigma, \tau \in \mathbb{R}$ with $\sigma > 1$, then*

$$\log|\zeta^3(\sigma)\zeta^4(\sigma + i\tau)\zeta(\sigma + 2i\tau)| \geq 0.$$

Proof. We start with the observation that

$$3 + 4\cos(\theta) + \cos(2\theta) = 2(1 + \cos(\theta))^2,$$

so in particular the left hand side is non-negative.

Now we write

$$\begin{aligned}
&\log|\zeta^3(\sigma)\zeta^4(\sigma + i\tau)\zeta(\sigma + 2i\tau)| \\
&\quad = 3\log|\zeta(\sigma)| + 4\log|\zeta(\sigma + i\tau)| + \log|\zeta(\sigma + 2i\tau)| \\
&\quad = 3\mathrm{Re}(\log(\zeta(\sigma))) + 4\mathrm{Re}(\log(\zeta(\sigma + i\tau))) + \mathrm{Re}(\log(\zeta(\sigma + 2i\tau))) \\
&\quad = \sum_{n=1}^{\infty} c_n n^{-\sigma}(3 + 4\cos(t\log n) + \cos(2(t\log n))) \\
&\quad \geq 0. \square
\end{aligned}$$

Proof of Theorem 17.3.1. This is a proof by contradiction; we assume that $\zeta(1 + i\tau) = 0$. We may assume $\tau \neq 0$ since there is a pole at $s = 1$.

Since the zero must be of order at least 1, we get

$$|\zeta(\sigma + i\tau)|^4 \leq C(\sigma - 1)^4 \qquad \text{as } \sigma \to 1.$$

Since the pole at $s = 1$ is simple, so

$$|\zeta(\sigma)|^3 \leq C'(\sigma - 1)^{-3} \qquad \text{as } \sigma \to 1.$$

Since ζ is holomorphic at $\sigma + 2i\tau$, we have that

$$|\zeta(\sigma + 2i\tau)| \leq C'' \qquad \text{as } \sigma \to 1.$$

Inserting this in the previous lemma, we get that

$$|\zeta^3(\sigma)\zeta^4(\sigma + i\tau)\zeta(\sigma + 2i\tau)| \to 0 \qquad \text{as } \sigma \to 1.$$

This means that its logarithm must become negative, contradicting the previous lemma. $\square$

17.3.1 Some additional estimates

In fact, we will need a quantative version of Theorem 17.3.1, bounding $1/|\zeta(s)|$ from above when $s = \sigma + i\tau$ with $\sigma \geq 1$ and $|\tau| \geq 1$.

Proposition 17.3.4. *For $\epsilon > 0$ we have $1/|\zeta(s)| \leq c'_\epsilon |\tau|^\epsilon$ when $s = \sigma + i\tau$ with $\sigma \geq 1$ and $|\tau| \geq 1$.*

Proof. By the lemma, we have that

$$|\zeta^3(\sigma)\zeta^4(\sigma + i\tau)\zeta(\sigma + 2i\tau)| \geq 1.$$

So we have that

$$|\zeta^4(\sigma + i\tau)| = \frac{1}{|\zeta^3(\sigma)|}\frac{1}{|\zeta(\sigma + 2i\tau)|}.$$

Now use that ζ has a simple at $s = 1$, and that last lecture we prove the estimate $|\zeta(\sigma + i\tau)| \leq c_\epsilon |\tau|^\epsilon$. Thus we can estimate the right hand side for $\sigma > 1$ as

$$\geq C(\sigma - 1)^{-3}|\tau|^{-\epsilon}.$$

Taking fourth roots we get

$$|\zeta(\sigma + i\tau)| \geq C'(\sigma - 1)^{-3/4}|\tau|^{-\epsilon/4}. \tag{17.1}$$

By the mean value theorem and the estimate $|\zeta'(\sigma + i\tau)| \leq c'_\epsilon |\tau|^\epsilon$ from the last lecture, we have

$$|\zeta(\sigma' + i\tau) - \zeta(\sigma + i\tau)| \leq C''|\sigma' - \sigma||\tau|^\epsilon.$$

Let us set $A = (C'/(2C''))^4$.

Now there are two cases:

(i) The first case is $\sigma - 1 \geq A|\tau|^{-5\epsilon}$ we have from (17.1) we get

$$|\zeta(\sigma + i\tau)| \geq A'|\tau|^{-4\epsilon},$$

which upon replacing 4ϵ by ϵ given by desired estimate.

(ii) The second case is $\sigma - 1 \leq A|\tau|^{-5\epsilon}$. We take $\sigma' - 1 = A|\tau|^{-5\epsilon}$ with the triangle inequality

$$|\zeta(\sigma + i\tau)| \geq |\zeta(\sigma' + i\tau)| - |\zeta(\sigma + i\tau) - \zeta(\sigma' + i\tau)|.$$

Above we estimate the second right hand term by $C''|\sigma' - \sigma||\tau|^\epsilon$, which it $\leq C''|\sigma' - 1||\tau|^\epsilon$.

Applying (17.1) for σ' we get

$$|\zeta(\sigma + i\tau)| \geq C'(\sigma' - 1)^{3/4}|\tau|^{-\epsilon/4} - C''|\sigma' - 1||\tau|^\epsilon.$$

Now A has been chosen such that $C'(\sigma' - 1)^{3/4}|\tau|^{-\epsilon/4} = 2C''|\sigma' - 1||\tau|^\epsilon$, and we get that the right hand side is $C''|\sigma' - 1||t|^\epsilon \geq A''|\tau|^{-4\epsilon}$ using the fact that $\sigma' - 1 = A|\tau|^{-5\epsilon}$.

∎

Chapter 18

Tchebychev's functions

Having proven that $\zeta(s)$ has no zeroes when $\mathrm{Re}(s) \geq$, we will use this to prove the prime number theorem. This is Section 7.2 of [SS03].

18.1 Tchebychev's functions

To relate the prime counting function

$$\begin{aligned} \pi \colon \mathbb{R}_{\geq 0} &\longrightarrow \mathbb{R} \\ x &\longmapsto \#\{\text{primes} \leq x\} \end{aligned}$$

to the Riemann zeta function we will use two functions ψ and ψ_1 introduced by Tchebychev.

Definition 18.1.1. We define the *Tchebychev ψ-function* by

$$\begin{aligned} \psi \colon \mathbb{R}_{\geq 0} &\longrightarrow \mathbb{R} \\ x &\longmapsto \sum_{p^m \leq x} \log p. \end{aligned}$$

It is easy to give alternatively expressions for $\psi(x)$, both of which we shall use later. Firstly, we can write it in terms of the *von Mangoldt Λ-function*

$$\begin{aligned} \Lambda \colon \mathbb{N}_{>0} &\longrightarrow \mathbb{R} \\ n &\longmapsto \begin{cases} \log(p) & \text{if } n = p^m, \\ 0 & \text{otherwise,} \end{cases} \end{aligned}$$

as follows:

$$\psi(x) = \sum_{1 \leq n \leq x} \Lambda(n).$$

Secondly, we can write it terms of the floor function $\lfloor u \rfloor$ as follows:

$$\psi(x) = \sum_{p \leq x} \left\lfloor \frac{\log(x)}{\log(p)} \right\rfloor \log(p).$$

As sounds plausible, the asymptotics of ψ are related to those of π.

Proposition 18.1.2. *If $\psi(x) \sim x$, then $\pi(x) \sim x/\log(x)$.*

Proof. The strategy is to bound $\pi(x)\log(x)/x$ as $x \to \infty$ by 1 from above and below. Let us first give the lower bound:

$$\psi(x) = \sum_{p \le x} \left\lfloor \frac{\log(x)}{\log(p)} \right\rfloor \log(p) \le \sum_{p \le x} \frac{\log(x)}{\log(p)} \log(p) = \pi(x)\log(x),$$

so we have that $\psi(x)/x \le \pi(x)\log(x)/x$. As $x \to \infty$ the left hand side goes to 1, so we get $1 \le \liminf_{x\to\infty} \pi(x)\log(x)/x$.

For the upper bound, we fix an $0 < \alpha < 1$ (with the intention of letting it go to 1) and write

$$\begin{aligned}\psi(x) = \sum_{p \le x} \left\lfloor \frac{\log(x)}{\log(p)} \right\rfloor \log(p) &\ge \sum_{p \le x} \log(p) \ge \sum_{x^\alpha \le p \le x} \log(p) \\ &\ge \sum_{x^\alpha \le p \le x} \log(x^\alpha) = (\pi(x) - \pi(x^\alpha))\log(x^\alpha).\end{aligned}$$

From this we get that

$$\frac{\psi(x) + \alpha\pi(x^\alpha)\log(x)}{x} \ge \frac{\alpha\pi(x)\log(x)}{x}.$$

Since $x^\alpha \ge \pi(x^\alpha)$ and $\alpha x^{\alpha-1}\log(x) \to 0$ as $x \to \infty$, the left hand side goes to 1 as $x \to \infty$. We conclude that $1 \ge \limsup_{x\to\infty} \alpha\pi(x)\log(x)/x$. Letting $\alpha \to 1$, we get the desired upper bound. □

It will turn out to be more convenient to study an integrated version of ψ:

$$\begin{aligned}\psi_1 \colon \mathbb{R}_{\ge 1} &\longrightarrow \mathbb{R} \\ x &\longmapsto \int_1^x \psi(t)dt.\end{aligned}$$

Observe we may as well have had the integral start at 0 instead of 1, as $\psi(t) = 0$ for $t < 1$.

Proposition 18.1.3. *If $\psi_1(x) \sim x^2/2$, then $\psi(x) \sim x$.*

Proof. As $\psi(x)$ is increasing, we have that for $0 < \alpha < 1 < \beta$ (for the sake of convenience, we will also assume $\alpha x \ge 1$) we can compare average values on the intervals $[\alpha x, x]$ and $[x, \beta x]$ to the value at x:

$$\frac{1}{(1-\alpha)x}\int_{\alpha x}^{x} \psi(t)dt \le \psi(x) \le \frac{1}{(\beta-1)x}\int_x^{\beta x} \psi(t)dt.$$

We can write the left hand side as $\frac{1}{(1-\alpha)x}(\psi_1(x) - \psi_1(\alpha x))$. Thus we get

$$\frac{1}{(1-\alpha)}\left(\frac{\psi_1(x)}{x^2} - \frac{\alpha^2\psi_1(\alpha x)}{(\alpha x)^2}\right) \le \frac{\psi(x)}{x}.$$

Letting $x \to \infty$ the left hand side goes to $\frac{1}{1-\alpha}(1/2 - \alpha^2/2) = (1+\alpha)/2$, so we get $(1+\alpha)/2 \leq \liminf_{x\to\infty} \psi(x)/x$. Letting $\alpha \to 1$ we get the lower bound of 1 for the lim inf. A similar argument with the right hand side gives an upper bound $\limsup_{x\to\infty} \psi(x)/x \leq 1$. $\square$

Remark 18.1.4. In fact, the following are equivalent:

- $\pi(x) \sim x/\log(x)$,
- $\psi(x) \sim x$,
- $\psi_1(x) \sim x^2/2$.

Remark 18.1.5. Another way to think of $f(x) \sim g(x)$ is that there is a function $c(x)$ such that $f(x) = c(x)g(x)$ and $c(x) \to 1$ as $x \to \infty$. Let's apply this to extract an interesting consequence from $\pi(x) \sim x/\log(x)$: taking logarithms we get

$$\log(\pi(x)) = \log(x) - \log\log(x) + \log(c(x)).$$

Dividing by $\log(x)$, we see that

$$\frac{\log(\pi(x))}{\log(x)} = 1 - \frac{\log\log(x)}{\log(x)} + \frac{\log(c(x))}{\log(x)},$$

and the right hand side goes to 1 as $x \to \infty$, as $\log\log(x)$ grows much slower than $\log(x)$ and $\log(c(x))$ already goes to 0. We conclude that $\log(\pi(x)) \sim \log(x)$.

This means that we can deduce from $\pi(x) \sim x/\log(x)$ that $\pi(x)\log(\pi(x)) \sim x$. Substituting the nth prime p_n for x, we get $n\log(n) \sim p_n$. In other words, we deduce an asymptotic for the nth prime number.

18.2 Relating Tchebychev's functions to ζ

The von Mangoldt Λ-function will provide the link between ψ and ψ_1 on the one hand, and ζ on the other hand. Recall that for $\mathrm{Re}(s) > 1$ we have

$$\log(\zeta(s)) = \sum_{m,p} \frac{p^{-ms}}{m} = \sum_{m,p} \frac{e^{-m\log(p)s}}{m},$$

so taking the derivative we get

$$\frac{\zeta'(s)}{\zeta(s)} = -\sum_{m,p} \log(p)p^{-ms} = -\sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s}.$$

We shall use this expression to prove that $\psi_1(x)$ can be computed by integrals of a function involving $\zeta'(s)/\zeta(s)$ over vertical lines in the complex plane.

Proposition 18.2.1. *For $c > 1$ we have*

$$\psi_1(x) = -\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{x^{s+1}}{s(s+1)}\frac{\zeta'(s)}{\zeta(s)}ds.$$

We first compute a simpler contour integral:

Lemma 18.2.2. *For $c > 1$ we have*

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{a^s}{s(s+1)}ds = \begin{cases} 0 & \text{if } a \in (0,1), \\ 1-1/a & \text{if } a \in [1,\infty). \end{cases}$$

Proof. The integral converges because we can estimate its absolute value as $\leq C\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{1}{(t+1)^2}dt$, using $|a^s| = a^c$ is independent of s and an estimate $|\frac{1}{s(s+1)}| \leq \frac{C'}{(t+1)^2}$ when s lies on the line $\mathrm{Re}(s) = c$ with $c > 1$.

There are two cases: the first is $a \in [1,\infty)$, in which case $\beta = \log\alpha \geq 0$. Then we write

$$f(s) = \frac{e^{s\beta}}{s(s+1)},$$

which has a pole at $s = 0$ with residue 1 and at $s = -1$ with residue $-e^{-\beta} = -1/\alpha$. We then take a contour given by a semi-circle $S(T)$ with vertical line segment $[c-iT, c+iT]$ and a semi-circle on the left encircling both poles (at least if T is large enough). The residue theorem gives

$$\frac{1}{2\pi i}\int_{S(T)} f(s)ds = 1 - 1/a.$$

The result then follows when we prove that the integral of the semi-circle goes to 0 as $T \to \infty$. This follows from the estimate that $|s(s+1)| \geq C'T^2$ when T is large and s is on this semi-circle, and $|e^{\beta s}| \leq e^{\beta c}$ as $\mathrm{Re}(s) \leq c$. Thus we can estimate

$$\left|\int_{\text{semicircle}} f(s)ds\right| \leq \frac{e^{\beta c}}{C'T^2}\pi T \to 0 \qquad \text{as } T \to \infty.$$

For $a \in (0,1)$ we take a similar contour but with semicircle to the right. This will contain no poles, so it again suffices to prove that the integral over the semi-circle goes to 0 as $T \to \infty$. This uses that now $\beta < 0$ and hence $|e^{\beta s}| \leq e^{\beta c}$ as $\mathrm{Re}(s) \geq c$. $\square$

Proof of Proposition 18.2.1. Recalling that $\frac{\zeta'(s)}{\zeta(s)} = -\sum_{n=1}^{\infty}\frac{\Lambda(n)}{n^s}$, let us interchange the sum and integral in

$$-\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{x^{s+1}}{s(s+1)}\frac{\zeta'(s)}{\zeta(s)}ds$$

to get

$$\sum_{n=1}^{\infty}\frac{x}{2\pi i}\int_{c-\infty}^{c+i\infty}\Lambda(n)\frac{(x/n)^s}{s(s+1)}ds.$$

(The interchange of sum and integral is justified by e.g. dominated convergence, and some estimates.) We can use the above lemma to identify this with

$$\sum_{n=1}^{\infty} x \sum_{x/n\geq 1}\Lambda(n)(1-n/x) = \sum_{n=1}^{\infty}\sum_{x/n\geq 1}\Lambda(n)(x-n).$$

We will now identify this with $\psi_1(x)$. To do so, recall that

$$\psi(t) = \sum_{n=1}^{\infty} \Lambda(n) f_n(t)$$

with $f_n(t)$ taking the value 1 on $[0, n]$ and 0 otherwise. Thus we have that

$$\begin{aligned}\psi_1(x) &= \int_0^x \psi(t)dt \\ &= \sum_{n=1}^{\infty} \int_0^x \Lambda(n) f_n(t)dt \\ &= \sum_{n=1}^{\infty} \Lambda(n)(x-n).\square\end{aligned}$$

18.3 Proving the asymptotics of ψ_1

Let's regroup: our goal is to prove that $\psi_1(x) \sim x^2/2$, which implies $\pi(x) \sim x/\log(x)$, using the following two facts: (I) for $c > 1$ we have

$$\psi_1(x) = -\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{x^{s+1}}{s(s+1)} \frac{\zeta'(s)}{\zeta(s)} ds,$$

and (II) $\zeta(s)$ has no zeroes on the line $\mathrm{Re}(s) = 1$ (or rather the estimates we gave on $\zeta'(s)$ and $1/\zeta(s)$ near that line).

Of course, the strategy will be to let $c \to 1$. The main difficulty is the pole of $\zeta(s)$ at $s = 1$, which will give the term $x^2/2$. The estimates will serve to rule out other contributions. We will do so in the next lecture.

Chapter 19

The proof of the prime number theorem

Having related ζ to ψ_1, we will now use estimate on the Riemann zeta function near the line $\mathrm{Re}(s) = 1$ to prove the prime number theorem. This is the end of Section 7.2 of [SS03]. We then discuss some stronger results relating the zeroes of the Riemann zeta function to the distribution of the prime numbers.

19.1 The asymptotics of $\psi_1(x)$

Recall that we want to prove $\pi(x) \sim x/\log(x)$, which we showed is implied by the estimate $\psi_1(x) \sim x^2/2$, with $\psi_1(x)$ given by

$$\psi_1(x) = -\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{x^{s+1}}{s(s+1)}\frac{\zeta'(s)}{\zeta(s)}ds.$$

To do so we will use the following estimate, a consequence of estimates in the previous three lectures:

Lemma 19.1.1. *For fixed $\eta > 0$, there is a constant $A > 0$ such that for $s = \sigma + i\tau$ with $\sigma \geq 1$ and $|\tau| \geq 1$ we have*

$$\left|\frac{\zeta'(s)}{\zeta(s)}\right| \leq A|\tau|^\eta.$$

Proof. The estimate we proved earlier were that, with same conditions on s, for any $\epsilon > 0$ we can find constants A', A'' such that:

- $|\zeta'(s)| \leq A'|\tau|^\epsilon$,
- $|1/\zeta(s)| \leq A''|\tau|^\epsilon$.

Taking $\epsilon = \eta/2$ and $A = A'A''$ gives the result. □

The strategy is to let $c \to 1$ in a smart way.

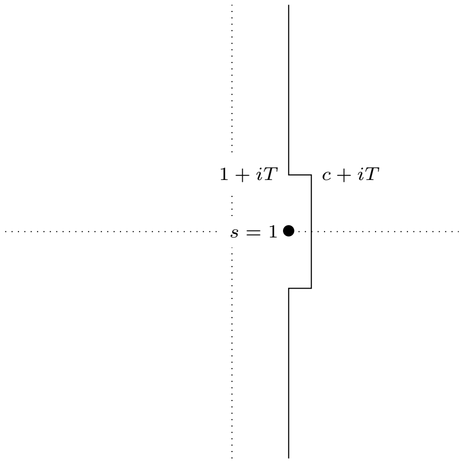
Theorem 19.1.2. *We have that $\psi_1(x) \sim x^2/2$.*

Proof. Let us introduce the notation

$$F(s) = -\frac{x^{s+1}}{s(s+1)}\frac{\zeta'(s)}{\zeta(s)}.$$

The above estimate for $|\zeta'(s)/\zeta(s)|$ implies the estimate $|F(s)| \leq C|\tau|^{\eta-2}$.

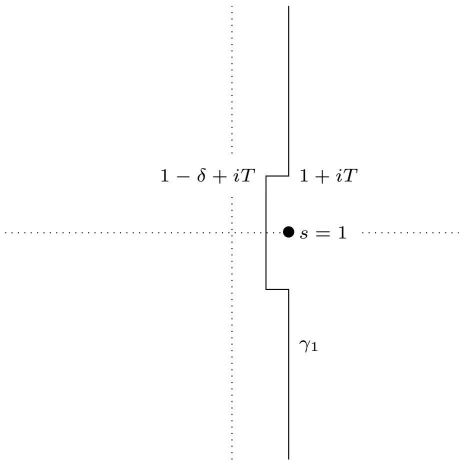
We first claim that the integral over the line $\mathrm{Re}(s) = c$ equals that of the contour $\gamma(T)$ with T large to be picked later given by:



To see this, we observe that $F(s)$ has no poles in the rectangle with corners $(1+iT, c+iT')$, $T' \gg T$. Hence by Cauchy's theorem the integral over the boundary of that rectangle vanishes. To get the result we want, we need to prove that the integral over the top edge vanishes at $T' \to \infty$. This is justified by the above estimate, taking $\eta = 1/2$ say. A similar argument works for the rectangle with corners $(1 - iT, c - iT')$, $T' \gg T$. We conclude that

$$\psi_1(x) = \frac{1}{2\pi i} \int_{\mathrm{Re}(s)=1} F(s)ds = \frac{1}{2\pi i} \int_{\gamma(T)} F(s)ds.$$

We now pass to the integral over the contour $\gamma(T, \delta)$ given by



Here $\delta > 0$ is chosen small enough so that the rectangle with corners $(1 - \delta - iT, 1 + iT)$ contains no zeroes of ζ. This is possible as there are no zeroes of ζ on the line $\mathrm{Re}(s) = 1$.

Since $F(s)$ has a simple pole at $s = 1$ with residue $-x^2/2$, the residue theorem tells us that

$$\frac{1}{2\pi i} \int_{\gamma(T)} F(s)ds - x^2/2 = \frac{1}{2\pi i} \int_{\gamma(T,\delta)} F(s)ds.$$

We now decompose the left hand side into integrals over the five straight line segments, $\gamma_1, \ldots, \gamma_5$ (consecutively ordered, starting at the bottom), and estimate each of these.

- Firstly, by adjusting T we will make

$$\left|\int_{\gamma_1} F(s)ds\right| \leq \epsilon x^2/2 \qquad \left|\int_{\gamma_5} F(s)ds\right| \leq \epsilon x^2/2.$$

  We will give the argument in the case of γ_5 only, as the case of γ_1 is identical. Observe that $|x^s| = x^2$ here, so that by the estimate $F(s) \leq C|\tau|^{-3/2}$ we get

$$\left|\int_{\gamma_1} F(s)ds\right| \leq C'x^2 \int_T^\infty \frac{1}{t^{3/2}}dt \leq C''x^2T^{-1/2},$$

  so letting T be large enough we get the desired estimate. (At this point we also fix our $\delta > 0$.)

- On γ_3 we have that $|x^{1+s}| = x^{2-\delta}$ so that we get a crude estimate (using the fact that remaining terms in $F(s)$ are bounded):

$$\left|\int_{\gamma_3} F(s)ds\right| \leq C'''x^{2-\delta}.$$

- On γ_4 (as before, γ_2 will be similar), we again use that all terms except x^{1+s} are bounded because they are continuous to get

$$\begin{aligned}\left|\int_{\gamma_4} F(s)ds\right| &\leq C'''' \int_{1-\delta}^1 x^{1+\sigma}d\sigma \\ &= C''''(x^2/\log(x) - (x^{2-\delta}/\log(x)) \leq C''''x^2/\log(x).\end{aligned}$$

In the end we get an estimate

$$\left|\psi_1(x) - x^2/2\right| = \left|\frac{1}{2\pi i}\int_{\gamma(T,\delta)} F(s)ds\right| \leq \epsilon x^2/2 + C'''x^{2-\delta} + C''''x^2/\log(x).$$

Dividing by $x^2/2$, this implies that

$$\left|2\psi_1(x)/x^2 - 1\right| \leq \epsilon + 2C'''x^{-\delta} + 2C''''1/\log(x)$$

So as $x \to \infty$, we can make $\limsup_{x\to\infty} \left|2\psi_1(x)/x^2 - 1\right|$ arbitrary small. This completes the proof that $\psi_1(x) \sim x^2/2$. $\square$

19.2 Zeroes of the Riemann zeta function and error terms for the prime number theorem

The following is much stronger estimate on $\psi_1(x)$ (which we will not prove):

$$\psi_1(x) = \frac{x^2}{2} - \sum_\rho \frac{x^{\rho+1}}{\rho(\rho+1)} - E(x),$$

where ρ runs over the zeroes of the Riemann zeta function in the critical strip, and $E(x) = c_1 x + c_0 + \sum_{k=1}^{\infty} \frac{x^{1-2k}}{2k(2k-1)}$ is an essentially linear term collecting the trivial zeroes.

Observe that $|x^{\rho+1}| = x^{\mathrm{Re}(\rho)+1}$. That there are no zeroes on the lines $\mathrm{Re}(s) = 1$ thus says that none of the terms has grows as quickly as $x^2/2$. Under the Riemann hypothesis, $\mathrm{Re}(s) = 1/2$ so we get even better error terms. There is an issue that maybe the infinitely many terms, each not growing as quickly as $x^2/2$, could add up something that does. To understand whether this can happen, let us make a further observation on the Riemann zeta function.

Proposition 19.2.1. $(s-1)\zeta(s)$ *is an entire function with order of growth* ≤ 1.

Sketch of proof. That it is entire follows from the fact that $\zeta(s)$ has a unique simple pole at $s = 1$. For the sake of estimate the order of growth, we may ignore the term $(s-1)$.

There are a couple of estimates to do: Let us first consider the situation when $\mathrm{Re}(s) \geq 1/2$ and $|s|$ is large. Firstly, since $|n^s| = n^{\mathrm{Re}(s)}$, $\zeta(s)$ is bounded on $\mathrm{Re}(s) \geq 2$. Secondly, from a previous lecture, we know that if $s = \sigma + i\tau$ with $\sigma \geq 1/2$ and $|\tau| \geq 1$, then $\zeta(s) \leq C|t|^{1/2+\epsilon}$ for any $\epsilon > 0$. Thus in general case we can estimate $|\zeta(s)| \leq A + B|s|$ under the above condition.

Let us use the function equation for $\xi(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s)$ the Riemann xi function, given by $\xi(s) = \xi(1-s)$, to write

$$\zeta(s) = \pi^{s-1/2} \frac{\Gamma((1-s)/2)}{\Gamma(s/2)} \zeta(1-s).$$

Let us recall that for $\mathrm{Re}(s) > 0$, $\Gamma(s)$ is bounded by $\Gamma(\mathrm{Re}(s))$ which can be estimated by $Ce^{C'\mathrm{Re}(s)\log(\mathrm{Re}(s))}$ and that $1/\Gamma(s)$ has order of growth 1. Thus for $\mathrm{Re}(s) \leq 1/2$ and $|s|$ large, we have that

$$|\zeta(s)| \leq Ce^{C'|s|^{1+\epsilon}}(A + B|s|)$$

for any $\epsilon > 0$. This shows that $\zeta(s)$ has order of growth ≤ 1. □

In particular, we have that $\sum_\rho \frac{1}{|\rho|^{1+\epsilon}} < \infty$ for all ϵ. This rules out that the sum $\sum_\rho \frac{x^{\rho+1}}{\rho(\rho+1)}$ grows too quickly: under the Riemann hypothesis

$$\left| \sum_\rho \frac{x^{\rho+1}}{\rho(\rho+1)} \right| \leq Cx^{3/2}.$$

In fact, the Riemann zeta function and the Riemann xi function both haves order of growth exactly 1, which we can use to show that ζ has infinitely many zeroes in the critical strip. Let us define $F(s) = (1/2 - s)(1/2 + s)\xi(s + 1/2)$, which has removable singularities and hence is entire.

Lemma 19.2.2. *There is a holomorphic function* G *such that* $G(s^2) = F(s + 1/2)$, *with order of growth* $1/2$.

Proof. Observe that the function equation $\xi(s) = \xi(1-s)$ translates to $F(s + 1/2) = F(-(s + 1/2))$, so $s \mapsto F(s + 1/2)$ is even. This implies that the existence of G; we read off from Taylor expansion that there are no odd degree terms. It is easy to see that G must then have half the order of growth of F. □

Proposition 19.2.3. *The Riemann zeta function has infinitely many zeroes in the critical strip.*

Proof. We proved before that a fractional order of growth implies that there are infinitely many zeroes, so G has infinitely many zeroes and hence so has $(1/2-s)(1/2+s)\xi(s+1/2)$ and thus $\xi(s)$. Now recall that $\xi(s)$ has no zeroes outside the critical strip; $\pi^{s/2}\Gamma(s/2)\zeta(s)$ doesn't for $\mathrm{Re}(s) > 1$ because none of the terms do, and by the functional equation the same is true for $\mathrm{Re}(s) < 0$. Thus ξ must have infinitely many zeroes in the critical strip, and hence so does $\zeta(s)$. $\square$

Chapter 20

Conformal equivalences

We have mostly been ignoring the role played by the domain Ω of a holomorphic function. If we mentioned its properties, e.g. whether it is connected or simply-connected, they appeared at conditions in results about holomorphic functions. In this and the next lecture we will study the geometry of Ω from the perspective of complex analysis. This is Section 8.1 of [SS03].

20.1 Conformal equivalences

Two open subsets $U, V \subset \mathbb{C}$ are the same from the point of view of complex analysis if there exists a holomorphic bijection $f\colon U \to V$ between them. We will soon see that inverse is also holomorphic. This is the correct notion, because it means that f induces a bijection between holomorphic functions $V \to \mathbb{C}$ and $U \to \mathbb{C}$. In this lecture we give some first examples, and state the Riemann mapping theorem.

Definition 20.1.1. A *conformal map* is a bijective holomorphic function $f\colon U \to V$.

This terminology comes from the early days of complex analysis: we saw in one of the first lectures

Proposition 20.1.2. *If $f\colon U \to V$ is holomorphic and injective, then $f'(z) \neq 0$ for all $z \in U$. More, $f^{-1}\colon f(U) \to U$ is also holomorphic.*

Proof. The proof is by contradiction, so suppose that $f'(z_0) = 0$. Then for z near z_0 we can write

$$f(z) - f(z_0) = a_k(z - z_0)^k + G(z)$$

for some $k \geq 2$, $a_k \neq 0$ and G holomorphic vanishing to order $k+1$.

Recall Rouché's theorem says that if F and G are holomorphic on an open set containing a circle C and its interior, and $|F(z)| > |G(z)|$ for all $z \in C$, then F and $F + G$ have the same number of zeroes inside C.

As the zero z_0 of f' is necessarily isolated, we can fix a sufficiently small circle C around z_0 such that (a) z_0 is the only point at which f' vanishes inside C, (b) $|a_k(z-z_0)^k| > |G(z)|$ for all $z \in C$. We will take $G(z)$ as above and $F(z) = a_k(z - z_0)^k - w$, with w non-zero small enough so that it is still true that $|F(z)| > |G(z)|$ for $z \in C$. Then Rouché's theorem says that because F has at least 2 zeroes in C, so does

$$F(z) + G(z) = f(z) - f(z_0) - w.$$

Since the zero z_0 of f' is isolated, on a small disk around z_0 the derivative of $f(z) - f(z_0) - w$ only vanish at z_0. However, as $w \neq 0$ the two zeroes of $f(z) - f(z_0) - w$ within C are not at z_0 and hence distinct. This proves that $f(z)$ takes the value $f(z_0) + w$ twice and hence is not injective.

Without loss generality $f(U) = V$, and we write $g := f^{-1}$. We will show that g is holomorphic at $w_0 = f(z_0) \in V$, which is unique by the first part of this proposition. To compute the relevant difference quotient, we take $w = f(z) \in V$ near w_0. We assume $w \neq w_0$ and write

$$\frac{g(w) - g(w_0)}{w - w_0} = \left(\frac{w - w_0}{g(w) - g(w_0)}\right)^{-1} = \left(\frac{f(z) - f(z_0)}{z - z_0}\right)^{-1}.$$

Since $f'(z_0) \neq 0$, we can let $z \to z_0$ to see that this converges. We conclude that g is holomorphic at w_0 with derivative $1/f'(z_0)$. $\square$

Corollary 20.1.3. *If f is a conformal map, then f^{-1} is also a conformal map.*

Thus the following is an equivalence relation on open subsets of $\mathbb{C}$:

Definition 20.1.4. We say that $U, V \subset \mathbb{C}$ are *conformally equivalent* if there is a conformal map $f \colon U \to V$.

We just saw that if f is injective then f' is non-vanishing. The converse is not true, e.g. $z \mapsto z^2$ on $\mathbb{C} \setminus \{0\}$. However, it is still the case that f is locally injective:

Theorem 20.1.5 (The holomorphic inverse function theorem)**.** *If $f \colon U \to V$ has an everywhere non-vanishing derivative, then for around every $z_0 \in U$ there exists a open disk D such that $f|_D$ is injective.*

Proof. We intend to apply Rouché again. Now, for z near z_0 we can write

$$f(z) - f(z_0) = a_1(z - z_0) + G(z)$$

with $G(z)$ vanishing to order 2 at z_0. We will take C small enough such that $|a_1(z-z_0)| > |G(z)|$ for all $z \in C$, $G(z)$ as above and $F(z) = a_1(z - z_0) - w$, with w non-zero small enough so that it is still true $|F(z)| > |G(z)|$ for $z \in C$. Then Rouché's theorem says that because F has a single zero in C, so does

$$F(z) + G(z) = f(z) - f(z_0) - w.$$

This proves that $f(z)$ takes the value $f(z_0) + w$ only a single time on the interior D of C when w is close enough to 0, and hence is injective near z_0. $\square$

20.2 Examples

To build some intuition for conformal equivalence, we give a few examples of conformally equivalent subsets of $\mathbb{C}$.

20.2.1 The half-plane is conformally equivalent to the unit disk

Our first examples are the upper half plane

$$\mathbb{H} := \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\},$$

and the open unit disk

$$\mathbb{D} := \{w \in \mathbb{C} \mid |w| < 1\}.$$

The maps we will use are examples of *Moebius transformation* or *fractional linear transformations*, which we saw appear as isomorphisms of $\mathbb{C}P^1$ before and which we will study in more detail in the next lecture.

Theorem 20.2.1. *The map*

$$\begin{aligned} F\colon \mathbb{H} &\longrightarrow \mathbb{D} \\ z &\longmapsto \frac{i-z}{i+z}, \end{aligned}$$

is a conformal equivalence with inverse

$$\begin{aligned} G\colon \mathbb{D} &\longrightarrow \mathbb{H} \\ w &\longmapsto i\frac{1-w}{1+w}. \end{aligned}$$

Proof. It is obvious that F and G are holomorphic, but we must verify that both maps have the desired codomain. For F, this follows because $-i$ is further from z than i is, so $|i-z| < |i+z|$. For G, we may equivalently check that $\mathrm{Re}(\frac{1-w}{1+w}) > 0$ for $w \in \mathbb{D}$. Writing $w = x + iy$ we have

$$\begin{aligned} \mathrm{Re}\left(\frac{1-w}{1+w}\right) &= \mathrm{Re}\left(\frac{1-x-iy}{1+x+iy}\right) \\ &= \mathrm{Re}\left(\frac{(1-x-iy)(1-x+iy)}{(1+x)^2+y^2}\right) \\ &= \frac{1-x^2-y^2}{(1+x)^2+y^2} > 0 \end{aligned}$$

because $x^2 + y^2 < 1$.

Finally we verify that $G(F(z)) = z$ (leaving $F(G(w)) = w$ for the book):

$$\begin{aligned} G(F(z)) &= i\frac{1 - \frac{i-z}{i+z}}{1 + \frac{i-z}{i+z}} \\ &= i\frac{i + z - (i - z)}{i + z + i - z} \\ &= i\frac{2z}{2i} \\ &= z. \square \end{aligned}$$

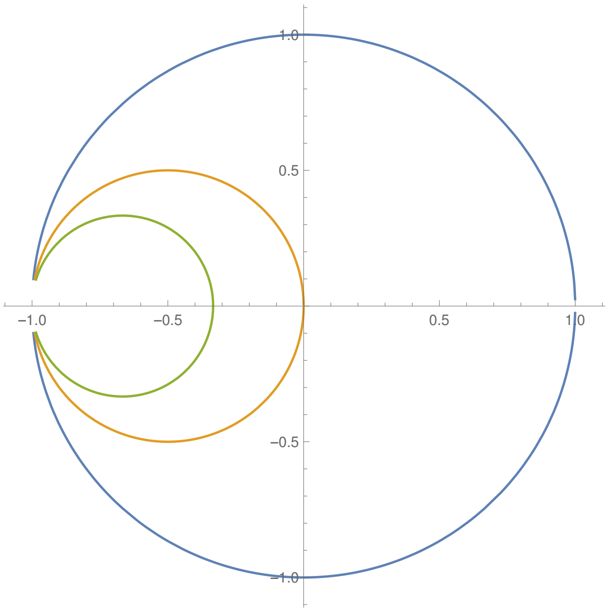


Figure 20.1 The images under F of $[-20, 20]$ (blue), $[-20+i, 20+i]$ (orange), $[-20+2i, 20+2i]$ (green).

20.2.2 Further examples

Some of these examples are pictured in Figure 1 on [SS03, p. 213].

Example 20.2.2 (Translations)*.* For $a \in \mathbb{C}$, the map

$$z \longmapsto z + a$$

has the effect of translating. Its inverse is given by $z \mapsto z - a$. For example, it gives a conformal equivalence from $\mathbb{D} = D_1(0)$ to $D_1(c)$. When a is real, it gives a conformal equivalence from $\mathbb{H}$ onto itself.

Example 20.2.3 (Dilation and rotation)*.* For $c \in \mathbb{C} \setminus \{0\}$, the map

$$z \longmapsto cz$$

has the effect of dilating by $|c|$ and rotating by $\arg(z)$. It inverse is given by $z \mapsto c^{-1}z$. For example, when $|c| = 1$ it gives a conformal equivalence from $\mathbb{D}$ onto itself. When c is real, it gives a conformal equivalence from $\mathbb{H}$ onto itself.

Example 20.2.4 (Powers). The map

$$\begin{aligned} S = \{z \in \mathbb{C} \mid 0 < \arg(z) < \pi/n\} &\longrightarrow \mathbb{H} \\ z &\longmapsto z^n \end{aligned}$$

is a conformal equivalence from the sector S onto the upper half plane. Its inverse is $z \mapsto z^{1/n}$ (whose definition requires a choice of branch of logarithm, we take the one where we delete the positive real axis). In fact, this works for αn with $0 < \alpha < 2$.

Example 20.2.5 (Logarithm and exponential). Taking the principal branch of the logarithm, the map

$$\begin{aligned} \mathbb{C} \setminus \{z \in \mathbb{R} \mid z \leq 0\} &\longrightarrow \{z \in \mathbb{C} \mid 0 < \operatorname{Im}(z) < \pi\} \\ z &\longmapsto \log(z) \end{aligned}$$

gives a conformal equivalence between a complex plane with the negative real axis deleted, and the vertical strip of complex numbers whose imaginary part lies in $(0, 1)$. Its inverse is of course $z \mapsto e^z$.

We can restrict to intersection of the domain with $\mathbb{D}$; its codomain is then the half-strip $\{z \in \mathbb{C} \mid 0 < \operatorname{Im}(z) < \pi, \text{ and } \operatorname{Re}(z) < 0\}$.

20.3 The Riemann mapping theorem

The Riemann theorem is a deep result in complex analysis, which describes all open subsets $U \subset \mathbb{C}$ which are conformally equivalent to the open unit disk $\mathbb{D}$. Recall that U is simply-connected if every closed curve is homotopy to a constant map.

Theorem 20.3.1 (Riemann mapping theorem)**.** *Suppose that U is connected, simply-connected and not equal to $\mathbb{C}$. Then U is conformally equivalent to $\mathbb{D}$. Moreover, if we fix $z_0 \in U$ then there is a* unique *conformal map $f: U \to \mathbb{D}$ such that $f(z_0) = 0$ and $f'(z_0)$ is a positive real number.*

Let us recall why there is no conformal equivalence $\mathbb{C} \to \mathbb{D}$. If this existed, we would have a bounded entire function, which is constant by Liouville's theorem and hence not injective.

It is a purely topological argument that if U is conformally equivalent to $\mathbb{D}$, then it needs to be simply-connected.

Lemma 20.3.2. *If we have a conformal equivalence $f: U \to \mathbb{D}$, then U is simply-connected.*

Proof. If γ is a closed curve in U, then $f(\gamma)$ is a closed curve in $\mathbb{D}$. Since $\mathbb{D}$ is simply-connected, there is a homotopy H from $f(\gamma)$ to a constant map. Then $f^{-1}H$ is a homotopy from $f^{-1}(f(\gamma)) = \gamma$ to a constant map. $\square$

Chapter 21

The Schwartz lemma and conformal automorphisms

In the last chapter we discussed conformal equivalence of open subsets of $\mathbb{C}$; U and V are conformally equivalent if there exists an bijective holomorphic function $U \to V$. Today we discuss what happens when U is equal to V, in the case that $U = \mathbb{D}$ or $V = \mathbb{D}$. Along the way we prove a very useful lemma. This is Section 8.2 of [SS03].

21.1 Schwartz lemma

The Schwartz lemma is an extremely useful result about holomorphic maps $\mathbb{D} \to \mathbb{D}$:

Theorem 21.1.1 (Schwartz lemma). *Suppose that $f\colon \mathbb{D} \to \mathbb{D}$ is a holomorphic map with $f(0) = 0$. Then we have*

(i) $|f(z)| \leq |z|$ for all $z \in \mathbb{D}$ and equality holds if and onlny if $z \mapsto c \cdot z$ with $|c| = 1$,

(ii) $|f'(0)| \leq 1$ and equality holds if and only f is given by $z \mapsto c \cdot z$ with $|c| = 1$.

In both case, $c = f'(0)$.

Proof. For part (i), we start with the observation that since f has a zero at 0, $f(z)/z$ has a removable singularity at 0 and hence extends to a holomorphic function. This extension takes the value $f'(0)$ at the origin. Since $|f(z)| \leq 1$ we have that

$$\left|\frac{f(z)}{z}\right| \leq \frac{1}{|z|}.$$

By the maximum modulus principle, the same inequality holds for any w with $|w| \leq |z|$:

$$\left|\frac{f(w)}{w}\right| \leq \frac{1}{|z|}.$$

Letting $|z| \to 1$ we obtain $|\frac{f(w)}{w}| \leq 1$ or equivalently $|f(w)| \leq |w|$. This completes the first part of (i). For the second part, observe that if $|f(z_0)| = |z_0|$ then $\frac{f(z)}{z}$ attains a local maximum in its interior and needs to be constant, necessarily given by some c of absolute value 1.

For part (ii), recall that the value of $f(z)/z$ at 0 is equal to $f'(0)$ and hence in absolute value ≤ 1. If it is equal to 1, we can once again apply the maximum modulus principle as above. $\square$

21.2 Conformal automorphisms of the upper half-plane and unit disk

21.2.1 Conformal automorphisms

Let $U \subset \mathbb{C}$ be a open subset, then we define

$$\mathrm{Aut}(U) = \{\text{conformal equivalences } f \colon U \to U\}.$$

This is a group under composition: its unit is provided by the identity function, we know that if f is a conformal equivalence so if f^{-1}, and if $f_1, f_2 \colon U \to U$ are holomorphic bijections then so is their composition $f_2 \circ f_1$.

Lemma 21.2.1. *If U and V are conformally equivalent, then there is an isomorphism of groups* $\mathrm{Aut}(U) \cong \mathrm{Aut}(V)$.

Proof. Let $g \colon U \to V$ be a conformal equivalence, then we have a map

$$\begin{aligned} g^* \colon \mathrm{Aut}(V) &\longrightarrow \mathrm{Aut}(U) \\ f &\longmapsto g^{-1} \circ f \circ g. \end{aligned}$$

This is a bijection because it has $(g^{-1})^*$ as an inverse. It is a group homomorphism because

$$g^*(\mathrm{id}_V) = g^{-1} \circ g = \mathrm{id}_U,$$

$$g^*(f_2 \circ f_1) = g^{-1} \circ f_2 \circ f_1 \circ g = (g^{-1} \circ f_2 \circ g) \circ (g^{-1} \circ f_1 \circ g) = g^*(f_2) \circ g^*(f_1). \qquad \square.$$

Example 21.2.2. Some time ago we studied the conformal automorphisms of $\mathbb{C}$, and proved that it is given by translation, dilation, and rotation:

$$\mathrm{Aut}(\mathbb{C}) = \{z \mapsto az + b \mid a \neq 0\}.$$

21.2.2 The conformal automorphisms of $\mathbb{D}$

We next use the Schwarz lemma to determine the conformal automorphisms of $\mathbb{D}$. Last lecture we saw that the rotations

$$z \longmapsto cz \qquad \text{for } |c| = 1$$

are examples. Further examples are given by the Blaschke factors for $|\alpha| < 1$

$$\begin{aligned} \beta_\alpha \colon \mathbb{D} &\longrightarrow \mathbb{D} \\ z &\longmapsto \frac{\alpha - z}{1 - \overline{\alpha} z}. \end{aligned}$$

You have proven its properties in the very beginning of the course, and we've used it a couple of times since. Let us recall these properties below and give alternative proofs:

Lemma 21.2.3.

(i) β_α *is indeed a holomorphic map* $\mathbb{D} \to \mathbb{D}$*,*

(ii) β *interchanges* α *and* 0*,*

(iii) $\beta_\alpha \circ \beta_\alpha = \mathrm{id}_{\mathbb{D}}$ *and hence it is bijective.*

Proof. Instead of a direct computation, we can prove (i) by observing that β_α is holomorphic on an open neighborhood of $\overline{\mathbb{D}}$ and that $\beta_\alpha(1) = \frac{\alpha-1}{1-\overline{\alpha}}$ has absolute value 1. By the maximum modulus principle we thus must have $|\beta_\alpha(z)| < 1$ if $|z| < 1$.

For (ii), we simply compute that

$$\beta_\alpha(\alpha) = 0 \qquad \beta_\alpha(0) = \alpha.$$

For (iii), we observe that $\beta_\alpha \circ \beta_\alpha$ is a map $\mathbb{D} \to \mathbb{D}$ which fixes the origin. Hence we can apply the Schwartz lemma; to prove our result we need to show that the derivative at 0 is 1. By the chain rule, this is the product of the derivatives of β_α at 0 and α:

$$\beta'_\alpha(z) = \frac{-1}{1-\overline{\alpha}z} + \overline{\alpha}\frac{\alpha - z}{(1-\overline{\alpha}z)^2}.$$

So we get $\beta_\alpha(0) = -1 + |\alpha|^2$ and $\beta_\alpha(\alpha) = -\frac{1}{1-|\alpha|^2}$, which multiply to 1. □

Thus we can get many automorphisms by composing Blaschke factors with rotations. It doesn't really matter in which order we do this, because

$$\beta_\alpha(cz) = \frac{\alpha - cz}{1-\bar{\alpha}cz} = c\frac{\alpha\bar{c} - z}{1-\bar{\alpha}cz} = c\beta_{\alpha\bar{c}}(z).$$

Theorem 21.2.4. *Every conformal equivalence of* $\mathbb{D}$ *is given by*

$$z \longmapsto c\beta_\alpha(z)$$

for c *with* $|c| = 1$ *and* α *with* $|\alpha| < 1$*.*

Proof. Suppose we are given a conformal automorphism $f\colon \mathbb{D} \to \mathbb{D}$ with $f(0) = \alpha$. Then $g = \beta_\alpha \circ f$ is another conformal automorphism fixing the origin. Applying the Schwartz lemma, we get that

$$|g(z)| \leq |z|.$$

1 Also applying the Schwartz lemma to g^{-1}, we get

$$|g^{-1}(w)| \leq |w|,$$

and writing $w = g(z)$ we see

$$|z| \leq |g(z)|.$$

We conclude that $|g(z)| = |z|$ and hence g is a rotation by the Schwartz lemma. □

Observing that $\beta_0(z) = -z$ is the only case when a rotation is a Blaschke factor, we see that the c and α in previous theorem are almost unique: if $f(0) = 0$, then there is a unique representation of f as $z \mapsto cz$. If $f(0) = \alpha$, then $\beta_\alpha \circ f$ has a unique such representation and we conclude that f is $z \mapsto c\beta_{\alpha\bar{c}}(z)$ with unique $|c| = 1$ and $0 < |\alpha| < 1$.

Setting $\alpha = 0$ we get:

Corollary 21.2.5. *The only conformal automorphisms of $\mathbb{D}$ which fix the origin are rotations.*

Example 21.2.6. We can use the classification of conformal automorphisms of $\mathbb{D}$ to prove the uniqueness part of the Riemann mapping theorem: this said that given a connected simply-connected open subset $U \subsetneq \mathbb{C}$ and $z_0 \in U$ there is a unique conformal equivalence $f : U \to V$ such that $f(z_0) = 0$ and $f'(z_0)$ is real and positive. For uniqueness, suppose that $f, \tilde{f} : U \to \mathbb{D}$ both have these properties. Then $\tilde{f} \circ f : \mathbb{D} \to \mathbb{D}$ is a conformal map fixing the origin, hence a rotation, with positive real derivative at the origin, hence the identity map.

Setting $z = 0$, we see that (of course this only uses the Blaschke factors):

Corollary 21.2.7. $\mathrm{Aut}(\mathbb{D})$ *acts transitively on* $\mathbb{D}$.

We still need to determine the group structure on the set (c, α) of coefficients of a conformal equivalence $z \mapsto c\beta_\alpha(z)$. We will do so by determining the conformal automorphisms of $\mathbb{H}$ instead.

21.2.3 The conformal automorphisms of $\mathbb{H}$

Since there is a conformal equivalence

$$\begin{aligned} F : \mathbb{H} &\longrightarrow \mathbb{D} \\ z &\longmapsto \frac{i - z}{i + z} \end{aligned}$$

we get an isomorphism

$$F^* : \mathrm{Aut}(\mathbb{D}) \xrightarrow{\sim} \mathrm{Aut}(\mathbb{H}).$$

In principle, one can use this to compute $\mathrm{Aut}(\mathbb{H})$. We'll take a slightly different route, first constructing many elements of $\mathrm{Aut}(\mathbb{H})$ and then using the above result to verify we have found all. Suppose we are given a (2×2)-matrix with real entries

$$M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

with $\det(M) = ad - bc = 1$. Then we can define a map, called a *fractional linear transformation*,

$$\begin{aligned} f_M : \mathbb{H} &\longrightarrow \mathbb{H} \\ z &\longmapsto \frac{az + b}{cz + d}. \end{aligned}$$

Clearly if two matrices differ by $-\mathrm{id}$, they give the same fractional linear transformation. This is in fact the only way that f_M can be equal to $f_{M'}$. I'll leave this as an exercise.

Theorem 21.2.8. *Every conformal equivalence of $\mathbb{H}$ is given by f_M, where M is unique up to replacing it by $-M$.*

Proof. We first verify that f_M maps $\mathbb{H}$ to $\mathbb{H}$: if $\mathrm{Im}(z) > 0$, then

$$\begin{aligned}\mathrm{Im}\left(\frac{az+b}{cz+d}\right) &= \mathrm{Im}\left(\frac{(az+b)(c\overline{z}+d)}{|cz+d|^2}\right)\\ &= \mathrm{Im}\left(\frac{ac|z|^2+bd+adz+bc\overline{z}}{|cz+d|^2}\right)\\ &= \frac{(ad-bc)\mathrm{Im}(z)}{|cz+d|^2} > 0.\end{aligned}$$

Next we prove that $f_{M_1} \circ f_{M_2} = f_{M_1 M_2}$ with $M_1 = \left[\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right]$, $M_2 = \left[\begin{smallmatrix} a' & b' \\ c' & d'\end{smallmatrix}\right]$:: it is given by

$$\begin{aligned}f_{M_2} f_{M_1}(z) &= \frac{a\frac{a'z+b'}{c'z+d'}+b}{c\frac{a'z+b'}{c'z+d'}+d}\\ &= \frac{a(a'z+b')+b(c'z+d')}{c(a'z+b')+d(c'z+d')}\\ &= \frac{(aa'+bc')z+(ab'+bd')}{(ca'+dc')z+(cb'+dd')}\\ &= f_{M_1M_2}(z).\end{aligned}$$

This shows that each f_M is a conformal equivalence with inverse $f_{M^{-1}}$.

Next we show that the f_M act transitively on $\mathbb{H}$. It suffices to prove at every $z \in \mathbb{H}$ can be mapped to i. We first note that setting $d = 0$ so that $bc = -1$, we get

$$\mathrm{Im}\left(\frac{az+b}{cz}\right) = \frac{\mathrm{Im}(z)}{|cz|^2}$$

so by adjusting c we can make the imaginary part i. That is, we shall take

$$M_1 = \begin{bmatrix} 0 & -c^{-1} \\ c & 0 \end{bmatrix}.$$

Then $f_{M_1}(z) = b + i$ for some $b \in \mathbb{R}$, and we can move this to i by applying f_{M_2} with

$$M_2 = \begin{bmatrix} 1 & -b \\ 0 & 1 \end{bmatrix}.$$

Next we investigate what happens when we transfer $f_{M_\theta} \in \mathrm{Aut}(\mathbb{H})$ with

$$M_\theta = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix}$$

to $\mathrm{Aut}(\mathbb{H})$ using the inverse $G\colon \mathbb{D} \to \mathbb{H}$ of F given by $z \mapsto i\frac{1-z}{1+z}$. Since $f_{M_\theta}(i) = \frac{\cos(\theta)i+\sin(\theta)}{-\sin(\theta)i+\cos(\theta)} = i$, $G^*(f_{M_\theta})$ fixes the origin and thus is a rotation. Its derivative at the origin can be computed using the chain rule as

$$(G^*(f_{M_\theta})'(0) = F'(i)f'_{M_\theta}(i)G'(0) = e^{2i\theta}$$

with

$$G'(z) = -i\frac{1}{1+z} - i\frac{1-z}{(1+z)^2} \qquad \text{so } G'(0) = -2i$$
$$F'(z) = -\frac{1}{i+z} - \frac{i-z}{(i+z)^2} \qquad \text{so } F'(i) = -\frac{2i}{.}$$

$$f'_{M_\theta}(z) = \frac{\cos(\theta)}{-\sin(\theta)z + \cos(\theta)} + \sin(\theta)\frac{\cos(\theta)z + \sin(\theta)(z)}{(-\sin(\theta)z + \cos(\theta))^2}$$
$$\text{so } f'_{M_\theta}(i) = \frac{\cos(\theta)}{-\sin(\theta)i + \cos(\theta)} + \frac{\sin(\theta)i}{-\sin(\theta)i + \cos(\theta)} = e^{2i\theta}.$$

Thus $G^*(f_{M_\theta})$ is rotation by $e^{2i\theta}$. Applying the inverse F^* of G^* we see that all rotations go to conformal equivalences of the form f_{M_θ}.

Under the map $F\colon \mathbb{D} \to \mathbb{H}$, the origin goes to i. If $f\colon \mathbb{H} \to \mathbb{H}$ is a conformal equivalence sending i to z, we can compose with an f_M to get a new conformal equivalence $g \coloneqq f_M \circ f$ which fixes i. Then under $G^*\colon \mathrm{Aut}(\mathbb{H}) \to \mathrm{Aut}(\mathbb{D})$ this gets mapped to a conformal equivalence of $\mathbb{D}$ fixing the origin, i.e. a rotation by $e^{2i\theta}$. But rotation $z \mapsto e^{2i\theta}z$ goes under $F^*\colon \mathrm{Aut}(\mathbb{H}) \to \mathrm{Aut}(\mathbb{D})$ to f_{M_θ}. This implies that $g = f_{M_\theta}$ and hence $f = f_{M^{-1}M_\theta}$.

For uniqueness, suppose that $f_{M_1} = f_{M_2}$. By composing with f_M moving $f_{M_1}(i) = z = f_{M_2}(i)$ to i, we get that $f_{MM_1} = f_{MM_2}$, both fixing the origin. By the above argument, this is equal to a rotation matrix f_{M_θ}. We conclude that $MM_1M_\theta^{-1} = MM_2M_\theta^{-1}$, up to a sign, and hence $M_1 = M_2$ up to a sign. □

Example 21.2.9. If an automorphism f of $\mathbb{H}$ fixes two points it is the identity. Indeed, by conjugating with an automorphism sending $f(z)$ to i, we may assume it fixes i. It hence corresponds to a rotation of $\mathbb{D}$ which fixes three points. But if a rotation fixes two points it is the identity.

The advantage of the previous proof is that allows us to understand $\mathrm{Aut}(\mathbb{H})$ as a group. It is given by:

Definition 21.2.10. $\mathrm{PSL}_2(\mathbb{R})$ is the quotient group of (2×2)-matrices with real entries

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

and determinant 1, under composition, by the central subgroup $\pm\mathrm{id}$.

So what we have just shown is that

$$\mathrm{Aut}(\mathbb{D}) \cong \mathrm{Aut}(\mathbb{H}) = \mathrm{PSL}_2(\mathbb{R}).$$

Example 21.2.11. An alternative description of $\mathrm{PSL}_2(\mathbb{R})$ is as follows. Let $\mathrm{SU}(1,1)$ denote the group of (2×2)-matrices with entries in $\mathbb{C}$ which fix the Hermitian form

$$\left\langle \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}, \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \right\rangle = z_1\overline{w_1} - z_2\overline{w_2},$$

under composition. (A Hermitian form is like a bilinear form over $\mathbb{C}$, expect it is linearity in the second entry involves a complex conjugation.)

We claim that $\mathrm{PSL}_2(\mathbb{R}) \cong \mathrm{SU}(1,1)$. Indeed, setting

$$J = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

the condition is that $\overline{A}^t JA = J$. Writing $A = \left[\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right]$, this gives

$$\begin{bmatrix} |\alpha|^2 - |\gamma|^2 & \overline{\alpha}\beta - \overline{\gamma}\delta \\ \overline{\gamma}\alpha - \overline{\delta}\beta & -|\delta|^2 + |\beta|^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

so we get $\gamma = \overline{\beta}$, $\delta = \overline{\alpha}$ and $|\alpha|^2 - |\beta|^2 = 1$. That is,

$$A = \begin{bmatrix} \alpha & \beta \\ \overline{\beta} & \overline{\alpha} \end{bmatrix}$$

We can construct a homomorphism

$$\begin{aligned} \mathrm{SU}(1,1) &\longrightarrow \mathrm{Aut}(\mathbb{D}) \\ \begin{bmatrix} \alpha & \beta \\ \overline{\beta} & \overline{\alpha} \end{bmatrix} &\longmapsto \left(z \mapsto \frac{\alpha z + \beta}{\overline{\beta} z + \overline{\alpha}} \right). \end{aligned}$$

I'll it to leave to verify that this preserves the unit disk. That it is a homomorphism is exactly the same computation as before. By picking $\alpha = e^{i\theta}$ and $\beta = 1$ we can realize all rotations. By picking $\alpha = -1$ and $|\beta| < 1$, we get $z \mapsto \frac{\beta - z}{1 - \overline{\beta} z}$ and hence can realize all Blaschke factors. Thus $\mathrm{SU}(1,1) \to \mathrm{Aut}(\mathbb{D})$ is surjective. I'll again leave injectivity to you, but it is essentially because α and β can be recovered from the value and derivative of the automorphism at the origin.

Thus we have isomorphisms of groups

$$\mathrm{SU}(1,1) \xrightarrow{\sim} \mathrm{Aut}(\mathbb{D}) \xrightarrow{\sim} \mathrm{Aut}(\mathbb{H}) \xleftarrow{\sim} \mathrm{PSL}_2(\mathbb{R}).$$

What is the explicit isomorphism?

Chapter 22

Several complex variables

We end with a short preview of functions of several complex variables. This is not part of the material for the exam. Our reference is [Ran03].[1]

22.1 Holomorphic functions of several complex variables

As the name suggests, functions of several complex variables are complex-valued functions defined on an open subset Ω of $\mathbb{C}^n$.

Definition 22.1.1. A function $f\colon \mathbb{C}^n \supset \Omega \to \mathbb{C}$ is *holomorphic* if it is holomorphic in each variable separately.

As in the case $n = 1$, the most important properties of such functions are derived from an integral formula. Let's study this in the case $n = 2$ for concreteness. For $z = (z_1, z_2) \in \mathbb{C}^2$ we can fix z_1 and apply the Cauchy integral formula in the second variable to write

$$f(z_1, z_2) = \frac{1}{2\pi i} \int_{|\zeta_2 - a_2| < r_2} \frac{f(z_1, \zeta_2)}{\zeta_2 - z_2} d\zeta_2,$$

for those $z = (z_1, z_2)$ satisfying $|\zeta_2 - a_2| < r_2$. Let us now also apply the Cauchy integral formula in the first variable:

$$f(z_1, z_2) = \frac{1}{(2\pi i)^2} \int_{|\zeta_1 - a_1| < r_1} \int_{|\zeta_2 - a_2| < r_2} \frac{f(\zeta_1, \zeta_2)}{(\zeta_1 - z_1)(\zeta_2 - z_2)} d\zeta_2 d\zeta_1,$$

for those $z = (z_1, z_2)$ satisfying $|\zeta_j - a_j| < r_j$ for $j = 1, 2$. Defining the *polydisk* $P(z, r) \subset \mathbb{C}^2$ to be $\{(z_1, z_2) \mid |\zeta_j - z_j| < r_j \text{ for } j = 1, 2\}$ (this is just a product of disks in each copy of $\mathbb{C}$), we can write this as

$$f(z_1, z_2) = \frac{1}{(2\pi i)^2} \int_{\partial_0 P(a, r)} \frac{f(\zeta_1, \zeta_2)}{(\zeta_1 - z_1)(\zeta_2 - z_2)} d\zeta_1 d\zeta_2,$$

with $\partial_0 P(z, r) = \{(z_1, z_2) \mid |\zeta_j - z_j| = r_j \text{ for } j = 1 \text{ or } j = 2\}$ (this looks like a product of circles). Thus we see that the Cauchy integral formula goes through, but with disks

[1] I have posted it to Canvas.

generalized in a somewhat surprising manner: not to higher-dimensional disks but to polydisks.

From this expression we get several generalizations of results we proved for holomorphic functions:

- By differentiating under the integral sign, we obtain that the partial derivatives of f are again holomorphic functions of two variables.
- Similarly, we obtain that locally f can be written as a Taylor series of two variables

$$f(z_1, z_2) = \sum_{i,j} \frac{1}{i!j!} \frac{\partial^i \partial^j f}{\partial z_1^i \partial z_2^j}(a_1, a_2) \cdot (z_1 - a_1)^i (z_j - a_j)^j.$$

  This at least converges on some polydisk $P(a, r)$ around $a = (a_1, a_2)$, but later we will see that understanding the precise domain of convergence of such power series is more subtle.
- As a consequence, the uniqueness of extensions holds. Using power series, this is easily deduced from the fact that if f vanishes on an infinite set of points which has a convergence point, it vanishes on the entire component of its domain containing the convergence point.

In other words, most of the theory for holomorphic functions generalizes. In the next section I will discuss one feature of the high-dimensional setting which is very different, which leads to a new phenomena which we discuss in the third section.

22.2 The Hartogs extension theorem

The most important feature distinguishing several complex variables from a single complex variable concerns the existence of extensions of holomorphic functions:

Theorem 22.2.1 (Hartogs extension theorem)**.** *Suppose that B is a bounded connected open subset of $\mathbb{C}^n$ with connected boundary ∂B. If $n \geq 2$ then every holomorphic function f defined on a connected open neighborhood of ∂B has a unique holomorphic extension to B.*

This is false for $n = 1$: $f(z) = 1/z$ is holomorphic near $\partial D_1(0)$ but does not extend to $D_1(0)$. Let's prove a lemma, which is also false for $n = 1$ (with same counterexample), that illustrates why $n \geq 2$ is important:

Lemma 22.2.2. *Suppose that $K \subset \mathbb{C}^n$ is compact and $n \geq 2$. Then every bounded holomorphic function $\mathbb{C}^n \setminus K$ is constant on the unbounded component of $\mathbb{C}^n \setminus K$. In particular it extends to all of $\mathbb{C}^n$.*

Proof. Let's again assume $n = 2$. Suppose that $|z| < R$ for $z \in K$. Then whenever $|z| \geq R$ we have that $z_1 \mapsto f(z_1, z_2)$ is a bounded entire function, hence constant by Liouville's theorem. This means that for $|z| \geq R$, f is independent of the second variable. Reversing the role of z_1 and z_2, we see that f is also independent of the first variables. This shows that f is constant on $|z| \geq R$. By the uniqueness of extensions, we see that f is constant on all of $\mathbb{C}^2 \setminus K$. ☐

To understand the proof in general, we recall a trick we used before to define Laurent expansions of holomorphic functions f defined on an open neighborhood of the annulus $\mathbb{A} = \{z \in \mathbb{C} \mid r \leq |z| \leq R\}$. If we apply the residue theorem to $\zeta \mapsto \frac{f(\zeta)}{\zeta - z}$ with a contour given by almost completing the two circles C_R and C_r and connecting them by a strip of width δ, and $\delta \to 0$, we get that

$$f(z) = \frac{1}{2\pi i} \int_{C_R} \frac{f(\zeta)}{\zeta - z} d\zeta - \frac{1}{2\pi i} \int_{C_r} \frac{f(\zeta)}{\zeta - z} d\zeta. \tag{22.1}$$

The first term defines holomorphic function on $D_R(0)$ Similarly, the second term $z \mapsto \frac{1}{2\pi i} \int_{C_r} \frac{f(\zeta)}{\zeta - z} d\zeta$ defines a holomorphic function on $\mathbb{C} \setminus \overline{D}_r(0)$ and is bounded as $|z| \to \infty$. Thus if Lemma 22.2.2 had held, the latter would in fact extend to all of $\mathbb{C}$ and we would have obtained our desired extension.

The idea of proof of Theorem 22.2.1 is that (22.1) generalizes to several variables, through the *Bochner–Martinelli formula*. As in this case Lemma 22.2.2 does hold, this proves the result for annuli, and for the general case, we just need to observe that we can replace the boundaries of higher-dimensional annuli with the boundaries of a more complicated shape adapted to B.

Example 22.2.3. The Hartogs extension theorem implies that in several complex variables *all* isolated singularities are removable. Here an isolated singularity such that f is defined on a neighborhood over it but not at the point itself, and removable means that the function in question extends over it. In fact, it just suffices that the singularities we are considering contained in some bounded subset. Indeed, it is codimension 1 analytic subsets instead of points that play the role of singularities. As soon as a singularity is codimension ≥ 2 it is removable.

22.3 Domains of holomorphy and convergence of power series

The curious consequence of the Hartogs extension theorem is that there are subsets $\Omega \subset \mathbb{C}^n$ such that *every* holomorphic function $f \colon \Omega \to \mathbb{C}^n$ extends to some strictly larger open subset. An Ω for which this is not the case has a name:

Definition 22.3.1. A *domain of holomorphy* is a subset $\Omega \subset \mathbb{C}^n$ such that at least one holomorphic function $f \colon \Omega \to \mathbb{C}$ cannot be extend to any strictly larger open subset.

It turns out to be equivalent to say that for all $b \in \partial\Omega$ there exists a holomorphic function $f_b \colon \Omega \to \mathbb{C}$ which does not extend to an open subset containing b.

Example 22.3.2. Every open subset is a domain of holomorphy when $n = 1$: for $b \in \partial\Omega$ take $f_b = \frac{1}{z-b}$. This is the reason we did not see domains of holomorphy appearing in our course; every open subset is one. Products of open subsets of $\mathbb{C}$ are domains of holomorphy for the same reason.

Example 22.3.3. Convex subsets Ω are domain of holomorphy. To see this, observe that by translating and rotation, we may assume that $b = 0$ and Ω lies in $\mathrm{Re}(z_1) > 0$ (this uses convexity). Then $1/z_1$ does not extend over b.

There is an interesting connection between domains of holomorphy to convergence of power series. Let us assume the power series we are investigating is centered at the origin, i.e. of the form $\sum_{i,j} a_{ij} z_1^i z_2^j$. Our leading question is: what can we say about the interior Ω of the subset of $\mathbb{C}^n$ where this converges absolutely? We call this the *domain of convergence*.

If $r_1, r_2 > 0$ are such that

$$\sup_{i,j} |a_{i,j}| r_1^i r_2^j < \infty,$$

then the sum $\sum_{i,j} a_{i,j} z_1^i z_2^j$ converges absolutely on the polydisc $P(r) = \{(z_1, z_2) \in \mathbb{C}^2 \mid |z_i| < r_i\}$. Indeed, we can then estimate our sums as

$$\left| \sum_{i,j} a_{i,j} z_1^i z_2^j \right| \leq \sum_{i,j} |a_{i,j}| |z_1|^i |z_2|^j \leq \sum_{i,j} C |z_1/r_1|^i |z_2/r_2|^j,$$

which converges. Thus we conclude that

Lemma 22.3.4. *If $p = (p_1, p_2)$ lies in the domain of convergence Ω, then Ω contains the polydisk $P(|p|)$ with $|p| = (|p_1|, |p_2|)$.*

This is quite reminiscent of the single-variable situation. However, there is a second more subtle property. Suppose we have two points $p, q \in \mathbb{C}^2$ in the domain of convergence. Then we can pick M such that

$$\sup_{i,j} |a_{ij}| |p_1|^i |p_2|^j \leq M \qquad \sup_{i,j} |a_{ij}| |q_1|^i |q_2|^j \leq M.$$

It is then easy to see that when we pick $\lambda \in [0, 1]$ and define $s \in \mathbb{C}^2$ by

$$s = (s_1, s_2) = (|p_1|^\lambda |q_1|^{1-\lambda}, |p_2|^\lambda |q_2|^{1-\lambda})$$

we have that

$$\sup_{ij} |a_{ij}| |s_1|^i |s_2|^j \leq M.$$

Thus the power series converges then and hence on the entire polydisk $P(s)$. We conclude that:

Lemma 22.3.5. *The domain of convergence Ω has the property that*

$$\{(\log |z_1|, \log |z_2|) \mid (z_1, z_2) \in \Omega\} \subset \mathbb{R}^2$$

is convex.

We will say that Ω is *logarithmically convex* if it satisfies the condition of the previous lemma.

Theorem 22.3.6. *The following are equivalent for Ω satisfying the condition in Lemma 22.3.4:*

- *Ω is a domain of convergence for some power series.*

- Ω *is logarithmically convex.*
- Ω *is a domain of holomorphy.*

Example 22.3.7. Since the unit disk $\{(z_1, z_2) \mid |z_1|^2 + |z_2|^2 < 1\} \subset \mathbb{C}^2$ satisfies the condition in Lemma 22.3.4 and is logarithmically convex, it is a domain of holomorphy and the domain of convergence for some power series (which one?). Of course, we already knew the former because it is convex.

Chapter 23

Final recap

To help you prepare for the final, let me recall what material we covered in the second half of the course. We covered Chapters 5, 6, and 7 of [SS03] as well as Sections 8.1 and 8.2. Before the midterm, we covered Chapters 1, 2, 3, with the exception of Sections 2.5.5 and 3.7.

23.1 Midterm material

This course was about holomorphic functions:

Definition 23.1.1. For $\Omega \subset \mathbb{C}$ an open subset, $f\colon \Omega \to \mathbb{C}$ is *holomorphic* if it is complex-differentiable at each point in $z \in \Omega$.

In the first part of the course we proved that these functions are very well-behaved, and in particular have the following basic properties:

- If $f\colon \Omega \to \mathbb{C}$ is holomorphic, it is infinitely many times complex differentiable. In particular, its derivative is again holomorphic.
- If $f\colon \Omega \to \mathbb{C}$ is the limit of a sequence of holomorphic functions which converges on uniformly on compact subsets, it is holomorphic. There are similar results for sums and integrals.
- Every holomorphic function f is locally equal to a convergent power series $\sum_{n=0} a_n(z-z_0)^n$.
- If $f, g\colon U \to \mathbb{C}$ are holomorphic and agree on an open subset V, they agree on all connected components of U which meet V.
- If $f\colon \Omega \to \mathbb{C}$ is holomorphic and $C \subset \Omega$ is toy contour whose interior is also contained in Ω, then $\int_C f(z)dz = 0$.
- More generally, if $\gamma_0, \gamma_1\colon [0,1] \to \Omega$ are piecewise smooth paths with the same endpoint which are homotopic while keeping the endpoints fixed and $f\colon \Omega \to \mathbb{C}$ is holomorphic, then $\int_{\gamma_0} f(z)dz = \int_{\gamma_1} f(z)dz$.

We obtained more advanced results in Chapter 3: the argument principle, Rouchéś theorem, the open mapping theorem, and the maximum modulus principle. However, there is one particular result I want to highlight, because of its prominent role in later chapters: the *residue theorem*.

Recall that if $f\colon \Omega \to \mathbb{C}$ has isolated singularities, these come in three types: (i) removable ones, (ii) poles, (iii) essential singularities. We are only interested in the case of poles; removable ones we will remove and essential ones are too hard. If $f\colon \Omega \to \mathbb{C}$ only has poles as isolated singularities we say it is *meromorphic*. If z_0 is a pole f, then near z_0 f can be written as

$$\frac{a_{-n}}{(z-z_0)^n} + \frac{a_{n-1}}{(z-z_0)^{n-1}} + \cdots + \frac{a_{-1}}{z-z_0} + g(z),$$

with $g(z)$ a holomorphic function which is defined not only near z_0 but also at z_0. The coefficient a_{-1} is called the *residue* of f at z_0.

Theorem 23.1.2 (Residue theorem). *Suppose that $f\colon \Omega \to \mathbb{C}$ is meromorphic and $C \subset \Omega$ is a toy contour whose interior is also contained in Ω and which avoids the poles of f. Then we have that*

$$\int_C f(z)dz = 2\pi i \sum_{\textit{poles in } \mathrm{int}(C)} \mathrm{res}_{z_0}(f).$$

This is a powerful tool for computing integrals, but also for proving general results about meromorphic functions.

23.2 Zeroes of entire functions

The residue theorem even has applications for *entire functions*, i.e. holomorphic functions $f\colon \mathbb{C} \to \mathbb{C}$. Indeed, if f is not constant equal to 0, then $z \mapsto 1/f(z)$ will have a pole wherever f has a zero. We used it to study what can be said about the set of zeroes of an entire function.

23.2.1 Jensen's formula

The most important estimate in this topic is Jensen's formula:

Theorem 23.2.1 (Jensen's formula). *Suppose f is holomorphic near $\overline{D}_R(0)$, $f(0) \neq 0$, and f doesn't vanishes on C_R. Then if $z_1, \ldots, z_r$ are the zeroes of f within C_R counted with multiplicity, we have*

$$\log|f(0)| = \sum_{k=1}^{r} \log\left(\frac{|z_k|}{R}\right) + \frac{1}{2\pi}\int_0^{2\pi} \log|f(Re^{i\theta})|d\theta.$$

This has a consequence for the *zero-counting function*

$$\begin{aligned} \mathfrak{n}\colon \mathbb{R}_{>0} &\longrightarrow \mathbb{N} \\ r &\longmapsto \text{circles in } D_r \end{aligned}$$

Corollary 23.2.2. *If $z_1, \ldots, x_r$ are the zeroes of f within D_R, then*

$$\int_0^R \frac{\mathfrak{n}(r)}{r} dr = \frac{1}{2\pi}\int_0^{2\pi} \log|f(Re^{i\theta})|d\theta - \log|f(0)|.$$

23.2.2 Functions of finite order

We say more if we impose a growth condition on the entire function f: it is *of order* $\leq \rho$ if $|f(z)| \leq Ae^{B|z|^\rho}$ for some $A, B > 0$.

Theorem 23.2.3. *If f is an entire function of order $\leq \rho$, then $\mathfrak{n}(r) \leq Cr^\rho$ for some $C > 0$ and r sufficiently, and if $z_1, z_2, \ldots$ are the zeroes of f which are not equal to 0 then for $s > \rho$ we have*

$$\sum_{k=1}^{\infty} \frac{1}{|z_k|^s} < \infty.$$

In fact, such functions are nearly determined by their zeroes, as polynomials are. Recall the *canonical factors*

$$E_k(z) := \begin{cases} 1 - z & \text{if } k = 0, \\ (1-z)e^{z+z^2/2+\cdots+z^k/k} & \text{if } k \geq 1. \end{cases}$$

(You can remember these are roughly $(1-z)e^{-\log(1-z)}$, but using only partial Taylor expansions for $-\log(1-z)$.) These are versions of $1 - z$ with more control on their growth behavior.

Theorem 23.2.4 (Hadamard factorization)**.** *Suppose f is entire and has order of growth ρ. Let k be an integer with $k \leq \rho_0 < k + 1$. If $a_1, a_2, \ldots$ denote the non-zero zeroes of f, then*

$$f(z) = e^{P(z)} z^m \prod_{n=1}^{\infty} E_k(z/a_n).$$

Here P is a polynomial of degree $\leq k$, and m is the order of the zero of f at 0 (this order can be 0).

Example 23.2.5. The function $\frac{\sin(\pi z)}{\pi}$ has order 1, and its Hadamard factorization simplifies to

$$\frac{\sin(\pi z)}{\pi} = z \prod_{n=1}^{\infty} (1 - z^2/n^2).$$

(The sine function has order 1, but the exponentials in the canonical factor for the zeroes n and $-n$ cancel.)

23.3 The Gamma function, Riemann zeta function, and the prime number theorem

The next goal was to prove the following result about the distribution of the prime numbers. This is phrased in terms of the prime counting function

$$\begin{aligned} \pi \colon \mathbb{R}_{>0} &\longrightarrow \mathbb{R} \\ x &\longmapsto \#\{\text{primes } p \leq x\}. \end{aligned}$$

Theorem 23.3.1. *We have that* $\pi(x) \sim x/\log(x)$*, which means that* $\lim_{x\to\infty} \frac{\pi(x)}{x/\log(x)}$ *exists and is equal to* 1*.*

You interpret this as $\pi(x)$ as growing like $x/\log(x)$ asymptotically. This result is proven by studying a pair of special holomorphic functions, the most important of which is the *Riemann zeta function*

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{\text{primes } p} \frac{1}{1-p^{-s}} \qquad \text{for } \mathrm{Re}(s) > 1.$$

We proved it has several properties:

- It extends uniquely to a holomorphic function $\zeta\colon \mathbb{C}\setminus\{1\} \to \mathbb{C}$ with a simple pole at $s = 1$.
- It satisfies a functional equation, best phrased in terms of $\xi(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s)$:
$$\xi(s) = \xi(1-s).$$
- The only zeroes outside the *critical strip* $0 < \mathrm{Re}(s) < 1$ are the "trivial zeroes" at $s = -2, -4, -6, \ldots$. In particular, there are no zeroes on $\mathrm{Re}(s) = 1$.
- Moreover, we can bound $1/\zeta(s)$ and $\zeta'(s)$ near the line $\mathrm{Re}(s) = 1$ in terms of $|\mathrm{Im}(s)|$.
- The Riemann hypothesis, one of the most famous open conjectures, says that the only non-trivial zeroes lie on the line $\mathrm{Re}(s) = 1/2$.

Let us focus our attention on the functional equation: it is used all over the place, e.g. in the continuation of $\zeta(s)$ to all of $\mathbb{C}\setminus\{1\}$ and in the results about zeroes. Furthermore, it involves another function that we studied: the *Gamma function*

$$\Gamma(s) = \int_0^\infty e^{-t}t^{s-1}dt \qquad \text{for } \mathrm{Re}(s) > 0,$$

which extends uniquely to a holomorphic function $\Gamma\colon \mathbb{C}\setminus\{0,-1,-2,\ldots\} \to \mathbb{C}$ with simple poles at the non-positive integers.

23.4 Conformal equivalences

At the end, we shifted our attention from holomorphic functions to their domains. We studied the following equivalence relation on open subsets $U \subset \mathbb{C}$:

Definition 23.4.1. We say that $U, V \subset \mathbb{C}$ are *conformally equivalent* if there is a holomorphic bijection $U \to V$.

To see this is indeed an equivalence relation, we proved that if $f\colon U \to V$ is a holomorphic bijection, then its inverse $f^{-1}\colon V \to U$ is also holomorphic. For example, the upper half-plane $\mathbb{H}$ is conformally equivalent to the open unit disk $\mathbb{D}$, and in fact the Riemann mapping theorem (which we didn't prove) says that any connected simply-connected open subset $U \subsetneq \mathbb{C}$ is conformally equivalent to $\mathbb{D}$. This makes it interesting to study the conformal automorphisms of $\mathbb{D}$ (or equivalently, of $\mathbb{H}$). Using the Schwartz lemma, we proved these are given by compositions of rotations and Blaschke factors.

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