University of Toronto

Faculty of Applied Science and Engineering

First name (please write as legibly as possible within the boxes)

Last name

Student number

ESC194F Calculus I
Final Exam
December 2019

No calculators or aids
There are 12 questions, each question is worth 10 marks

Examiners: P.C. Stangeby and J.W. Davis

$x \to x_0$ $\lim_{x \to x_0} f(x) = A$

Equivalent Infinitesimal replacement

$x \to 0$ $\sin x \sim \tan x \sim \arcsin x \sim \arctan x \sim x$

原则: 只能替换乘除因子

$\sin x \sim \tan x$
$\sim \arctan x \sim \arcsin x \sim x$

1) ~~Evaluate the following~~ limits:

a) $\lim_{x\to 0}(\csc x - \cot x)$

b) $\lim_{x\to 0}(1-2x)^{1/x}$

c) $\lim_{x\to 0}\frac{\sin^{-1}x}{x}$

"$\infty - \infty$"

$x \sim x$ $\qquad \frac{0}{x^2}$

a) $\lim_{x\to 0}\left(\frac{1}{\sin x} - \frac{1}{\tan x}\right) = \lim_{x\to 0}\frac{\tan x - \sin x}{\underset{x}{\underset{\wr}{\sin x}}\;\underset{x}{\underset{\wr}{\tan x}}}$ $\sim \frac{1}{2}x^3$

$= \lim_{x\to 0}\frac{\frac{1}{2}x^3}{x^2} = \lim_{x\to 0}\frac{1}{2}x = 0$

arctan $\qquad$ sin x $\qquad$ x $\qquad$ arcsin x $\qquad$ tan x

$\frac{1}{6}x^3$

Exercise 1. $\lim_{x\to 0}\frac{\arcsin x - \sin x}{3(\sin x)^3} = \lim_{x\to 0}\frac{\frac{1}{3}x^3}{3x^3} = \frac{1}{9}$

$\lim_{x\to\infty}\frac{\sin x}{x} = 0$ (无穷小量 × 有界 = 0)

$\lim_{x\to 0}\frac{\sin x}{x} = 1$

"1^{∞}"

$$\lim_{x\to 0}(1+x)^{\frac{1}{x}} = e.$$

b) $$\lim_{x\to 0}(1-2x)^{\frac{1}{x}}$$

~1 ~∞

$$= \lim_{x\to 0}\left[(1-2x)^{-\frac{1}{2x}}\right]^{-2} = e^{-2}.$$

$$A = \lim_{x\to 0} -2x\cdot\frac{1}{x} = -2$$

对于1^{∞}，三部曲：

① 化为标准形式：$\lim_{x\to\square}[1+f(x)]^{g(x)}$

② 令 $\lim_{x\to\square}[1+f(x)]^{g(x)} = e^{A}$

③ $A = \lim_{x\to\square} f(x)\cdot g(x)$

$$e^{\frac{1}{6}}$$

Exercise 2. $$\lim_{x\to 0}\left(\frac{\arcsin x}{x}\right)^{\frac{1}{x^2}} = ?$$

① $$\lim_{x\to 0}\left(1+\frac{\arcsin x - x}{x}\right)^{\frac{1}{x^2}} = e^{A}$$

③ $$A = \lim_{x\to 0}\frac{\arcsin x - x}{x^3} = \frac{1}{6}$$

c) $$\lim_{x\to 0}\frac{\arcsin x}{x} = 1$$

Example. $$\lim_{x\to\infty}\frac{x^4+3x^2+2x+6}{x^3+2x^2+x+1}$$ (not exist)

分子分母同时除以最高次数

$2x^4+3x^3+x+1$ (抓大头)

$$= \lim_{x\to\infty}\frac{1+\frac{3}{x^2}+\frac{2}{x^3}+\frac{6}{x^4}}{2+\frac{3}{x}+\frac{1}{x^3}+\frac{1}{x^4}} = \frac{1}{2}$$

$$\lim_{x\to 0}\frac{x^4+3x^2+2x}{x^3+2x^2+x}=2$$ (抓小头)

分子分母同时除以最低次数

$$\lim_{x\to 0}\frac{x^3+3x+2}{x^2+2x+1}=2$$

连续　$f(x)$　$x=x_0$ 是否连续？

$$\lim_{x\to x_0}f(x)=f(x_0)$$

$f(x)$　$x=x_0$ 是否可导？

$f(x)$ 在 $x=x_0$ 处导数：$\lim_{x\to x_0}\frac{f(x)-f(x_0)}{x-x_0}$ $\left(\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}\right)$

导数定义　是否存在

$\tan\frac{\pi}{3}$

$$\lim_{x\to\frac{\pi}{3}}\frac{\tan x-\sqrt{3}}{x-\frac{\pi}{3}}=?$$

$f'(x_0)=3$, $f(x_0)=1$

$$\lim_{x\to x_0}4\frac{f(x)-1}{x-x_0}=?$$

12

2) Find the derivative of: $3x^3$, $cos(3x)$, $ln(x^{1/2})$, e^{-x^2}, 3^{x^2}.

$(kx^n)' = k \cdot n \cdot x^{n-1}$

$(\sin x)' = \cos x$

$(\cos x)' = -\sin x$

$(\tan x)' = \sec^2 x$

$(\arcsin x)' = \frac{1}{\sqrt{1-x^2}}$

$(\arccos x)' = -\frac{1}{\sqrt{1-x^2}}$

$(\arctan x)' = \frac{1}{1+x^2}$

$(e^x)' = e^x$

$(\ln x)' = \frac{1}{x}$

$(a^x)' = a^x \cdot \ln a$

$(\log_a x)' = \frac{1}{x \cdot \ln a}$

$f(x) \quad g(x)$

$[f(g(x))]'$

Chain Rule $\frac{d\, f(g(x))}{dx} = f'(g(x)) \cdot g'(x)$

$y = \arcsin x$

$\cos^2 y + \sin^2 y = 1$

两边同时取sin ⇒ $\boxed{\sin y = x}$

两边同时求导 $\cos y \cdot y' = 1 \Rightarrow y' = \frac{1}{\cos y} = \frac{1}{\sqrt{1-\sin^2 y}} = \frac{1}{\sqrt{1-x^2}}$

导数 和微分.

3) Find the anti-derivative of: $3x^3$, $cos(3x)$, xe^{-x^2}, $(9+x^2)^{-1}$, 3^x .

求导

$\frac{df(x)}{dx} = f'(x)$

微分

$df(x) = f'(x)dx$

$\Updownarrow$

凑微分

$\int 3x^3 dx = \frac{3}{4}x^4 + c$

$\int x \cdot e^{-x^2} dx = -\frac{1}{2} e^{-x^2} + c$

$\int \frac{1}{9+x^2} = \frac{1}{3}\arctan(\frac{x}{3}) + c$

$\int 3^x = \frac{1}{\ln 3} \cdot 3^x + c$

$\int \cos 3x \, dx$

$= \frac{1}{3}\int \cos 3x \, d3x$

$= \frac{1}{3}\sin 3x + C$

$\int \cos u \, du$

$= \sin u + c$

$e^{-x^2} d(-x^2)$

$d(-x^2) = -2x dx$

$d3x \neq dx$

$d3x = 3dx$

$\frac{1}{3}d3x = dx$

Example: $\underline{x^3y^7} - x^2 + y^3 = 0$ 求 $\frac{dy}{dx} = ?$

两边同时x求导: $3x^2y^7 + x^3\, 7y \cdot \frac{dy}{dx} - 2x + 3y^2 \cdot \frac{dy}{dx} = 0$

Applications of Derivatives:

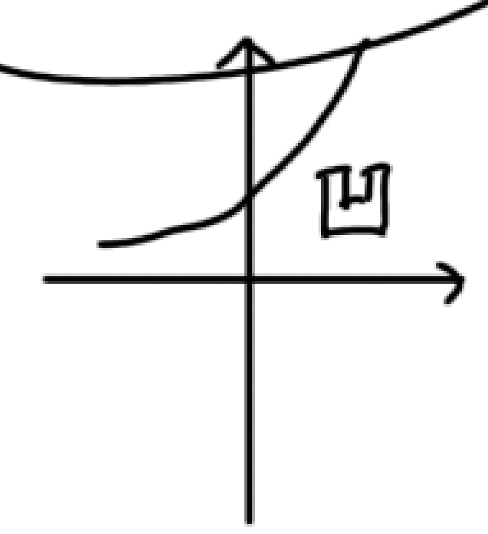
$$\begin{cases} f'(x) > 0 \Rightarrow f(x) \text{ increase} \\ f'(x) < 0 \Rightarrow f(x) \text{ decrease} \\ f'(x) = 0 \Rightarrow f(x) = C \end{cases}$$

单调性

$$\begin{cases} f''(x) < 0 \Rightarrow f(x) \text{ 凸} \\ f''(x) > 0 \Rightarrow f(x) \text{ 凹} \\ f''(x) = 0 \Rightarrow \text{拐点} \end{cases}$$

凹凸性

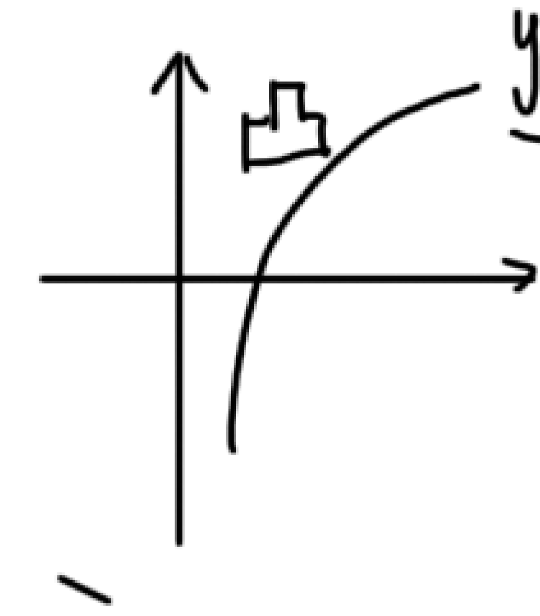
$y = e^x \quad y'' = e^x$



$y = \ln x \quad y' = \frac{1}{x}$

$y'' = -\frac{1}{x^2}$

< 0



The Mean value Theorem.

Rolle's Theorem
$$\left.\begin{array}{l} [a,b] \text{ 连续} \\ (a,b) \text{ 可导} \\ f(a) = f(b) \end{array}\right\} \Rightarrow \exists\, c \in (a,b),\ f'(c) = 0.$$

Lagrange's Theorem
$$\left.\begin{array}{l} [a,b] \text{ 连续} \\ (a,b) \text{ 可导} \end{array}\right\} \Rightarrow \exists\, c \in (a,b),\ f'(c) = \frac{f(b) - f(a)}{b - a}$$

domain: $(0,+\infty)$

range: $(-\infty, \frac{1}{e}]$

Example: $f(x)=\dfrac{\ln x}{x}$

1) 画图 ✓ 2) 找极值 ✓

3) 求渐近线 ✓

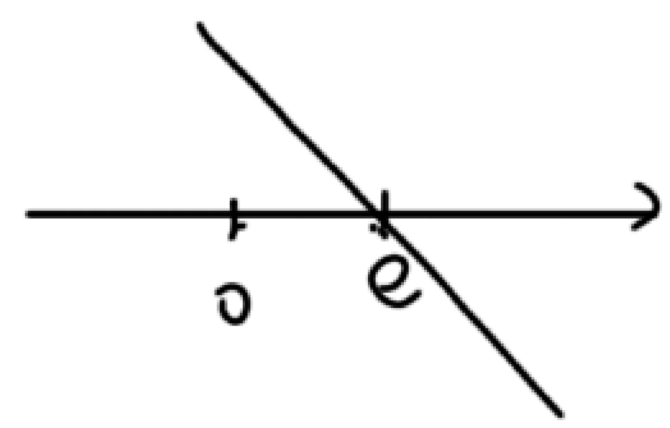
$\dfrac{\ln x}{x}=0$

$\ln x=0 \Rightarrow x=1$

一、求导 $f'(x)=\dfrac{1-\ln x}{x^2}$ $+$

二、找根 $f'(x)=0 \Rightarrow x=e$

三、画导数的草图.



单调性 ⇔ 导数正负

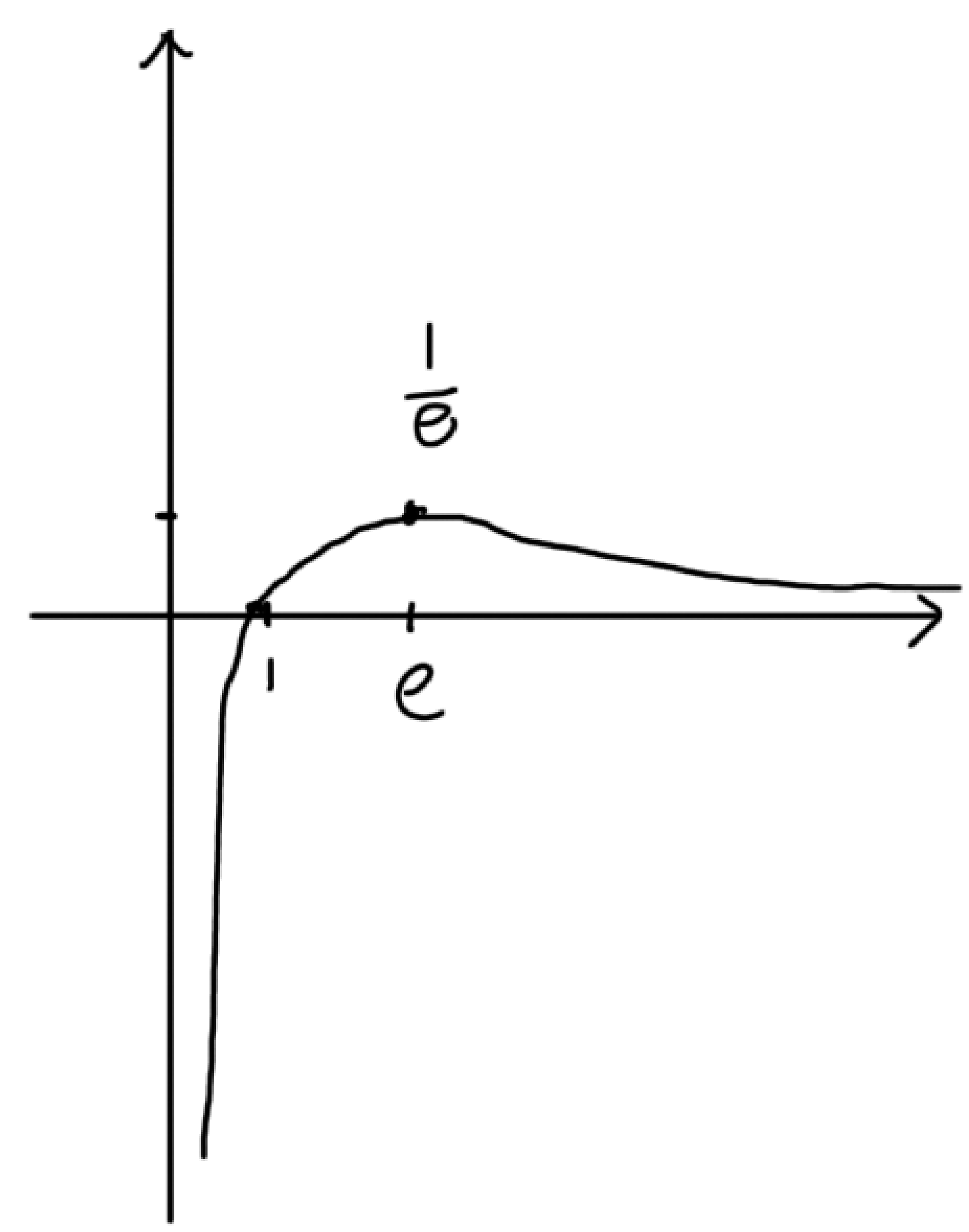
$1-\ln x \downarrow$

令 $f(x)=0 \Rightarrow x=1$

四、画表格

x	$(0,e)$	e	$(e,+\infty)$
$f'(x)$	$+$	0	$-$
$f(x)$	$\nearrow$	极大	$\searrow$

$f(x)_{max}=f(e)=\dfrac{1}{e}$



$x\to+\infty$

$\ln x \ll x^{\alpha} \ll a^{x}$

$x\to+\infty$, $\lim\limits_{x\to+\infty}\dfrac{\ln x}{x}=0$

对数 ≪ 幂 ≪ 指

垂直渐近线　当 $x\to x_0$, $f(x)\to\infty$　将 $x=x_0$ 称为 $f(x)$ 的垂直渐近线

$$\lim_{x\to x_0} f(x)=\infty$$

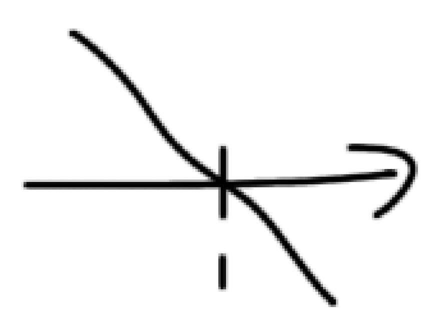
水平渐近线　当 $x\to +\infty$, $f(x)\to A$　将 $y=A$ 称为 $f(x)$ 的水平渐近线

斜渐近线　$\lim_{x\to\infty}[f(x)-(ax+b)]=0$　将 $y=ax+b$ 称为 $f(x)$ 的斜渐近线.

Exercise: $f(x)=\frac{x}{e^x}$　　$f(x)=0 \Rightarrow x=0$.

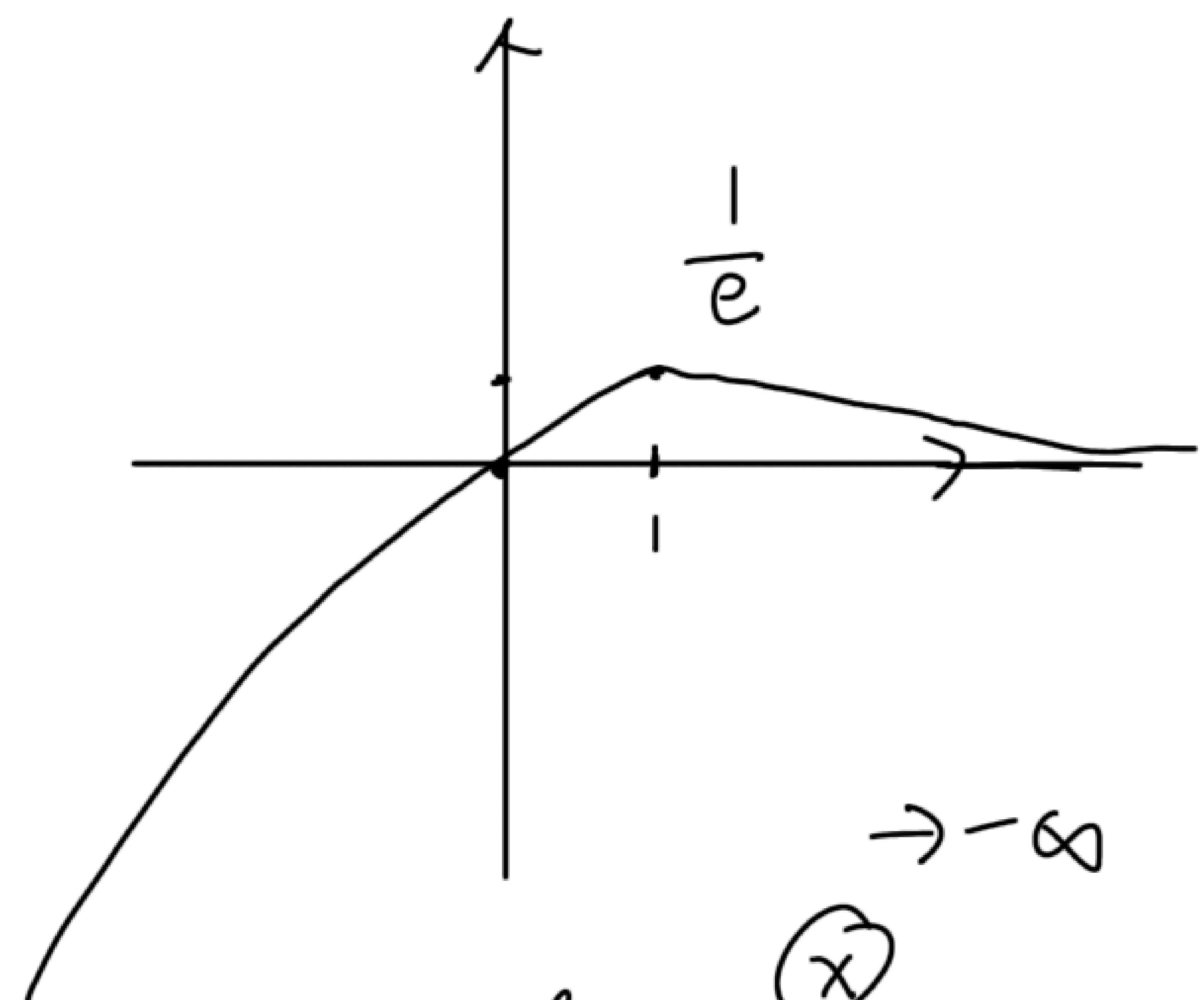
$$f'(x)=\frac{1-x}{e^x}$$

令 $f'(x)=0$, $x=1$

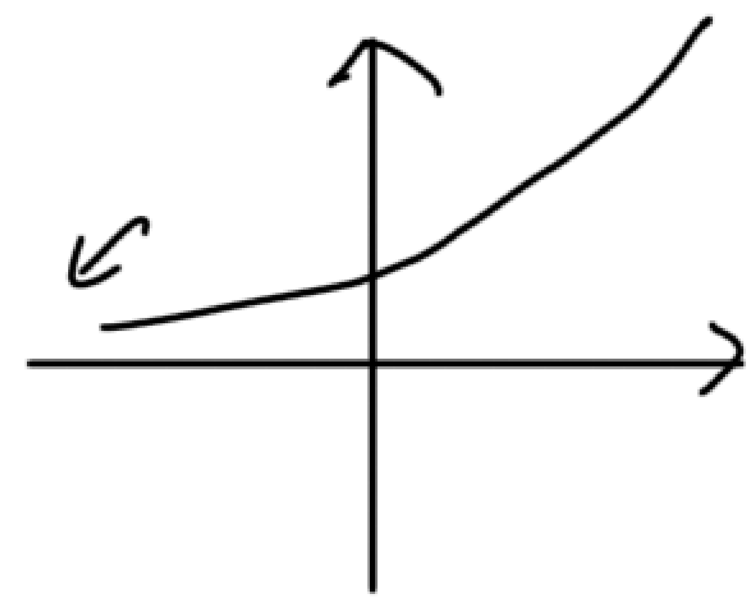


x	$(-\infty,1)$	1	$(1,+\infty)$
$f'(x)$	$+$	0	$-$
$f(x)$	↗	极大	↘

$f(x)_{max}=f(1)=\frac{1}{e}$



$x\to-\infty$, $\lim_{x\to-\infty}\frac{x}{e^x}$



$x\to+\infty$, $\lim_{x\to+\infty}\frac{x}{e^x}=0$

4) At noon, ship A is 150 km west of ship B. Ship A is sailing east at 35 km/hr and ship B is sailing north at 25 km/hr. How fast is the distance between the ships changing at 4:00 pm?

5) a) Find functions f and g such that each function is continuous at $x = 0$, but the composite function, $f \circ g$, is not continuous at 0.

b) What value of b maximizes the integral: $\int_{-1}^{b} x^2(3 - x)dx,\ b > -1$?

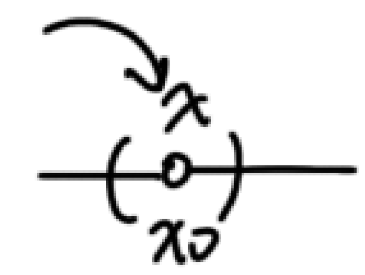
6) A wedge is cut out of a circular cylinder of radius 4 by two planes. One plane is perpendicular to the axis of the cylinder. The other intersects the first at an angle of 30° along a diameter of the cylinder. Find the volume of the wedge.

7) a) **[7 marks]** Provide a $\delta - \varepsilon$ proof that $\lim_{x \to 3} x^2 = 9$.

b) **[3 marks]** Given $f(x) = x^2, c = 3, \varepsilon = 7$, what is the largest δ that will ensure that when $0 < |x - c| < \delta$ then $|f(x) - f(c)| < \epsilon$? Is there a smallest δ?

极限证明: $\lim_{x \to x_0} f(x) = L$. $x_0 - \delta < x < x_0 + \delta$

① $x \to x_0$: $\forall \delta > 0$ $|x - x_0| < \delta$.



② $x \to x_0$, $f(x) \to L$: $|f(x) - L| < \varepsilon$

$L - \varepsilon < f(x) < L + \varepsilon$

8) Find the solution of the differential equation that satisfies the given initial condition:

a) $y' tanx = a + y, \quad y\left(\frac{\pi}{3}\right) = a, \quad 0 < x < \frac{\pi}{2}$

b) $(x^2 + 1)y' + 3x(y - 1) = 0 \quad y(0) = 2$

9) Use the method of undetermined coefficients to find the general solution to the 2nd order DE:

$$y'' - 3y' + 2y = coshx = \tfrac{1}{2}(e^{x}+e^{-x})$$

10) Suppose $f'(x) < 0 < f''(x)$ for $x < a$ and $f'(x) > 0 > f''(x)$ for $x > a$. Prove that f is not differentiable at a.
Hint: Assume that f is differentiable at a, and apply the Mean Value Theorem.

11) a) Prove that $e^{\pi} > \pi^{e}$ by first finding the maximum value of $f(x) = \frac{lnx}{x}$.

b) Sketch a graph of $f(t) = e^{t}$ on an arbitrary interval $[a, b]$. Use the graph and compare areas of regions to prove that: $e^{(a+b)/2} < \frac{e^{b}-e^{a}}{b-a}$.

12) Directly calculate the limit of a Riemann sum to evaluate the area of the region between $f(x) = \sqrt{x},\ x \in [0,2]$ and the x-axis.
Hint 1: Use the non-uniform partition: $x_i = i^2 \frac{2}{n^2}$
Hint 2: $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$