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18.02 Multivariable Calculus
Fall 2007

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18.02 Lecture 21. – Tue, Oct 30, 2007

Test for gradient fields.

Observe: if $\vec{F} = M\hat{i} + N\hat{j}$ is a gradient field then $N_x = M_y$. Indeed, if $\vec{F} = \nabla f$ then $M = f_x$, $N = f_y$, so $N_x = f_{yx} = f_{xy} = M_y$.

Claim: Conversely, if \vec{F} is defined and differentiable at every point of the plane, and $N_x = M_y$, then $\vec{F} = M\hat{i} + N\hat{j}$ is a gradient field.

Example: $\vec{F} = -y\hat{i} + x\hat{j}$: $N_x = 1$, $M_y = -1$, so \vec{F} is not a gradient field.

Example: for which value(s) of a is $\vec{F} = (4x^2 + axy)\hat{i} + (3y^2 + 4x^2)\hat{j}$ a gradient field? Answer: $N_x = 8x$, $M_y = ax$, so $a = 8$.

Finding the potential: if above test says \vec{F} is a gradient field, we have 2 methods to find the potential function f . Illustrated for the above example (taking $a = 8$):

Method 1: using line integrals (FTC backwards):

We know that if C starts at $(0, 0)$ and ends at (x_1, y_1) then $f(x_1, y_1) - f(0, 0) = \int_C \vec{F} \cdot d\vec{r}$. Here $f(0, 0)$ is just an integration constant (if f is a potential then so is $f + c$). Can also choose the simplest curve C from $(0, 0)$ to (x_1, y_1) .

Simplest choice: take $C =$ portion of x -axis from $(0, 0)$ to $(x_1, 0)$, then vertical segment from $(x_1, 0)$ to (x_1, y_1) (picture drawn).

Then $\int_C \vec{F} \cdot d\vec{r} = \int_{C_1+C_2} (4x^2 + 8xy) dx + (3y^2 + 4x^2) dy$:

$$\text{Over } C_1, 0 \leq x \leq x_1, y = 0, dy = 0: \int_{C_1} = \int_0^{x_1} (4x^2 + 8x \cdot 0) dx = \left[\frac{4}{3}x^3 \right]_0^{x_1} = \frac{4}{3}x_1^3.$$

$$\text{Over } C_2, 0 \leq y \leq y_1, x = x_1, dx = 0: \int_{C_2} = \int_0^{y_1} (3y^2 + 4x_1^2) dy = [y^3 + 4x_1^2 y]_0^{y_1} = y_1^3 + 4x_1^2 y_1.$$

$$\text{So } f(x_1, y_1) = \frac{4}{3}x_1^3 + y_1^3 + 4x_1^2 y_1 \text{ (+constant).}$$

Method 2: using antiderivatives:

We want $f(x, y)$ such that (1) $f_x = 4x^2 + 8xy$, (2) $f_y = 3y^2 + 4x^2$.

Taking antiderivative of (1) w.r.t. x (treating y as a constant), we get $f(x, y) = \frac{4}{3}x^3 + 4x^2 y +$ integration constant (independent of x). The integration constant still depends on y , call it $g(y)$.

So $f(x, y) = \frac{4}{3}x^3 + 4x^2 y + g(y)$. Take partial w.r.t. y , to get $f_y = 4x^2 + g'(y)$.

Comparing this with (2), we get $g'(y) = 3y^2$, so $g(y) = y^3 + c$.

Plugging into above formula for f , we finally get $f(x, y) = \frac{4}{3}x^3 + 4x^2 y + y^3 + c$.

Curl.

Now we have: $N_x = M_y \Leftrightarrow^* \vec{F}$ is a gradient field $\Leftrightarrow \vec{F}$ is conservative: $\oint_C \vec{F} \cdot d\vec{r} = 0$ for any closed curve.

(*): \Rightarrow only holds if \vec{F} is defined everywhere, or in a “simply-connected” region – see next week.

Failure of conservativeness is given by the *curl* of \vec{F} :

Definition: $\text{curl}(\vec{F}) = N_x - M_y$.

Interpretation of curl: for a velocity field, curl = (twice) angular velocity of the rotation component of the motion.

(Ex: $\vec{F} = \langle a, b \rangle$ uniform translation, $\vec{F} = \langle x, y \rangle$ expanding motion have curl zero; whereas $\vec{F} = \langle -y, x \rangle$ rotation at unit angular velocity has curl = 2).

For a force field, $\text{curl } \vec{F}$ = torque exerted on a test mass, measures how \vec{F} imparts rotation motion.

$$\text{For translation motion: } \frac{\text{Force}}{\text{Mass}} = \text{acceleration} = \frac{d}{dt}(\text{velocity}).$$

$$\text{For rotation effects: } \frac{\text{Torque}}{\text{Moment of inertia}} = \text{angular acceleration} = \frac{d}{dt}(\text{angular velocity}).$$

18.02 Lecture 22. – Thu, Nov 1, 2007

Handouts: PS8 solutions, PS9, practice exams 3A and 3B.

Green's theorem.

If C is a positively oriented closed curve enclosing a region R , then

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_R \text{curl } \vec{F} dA \quad \text{which means} \quad \oint_C M dx + N dy = \iint_R (N_x - M_y) dA.$$

Example (reduce a complicated line integral to an easy \iint):

Let C = unit circle centered at $(2,0)$, counterclockwise. R = unit disk at $(2,0)$. Then

$$\oint_C ye^{-x} dx + \left(\frac{1}{2}x^2 - e^{-x}\right) dy = \iint_R N_x - M_y dA = \iint_R (x + e^{-x}) - e^{-x} dA = \iint_R x dA.$$

This is equal to area $\cdot \bar{x} = \pi \cdot 2 = 2\pi$ (or by direct computation of the iterated integral). (Note: direct calculation of the line integral would probably involve setting $x = 2 + \cos \theta$, $y = \sin \theta$, but then calculations get really complicated.)

Application: proof of our criterion for gradient fields.

Theorem: if $\vec{F} = M\hat{i} + N\hat{j}$ is defined and continuously differentiable in the whole plane, then $N_x = M_y \Rightarrow \vec{F}$ is conservative ($\Leftrightarrow \vec{F}$ is a gradient field).

If $N_x = M_y$ then by Green, $\oint_C \vec{F} \cdot d\vec{r} = \iint_R \text{curl } \vec{F} dA = \iint_R 0 dA = 0$. So \vec{F} is conservative.

Note: this only works if \vec{F} and its curl are defined everywhere inside R . For the vector field on PS8 Problem 2, we can't do this if the region contains the origin – for example, the line integral along the unit circle is non-zero even though $\text{curl}(\vec{F})$ is zero wherever it's defined.

Proof of Green's theorem.

2 preliminary remarks:

1) the theorem splits into two identities, $\oint_C M dx = -\iint_R M_y dA$ and $\oint_C N dy = \iint_R N_x dA$.

2) additivity: if theorem is true for R_1 and R_2 then it's true for the union $R = R_1 \cup R_2$ (picture shown): $\oint_C = \oint_{C_1} + \oint_{C_2}$ (the line integrals along inner portions cancel out) and $\iint_R = \iint_{R_1} + \iint_{R_2}$.

Main step in the proof: prove $\oint_C M dx = -\iint_R M_y dA$ for “vertically simple” regions: $a < x < b$, $f_0(x) < y < f_1(x)$. (picture drawn). This involves calculations similar to PS5 Problem 3.

LHS: break C into four sides (C_1 lower, C_2 right vertical segment, C_3 upper, C_4 left vertical segment); $\int_{C_2} M dx = \int_{C_4} M dx = 0$ since $x = \text{constant}$ on C_2 and C_4 . So

$$\oint_C = \int_{C_1} + \int_{C_3} = \int_a^b M(x, f_0(x)) dx - \int_a^b M(x, f_1(x)) dx$$

(using along C_1 : parameter $a \leq x \leq b$, $y = f_0(x)$; along C_2 , x from b to a , hence – sign; $y = f_1(x)$).

$$\text{RHS: } - \iint_R M_y \, dA = - \int_a^b \int_{f_0(x)}^{f_1(x)} M_y \, dy \, dx = - \int_a^b (M(x, f_1(x)) - M(x, f_0(x))) \, dx \quad (= \text{LHS}).$$

Finally observe: any region R can be subdivided into vertically simple pieces (picture shown); for each piece $\oint_{C_i} M \, dx = - \iint_{R_i} M_y \, dA$, so by additivity $\oint_C M \, dx = - \iint_R M_y \, dA$.

Similarly $\oint_C N \, dy = \iint_R N_x \, dA$ by subdividing into horizontally simple pieces. This completes the proof.

Example. The area of a region R can be evaluated using a line integral: for example, $\oint_C x \, dy = \iint_R 1 \, dA = \text{area}(R)$.

This idea was used to build mechanical devices that measure area of arbitrary regions on a piece of paper: planimeters (photo of the actual object shown, and principle explained briefly: as one moves its arm along a closed curve, the planimeter calculates the line integral of a suitable vector field by means of an ingenious mechanism; at the end of the motion, one reads the area).

18.02 Lecture 23. – Fri, Nov 2, 2007

Flux. The flux of a vector field \vec{F} across a plane curve C is $\int_C \vec{F} \cdot \hat{\mathbf{n}} \, ds$, where $\hat{\mathbf{n}}$ = normal vector to C , rotated 90° clockwise from $\hat{\mathbf{T}}$.

We now have two types of line integrals: work, $\int \vec{F} \cdot \hat{\mathbf{T}} \, ds$, sums $\vec{F} \cdot \hat{\mathbf{T}}$ = component of \vec{F} in direction of C , along the curve C . Flux, $\int \vec{F} \cdot \hat{\mathbf{n}} \, ds$, sums $\vec{F} \cdot \hat{\mathbf{n}}$ = component of \vec{F} perpendicular to C , along the curve.

If we break C into small pieces of length Δs , the flux is $\sum_i (\vec{F} \cdot \hat{\mathbf{n}}) \Delta s_i$.

Physical interpretation: if \vec{F} is a velocity field (e.g. flow of a fluid), flux measures how much matter passes through C per unit time.

Look at a small portion of C : locally \vec{F} is constant, what passes through portion of C in unit time is contents of a parallelogram with sides Δs and \vec{F} (picture shown with \vec{F} horizontal, and portion of curve = diagonal line segment). The area of this parallelogram is $\Delta s \cdot \text{height} = \Delta s (\vec{F} \cdot \hat{\mathbf{n}})$. (picture shown rotated with portion of C horizontal, at base of parallelogram). Summing these contributions along all of C , we get that $\int (\vec{F} \cdot \hat{\mathbf{n}}) \, ds$ is the total flow through C per unit time; counting positively what flows towards the right of C , negatively what flows towards the left of C , as seen from the point of view of a point travelling along C .

Example: C = circle of radius a counterclockwise, $\vec{F} = x\hat{i} + y\hat{j}$ (picture shown): along C , $\vec{F} \parallel \hat{\mathbf{n}}$, and $|\vec{F}| = a$, so $\vec{F} \cdot \hat{\mathbf{n}} = a$. So

$$\int_C \vec{F} \cdot \hat{\mathbf{n}} \, ds = \int_C a \, ds = a \, \text{length}(C) = 2\pi a^2.$$

Meanwhile, the flux of $-y\hat{i} + x\hat{j}$ across C is zero (field tangent to C).

That was a geometric argument. What about the general situation when calculation of the line integral is required?

Observe: $d\vec{r} = \hat{\mathbf{T}} \, ds = \langle dx, dy \rangle$, and $\hat{\mathbf{n}}$ is $\hat{\mathbf{T}}$ rotated 90° clockwise; so $\hat{\mathbf{n}} \, ds = \langle dy, -dx \rangle$.

So, if $\vec{F} = P\hat{i} + Q\hat{j}$ (using new letters to make things look different; of course we could call the components M and N), then

$$\int_C \vec{F} \cdot \hat{\mathbf{n}} \, ds = \int_C \langle P, Q \rangle \cdot \langle dy, -dx \rangle = \int_C -Q \, dx + P \, dy.$$

(or if $\vec{F} = \langle M, N \rangle$, $\int_C -N dx + M dy$).

So we can compute flux using the usual method, by expressing x, y, dx, dy in terms of a parameter variable and substituting (no example given).

Green's theorem for flux. If C encloses R counterclockwise, and $\vec{F} = P\hat{i} + Q\hat{j}$, then

$$\oint_C \vec{F} \cdot \hat{n} ds = \iint_R \operatorname{div}(\vec{F}) dA, \quad \text{where } \operatorname{div}(\vec{F}) = P_x + Q_y \text{ is the divergence of } \vec{F}.$$

Note: the counterclockwise orientation of C means that we count flux of \vec{F} *out* of R through C .

Proof: $\oint_C \vec{F} \cdot \hat{n} ds = \oint_C -Q dx + P dy$. Call $M = -Q$ and $N = P$, then apply usual Green's theorem $\oint_C M dx + N dy = \iint_R (N_x - M_y) dA$ to get

$$\oint_C -Q dx + P dy = \iint_R (P_x - (-Q_y)) dA = \iint_R \operatorname{div}(\vec{F}) dA.$$

This proof by “renaming” the components is why we called the components P, Q instead of M, N . If we call $\vec{F} = \langle M, N \rangle$ the statement becomes $\oint_C -N dx + M dy = \iint_R (M_x + N_y) dA$.

Example: in the above example ($x\hat{i} + y\hat{j}$ across circle), $\operatorname{div} \vec{F} = 2$, so $\operatorname{flux} = \iint_R 2 dA = 2 \operatorname{area}(R) = 2\pi a^2$. If we translate C to a different position (not centered at origin) (picture shown) then direct calculation of flux is harder, but total flux is still $2\pi a^2$.

Physical interpretation: in an incompressible fluid flow, divergence measures source/sink density/rate, i.e. how much fluid is being added to the system per unit area and per unit time.