University of Toronto Scarborough
Department of Computer & Mathematical Sciences

FINAL EXAMINATION

MATB41H – Techniques of the Calculus of Several Variables I

Examiner: E. Moore

Date: December 15, 2017
Start Time: 9:00AM
Duration: 3 hours

FAMILY NAME: ______________________________

GIVEN NAMES: ______________________________

STUDENT NUMBER: ______________________________

SIGNATURE:______________________________

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The University of Toronto's *Code of Behaviour on Academic Matters* applies to all University of Toronto Scarborough students. The *Code* prohibits all forms of academic dishonesty including, but not limited to, cheating, plagiarism, and the use of unauthorized aids. Students violating the *Code* may be subject to penalties up to and including suspension or expulsion from the University.

NOTES:

- Your signature above indicates that you have abided by the UofT Code of Behaviour while writing this exam.
- NO AIDS.
- No electronic devices of any kind (e.g. calculators, smart phones, smart watches, tablets, etc.) allowed.
- There are 13 numbered pages in the exam. It is your responsibility to ensure that, at the start of the exam, this booklet has all its pages.
- Answer all questions. Explain and justify your answers.
- **Show all your work**. Credit will not be given for numerical answers if the work is not shown. If you need more space use the back of the page or the last page and indicate clearly the location of your continuing work.

question	1	2	3	4	5	6	7	8	9	10	11	12	13	total
marks	12	5	5	8	6	12	8	9	15	6	8	9	7	110

1. **[12 points]**

   (a) Carefully complete the following definition:

   Let $f : U \subset \mathbb{R}^n \to \mathbb{R}^k$ be a given function. We say that f is *differentiable at* $\boldsymbol{a} \in U$ if $\cdots$

   (b) Carefully state the following theorems. Make sure you define your terms.

   i. The Chain Rule for functions of more than one variable.

   ii. The Extreme Value Theorem for real valued functions of several variables.

   iii. The Change of Variables Theorem for multiple integrals.

2. **[5 points]** Let $f : \mathbb{R}^2 \to \mathbb{R}$ be given by

$$f(x, y) = \begin{cases} \dfrac{\sin(x\,y)}{x^2 + y^2} & , \text{ if } (x, y) \neq (0, 0) \\ 0 & , \text{ if } (x, y) = (0, 0) \ . \end{cases}$$

Determine the set of points (x, y) where $f(x, y)$ is continuous.

3. **[5 points]** Give the 4^{th} degree Taylor polynomial about the origin of

$$f(x, y) = \frac{\sin(x\,y)}{1 + x + y} \ .$$

4. **[8 points]** Let $f(x, y) = e^{xy}\sin(x + y)$.

   (a) In what direction(s), starting at $\left(0, \frac{\pi}{2}\right)$, is f increasing the fastest?

   (b) In what direction(s), starting at $\left(0, \frac{\pi}{2}\right)$, is f changing at 50% of its maximum rate?

5. **[6 points]** Let $f : \mathbb{R}^3 \to \mathbb{R}^2$ be given by $f(x, y, z) = (x^2y,\ yz^2)$ and let $g : \mathbb{R}^2 \to \mathbb{R}^5$ be given by $g(x, y) = (xy,\ 2x^2,\ x + y,\ -x,\ y)$. Find Df and Dg and use the Chain Rule to find $D(g \circ f)$.

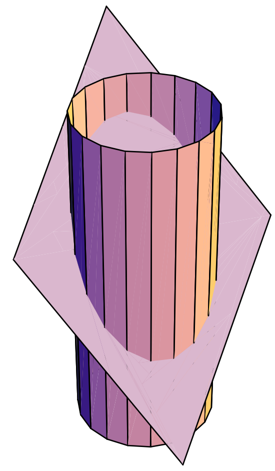
6. **[12 points]** Let $f : \mathbb{R}^2 \to \mathbb{R}$ be given by $f(x, y) = 1 - (x^2 + y^2 - 1)^2$.

(a) Characterize and sketch several level curves of f being careful to indicate where f is zero, positive, negative and not defined. What is the range of f ?

(b) Find the critical points of f and determine the local and global extrema of f or explain why such extrema do not exist.

(c) Find the equation of the tangent plane to the graph of f at $(1, 1, f(1, 1))$?

7. **[8 points]** Let $f(x, y) = 2x^4 - xy^2 + 2y^2$. Find all the critical points of f. For each critical point, determine if that point is a local minimum, a local maximum or a saddle.



8. **[9 points]** Let $f(x, y, z) = x + 2y + 3z$. Find the global extrema of f on the intersection of the surfaces $x^2+y^2 = 1$ and $x - y + z = 1$.
   Justify your answer including an explanation of why global extrema do exist.

9. **[15 points]**

(a) Compute $\displaystyle\iint_D (1 + 2y\cos x)\,dA$, where D is the region bounded by the curve $y = \sqrt{x}$ and the lines $x = 0$ and $y = 3$.

(b) Rewrite the integral $\displaystyle\int_{1/\sqrt{2}}^{1} \int_{\sqrt{1-x^2}}^{x} f(x,y)\,dy\,dx$ with the order of integration reversed.

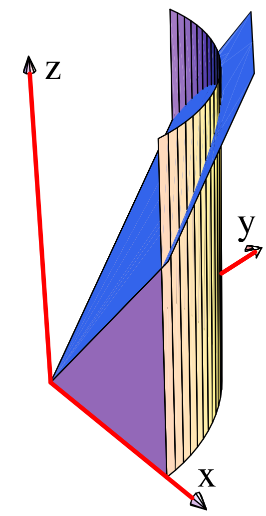
Question 9. (cont'd)

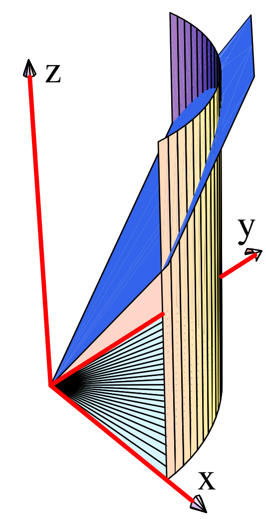
(c) Give an integral in polar coordinates $(r,\ \theta)$ which is equivalent to $\displaystyle\int_0^4 \int_3^{\sqrt{25-x^2}} dy\, dx.$

10. [**6 points**] Sketch the curve given by the polar equation $r = 1 + 2\cos(2\,\theta)$.

11. [**8 points**] Find, in terms of a, the volume of the first octant region bounded above by the plane $z = x + y$ and bounded on one side by the cylinder $x^2 + y^2 = a^2$, where $a > 0$.

    (The enclosed region is shown in the top figure on the right. In the bottom figure the region is shown with the face in the xz–plane removed.)



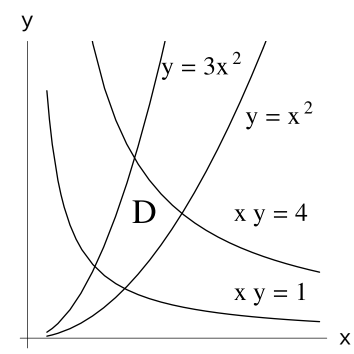


12. **[9 points]** Let B be the interior of the unit sphere, $x^2 + y^2 + z^2 = 1$.

(a) Evaluate $\int_B (x^2 + y^2 + z^2)\, dV$.

(b) Explain why this should or should not give the same answer as $\int_B 1\ dV$.

13. [**7 points**] Use a change of variable to evaluate $\displaystyle\iint_D xy\,dA$, where D is the first quadrant region bounded by $xy = 1$, $xy = 4$, $y = x^2$ and $y = 3x^2$.



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