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18.02 Multivariable Calculus  
Fall 2007

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## 18.02 Lecture 21. – Tue, Oct 30, 2007

### Test for gradient fields.

Observe: if  $\vec{F} = M\hat{i} + N\hat{j}$  is a gradient field then  $N_x = M_y$ . Indeed, if  $\vec{F} = \nabla f$  then  $M = f_x$ ,  $N = f_y$ , so  $N_x = f_{yx} = f_{xy} = M_y$ .

Claim: Conversely, if  $\vec{F}$  is defined and differentiable at every point of the plane, and  $N_x = M_y$ , then  $\vec{F} = M\hat{i} + N\hat{j}$  is a gradient field.

Example:  $\vec{F} = -y\hat{i} + x\hat{j}$ :  $N_x = 1$ ,  $M_y = -1$ , so  $\vec{F}$  is not a gradient field.

Example: for which value(s) of  $a$  is  $\vec{F} = (4x^2 + axy)\hat{i} + (3y^2 + 4x^2)\hat{j}$  a gradient field? Answer:  $N_x = 8x$ ,  $M_y = ax$ , so  $a = 8$ .

**Finding the potential:** if above test says  $\vec{F}$  is a gradient field, we have 2 methods to find the potential function  $f$ . Illustrated for the above example (taking  $a = 8$ ):

**Method 1:** using line integrals (FTC backwards):

We know that if  $C$  starts at  $(0, 0)$  and ends at  $(x_1, y_1)$  then  $f(x_1, y_1) - f(0, 0) = \int_C \vec{F} \cdot d\vec{r}$ . Here  $f(0, 0)$  is just an integration constant (if  $f$  is a potential then so is  $f + c$ ). Can also choose the simplest curve  $C$  from  $(0, 0)$  to  $(x_1, y_1)$ .

Simplest choice: take  $C =$  portion of  $x$ -axis from  $(0, 0)$  to  $(x_1, 0)$ , then vertical segment from  $(x_1, 0)$  to  $(x_1, y_1)$  (picture drawn).

Then  $\int_C \vec{F} \cdot d\vec{r} = \int_{C_1+C_2} (4x^2 + 8xy) dx + (3y^2 + 4x^2) dy$ :

Over  $C_1$ ,  $0 \leq x \leq x_1$ ,  $y = 0$ ,  $dy = 0$ :  $\int_{C_1} = \int_0^{x_1} (4x^2 + 8x \cdot 0) dx = \left[ \frac{4}{3}x^3 \right]_0^{x_1} = \frac{4}{3}x_1^3$ .

Over  $C_2$ ,  $0 \leq y \leq y_1$ ,  $x = x_1$ ,  $dx = 0$ :  $\int_{C_2} = \int_0^{y_1} (3y^2 + 4x_1^2) dy = [y^3 + 4x_1^2y]_0^{y_1} = y_1^3 + 4x_1^2y_1$ .

So  $f(x_1, y_1) = \frac{4}{3}x_1^3 + y_1^3 + 4x_1^2y_1$  (+constant).

**Method 2:** using antiderivatives:

We want  $f(x, y)$  such that (1)  $f_x = 4x^2 + 8xy$ , (2)  $f_y = 3y^2 + 4x^2$ .

Taking antiderivative of (1) w.r.t.  $x$  (treating  $y$  as a constant), we get  $f(x, y) = \frac{4}{3}x^3 + 4x^2y +$  integration constant (independent of  $x$ ). The integration constant still depends on  $y$ , call it  $g(y)$ .

So  $f(x, y) = \frac{4}{3}x^3 + 4x^2y + g(y)$ . Take partial w.r.t.  $y$ , to get  $f_y = 4x^2 + g'(y)$ .

Comparing this with (2), we get  $g'(y) = 3y^2$ , so  $g(y) = y^3 + c$ .

Plugging into above formula for  $f$ , we finally get  $f(x, y) = \frac{4}{3}x^3 + 4x^2y + y^3 + c$ .

### Curl.

Now we have:  $N_x = M_y \Leftrightarrow^* \vec{F}$  is a gradient field  $\Leftrightarrow \vec{F}$  is conservative:  $\oint_C \vec{F} \cdot d\vec{r} = 0$  for any closed curve.

(\*):  $\Rightarrow$  only holds if  $\vec{F}$  is defined everywhere, or in a “simply-connected” region – see next week.

Failure of conservativeness is given by the *curl* of  $\vec{F}$ :

Definition:  $\text{curl}(\vec{F}) = N_x - M_y$ .

**Interpretation of curl:** for a velocity field,  $\text{curl} =$  (twice) angular velocity of the rotation component of the motion.

(Ex:  $\vec{F} = \langle a, b \rangle$  uniform translation,  $\vec{F} = \langle x, y \rangle$  expanding motion have curl zero; whereas  $\vec{F} = \langle -y, x \rangle$  rotation at unit angular velocity has curl = 2).

For a force field, curl  $\vec{F}$  = torque exerted on a test mass, measures how  $\vec{F}$  imparts rotation motion.

For translation motion:  $\frac{\text{Force}}{\text{Mass}} = \text{acceleration} = \frac{d}{dt}(\text{velocity})$ .

For rotation effects:  $\frac{\text{Torque}}{\text{Moment of inertia}} = \text{angular acceleration} = \frac{d}{dt}(\text{angular velocity})$ .

## 18.02 Lecture 22. – Thu, Nov 1, 2007

Handouts: PS8 solutions, PS9, practice exams 3A and 3B.

### Green's theorem.

If  $C$  is a positively oriented closed curve enclosing a region  $R$ , then

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_R \text{curl } \vec{F} \, dA \quad \text{which means} \quad \oint_C M \, dx + N \, dy = \iint_R (N_x - M_y) \, dA.$$

Example (reduce a complicated line integral to an easy  $\iint$ ):

Let  $C$  = unit circle centered at  $(2,0)$ , counterclockwise.  $R$  = unit disk at  $(2,0)$ . Then

$$\oint_C y e^{-x} \, dx + \left(\frac{1}{2}x^2 - e^{-x}\right) dy = \iint_R N_x - M_y \, dA = \iint_R (x + e^{-x}) - e^{-x} \, dA = \iint_R x \, dA.$$

This is equal to area  $\cdot \bar{x} = \pi \cdot 2 = 2\pi$  (or by direct computation of the iterated integral). (Note: direct calculation of the line integral would probably involve setting  $x = 2 + \cos \theta$ ,  $y = \sin \theta$ , but then calculations get really complicated.)

**Application:** proof of our criterion for gradient fields.

Theorem: if  $\vec{F} = M\hat{i} + N\hat{j}$  is defined and continuously differentiable in the whole plane, then  $N_x = M_y \Rightarrow \vec{F}$  is conservative ( $\Leftrightarrow \vec{F}$  is a gradient field).

If  $N_x = M_y$  then by Green,  $\oint_C \vec{F} \cdot d\vec{r} = \iint_R \text{curl } \vec{F} \, dA = \iint_R 0 \, dA = 0$ . So  $\vec{F}$  is conservative.

Note: this only works if  $\vec{F}$  and its curl are defined everywhere inside  $R$ . For the vector field on PS8 Problem 2, we can't do this if the region contains the origin – for example, the line integral along the unit circle is non-zero even though  $\text{curl}(\vec{F})$  is zero wherever it's defined.

**Proof of Green's theorem.** 2 preliminary remarks:

1) the theorem splits into two identities,  $\oint_C M \, dx = -\iint_R M_y \, dA$  and  $\oint_C N \, dy = \iint_R N_x \, dA$ .

2) additivity: if theorem is true for  $R_1$  and  $R_2$  then it's true for the union  $R = R_1 \cup R_2$  (picture shown):  $\oint_C = \oint_{C_1} + \oint_{C_2}$  (the line integrals along inner portions cancel out) and  $\iint_R = \iint_{R_1} + \iint_{R_2}$ .

Main step in the proof: prove  $\oint_C M \, dx = -\iint_R M_y \, dA$  for “vertically simple” regions:  $a < x < b$ ,  $f_0(x) < y < f_1(x)$ . (picture drawn). This involves calculations similar to PS5 Problem 3.

LHS: break  $C$  into four sides ( $C_1$  lower,  $C_2$  right vertical segment,  $C_3$  upper,  $C_4$  left vertical segment);  $\int_{C_2} M \, dx = \int_{C_4} M \, dx = 0$  since  $x = \text{constant}$  on  $C_2$  and  $C_4$ . So

$$\oint_C = \int_{C_1} + \int_{C_3} = \int_a^b M(x, f_0(x)) \, dx - \int_a^b M(x, f_1(x)) \, dx$$

(using along  $C_1$ : parameter  $a \leq x \leq b$ ,  $y = f_0(x)$ ; along  $C_2$ ,  $x$  from  $b$  to  $a$ , hence  $-$  sign;  $y = f_1(x)$ ).

$$\text{RHS: } - \iint_R M_y dA = - \int_a^b \int_{f_0(x)}^{f_1(x)} M_y dy dx = - \int_a^b (M(x, f_1(x)) - M(x, f_0(x))) dx (= \text{LHS}).$$

Finally observe: any region  $R$  can be subdivided into vertically simple pieces (picture shown); for each piece  $\oint_{C_i} M dx = - \iint_{R_i} M_y dA$ , so by additivity  $\oint_C M dx = - \iint_R M_y dA$ .

Similarly  $\oint_C N dy = \iint_R N_x dA$  by subdividing into horizontally simple pieces. This completes the proof.

**Example.** The area of a region  $R$  can be evaluated using a line integral: for example,  $\oint_C x dy = \iint_R 1 dA = \text{area}(R)$ .

This idea was used to build mechanical devices that measure area of arbitrary regions on a piece of paper: planimeters (photo of the actual object shown, and principle explained briefly: as one moves its arm along a closed curve, the planimeter calculates the line integral of a suitable vector field by means of an ingenious mechanism; at the end of the motion, one reads the area).

### 18.02 Lecture 23. – Fri, Nov 2, 2007

**Flux.** The flux of a vector field  $\vec{F}$  across a plane curve  $C$  is  $\int_C \vec{F} \cdot \hat{n} ds$ , where  $\hat{n}$  = normal vector to  $C$ , rotated 90° clockwise from  $\hat{T}$ .

We now have two types of line integrals: work,  $\int \vec{F} \cdot \hat{T} ds$ , sums  $\vec{F} \cdot \hat{T}$  = component of  $\vec{F}$  in direction of  $C$ , along the curve  $C$ . Flux,  $\int \vec{F} \cdot \hat{n} ds$ , sums  $\vec{F} \cdot \hat{n}$  = component of  $\vec{F}$  perpendicular to  $C$ , along the curve.

If we break  $C$  into small pieces of length  $\Delta s$ , the flux is  $\sum_i (\vec{F} \cdot \hat{n}) \Delta s_i$ .

**Physical interpretation:** if  $\vec{F}$  is a velocity field (e.g. flow of a fluid), flux measures how much matter passes through  $C$  per unit time.

Look at a small portion of  $C$ : locally  $\vec{F}$  is constant, what passes through portion of  $C$  in unit time is contents of a parallelogram with sides  $\Delta s$  and  $\vec{F}$  (picture shown with  $\vec{F}$  horizontal, and portion of curve = diagonal line segment). The area of this parallelogram is  $\Delta s \cdot \text{height} = \Delta s (\vec{F} \cdot \hat{n})$ . (picture shown rotated with portion of  $C$  horizontal, at base of parallelogram). Summing these contributions along all of  $C$ , we get that  $\int (\vec{F} \cdot \hat{n}) ds$  is the total flow through  $C$  per unit time; counting positively what flows towards the right of  $C$ , negatively what flows towards the left of  $C$ , as seen from the point of view of a point travelling along  $C$ .

**Example:**  $C$  = circle of radius  $a$  counterclockwise,  $\vec{F} = x\hat{i} + y\hat{j}$  (picture shown): along  $C$ ,  $\vec{F} // \hat{n}$ , and  $|\vec{F}| = a$ , so  $\vec{F} \cdot \hat{n} = a$ . So

$$\int_C \vec{F} \cdot \hat{n} ds = \int_C a ds = a \text{ length}(C) = 2\pi a^2.$$

Meanwhile, the flux of  $-y\hat{i} + x\hat{j}$  across  $C$  is zero (field tangent to  $C$ ).

That was a geometric argument. What about the general situation when calculation of the line integral is required?

Observe:  $d\vec{r} = \hat{T} ds = \langle dx, dy \rangle$ , and  $\hat{n}$  is  $\hat{T}$  rotated 90° clockwise; so  $\hat{n} ds = \langle dy, -dx \rangle$ .

So, if  $\vec{F} = P\hat{i} + Q\hat{j}$  (using new letters to make things look different; of course we could call the components  $M$  and  $N$ ), then

$$\int_C \vec{F} \cdot \hat{n} ds = \int_C \langle P, Q \rangle \cdot \langle dy, -dx \rangle = \int_C -Q dx + P dy.$$

(or if  $\vec{F} = \langle M, N \rangle$ ,  $\int_C -N dx + M dy$ ).

So we can compute flux using the usual method, by expressing  $x, y, dx, dy$  in terms of a parameter variable and substituting (no example given).

**Green's theorem for flux.** If  $C$  encloses  $R$  counterclockwise, and  $\vec{F} = P\hat{i} + Q\hat{j}$ , then

$$\oint_C \vec{F} \cdot \hat{n} ds = \iint_R \operatorname{div}(\vec{F}) dA, \quad \text{where} \quad \operatorname{div}(\vec{F}) = P_x + Q_y \quad \text{is the divergence of } \vec{F}.$$

Note: the counterclockwise orientation of  $C$  means that we count flux of  $\vec{F}$  *out* of  $R$  through  $C$ .

Proof:  $\oint_C \vec{F} \cdot \hat{n} ds = \oint_C -Q dx + P dy$ . Call  $M = -Q$  and  $N = P$ , then apply usual Green's theorem  $\oint_C M dx + N dy = \iint_R (N_x - M_y) dA$  to get

$$\oint_C -Q dx + P dy = \iint_R (P_x - (-Q_y)) dA = \iint_R \operatorname{div}(\vec{F}) dA.$$

This proof by “renaming” the components is why we called the components  $P, Q$  instead of  $M, N$ .

If we call  $\vec{F} = \langle M, N \rangle$  the statement becomes  $\oint_C -N dx + M dy = \iint_R (M_x + N_y) dA$ .

**Example:** in the above example ( $x\hat{i} + y\hat{j}$  across circle),  $\operatorname{div} \vec{F} = 2$ , so  $\text{flux} = \iint_R 2 dA = 2 \text{area}(R) = 2\pi a^2$ . If we translate  $C$  to a different position (not centered at origin) (picture shown) then direct calculation of flux is harder, but total flux is still  $2\pi a^2$ .

Physical interpretation: in an incompressible fluid flow, divergence measures source/sink density/rate, i.e. how much fluid is being added to the system per unit area and per unit time.