18.02 Multivariable Calculus
Fall 2007

**18.02 Lecture 26. – Tue, Nov 13, 2007**

**Spherical coordinates** $(\rho, \phi, \theta)$.

$\rho$ = rho = distance to origin. $\phi = \varphi$ = phi = angle down from $z$-axis. $\theta$ = same as in cylindrical coordinates. Diagram drawn in space, and picture of 2D slice by vertical plane with $z, r$ coordinates.

Formulas to remember: $z = \rho\cos\phi$, $r = \rho\sin\phi$ (so $x = \rho\sin\phi\cos\theta$, $y = \rho\sin\phi\sin\theta$).

$\rho = \sqrt{x^2+y^2+z^2} = \sqrt{r^2+z^2}$. The equation $\rho = a$ defines the sphere of radius $a$ centered at 0.

On the surface of the sphere, $\phi$ is similar to *latitude*, except it's 0 at the north pole, $\pi/2$ on the equator, $\pi$ at the south pole. $\theta$ is similar to *longitude*.

$\phi = \pi/4$ is a cone (asked using flash cards) ($z = r = \sqrt{x^2+y^2}$). $\phi = \pi/2$ is the xy-plane.

**Volume element:** $dV = \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$.

To understand this formula, first study surface area on sphere of radius $a$: picture shown of a "rectangle" corresponding to $\Delta\phi$, $\Delta\theta$, with sides = portion of circle of radius $a$, of length $a\Delta\phi$, and portion of circle of radius $r = a\sin\phi$, of length $r\Delta\theta = a\sin\phi\Delta\theta$. So $\Delta S \approx a^2\sin\phi\,\Delta\phi\Delta\theta$, which gives the surface element $dS = a^2\sin\phi\,d\phi d\theta$.

The volume element follows: for a small "box", $\Delta V = \Delta S\,\Delta\rho$, so $dV = d\rho\,dS = \rho^2\sin\phi\,d\rho d\phi d\theta$.

**Example:** recall the complicated example at end of Friday's lecture (region sliced by a plane inside unit sphere). After rotating coordinate system, the question becomes: volume of the portion of unit sphere above the plane $z = 1/\sqrt{2}$? (picture drawn). This can be set up in cylindrical (left as exercise) or spherical coordinates.

For fixed $\phi, \theta$ we are slicing our region by rays straight out of the origin; $\rho$ ranges from its value on the plane $z = 1/\sqrt{2}$ to its value on the sphere $\rho = 1$. Spherical coordinate equation of the plane: $z = \rho\cos\phi = 1/\sqrt{2}$, so $\rho = \sec\phi/\sqrt{2}$. The volume is:

$$\int_0^{2\pi}\int_0^{\pi/4}\int_{\frac{1}{\sqrt{2}}\sec\phi}^{1} \rho^2\sin\phi\,d\rho\,d\phi\,d\theta.$$

(Bound for $\phi$ explained by looking at a slice by vertical plane $\theta$ =constant: the edge of the region is at $z = r = \frac{1}{\sqrt{2}}$).

Evaluation: not done. Final answer: $\dfrac{2\pi}{3} - \dfrac{5\pi}{6\sqrt{2}}$.

**Application to gravitation.**

Gravitational force exerted on mass $m$ at origin by a mass $\Delta M$ at $(x, y, z)$ (picture shown) is given by $|\vec{F}| = \dfrac{G\,\Delta M\,m}{\rho^2}$, $dir(\vec{F}) = \dfrac{\langle x,y,z\rangle}{\rho}$, i.e. $\vec{F} = \dfrac{G\,\Delta M\,m}{\rho^3}\langle x,y,z\rangle$. ($G$ = gravitational constant).

If instead of a point mass we have a solid with density $\delta$, then we must integrate contributions to gravitational attraction from small pieces $\Delta M = \delta\,\Delta V$. So

$$\vec{F} = \iiint_R \frac{Gm\,\langle x,y,z\rangle}{\rho^3}\,\delta\,dV, \quad \text{i.e. } z\text{-component is} \quad F_z = Gm\iiint_R \frac{z}{\rho^3}\delta\,dV, \ \ldots$$

If we can set up to use symmetry, then $F_z$ can be computed nicely using spherical coordinates.

**General setup:** place the mass $m$ at the origin (so integrand is as above), and place the solid so that the $z$-axis is an axis of symmetry. Then $\vec{F} = \langle 0, 0, F_z\rangle$ by symmetry, and we have only one component to compute. Then

$$F_z = Gm \iiint_R \frac{z}{\rho^3}\,\delta\,dV = Gm \iiint_R \frac{\rho\cos\phi}{\rho^3}\,\delta\,\rho^2\sin\phi\,d\rho\,d\phi\,d\theta = Gm \iiint_R \delta\cos\phi\sin\phi\,d\rho\,d\phi\,d\theta.$$

Example: Newton's theorem: the gravitational attraction of a spherical planet with uniform density $\delta$ is the same as that of the equivalent point mass at its center.

[[Setup: the sphere has radius $a$ and is centered on the positive z-axis, tangent to $xy$-plane at the origin; the test mass is $m$ at the origin. Then

$$F_z = Gm \iiint_R \frac{z}{\rho^3}\,\delta\,dV = Gm \int_0^{2\pi}\int_0^{\pi/2}\int_0^{2a\cos\phi} \delta\,\cos\phi\,\sin\phi\,d\rho\,d\phi\,d\theta = \cdots = \frac{4}{3}Gm\delta\,\pi a = \frac{GMm}{a^2}$$

where $M =$ mass of the planet $= \frac{4}{3}\pi a^3\delta$. (The bounds for $\rho$ and $\phi$ need to be explained carefully, by drawing a diagram of a vertical slice with $z$ and $r$ coordinate axes, and the inscribed right triangle with vertices the two poles of the sphere + a point on its surface, the hypothenuse is the diameter $2a$ and we get $\rho = 2a\cos\phi$ for the spherical coordinate equation of the sphere).]]

## 18.02 Lecture 27. – Thu, Nov 15, 2007

Handouts: PS10 solutions, PS11

**Vector fields in space.**

At every point in space, $\vec{F} = P\hat{\imath} + Q\hat{\jmath} + R\hat{k}$, where $P, Q, R$ are functions of $x, y, z$.

Examples: force fields (gravitational force $\vec{F} = -c\langle x, y, z\rangle/\rho^3$; electric field $\mathbf{E}$, magnetic field $\mathbf{B}$); velocity fields (fluid flow, $\boldsymbol{v} = \boldsymbol{v}(x, y, z)$); gradient fields (e.g. temperature and pressure gradients).

**Flux.**

Recall: in 2D, flux of a vector field $\vec{F}$ across a curve $C = \int_C \vec{F}\cdot\hat{\boldsymbol{n}}\,ds$.

In 3D, flux of a vector field is a *double* integral: flux through a *surface*, not a curve!

$\vec{F}$ vector field, $S$ surface, $\hat{\boldsymbol{n}}$ unit normal vector: Flux $= \iint \vec{F}\cdot\hat{\boldsymbol{n}}\,dS$.

Notation: $d\vec{S} = \hat{\boldsymbol{n}}\,dS$. (We'll see that $d\vec{S}$ is often easier to compute than $\hat{\boldsymbol{n}}$ and $dS$).

Remark: there are 2 choices for $\hat{\boldsymbol{n}}$ (choose which way is counted positively!)

**Geometric interpretation of flux:**

As in 2D, if $\vec{F} =$ velocity of a fluid flow, then flux = flow per unit time across $S$.

Cut $S$ into small pieces, then over each small piece: what passes through $\Delta S$ in unit time is the contents of a parallelepiped with base $\Delta S$ and third side given by $\vec{F}$.

Volume of box = base × height = $(\vec{F}\cdot\hat{\boldsymbol{n}})\,\Delta S$.

• Examples:

1) $\vec{F} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$ through sphere of radius $a$ centered at 0.

$\hat{\boldsymbol{n}} = \frac{1}{a}\langle x, y, z\rangle$ (other choice: $-\frac{1}{a}\langle x, y, z\rangle$; traditionally choose $\hat{\boldsymbol{n}}$ pointing out).

$\vec{F}\cdot\hat{\boldsymbol{n}} = \langle x, y, z\rangle\cdot\hat{\boldsymbol{n}} = \frac{1}{a}(x^2 + y^2 + z^2) = a$, so $\iint_S \vec{F}\cdot\hat{\boldsymbol{n}}dS = \iint_S a\,dS = a\,(4\pi a^2)$.

2) Same sphere, $\vec{H} = z\hat{\boldsymbol{k}}$: $\vec{H}\cdot\hat{\boldsymbol{n}} = \frac{z^2}{a}$.

$$\iint_S \vec{H}\cdot d\vec{S} = \iint_S \frac{z^2}{a}\,dS = \int_0^{2\pi}\int_0^{\pi} \frac{a^2\cos^2\phi}{a}\,a^2\sin\phi\,d\phi d\theta = 2\pi a^3\int_0^{\pi}\cos^2\phi\sin\phi\,d\phi = \frac{4}{3}\pi a^3.$$

**Setup.** Sometimes we have an easy geometric argument, but in general we must compute the surface integral. The setup requires the use of two parameters to describe the surface, and $\vec{F}\cdot\hat{\boldsymbol{n}}\,dS$ must be expressed in terms of them. How to do this depends on the type of surface. For now, formulas to remember:

0) plane $z = a$ parallel to $xy$-plane: $\hat{\boldsymbol{n}} = \pm\hat{\boldsymbol{k}}$, $dS = dx\,dy$. (similarly for planes // $xz$ or $yz$-plane).

1) sphere of radius $a$ centered at origin: use $\phi, \theta$ (substitute $\rho = a$ for evaluation); $\hat{\boldsymbol{n}} = \frac{1}{a}\langle x, y, z\rangle$, $dS = a^2\sin\phi\,d\phi\,d\theta$.

2) cylinder of radius $a$ centered on $z$-axis: use $z, \theta$ (substitute $r = a$ for evaluation): $\hat{\boldsymbol{n}}$ is radially out in horizontal directions away from $z$-axis, i.e. $\hat{\boldsymbol{n}} = \frac{1}{a}\langle x, y, 0\rangle$; and $dS = a\,dz\,d\theta$ (explained by drawing a picture of a "rectangular" piece of cylinder, $\Delta S = (\Delta z)\,(a\Delta\theta)$).

3) graph $z = f(x, y)$: use $x, y$ (substitute $z = f(x, y)$). We'll see on Friday that $\hat{\boldsymbol{n}}$ and $dS$ separately are complicated, but $\hat{\boldsymbol{n}}\,dS = \langle -f_x, -f_y, 1\rangle\,dx\,dy$.

**18.02 Lecture** 28. – **Fri, Nov 16, 2007**

Last time, we defined the flux of $\vec{F}$ through surface $S$ as $\iint \vec{F}\cdot\hat{\boldsymbol{n}}\,dS$, and saw how to set up in various cases. Continue with more:

**Flux through a graph.** If $S$ is the graph of some function $z = f(x, y)$ over a region $R$ of $xy$-plane: use $x$ and $y$ as variables. Contribution of a small piece of $S$ to flux integral?

Consider portion of $S$ lying above a small rectangle $\Delta x\,\Delta y$ in $xy$-plane. In linear approximation it is a parallelogram. (picture shown)

The vertices are $(x, y, f(x, y))$; $(x + \Delta x, y, f(x + \Delta x, y))$; $(x, y + \Delta y, f(x, y + \Delta y))$; etc. Linear approximation: $f(x + \Delta x, y) \simeq f(x, y) + \Delta x\, f_x(x, y)$, and $f(x, y + \Delta y) \simeq f(x, y) + \Delta y\, f_y(x, y)$.

So the sides of the parallelogram are $\langle \Delta x, 0, \Delta x\, f_x\rangle$ and $\langle 0, \Delta y, \Delta y\, f_y\rangle$, and

$$\Delta\vec{S} = (\Delta x\,\langle 1, 0, f_x\rangle)\times(\Delta y\,\langle 0, 1, f_y\rangle) = \Delta x\Delta y\begin{vmatrix}\hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\boldsymbol{k}}\\ 1 & 0 & f_x\\ 0 & 1 & f_y\end{vmatrix} = \langle -f_x, -f_y, 1\rangle\Delta x\Delta y.$$

So $d\vec{S} = \pm\langle -f_x, -f_y, 1\rangle dx\,dy$.

(From this we can get $\hat{\boldsymbol{n}} = \mathrm{dir}(d\vec{S}) = \dfrac{\langle -f_x, -f_y, 1\rangle}{\sqrt{f_x^2 + f_y^2 + 1}}$ and $dS = |d\vec{S}| = \sqrt{f_x^2 + f_y^2 + 1}\,dx\,dy$. The conversion factor $\sqrt{\cdots}$ between $dS$ and $dA$ relates area on $S$ to area of projection in $xy$-plane.)

• Example: flux of $\vec{F} = z\hat{\boldsymbol{k}}$ through $S$ = portion of paraboloid $z = x^2 + y^2$ above unit disk, oriented with normal pointing up (and into the paraboloid): geometrically flux should be $> 0$ (asked using flashcards). We have $\hat{\boldsymbol{n}}\,dS = \langle -2x, -2y, 1\rangle\,dx\,dy$, and

$$\iint_S \vec{F}\cdot d\vec{S} = \iint_S z\,dx\,dy = \iint_S (x^2 + y^2)\,dx\,dy = \int_0^{2\pi}\int_0^1 r^2\,r\,dr\,d\theta = \pi/2.$$

**Parametric surfaces.** If we can describe $S$ by parametric equations $x = x(u, v)$, $y = y(u, v)$, $z = z(u, v)$ (i.e. $\vec{r} = \vec{r}(u, v)$), then we can set up flux integrals using variables $u, v$. To find $d\vec{S}$, consider a small portion of surface corresponding to changes $\Delta u$ and $\Delta v$ in parameters, it's a parallelogram with sides $\vec{r}(u+\Delta u, v) - \vec{r}(u,v) \approx (\partial\vec{r}/\partial u)\,\Delta u$ and $(\partial\vec{r}/\partial v)\,\Delta v$, so

$$\Delta\vec{S} = \pm\left(\frac{\partial\vec{r}}{\partial u}\Delta u\right)\times\left(\frac{\partial\vec{r}}{\partial v}\Delta v\right), \qquad d\vec{S} = \pm\left(\frac{\partial\vec{r}}{\partial u}\times\frac{\partial\vec{r}}{\partial v}\right)du\,dv.$$

(This generalizes all formulas previously seen; but won't be needed on exam).

**Implicit surfaces:** If we have an implicitly defined surface $g(x,y,z)=0$, then we have a (non-unit) normal vector $\mathbf{N} = \nabla g$. (similarly for a slanted plane, from equation $ax+by+cz=d$ we get $\mathbf{N} = \langle a,b,c\rangle$).

Unit normal $\hat{\boldsymbol{n}} = \pm\mathbf{N}/|\mathbf{N}|$; surface element $\Delta S = ?$ Look at projection to $xy$-plane: $\Delta A = \Delta S\cos\alpha = (\mathbf{N}\cdot\hat{\boldsymbol{k}}/|\mathbf{N}|)\,\Delta S$ (where $\alpha$ = angle between slanted surface element and horizontal: projection shrinks one direction by factor $\cos\alpha = (\mathbf{N}\cdot\hat{\boldsymbol{k}})/|\mathbf{N}|$, preserves the other).

Hence $dS = \dfrac{|\mathbf{N}|}{\mathbf{N}\cdot\hat{\boldsymbol{k}}}\,dA$, and $\hat{\boldsymbol{n}}\,dS = \dfrac{|\mathbf{N}|\hat{\boldsymbol{n}}}{\mathbf{N}\cdot\hat{\boldsymbol{k}}}\,dx\,dy = \pm\dfrac{\mathbf{N}}{\mathbf{N}\cdot\hat{\boldsymbol{k}}}\,dx\,dy$.

(In fact the first formula should be $dS = \dfrac{|\mathbf{N}|}{|\mathbf{N}\cdot\hat{\boldsymbol{k}}|}\,dA$, I forgot the absolute value).

Note: if $S$ is vertical then the denominator is zero, can't project to $xy$-plane any more (but one could project e.g. to the $xz$-plane).

Example: if $S$ is a graph, $g(x,y,z) = z - f(x,y) = 0$, then $\mathbf{N} = \langle g_x, g_y, g_z\rangle = \langle -f_x, -f_y, 1\rangle$, $\mathbf{N}\cdot\hat{\boldsymbol{k}} = 1$, so we recover the formula $d\vec{S} = \langle -f_x, -f_y, 1\rangle dx\,dy$ seen before.

**Divergence theorem.** ("Gauss-Green theorem") – 3D analogue of Green theorem for flux.

If $S$ is a closed surface bounding a region $D$, with normal pointing outwards, and $\vec{F}$ vector field defined and differentiable over all of $D$, then

$$\iint_S \vec{F}\cdot d\vec{S} = \iiint_D \operatorname{div}\vec{F}\,dV, \quad \text{where} \quad \operatorname{div}(P\hat{\boldsymbol{\imath}} + Q\hat{\boldsymbol{\jmath}} + R\hat{\boldsymbol{k}}) = P_x + Q_y + R_z.$$

Example: flux of $\vec{F} = z\hat{\boldsymbol{k}}$ out of sphere of radius $a$ (seen Thursday): $\operatorname{div}\vec{F} = 0+0+1 = 1$, so $\iint_S \vec{F}\cdot d\vec{S} = 3\,\mathrm{vol}(D) = 4\pi a^3/3$.

**Physical interpretation** (mentioned very quickly and verbally only): $\operatorname{div}\vec{F}$ = source rate = flux generated per unit volume. So the divergence theorem says: the flux outwards through $S$ (net amount leaving $D$ per unit time) is equal to the total amount of sources in $D$.