

### Assignment 5 - Singular Value Decomposition

1. Suppose that  $T \in \mathcal{L}(V, W)$ . Show that  $T$  and  $T^*$  have the same positive singular values.
2. Prove that a normal operator on a complex inner product space is skew ( $T = -T^*$ ) if and only if all its eigenvalues are purely imaginary (meaning that they have real part equal to 0).
3. Suppose that  $V$  is a complex inner product space and  $T \in \mathcal{L}(V)$  is a normal operator such that  $T^9 = T^8$ . Prove that  $T$  is self-adjoint and  $T^2 = T$ . Then give an example of an  $S \in \mathcal{L}(V)$  that is not normal where  $T^9 = T^8$  but  $T^2 \neq T$ .
4. Suppose that  $T \in \mathcal{L}(V, W)$ . Prove that  $(T^*)^\dagger = (T^\dagger)^*$ .
5. Suppose that  $S, T \in \mathcal{L}(V)$  are positive operators. Show that

$$\|S - T\| \leq \max\{\|S\|, \|T\|\} \leq \|S + T\|.$$

6. For each linear operator  $T$  on an inner product space  $V$ , determine and justify whether  $T$  is normal, Hermitian (self-adjoint), or neither.
  - (a)  $V = \mathbb{R}^3$  and  $T(a, b, c) = (-a + b, 5b, 4a - 2b + 5c)$
  - (b)  $V = C^2$  and  $T(a, b) = (2a + ib, a + 2b)$
  - (c)  $V = M_{2 \times 2}(\mathbb{R})$  and  $T \left( \begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) = \begin{bmatrix} c & d \\ a & b \end{bmatrix}$ .
7. Let  $T \in \mathcal{L}(V, W)$  be a linear transformation of rank  $r$ , where  $V$  and  $W$  are finite-dimensional inner product spaces. In each of the following, find orthonormal bases  $\{v_1, \dots, v_n\}$  for  $V$  and  $\{u_1, \dots, u_m\}$  for  $W$ , and the nonzero singular values  $\sigma_1 \geq \dots \geq \sigma_r$  of  $T$  such that  $T(v_i) = \sigma_i u_i$  for  $1 \leq i \leq r$ .
  - (a)  $T : \mathbb{R}^2 \rightarrow \mathbb{R}^3$  with  $T(x_1, x_2) = (x_1, x_1 + x_2, x_1 - x_2)$
  - (b)  $T : P_2(\mathbb{R}) \rightarrow P_1(\mathbb{R})$  where  $T(f(x)) = f''(x)$ , and the inner products are given by  $\langle f, g \rangle = \int_{-1}^1 f(t)g(t)dt$ .
  - (c)  $T : \mathbb{C}^2 \rightarrow \mathbb{C}^2$  where  $T(z_1, z_2) = ((1-i)z_2, (1+i)z_1 + z_2)$