

Assignment 5 - Singular Value Decomposition

1. Suppose that $T \in \mathcal{L}(V, W)$. Show that T and T^* have the same positive singular values.
2. Prove that a normal operator on a complex inner product space is skew ($T = -T^*$) if and only if all its eigenvalues are purely imaginary (meaning that they have real part equal to 0).
3. Suppose that V is a complex inner product space and $T \in \mathcal{L}(V)$ is a normal operator such that $T^9 = T^8$. Prove that T is self-adjoint and $T^2 = T$. Then give an example of an $S \in \mathcal{L}(V)$ that is not normal where $T^9 = T^8$ but $T^2 \neq T$.
4. Suppose that $T \in \mathcal{L}(V, W)$. Prove that $(T^*)^\dagger = (T^\dagger)^*$.
5. Suppose that $S, T \in \mathcal{L}(V)$ are positive operators. Show that

$$\|S - T\| \leq \max\{\|S\|, \|T\|\} \leq \|S + T\|.$$

6. For each linear operator T on an inner product space V , determine and justify whether T is normal, Hermitian (self-adjoint), or neither.
 - (a) $V = \mathbb{R}^3$ and $T(a, b, c) = (-a + b, 5b, 4a - 2b + 5c)$
 - (b) $V = C^2$ and $T(a, b) = (2a + ib, a + 2b)$
 - (c) $V = M_{2 \times 2}(\mathbb{R})$ and $T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} c & d \\ a & b \end{bmatrix}$.
7. Let $T \in \mathcal{L}(V, W)$ be a linear transformation of rank r , where V and W are finite-dimensional inner product spaces. In each of the following, find orthonormal bases $\{v_1, \dots, v_n\}$ for V and $\{u_1, \dots, u_m\}$ for W , and the nonzero singular values $\sigma_1 \geq \dots \geq \sigma_r$ of T such that $T(v_i) = \sigma_i u_i$ for $1 \leq i \leq r$.
 - (a) $T : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ with $T(x_1, x_2) = (x_1, x_1 + x_2, x_1 - x_2)$
 - (b) $T : P_2(\mathbb{R}) \rightarrow P_1(\mathbb{R})$ where $T(f(x)) = f''(x)$, and the inner products are given by $\langle f, g \rangle = \int_{-1}^1 f(t)g(t)dt$.
 - (c) $T : \mathbb{C}^2 \rightarrow \mathbb{C}^2$ where $T(z_1, z_2) = ((1 - i)z_2, (1 + i)z_1 + z_2)$