Two-Scale Asymptotic Homogenization in Linear Elasticity

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Summary

This handout provides a concise overview of periodic homogenization in linear elasticity. It covers the governing equations, the two-scale expansion approach, the local (cell) problem, and the computation of the homogenized elasticity tensor.
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Strong Form of Linear Elasticity

The strong form of linear elasticity consists of the following field equations, valid for $x \in \Omega$:

$$\text{equilibrium} \qquad \nabla \cdot \sigma = -f \tag{1}$$

$$\text{kinematics} \qquad e(u) = \frac{1}{2}\left(\nabla u + \nabla u^{\top}\right) = \nabla^{s} u \tag{2}$$

$$\text{material law} \qquad \sigma = \mathbb{C} : e(u) \tag{3}$$

with displacement vector field u, stress tensor σ, and the fourth order elasticity tensor $\mathbb{C}$. Inserting (3) and (2) into (1), eventually yields

$$\nabla \cdot [\mathbb{C} : e(u)] = -f \quad \text{in } \Omega \tag{4}$$

For a well-posed problem, appropriate boundary conditions are required. Common choices include:

$$u = 0 \quad \text{on } \partial\Omega \quad \text{(Dirichlet)}, \quad \text{or} \quad \sigma \cdot n = t \quad \text{on } \partial\Omega \quad \text{(Neumann)}.$$

Periodically varying elastic properties

We now consider a linear elasticity problem with a heterogeneous, periodically varying elasticity tensor. To this end, we introduce a fast spatial variable $y = \frac{x}{\epsilon}$ in addition to the macroscopic variable x. The elasticity tensor $\mathbb{C}^{\epsilon} = \mathbb{C}\left(\frac{x}{\epsilon}\right)$ is Y-periodic in the fast variable y and satisfies the usual symmetry and ellipticity conditions.

Let $u^{\epsilon} : \Omega \to \mathbb{R}^{d}$ denote the displacement field in a composite medium with periodic microstructure. Then, the strong form of the equilibrium problem reads:

$$-\nabla \cdot [\mathbb{C}^{\epsilon} : \nabla^{s} u^{\epsilon}] = f(x) \quad \text{in } \Omega, \tag{5}$$

where u^{ϵ} is supposed to depend on x and y, but the right hand side does not vary with y.

The mechanical response of a structure or structural element under a given load typically does not require resolving the full detail of its microstructure. A direct numerical simulation at the microscale would also be highly demanding in terms of computational time and resources. A more efficient approach is to replace the heterogeneous material by an equivalent homogeneous medium characterized by an effective stiffness tensor, as illustrated in Fig. 1.

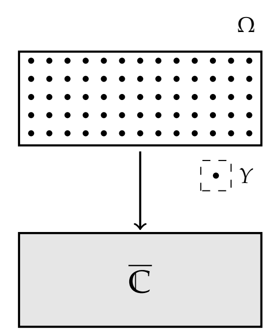

Figure 1: Periodic microstructure occupying the domain Ω (top), periodic cell with domain Y, and homogenized material with effective tensor $\overline{\mathbb{C}}$ (bottom).

Two-Scale Expansion

To derive the homogenized problem, we use the ansatz:

$$u^\epsilon = u_0\left(x, \frac{x}{\epsilon}\right) + \epsilon u_1\left(x, \frac{x}{\epsilon}\right) + \epsilon^2 u_2\left(x, \frac{x}{\epsilon}\right) + \cdots . \tag{6}$$

Due to the two-scale ansatz, all differential operators must be rewritten using the chain rule. In particular, the gradient operators become:

$$\nabla \mapsto \nabla_x + \frac{1}{\epsilon}\nabla_y, \qquad \nabla^s \mapsto \nabla^s_x + \frac{1}{\epsilon}\nabla^s_y, \tag{7}$$

where ∇_x, ∇^s_x and ∇_y, ∇^s_y denote the gradients with respect to the slow variable x and the fast variable $y = \frac{x}{\epsilon}$, respectively.

Inserting (6) into (5), using (7), and matching orders in ϵ gives a hierarchy of equations.

$$\varepsilon^{-2}: \quad A_0 u_0 = 0, \tag{8}$$

$$\varepsilon^{-1}: \quad A_0 u_1 = -A_1 u_0, \tag{9}$$

$$\varepsilon^{0}: \quad A_0 u_2 = -A_1 u_1 - A_2 u_0 + f(x). \tag{10}$$

with the individual differential operators given by

$$A_0 := -\nabla_y \cdot \left(\mathbb{C}^\epsilon : \nabla^s_y(\cdot)\right), \tag{11}$$

$$A_1 := -\nabla_y \cdot \left(\mathbb{C}^\epsilon : \nabla^s_x(\cdot)\right) - \nabla_x \cdot \left(\mathbb{C}^\epsilon : \nabla^s_y(\cdot)\right), \tag{12}$$

$$A_2 := -\nabla_x \cdot \left(\mathbb{C}^\epsilon : \nabla^s_x(\cdot)\right), \tag{13}$$

where the $(\cdot)$ is a placeholder for the argument.

Inspecting the problem at order ϵ^{-2} reveals, that u_0 depends only on x and not on y, i.e.,

$$u_0 = u_0(x) =: u\,. \tag{14}$$

Taking into account, that $\nabla^s_x u = \mathbb{E}$, for some macroscopic strain $\mathbb{E}$, we obtain the so-called cell problem at order ϵ^{-1} from (9) for the unknown fluctuation u_1.

Find $u_1 : Y \to \mathbb{R}^d$, such that

$$\nabla_y \cdot \left(\mathbb{C}^\epsilon : \nabla^s_y u_1\right) = -\nabla_y \cdot (\mathbb{C}^\epsilon : \mathbb{E}) \quad \text{in } Y \tag{15}$$

with u_1 Y-periodic, i.e., $\int_Y u_1 \,\mathrm{d}y = 0\,.$

Since the problem is linear, u_1 can be expressed as $u_1 = \mathbb{E} \cdot \chi$, which yields the standard cell problem.

Find $\chi : Y \to \mathbb{R}^d$, such that

$$\nabla_y \cdot \left(\mathbb{C}^\epsilon : \left[\mathbb{E} + \nabla^s_y \chi\right]\right) = 0 \quad \text{in } Y \tag{16}$$

with χ Y-periodic, i.e., $\int_Y \chi \, \mathrm{d}y = 0$.

Furthermore, for any given strain tensor $\mathbb{E}$, say $\mathbb{E}^{op}$, there is a corresponding solution χ^{op}.

Effective Stiffness Tensor

The effective stiffness tensor $\overline{\mathbb{C}}$ can be obtained from energy equivalence. We require that the homogenized material stores the same strain energy under a macroscopic strain $\mathbb{E}$ as the heterogeneous material averaged over the unit cell.

The strain energy for the unit cell is given by

$$W = \frac{1}{2}\int_Y e\left(u^\epsilon\right) : \mathbb{C}^\epsilon : e\left(u^\epsilon\right) \mathrm{d}y = \frac{1}{2}\int_Y \nabla^s u^\epsilon : \mathbb{C}^\epsilon : \nabla^s u^\epsilon \, \mathrm{d}y \tag{17}$$

Taking into account (7), we obtain

$$W = \frac{1}{2}\int_Y \left[\mathbb{E} + \nabla^s \chi\right] : \mathbb{C} : \left[\mathbb{E} + \nabla^s \chi\right] \, \mathrm{d}y\,. \tag{18}$$

On the other hand, this strain energy must be equal to

$$\overline{W} = \frac{1}{2}\mathbb{E} : \overline{\mathbb{C}} : \mathbb{E}\,. \tag{19}$$

The effective stress can be computed from

$$\Sigma = \frac{\mathrm{d}W}{\mathrm{d}\mathbb{E}} = \overline{\mathbb{C}} : \mathbb{E} = \frac{\mathrm{d}\overline{W}}{\mathrm{d}\mathbb{E}}. \tag{20}$$

Therefore

$$\overline{\mathbb{C}} : \mathbb{E} = \int_Y \mathbb{C} : \left[\mathbb{E} + \nabla^s \chi\right] \, \mathrm{d}y\,. \tag{21}$$

Again, these equations hold for any specific choice of $\mathbb{E}$. Selecting particular instances of $\mathbb{E}$, enables an efficient computation of the components of the effective stiffness tensor $\overline{\mathbb{C}}$. A natural choice is to use an orthogonal basis of the space of symmetric second-order tensors

$$\mathbb{E}^{op} = \frac{1}{2}\left[e_o \otimes e_p + e_p \otimes e_o\right] \tag{22}$$

with inner product between two second order tensors defined by[1]

[1] The double dot for tensor inner products is common for historical reasons, but not necessary, since the vector space inner product makes the single dot unambiguous.

$$\mathbb{A} \cdot \mathbb{B} = \mathbb{A} : \mathbb{B} = \left[A_{ij} e_i \otimes e_j\right] : \left[B_{kl} e_k \otimes e_l\right] = A_{ij} B_{kl} \delta_{ik} \delta_{jl} = A_{ij} B_{ij} \,. \quad (23)$$

Solving the cell problem for a particular base element $\mathbb{E}^{op}$ and multiplying this solution with a base element $\mathbb{E}^{rs}$ gives the corresponding element of $\overline{\mathbb{C}}$

$$\overline{C}_{rsop} = \mathbb{E}^{rs} : \overline{\mathbb{C}} : \mathbb{E}^{op} = \mathbb{E}^{rs} : \int\limits_{Y} \mathbb{C} : \left[\mathbb{E}^{op} + \nabla^s \chi^{op}\right] \, \mathrm{d}y \,. \quad (24)$$

Weak Formulation

The weak form of the cell problem can be stated as follows.

Find $\chi \in V := \left\{ u \in H^1(Y) \;\middle|\; u \text{ is } Y\text{-periodic} \right\}$ such that

$$\int_\Omega \nabla_y^s v : \mathbb{C}^\epsilon : \left[\mathbb{E} + \nabla_y^s \chi\right] \, \mathrm{d}y = 0 \quad \forall v \in V.$$