

# Rabi – I

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## Section 1: 背景知识

### 1.1 Baker-Campbell-Hausdorff 公式

Baker-Campbell-Hausdorff 公式，又称 BCH 公式，是一个用于计算么正变换公式，其形式如下：

$$\begin{aligned} e^A B e^{-A} &= \sum_{n=1} \frac{C_n}{n!} \\ &= B + [A, B] + \frac{1}{2!}[A, [A, B]] + \frac{1}{3!}[A, [A, [A, B]]] + \dots \end{aligned} \quad (1)$$

其中  $C_0 = B$ ,  $C_n = [A, C_{n-1}]$ .

### 1.2 Rabi 模型

Rabi 模型是一个描述原子与光场相互作用的模型，其哈密顿量为

$$H = \omega \hat{a}^\dagger \hat{a} + \frac{\Omega}{2} \hat{\sigma}_x + g \hat{\sigma}_z (\hat{a}^\dagger + \hat{a}) \quad (2)$$

其中第一项的物理含义是光子的能量，第二项的物理含义是原子的能级，第三项的物理含义是原子的自旋与光子的耦合。

### 1.3 么正算符

么正算符是一个满足  $U^\dagger U = I$  的算符，其中  $I$  是单位算符，这次推导选择的么正算符为

$$U = \exp A = \exp(\lambda \hat{\sigma}_z (\hat{a} - \hat{a}^\dagger)) \quad (3)$$

即  $A = \lambda \hat{\sigma}_z (\hat{a} - \hat{a}^\dagger)$ ，其中  $\lambda$  是一个实数。

在这个推导中，将会计算  $U H U^\dagger$ ，根据 BCH 公式，有

$$\begin{aligned} U H U^\dagger &= \exp(\lambda \hat{\sigma}_z (\hat{a} - \hat{a}^\dagger)) H \exp(-\lambda \hat{\sigma}_z (\hat{a} - \hat{a}^\dagger)) \\ &= H + [\lambda \hat{\sigma}_z (\hat{a} - \hat{a}^\dagger), H] + \frac{1}{2!} [\lambda \hat{\sigma}_z (\hat{a} - \hat{a}^\dagger), [\lambda \hat{\sigma}_z (\hat{a} - \hat{a}^\dagger), H]] + \dots \end{aligned} \quad (4)$$

在这里（以及下文）， $A = \lambda \hat{\sigma}_z (\hat{a} - \hat{a}^\dagger)$ ， $B = H$ ， $C_n$  是 BCH 公式中的系数。

由于哈密顿量  $H$  的每一项是线性相加的，因此可以分别计算每一项的变换，然后再合并。

### 1.4 常用对易关系

对于升降算符  $a$  和  $a^\dagger$ ，有以下对易关系：

$$[a, a^\dagger] = 1 \quad (5)$$

对于 Pauli 矩阵  $\hat{\sigma}_x, \hat{\sigma}_y, \hat{\sigma}_z$ ，有以下对易关系：

$$\begin{aligned}
[\hat{\sigma}_x, \hat{\sigma}_y] &= 2i\hat{\sigma}_z \\
[\hat{\sigma}_y, \hat{\sigma}_z] &= 2i\hat{\sigma}_x \\
[\hat{\sigma}_z, \hat{\sigma}_x] &= 2i\hat{\sigma}_y \\
\hat{\sigma}_x^2 &= \hat{\sigma}_y^2 = \hat{\sigma}_z^2 = I \\
\hat{\sigma}_z &= \hat{\sigma}_+ + \hat{\sigma}_- \\
\hat{\sigma}_y &= i(\hat{\sigma}_+ - \hat{\sigma}_-)
\end{aligned} \tag{6}$$

同时，升降算符和 Pauli 矩阵之间对易.

## Section 2: 推导

### 2.1 对 $\hat{a}$ 进行 $U$ 的么正变换

在这种情况下,  $B = \hat{a}$

$$\begin{aligned}
 C_0 &= B = \hat{a} \\
 C_1 &= [A, B] = [\lambda\hat{\sigma}_z(\hat{a} - \hat{a}^\dagger), \hat{a}] = \lambda\hat{\sigma}_z([\hat{a}, \hat{a}] - [\hat{a}^\dagger, \hat{a}]) = \lambda\hat{\sigma}_z \\
 C_2 &= [A, C_1] = [\lambda\hat{\sigma}_z(\hat{a} - \hat{a}^\dagger), \lambda\hat{\sigma}_z] = 0 \\
 &\Rightarrow C_n = 0 (n \geq 2)
 \end{aligned} \tag{7}$$

因此, 对于  $a$ , 有

$$\begin{aligned}
 U\hat{a}U^\dagger &= \sum_n \frac{C_n}{n!} \\
 &= \hat{a} + \lambda\hat{\sigma}_z
 \end{aligned} \tag{8}$$

它的物理意义是, 对于算符  $\hat{a}$  做了一个平移.

### 2.2 对 $\hat{a}^\dagger$ 进行 $U$ 的么正变换

在这种情况下,  $B = \hat{a}^\dagger$

$$\begin{aligned}
 C_0 &= B = \hat{a}^\dagger \\
 C_1 &= [A, B] = [\lambda\hat{\sigma}_z(\hat{a} - \hat{a}^\dagger), \hat{a}^\dagger] = \lambda\hat{\sigma}_z([\hat{a}, \hat{a}^\dagger] - [\hat{a}^\dagger, \hat{a}^\dagger]) = \lambda\hat{\sigma}_z \\
 C_2 &= [A, C_1] = [\lambda\hat{\sigma}_z(\hat{a} - \hat{a}^\dagger), \lambda\hat{\sigma}_z] = 0 \\
 &\Rightarrow C_n = 0 (n \geq 2)
 \end{aligned} \tag{9}$$

因此, 对于  $\hat{a}^\dagger$ , 有

$$\begin{aligned}
 U\hat{a}^\dagger U^\dagger &= \sum_n \frac{C_n}{n!} \\
 &= \hat{a}^\dagger + \lambda\hat{\sigma}_z
 \end{aligned} \tag{10}$$

### 2.3 对 $\hat{\sigma}_x$ 进行 $U$ 的么正变换

在这种情况下,  $B = \hat{\sigma}_x$

$$\begin{aligned}
C_0 &= B \\
&= \hat{\sigma}_x \\
C_1 &= [A, B] = [\lambda \hat{\sigma}_z (\hat{a} - \hat{a}^\dagger), \hat{\sigma}_x] = \lambda (\hat{a} - \hat{a}^\dagger) [\hat{\sigma}_z, \hat{\sigma}_x] \\
&= 2i\lambda \hat{\sigma}_y (\hat{a} - \hat{a}^\dagger) \\
C_2 &= [A, C_1] = [\lambda \hat{\sigma}_z (\hat{a} - \hat{a}^\dagger), 2i\lambda \hat{\sigma}_y (\hat{a} - \hat{a}^\dagger)] \\
&= 2i\lambda^2 (\hat{a} - \hat{a}^\dagger)^2 [\hat{\sigma}_z, \hat{\sigma}_y] \\
&= 4\lambda^2 \hat{\sigma}_x (\hat{a} - \hat{a}^\dagger)^2 \\
C_3 &= [A, C_2] = [\lambda \hat{\sigma}_z (\hat{a} - \hat{a}^\dagger), 4\lambda^2 \hat{\sigma}_x (\hat{a} - \hat{a}^\dagger)^2] \\
&= 4\lambda^3 (\hat{a} - \hat{a}^\dagger)^3 [\hat{\sigma}_z, \hat{\sigma}_x] \\
&= 8i\lambda^3 \hat{\sigma}_y (\hat{a} - \hat{a}^\dagger)^3 \\
C_4 &= [A, C_3] = [\lambda \hat{\sigma}_z (\hat{a} - \hat{a}^\dagger), 8i\lambda^3 \hat{\sigma}_y (\hat{a} - \hat{a}^\dagger)^3] \\
&= 8i\lambda^4 (\hat{a} - \hat{a}^\dagger)^4 [\hat{\sigma}_z, \hat{\sigma}_y] \\
&= 16\lambda^4 \hat{\sigma}_x (\hat{a} - \hat{a}^\dagger)^4 \\
C_n &= \dots
\end{aligned} \tag{11}$$

可以总结出

$$\begin{aligned}
U \hat{\sigma}_x U^\dagger &= \sum_n \frac{C_n}{n!} \\
&= \hat{\sigma}_x + \frac{2}{1!} i\lambda \hat{\sigma}_y (\hat{a} - \hat{a}^\dagger) + \frac{4}{2!} \lambda^2 \hat{\sigma}_x (\hat{a} - \hat{a}^\dagger)^2 + \\
&\quad \frac{8}{3!} i\lambda^3 \hat{\sigma}_y (\hat{a} - \hat{a}^\dagger)^3 + \frac{16}{4!} \lambda^4 \hat{\sigma}_x (\hat{a} - \hat{a}^\dagger)^4 + \dots \\
&= \sum_{n=0} \left[ \hat{\sigma}_x \frac{(2\lambda(\hat{a} - \hat{a}^\dagger))^{2n}}{2n!} + i\hat{\sigma}_y \frac{(2\lambda(\hat{a} - \hat{a}^\dagger))^{2n+1}}{(2n+1)!} \right] \\
&= \hat{\sigma}_x \cosh(2\lambda(\hat{a} - \hat{a}^\dagger)) + i\hat{\sigma}_y \sinh(2\lambda(\hat{a} - \hat{a}^\dagger))
\end{aligned} \tag{12}$$

因此，对于  $\hat{\sigma}_x$ ，有

$$U \hat{\sigma}_x U^\dagger = \hat{\sigma}_x \cosh(2\lambda(\hat{a} - \hat{a}^\dagger)) + i\hat{\sigma}_y \sinh(2\lambda(\hat{a} - \hat{a}^\dagger)) \tag{13}$$

## 2.4 对 $\hat{\sigma}_z$ 进行 $U$ 的么正变换

在这种情况下， $B = \hat{\sigma}_z$

$$\begin{aligned}
C_0 &= B \\
&= \hat{\sigma}_z \\
C_1 &= [A, B] = [\lambda \hat{\sigma}_z (\hat{a} - \hat{a}^\dagger), \hat{\sigma}_z] = \lambda (\hat{a} - \hat{a}^\dagger) [\hat{\sigma}_z, \hat{\sigma}_z] \\
&= 0 \\
C_n &= 0 (n \geq 1)
\end{aligned} \tag{14}$$

因此，对于  $\hat{\sigma}_z$ ，有

$$\begin{aligned}
 U\hat{\sigma}_zU^\dagger &= \sum_n \frac{C_n}{n!} \\
 &= \hat{\sigma}_z
 \end{aligned}
 \tag{15}$$

### Section 3: 对 $H$ 进行 $U$ 的么正变换

对于哈密顿量  $H$ ，可以分解为三个部分

$$H = \omega \hat{a}^\dagger \hat{a} + \frac{\Omega}{2} \hat{\sigma}_x + g \hat{\sigma}_z (\hat{a}^\dagger + \hat{a}) \quad (16)$$

分别求解

$$\begin{aligned} H_1 &= \omega \hat{a}^\dagger \hat{a} \\ H_2 &= \frac{\Omega}{2} \hat{\sigma}_x \\ H_3 &= g \hat{\sigma}_z (\hat{a}^\dagger + \hat{a}) \end{aligned} \quad (17)$$

#### 3.1 对 $H_1$ 进行 $U$ 的么正变换

可以通过么正算符  $U^\dagger U = I$  的性质和 式 8 和 式 10 的结果求解

$$\begin{aligned} U H_1 U^\dagger &= U \omega \hat{a}^\dagger \hat{a} U^\dagger \\ &= \omega U \hat{a}^\dagger U^\dagger U \hat{a} U^\dagger \\ &= \omega (\hat{a}^\dagger + \lambda \hat{\sigma}_z) (\hat{a} + \lambda \hat{\sigma}_z) \\ &= \omega \hat{a}^\dagger \hat{a} + \omega \lambda \hat{\sigma}_z (\hat{a} + \hat{a}^\dagger) + \omega \lambda^2 \end{aligned} \quad (18)$$

#### 3.2 对 $H_2$ 进行 $U$ 的么正变换

根据 式 13 的结果，可以求解

$$\begin{aligned} U H_2 U^\dagger &= U \frac{\Omega}{2} \hat{\sigma}_x U^\dagger \\ &= \frac{\Omega}{2} [\hat{\sigma}_x \cosh(2\lambda(\hat{a} - \hat{a}^\dagger)) + i \hat{\sigma}_y \sinh(2\lambda(\hat{a} - \hat{a}^\dagger))] \end{aligned} \quad (19)$$

#### 3.3 对 $H_3$ 进行 $U$ 的么正变换

根据 式 8 和 式 10 的结果求解

$$\begin{aligned} U H_3 U^\dagger &= U g \hat{\sigma}_z (\hat{a}^\dagger + \hat{a}) U^\dagger \\ &= g \hat{\sigma}_z (U \hat{a}^\dagger U^\dagger + U \hat{a} U^\dagger) \\ &= g \hat{\sigma}_z (\hat{a}^\dagger + \lambda \hat{\sigma}_z + \hat{a} + \lambda \hat{\sigma}_z) \\ &= g \hat{\sigma}_z (\hat{a}^\dagger + \hat{a}) + 2g\lambda \end{aligned} \quad (20)$$

#### 3.4 合并得到

将以上结果合并，得到

$$\begin{aligned}
\tilde{H} &= U H U^\dagger = U H_1 U^\dagger + U H_2 U^\dagger + U H_3 U^\dagger \\
&= \omega \hat{a}^\dagger \hat{a} + \omega \lambda \hat{\sigma}_z (\hat{a} + \hat{a}^\dagger) + \omega \lambda^2 + \\
&\quad \frac{\Omega}{2} [\hat{\sigma}_x \cosh(2\lambda(\hat{a} - \hat{a}^\dagger)) + i \hat{\sigma}_y \sinh(2\lambda(\hat{a} - \hat{a}^\dagger))] + \\
&\quad g \hat{\sigma}_z (\hat{a}^\dagger + \hat{a}) + 2g\lambda \\
&= \omega \hat{a}^\dagger \hat{a} + (\omega \lambda + g) \hat{\sigma}_z (\hat{a} + \hat{a}^\dagger) + (\omega \lambda^2 + 2g\lambda) + \\
&\quad \frac{\Omega}{2} [\hat{\sigma}_x \cosh(2\lambda(\hat{a} - \hat{a}^\dagger)) + i \hat{\sigma}_y \sinh(2\lambda(\hat{a} - \hat{a}^\dagger))]
\end{aligned} \tag{21}$$

至此，推导结束.

## Section 4: 连带拉盖尔多项式

连带拉盖尔多项式是一个用于求解量子力学问题的数学工具，其定义如下：

$$\begin{aligned} L_n^\mu(x) &= \sum_{i=0}^{\infty} (-1)^i \frac{(n+\mu)!}{(n-i)!(\mu+i)!} \frac{x^i}{i!} \\ &= \frac{(n+\mu)!}{n!} \sum_{i=0}^{\infty} (-1)^i \frac{n!}{(n-i)!(\mu+i)!} \frac{x^i}{i!} \end{aligned} \quad (22)$$

### 4.1 BHC 公式的另一个形式

BHC 公式的另一个形式如下：

$$e^A e^B = e^{A+B+\frac{1}{2}[A,B]+\frac{1}{12}[A,[A,B]]-\frac{1}{12}[B,[A,B]]+\dots} \quad (23)$$

当  $[A, [A, B]] = [B, [A, B]] = 0$  时，BHC 公式的另一个形式可以简化为

$$e^{A+B} = e^A e^B e^{-\frac{1}{2}[A,B]} \quad (24)$$

### 4.2 展开 sinh cosh 项

先要解决一个问题，升降算符的幂乘积的问题，即  $(\hat{a}^\dagger)^m (\hat{a})^n$  的简化问题，首先定义算符  $\hat{N} = \hat{a}^\dagger \hat{a}$ 。它与  $\hat{a}^\dagger$  和  $\hat{a}$  的对易关系为

$$\begin{aligned} [\hat{N}, \hat{a}^\dagger] &= [\hat{a}^\dagger \hat{a}, \hat{a}^\dagger] = \hat{a}^\dagger \hat{a} \hat{a}^\dagger - \hat{a}^\dagger \hat{a}^\dagger \hat{a} = \hat{a}^\dagger [\hat{a}, \hat{a}^\dagger] = \hat{a}^\dagger \\ [\hat{N}, \hat{a}] &= [\hat{a}^\dagger \hat{a}, \hat{a}] = \hat{a}^\dagger \hat{a} \hat{a} - \hat{a} \hat{a}^\dagger \hat{a} = \hat{a} [\hat{a}^\dagger, \hat{a}] = -\hat{a} \end{aligned} \quad (25)$$

先让  $m \geq n$ ，有

$$\begin{aligned} (\hat{a}^\dagger)^m (\hat{a})^n &= \underbrace{\hat{a}^\dagger \hat{a}^\dagger \hat{a}^\dagger \dots \hat{a}^\dagger}_m \overbrace{\hat{a} \hat{a} \hat{a} \dots \hat{a}}^n \\ &= \underbrace{\hat{a}^\dagger \hat{a}^\dagger \hat{a}^\dagger \dots \hat{a}^\dagger}_{m-1} \hat{N} \overbrace{\hat{a} \hat{a} \hat{a} \dots \hat{a}}^{n-1} \\ &= \underbrace{\hat{a}^\dagger \hat{a}^\dagger \hat{a}^\dagger \dots \hat{a}^\dagger}_{m-1} \hat{a} (\hat{N} - 1) \overbrace{\hat{a} \hat{a} \hat{a} \dots \hat{a}}^{n-2} \\ &= \underbrace{\hat{a}^\dagger \hat{a}^\dagger \hat{a}^\dagger \dots \hat{a}^\dagger}_{m-2} \hat{N} (\hat{N} - 1) \overbrace{\hat{a} \hat{a} \hat{a} \dots \hat{a}}^{n-2} \\ &= \underbrace{\hat{a}^\dagger \hat{a}^\dagger \hat{a}^\dagger \dots \hat{a}^\dagger}_{m-2} \hat{N} (\hat{N} - 1) (\hat{N} - 2) \overbrace{\hat{a} \hat{a} \hat{a} \dots \hat{a}}^{n-3} \\ &= \underbrace{\hat{a}^\dagger \hat{a}^\dagger \hat{a}^\dagger \dots \hat{a}^\dagger}_{m-2} \hat{N} a (\hat{N} - 2) \overbrace{\hat{a} \hat{a} \hat{a} \dots \hat{a}}^{n-3} \\ &= \underbrace{\hat{a}^\dagger \hat{a}^\dagger \hat{a}^\dagger \dots \hat{a}^\dagger}_{m-2} \hat{N} a (\hat{N} - 1) (\hat{N} - 2) \overbrace{\hat{a} \hat{a} \hat{a} \dots \hat{a}}^{n-3} \\ &= \underbrace{\hat{a}^\dagger \hat{a}^\dagger \hat{a}^\dagger \dots \hat{a}^\dagger}_{m-3} \hat{N} (\hat{N} - 1) (\hat{N} - 2) \overbrace{\hat{a} \hat{a} \hat{a} \dots \hat{a}}^{n-3} \\ &= \dots \end{aligned} \quad (26)$$



可以总结出规律, 即  $(\hat{a}^\dagger)^m (\hat{a})^n = (\hat{a}^\dagger)^{m-n} \hat{N}(\hat{N}-1)\dots(\hat{N}-n+1)$ , 其中  $\hat{N} = \hat{a}^\dagger \hat{a}$ , 令连乘项为函数  $h_n(\hat{N}) = \hat{N}(\hat{N}-1)\dots(\hat{N}-n+1)$ .

对于  $m < n$  的情况, 同样可以得到  $(\hat{a}^\dagger)^m (\hat{a})^n = h_m(\hat{N}) \hat{a}^{n-m}$ .

#### 4.2.a 展开 cosh 项

令  $\nu = -2\lambda$ , 式 21 的 cosh 项可以展开为

$$\begin{aligned}
 \cosh(\nu(\hat{a}^\dagger - \hat{a})) &= \frac{1}{2} (e^{\nu(\hat{a}^\dagger - \hat{a})} + e^{-\nu(\hat{a}^\dagger - \hat{a})}) \\
 &= \frac{1}{2} (e^{\nu\hat{a}^\dagger} e^{-\nu\hat{a}} e^{-\frac{1}{2}[\nu\hat{a}^\dagger, -\nu\hat{a}]} + e^{-\nu\hat{a}^\dagger} e^{\nu\hat{a}} e^{-\frac{1}{2}[-\nu\hat{a}^\dagger, \nu\hat{a}]} ) \\
 &= \frac{1}{2} (e^{\nu\hat{a}^\dagger} e^{-\nu\hat{a}} e^{-\frac{\nu^2}{2}} + e^{-\nu\hat{a}^\dagger} e^{\nu\hat{a}} e^{-\frac{\nu^2}{2}} ) \\
 &= \frac{1}{2} e^{-\frac{\nu^2}{2}} (e^{\nu\hat{a}^\dagger} e^{-\nu\hat{a}} + e^{-\nu\hat{a}^\dagger} e^{\nu\hat{a}}) \\
 &= \frac{1}{2} e^{-\frac{\nu^2}{2}} \sum_{m,n} \frac{1}{m!n!} [\nu^m (-\nu)^n + (-\nu)^m \nu^n] (\hat{a}^\dagger)^m \hat{a}^n
 \end{aligned} \tag{27}$$

这里我们先假定了  $[A, [A, B]] = [B, [A, B]] = 0$ , 实际情况

$$[A, B] = [\nu\hat{a}^\dagger, -\nu\hat{a}] = \nu^2 \equiv \text{常数} \tag{28}$$

常数与任何算符对易, 假定成立.

对于  $m - n = 2k \geq 0$  的情况 (由于 cosh 只有偶次幂项), 有

$$\begin{aligned}
 I_x^+ &= \frac{1}{2} e^{-\frac{\nu^2}{2}} \sum_{m,n} \frac{1}{m!n!} [\nu^m (-\nu)^n + (-\nu)^m \nu^n] (\hat{a}^\dagger)^m \hat{a}^n \\
 &= \frac{1}{2} e^{-\frac{\nu^2}{2}} \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} \frac{1}{(2k+n)!n!} [\nu^{n+2k} (-\nu)^n + (-\nu)^{n+2k} \nu^n] (\hat{a}^\dagger)^{n+2k} \hat{a}^n \\
 &= \frac{1}{2} e^{-\frac{\nu^2}{2}} \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} \frac{(-)^n h_n(\hat{N})}{(2k+n)!n!} 2\nu^{2n+2k} (\hat{a}^\dagger)^{2k} \\
 &= e^{-\frac{\nu^2}{2}} \sum_{k=0}^{\infty} (\hat{a}^\dagger)^{2k} \nu^{2k} \sum_{n=0}^{\infty} \frac{(-)^n h_n(\hat{N})}{(2k+n)!n!} \nu^{2n} \\
 &= e^{-\frac{\nu^2}{2}} \sum_{k=0}^{\infty} (\hat{a}^\dagger)^{2k} \nu^{2k} \sum_{n=0}^{\infty} \frac{(-)^n \hat{N}!}{(2k+n)!(\hat{N}-n)!} \frac{(\nu^2)^n}{n!} \\
 &= e^{-\frac{\nu^2}{2}} \sum_{k=0}^{\infty} (\hat{a}^\dagger)^{2k} \nu^{2k} \frac{\hat{N}!}{(\hat{N}+2k)!} \sum_{n=0}^{\infty} \frac{(-)^n (\hat{N}+2k)!}{(2k+n)!(\hat{N}-n)!} \frac{(\nu^2)^n}{n!} \\
 &= e^{-\frac{\nu^2}{2}} \sum_{k=0}^{\infty} (\hat{a}^\dagger)^{2k} \nu^{2k} \frac{\hat{N}!}{(\hat{N}+2k)!} L_{\hat{N}}^{2k}(\nu^2)
 \end{aligned} \tag{29}$$

对于  $n - m = 2k \geq 0$  的情况, 有

$$\begin{aligned}
I_x^- &= \frac{1}{2} e^{-\frac{\nu^2}{2}} \sum_{m,n} \frac{1}{m!n!} [\nu^m (-\nu)^n + (-\nu)^m \nu^n] (\hat{a}^\dagger)^m \hat{a}^n \\
&= \frac{1}{2} e^{-\frac{\nu^2}{2}} \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \frac{1}{(m+2k)!2k!} [\nu^{m+2k} (-\nu)^m + (-\nu)^{m+2k} \nu^m] (\hat{a}^\dagger)^m \hat{a}^{m+2k} \\
&= \frac{1}{2} e^{-\frac{\nu^2}{2}} \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \frac{(-)^m h_m(\hat{N})}{(m+2k)!2k!} 2\nu^{m+2k} (\hat{a}^\dagger)^m \hat{a}^{m+2k} \\
&= e^{-\frac{\nu^2}{2}} \sum_{k=0}^{\infty} \nu^{2k} \sum_{m=0}^{\infty} \frac{(-)^m h_m(\hat{N})}{(m+2k)!2k!} \nu^m \hat{a}^{2k} \\
&= e^{-\frac{\nu^2}{2}} \sum_{k=0}^{\infty} \nu^{2k} \sum_{m=0}^{\infty} \frac{(-)^m \hat{N}!}{(m+2k)!(\hat{N}-m)!} \frac{(\nu^2)^m}{m!} \hat{a}^{2k} \\
&= e^{-\frac{\nu^2}{2}} \sum_{k=0}^{\infty} \nu^{2k} \frac{\hat{N}!}{(\hat{N}+2k)!} \sum_{m=0}^{\infty} \frac{(-)^m (\hat{N}+2k)!}{(m+2k)!(\hat{N}-m)!} \frac{(\nu^2)^m}{m!} \hat{a}^{2k} \\
&= e^{-\frac{\nu^2}{2}} \sum_{k=0}^{\infty} \nu^{2k} \frac{\hat{N}!}{(\hat{N}+2k)!} L_{\hat{N}}^{2k}(\nu^2) \hat{a}^{2k}
\end{aligned} \tag{30}$$

定义一个函数

$$f(\nu, \hat{N}, m) = e^{-\frac{\nu^2}{2}} \nu^m \frac{\hat{N}!}{(\hat{N}+m)!} L_{\hat{N}}^m(\nu^2) \tag{31}$$

可以得到

$$\begin{aligned}
I_x^+ &= \sum_{k=0}^{\infty} (\hat{a}^\dagger)^{2k} f(\nu, \hat{N}, 2k) \\
I_x^- &= \sum_{k=0}^{\infty} f(\nu, \hat{N}, 2k) \hat{a}^{2k}
\end{aligned} \tag{32}$$

总结得到

$$\cosh(\nu(\hat{a} - \hat{a}^\dagger)) = I_x^+ + I_x^- = f(\nu, \hat{N}, 0) + \sum_{k=1}^{\infty} [(\hat{a}^\dagger)^{2k} f(\nu, \hat{N}, 2k) + f(\nu, \hat{N}, 2k) \hat{a}^{2k}]. \tag{33}$$

#### 4.2.b 展开 sinh 项

同理，对于 sinh 项，有

$$\sinh(\nu(\hat{a} - \hat{a}^\dagger)) = \sum_{k=0}^{\infty} [(\hat{a}^\dagger)^{2k+1} f(\nu, \hat{N}, 2k+1) - f(\nu, \hat{N}, 2k+1) \hat{a}^{2k+1}]. \tag{34}$$

## Section 5: $\tilde{H}$ 的绝热形式

$$\begin{aligned}\tilde{H} = & \omega \hat{a}^\dagger \hat{a} + (\omega\lambda + g) \hat{\sigma}_z (\hat{a} + \hat{a}^\dagger) + (\omega\lambda^2 + 2g\lambda) + \\ & \frac{\Omega}{2} [\hat{\sigma}_x \cosh(2\lambda(\hat{a} - \hat{a}^\dagger)) + i\hat{\sigma}_y \sinh(2\lambda(\hat{a} - \hat{a}^\dagger))] \end{aligned} \quad (35)$$

首先, 令  $\lambda = -\frac{g}{\omega}$ , 有

$$\begin{aligned}\tilde{H} = & \omega \hat{a}^\dagger \hat{a} + \left( \omega * \left( -\frac{g}{\omega} \right) + g \right) \hat{\sigma}_z (\hat{a} + \hat{a}^\dagger) + \left( \omega \left( -\frac{g}{\omega} \right)^2 + 2g * \left( -\frac{g}{\omega} \right) \right) + \\ & \frac{\Omega}{2} [\hat{\sigma}_x \cosh(2\lambda(\hat{a} - \hat{a}^\dagger)) + i\hat{\sigma}_y \sinh(2\lambda(\hat{a} - \hat{a}^\dagger))] \\ = & \omega \hat{a}^\dagger \hat{a} - \frac{g^2}{\omega} + \frac{\Omega}{2} [\hat{\sigma}_x \cosh(2\lambda(\hat{a} - \hat{a}^\dagger)) + i\hat{\sigma}_y \sinh(2\lambda(\hat{a} - \hat{a}^\dagger))] \end{aligned} \quad (36)$$

绝热近似只取  $\frac{\Omega}{2}$  的展开零阶项, 即

$$\begin{aligned}\tilde{H}_{AA} \approx & \omega \hat{a}^\dagger \hat{a} - \frac{g^2}{\omega} + \frac{\Omega}{2} \hat{\sigma}_x f(-2\lambda, \hat{N}, 0) \\ = & \begin{pmatrix} \omega \hat{a}^\dagger \hat{a} - \frac{g^2}{\omega} & \frac{\Omega}{2} f(-2\lambda, \hat{N}, 0) \\ \frac{\Omega}{2} f(-2\lambda, \hat{N}, 0) & \omega \hat{a}^\dagger \hat{a} - \frac{g^2}{\omega} \end{pmatrix} \end{aligned} \quad (37)$$

求本征值, 有

$$E_{AA}^{\pm, N} = \omega \hat{a}^\dagger \hat{a} - \frac{g^2}{\omega} \pm \frac{\Omega}{2} f(-2\lambda, \hat{N}, 0) \quad (38)$$

本征函数为

$$|\tilde{\Psi}_{AA}^{\pm, N}\rangle = |\pm_x, N\rangle \quad (39)$$

## Section 6: 广义旋转波近似

取一阶项  $f(-2\lambda, \hat{N}, 1)$

哈密顿量写为

$$\tilde{H}_{GRWA} = \tilde{H}_{AA} + \frac{\Omega}{2} [i\hat{\sigma}_y \hat{a}^\dagger f(-2\lambda, \hat{N}, 1) + \dots] \quad (40)$$

根据式6可以简化为

$$\tilde{H}_{GRWA} \approx \tilde{H}_{AA} - \frac{\Omega}{2} [(\hat{\sigma}_+ - \hat{\sigma}_-) \hat{a}^\dagger f(-2\lambda, \hat{N}, 1) - (\hat{\sigma}_+ - \hat{\sigma}_-) f(-2\lambda, \hat{N}, 1) \hat{a}] \quad (41)$$

做旋转波近似

$$\tilde{H}_{GRWA} \approx \tilde{H}_{AA} + \frac{\Omega}{2} [\hat{\sigma}_- \hat{a}^\dagger f(-2\lambda, \hat{N}, 1) + \hat{\sigma}_+ f(-2\lambda, \hat{N}, 1) \hat{a}] \quad (42)$$

$\hat{a}^\dagger, \hat{a}$ 对本征函数的作用:

$$\begin{aligned} \hat{a}^\dagger |N\rangle &= \sqrt{N+1} |N+1\rangle \\ \hat{a} |N\rangle &= \sqrt{N} |N-1\rangle \end{aligned} \quad (43)$$

用  $|\pm_x, N\rangle$  作为本征函数求它的非对角矩阵元有

$$\begin{aligned} \langle -_x, N | \frac{\Omega}{2} \hat{\sigma}_- \hat{a}^\dagger f(-2\lambda, \hat{N}, 1) |+_x, N-1\rangle &= \frac{\Omega}{2} f(-2\lambda, N-1, 1) \langle -_x, N | \hat{\sigma}_- \hat{a}^\dagger |+_x, N-1\rangle \\ &= \frac{\Omega}{2} \sqrt{N} f(-2\lambda, N-1, 1) \langle -_x, N | \hat{\sigma}_- |+_x, N\rangle \\ &= \frac{\Omega}{2} \sqrt{N} f(-2\lambda, N-1, 1) \cdot \langle -_x, N | -_x, N\rangle \\ &= \frac{\Omega}{2} \sqrt{N} f(-2\lambda, N-1, 1) \\ \langle +_x, N-1 | \frac{\Omega}{2} \hat{\sigma}_+ f(-2\lambda, \hat{N}, 1) \hat{a} | -_x, N\rangle &= \frac{\Omega}{2} \sqrt{N} \langle +_x, N-1 | \hat{\sigma}_+ f(-2\lambda, \hat{N}, 1) | -_x, N-1\rangle \\ &= \frac{\Omega}{2} \sqrt{N} f(-2\lambda, N-1, 1) \langle +_x, N-1 | \hat{\sigma}_+ | -_x, N-1\rangle \\ &= \frac{\Omega}{2} \sqrt{N} f(-2\lambda, N-1, 1) \cdot \langle +_x, N-1 | +_x, N-1\rangle \\ &= \frac{\Omega}{2} \sqrt{N} f(-2\lambda, N-1, 1) \end{aligned} \quad (44)$$

块矩阵元为

$$\tilde{H}_{GRWA}^{\text{BLOCK}} = \begin{pmatrix} E_{AA}^{+_x, N-1} & \frac{\Omega}{2} \sqrt{N} f(-2\lambda, N-1, 1) \\ \frac{\Omega}{2} \sqrt{N} f(-2\lambda, N-1, 1) & E_{AA}^{-_x, N} \end{pmatrix} \quad (45)$$

本征值求解得到

$$E_{\text{GRWA}}^{\pm, N} = \frac{1}{2} \left[ E_{AA}^{+, N-1} + E_{AA}^{-, N} \pm \sqrt{4 \cdot \left( \frac{\Omega}{2} \sqrt{N} f(-2\lambda, N-1, 1) \right)^2 + \left( E_{AA}^{+, N-1} - E_{AA}^{-, N} \right)^2} \right] \quad (46)$$

对应文献[1]的参数, 有

$$g \rightarrow \lambda, \quad \lambda \rightarrow \frac{\lambda}{\omega_0} \quad N \rightarrow N+1 \quad (47)$$

文献[1]的形式为

$$E_{\pm, N}^{\text{GRWA}} = \left( N + \frac{1}{2} \right) \omega_0 - \frac{\lambda^2}{\omega_0} + \frac{\Omega}{4} e^{-2\lambda^2/\omega_0^2} [L_N(4\lambda^2/\omega_0^2) - L_{N+1}(4\lambda^2/\omega_0^2)] \pm \left( \left[ \frac{1}{2} \omega_0 - \frac{1}{4} \Omega e^{-2\lambda^2/\omega_0^2} [L_N(4\lambda^2/\omega_0^2) + L_{N+1}(4\lambda^2/\omega_0^2)] \right]^2 + \frac{\lambda^2 \Omega^2}{\omega_0^2 (N+1)} e^{-4\lambda^2/\omega_0^2} [L_N^1(4\lambda^2/\omega_0^2)]^2 \right)^{1/2}. \quad (20)$$

我们的形式为

$$\pm \frac{1}{2} \sqrt{\frac{4 \lambda^2 (N+1) \Omega^2 (N!)^2 e^{-\frac{4\lambda^2}{\omega^2}} L_N^2\left(\frac{4\lambda^2}{\omega^2}\right)}{\omega^2 ((N+1)!)^2} + \left( \frac{1}{2} \Omega e^{-\frac{2\lambda^2}{\omega^2}} L_N\left(\frac{4\lambda^2}{\omega^2}\right) + \frac{1}{2} \Omega e^{-\frac{2\lambda^2}{\omega^2}} L_{N+1}\left(\frac{4\lambda^2}{\omega^2}\right) + N\omega - (N+1)\omega \right)^2 - \frac{\lambda^2}{\omega} + \frac{1}{4} \Omega e^{-\frac{2\lambda^2}{\omega^2}} L_N\left(\frac{4\lambda^2}{\omega^2}\right) - \frac{1}{4} \Omega e^{-\frac{2\lambda^2}{\omega^2}} L_{N+1}\left(\frac{4\lambda^2}{\omega^2}\right) + N\omega + \frac{\omega}{2}} \quad \text{数值运算}$$

每项完全一致, 认为我们的推导是正确的.

令  $\omega_0 = \frac{3}{4}\Omega$ , 绘制不同能级的  $\frac{E}{\Omega}$  的图像

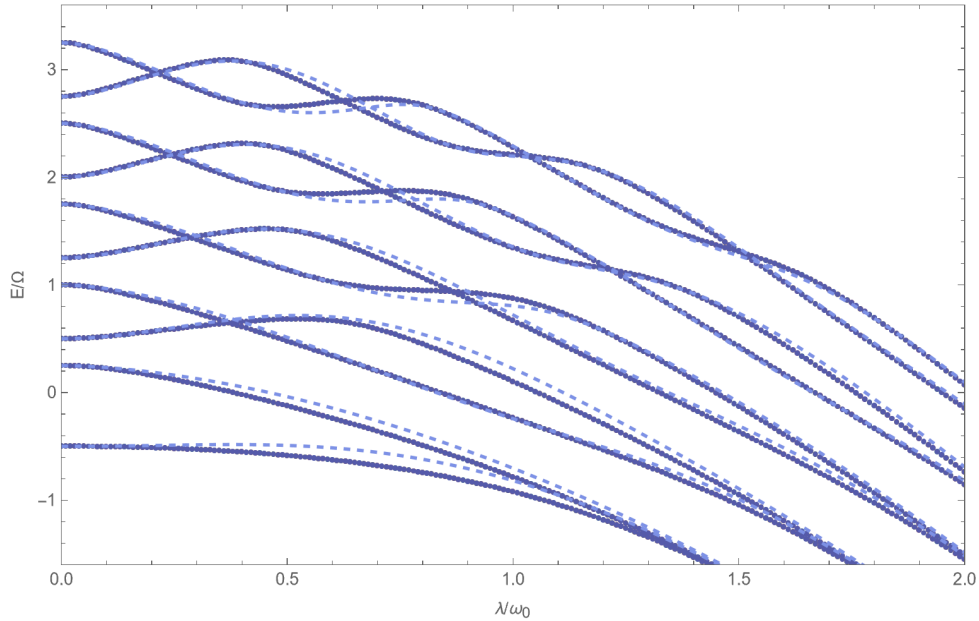


Figure 1: 广义旋转波近似下不同能级的能量图像,  
虚线为 GRWA 近似的结果, 实线为数值解的结果  
选择参数关系  $\omega_0 = 0.75\Omega$

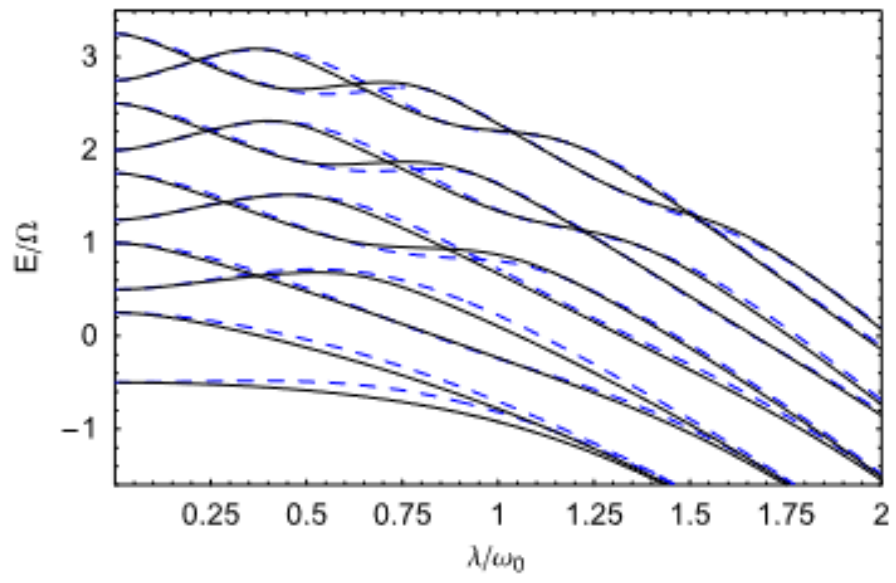


Figure 2: 文献[1]的结果

## Bibliography

- [1] E. K. Irish, “Generalized Rotating-Wave Approximation for Arbitrarily Large Coupling,” *Physical Review Letters*, vol. 99, no. 17, p. 173601–173602, Oct. 2007, doi: 10/dj5z3b.