Rabi – I

江源

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Section 1: 背景知识

1.1 Baker-Campbell-Hausdorff 公式

Baker-Campbell-Hausdorff 公式，又称 BCH 公式，是一个用于计算幺正变换公式，其形式如下：

$$
\begin{aligned}
e^{A} B e^{-A} &= \sum_{n=1} \frac{C_n}{n!} \\
&= B + [A, B] + \frac{1}{2!}[A, [A, B]] + \frac{1}{3!}[A, [A, [A, B]]] + \cdots
\end{aligned}
\tag{1}
$$

其中$C_0 = B$，$C_n = [A, C_{n-1}]$.

1.2 Rabi 模型

Rabi 模型是一个描述原子与光场相互作用的模型，其哈密顿量为

$$
H = \omega \hat{a}^{\dagger} \hat{a} + \frac{\Omega}{2} \hat{\sigma}_x + g \hat{\sigma}_z \left(\hat{a}^{\dagger} + \hat{a}\right)
\tag{2}
$$

其中第一项的物理含义是光子的能量，第二项的物理含义是原子的能级，第三项的物理含义是原子的自旋与光子的耦合.

1.3 幺正算符

幺正算符是一个满足$U^{\dagger} U = I$的算符，其中I是单位算符，这次推导选择的幺正算符为

$$
U = \exp A = \exp\left(\lambda \hat{\sigma}_z \left(\hat{a} - \hat{a}^{\dagger}\right)\right)
\tag{3}
$$

即$A = \lambda \hat{\sigma}_z \left(\hat{a} - \hat{a}^{\dagger}\right)$，其中$\lambda$是一个实数.

在这个推导中，将会计算$U H U^{\dagger}$，根据 BCH 公式，有

$$
\begin{aligned}
U H U^{\dagger} &= \exp\left(\lambda \hat{\sigma}_z \left(\hat{a} - \hat{a}^{\dagger}\right)\right) H \exp\left(-\lambda \hat{\sigma}_z \left(\hat{a} - \hat{a}^{\dagger}\right)\right) \\
&= H + \left[\lambda \hat{\sigma}_z \left(\hat{a} - \hat{a}^{\dagger}\right), H\right] + \frac{1}{2!}\left[\lambda \hat{\sigma}_z \left(\hat{a} - \hat{a}^{\dagger}\right), \left[\lambda \hat{\sigma}_z \left(\hat{a} - \hat{a}^{\dagger}\right), H\right]\right] + \cdots
\end{aligned}
\tag{4}
$$

在这里（以及下文），$A = \lambda \hat{\sigma}_z \left(\hat{a} - \hat{a}^{\dagger}\right)$，$B = H$，$C_n$是 BCH 公式中的系数.

由于哈密顿量H的每一项是线性相加的，因此可以分别计算每一项的变换，然后再合并.

1.4 常用对易关系

对于升降算符a和$a^{\dagger}$，有以下对易关系：

$$
\left[a, a^{\dagger}\right] = 1
\tag{5}
$$

对于 Pauli 矩阵$\hat{\sigma}_x, \hat{\sigma}_y, \hat{\sigma}_z$，有以下对易关系：

$$
\begin{aligned}
\left[\hat{\sigma}_x, \hat{\sigma}_y\right] &= 2\mathrm{i}\hat{\sigma}_z \\
\left[\hat{\sigma}_y, \hat{\sigma}_z\right] &= 2\mathrm{i}\hat{\sigma}_x \\
\left[\hat{\sigma}_z, \hat{\sigma}_x\right] &= 2\mathrm{i}\hat{\sigma}_y \\
{\hat{\sigma}_x}^2 = {\hat{\sigma}_y}^2 &= {\hat{\sigma}_z}^2 = I \\
\hat{\sigma}_z &= \hat{\sigma}_+ + \hat{\sigma}_- \\
\hat{\sigma}_y &= i\left(\hat{\sigma}_+ - \hat{\sigma}_-\right)
\end{aligned}
\tag{6}
$$

同时，升降算符和 Pauli 矩阵之间对易.

Section 2: 推导

2.1 对 $\hat{a}$ 进行 U 的幺正变换

在这种情况下，$B = \hat{a}$

$$\begin{aligned} C_0 &= B = \hat{a} \\ C_1 &= [A, B] = \left[\lambda\hat{\sigma}_z\left(\hat{a} - \hat{a}^\dagger\right), \hat{a}\right] = \lambda\hat{\sigma}_z\left([\hat{a}, \hat{a}] - \left[\hat{a}^\dagger, \hat{a}\right]\right) = \lambda\hat{\sigma}_z \\ C_2 &= [A, C_1] = \left[\lambda\hat{\sigma}_z\left(\hat{a} - \hat{a}^\dagger\right), \lambda\hat{\sigma}_z\right] = 0 \\ \Rightarrow C_n &= 0 (n \geq 2) \end{aligned} \tag{7}$$

因此，对于a，有

$$\begin{aligned} U\hat{a}U^\dagger &= \sum_n \frac{C_n}{n!} \\ &= \hat{a} + \lambda\hat{\sigma}_z \end{aligned} \tag{8}$$

它的物理意义是，对于算符$\hat{a}$做了一个平移.

2.2 对 $\hat{a}^\dagger$ 进行 U 的幺正变换

在这种情况下，$B = \hat{a}^\dagger$

$$\begin{aligned} C_0 &= B = \hat{a}^\dagger \\ C_1 &= [A, B] = \left[\lambda\hat{\sigma}_z\left(\hat{a} - \hat{a}^\dagger\right), \hat{a}^\dagger\right] = \lambda\hat{\sigma}_z\left(\left[\hat{a}, \hat{a}^\dagger\right] - \left[\hat{a}^\dagger, \hat{a}^\dagger\right]\right) = \lambda\hat{\sigma}_z \\ C_2 &= [A, C_1] = \left[\lambda\hat{\sigma}_z\left(\hat{a} - \hat{a}^\dagger\right), \lambda\hat{\sigma}_z\right] = 0 \\ \Rightarrow C_n &= 0 (n \geq 2) \end{aligned} \tag{9}$$

因此，对于$\hat{a}^\dagger$，有

$$\begin{aligned} U\hat{a}^\dagger U^\dagger &= \sum_n \frac{C_n}{n!} \\ &= \hat{a}^\dagger + \lambda\hat{\sigma}_z \end{aligned} \tag{10}$$

2.3 对 $\hat{\sigma}_x$ 进行 U 的幺正变换

在这种情况下，$B = \hat{\sigma}_x$

$$
\begin{aligned}
C_0 &= B \\
&= \hat{\sigma}_x \\
C_1 &= [A,B] = \left[\lambda\hat{\sigma}_z\left(\hat{a}-\hat{a}^\dagger\right),\hat{\sigma}_x\right] = \lambda\left(\hat{a}-\hat{a}^\dagger\right)[\hat{\sigma}_z,\hat{\sigma}_x] \\
&= 2i\lambda\hat{\sigma}_y\left(\hat{a}-\hat{a}^\dagger\right) \\
C_2 &= [A,C_1] = \left[\lambda\hat{\sigma}_z\left(\hat{a}-\hat{a}^\dagger\right),2i\lambda\hat{\sigma}_y\left(\hat{a}-\hat{a}^\dagger\right)\right] \\
&= 2i\lambda^2\left(\hat{a}-\hat{a}^\dagger\right)^2[\hat{\sigma}_z,\hat{\sigma}_y] \\
&= 4\lambda^2\hat{\sigma}_x\left(\hat{a}-\hat{a}^\dagger\right)^2 \\
C_3 &= [A,C_2] = \left[\lambda\hat{\sigma}_z\left(\hat{a}-\hat{a}^\dagger\right),4\lambda^2\hat{\sigma}_x\left(\hat{a}-\hat{a}^\dagger\right)^2\right] \\
&= 4\lambda^3\left(\hat{a}-\hat{a}^\dagger\right)^3[\hat{\sigma}_z,\hat{\sigma}_x] \\
&= 8i\lambda^3\hat{\sigma}_y\left(\hat{a}-\hat{a}^\dagger\right)^3 \\
C_4 &= [A,C_3] = \left[\lambda\hat{\sigma}_z\left(\hat{a}-\hat{a}^\dagger\right),8i\lambda^3\hat{\sigma}_y\left(\hat{a}-\hat{a}^\dagger\right)^3\right] \\
&= 8i\lambda^4\left(\hat{a}-\hat{a}^\dagger\right)^4[\hat{\sigma}_z,\hat{\sigma}_y] \\
&= 16\lambda^4\hat{\sigma}_x\left(\hat{a}-\hat{a}^\dagger\right)^4 \\
C_n &= \cdots
\end{aligned}
\tag{11}
$$

可以总结出

$$
\begin{aligned}
U\hat{\sigma}_xU^\dagger &= \sum_n \frac{C_n}{n!} \\
&= \hat{\sigma}_x + \frac{2}{1!}i\lambda\hat{\sigma}_y\left(\hat{a}-\hat{a}^\dagger\right) + \frac{4}{2!}\lambda^2\hat{\sigma}_x\left(\hat{a}-\hat{a}^\dagger\right)^2 + \\
&\frac{8}{3!}i\lambda^3\hat{\sigma}_y\left(\hat{a}-\hat{a}^\dagger\right)^3 + \frac{16}{4!}\lambda^4\hat{\sigma}_x\left(\hat{a}-\hat{a}^\dagger\right)^4 + \cdots \\
&= \sum_{n=0}\left[\hat{\sigma}_x\frac{\left(2\lambda\left(\hat{a}-\hat{a}^\dagger\right)\right)^{2n}}{2n!} + i\hat{\sigma}_y\frac{\left(2\lambda\left(\hat{a}-\hat{a}^\dagger\right)\right)^{2n+1}}{(2n+1)!}\right] \\
&= \hat{\sigma}_x\cosh\left(2\lambda\left(\hat{a}-\hat{a}^\dagger\right)\right) + i\hat{\sigma}_y\sinh\left(2\lambda\left(\hat{a}-\hat{a}^\dagger\right)\right)
\end{aligned}
\tag{12}
$$

因此，对于$\hat{\sigma}_x$，有

$$
U\hat{\sigma}_xU^\dagger = \hat{\sigma}_x\cosh\left(2\lambda\left(\hat{a}-\hat{a}^\dagger\right)\right) + i\hat{\sigma}_y\sinh\left(2\lambda\left(\hat{a}-\hat{a}^\dagger\right)\right) \tag{13}
$$

2.4 对 $\hat{\sigma}_z$ 进行 U 的幺正变换

在这种情况下，$B = \hat{\sigma}_z$

$$
\begin{aligned}
C_0 &= B \\
&= \hat{\sigma}_z \\
C_1 &= [A,B] = \left[\lambda\hat{\sigma}_z\left(\hat{a}-\hat{a}^\dagger\right),\hat{\sigma}_z\right] = \lambda\left(\hat{a}-\hat{a}^\dagger\right)[\hat{\sigma}_z,\hat{\sigma}_z] \\
&= 0 \\
C_n &= 0 (n \geq 1)
\end{aligned}
\tag{14}
$$

因此，对于$\hat{\sigma}_z$，有

$$
\begin{aligned}
U\hat{\sigma}_z U^\dagger &= \sum_n \frac{C_n}{n!} \\
&= \hat{\sigma}_z
\end{aligned}
\tag{15}
$$

Section 3: 对 H 进行 U 的幺正变换

对于哈密顿量H，可以分解为三个部分

$$H = \omega\hat{a}^\dagger\hat{a} + \frac{\Omega}{2}\hat{\sigma}_x + g\hat{\sigma}_z\left(\hat{a}^\dagger + \hat{a}\right) \tag{16}$$

分别求解

$$\begin{aligned} H_1 &= \omega\hat{a}^\dagger\hat{a} \\ H_2 &= \frac{\Omega}{2}\hat{\sigma}_x \\ H_3 &= g\hat{\sigma}_z\left(\hat{a}^\dagger + \hat{a}\right) \end{aligned} \tag{17}$$

3.1 对 H_1 进行 U 的幺正变换

可以通过幺正算符$U^\dagger U = I$的性质和 式 8 和 式 10 的结果求解

$$\begin{aligned} UH_1U^\dagger &= U\omega\hat{a}^\dagger\hat{a}U^\dagger \\ &= \omega U\hat{a}^\dagger U^\dagger U\hat{a}U^\dagger \\ &= \omega\left(\hat{a}^\dagger + \lambda\hat{\sigma}_z\right)\left(\hat{a} + \lambda\hat{\sigma}_z\right) \\ &= \omega\hat{a}^\dagger\hat{a} + \omega\lambda\hat{\sigma}_z\left(\hat{a} + \hat{a}^\dagger\right) + \omega\lambda^2 \end{aligned} \tag{18}$$

3.2 对 H_2 进行 U 的幺正变换

根据 式 13 的结果，可以求解

$$\begin{aligned} UH_2U^\dagger &= U\frac{\Omega}{2}\hat{\sigma}_xU^\dagger \\ &= \frac{\Omega}{2}\left[\hat{\sigma}_x\cosh\left(2\lambda\left(\hat{a} - \hat{a}^\dagger\right)\right) + i\hat{\sigma}_y\sinh\left(2\lambda\left(\hat{a} - \hat{a}^\dagger\right)\right)\right] \end{aligned} \tag{19}$$

3.3 对 H_3 进行 U 的幺正变换

根据 式 8 和 式 10 的结果求解

$$\begin{aligned} UH_3U^\dagger &= Ug\hat{\sigma}_z\left(\hat{a}^\dagger + \hat{a}\right)U^\dagger \\ &= g\hat{\sigma}_z\left(U\hat{a}^\dagger U^\dagger + U\hat{a}U^\dagger\right) \\ &= g\hat{\sigma}_z\left(\hat{a}^\dagger + \lambda\hat{\sigma}_z + \hat{a} + \lambda\hat{\sigma}_z\right) \\ &= g\hat{\sigma}_z\left(\hat{a}^\dagger + \hat{a}\right) + 2g\lambda \end{aligned} \tag{20}$$

3.4 合并得到

将以上结果合并，得到

$$
\begin{aligned}
\tilde{H} = UHU^\dagger &= UH_1U^\dagger + UH_2U^\dagger + UH_3U^\dagger \\
&= \omega\hat{a}^\dagger\hat{a} + \omega\lambda\hat{\sigma}_z\left(\hat{a} + \hat{a}^\dagger\right) + \omega\lambda^2 + \\
&\frac{\Omega}{2}\left[\hat{\sigma}_x \cosh\left(2\lambda\left(\hat{a} - \hat{a}^\dagger\right)\right) + i\hat{\sigma}_y \sinh\left(2\lambda\left(\hat{a} - \hat{a}^\dagger\right)\right)\right] + \\
&g\hat{\sigma}_z\left(\hat{a}^\dagger + \hat{a}\right) + 2g\lambda \\
&= \omega\hat{a}^\dagger\hat{a} + (\omega\lambda + g)\hat{\sigma}_z\left(\hat{a} + \hat{a}^\dagger\right) + \left(\omega\lambda^2 + 2g\lambda\right) + \\
&\frac{\Omega}{2}\left[\hat{\sigma}_x \cosh\left(2\lambda\left(\hat{a} - \hat{a}^\dagger\right)\right) + i\hat{\sigma}_y \sinh\left(2\lambda\left(\hat{a} - \hat{a}^\dagger\right)\right)\right]
\end{aligned}
\tag{21}
$$

至此，推导结束.

Section 4: 连带拉盖尔多项式

连带拉盖尔多项式是一个用于求解量子力学问题的数学工具，其定义如下：

$$
\begin{aligned}
L_n^\mu(x) &= \sum_{i=0}^{\infty}(-1)^i\frac{(n+\mu)!}{(n-i)!(\mu+i)!}\frac{x^i}{i!} \\
&= \frac{(n+\mu)!}{n!}\sum_{i=0}^{\infty}(-1)^i\frac{n!}{(n-i)!(\mu+i)!}\frac{x^i}{i!}
\end{aligned}
\tag{22}
$$

4.1 BHC 公式的另一个形式

BHC 公式的另一个形式如下：

$$
e^A e^B = e^{A+B+\frac{1}{2}[A,B]+\frac{1}{12}[A,[A,B]]-\frac{1}{12}[B,[A,B]]+\cdots} \tag{23}
$$

当$[A,[A,B]]=[B,[A,B]]=0$时，BHC 公式的另一个形式可以简化为

$$
e^{A+B} = e^A e^B e^{-\frac{1}{2}[A,B]} \tag{24}
$$

4.2 展开 sinh　cosh项

先要解决一个问题，升降算符的幂乘积的问题，即 $\left(\hat{a}^\dagger\right)^m(\hat{a})^n$ 的简化问题，首先定义算符 $\hat{N}=\hat{a}^\dagger\hat{a}$。它与 $\hat{a}^\dagger$ 和 $\hat{a}$ 的对易关系为

$$
\begin{aligned}
\left[\hat{N},\hat{a}^\dagger\right] &= \left[\hat{a}^\dagger\hat{a},\hat{a}^\dagger\right] = \hat{a}^\dagger\hat{a}\hat{a}^\dagger - \hat{a}^\dagger\hat{a}^\dagger\hat{a} = \hat{a}^\dagger\left[\hat{a},\hat{a}^\dagger\right] = \hat{a}^\dagger \\
\left[\hat{N},\hat{a}\right] &= \left[\hat{a}^\dagger\hat{a},\hat{a}\right] = \hat{a}^\dagger\hat{a}\hat{a} - \hat{a}\hat{a}^\dagger\hat{a} = \hat{a}\left[\hat{a}^\dagger,\hat{a}\right] = -\hat{a}
\end{aligned}
\tag{25}
$$

先让 $m \geq n$，有

$$
\begin{aligned}
\left(\hat{a}^\dagger\right)^m(\hat{a})^n &= \underbrace{\hat{a}^\dagger\hat{a}^\dagger\hat{a}^\dagger...\hat{a}^\dagger}_{m}\overbrace{\hat{a}\hat{a}\hat{a}...\hat{a}}^{n} \\
&= \underbrace{\hat{a}^\dagger\hat{a}^\dagger\hat{a}^\dagger...\hat{a}^\dagger}_{\text{m-1}}\hat{N}\overbrace{\hat{a}\hat{a}\hat{a}...\hat{a}}^{\text{n-1}} \\
&= \underbrace{\hat{a}^\dagger\hat{a}^\dagger\hat{a}^\dagger...\hat{a}^\dagger}_{\text{m-1}}\hat{a}\left(\hat{N}-1\right)\overbrace{\hat{a}\hat{a}\hat{a}...\hat{a}}^{\text{n-2}} \\
&= \underbrace{\hat{a}^\dagger\hat{a}^\dagger\hat{a}^\dagger...\hat{a}^\dagger}_{\text{m-2}}\hat{N}\left(\hat{N}-1\right)\overbrace{\hat{a}\hat{a}\hat{a}...\hat{a}}^{\text{n-2}} \\
&= \underbrace{\hat{a}^\dagger\hat{a}^\dagger\hat{a}^\dagger...\hat{a}^\dagger}_{\text{m-2}}\hat{N}\left(a\left(\hat{N}-1\right)-a\right)\overbrace{\hat{a}\hat{a}\hat{a}...\hat{a}}^{\text{n-3}} \\
&= \underbrace{\hat{a}^\dagger\hat{a}^\dagger\hat{a}^\dagger...\hat{a}^\dagger}_{\text{m-2}}\hat{N}a\left(\hat{N}-2\right)\overbrace{\hat{a}\hat{a}\hat{a}...\hat{a}}^{\text{n-3}} \\
&= \underbrace{\hat{a}^\dagger\hat{a}^\dagger\hat{a}^\dagger...\hat{a}^\dagger}_{\text{m-2}}a\left(\hat{N}-1\right)\left(\hat{N}-2\right)\overbrace{\hat{a}\hat{a}\hat{a}...\hat{a}}^{\text{n-3}} \\
&= \underbrace{\hat{a}^\dagger\hat{a}^\dagger\hat{a}^\dagger...\hat{a}^\dagger}_{\text{m-3}}\hat{N}\left(\hat{N}-1\right)\left(\hat{N}-2\right)\overbrace{\hat{a}\hat{a}\hat{a}...\hat{a}}^{\text{n-3}} \\
&= \cdots
\end{aligned}
\tag{26}
$$

可以总结出规律，即 $\left(\hat{a}^{\dagger}\right)^{m}(\hat{a})^{n}=\left(\hat{a}^{\dagger}\right)^{m-n}\hat{N}\left(\hat{N}-1\right)...\left(\hat{N}-n+1\right)$，其中 $\hat{N}=\hat{a}^{\dagger}\hat{a}$，令连乘项为函数 $h_{n}\left(\hat{N}\right)=\hat{N}\left(\hat{N}-1\right)...\left(\hat{N}-n+1\right)$.

对于 $m<n$ 的情况，同样可以得到 $\left(\hat{a}^{\dagger}\right)^{m}(\hat{a})^{n}=h_{m}\left(\hat{N}\right)\hat{a}^{n-m}$。

4.2.a 展开 cosh 项

令 $\nu=-2\lambda$，式 21 的 cosh 项可以展开为

$$\begin{aligned}\cosh\left(\nu\left(\hat{a}^{\dagger}-\hat{a}\right)\right)&=\frac{1}{2}\left(e^{\nu\left(\hat{a}^{\dagger}-\hat{a}\right)}+e^{-\nu\left(\hat{a}^{\dagger}-\hat{a}\right)}\right)\\&=\frac{1}{2}\left(e^{\nu\hat{a}^{\dagger}}e^{-\nu\hat{a}}e^{-\frac{1}{2}\left[\nu\hat{a}^{\dagger},-\nu\hat{a}\right]}+e^{-\nu\hat{a}^{\dagger}}e^{\nu\hat{a}}e^{-\frac{1}{2}\left[-\nu\hat{a}^{\dagger},\nu\hat{a}\right]}\right)\\&=\frac{1}{2}\left(e^{\nu\hat{a}^{\dagger}}e^{-\nu\hat{a}}e^{-\frac{\nu^{2}}{2}}+e^{-\nu\hat{a}^{\dagger}}e^{\nu\hat{a}}e^{-\frac{\nu^{2}}{2}}\right)\\&=\frac{1}{2}e^{-\frac{\nu^{2}}{2}}\left(e^{\nu\hat{a}^{\dagger}}e^{-\nu\hat{a}}+e^{-\nu\hat{a}^{\dagger}}e^{\nu\hat{a}}\right)\\&=\frac{1}{2}e^{-\frac{\nu^{2}}{2}}\sum_{m,n}^{\infty}\frac{1}{m!n!}\left[\nu^{m}(-\nu)^{n}+(-\nu)^{m}\nu^{n}\right]\left(\hat{a}^{\dagger}\right)^{m}\hat{a}^{n}\end{aligned}\tag{27}$$

这里我们先假定了$[A,[A,B]]=[B,[A,B]]=0$，实际情况

$$[A,B]=\left[\nu\hat{a}^{\dagger},-\nu\hat{a}\right]=\nu^{2}\equiv\text{常数}\tag{28}$$

常数与任何算符对易，假定成立.

对于 $m-n=2k\geq0$ 的情况（由于 cosh只有偶次幂项），有

$$\begin{aligned}I_{x}^{+}&=\frac{1}{2}e^{-\frac{\nu^{2}}{2}}\sum_{m,n}^{\infty}\frac{1}{m!n!}\left[\nu^{m}(-\nu)^{n}+(-\nu)^{m}\nu^{n}\right]\left(\hat{a}^{\dagger}\right)^{m}\hat{a}^{n}\\&=\frac{1}{2}e^{-\frac{\nu^{2}}{2}}\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\frac{1}{(2k+n)!n!}\left[\nu^{n+2k}(-\nu)^{n}+(-\nu)^{n+2k}\nu^{n}\right]\left(\hat{a}^{\dagger}\right)^{n+2k}\hat{a}^{n}\\&=\frac{1}{2}e^{-\frac{\nu^{2}}{2}}\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-)^{n}h_{n}\left(\hat{N}\right)}{(2k+n)!n!}2\nu^{2n+2k}\left(\hat{a}^{\dagger}\right)^{2k}\\&=e^{-\frac{\nu^{2}}{2}}\sum_{k=0}^{\infty}\left(\hat{a}^{\dagger}\right)^{2k}\nu^{2k}\sum_{n=0}^{\infty}\frac{(-)^{n}h_{n}\left(\hat{N}\right)}{(2k+n)!n!}\nu^{2n}\\&=e^{-\frac{\nu^{2}}{2}}\sum_{k=0}^{\infty}\left(\hat{a}^{\dagger}\right)^{2k}\nu^{2k}\sum_{n=0}^{\infty}\frac{(-)^{n}\hat{N}!}{(2k+n)!\left(\hat{N}-n\right)!}\frac{\left(\nu^{2}\right)^{n}}{n!}\\&=e^{-\frac{\nu^{2}}{2}}\sum_{k=0}^{\infty}\left(\hat{a}^{\dagger}\right)^{2k}\nu^{2k}\frac{\hat{N}!}{\left(\hat{N}+2k\right)!}\sum_{n=0}^{\infty}\frac{(-)^{n}\left(\hat{N}+2k\right)!}{(2k+n)!\left(\hat{N}-n\right)!}\frac{\left(\nu^{2}\right)^{n}}{n!}\\&=e^{-\frac{\nu^{2}}{2}}\sum_{k=0}^{\infty}\left(\hat{a}^{\dagger}\right)^{2k}\nu^{2k}\frac{\hat{N}!}{\left(\hat{N}+2k\right)!}L_{\hat{N}}^{2k}\left(\nu^{2}\right)\end{aligned}\tag{29}$$

对于 $n-m=2k\geq0$ 的情况，有

$$
\begin{aligned}
I_x^- &= \frac{1}{2}e^{-\frac{\nu^2}{2}}\sum_{m,n}^{\infty}\frac{1}{m!n!}\left[\nu^m(-\nu)^n+(-\nu)^m\nu^n\right]\left(\hat{a}^\dagger\right)^m\hat{a}^n \\
&= \frac{1}{2}e^{-\frac{\nu^2}{2}}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\frac{1}{(m+2k)!2k!}\left[\nu^{m+2k}(-\nu)^m+(-\nu)^{m+2k}\nu^m\right]\left(\hat{a}^\dagger\right)^m\hat{a}^{m+2k} \\
&= \frac{1}{2}e^{-\frac{\nu^2}{2}}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-)^m h_m\left(\hat{N}\right)}{(m+2k)!2k!}2\nu^{m+2k}\left(\hat{a}^\dagger\right)^m\hat{a}^{m+2k} \\
&= e^{-\frac{\nu^2}{2}}\sum_{k=0}^{\infty}\nu^{2k}\sum_{m=0}^{\infty}\frac{(-)^m h_m\left(\hat{N}\right)}{(m+2k)!2k!}\nu^m\hat{a}^{2k} \\
&= e^{-\frac{\nu^2}{2}}\sum_{k=0}^{\infty}\nu^{2k}\sum_{m=0}^{\infty}\frac{(-)^m\hat{N}!}{(m+2k)!\left(\hat{N}-m\right)!}\frac{\left(\nu^2\right)^m}{m!}\hat{a}^{2k} \\
&= e^{-\frac{\nu^2}{2}}\sum_{k=0}^{\infty}\nu^{2k}\frac{\hat{N}!}{\left(\hat{N}+2k\right)!}\sum_{m=0}^{\infty}\frac{(-)^m\left(\hat{N}+2k\right)!}{(m+2k)!\left(\hat{N}-m\right)!}\frac{\left(\nu^2\right)^m}{m!}\hat{a}^{2k} \\
&= e^{-\frac{\nu^2}{2}}\sum_{k=0}^{\infty}\nu^{2k}\frac{\hat{N}!}{\left(\hat{N}+2k\right)!}L_{\hat{N}}^{2k}\left(\nu^2\right)\hat{a}^{2k}
\end{aligned}
\tag{30}
$$

定义一个函数

$$
f\left(\nu,\hat{N},m\right)=e^{-\frac{\nu^2}{2}}\nu^m\frac{\hat{N}!}{\left(\hat{N}+m\right)!}L_{\hat{N}}^m\left(\nu^2\right)
\tag{31}
$$

可以得到

$$
\begin{aligned}
I_x^+ &= \sum_{k=0}^{\infty}\left(\hat{a}^\dagger\right)^{2k}f\left(\nu,\hat{N},2k\right) \\
I_x^- &= \sum_{k=0}^{\infty}f\left(\nu,\hat{N},2k\right)\hat{a}^{2k}
\end{aligned}
\tag{32}
$$

总结得到

$$
\cosh\left(\nu\left(\hat{a}-\hat{a}^\dagger\right)\right)=I_x^++I_x^-=f\left(\nu,\hat{N},0\right)+\sum_{k=1}^{\infty}\left[\left(\hat{a}^\dagger\right)^{2k}f\left(\nu,\hat{N},2k\right)+f\left(\nu,\hat{N},2k\right)\hat{a}^{2k}\right].
\tag{33}
$$

4.2.b 展开 sinh 项

同理，对于 sinh 项，有

$$
\sinh\left(\nu\left(\hat{a}-\hat{a}^\dagger\right)\right)=\sum_{k=0}^{\infty}\left[\left(\hat{a}^\dagger\right)^{2k+1}f\left(\nu,\hat{N},2k+1\right)-f\left(\nu,\hat{N},2k+1\right)\hat{a}^{2k+1}\right].
\tag{34}
$$

Section 5: $\tilde{H}$的绝热形式

$$\begin{aligned}\tilde{H} = \omega\hat{a}^\dagger\hat{a} + (\omega\lambda + g)\hat{\sigma}_z\left(\hat{a} + \hat{a}^\dagger\right) + \left(\omega\lambda^2 + 2g\lambda\right) + \\ \frac{\Omega}{2}\left[\hat{\sigma}_x \cosh\left(2\lambda\left(\hat{a} - \hat{a}^\dagger\right)\right) + i\hat{\sigma}_y \sinh\left(2\lambda\left(\hat{a} - \hat{a}^\dagger\right)\right)\right]\end{aligned} \tag{35}$$

首先，令 $\lambda = -\frac{g}{\omega}$，有

$$\begin{aligned}\tilde{H} &= \omega\hat{a}^\dagger\hat{a} + \left(\omega * \left(-\frac{g}{\omega}\right) + g\right)\hat{\sigma}_z\left(\hat{a} + \hat{a}^\dagger\right) + \left(\omega\left(-\frac{g}{\omega}\right)^2 + 2g * \left(-\frac{g}{\omega}\right)\right) + \\ &\frac{\Omega}{2}\left[\hat{\sigma}_x \cosh\left(2\lambda\left(\hat{a} - \hat{a}^\dagger\right)\right) + i\hat{\sigma}_y \sinh\left(2\lambda\left(\hat{a} - \hat{a}^\dagger\right)\right)\right] \\ &= \omega\hat{a}^\dagger\hat{a} - \frac{g^2}{\omega} + \frac{\Omega}{2}\left[\hat{\sigma}_x \cosh\left(2\lambda\left(\hat{a} - \hat{a}^\dagger\right)\right) + i\hat{\sigma}_y \sinh\left(2\lambda\left(\hat{a} - \hat{a}^\dagger\right)\right)\right]\end{aligned} \tag{36}$$

绝热近似只取 $\frac{\Omega}{2}$ 的展开零阶项，即

$$\begin{aligned}\tilde{H}_{AA} &\approx \omega\hat{a}^\dagger\hat{a} - \frac{g^2}{\omega} + \frac{\Omega}{2}\hat{\sigma}_x f\left(-2\lambda, \hat{N}, 0\right) \\ &= \begin{pmatrix} \omega\hat{a}^\dagger\hat{a} - \frac{g^2}{\omega} & \frac{\Omega}{2} f\left(-2\lambda, \hat{N}, 0\right) \\ \frac{\Omega}{2} f\left(-2\lambda, \hat{N}, 0\right) & \omega\hat{a}^\dagger\hat{a} - \frac{g^2}{\omega} \end{pmatrix}\end{aligned} \tag{37}$$

求本征值，有

$$E_{AA}^{\pm,N} = \omega\hat{a}^\dagger\hat{a} - \frac{g^2}{\omega} \pm \frac{\Omega}{2} f\left(-2\lambda, \hat{N}, 0\right) \tag{38}$$

本征函数为

$$\left|\tilde{\Psi}_{AA}^{\pm,N}\right\rangle = \left|\pm_x, N\right\rangle \tag{39}$$

Section 6: 广义旋转波近似

取一阶项 $f\Big(-2\lambda,\hat{N},1\Big)$

哈密顿量写为

$$\tilde{H}_{GRWA}=\tilde{H}_{AA}+\frac{\Omega}{2}\Big[i\hat{\sigma}_y\hat{a}^\dagger f\Big(-2\lambda,\hat{N},1\Big)+\cdots\Big] \tag{40}$$

根据 式 6 可以简化为

$$\tilde{H}_{GRWA}\approx\tilde{H}_{AA}-\frac{\Omega}{2}\Big[(\hat{\sigma}_+-\hat{\sigma}_-)\hat{a}^\dagger f\Big(-2\lambda,\hat{N},1\Big)-(\hat{\sigma}_+-\hat{\sigma}_-)f\Big(-2\lambda,\hat{N},1\Big)\hat{a}\Big] \tag{41}$$

做旋转波近似

$$\tilde{H}_{GRWA}\approx\tilde{H}_{AA}+\frac{\Omega}{2}\Big[\hat{\sigma}_-\hat{a}^\dagger f\Big(-2\lambda,\hat{N},1\Big)+\hat{\sigma}_+f\Big(-2\lambda,\hat{N},1\Big)\hat{a}\Big] \tag{42}$$

$\hat{a}^\dagger,\hat{a}$对本征函数的作用：

$$\begin{aligned}\hat{a}^\dagger|N\rangle&=\sqrt{N+1}|N+1\rangle\\ \hat{a}|N\rangle&=\sqrt{N}|N-1\rangle\end{aligned} \tag{43}$$

用 $|\pm_x,N\rangle$ 作为本征函数求它的非对角矩阵元有

$$\begin{aligned}\langle-_x,N|\frac{\Omega}{2}\hat{\sigma}_-\hat{a}^\dagger f\Big(-2\lambda,\hat{N},1\Big)|+_x,N-1\rangle&=\frac{\Omega}{2}f(-2\lambda,N-1,1)\langle-_x,N|\hat{\sigma}_-\hat{a}^\dagger|+_x,N-1\rangle\\ &=\frac{\Omega}{2}\sqrt{N}f(-2\lambda,N-1,1)\langle-_x,N|\hat{\sigma}_-|+_x,N\rangle\\ &=\frac{\Omega}{2}\sqrt{N}f(-2\lambda,N-1,1)\cdot\langle-_x,N|-_x,N\rangle\\ &=\frac{\Omega}{2}\sqrt{N}f(-2\lambda,N-1,1)\\ \langle+_x,N-1|\frac{\Omega}{2}\hat{\sigma}_+f\Big(-2\lambda,\hat{N},1\Big)\hat{a}|-_x,N\rangle&=\frac{\Omega}{2}\sqrt{N}\langle+_x,N-1|\hat{\sigma}_+f\Big(-2\lambda,\hat{N},1\Big)|-_x,N-1\rangle\\ &=\frac{\Omega}{2}\sqrt{N}f(-2\lambda,N-1,1)\langle+_x,N-1|\hat{\sigma}_+|-_x,N-1\rangle\\ &=\frac{\Omega}{2}\sqrt{N}f(-2\lambda,N-1,1)\cdot\langle+_x,N-1|+_x,N-1\rangle\\ &=\frac{\Omega}{2}\sqrt{N}f(-2\lambda,N-1,1)\end{aligned} \tag{44}$$

块矩阵元为

$$\tilde{H}^{\mathrm{BLOCK}}_{\mathrm{GRWA}}=\begin{pmatrix}E_{AA}^{+_x,N-1} & \frac{\Omega}{2}\sqrt{N}f(-2\lambda,N-1,1)\\ \frac{\Omega}{2}\sqrt{N}f(-2\lambda,N-1,1) & E_{AA}^{-_x,N}\end{pmatrix} \tag{45}$$

本征值求解得到

$$
\begin{aligned}
E_{\text{GRWA}}^{\pm,N} = \frac{1}{2}\Bigg[& E_{AA}^{+_x,N-1} + E_{AA}^{-_x,N} \\
& \pm\sqrt{4\cdot\left(\frac{\Omega}{2}\sqrt{N}f(-2\lambda,N-1,1)\right)^2 + \left(E_{AA}^{+_x,N-1} - E_{AA}^{-_x,N}\right)^2}\Bigg]
\end{aligned}
\tag{46}
$$

对应文献[1]的参数，有

$$
g\to\lambda,\quad \lambda\to\frac{\lambda}{\omega_0}\quad N\to N+1 \tag{47}
$$

文献[1]的形式为

$$
\begin{aligned}
E_{\pm,N}^{\text{GRWA}} = & \left(N+\frac{1}{2}\right)\omega_0 - \frac{\lambda^2}{\omega_0} + \frac{\Omega}{4}e^{-2\lambda^2/\omega_0^2}[L_N(4\lambda^2/\omega_0^2) - L_{N+1}(4\lambda^2/\omega_0^2)] \\
& \pm\left(\left\{\frac{1}{2}\omega_0 - \frac{1}{4}\Omega e^{-2\lambda^2/\omega_0^2}[L_N(4\lambda^2/\omega_0^2) + L_{N+1}(4\lambda^2/\omega_0^2)]\right\}^2 + \frac{\lambda^2\Omega^2}{\omega_0^2(N+1)}e^{-4\lambda^2/\omega_0^2}[L_N^1(4\lambda^2/\omega_0^2)]^2\right)^{1/2}.
\end{aligned}
\tag{20}
$$

我们的形式为

$$
\pm\frac{1}{2}\sqrt{\frac{4\,\lambda^2\,(N+1)\,\Omega^2\,(N!)^2\,e^{-\frac{4\lambda^2}{\omega^2}}\,L_N^1\left(\frac{4\lambda^2}{\omega^2}\right)^2}{\omega^2\,((N+1)!)^2} + \left(\frac{1}{2}\,\Omega\,e^{-\frac{2\lambda^2}{\omega^2}}\,L_N\left(\frac{4\lambda^2}{\omega^2}\right) + \frac{1}{2}\,\Omega\,e^{-\frac{2\lambda^2}{\omega^2}}\,L_{N+1}\left(\frac{4\lambda^2}{\omega^2}\right) + N\,\omega - (N+1)\,\omega\right)^2} - \frac{\lambda^2}{\omega} + \frac{1}{4}\,\Omega\,e^{-\frac{2\lambda^2}{\omega^2}}\,L_N\left(\frac{4\lambda^2}{\omega^2}\right) - \frac{1}{4}\,\Omega\,e^{-\frac{2\lambda^2}{\omega^2}}\,L_{N+1}\left(\frac{4\lambda^2}{\omega^2}\right) + N\,\omega + \frac{\omega}{2}
$$

每项完全一致，认为我们的推导是正确的.

令 $\omega_0 = \frac{3}{4}\Omega$，绘制不同能级$\frac{E}{\Omega}$的图像

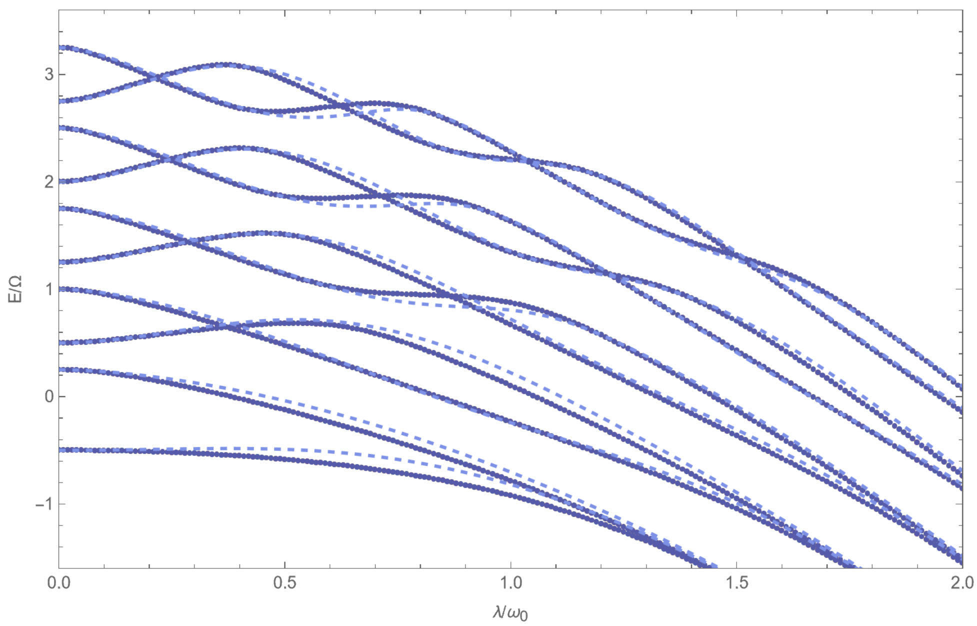

Figure 1: 广义旋转波近似下不同能级的能量图像，
虚线为 GRWA 近似的结果，实线为数值解的结果
选择参数关系 $\omega_0 = 0.75\Omega$

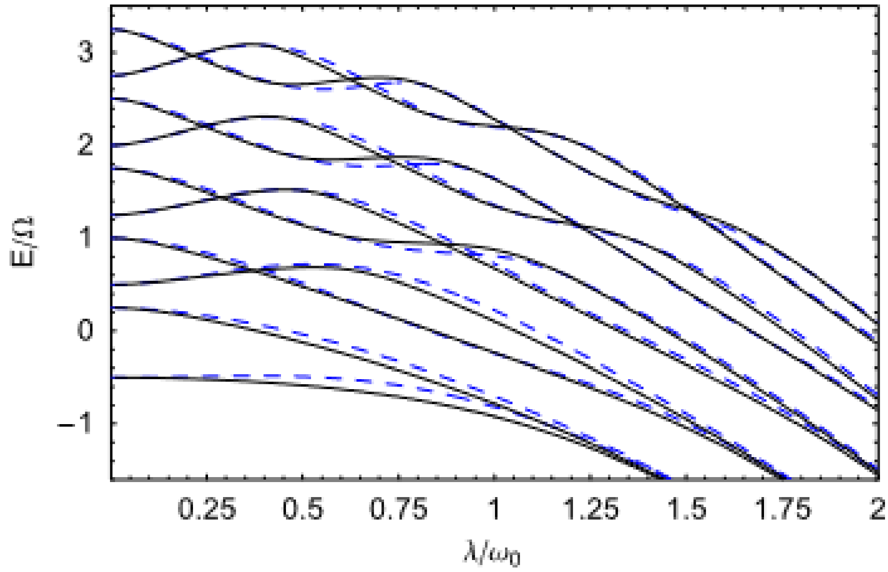

Figure 2: 文献[1]的结果

Bibliography

[1] E. K. Irish, "Generalized Rotating-Wave Approximation for Arbitrarily Large Coupling," *Physical Review Letters*, vol. 99, no. 17, p. 173601–173602, Oct. 2007, doi: 10/dj5z3b.