Rabi – IV

江源

2024-06-14

Section 1: Schrieffer Wolff transformation

We consider the Hamiltonian of the Rabi model

$$H = \omega \hat{a}^\dagger \hat{a} + \frac{\Omega}{2}\hat{\sigma}_z - \lambda\left(\hat{a}^\dagger + \hat{a}\right)\hat{\sigma}_x \tag{1}$$

where $\lambda = \frac{\sqrt{\Omega\omega}}{2}g$. We want to perform a Schrieffer-Wolff transformation to obtain an effective Hamiltonian in the limit of $\eta \gg 1$.

$$H_d = \frac{1}{\eta}\hat{a}^\dagger \hat{a} + \frac{1}{2}\hat{\sigma}_z - \frac{1}{2}\frac{g}{\sqrt{\eta}}\left(\hat{a}^\dagger + \hat{a}\right)\hat{\sigma}_x \tag{2}$$

1.1 The transformed Hamiltonian

The transformed Hamiltonian can be written as

$$\begin{aligned} H_{d'} \approx & \frac{1}{\eta}\hat{a}^\dagger \hat{a} + \frac{1}{2}\hat{\sigma}_z \\ & + \frac{1}{4\eta}g^2\left(\hat{a}^\dagger + \hat{a}\right)^2\hat{\sigma}_z + O\left(\frac{1}{\eta^2}\right) \\ & - \frac{1}{16\eta^2}g^4\left(\hat{a}^\dagger + \hat{a}\right)^4\hat{\sigma}_z + O\left(\frac{1}{\eta^3}\right) \end{aligned} \tag{3}$$

By projecting the above Hamiltonian to $H_\downarrow$, we obtain

$$H_\downarrow = \frac{1}{\eta}\hat{a}^\dagger \hat{a} - \frac{1}{2} + \frac{1}{4\eta}g^2\left(\hat{a}^\dagger + \hat{a}\right)^2 - \frac{1}{16\eta^2}g^4\left(\hat{a}^\dagger + \hat{a}\right)^4 \tag{4}$$

When $\eta \gg 1$, $H_\downarrow$ can be further simplified with first order $O\left(\frac{1}{\eta}\right)$ approximation

$$H_\downarrow = \frac{1}{\eta}\hat{a}^\dagger \hat{a} - \frac{1}{2} + \frac{1}{4\eta}g^2\left(\hat{a}^\dagger + \hat{a}\right)^2 \tag{5}$$

1.2 The superradiant phase

Take the unitary transformation

$$D[\alpha] = \exp\left(\alpha\left(\hat{a}^\dagger - \hat{a}\right)\right) \tag{6}$$

where α is a real number. The transformed Hamiltonian is

$$\tilde{H} = D^\dagger[\alpha]H_d D[\alpha] = H_d + \alpha\left[H_d, \hat{a}^\dagger - \hat{a}\right] + \frac{1}{2}\alpha^2\left[H_d, \left[H, \hat{a}^\dagger - \hat{a}\right]\right] + \ldots \tag{7}$$

We have

1.2.a $\left[\hat{a}^\dagger, \hat{a}^\dagger - \hat{a}\right]$

$$\left[\hat{a}^\dagger, \hat{a}^\dagger - \hat{a}\right] = 1 \tag{8}$$

1.2.b $\left[\hat{a}, \hat{a}^\dagger - \hat{a}\right]$

$$\left[\hat{a}, \hat{a}^\dagger - \hat{a}\right] = 1 \tag{9}$$

1.2.c $\left[\hat{a}^\dagger + \hat{a}, \hat{a}^\dagger - \hat{a}\right]$

$$\left[\hat{a}^\dagger + \hat{a}, \hat{a}^\dagger - \hat{a}\right] = 2 \tag{10}$$

1.2.d $\left[\hat{a}^\dagger \hat{a}, \hat{a}^\dagger - \hat{a}\right]$

$$\left[\hat{a}^\dagger \hat{a}, \hat{a}^\dagger - \hat{a}\right] = \hat{a}^\dagger + \hat{a} \tag{11}$$

1.2.e $\left[\left[\hat{a}^\dagger \hat{a}, \hat{a}^\dagger - \hat{a}\right], \hat{a}^\dagger - \hat{a}\right]$

$$\left[\left[\hat{a}^\dagger \hat{a}, \hat{a}^\dagger - \hat{a}\right], \hat{a}^\dagger - \hat{a}\right] = 2 \tag{12}$$

1.2.f $D^\dagger[\alpha]\hat{a}^\dagger D[\alpha]$

$$D^\dagger[\alpha]\hat{a}^\dagger D[\alpha] = \hat{a}^\dagger + \alpha \tag{13}$$

1.2.g $D^\dagger[\alpha]\hat{a} D[\alpha]$

$$D^\dagger[\alpha]\hat{a} D[\alpha] = \hat{a} + \alpha \tag{14}$$

1.3 Eigenstate

Therefore, the transformed Hamiltonian is

$$\begin{aligned} \tilde{H} &= D^\dagger[\alpha] H_d D[\alpha] \\ &= D^\dagger[\alpha]\left(\frac{1}{\eta}\hat{a}^\dagger\hat{a} + \frac{1}{2}\hat{\sigma}_z - g\frac{1}{2\sqrt{\eta}}(\hat{a}^\dagger + \hat{a})\hat{\sigma}_x\right) D[\alpha] \\ &= \frac{1}{\eta}(\hat{a}^\dagger + \alpha)(\hat{a} + \alpha) - \frac{g}{2\sqrt{\eta}}(\hat{a}^\dagger + \hat{a})\hat{\sigma}_x + \frac{1}{2}\hat{\sigma}_z - \frac{\alpha g}{\sqrt{\eta}}\hat{\sigma}_x \end{aligned} \tag{15}$$

1.4 The atomic part of the Hamiltonian

$$H_a = \frac{1}{2}\hat{\sigma}_z - \frac{\alpha g}{\sqrt{\eta}}\hat{\sigma}_x \tag{16}$$

choose basis as $|\hat{\sigma}_z\rangle \in \{|\uparrow\rangle, |\downarrow\rangle\}$, the corresponding Hamiltonian matrix is

$$H_a = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} - \frac{\alpha g}{\sqrt{\eta}}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & -\frac{\alpha g}{\sqrt{\eta}} \\ -\frac{\alpha g}{\sqrt{\eta}} & -\frac{1}{2} \end{pmatrix} \tag{17}$$

The eigenvalues of H_a are

$$E_{\{\pm\}} = \pm\frac{1}{2}\sqrt{1 + 4\alpha^2 g^2/\eta} \tag{18}$$

The eigenstates of the atomic part of the Hamiltonian, i.e. of $\frac{1}{2}\hat{\sigma}_z - \frac{\alpha g}{\sqrt{\eta}}\hat{\sigma}_x$

$$\begin{aligned} |\tilde{\uparrow}\rangle &= \cos(\theta)|\uparrow\rangle + \sin(\theta)|\downarrow\rangle \\ |\tilde{\downarrow}\rangle &= -\sin(\theta)|\uparrow\rangle + \cos(\theta)|\downarrow\rangle \end{aligned} \tag{19}$$

Also

$$
\begin{aligned}
|\uparrow\rangle &= \cos(\theta)\big|\tilde{\uparrow}\big\rangle - \sin(\theta)\big|\tilde{\downarrow}\big\rangle \\
|\downarrow\rangle &= \sin(\theta)\big|\tilde{\uparrow}\big\rangle + \cos(\theta)\big|\tilde{\downarrow}\big\rangle
\end{aligned}
\tag{20}
$$

To evaluate this equation, we have

$$
\begin{aligned}
H_a\big|\tilde{\uparrow}\big\rangle &= E_+\big|\tilde{\uparrow}\big\rangle \\
H_a\big|\tilde{\downarrow}\big\rangle &= E_-\big|\tilde{\downarrow}\big\rangle
\end{aligned}
\tag{21}
$$

Therefore, we have

$$
\begin{aligned}
&\left(\frac{1}{2}\hat{\sigma}_z - \frac{\alpha g}{\sqrt{\eta}}\hat{\sigma}_x\right)\big|\tilde{\uparrow}\big\rangle = \frac{1}{2}\sqrt{1+4\alpha^2 g^2/\eta}\big|\tilde{\uparrow}\big\rangle \\
\Rightarrow &\begin{cases}
\left(\frac{1}{2}\cos(\theta) - \frac{\alpha g}{\sqrt{\eta}}\sin(\theta)\right)|\uparrow\rangle &= \frac{1}{2}\sqrt{1+4\alpha^2 g^2/\eta}\cos(\theta)|\uparrow\rangle \\
\left(-\frac{1}{2}\sin(\theta) - \frac{\alpha g}{\sqrt{\eta}}\cos(\theta)\right)|\downarrow\rangle &= \frac{1}{2}\sqrt{1+4\alpha^2 g^2/\eta}\sin(\theta)|\downarrow\rangle
\end{cases} \\
&\Rightarrow \frac{-\frac{1}{2}\sin(\theta) - \frac{\alpha g}{\sqrt{\eta}}\cos(\theta)}{\frac{1}{2}\cos(\theta) - \frac{\alpha g}{\sqrt{\eta}}\sin(\theta)} = \frac{\sin(\theta)}{\cos(\theta)} = \tan(\theta) \\
&\Rightarrow \frac{-\tan(\theta) - \frac{2\alpha g}{\sqrt{\eta}}}{1 - \frac{2\alpha g}{\sqrt{\eta}}\tan(\theta)} = \tan(\theta) \\
&\Rightarrow -\tan(\theta) - \frac{2\alpha g}{\sqrt{\eta}} = \tan(\theta) - \frac{2\alpha g}{\sqrt{\eta}}\tan^2(\theta) \\
&\Rightarrow \frac{2\tan(\theta)}{1-\tan^2(\theta)} = \tan(2\theta) = -\frac{2\alpha g}{\sqrt{\eta}}
\end{aligned}
\tag{22}
$$

And, the new transition dimensionless coupling constant is defined as $\gamma = 1 + 4\alpha^2 g^2/\eta$

Therefore, we can obtain the new Hamiltonian with new Pauli matrices τ_x, τ_y, τ_z in term of the new basis $\big|\tilde{\uparrow}\big\rangle, \big|\tilde{\downarrow}\big\rangle$

$$
\begin{aligned}
\tilde{H}\big|\tilde{\uparrow}\big\rangle &= \frac{1}{\eta}(\hat{a}^\dagger + \alpha)(\hat{a} + \alpha)\big|\tilde{\uparrow}\big\rangle - \frac{g}{2\sqrt{\eta}}(\hat{a}^\dagger + \hat{a})\hat{\sigma}_x\big|\tilde{\uparrow}\big\rangle + H_a\big|\tilde{\uparrow}\big\rangle \\
&= \frac{1}{\eta}(\hat{a}^\dagger + \alpha)(\hat{a} + \alpha)\big|\tilde{\uparrow}\big\rangle \\
&- \frac{g}{2\sqrt{\eta}}(\hat{a}^\dagger + \hat{a})\hat{\sigma}_x\big|\tilde{\uparrow}\big\rangle \\
&+ E_+\big|\tilde{\uparrow}\big\rangle
\end{aligned}
\tag{23}
$$

We can focus on the term $-\frac{g}{2\sqrt{\eta}}(\hat{a}^\dagger + \hat{a})\hat{\sigma}_x\big|\tilde{\uparrow}\big\rangle$

$$
\begin{aligned}
&-\frac{g}{2\sqrt{\eta}}\left(\hat{a}^\dagger+\hat{a}\right)\hat{\sigma}_x\left|\tilde{\uparrow}\right\rangle \\
&=-\frac{g}{2\sqrt{\eta}}\left(\hat{a}^\dagger+\hat{a}\right)\hat{\sigma}_x(\cos(\theta)|\uparrow\rangle+\sin(\theta)|\downarrow\rangle) \\
&=-\frac{g}{2\sqrt{\eta}}\left(\hat{a}^\dagger+\hat{a}\right)(\cos(\theta)|\downarrow\rangle+\sin(\theta)|\uparrow\rangle) \\
&=-\frac{g}{2\sqrt{\eta}}\left(\hat{a}^\dagger+\hat{a}\right)\left[\cos(\theta)\left(\sin(\theta)\left|\tilde{\uparrow}\right\rangle+\cos(\theta)\left|\tilde{\downarrow}\right\rangle\right)+\sin(\theta)\left(\cos(\theta)\left|\tilde{\uparrow}\right\rangle-\sin(\theta)\left|\tilde{\downarrow}\right\rangle\right)\right] \\
&=-\frac{g}{2\sqrt{\eta}}\left(\hat{a}^\dagger+\hat{a}\right)\left[2\cos(\theta)\sin(\theta)\left|\tilde{\uparrow}\right\rangle+\left(\cos^2(\theta)-\sin^2(\theta)\right)\left|\tilde{\downarrow}\right\rangle\right] \\
&=-\frac{g}{2\sqrt{\eta}}\left(\hat{a}^\dagger+\hat{a}\right)\left[\sin(2\theta)\left|\tilde{\uparrow}\right\rangle+\cos(2\theta)\left|\tilde{\downarrow}\right\rangle\right]
\end{aligned}
\tag{24}
$$

Therefore, the following equation is obtained

$$
\begin{aligned}
\tilde{H}\left|\tilde{\uparrow}\right\rangle = \\
&=\frac{1}{\eta}\left(\hat{a}^\dagger+\alpha\right)(\hat{a}+\alpha)\left|\tilde{\uparrow}\right\rangle \\
&-\frac{g}{2\sqrt{\eta}}\left(\hat{a}^\dagger+\hat{a}\right)\left[\sin(2\theta)\left|\tilde{\uparrow}\right\rangle+\cos(2\theta)\left|\tilde{\downarrow}\right\rangle\right] \\
&+E_+\left|\tilde{\uparrow}\right\rangle
\end{aligned}
\tag{25}
$$

Also, we can obtain the equation for $\left|\tilde{\downarrow}\right\rangle$

$$
\begin{aligned}
\tilde{H}\left|\tilde{\downarrow}\right\rangle = \\
&=\frac{1}{\eta}\left(\hat{a}^\dagger+\alpha\right)(\hat{a}+\alpha)\left|\tilde{\downarrow}\right\rangle \\
&-\frac{g}{2\sqrt{\eta}}\left(\hat{a}^\dagger+\hat{a}\right)\left[\sin(2\theta)\left|\tilde{\uparrow}\right\rangle-\cos(2\theta)\left|\tilde{\downarrow}\right\rangle\right] \\
&+E_-\left|\tilde{\downarrow}\right\rangle
\end{aligned}
\tag{26}
$$

Therefore, the new Hamiltonian in the new basis $\left|\tilde{\uparrow}\right\rangle, \left|\tilde{\downarrow}\right\rangle$ is

$$
\begin{aligned}
\tilde{H}(\pm\alpha) &= \frac{1}{\eta}\left(\hat{a}^\dagger+\alpha\right)(\hat{a}+\alpha)+\frac{\gamma}{2}\tau_z-\frac{g}{2\sqrt{\eta}}\left(\hat{a}^\dagger+\hat{a}\right)\sin(2\theta)\tau_z-\frac{g}{2\sqrt{\eta}}\left(\hat{a}^\dagger+\hat{a}\right)\cos(2\theta)\tau_x \\
&=\frac{1}{\eta}\hat{a}^\dagger\hat{a}+\frac{\alpha^2}{\eta}+\frac{\alpha}{\eta}\left(\hat{a}^\dagger+\hat{a}\right)+\frac{\gamma}{2}\tau_z-\frac{g}{2\sqrt{\eta}}\left(\hat{a}^\dagger+\hat{a}\right)\sin(2\theta)\tau_z-\frac{g}{2\sqrt{\eta}}\left(\hat{a}^\dagger+\hat{a}\right)\cos(2\theta)\tau_x \\
&=\frac{1}{\eta}\hat{a}^\dagger\hat{a}-\frac{g}{2\sqrt{\eta}}\left(\hat{a}^\dagger+\hat{a}\right)\cos(2\theta)\tau_x+\frac{\gamma}{2}\tau_z+\frac{\alpha^2}{\eta}+\left(\frac{\alpha}{\eta}-\frac{g}{2\sqrt{\eta}}\sin(2\theta)\tau_z\right)\left(\hat{a}^\dagger+\hat{a}\right)
\end{aligned}
\tag{27}
$$

where $\gamma = 1+4\alpha^2\frac{g^2}{\eta}$

Now, we require that the block-diagonal perturbation term in the above equation, $\tilde{V}_d = \left(\frac{\alpha}{\eta} - \frac{g}{2\sqrt{\eta}}\sin(2\theta)\tau_z\right)\left(\hat{a}^\dagger+\hat{a}\right)$, vanishes upon the projection to the $H_{\tilde{\downarrow}}$, this is, $\frac{\alpha}{\eta}+\frac{g}{2\sqrt{\eta}}\sin(2\theta)=0$, whose nontrivial solutions are

$$
\begin{aligned}
\alpha &= -\frac{g}{2}\sqrt{\eta}\sin(2\theta) = \frac{\alpha g^2\sqrt{\eta}}{\sqrt{\eta + 4\alpha^2 g^2}} \\
\eta + 4\alpha^2 g^2 &= g^4\eta \\
\alpha^2 &= \frac{(g^4-1)\eta}{4g^2} \\
\alpha_g &= \pm\frac{\sqrt{(g^4-1)\eta}}{2g}
\end{aligned}
\tag{28}
$$

With the above choice of α, the block-diagonal perturbation becomes$\tilde{V}_d = \pm 2\frac{\alpha_g}{\eta}(\hat{a}^\dagger + \hat{a})|\tilde{\uparrow}\rangle\langle\tilde{\uparrow}|$, substituting this into γ, we obtain

$$
\begin{aligned}
\gamma &= 1 + 4\alpha_g^2\frac{g^2}{\eta} = 1 + 4\frac{(g^4-1)\eta}{4g^2}\frac{g^2}{\eta} \\
&= 1 + g^2 - 1 = g^2
\end{aligned}
\tag{29}
$$

And we can transform $\frac{g}{2\sqrt{\eta}}\cos(2\theta)$

$$
\begin{aligned}
\frac{g}{2\sqrt{\eta}}\cos(2\theta) &= \frac{g}{2\sqrt{\eta}}\frac{\sqrt{\eta}}{\sqrt{4\alpha_g^2 g^2 + \eta}} \\
&= \frac{g}{2\sqrt{\eta}}\frac{\sqrt{\eta}}{\sqrt{4\frac{(g^4-1)\eta}{4g^2}g^2 + \eta}} \\
&= \frac{g}{2\sqrt{\eta}}\frac{\sqrt{\eta}}{\sqrt{(g^4-1)\eta + \eta}} \\
&= \frac{1}{2g\sqrt{\eta}}
\end{aligned}
\tag{30}
$$

Therefore, within the subspace of $H_{\tilde{\downarrow}}$, the effective Hamiltonian is

$$
\tilde{H}(\pm\alpha_g) = \frac{1}{\eta}\hat{a}^\dagger\hat{a} - \frac{1}{2g\sqrt{\eta}}(\hat{a}^\dagger + \hat{a})\tau_x + \frac{g^2}{2}\tau_z + \frac{\alpha_g^2}{\eta}
\tag{31}
$$