Rabi – III

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Section 1: Schrieffer Wolff transformation

1.1 Rabi Hamiltonian

The Rabi Hamiltonian is given by

$$H = \omega \hat{a}^{\dagger} \hat{a} + \frac{\Omega}{2} \hat{\sigma}_z - \lambda \left(\hat{a}^{\dagger} + \hat{a} \right) \hat{\sigma}_x \tag{1}$$

It can be rewritten as

$$H = H_0 - \lambda V \tag{2}$$

where

$$H_0 = \omega \hat{a}^{\dagger} \hat{a} + \frac{\Omega}{2} \hat{\sigma}_z \tag{3}$$

and

$$V = \left(\hat{a}^{\dagger} + \hat{a} \right) \hat{\sigma}_x \tag{4}$$

.

With top four energy levels $|-, 0\rangle$, $|+, 0\rangle$, $|-, 1\rangle$, $|+, 1\rangle$, we can get the matrix form of H_0 as

$$H_0 = \begin{pmatrix} -\frac{\Omega}{2} & 0 & 0 & 0 \\ 0 & \frac{\Omega}{2} & 0 & 0 \\ 0 & 0 & \omega - \frac{\Omega}{2} & 0 \\ 0 & 0 & 0 & \omega + \frac{\Omega}{2} \end{pmatrix} \tag{5}$$

and the matrix form of the off-diagonal part $V = \left(\hat{a}^{\dagger} + \hat{a} \right) \hat{\sigma}_x$ as

$$V = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \tag{6}$$

1.2 Unitary transformation

We consider a unitary transformation $U = e^S$, where the generator S should be anti-Hermitian $S^{\dagger} = -S$. The transformed Hamiltonian can be written as

$$H' = U^{\dagger} H U = H + [H, S] + \frac{1}{2!} [[H, S], S] + \ldots \tag{7}$$

The transformed Hamiltonian can be divided into diagonal part and off-diagonal part using the fact that S is block off-diagonal and V is block-diagonal, and denoted as H'_{od} and H'_d.

$$
\begin{aligned}
H_d' &= \sum_{k=0}^{\infty} \frac{1}{(2k)!}[H_0, S]^{(2k)} - \sum_{k=0}^{\infty} \frac{1}{(2k+1)!}[\lambda V, S]^{(2k+1)} \\
H_{\text{od}}' &= \sum_{k=0}^{\infty} \frac{1}{(2k+1)!}[H_0, S]^{(2k+1)} - \sum_{k=0}^{\infty} \frac{1}{(2k)!}[\lambda V, S]^{(2k)}
\end{aligned} \tag{8}
$$

The off-diagonal part H_{od}' must be zero to diagonalize H. We can expand S as

$$
S = \lambda S^{(1)} + \lambda^2 S^{(2)} + \ldots \tag{9}
$$

.

So we can get the every order of S by solving the equation

$$
[H_0, S] + \lambda V - \frac{1}{2}\lambda[[V, S], S] + \cdots = 0 \tag{10}
$$

The $O(\lambda)$ term gives

$$
\begin{aligned}
&\left[H_0, S^{(1)}\right] - V = 0 \\
\Rightarrow &\left[H_0, S^{(1)}\right] = V
\end{aligned} \tag{11}
$$

The $O(\lambda^2)$ term gives

$$
\left[H_0, S^{(2)}\right] = 0 \tag{12}
$$

The $O(\lambda^3)$ term gives

$$
\begin{aligned}
&\left[H_0, S^{(3)}\right] - \frac{1}{2}\left[\left[V, S^{(1)}\right], S^{(1)}\right] + \frac{1}{3!}\left[\left[\left[H_0, S^{(1)}\right], S^{(1)}\right], S^{(1)}\right] = 0 \\
\Rightarrow &\left[H_0, S^{(3)}\right] = \frac{1}{3}\left[\left[V, S^{(1)}\right], S^{(1)}\right]
\end{aligned} \tag{13}
$$

The $O(\lambda^4)$ term gives

$$
\begin{aligned}
&\left[H_0, S^{(4)}\right] - \frac{1}{2}\left[\left[V, S^{(1)}\right], S^{(2)}\right] - \frac{1}{2}\left[\left[V, S^{(2)}\right], S^{(1)}\right] = 0 \\
\Rightarrow &\left[H_0, S^{(4)}\right] = 0
\end{aligned} \tag{14}
$$

1.2.a First order

The first order of S can be obtained by solving the equationeq. 11. Using the basis $|z_-, n\rangle$ and $|z_+, m\rangle$, we can get the matrix form of $S^{(1)}$ as

$$
\begin{aligned}
\left[H_0, S^{(1)}\right] &= V \\
\Rightarrow \left\langle z_-, n \middle| \left[H_0, S^{(1)}\right] \middle| z_+, m \right\rangle &= \langle z_-, n | V | z_+, m\rangle \\
\left\langle z_-, n \middle| H_0 S^{(1)} \middle| z_+, m\right\rangle - \left\langle z_-, n \middle| S^{(1)} H_0 \middle| z_+, m\right\rangle &= \langle z_-, n | V | z_+, m\rangle \\
\left\langle z_-, n \middle| \left(\omega \hat{a}^\dagger \hat{a} + \frac{\Omega}{2}\hat{\sigma}_z\right) S^{(1)} \middle| z_+, m\right\rangle - \left\langle z_-, n \middle| S^{(1)} \left(\omega \hat{a}^\dagger \hat{a} + \frac{\Omega}{2}\hat{\sigma}_z\right) \middle| z_+, m\right\rangle &= \langle z_-, n | V | z_+, m\rangle
\end{aligned} \tag{15}
$$

The first term of the left side is

$$
\begin{aligned}
&\left\langle z_-, n\middle|\left(\omega\hat{a}^\dagger\hat{a} + \frac{\Omega}{2}\hat{\sigma}_z\right)S^{(1)}\middle|z_+, m\right\rangle \\
=&\left\langle z_-, n\middle|\left[\left(\omega\hat{a}^\dagger\hat{a} + \frac{\Omega}{2}\hat{\sigma}_z\right)\sum_{n',m'}\left(|z_-, n'\rangle\langle(z_-, n')| + |z_+, m'\rangle\langle(z_+, m')|\right)S^{(1)}\right]\middle|z_+, m\right\rangle \\
=&\left\langle z_-, n\middle|\left[\sum_{n',m'}\left(\left(\omega\hat{a}^\dagger\hat{a} - \frac{\Omega}{2}\right)|z_-, n'\rangle\langle(z_-, n')| + \left(\omega\hat{a}^\dagger\hat{a} + \frac{\Omega}{2}\right)|z_+, m'\rangle\langle(z_+, m')|\right)S^{(1)}\right]\middle|z_+, m\right\rangle \\
=&\left\langle z_-, n\middle|\left[\sum_{n',m'}\left(\left(\omega n' - \frac{\Omega}{2}\right)|z_-, n'\rangle\langle(z_-, n')|S^{(1)} + \left(\omega m' + \frac{\Omega}{2}\right)|z_+, m'\rangle\langle(z_+, m')|S^{(1)}\right)\right]\middle|z_+, m\right\rangle \\
=&\left(\omega n' - \frac{\Omega}{2}\right)\delta_{n,n'}\langle z_-, n'|S^{(1)}|z_+, m\rangle \\
=&\left(\omega n - \frac{\Omega}{2}\right)\langle z_-, n|S^{(1)}|z_+, m\rangle
\end{aligned}
\tag{16}
$$

As the same way, the second term of the left side is

$$
\begin{aligned}
&\left\langle z_-, n\middle|S^{(1)}\left(\omega\hat{a}^\dagger\hat{a} + \frac{\Omega}{2}\hat{\sigma}_z\right)\middle|z_+, m\right\rangle \\
=&\left(\omega m + \frac{\Omega}{2}\right)\langle z_-, n|S^{(1)}|z_+, m\rangle
\end{aligned}
\tag{17}
$$

So the left side is

$$
\begin{aligned}
&\left\langle z_-, n\middle|\left(\omega\hat{a}^\dagger\hat{a} + \frac{\Omega}{2}\hat{\sigma}_z\right)S^{(1)}\middle|z_+, m\right\rangle - \left\langle z_-, n\middle|S^{(1)}\left(\omega\hat{a}^\dagger\hat{a} + \frac{\Omega}{2}\hat{\sigma}_z\right)\middle|z_+, m\right\rangle \\
=&\left(\omega n - \frac{\Omega}{2}\right)\langle z_-, n|S^{(1)}|z_+, m\rangle - \left(\omega m + \frac{\Omega}{2}\right)\langle z_-, n|S^{(1)}|z_+, m\rangle \\
=&\left(\omega n - \frac{\Omega}{2} - \omega m - \frac{\Omega}{2}\right)\langle z_-, n|S^{(1)}|z_+, m\rangle \\
=&(\omega(n-m) - \Omega)\langle z_-, n|S^{(1)}|z_+, m\rangle
\end{aligned}
\tag{18}
$$

The right side is

$$
\begin{aligned}
&\langle z_-, n|V|z_+, m\rangle \\
=&\langle z_-, n|(\hat{a}^\dagger + \hat{a})\hat{\sigma}_x|z_+, m\rangle \\
=&\langle z_-, n|(\hat{a}^\dagger + \hat{a})|z_-, m\rangle \\
=&\langle z_-, n|\hat{a}^\dagger|z_-, m\rangle + \langle z_-, n|\hat{a}|z_-, m\rangle \\
=&\delta_{n,m+1}\sqrt{m+1} + \delta_{n,m-1}\sqrt{m}
\end{aligned}
\tag{19}
$$

Summarizing the above results, we can get

$$
\begin{aligned}
&(\omega(n-m) - \Omega)\langle z_-, n|S^{(1)}|z_+, m\rangle = \delta_{n,m+1}\sqrt{n} + \delta_{n,m-1}\sqrt{m} \\
\Rightarrow&\ \langle z_-, n|S^{(1)}|z_+, m\rangle = \frac{1}{\omega(n-m) - \Omega}\left(\delta_{n,m+1}\sqrt{n} + \delta_{n,m-1}\sqrt{m}\right)
\end{aligned}
\tag{20}
$$

Because $\Omega \gg \omega$, we can get

$$\frac{1}{\omega(n-m)-\Omega} \sim -\frac{1}{\Omega}\left(1+\frac{\omega}{\Omega}(n-m)+O\left(\frac{\omega^2}{\Omega^2}\right)\right) \tag{21}$$

So the eq. 20 can be simplified as

$$\langle z_-, n|S^{(1)}|z_+, m\rangle \approx -\frac{1}{\Omega}\left(\delta_{n,m+1}\sqrt{n}+\delta_{n,m-1}\sqrt{m}\right) \tag{22}$$

Since the $S^{(1)}$ is block off-diagonal, we can get the other elements as

$$\langle z_+, n|S^{(1)}|z_-, m\rangle \approx \frac{1}{\Omega}\left(\delta_{n,m+1}\sqrt{m}+\delta_{n,m-1}\sqrt{n}\right) \tag{23}$$

It is easy to get the operator form of $S^{(1)}$ as

$$S^{(1)} = \frac{1}{\Omega}\left(\hat{a}^\dagger+\hat{a}\right)\left(\hat{\sigma}_+-\hat{\sigma}_-\right)+O\left(\frac{\omega}{\Omega^2}\right) \tag{24}$$

1.2.b Second order

The second order of S can be obtained by solving the equation

$$\left[H_0, S^{(2)}\right] = 0 \tag{25}$$

It is easy to get the operator form of $S^{(2)}$ as

$$S^{(2)} = 0 \tag{26}$$

1.2.c Third order

The third order of S can be obtained by solving the equation

$$\left[H_0, S^{(3)}\right] = \frac{1}{3}\left[\left[V, S^{(1)}\right], S^{(1)}\right] \tag{27}$$

The left side is the same as the first order as

$$\left(\omega(n-m)-\Omega\right)\langle z_-, n|S^{(3)}|z_+, m\rangle \tag{28}$$

To solve the right side, we need to calculate the commutator $\left[\left[V, S^{(1)}\right], S^{(1)}\right]$, firstly we need to calculate the commutator $\left[V, S^{(1)}\right]$

$$\begin{aligned}\left[V, S^{(1)}\right] &= \left[\left(\hat{a}^\dagger+\hat{a}\right)\hat{\sigma}_x, \frac{1}{\Omega}\left(\hat{a}^\dagger+\hat{a}\right)\left(\hat{\sigma}_+-\hat{\sigma}_-\right)\right] \\ = \frac{1}{\Omega}\left[\left(\hat{a}^\dagger+\hat{a}\right)\hat{\sigma}_x, \hat{a}^\dagger+\hat{a}\right]&\left(\hat{\sigma}_+-\hat{\sigma}_-\right)+\frac{1}{\Omega}\left(\hat{a}^\dagger+\hat{a}\right)\left[\left(\hat{a}^\dagger+\hat{a}\right)\hat{\sigma}_x, \hat{\sigma}_+-\hat{\sigma}_-\right]\end{aligned} \tag{29}$$

We define the $\hat{\sigma}_+$ and $\hat{\sigma}_-$ as

$$\begin{aligned}\hat{\sigma}_+ &= \frac{\hat{\sigma}_x+i\hat{\sigma}_y}{2} \\ \hat{\sigma}_- &= \frac{\hat{\sigma}_x-i\hat{\sigma}_y}{2}\end{aligned} \tag{30}$$

So the eq. 29 can be simplified as

$$\begin{aligned}[V, S^{(1)}] &= \frac{1}{\Omega}[\hat{a}^\dagger + \hat{a}, \hat{a}^\dagger + \hat{a}]\hat{\sigma}_x(i\hat{\sigma}_y) + \frac{1}{\Omega}(\hat{a}^\dagger + \hat{a})^2[\hat{\sigma}_x, i\hat{\sigma}_y] \\ &= \frac{1}{\Omega}(\hat{a}^\dagger + \hat{a})^2 2i^2\hat{\sigma}_z \\ &= -\frac{2}{\Omega}(\hat{a}^\dagger + \hat{a})^2\hat{\sigma}_z\end{aligned} \tag{31}$$

The commutator $[[V, S^{(1)}], S^{(1)}]$ is

$$\begin{aligned}[[V, S^{(1)}], S^{(1)}] &= -\frac{2}{\Omega}\left[(\hat{a}^\dagger + \hat{a})^2\hat{\sigma}_z, \frac{1}{\Omega}(\hat{a}^\dagger + \hat{a})(i\hat{\sigma}_y)\right] \\ &= -\frac{2}{\Omega^2}(\hat{a}^\dagger + \hat{a})^3[\hat{\sigma}_z, (i\hat{\sigma}_y)] \\ &= -\frac{2}{\Omega^2}(\hat{a}^\dagger + \hat{a})^3(-2i^2\hat{\sigma}_x) \\ &= -\frac{4}{\Omega^2}(\hat{a}^\dagger + \hat{a})^3(\hat{\sigma}_+ + \hat{\sigma}_-)\end{aligned} \tag{32}$$

So the right side of the equation is

$$\frac{1}{3}[[V, S^{(1)}], S^{(1)}] = -\frac{4}{3\Omega^3}(\hat{a}^\dagger + \hat{a})^3\hat{\sigma}_x \tag{33}$$

Now, we can solve the eq. 27 by using the eq. 28 and eq. 33, as

$$\begin{aligned}(\omega(n-m) - \Omega)\langle z_-, n|S^{(3)}|z_+, m\rangle &= \left\langle z_-, n\middle| -\frac{4}{3\Omega^2}(\hat{a}^\dagger + \hat{a})^3(\hat{\sigma}_+ + \hat{\sigma}_-)\middle| z_+, m\right\rangle \\ &= \left\langle z_-, n\middle| -\frac{4}{3\Omega^2}(\hat{a}^\dagger + \hat{a})^3(\hat{\sigma}_+ + \hat{\sigma}_-)\middle| z_+, m\right\rangle \\ &= \left\langle z_-, n\middle| -\frac{4}{3\Omega^2}(\hat{a}^\dagger + \hat{a})^3(\hat{\sigma}_-)\middle| z_+, m\right\rangle \\ \Rightarrow \langle z_-, n|S^{(3)}|z_+, m\rangle &= -\frac{1}{\Omega}\left\langle z_-, n\middle| -\frac{4}{3\Omega^2}(\hat{a}^\dagger + \hat{a})^3(\hat{\sigma}_-)\middle| z_+, m\right\rangle + O\left(\frac{\omega}{\Omega^4}\right)\end{aligned} \tag{34}$$

Like eq. 23 we can get the other elements as

$$\langle z_+, n|S^{(3)}|z_-, m\rangle = \frac{1}{\Omega}\left\langle z_+, n\middle| -\frac{4}{3\Omega^2}(\hat{a}^\dagger + \hat{a})^3(\hat{\sigma}_+)\middle| z_-, m\right\rangle + O\left(\frac{\omega}{\Omega^4}\right) \tag{35}$$

The operator form of $S^{(3)}$ is

$$S^{(3)} = -\frac{4}{3\Omega^3}(\hat{a}^\dagger + \hat{a})^3(\hat{\sigma}_+ - \hat{\sigma}_-) + O\left(\frac{\omega}{\Omega^4}\right) \tag{36}$$

1.3 The transformed Hamiltonian

The transformed Hamiltonian can be written as

$$
\begin{aligned}
H_{d'} &= \sum_{k=0}^{\infty} \frac{1}{(2k)!}[H_0, S]^{(2k)} - \sum_{k=0}^{\infty} \frac{1}{(2k+1)!}[\lambda V, S]^{(2k+1)} \\
&= [H_0, S]^{(0)} + \frac{1}{(2)!}[H_0, S]^{(2)} + \frac{1}{(4)!}[H_0, S]^{(4)} \\
&-\lambda[V, S]^{(1)} - \frac{1}{(3)!}[\lambda V, S]^{(3)} \\
\Rightarrow H_{d'} &\approx H_0 \\
&+\frac{1}{2}\left[H_0, \lambda S^{(1)}\right]^{(2)} \\
&+\frac{1}{2}\left[\left[H_0, \lambda S^{(1)}\right], \lambda^3 S^{(3)}\right] + \frac{1}{2}\left[\left[H_0, \lambda^3 S^{(3)}\right], \lambda S^{(1)}\right] \\
&+\frac{1}{2}\left[H_0, \lambda S^{(1)}\right]^{(4)} \\
&-\left[\lambda V, \lambda S^{(1)}\right] - \left[\lambda V, \lambda^3 S^{(3)}\right] \\
&-\frac{1}{3!}\left[\lambda V, \lambda S^{(1)}\right]^{(3)}
\end{aligned}
\tag{37}
$$

Using eq. 11 and eq. 13, we can simplify the eq. 37 as

$$
\begin{aligned}
H_{d'} &\approx H_0 \\
&+\frac{1}{2}\left[H_0, \lambda S^{(1)}\right]^{(2)} \\
&+\frac{1}{2}\left[\left[H_0, \lambda S^{(1)}\right], \lambda^3 S^{(3)}\right] + \frac{1}{2}\left[\left[H_0, \lambda^3 S^{(3)}\right], \lambda S^{(1)}\right] \\
&+\frac{1}{4!}\left[H_0, \lambda S^{(1)}\right]^{(4)} \\
&-\left[\lambda V, \lambda S^{(1)}\right] - \left[\lambda V, \lambda^3 S^{(3)}\right] \\
&-\frac{1}{3!}\left[\lambda V, \lambda S^{(1)}\right]^{(3)}
\end{aligned}
\tag{38}
$$

1.4 η replace $\frac{\omega}{\Omega}$

We define $\eta = \frac{\Omega}{\omega}$, $\lambda = \frac{\sqrt{\Omega\omega}}{2} g$ and rewrite the transformed Hamiltonian as

$$H = \frac{H'}{\Omega} = \frac{1}{\eta}\hat{a}^\dagger\hat{a} + \frac{1}{2}\hat{\sigma}_z - \frac{1}{2\sqrt{\eta}} g\hat{\sigma}_x\left(\hat{a}^\dagger + \hat{a}\right) \tag{39}$$

Now we can get the H_0 and V as

$$\begin{aligned} H_0 &= \frac{1}{\eta}\hat{a}^\dagger\hat{a} + \frac{1}{2}\hat{\sigma}_z \\ V &= \frac{1}{2\sqrt{\eta}}\hat{\sigma}_x\left(\hat{a}^\dagger + \hat{a}\right) \end{aligned} \tag{40}$$

From eq. 10, eq. 11, eq. 12, eq. 13, eq. 14, we can get the S as

$$\begin{aligned} \left[H_0, S^{(1)}\right] &= V \\ \left[H_0, S^{(2)}\right] &= 0 \\ \left[H_0, S^{(3)}\right] &= \frac{1}{3}\left[\left[V, S^{(1)}\right], S^{(1)}\right] \\ \left[H_0, S^{(4)}\right] &= 0 \end{aligned} \tag{41}$$

1.4.a First order

Using the basis $|z_-, n\rangle$ and $|z_+, m\rangle$, we can get the matrix form of $S^{(1)}$ as

$$\langle z_-, n|\left[H_0, S^{(1)}\right]|z_+, m\rangle = \langle z_-, n|V|z_+, m\rangle \tag{42}$$

The left side is

$$\begin{aligned} \left\langle z_-, n\middle|\left[\frac{1}{\eta}\hat{a}^\dagger\hat{a} + \frac{1}{2}\hat{\sigma}_z, S^{(1)}\right]\middle|z_+, m\right\rangle &= \left\langle z_-, n\middle|\left[\frac{1}{\eta}\hat{a}^\dagger\hat{a}, S^{(1)}\right]\middle|z_+, m\right\rangle + \left\langle z_-, n\middle|\left[\frac{1}{2}\hat{\sigma}_z, S^{(1)}\right]\middle|z_+, m\right\rangle \\ &= \frac{1}{\eta}(n-m)\langle z_-, n|S^{(1)}|z_+, m\rangle - \langle z_-, n|S^{(1)}|z_+, m\rangle \\ &= \left(\frac{1}{\eta}(n-m) - 1\right)\langle z_-, n|S^{(1)}|z_+, m\rangle \end{aligned} \tag{43}$$

The right side is

$$\begin{aligned} \langle z_-, n|V|z_+, m\rangle &= \left\langle z_-, n\middle|\frac{1}{2\sqrt{\eta}}\hat{\sigma}_x\left(\hat{a}^\dagger + \hat{a}\right)\middle|z_+, m\right\rangle \\ &= \frac{1}{2\sqrt{\eta}}\left(\delta_{n,m+1}\sqrt{n} + \delta_{n,m-1}\sqrt{m}\right) \end{aligned} \tag{44}$$

So we can get

$$\begin{aligned} \left(\frac{1}{\eta}(n-m) - 1\right)\langle z_-, n|S^{(1)}|z_+, m\rangle &= \frac{1}{2\sqrt{\eta}}\left(\delta_{n,m+1}\sqrt{n} + \delta_{n,m-1}\sqrt{m}\right) \\ \Rightarrow \langle z_-, n|S^{(1)}|z_+, m\rangle &= \frac{1}{2\sqrt{\eta}\left(\frac{1}{\eta}(n-m) - 1\right)}\left(\delta_{n,m+1}\sqrt{n} + \delta_{n,m-1}\sqrt{m}\right) \end{aligned} \tag{45}$$

Because $\eta \gg 1$, we can expand the above formula as

$$\frac{1}{2\sqrt{\eta}\left(\frac{1}{\eta}(n-m)-1\right)} \sim -\frac{1}{2\sqrt{\eta}}\left(1+\frac{1}{\eta}(n-m)+O\left(\frac{1}{\eta^2}\right)\right) \tag{46}$$

Choose the first order of upper formula to eq. 45, we can get

$$\langle z_-, n|S^{(1)}|z_+, m\rangle \approx -\frac{1}{2\sqrt{\eta}}\left(\delta_{n,m+1}\sqrt{n}+\delta_{n,m-1}\sqrt{m}\right) \tag{47}$$

Since the $S^{(1)}$ is block off-diagonal, we can get the other elements as

$$\langle z_+, n|S^{(1)}|z_-, m\rangle \approx \frac{1}{2\sqrt{\eta}}\left(\delta_{n,m+1}\sqrt{m}+\delta_{n,m-1}\sqrt{n}\right) \tag{48}$$

It is easy to get the operator form of $S^{(1)}$ as

$$S^{(1)} = \frac{1}{2\sqrt{\eta}}\left(\hat{a}^\dagger+\hat{a}\right)\left(\hat{\sigma}_+-\hat{\sigma}_-\right)+O\left(\frac{1}{\eta}\right) \tag{49}$$

1.4.b Second order

The operator form of $S^{(2)}$ is

$$S^{(2)} = 0 \tag{50}$$

1.4.c Third order

The third order of S can be obtained by solving the equation

$$\left[H_0, S^{(3)}\right] = \frac{1}{3}\left[\left[V, S^{(1)}\right], S^{(1)}\right] \tag{51}$$

The left side is the same as the first order as

$$\left(\frac{1}{\eta}(n-m)-1\right)\langle z_-, n|S^{(3)}|z_+, m\rangle \tag{52}$$

To solve the right side, we need to calculate the commutator $\left[\left[V, S^{(1)}\right], S^{(1)}\right]$, firstly we need to calculate the commutator $\left[V, S^{(1)}\right]$

$$\begin{aligned} \left[V, S^{(1)}\right] &= \left[\frac{1}{2\sqrt{\eta}}\hat{\sigma}_x\left(\hat{a}^\dagger+\hat{a}\right), \frac{1}{2\sqrt{\eta}}\left(\hat{a}^\dagger+\hat{a}\right)\left(\hat{\sigma}_+-\hat{\sigma}_-\right)+O\left(\frac{1}{\eta}\right)\right] \\ &= \frac{1}{4\eta}\left(\hat{a}^\dagger+\hat{a}\right)^2\left[\hat{\sigma}_x, i\hat{\sigma}_y\right]+O\left(\frac{1}{\eta\sqrt{\eta}}\right) \\ &= -\frac{1}{2\eta}\left(\hat{a}^\dagger+\hat{a}\right)^2\hat{\sigma}_z+O\left(\frac{1}{\eta\sqrt{\eta}}\right) \end{aligned} \tag{53}$$

The commutator $\left[\left[V, S^{(1)}\right], S^{(1)}\right]$ is

$$\begin{aligned}
\left[\left[V, S^{(1)}\right], S^{(1)}\right] &= \left[-\frac{1}{2\eta}\left(\hat{a}^\dagger + \hat{a}\right)^2 \hat{\sigma}_z + O\left(\frac{1}{\eta\sqrt{\eta}}\right), \frac{1}{2\sqrt{\eta}}\left(\hat{a}^\dagger + \hat{a}\right)\left(\hat{\sigma}_+ - \hat{\sigma}_-\right) + O\left(\frac{1}{\eta}\right)\right] \\
&= -\frac{1}{4\eta\sqrt{\eta}}\left(\hat{a}^\dagger + \hat{a}\right)^3 \left[\hat{\sigma}_z, i\hat{\sigma}_y\right] + O\left(\frac{1}{\eta^2}\right) \\
&= -\frac{1}{2\eta\sqrt{\eta}}\left(\hat{a}^\dagger + \hat{a}\right)^3 \left(\hat{\sigma}_+ + \hat{\sigma}_-\right) + O\left(\frac{1}{\eta^2}\right)
\end{aligned} \tag{54}$$

So the right side of the equation is

$$\frac{1}{3}\left[\left[V, S^{(1)}\right], S^{(1)}\right] = -\frac{1}{6\eta\sqrt{\eta}}\left(\hat{a}^\dagger + \hat{a}\right)^3 \left(\hat{\sigma}_+ + \hat{\sigma}_-\right) + O\left(\frac{1}{\eta^2}\right) \tag{55}$$

Now, we can solve the eq. 51 by using the eq. 52 and eq. 55, as

$$\begin{aligned}
\left(\frac{1}{\eta}(n-m) - 1\right)\left\langle z_-, n\middle|S^{(3)}\middle|z_+, m\right\rangle &= \left\langle z_-, n\middle|-\frac{1}{6\eta\sqrt{\eta}}\left(\hat{a}^\dagger + \hat{a}\right)^3 \left(\hat{\sigma}_+ + \hat{\sigma}_-\right) + O\left(\frac{1}{\eta^2}\right)\middle|z_+, m\right\rangle \\
&= \left\langle z_-, n\middle|-\frac{1}{6\eta\sqrt{\eta}}\left(\hat{a}^\dagger + \hat{a}\right)^3 \left(\hat{\sigma}_+ + \hat{\sigma}_-\right)\middle|z_+, m\right\rangle + O\left(\frac{1}{\eta^2}\right) \\
&= \left\langle z_-, n\middle|-\frac{1}{6\eta\sqrt{\eta}}\left(\hat{a}^\dagger + \hat{a}\right)^3 \left(\hat{\sigma}_-\right)\middle|z_+, m\right\rangle + O\left(\frac{1}{\eta^2}\right) \\
\Rightarrow \left\langle z_-, n\middle|S^{(3)}\middle|z_+, m\right\rangle &= -\left\langle z_-, n\middle|-\frac{1}{6\eta\sqrt{\eta}}\left(\hat{a}^\dagger + \hat{a}\right)^3 \left(\hat{\sigma}_-\right)\middle|z_+, m\right\rangle + O\left(\frac{1}{\eta^2}\right)
\end{aligned} \tag{56}$$

Therefore, we can get the operator form of $S^{(3)}$ as

$$S^{(3)} = -\frac{1}{6\eta\sqrt{\eta}}\left(\hat{a}^\dagger + \hat{a}\right)^3 \left(\hat{\sigma}_+ - \hat{\sigma}_-\right) + O\left(\frac{1}{\eta^2}\right) \tag{57}$$

1.5 The transformed Hamiltonian

The transformed Hamiltonian can be written like eq. 37, but replace $\frac{\omega}{\Omega}$ by η.

$$\begin{aligned}
H_{d'} &= \sum_{k=0}^{\infty} \frac{1}{(2k)!} [H_0, S]^{(2k)} - \sum_{k=0}^{\infty} \frac{1}{(2k+1)!} [gV, S]^{(2k+1)} \\
&= [H_0, S]^{(0)} + \frac{1}{(2)!} [H_0, S]^{(2)} + \frac{1}{(4)!} [H_0, S]^{(4)} \\
&\quad -g[V, S]^{(1)} - \frac{1}{(3)!} [gV, S]^{(3)} \\
\Rightarrow H_{d'} &\approx H_0 \\
&\quad + \frac{1}{2} \left[H_0, gS^{(1)}\right]^{(2)} \\
&\quad + \frac{1}{2} \left[\left[H_0, gS^{(1)}\right], g^3 S^{(3)}\right] + \frac{1}{2} \left[\left[H_0, g^3 S^{(3)}\right], gS^{(1)}\right] \\
&\quad + \frac{1}{4!} \left[H_0, gS^{(1)}\right]^{(4)} \\
&\quad - \left[gV, gS^{(1)}\right] - \left[gV, g^3 S^{(3)}\right] \\
&\quad - \frac{1}{3!} \left[gV, gS^{(1)}\right]^{(3)} \\
\Rightarrow H_{d'} &\approx H_0 \\
&\quad + \frac{1}{2} g^2 \left[V, S^{(1)}\right] \\
&\quad + \frac{1}{2} g^4 \left[V, S^{(3)}\right] + \frac{1}{6} g^4 \left[V, S^{(1)}\right]^{(3)} \\
&\quad + \frac{1}{4!} g^4 \left[V, S^{(1)}\right]^{(3)} \\
&\quad - g^2 \left[V, S^{(1)}\right] - g^4 \left[V, S^{(3)}\right] \\
&\quad - \frac{1}{3!} g^4 \left[V, S^{(1)}\right]^{(3)} \\
\Rightarrow H_{d'} &\approx H_0 \\
&\quad - \frac{1}{2} g^2 \left[V, S^{(1)}\right] \\
&\quad - \frac{1}{2} g^4 \left[V, S^{(3)}\right] \\
&\quad + \frac{1}{24} g^4 \left[V, S^{(1)}\right]^{(3)}
\end{aligned} \tag{58}$$

1.5.a $\left[V, S^{(1)}\right]$

The commutator $\left[V, S^{(1)}\right]$ is

$$\begin{aligned}
\left[V, S^{(1)}\right] &= \left[\frac{1}{2\sqrt{\eta}} \hat{\sigma}_x \left(\hat{a}^\dagger + \hat{a}\right), \frac{1}{2\sqrt{\eta}} \left(\hat{a}^\dagger + \hat{a}\right)\left(\hat{\sigma}_+ - \hat{\sigma}_-\right)\right] \\
&= \frac{1}{4\eta} \left(\hat{a}^\dagger + \hat{a}\right)^2 \left[\hat{\sigma}_x, i\hat{\sigma}_y\right] \\
&= -\frac{1}{2\eta} \left(\hat{a}^\dagger + \hat{a}\right)^2 \hat{\sigma}_z
\end{aligned} \tag{59}$$

1.5.b $\left[V, S^{(3)}\right]$

The commutator $\left[V, S^{(3)}\right]$ is

$$\begin{aligned}
\left[V, S^{(3)}\right] &= \left[\frac{1}{2\sqrt{\eta}}\hat{\sigma}_x\left(\hat{a}^\dagger + \hat{a}\right), -\frac{1}{6\eta\sqrt{\eta}}\left(\hat{a}^\dagger + \hat{a}\right)^3\left(\hat{\sigma}_+ - \hat{\sigma}_-\right)\right] \\
&= -\frac{1}{12\eta^2}\left(\hat{a}^\dagger + \hat{a}\right)^4\left[\hat{\sigma}_x, i\hat{\sigma}_y\right] \\
&= \frac{1}{6\eta^2}\left(\hat{a}^\dagger + \hat{a}\right)^4\hat{\sigma}_z
\end{aligned} \tag{60}$$

1.5.c $\left[V, S^{(1)}\right]^{(3)}$

The commutator $\left[V, S^{(1)}\right]^{(3)}$ is

$$\begin{aligned}
\left[V, S^{(1)}\right]^{(3)} &= \left[\left[\left[V, S^{(1)}\right], S^{(1)}\right], S^{(1)}\right] \\
&= \left[-\frac{1}{2\eta\sqrt{\eta}}\left(\hat{a}^\dagger + \hat{a}\right)^3\left(\hat{\sigma}_+ + \hat{\sigma}_-\right), \frac{1}{2\sqrt{\eta}}\left(\hat{a}^\dagger + \hat{a}\right)\left(\hat{\sigma}_+ - \hat{\sigma}_-\right)\right] \\
&= -\frac{1}{4\eta^2}\left(\hat{a}^\dagger + \hat{a}\right)^4\left[\hat{\sigma}_x, i\hat{\sigma}_y\right] \\
&= \frac{1}{2\eta^2}\left(\hat{a}^\dagger + \hat{a}\right)^4\hat{\sigma}_z
\end{aligned} \tag{61}$$

1.5.d The transformed Hamiltonian

The transformed Hamiltonian can be written as

$$
\begin{aligned}
H_{d'} \approx\ & H_0 \\
& -\frac{1}{2} g^2 \left[V, S^{(1)}\right] \\
& -\frac{1}{2} g^4 \left[V, S^{(3)}\right] \\
& +\frac{1}{24} g^4 \left[V, S^{(1)}\right]^{(3)} \\
\Rightarrow H_{d'} \approx\ & \frac{1}{\eta} \hat{a}^\dagger \hat{a} + \frac{1}{2} \hat{\sigma}_z \\
& -\frac{1}{2} g^2 \left(-\frac{1}{2\eta} \left(\hat{a}^\dagger + \hat{a}\right)^2 \hat{\sigma}_z\right) \\
& -\frac{1}{2} g^4 \left(\frac{1}{6\eta^2} \left(\hat{a}^\dagger + \hat{a}\right)^4 \hat{\sigma}_z\right) \\
& +\frac{1}{24} g^4 \left(\frac{1}{2\eta^2} \left(\hat{a}^\dagger + \hat{a}\right)^4 \hat{\sigma}_z\right) \\
\Rightarrow H_{d'} \approx\ & \frac{1}{\eta} \hat{a}^\dagger \hat{a} + \frac{1}{2} \hat{\sigma}_z \\
& +\frac{1}{4\eta} g^2 \left(\hat{a}^\dagger + \hat{a}\right)^2 \hat{\sigma}_z + O\left(\frac{1}{\eta^2}\right) \\
& -\frac{1}{16\eta^2} g^4 \left(\hat{a}^\dagger + \hat{a}\right)^4 \hat{\sigma}_z + O\left(\frac{1}{\eta^3}\right)
\end{aligned}
\tag{62}
$$

Bibliography