Name(s):

EXERCISE 6B: BLACK BODY RADIATION FLUX

Objectives:

- Finish deriving the **Planck Law of Radiation**
- Derive the **Stefan-Boltzmann Law** and the radiant energy flux of a black body
- Understand the relation between **absorptivity** and **emissivity**

Useful results from last time:

- Planck distribution for photon number s in a single mode: $\langle s \rangle_\omega = \frac{1}{e^{\hbar\omega/\tau}-1}$
- Density of states of radiation: $\mathcal{D}(\omega) = \frac{\omega^2 V}{\pi^2 c^3}$

1. *Planck Law of Radiation.*

   a. The radiant energy per unit volume U/V can be found by integrating the **spectral density of radiation** u_ω over all frequencies:

   $$U/V = \int d\omega u_\omega. \tag{1}$$

   b. Give an expression for u_ω in terms of the density of states $\mathcal{D}(\omega)$ and the average number $\langle s \rangle_\omega$ of photons in the mode.

   c. Plug in your results for $\mathcal{D}(\omega)$ and $\langle s \rangle_\omega$ to obtain an explicit expression for the spectral density of radiation at temperature τ. The result is the **Planck radiation law**.

d. Sketch the Planck spectrum u_ω. Which portion of this spectrum could have been correctly predicted by classical theory? Which range of parameters requires the quantum mechanical description?

e. Explain physically the behavior of the black-body spectrum in the low- and high-frequency limits.

f. The peak of the Planck black body spectrum is at $\hbar\omega_{\text{max}} \approx 2.82 k_B T$. What wavelength does this correspond to...

   i. ...for the sun?

   ii. ...for the earth?

   What are the implications for the amount of radiation emitted in the *visible* regime of the spectrum?

g. Evaluate the integral in Eq. 1 to obtain the **Stefan-Boltzmann law of radiation** describing the energy density U/V. You will need the definite integral:

$$\int_0^\infty dx \frac{x^3}{e^x - 1} = \frac{\pi^4}{15} \tag{2}$$

h. Give a few physical reasons for why the energy density scales so strongly with temperature.

2. The Stefan-Boltzmann Law tells us the radiant energy density inside a black body at temperature τ. How do we determine the **radiant energy flux** J, i.e., the power emitted per unit surface area?

   a. Consider a small hole in the surface of a black body at temperature τ. Radiation can exit this hole at a variety of different angles θ, ϕ, where θ is measured relative to the surface normal.

      i. Draw a sketch illustrating the angle θ.

      ii. What is the relevant range of values θ, ϕ?

   b. Let du_ω denote the infinitesimal spectral density of radiant **energy density** that is directed into an infinitesimal solid angle $d\Omega = \sin\theta d\theta d\phi$ centered about (θ, ϕ).

      i. Express du_ω in terms of the total spectral density u_ω and $d\Omega$.

      ii. What are the dimensions of du_ω, in terms of energy, length, and time?

c. Let dj_ω denote the infinitesimal spectral density of radiant energy **flux** that is directed into the solid angle $d\Omega$ centered about (θ, ϕ).
   i. What are the dimensions of dj_ω, in terms of energy, length, and time?
   ii. Express dj_ω in terms of du_ω, the speed of light c, and the angles θ and/or ϕ.

d. To calculate the spectral density of radiant energy **flux** j_ω through the hole, we must integrate over all possible angles at which the radiation can exit the hole.
   i. Write down an integral expression for j_ω in terms of u_ω, θ, and ϕ.
   ii. Do the integral to calculate j_ω in terms of u_ω.

e. The total flux at the surface, $J = \int u_\omega d\omega$, is often expressed as

$$J_U = \sigma_B T^4, \tag{3}$$

where σ_B is called the **Stefan-Boltzmann constant**. What is the value of σ_B?

f. In the homework, you will calculate the relationship between the temperature of the sun and the temperature of the earth. For now:

   i. How much higher is the radiant energy flux emitted from the surface of the sun ($T \sim 6000$ K) than from the surface of the earth ($T \sim 300$ K)?

   ii. What other factors affect how bright the sun and earth appear to an observer comparing the two from the same distance?

3. *Kirchhoff's law.* The container in Fig. 1 is filled with hot water. Its left side is silvered on the outside, whereas its right side is blackened on the outside. Which thermometer reads a higher temperature? Explain.

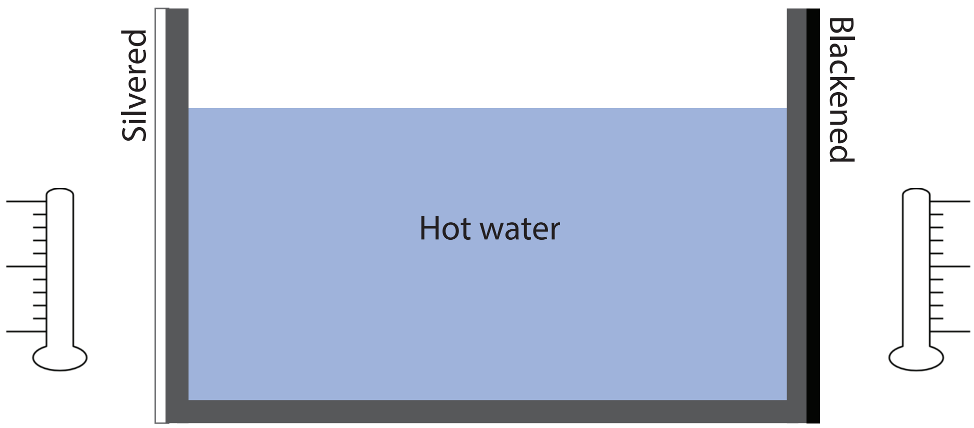

FIG. 1. Which thermometer reads a higher temperature?