Names:

EXERCISE 9B: BOSE-EINSTEIN CONDENSATION

Objective:

- Calculate the critical temperature for **Bose-Einstein condensation**
- Understand what determines the **condensate fraction** and how it is measured

References: Kittel & Kroemer, Ch. 7, pp. 199-206

Useful past results:

- Fermi-Dirac (+) and Bose-Einstein (−) distributions

$$f_{\pm}(\varepsilon) = \frac{1}{e^{\beta(\varepsilon-\mu)} \pm 1}$$

  describing the mean occupation of a single orbital of energy ε.

- Density of states

$$\mathcal{D}(\varepsilon) = \frac{gV}{4\pi^2\hbar^3}(2m)^{3/2}\varepsilon^{1/2}$$

  for particles of mass m with multiplicity g of internal states, in a box of volume V.

1. *Bose-Einstein condensation.* For a gas of non-interacting bosons, the many-particle ground state consists of all particles occupying the orbital of lowest energy, which we shall take to be at $\varepsilon = 0$. How low must the temperature be in order to accrue a macroscopic occupation of the ground state?

   a. Let N_0 denote the number of bosons in the ground state and N_e the population of all excited states combined. Write down expressions for N_0 and N_e, the latter in terms of the density of states.

   b. What condition must be imposed on the chemical potential for your expressions in a. to be physically reasonable?

c. Assume that the gas consists of sufficiently many atoms that $N_0 \gg 1$, without making any assumption about the fraction N_0/N_e of ground- to excited-state atoms.

   i. What does the condition $N_0 \gg 1$ imply about the absolute value of the chemical potential $|\mu|$?

   ii. Use the assumption $N_0 \gg 1$ to give an approximate expression for N_e that is independent of the chemical potential μ.

d. Evaluate N_e as a function of temperature τ for spinless bosons in a three-dimensional box of volume V. Your result will include a definite integral that you should express in dimensionless form and leave unevaluated.

e. Does the definite integral from part d. converge or diverge...

   i. ...at $x = 0$?

   ii. ...as $x \to \infty$?

f. Physically, how do you interpret the convergence or divergence of the integral from part d.?

g. For a given total atom number N, let τ_E denote the lowest temperature for which all atoms can be accommodated in excited states; this is the **critical temperature for Bose-Einstein condensation**. Below this temperature, there must be a macroscopic occupation N_0 of the ground state.

   i. Express the condition for Bose-Einstein condensation in terms of the density n and the thermal de Broglie wavelength $\lambda_\tau = h/\sqrt{2\pi m\tau}$.

   ii. How does the critical temperature τ_E scale with density n?

   iii. Let Δp represent the spread of the momentum distribution at temperature τ. What do your results imply about the critical *phase space density* $n/(\Delta p)^3$ for Bose-Einstein condensation?

h. How does the critical temperature for Bose-Einstein condensation compare with the energy of the first excited state? Is the result surprising? Why or why not?

i. Calculate the condensate fraction N_0/N as a function of density $n = N/V$ and temperature $\tau < \tau_E$. Then sketch it, as follows:
   i. Sketch N_0/N vs. temperature $\tau < \tau_E$ at fixed density.
   ii. Sketch N_0/N vs. density n at fixed temperature $\tau < \tau_E$.