EXERCISE 9A: FERMI DEGENERACY

Objectives:

- Calculate the **density of states** of electrons in a metal
- Define and calculate the **Fermi energy**
- Examine the physical significance of the Fermi energy and **Fermi momentum**

References: Kittel & Kroemer, Ch. 7, pp. 181-199

Useful past results:

- Fermi-Dirac (+) and Bose-Einstein (−) distributions

$$f_{\pm}(\varepsilon) = \frac{1}{e^{\beta(\varepsilon-\mu)} \pm 1}$$

describing the mean occupation of a single orbital of energy ε.

1. *Fermi Degeneracy.* Today, we will begin analyzing the implications of the Fermi-Dirac distribution for a realistic system with many orbitals characterized by a density of states $\mathcal{D}(\varepsilon)$, focusing on the regime of **quantum degeneracy**.

   a. List some physical systems can be modeled as gases of degenerate fermions.

   b. For a gas of identical fermions, write down an integral expression for the particle number N in terms of the Fermi-Dirac distribution and the density of states.

c. Simplify your expression in the zero-temperature limit $\tau \to 0$. The Fermi-Dirac distribution should drop out. What quantities determine N in this limit?

2. *Density of states for massive particles in a box.* Calculate the density of states $\mathcal{D}(\varepsilon)$ for a non-relativistic particle of mass m in a three-dimensional box of volume $V = L \times L \times L$. Allow for the possibility that the particle has g internal states that are independent of its spatial wave function (e.g., for a spin-1/2 particle, $g = 2$).

   a. We previously calculated the density of states $\mathcal{D}(\varepsilon)$ for photons and phonons. For purposes of such a calculation, what are the key similarities and differences between those systems and the gas of fermions?

   b. The eigenstates can be visualized as a grid of points in $\mathbf{k}$-space, where $\hbar\mathbf{k}$ is the momentum. Sketch this grid and indicate the spacing between the points in terms of L.

c. Calculate the following quantities in terms of the volume V:
   i. the number $\mathcal{N}(k)$ of quantum states of momentum $\leq \hbar k$
   ii. the number $\mathcal{N}(\varepsilon)$ of quantum states of energy $\leq \varepsilon$.

d. Show that the density of states $\mathcal{D}(\varepsilon) = d\mathcal{N}/d\varepsilon$ is directly proportional to $\mathcal{N}/\varepsilon$, and find the constant of proportionality.
   i. Derive a relation between $d\mathcal{N}$ and $d\varepsilon$ by first relating $\ln \mathcal{N}$ to $\ln \varepsilon$.
   ii. Rearrange your expression from i. to obtain $\mathcal{D}(\varepsilon)$ in terms of $\mathcal{N}$ and ε.

e. Evaluate $\mathcal{D}(\varepsilon)$ for a gas of spin-1/2 particles as a function of volume V, mass m, and energy ε.

3. *Fermi energy and Fermi momentum.* The **Fermi energy** ε_F for a gas of N particles is defined by the relation

$$N = \int_0^{\varepsilon_F} \mathcal{D}(\varepsilon) d\varepsilon. \tag{1}$$

a. How is the **Fermi energy** related to the **Fermi level** (chemical potential) $\mu(n, \tau)$?

b. Calculate the Fermi energy as a function of number density $n = N/V$ for a system of spin-1/2 fermions. Assume that ε_F is the same for both spin states.
Hint: use your calculation of $\mathcal{N}(\varepsilon)$ from problem 2.c.

c. In terms of the Fermi energy ε_F, define the **Fermi momentum** $\hbar k_F = \sqrt{2m\varepsilon_F}$. Calculate $\hbar k_F$ in terms of the density n.

d. Explain the physical significance of the Fermi momentum in words. You might supplement your explanation with sketches in real space and/or momentum space.

e. Calculate the de Broglie wavelength of particles at the Fermi level ε_F in terms of the interparticle spacing $n^{-1/3}$. How do you interpret your result?

f. Calculate the Fermi temperature $T_F = \varepsilon_F / k_B$ and the ratio T/T_F for the following systems:

   i. A gas of electrons in copper at room temperature; the density of conduction electrons is $n = (0.23\ \text{nm})^{-3}$.

   ii. A white dwarf star composed of hydrogen atoms at a density $n = (1\ \text{pm})^{-3}$. At such a high density, the electrons are not bound to the protons; consider each in turn. The temperature of such a star is believed to be on the order of $T \sim 10^7$ K.

   iii. A spin-polarized gas of potassium-40 atoms at a density $n = (1\ \mu\text{m})^{-3}$ and a temperature of 10 nK.

4. *Energy and heat capacity of an electron gas.*

   a. Write down an integral expression for the kinetic energy E of an electron gas in terms of $\mathcal{D}(\varepsilon)$ and $f_+(\varepsilon)$.

   b. Calculate the kinetic energy E_0 of an electron gas in the zero-temperature limit. Express it in terms of the Fermi energy ε_F and particle number N.

      i. The density of states is of the form $\mathcal{D}(\varepsilon) = \alpha\varepsilon^\zeta$. What is ζ? To minimize algebra, leave α unevaluated.

      ii. Find E_0 in terms of α and the Fermi energy ε_F.

      iii. Use your result from 2.d. to express the particle number N in terms of α and the Fermi energy ε_F.

      iv. Re-express E_0 in terms of only N, ε_F, and a numerical prefactor.

c. In section, you will show that the heat capacity of an electron gas for $\tau \ll \tau_F$ is

$$C_{\text{el}} = \frac{\pi^2}{3}\mathcal{D}(\varepsilon_F)\tau. \tag{2}$$

For now, let us find a physical explanation for the dependence of C_{el} on $\mathcal{D}(\varepsilon_F)$ and τ. Don't worry about numerical factors.

i. Sketch the Fermi-Dirac distribution at $\tau = 0$ and at a small but non-zero temperature $\tau \ll \tau_F$.

ii. Based on your sketch, estimate the energy difference $\Delta E \equiv E_\tau - E_0$ between the finite-temperature and zero-temperature cases.

iii. Estimate the heat capacity C_{el} associated with the kinetic energy of the electron gas. Is the scaling with temperature consistent with Eq. 2?