Name(s):

EXERCISE 6B: BLACK BODY RADIATION FLUX

Objectives:

- Finish deriving the Planck Law of Radiation
- Derive the Stefan-Boltzmann Law and Stefan-Boltzmann Constant
- Understand the relation between **absorptivity** and **emissivity**

Useful results from last time:

- Planck distribution for photon number s in a single mode: $\langle s\rangle_\omega = \frac{1}{e^{\hbar\omega/\tau}-1}$
- Density of states of radiation: $\mathcal{D}(\omega) = \frac{\omega^2 V}{\pi^2 c^3}$

1. *Planck Law of Radiation.*

   a. The radiant energy per unit volume U/V can be found by integrating the **spectral density of radiation** u_ω over all frequencies:

$$U/V = \int d\omega u_\omega. \tag{1}$$

   b. Give an expression for u_ω in terms of the density of states $\mathcal{D}(\omega)$ and the average number $\langle s\rangle_\omega$ of photons in the mode.

$$u_\omega = \underbrace{\langle s\rangle_\omega \cdot \hbar\omega}_{\text{energy per mode}} \cdot \mathcal{D}(\omega)/V$$

   c. Plug in your results for $\mathcal{D}(\omega)$ and $\langle s\rangle_\omega$ to obtain an explicit expression for the spectral density of radiation at temperature τ. The result is the **Planck radiation law**.

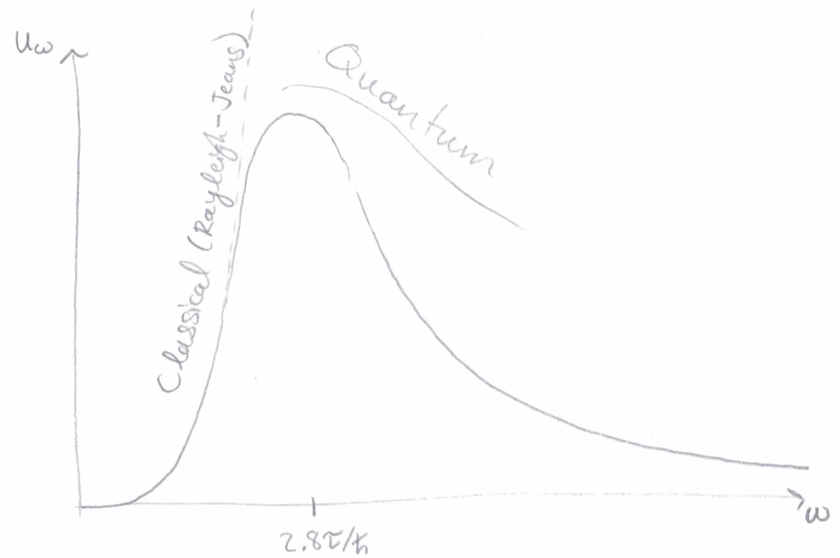
$$u_\omega = \frac{\hbar\omega}{e^{\hbar\omega/\tau}-1} \cdot \frac{\omega^2}{\pi^2 c^3} = \frac{\hbar\omega^3}{\pi^2 c^3 (e^{\hbar\omega/\tau}-1)}$$

<u>Limits</u>

$$\xrightarrow[\hbar\omega \ll \tau]{} \frac{\hbar\omega^3}{\pi^2 c^3 (\hbar\omega/\tau)} = \frac{\omega^2 \tau}{\pi^2 c^3}$$

$$\xrightarrow[\hbar\omega \gg \tau]{} \frac{\hbar\omega^3}{\pi^2 c^3} e^{-\hbar\omega/\tau}$$

d. Sketch the Planck spectrum u_ω. Which portion of this spectrum could have been correctly predicted by classical theory? Which range of parameters requires the quantum mechanical description?



Classical: $\tau \gg \hbar\omega$

e. Explain physically the behavior of the black-body spectrum in the low- and high-frequency limits.

* At low frequency $\hbar\omega \ll \tau$, the quantization of light can be neglected and the spectrum is well described simply by taking into account the scaling $D(\omega) \propto \omega^2$ of density of states and assuming energy $\sim\tau$ per mode. So $u_\omega \to 0$ as $\omega \to 0$ just because the sphere of possible states in k-space becomes vanishingly small.

* At high frequency, the occupation of modes with energy per photon $\hbar\omega \gg \tau$ is exponentially suppressed by Boltzmann factor — hence the exponential decay of u_ω as $\omega \to \infty$ is explained only by accounting for quantization of light.

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f. The peak of the Planck black body spectrum is at $\hbar\omega_{\text{max}} \approx 2.82 k_B T$. What wavelength does this correspond to...

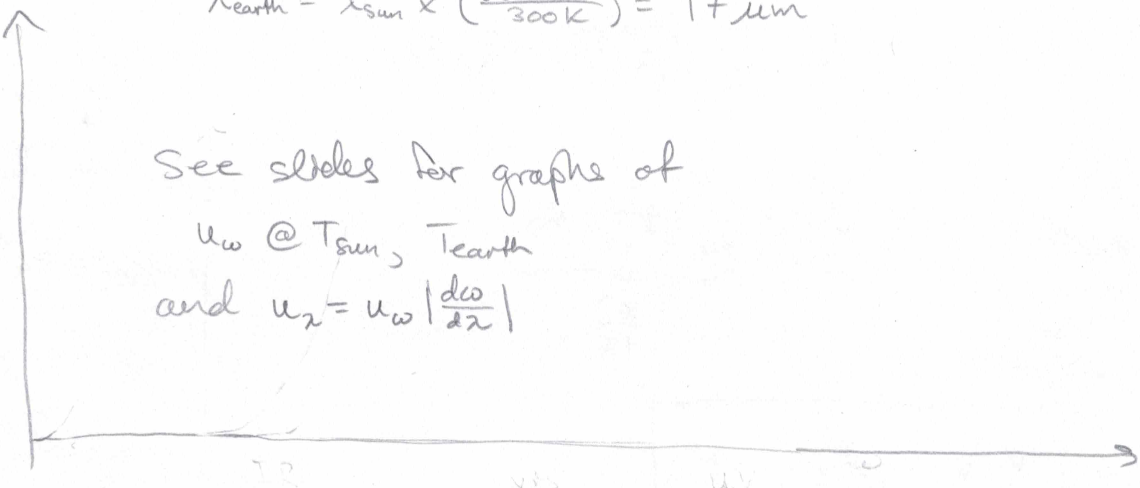
i. ...for the sun? $(T = 5800\,K)$

ii. ...for the earth? $(T \sim 300\,K)$

What are the implications for the amount of radiation emitted in the *visible* regime of the spectrum?

$$\lambda_{sun} = \frac{2\pi c}{\omega_{sun}} = \frac{2\pi \hbar c}{2.82\,k_B T} = \frac{hc}{2.82\,k_B T} = 880\text{ nm}$$

$$\lambda_{earth} = \lambda_{sun} \times \left(\frac{5800\,K}{300\,K}\right) = 17\,\mu m$$



See slides for graphs of

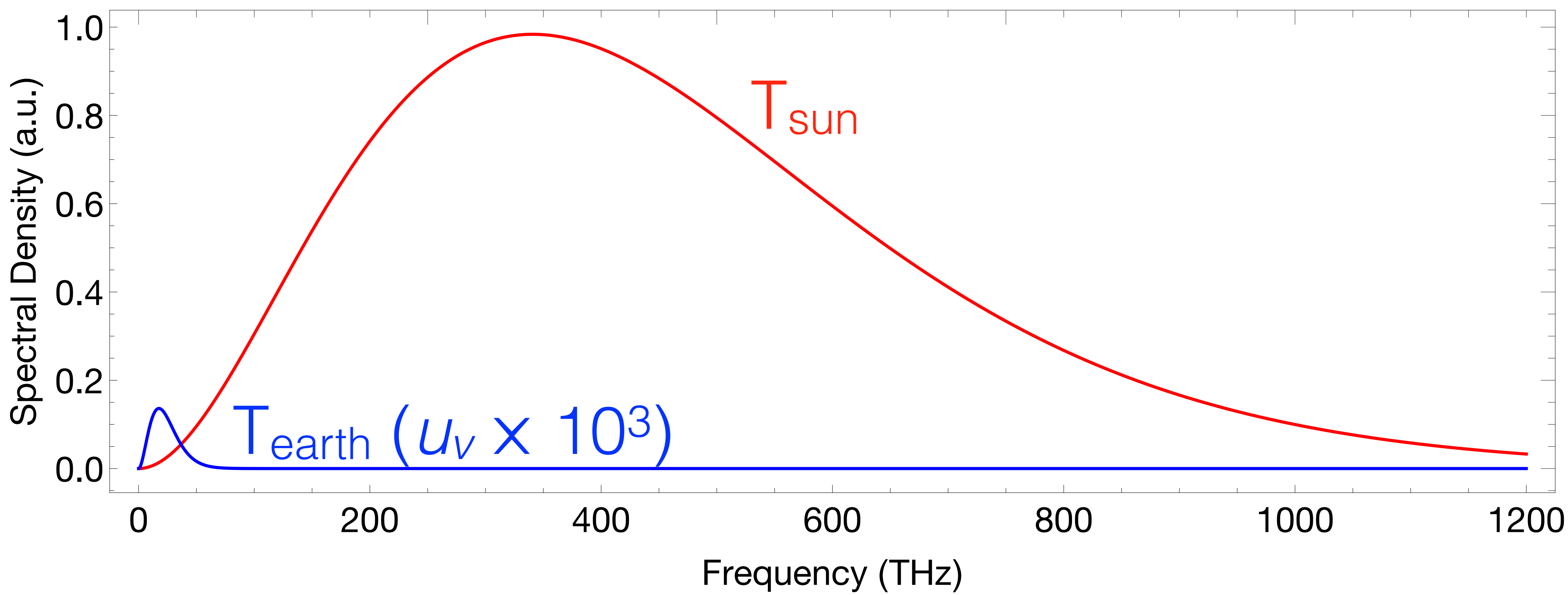
u_ω @ T_{sun}, T_{earth}

and $u_\lambda = u_\omega \left|\frac{d\omega}{d\lambda}\right|$

Note: if we instead look @ spectrum vs. wavelength (u_λ), the sun peaks @ 550 nm (Wien displacement law)

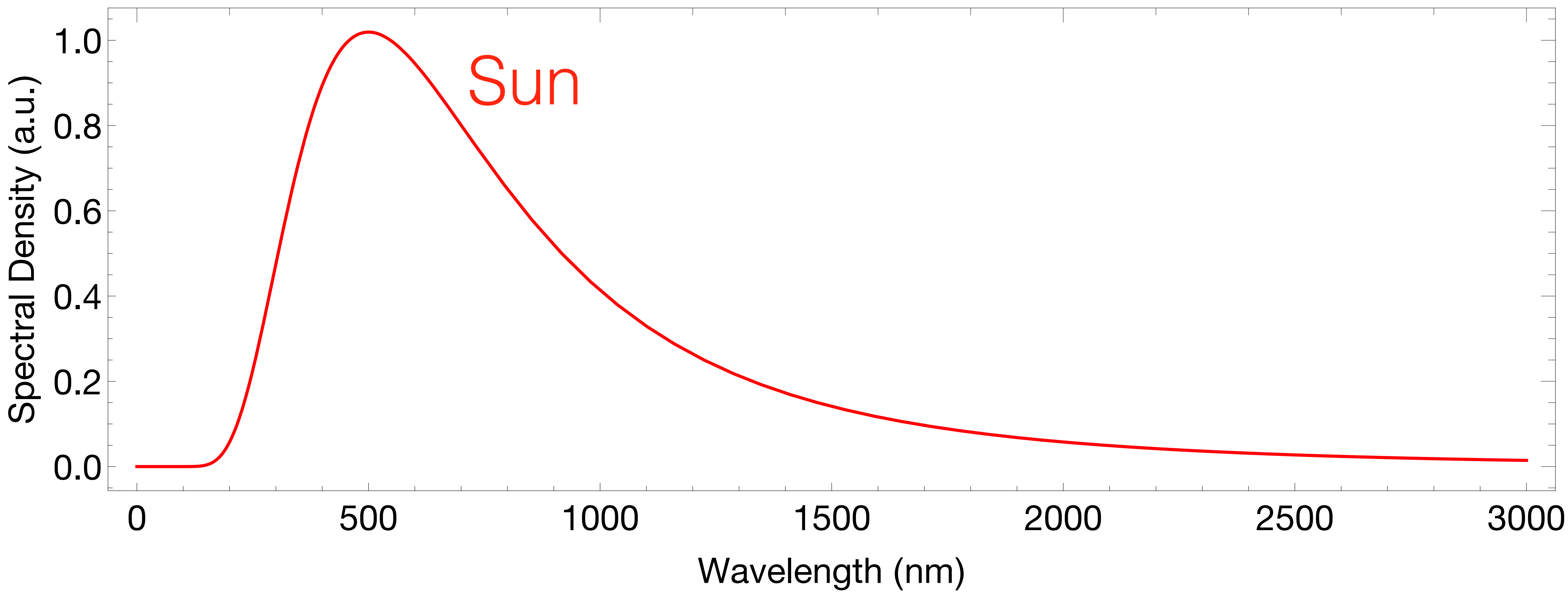
Black-Body Spectra

Spectral density vs. frequency:



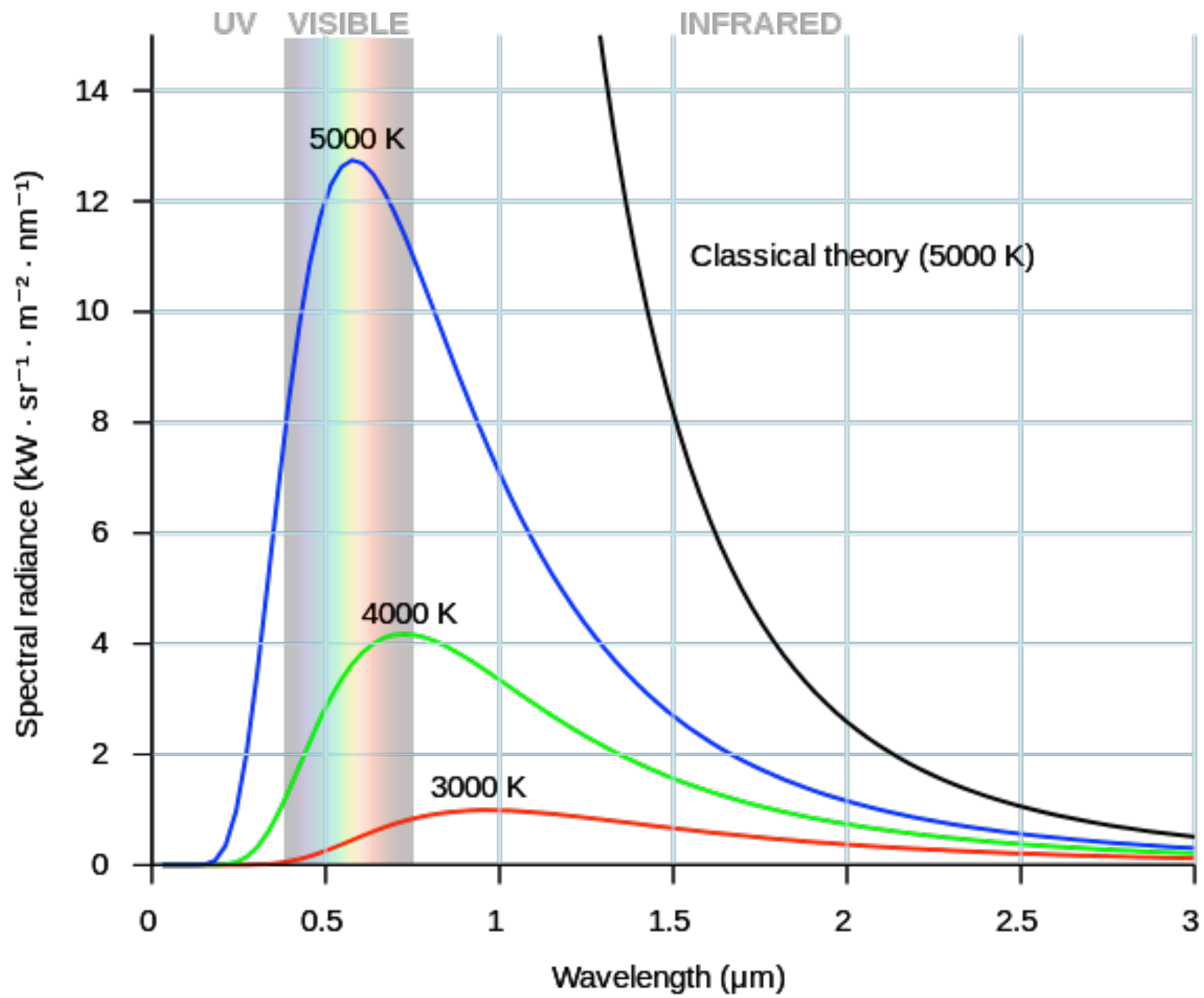
Black-Body Spectra

Spectral density per unit wavelength:



Blackbody Radiation

E.g., emission spectrum of the sun.



g. Evaluate the integral in Eq. 1 to obtain the **Stefan-Boltzmann law of radiation**. You will need the definite integral:

$$\int_0^\infty dx \frac{x^3}{e^x - 1} = \frac{\pi^4}{15} \tag{2}$$

$$U/V = \int d\omega \cdot u_\omega = \int_0^\infty \frac{\hbar \omega^3 d\omega}{\pi^2 c^3 (e^{\hbar\omega/\tau} - 1)}$$

$x = \hbar\omega/\tau$
$dx = \frac{\hbar}{\tau} d\omega$

$$= \frac{\hbar}{\pi^2 c^3} \int_0^\infty \frac{x^3 \cdot \tau^3/\hbar^3}{e^x - 1} \cdot \frac{\tau}{\hbar} dx$$

$$= \frac{\tau^4}{\pi^2 \hbar^2 c^3} \underbrace{\int_0^\infty \frac{x^3}{e^x - 1} dx}_{\frac{\pi^4}{15}}$$

$$\boxed{U/V = \frac{\pi^2}{15 \hbar^3 c^3} \cdot \tau^4}$$

h) Strong scaling of energy density w/ temperature!
— more modes accessible, increasing density of modes,
higher energy per mode.

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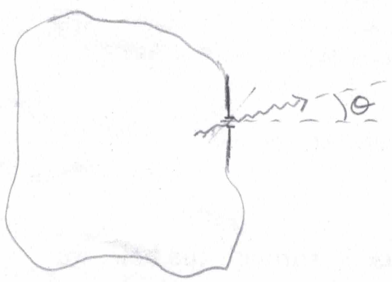
2. The Stefan-Boltzmann Law tells us the radiant energy density inside a black body at temperature τ. How do we determine the **radiant energy flux** J, i.e., the power emitted per unit surface area?

   a. Consider a small hole in the surface of a black body at temperature τ. Radiation can exit this hole at a variety of different angles θ, ϕ, where θ is measured relative to the surface normal.

      i. Draw a sketch illustrating ~~the angle θ.~~ the black body, its surface, and a photon exiting at angle θ.

      ii. What is the relevant range of values θ, ϕ?



To consider the full half-sphere of angles, we can take

$$0 \leq \theta \leq \frac{\pi}{2}$$

$$0 \leq \varphi \leq 2\pi$$

   b. Let du_ω denote the infinitesimal spectral density of radiant **energy density** that is directed into an infinitesimal solid angle $d\Omega = \sin\theta d\theta d\phi$ centered about (θ, ϕ).

      i. Express du_ω in terms of the total spectral density u_ω and $d\Omega$.

      ii. What are the dimensions of du_ω, in terms of energy, length, and time?

i. The radiation is distributed isotropically, so we simply have

$$du_\omega = u_\omega \cdot \frac{d\Omega}{4\pi}$$

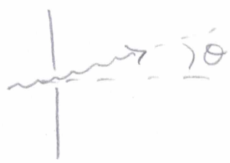
ii. $$[du_\omega] = [u_\omega] = \frac{\text{Energy}}{L^3/T} = \frac{E \cdot T}{L^3}$$

c. Let dj_ω denote the infinitesimal spectral density of radiant energy **flux** that is directed into the solid angle $d\Omega$ centered about (θ, ϕ).

i. What are the dimensions of dj_ω, in terms of energy, length, and time?

ii. Express dj_ω in terms of du_ω, the speed of light c, and the angles θ and/or ϕ.

i. $[dj_\omega] = [j_\omega] = \dfrac{\text{Power}}{\text{Area} \times \text{Frequency}} = \dfrac{E/T}{L^2/T} = \dfrac{E}{L^2}$



$dj_\omega = du_\omega \cdot \underbrace{c \cdot \cos\theta}_{\text{velocity normal to surface}}$

$\Rightarrow$ dimensions make sense ✓

d. To calculate the spectral density of radiant energy **flux** j_ω through the hole, we must integrate over all possible angles at which the radiation can exit the hole.

i. Write down an integral expression for j_ω in terms of u_ω, θ, and ϕ.

ii. Do the integral to calculate j_ω in terms of u_ω.

i. $j_\omega = \int dj_\omega$ where $dj_\omega = c \cdot \cos\theta \cdot \frac{u_\omega}{4\pi} \cdot \sin\theta \, d\theta \, d\phi$

$$= \int_0^{2\pi} d\phi \int_0^{\pi/2} d\theta \cdot \frac{u_\omega \cdot c \cdot \cos\theta \sin\theta}{4\pi}$$

$$= \frac{2\pi c}{4\pi} u_\omega \int_0^{\pi/2} d\theta \cdot \frac{1}{2}\sin(2\theta) = \frac{c u_\omega}{4} \underbrace{\left[-\frac{1}{2}\cos(2\theta)\right]_0^{\pi/2}}_{=1}$$

$$\boxed{j_\omega = \frac{c u_\omega}{4}}$$