Problem Set 8 Solutions

Problem 1

a. By the first law, the total work done is equal to the difference between the heat removed from the hot reservoir and the heat added to the cold reservoir: $W = Q_2 - Q_1$. The total heat transferred from the hot reservoir is $Q_2 = C_p(\tau_2 - \tau_f)$; likewise, the total heat transferred to the cold reservoir is $Q_1 = C_p(\tau_f - \tau_1)$. So we have

$$W = C_p(\tau_1 + \tau_2 - 2\tau_f).$$

b. In each infinitesimal cycle of the engine, the change in entropy must be non-negative. The infinitesimal change in the entropy of each reservoir is given by

$$d\sigma = \frac{dQ}{\tau} = C_p \frac{d\tau}{\tau},$$

so the total change in entropy of each reservoir is

$$\Delta\sigma_2 = \int_{\tau_2}^{\tau_f} C_p \frac{d\tau}{\tau} = C_p \ln\left(\frac{\tau_f}{\tau_2}\right)$$

and

$$\Delta\sigma_1 = \int_{\tau_1}^{\tau_f} C_p \frac{d\tau}{\tau} = C_p \ln\left(\frac{\tau_f}{\tau_1}\right).$$

By the second law, $\Delta\sigma_1 + \Delta\sigma_2 \geq 0$, which means that

$$C_p \ln\left(\frac{\tau_f^2}{\tau_1\tau_2}\right) \geq 0$$

or

$$\tau_f^2 \geq \tau_1\tau_2.$$

c. The engine does the most work when the entropy does not increase at all, which happens for $\tau_f = \sqrt{\tau_1\tau_2}$ as shown in part b. From part a, the maximum work is given by

$$W_{\max} = C_p(\tau_1 + \tau_2 - 2\sqrt{\tau_1\tau_2}) = C_p(\sqrt{\tau_2} - \sqrt{\tau_1})^2.$$

Problem 2

a. We begin with the thermodynamic identity

$$dE = \tau d\sigma - pdV + \mu dN.$$

Note that the energy in this expression is a function of the entropy, volume, and particle number: $E = E(\sigma, V, N)$. We want to define a new thermodynamic energy G that is a function of the intensive quantities: $G = G(\tau, p, \mu)$. This can be accomplished with Legendre transformations. Defining

$$G \equiv E - \sigma\tau + pV - \mu N$$

we have

$$dG = dE - d(\sigma\tau) + d(pV) - d(\mu N) = dE - \sigma d\tau - \tau d\sigma + pdV + Vdp - \mu dN - Nd\mu.$$

Substituting for dE, we obtain

$$dG = -\sigma d\tau + Vdp - Nd\mu.$$

In equilibrium, $dG = 0$, so

$$Vdp = \sigma d\tau + Nd\mu.$$

Dividing by V gives us the desired result,

$$dp = sd\tau + nd\mu.$$

b. If the gas is in the ground state, then $\tau = 0$. We know that the pressure is zero if $N = 0$ because then there are no particles to exert forces on the boundary of the volume. At $\tau = 0$, $N = 0$ implies $\mu = 0$. To calculate $p(\tau = 0, \mu)$, we use the expression from part a:

$$p(0, \mu) = p(0, 0) + \int_0^\mu \frac{\partial p}{\partial \mu'} d\mu' = \int_0^\mu \frac{N(\mu')}{V} d\mu'.$$

In class, we showed that $N = \frac{V}{3\pi^2} \frac{(2m\epsilon_F)^{3/2}}{\hbar^3}$, where ϵ_F is the Fermi energy, and also that $\mu = \epsilon_F$. Substituting, we have

$$p(0, \mu) = \int_0^\mu \frac{1}{V} \cdot \frac{V}{3\pi^2} \frac{(2m\mu')^{3/2}}{\hbar^3} d\mu' = \frac{(2m)^{3/2}}{3\pi^2} \cdot \frac{1}{\hbar^3} \cdot \frac{2}{5} \mu^{5/2}.$$

Solving for $\mu(N)$ and substituting, we have

$$p = \frac{(3\pi)^{2/3}}{5} \frac{\hbar^2}{m} \left(\frac{N}{V}\right)^{5/3}$$

as desired.

Problem 3

a. To get the correct scaling, we will assume spherical symmetry and constant mass density ρ. The binding energy of a spherical shell at radius r is given by

$$dE(r) = -\frac{GM_{\text{in}}}{r}dm = -\frac{G}{r}\left(\rho \cdot \frac{4}{3}\pi r^3\right) \cdot \rho \cdot 4\pi r^2 \, dr.$$

Here $M_{\text{in}} = \rho \cdot \frac{4}{3}\pi r^3$ is the mass contained within radius r, and we have used $dm = \rho \cdot 4\pi r^2 \, dr$. The total binding energy is given by

$$E = \int_0^R \frac{dE(r)}{dr} \, dr = -\frac{16\pi^2}{3}G\rho^2 \int_0^R r^4 \, dr = -\frac{16\pi^2}{15}G\rho^2 R^5$$
$$= -\frac{G}{R} \cdot \frac{3}{5}\left(\rho \cdot \frac{4}{3}\pi R^3\right)\left(\rho \cdot \frac{4}{3}\pi R^3\right) = -\frac{3}{5}\frac{GM^2}{R}.$$

So if the density is reasonably uniform, we expect the gravitational binding energy to be on the order of $-GM^2/R$.

b. For large N, the total kinetic energy of the electrons in the ground state is just N times the Fermi energy (since almost all electrons occupy states with energies near the Fermi energy). This is

$$N\epsilon_F = N \cdot \frac{\hbar^2(3\pi^2)^{2/3}}{V^{2/3}}\frac{N^{2/3}}{2m}$$

if we use the Fermi energy for electrons in a 3D box as calculated in class. Although the spherical star has a different geometry than a 3D box, we expect the volume scaling to be the same. Ignoring the numerical factors, we have

$$N\epsilon_F \sim N \cdot \frac{\hbar^2 N^{2/3}}{mR^2} = \frac{\hbar^2 N^{5/3}}{mR^2}.$$

Since almost all of the mass of the star is in the protons, and there are the same number of protons as electrons, $M = NM_H$. Substituting for N, we find that the total kinetic energy is

$$N\epsilon_F \sim \frac{\hbar^2 M^{5/3}}{mM_H^{5/3}R^2}.$$

c. Equating the gravitational energy from part a and the kinetic energy from part b, we have

$$\frac{GM^2}{R} \approx \frac{\hbar^2 M^{5/3}}{mM_H^{5/3}R^2} \implies M^{1/3}R \approx \frac{\hbar^2}{GmM_H^{5/3}} \approx 10^{20}\text{g}^{1/3}\text{ cm}$$

d. If $M = 2 \times 10^{33}$ g, the radius is

$$R = \frac{10^{20}\text{g}^{1/3}\text{ cm}}{(2 \cdot 10^{33}\text{g})^{1/3}} = 8 \cdot 10^8\text{cm}$$

and the density is

$$\frac{M}{\frac{4\pi}{3}R^3} = 10^6\frac{\text{g}}{\text{cm}^3}$$

e. For a neutron star, we replace the mass of the electron with the mass of the neutron (which is about equal to M_H) to get

$$(M^{1/3}R)_{\text{neutron}} = (M^{1/3}R)_{electron} \cdot \frac{m}{M_H} \approx 10^{17}\text{g}^{1/3}\text{ cm}$$

and similarly $R_{\text{neutron}} \approx 8 \cdot 10^5$ cm $= 8$ km.

Problem 4

In a 1D box, the energy of the n^{th} state is given by

$$\epsilon_n = \frac{\hbar^2 \pi^2}{2mL^2} n^2,$$

where n is a positive integer. If each boson has g internal states, then the number of different states with energy less than ϵ is given by

$$N(\epsilon) = gn = g\sqrt{\frac{2mL^2}{\hbar^2\pi^2}}\epsilon^{1/2}.$$

The density of states is given by

$$D(\epsilon) = \frac{dN}{d\epsilon} = g\sqrt{\frac{mL^2}{2\hbar^2\pi^2}}\epsilon^{-1/2}$$

and the number of particles in the excited states for $\mu = 0$ is

$$N_e = \int_0^\infty D(\epsilon) f_-(\epsilon)\, d\epsilon = g\sqrt{\frac{mL^2}{2\hbar^2\pi^2}} \int_0^\infty \epsilon^{-1/2} \frac{1}{e^{\epsilon/\tau} - 1}\, d\epsilon.$$

For $\epsilon/\tau \ll 1$, we can Taylor expand the integrand to get

$$\epsilon^{-1/2} \frac{1}{e^{\epsilon/\tau} - 1} \approx \frac{\epsilon^{-1/2}}{\epsilon/\tau} = \frac{\tau}{\epsilon^{3/2}}.$$

This integral does not converge as $\epsilon \to 0$, so the number of particles in the excited states diverges when $\mu/\tau \to 0$. This might make us doubt that a BEC can be formed in one dimension.

Problem 5

The bosons in the ground state have no energy. The energy of the bosons in the excited states is

$$E = \int_0^\infty \epsilon D(\epsilon) f_-(\epsilon)\, d\epsilon \approx \frac{V}{4\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2} \int_0^\infty \frac{\epsilon^{3/2}}{e^{\epsilon/\tau}-1}\, d\epsilon = \frac{V}{4\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2} \tau^{5/2} \int_0^\infty \frac{x^{3/2}}{e^x - 1} dx$$

for $x = \epsilon/\tau$. Similarly, the number of bosons in the excited states is

$$N_e = \int_0^\infty D(\epsilon) f_-(\epsilon)\, d\epsilon \approx \frac{V}{4\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2} \tau^{3/2} \int_0^\infty \frac{x^{1/2}}{e^x - 1} dx.$$

The total number of bosons is related to the number of bosons in the excited states by

$$\frac{N_e}{N} = \left(\frac{\tau}{\tau_E}\right)^{3/2},$$

so the energy is

$$E = NR\left(\frac{\tau^{5/2}}{\tau_E^{3/2}}\right)$$

where

$$R \equiv \left(\int_0^\infty \frac{x^{3/2}}{e^x - 1} dx\right)\left(\int_0^\infty \frac{x^{1/2}}{e^x - 1} dx\right)^{-1}.$$

The heat capacity is

$$C_V = \left(\frac{\partial E}{\partial \tau}\right)_V = \frac{5}{2} NR \left(\frac{\tau}{\tau_E}\right)^{3/2}$$

and the entropy is given by

$$C_V = \tau \frac{\partial \sigma}{\partial \tau} \implies \sigma = \int_0^\tau \frac{C_V}{\tau} d\tau = \int_0^\tau \frac{5}{2} NR \frac{\tau^{1/2}}{\tau_E^{3/2}} d\tau = \frac{5}{3} NR \left(\frac{\tau}{\tau_E}\right)^{3/2}.$$

Problem 6

a. For fermions, $N = 0$ or $N = 1$. Thus $N^2 = N$ and $\langle N^2 \rangle = \langle N \rangle$. So

$$\langle (\Delta N)^2 \rangle = \langle N^2 \rangle - \langle N \rangle^2 = \langle N \rangle - \langle N \rangle^2 = \langle N \rangle (1 - \langle N \rangle).$$

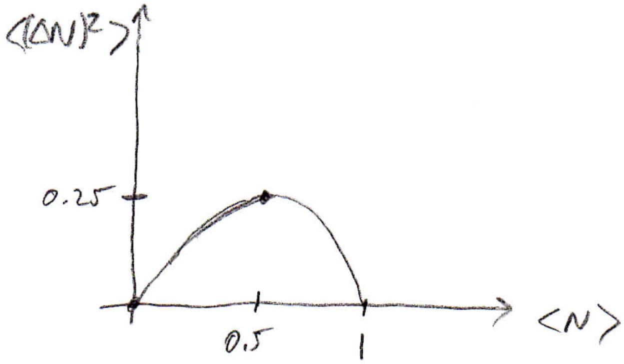
b. For bosons, we compute

$$\langle (\Delta N)^2 \rangle = \tau \frac{\partial \langle N \rangle}{\partial \mu} = \frac{e^{(\epsilon-\mu)/\tau}}{(e^{(\epsilon-\mu)/\tau} - 1)^2} = \langle N \rangle (1 + \langle N \rangle).$$

c. See sketches on next page.

6. c.

Fermions:



Bosons: