EXERCISE 1B: FROM BINOMIAL TO GAUSSIAN STATISTICS

New concepts and mathematical tools:

- The model system of the **paramagnet**
- **Microstates**, **macrostates**, and the **thermodynamic limit**
- **Stirling's approximation**

Useful results from last time:

- $P_N(X) = \frac{N!}{\left(\frac{N-X}{2}\right)!\left(\frac{N+X}{2}\right)!} p^{\frac{N+X}{2}} (1-p)^{\frac{N-X}{2}}$
- $\langle X \rangle = (2p-1)N$
- $\Delta X = 2\sqrt{Np(1-p)}$

1. Last time, you derived the *binomial distribution*, the probability distribution for the outcome of a random walk in one dimension. List a few other scenarios or physical phenomena that are ...

   a. ... described by the binomial distribution:

   b. ... analogous to random walks in higher dimensions:

2. **Microstates and macrostates**. Consider a simple model of a *paramagnet*, consisting of N non-interacting spins, each of magnetic moment μ, in a magnetic field $B\hat{\mathbf{z}}$. A **microstate** of the system can be specified by the orientations $\uparrow, \downarrow$ of the N spins with respect to the field.

   a. How many different **microstates** are available to the system?

   b. Many of these microstates are, for all practical purposes, equivalent. Suggest a macroscopic property that could be used to provide a simpler description of the state of the paramagnet. (*Note:* there is more than one correct answer!)

   c. Sketch all microstates of the paramagnet for $N = 3$. Based on your answer in b., circle groups of microstates that correspond to the same **macrostate**.

d. For general N, how many **macrostates** are available to the paramagnet?

e. Suppose that we know the total magnetization $M = \mu(N_\uparrow - N_\downarrow)$ and hence the spin excess $X \equiv N_\uparrow - N_\downarrow$. Write down the **multiplicity** $g_N(X)$ of microstates corresponding to this macrostate.

f. If each individual spin has probability p of pointing along $+\hat{\mathbf{z}}$, how is the probability $P_N(X)$ of finding the paramagnet in a state of spin excess X related to the multiplicity $g_N(X)$?

g. How does the relationship between probability $P_N(X)$ and multiplicity $g_N(X)$ simplify if $p = 1/2$? Explain.

h. Sketch the multiplicity $g_N(M)$ as a function of magnetization M for a paramagnet of N spins over the full domain $-N\mu \leq M \leq N\mu$. Show three different cases: $N = 3$, $N = 100$, and $N = 10^{23}$.

3. The probability distribution $P_N(X)$ is unwieldy. Let's find a simplified expression for the large-N (thermodynamic) limit.

   a. *Stirling's approximation*:
   Which of the following is the best approximation, for $N \gg 1$?

   A) $\ln N! \approx N \ln N$

   B) $\ln N! \approx N \ln N + N$

   C) $\ln N! \approx N \ln N - N$

   D) $\ln N! \approx N \ln N + N - \ln N$

b. Using Stirling's approximation, find an approximation to the binomial distribution $P_N(X)$ in the large-N limit.

   i. Write down an approximation for $\ln P_N(X)$. Simplify your expression into the form
$$\frac{N+X}{2}\ln(A)+\frac{N-X}{2}\ln(B), \tag{1}$$
   where A and B depend on N, p, q, and X.

   ii. You have shown that the typical fluctuation from the mean scales as $\Delta X \propto \sqrt{N}$. Thus, $\xi \equiv (X - \langle X\rangle)/N$ is a small parameter wherever the probability distribution is not negligibly small. Evaluate the probability distribution $P_N(X)$ by expanding for small ξ. You may set $p = 1/2$ to minimize algebra.

   Hint: You will need the Taylor series expansion $\ln(1+u) \approx u - u^2/2 + O(u^3)$.

iii. Is your result for $P_N(X)$ properly normalized? Why or why not? If not, find the normalization factor and write down the normalized distribution.
Hint: You will need to evaluate a Gaussian integral of the form: $\int_{-\infty}^{\infty} e^{-\alpha x^2}\, dx$.

iv. Find the standard deviation of the distribution $P_N(X)$ under the approximations you have made above. Is the result consistent with the one you obtained last class from the binomial distribution?

You have just seen an example of a powerful statistical result known as the **Central Limit Theorem**, which explains why the Gaussian distribution is so ubiquitous in nature. You will prove the central limit theorem in the homework and examine how it applies to the paramagnet in class on Monday.