EXAM REVIEW PROBLEMS

1. *Lifting water with an elastic band.* An elastic band consists of a tangle of long molecules with no particular orientation. Consider a toy model of the elastic band as consisting of N rods of length ℓ, each of which can point either up or down (Fig. 1a).
   a. A bucket of weight w is suspended by the elastic band. Calculate the equilibrium length L of the band as a function of its temperature τ and the weight w.
   b. Suppose you were to heat the elastic band: would the bucket move up, move down, or remain fixed? Give an entropic argument for your answer.
   c. Design a Carnot cycle to do work by adjusting the temperature of the band and the weight of the bucket (e.g., by pouring water in and out of the bucket).
      i. Mark arrows on Fig. 1(b) showing in which direction the cycle should go in order to form a heat engine.
      ii. Sketch the adiabats and isotherms in the $L - w$ plane (Fig. 1(c)) and label the points **A**,**B**,**C**, and **D** that match the corresponding points in Fig. 1(b). Again mark arrows showing the direction of the cycle.

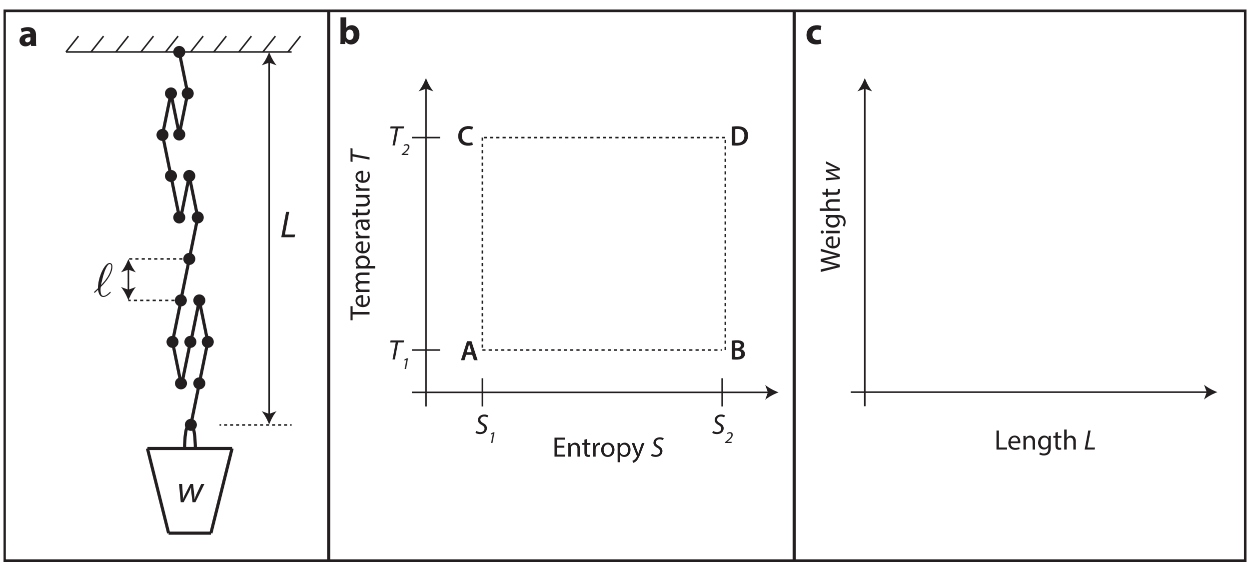

FIG. 1. Rubber band and Carnot cycle.

2. *2D electron gas.* A two-dimensional electron gas (2DEG) can be formed at the interface between two semiconductor materials. Consider an area $A = L \times L$ containing N free electrons, and let $n = N/A$ represent the electron density.

   Note: The following approximation may be helpful in this problem:

$$\int_0^\infty dx \frac{x}{ye^x + 1} \approx \frac{\pi^2}{6} + \frac{(\log y)^2}{2} - y + O(y^2)$$

   a. Calculate the Fermi energy ε_F as a function of electron density in the zero-temperature limit.
   b. *Magnetic susceptibility.* Suppose we apply a magnetic field B that shifts the energies of spin-up and spin-down electrons by $\mp\gamma B$, with $|\gamma B| \ll \varepsilon_F$. ($\gamma$ includes the electron's intrinsic magnetic moment and a contribution from orbital motion induced by the field.) In terms of N, γ, B, and ε_F, calculate
      i. the magnetization M
      ii. the magnetic susceptibility χ
   c. *Heat capacity.* Now consider the 2DEG at $B = 0$ and at low but non-zero temperature $\tau \ll \tau_F$, where τ_F is the Fermi temperature. Calculate the specific heat c_V—i.e., the heat capacity per electron—to lowest order in τ/τ_F. Compare your result quantitatively to the heat capacity of a two-dimensional ideal gas.

3. K&K 4.10 (*Heat capacity of intergalactic space*)
4. K&K 6.3 (*Distribution function for double occupancy statistics*)
5. K&K 6.6 (*Gas of atoms with internal degrees of freedom*)
6. K&K 7.2 (*Energy of a relativistic Fermi gas*)
7. K&K 7.10 (*Relativistic white dwarf stars*)