PROBLEM SET 2

Reading: Kittel & Kroemer, Ch. 2

1. Consider the simulation of Ehrenfest's urns that we observed in class, which is a toy model for particle diffusion between two chambers. Initially, N balls are placed into urn A. At each time-step a random ball is switched between urn A and urn B, so that the system equilibrates to a configuration with typically $\sim N/2$ balls in each urn.

   a. Consider a specific one of the individually labelled balls, and let n denote the number of times the ball has switched urns after M time-steps. By expressing n as a sum of M independent random variables, determine the mean $\langle n \rangle$ and standard deviation Δn.

   b. Based on your expression for Δn, argue that after $M \gg N$ time-steps, all microstates of the system are equally probable.

   c. Let $N = 100$, and suppose that each ball switches urns on average once every millisecond. Roughly how many years will it take for the system to return to the initial configuration with all balls in urn A? (No need for a rigorous proof; just give an estimate based on the argument of part b.)

2. A rubber band is fastened at one end to a peg, and supports from its other end a weight w. Assume as a simple microscopic model of the rubber band that it consists of a linked polymer chain of N segments joined end to end; each segment has length a and can be oriented either parallel or antiparallel to the vertical direction.

   a. How is the resultant length l of the rubber band related to its energy E? (Neglect the kinetic energies or weights of the segments themselves, or any interaction between the segments.)

   b. Give expressions for the multiplicity $g(E)$ and entropy $\sigma(E)$.

   c. Simplify your expression for the entropy using Stirling's approximation. Assume that $N \gg 1$ and that the rubber band is not fully extended, and explain where in your analysis these assumptions are necessary.

   d. Find an expression for the temperature τ in terms of E, a, w, and N.

   e. What is the length $l(\tau)$ of the rubber band at temperature τ?

   f. At what temperature(s) do the approximations in c. break down, invalidating your expression for $l(\tau)$? Explain.

3. Do problem 2.2 (***Paramagnetism***) in Kittel and Kroemer.

   a. Under what circumstances is the temperature τ of the paramagnet negative, and why?

   b. If the paramagnet is initially at $\tau < 0$ and is then placed into thermal contact with a bath at positive temperature $\tau > 0$, which way will energy flow?

4. Do problem 2.3 (***Quantum harmonic oscillator***) in Kittel and Kroemer.

5. Consider two spin systems $\mathcal{A}_1$ and $\mathcal{A}_2$ placed in an external field B. Each system $\mathcal{A}_i$ consists of weakly interacting localized particles of spin $\frac{1}{2}$ and magnetic moment μ_i. The two systems are initially isolated with respective total energies $b_i N_i \mu_i B$. (Here, b_i denotes the fractional magnetization of system $\mathcal{A}_i$.) They are then placed in thermal contact with each other. Suppose that $|b_1| \ll 1$ and $|b_2| \ll 1$ so that Gaussian approximations can be used for the densities of states of the two systems.

   a. In the most probable situation corresponding to the final thermal equilibrium, how is the energy $\hat{E}_1$ of system $\mathcal{A}_1$ reated to the energy $\hat{E}_2$ of system $\mathcal{A}_2$?

   b. What is the value of the energy $\hat{E}_1$ of system $\mathcal{A}_1$?

   c. How much energy is absorbed by system $\mathcal{A}_1$ in going from the initial situation to the final situation when it is in equilibrium with $\mathcal{A}_2$?

   d. What is the variance ΔE_1^2 of the energy of system $\mathcal{A}_1$ in the final equilibrium situation?

   e. What is the value of the relative energy spread $\Delta E_1/\hat{E}_1$ in the case when $N_2 \gg N_1$?

6. Do problem 2.5 (***Additivity of entropy for two spin systems***) in Kittel and Kroemer.

7. A solid at absolute temperature T is placed in an external magnetic field $B = 1$ tesla $= 1 \times 10^4$ gauss. The solid contains weakly interacting paramagnetic atoms of spin 1/2, so that the energy of each atom is $\pm\mu B$.

   a. If the magnetic moment μ is equal to one Bohr magneton, i.e., $\mu = h \times 1.4$ MHz/gauss, below what temperature must one cool the solid so that more than 75 % of the atoms are polarized with their spins parallel to the external magnetic field?

   b. Suppose that one considered instead a solid which is free of paramagnetic atoms but contains many protons (e.g., paraffin). Each proton has spin 1/2 and a magnetic moment $\mu_p = h \times 2.1$ kHz/gauss. Below what temperature must one cool this solid so that more than 75 % of the protons have their spins aligned parallel to the external magnetic field?