EXERCISE 8B: HEAT AND WORK IN AN IDEAL GAS

Objectives:

- Calculate the form of **adiabats** relating p and V in an ideal gas at constant entropy
- Understand the basic operating principle of a heat engine
- Calculate the **Carnot limit** on the efficieny of a heat engine

References: Kittel & Kroemer, Ch. 6

Useful result from Ex. 8A:

- Free energy of an ideal gas: $F = F_0 + F_{\text{int}}$,
  where $F_{\text{int}} = -N\tau \ln Z_{\text{int}}$ and F_0 is the free energy of the monatomic ideal gas.

1. Which of the following properties of the ideal gas are modified by the internal degrees of freedom, and how? Explain.

   a. The equation of state $p(N, \tau, V)$
   b. The entropy $\sigma(N, \tau, V)$
   c. The heat capacity $C_V = \tau(\partial\sigma/\partial\tau)_V$ at constant volume.
   d. The heat capacity $C_p = \tau(\partial\sigma/\partial\tau)_p$ at constant pressure.
   e. The isothermal compressibility $\kappa_\tau = -V^{-1}(\partial V/\partial p)_\tau$
   f. The adiabatic compressibility $\kappa_\sigma = -V^{-1}(\partial V/\partial p)_\sigma$

(cont'd)

2. *Adiabatic expansion or compression of an ideal gas.* Show that for any ideal gas, the adiabats are of the form $pV^\gamma =$ constant, where $\gamma = C_p/C_V$ is the **heat capacity ratio**.

   a. Write down the fundamental thermodynamic relation and simplify it for the case of constant entropy and constant particle number.

   b. Explain why the following relations hold for any ideal gas at fixed N:

$$N d\tau = p dV + V dp. \tag{1}$$

$$dE = C_V d\tau \tag{2}$$

   c. Use the above results to derive a relation of the form

$$V dp = -\gamma p dV. \tag{3}$$

   What is the value of γ in terms of C_V?

d. Show that in an adiabatic expansion or compression of an ideal gas, pV^γ is constant.

e. The relation $pV^\gamma =$ constant defines a family of curves called **adiabats**. Sketch adiabats (showing p as a function of V) for a monatomic ideal gas at two different entropies $\sigma_1 < \sigma_2$.

f. Add to your sketch two **isotherms**, i.e., curves of p vs V at two different constant temperatures $\tau_1 < \tau_2$. What is their functional form?

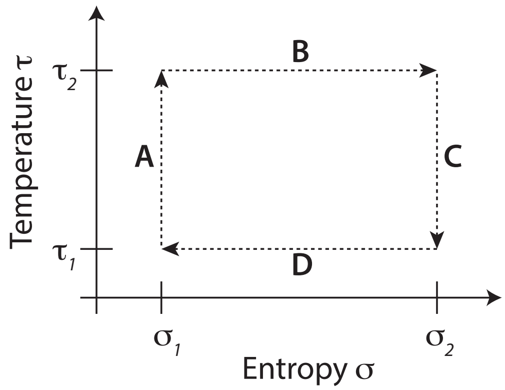

FIG. 1. Carnot cycle composed of adiabatic and isothermal quasi-static processes.

g. Consider a cycle of alternating isothermal and adiabatic processes that traces out the loop in the σ-τ plane shown in Fig. 2.
   i. In your plot in e., label the curves in the p-V plane corresponding to the trajectories A, B, C, and D, and shade in a region representing the amount of work done by the gas in one such cycle.
   ii. Is the net work done by the gas positive or negative?
   iii. How much heat is absorbed by the gas in the same cycle?

h. Show that the exponent γ is equal to the **heat capacity ratio**: $\gamma = C_p/C_V$. You will need the fundamental thermodynamic relation and the result of part c.

i. Explain in words how your sketch would change for a gas of molecules with internal degrees of freedom.

3. **Carnot efficiency.** *How efficiently can we generate work from heat?*
Figure 2(a) illustrates a generic process wherein a system $\mathcal{A}$ absorbs heat q_2 from a reservoir at temperature τ_2 and outputs work w, dumping heat into a reservoir at temperature $\tau_1 < \tau_2$. Assume that the process is cyclic, so that it can be repeated indefinitely.

   a. Express the **second law of thermodynamics** in terms of the quantities labelled in Fig. 2(a).

   b. Express the **first law of thermodynamics** in terms of the quantities labelled in Fig. 2(a).

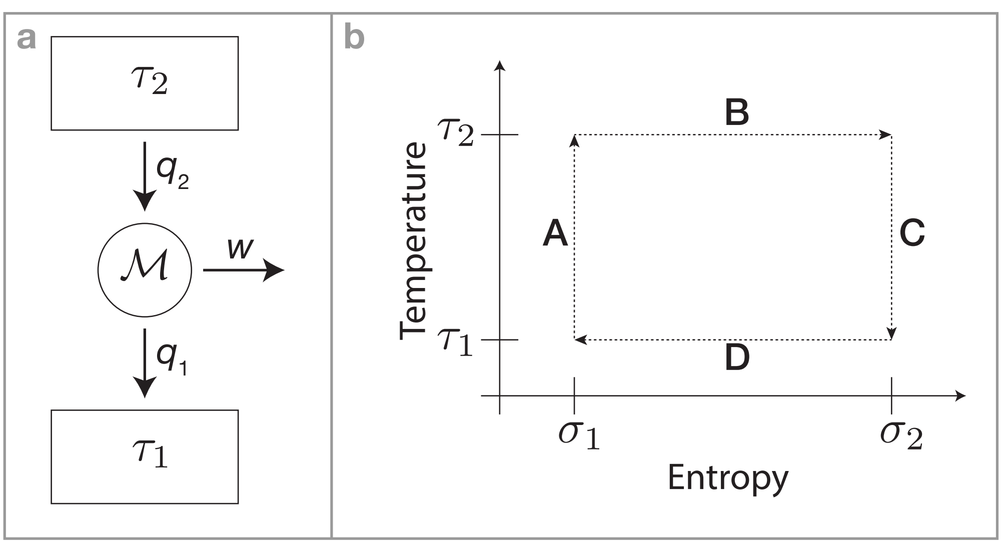

FIG. 2. **(a)** Schematic diagram of a heat engine. **(b)** Carnot cycle composed of adiabatic and isothermal quasi-static processes.

c. The **efficiency** of the heat engine is defined as the ratio $\eta \equiv w/q_2$ of work output by the system to heat input from the high-temperature reservoir. Calculate a fundamental limit on the efficiency of a heat engine operating between the two reservoirs at temperatures $\tau_2 > \tau_1$.