PROBLEM SET 1

Reading: Kittel & Kroemer, Chapter 1.
You must show and explain your work to receive full credit.

1. Algebraic derivation of the binomial expansion.

   The coefficients a_n in the expansion

   $$(x+y)^N = \sum_{n=0}^{N} a_n x^n y^{(N-n)} \tag{1}$$

   are the *binomial coefficients*.

   a. Briefly explain why calculating a_n is equivalent to counting the number of different ways for a coin to land heads n times in N tosses.

   b. Show that
   $$\left(\frac{\partial}{\partial x}\right)^n (x+y)^N|_{x=0} = Cy^{N-n}, \tag{2}$$
   where C can be related to a_n. Use this relation to determine the value of a_n.

2. In the game of Russian roulette (*not* recommended), one inserts a cartridge into the drum of a revolver, leaving the other five chambers of the drum empty. One then spins the drum, aims at one's head, and pulls the trigger.

   a. What is the probability of being still alive after playing the game N times?

   b. What is the probability of surviving $(N-1)$ turns in this game and then being shot the N^{th} time one pulls the trigger?

   c. What is the mean number of times a player gets the opportunity of pulling the trigger in this macabre game?

3. *Law of rare events.* The probability $P(n)$ that an event characterized by a probability p occurs n times in N trials is given by the binomial distribution, which we derived in class. Consider a situation where the probability p is small ($p \ll 1$).

   a. Explain why, if evaluating $P(n)$ for $p \ll 1$, one is typically interested the regime $n \ll N$. In this case, several approximations can be made to reduce the binomial distribution to a simpler form.

   b. Using the result $\ln(1-p) \approx -p$, show that $(1-p)^{N-n} \approx e^{-Np}$.

   c. Show that $N!/(N-n)! \approx N^n$.

d. Hence show that the binomial distribution reduces to

$$P(n) = \frac{\lambda^n}{n!} e^{-\lambda}, \tag{3}$$

where $\lambda \equiv Np$ is the mean number of events. The distribution in Eq. 3 is called the **Poisson distribution**.

e. Give an example of a real-world scenario described by Poisson statistics.

4. Consider the Poisson distribution of the preceding problem.

   a. Show that it is properly normalized, in the sense that $\sum_{n=0}^{N} P(n) = 1$. (The sum can be extended to infinity to an excellent approximation. Justify this approximation.)

   b. Use the Poisson distribution to calculate the mean $\langle n \rangle$.

   c. Use the Poisson distribution to calculate the variance $\Delta n^2 = \langle (n - \langle n \rangle)^2 \rangle$.

5. Consider a gas of N_0 noninteracting molecules enclosed in a container of volume V_0. Focus attention on any subvolume V of this container and denote by N the number of molecules located within this subvolume. Each molecule is equally likely to be located anywhere within the container; hence the probability that a given molecule is located within the subvolume V is simply equal to V/V_0.

   a. What is the mean number $\langle N \rangle$ of molecules located within V? Express your answer in terms of N_0, V_0, and V.

   b. Find the fractional fluctuation $\Delta N / \langle N \rangle$ in the number of molecules located within V. Express your answer in terms of $\langle N \rangle$, V, and V_0.

   c. What does the answer to part b. become when $V \ll V_0$?

   d. What value should the variance ΔN^2 assume when $V \to V_0$? Does the answer to part b. agree with this expectation?

6. Suppose that in the preceding problem the volume V under consideration is such that $0 \ll V/V_0 \ll 1$. Let $P(N)dN$ denote the probability that the number of molecules in this volume is between N and $N + dN$. Give an expression for $P(N)$ in terms of $\langle N \rangle$ and/or ΔN. Explain your result, justifying any approximations.

7. *Central Limit Theorem*, or, "Why is everything Gaussian?"

   a. Consider a set of independent random variables x_i, each of which is taken from some probability distribution $p(x_i)$. Then the probability distribution $P(Y)$ for the sum $Y = \sum_{i=1}^{N} x_i$ of these random variables can be written as

   $$P(Y) = \int \ldots \int \delta\left(Y - \sum_{i=1}^{N} x_i\right) \prod_{i=1}^{N} p(x_i)\, dx_i \tag{4}$$

   Explain Equation 4 in words. What is the role of the Dirac delta function in the integrand?

   b. The Dirac delta function is the Fourier transform of a constant function:

   $$2\pi\delta(u) = \int_{-\infty}^{\infty} dk\, e^{-iku}. \tag{5}$$

   Use this relation to express $P(Y)$ in terms of the Fourier transform

   $$Q(k) = \int_{-\infty}^{\infty} dx\, e^{ikx} p(x). \tag{6}$$

   c. Assume that $p(x)$ is a reasonably smooth function, so that $Q(k)$ is well approximated by an expansion for small "wavenumber" k. Write down such an expansion, and express your result in terms of moments $\langle x^m \rangle$ of x up to $m = 2$.

   d. We are now equipped to find a simple approximation for $Q^N(k)$. This is easiest to do by first evaluating $\ln Q^N(k)$ in the small-k limit, keeping all terms up to order k^2.

   e. Determine $P(Y)$ by substituting your expression for $Q^N(k)$ into your result from part b. and evaluating the integral.

   f. Express the mean $\langle Y \rangle$ and standard deviation ΔY in terms of the mean $\langle x \rangle$ and standard deviation Δx of the summands x_i.

   g. Suppose that the independent random variables x_i are uniformly distributed between 0 and 1. Sketch the following, annotating each sketch with the mean and (approximate) width of the distribution:

      i. $p(x_i)$
      ii. $|Q(k)|$
      iii. $|Q^N(k)|$
      iv. $P(Y)$