

Physics 170 Final Exam Solutions

1. Diatomic molecules

15 points total.

- a. 2 points. Only one if ℓ is included in answer.

In order for the quantum mechanics not to matter we need the average angular quantum number $\langle \ell \rangle$ to be much greater than 1. In other words the energy scale of the quantum mechanics \hbar^2/I must be much smaller than the temperature:

$$\tau \gg \frac{\hbar^2}{I}$$

- b. i. 7 points total. 2 for writing correct expression for Z . 2 points for evaluating integral/sum. 1 point for correct Z result. 1 point for $F = -\tau \ln(Z)$. 1 point for correct F .

The Helmholtz free energy is given by

$$F = -\tau \ln(Z)$$

where the partition function

$$Z = \sum_{\ell=0}^{\infty} (2\ell + 1) e^{-\beta \epsilon_{\ell}}$$

where

$$\epsilon_{\ell} = \frac{\ell(\ell + 1)\hbar^2}{2I}$$

The partition function can be approximated by

$$Z \approx \int_{\ell=0}^{\infty} 2\ell \exp\left(-\frac{\ell^2 \hbar^2 \beta}{2I}\right) d\ell$$

Define $u \equiv \ell^2 \hbar^2 \beta / 2I$ so that $du = \hbar^2 \beta \ell d\ell / I$ and

$$\begin{aligned} Z &\approx \frac{2I}{\hbar^2 \beta} \int_{u=0}^{\infty} \exp(-u) du \\ &= \frac{2I}{\hbar^2 \beta} \end{aligned}$$

Hence

$$F = -\tau \ln\left(\frac{2I\tau}{\hbar^2}\right)$$

- ii. 2 points. 1 point for method, 1 point for answer.

To get the entropy we recall that (if you don't remember it, it can be easily derived from $dF = -pdV - \sigma d\tau$ where there are no volume changes in the system)

$$\begin{aligned}
\sigma &= -\frac{\partial F}{\partial \tau} \\
&= \tau \left(\frac{\hbar^2}{2I\tau} \right) \frac{2I}{\hbar^2} + \ln \left(\frac{2I\tau}{\hbar^2} \right) \\
&= 1 + \ln \left(\frac{2I\tau}{\hbar^2} \right)
\end{aligned}$$

iii. 2 points. 1 point for method, 1 point for answer.

By definition $F = E - \tau\sigma$ so $E = F + \tau\sigma$. Using the above results

$$E = \tau$$

which agrees with what we would get from the equipartition theorem. (Note that the answer is not $5\tau/2$ as for an ideal gas of diatomic molecules because we aren't counting translational degrees of freedom in this problem. They will be added back in in the next problem.)

c. 2 points. 1 point for answer, 1 point for correct explanation (a.k.a. single ground state).

Because there is a single ground state, $\ell = m = 0$, the entropy is $\sigma = \ln(1) = 0$.

2. Hydrogen in equilibrium

25 points total.

- a. 2 points. Only one point if missing factor of 2.

Because the reaction removes 2 atoms for every molecule in order for it to be at equilibrium we must have

$$\mu_M = 2\mu_A$$

so that removing two atoms and introducing one molecule doesn't change the free energy. (Note that if you choose the convention where you include the binding energy in the difference between the chemical potentials then $\mu_M = 2\mu_A - E_b$).

- b. 2 points. Only one point if missing square root.

The thermal de-Broglie wavelength is given by

$$\lambda = \frac{h}{\sqrt{2\pi m\tau}}.$$

Because the only variable here that is different between an atom and a molecule is the mass we get

$$\frac{\lambda_{\tau,M}}{\lambda_{\tau,A}} = \frac{1}{\sqrt{2}}.$$

Note that the *bigger* particle has a *smaller* de-Broglie wavelength.

- c. 7 points. 2, 3, and 2 points for respective terms. Partial credit if term is incorrect. Give points if reasonable expression is used that is consistent with problem 1.

The chemical potential has three parts: the chemical potential of a gas particle, the free energy from the internal degrees of freedom, and the binding energy. Energies are additive:

$$\mu_M = \tau \ln \left(\frac{N_M \lambda_M^3}{V} \right) - \tau \ln \left(\frac{2I\tau}{\hbar^2} \right) - E_b.$$

- d. 5 points. Two for equating chemical potentials consistent with part a. 1 point for solving for ratio of N's (even if missing square). 2 points for correct answer.

Plugging this into the result from part a, we find:

$$\tau \ln \left(\frac{N_M \lambda_M^3}{V} \right) - \tau \ln \left(\frac{2I\tau}{\hbar^2} \right) - E_b = 2\tau \ln \left(\frac{N_A \lambda_A^3}{V} \right).$$

Dividing by τ and exponentiating:

$$\frac{N_M \lambda_M^3}{V} \frac{\hbar^2}{2I\tau} e^{-\beta E_b} = \left(\frac{N_A \lambda_A^3}{V} \right)^2$$

$$\frac{N_M}{N_A^2} = \frac{2^{5/2} I \tau}{\hbar^2} \frac{\lambda_A^3}{V} e^{\beta E_b}$$

- e. i. 3 points. 1 point for answer. 2 for explanation (only one for shaky explanation).

As volume increases, N_M/N_A^2 decreases. The larger the volume, the more *entropically* favorable it becomes for the molecules to dissociate into atoms, since there are more ways to choose the positions of two atoms than that of a single molecule. Viewed another way, the probability for two atoms to find each other to form a H_2 molecule decreases with increasing volume.

- ii. 3 points. 1 point for answer. 2 for explanation (only one for shaky explanation).

A higher binding energy increases N_M/N_A^2 . The binding *energy* makes the molecule more favorable.

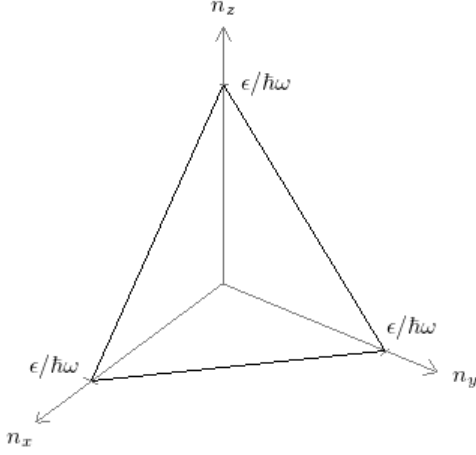
iii. 3 points. 1 point for answer. 2 for explanation (only one for shaky explanation).

A higher moment of inertia increases N_M/N_A^2 . Higher moment of inertia means a closer energy spacing of the rotational energy states and therefore more states accessible at a temperature τ . More accessible states means more *entropy*, so the higher moment of inertia shifts the equilibrium towards more molecules. Alternatively, at fixed atomic mass, a higher I means a larger spacing between the atoms within the molecule, and therefore more volume in phase space for the bound state.

3. Bose-Fermi Mixture

a. **3 points.**

The surfaces of constant energy ε are planes defined by $n_x + n_y + n_z = \varepsilon/(\hbar\omega)$. These intersect each axis at the same value $n_{x,y,z} = \varepsilon/(\hbar\omega)$.



b. **3 points.**

The density of states is proportional to the area of the triangle in part a). The side length of the triangle is proportional to $\varepsilon/(\hbar\omega)$. The density of states is a number of states per unit energy,

$$\mathcal{D}(\epsilon) \propto \frac{\epsilon^2}{(\hbar\omega)^3}$$

Another way to see this is to note that the total number of states $\mathcal{N}(\varepsilon)$ up to energy ε scales as the volume of a pyramid with side length $\varepsilon/(\hbar\omega)$. From $\mathcal{N}(\varepsilon) \propto (\varepsilon/(\hbar\omega))^3$, we obtain $\mathcal{D}(\varepsilon) = d\mathcal{N}/d\varepsilon \propto \varepsilon^2/(\hbar\omega)^3$.

c. i. **3 points.**

The Fermi energy is the energy such that N_f particles fit inside the tetrahedron in part a:

$$N_f \propto \left(\frac{\epsilon_F}{\hbar\omega}\right)^3$$

or

$$\epsilon_F = N_f^{1/3} \hbar\omega.$$

ii. **2 points.**

The temperature must be much smaller than the Fermi level so that (most of) the states below the Fermi level don't have enough thermal energy to rise above it:

$$\tau \ll \epsilon_F.$$

d. i. **4 points.**

In order for a BE condensate to occur, the occupation of the excited states when $\mu = 0$ must be finite. In this case a large number of bosons will lead to a high occupancy of the ground state as $\mu \rightarrow 0$, while the number of particles N_e in all excited states remains finite. The population of the excited states is

$$\begin{aligned} N_e &= \int_0^\infty \mathcal{D}(\epsilon) f(\epsilon) d\epsilon \\ &\approx \int_0^\infty \frac{1}{e^{\beta\epsilon} - 1} \frac{\epsilon^2}{(\hbar\omega)^3} d\epsilon. \end{aligned}$$

Defining $x = \epsilon\beta$, we have

$$N_e \approx \left(\frac{\tau}{\hbar\omega}\right)^3 \int_0^\infty \frac{x^2}{e^x - 1} dx$$

The integral will be some finite number: at low x it is roughly $\int x dx$, and at large x the integral is roughly $\int x^2 e^{-x} dx$, both of which converge in their respective limits. Thus, N_e is finite, such that if $N > N_e$ there will be a macroscopic occupation of the ground state (Bose-Einstein condensate).

ii. **2 points.**

The dimensionless integral above is some finite number. (To be precise, its value is $2\zeta(3) \approx 2.4$.) The critical temperature for Bose-Einstein condensation is the temperature at which $N = N_e$:

$$\tau_E \propto N_e^{1/3} \hbar\omega$$

Alternative approach: Some of you tried to derive τ_E from a phase-space density argument. This is indeed possible. For a harmonic oscillator at $\tau \gg \hbar\omega$, we have $m\omega^2 \langle x^2 \rangle \approx \tau$ and $\langle p^2 \rangle / m \approx \tau$. Bose-Einstein condensation occurs when the phase-space density exceeds $1/h^3$:

$$\frac{N_b}{\langle x^2 \rangle^{3/2} \langle p^2 \rangle^{3/2}} \approx \frac{N_b \omega^3}{\tau^3} > \frac{1}{h^3},$$

yielding a critical temperature $\tau_E \propto N_b^{1/3} \hbar\omega$.

e. i. **3 points.**

Only the excited bosons contribute to the energy. They have an average energy on the order of τ and there are on the order of $(\tau/\hbar\omega)^3$ of them so

$$\langle E \rangle \propto \frac{\tau^4}{(\hbar\omega)^3}$$

ii. **2 points.**

$$C_b = \left(\frac{dE}{d\tau} \right)_{N,\omega} \propto \frac{\tau^3}{(\hbar\omega)^3}$$

iii. **2 points.**

The average energy per particle is proportional to ϵ_F , so the total energy scales as

$$\langle E \rangle \approx N_f \epsilon_f \approx N_f^{4/3} \hbar\omega$$

iv. **3 points.**

We can approximate the density of states as constant over the range $\epsilon_F \pm \tau$ for $\tau \ll \epsilon_F$. Thus, the number of particles excited from the zero temperature configuration is roughly $\tau \mathcal{D}(\epsilon_F)$. The excitation energy per particle is on the order of τ , so

$$E - E_0 \approx \tau^2 \mathcal{D}(\epsilon_F) \approx \tau^2 \frac{\epsilon_F^2}{(\hbar\omega)^3} \approx \frac{(N_f^{1/3} \hbar\omega)^2}{(\hbar\omega)^3} \tau^2 \approx N_f^{2/3} \frac{\tau^2}{\hbar\omega}$$

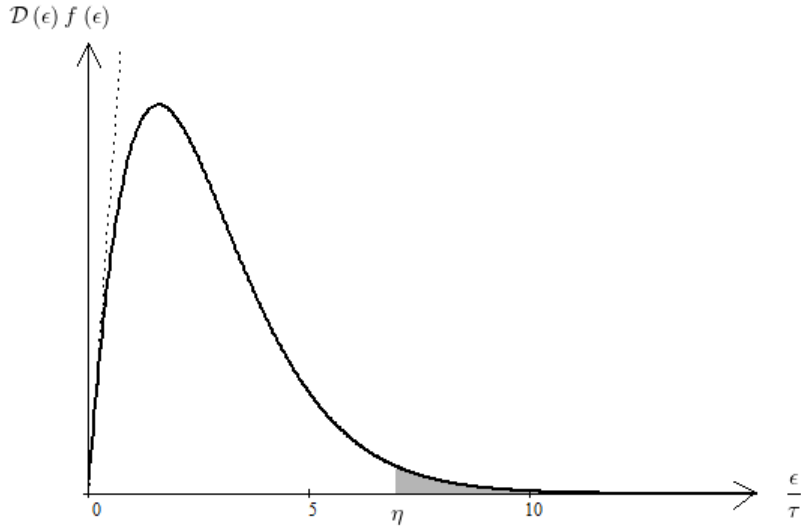
$$C_f = \left(\frac{dE}{d\tau} \right)_{N,\omega} \propto N_f^{2/3} \frac{\tau}{\hbar\omega}$$

f. **4 points.** (2 points for general shape with correct limits. 1 point for correctly shaded region. 1 points for $\eta \gg 1$. +1 bonus point for explicitly showing linear scaling at low energy.)

The number density per unit energy is the product of the density of states and their occupation:

$$\mathcal{D}(\epsilon) f(\epsilon) = \frac{\epsilon^2}{e^{\epsilon/\tau} - 1}$$

At low ϵ/τ we can approximate $\mathcal{D}(\epsilon) f(\epsilon) \approx \epsilon\tau$ and it must fall off exponentially at large ϵ/τ . At some point in the middle there must be a peak.



As long as $\eta \gg 1$ the fraction that leave $\Delta N_b/N_b$ will be small.

- g. i. **4 points. 2 points for C_f/C_b . 1 point for correct answer. 1 point for in terms of correct variables.**

The change in temperature is the same for both gases, so

$$\frac{\Delta E_f}{\Delta E_b} \approx \frac{C_f}{C_b} \approx \frac{N_f^{2/3} \frac{\tau}{\hbar\omega}}{\frac{\tau^3}{(\hbar\omega)^3}} = N_f^{2/3} (\hbar\omega)^2 / \tau^2.$$

We want this in terms of $\epsilon_F = N_f^{1/3} \hbar\omega$:

$$\frac{\Delta E_f}{\Delta E_b} \approx \frac{\epsilon_F^2}{\tau^2}$$

- ii. **2 points. No credit for guessing without giving justification.**

Because $\tau \ll \epsilon_F$, the change in energy is much more for the **fermions**.

- iii. **3 points.**

$$\Delta\tau \approx \frac{\epsilon_1 \Delta N_b}{C_f + C_b} \approx \frac{\epsilon_1 \Delta N_b}{C_f} \approx \frac{\epsilon_1 \Delta N_b}{N_f^{2/3} \frac{\tau}{\hbar\omega}} \approx \frac{\epsilon_1 \epsilon_F \Delta N_b}{N_f \tau}$$

,

and hence

$$\frac{\Delta\tau}{\epsilon_F} \approx \frac{\epsilon_1 \Delta N_b}{N_f \tau}$$

,

- h. **Extra credit: +3 points.**

Ideally, we should choose ϵ_{\max} as high as possible to maximize the energy ϵ_1 removed per boson removed. (The practical limitation is that the cooling will be slower the higher we make ϵ_{\max} , and we may run into competing heating or loss rates.) In addition, for a fixed fraction of bosons removed, the cooling will be stronger the larger the ratio N_b/N_f of bosons to fermions, i.e., we would like a large bath of bosons to sympathetically cool the fermions.

4. The Otto Cycle

a. 5 points.

Method 1: The heat capacity of an ideal gas is $C_V = 3N/2$. Since

$$C_V = \tau \left(\frac{d\sigma}{d\tau} \right)_{N,V},$$

we can calculate the difference in entropies at two different temperatures τ_1, τ_2 by integrating:

$$\sigma_2 - \sigma_1 = \int \frac{C_V}{\tau} d\tau = \frac{3N}{2} \ln \left(\frac{\tau_2}{\tau_1} \right).$$

It follows that

$$\frac{\tau_2}{\tau_1} = \frac{e^{2\sigma_2/(3N)}}{e^{2\sigma_1/(3N)}},$$

and hence

$$\tau = \alpha(N, V) \cdot e^{2\sigma/(3N)}$$

for some α that depends only on N and V .

Method 2: Alternatively, we could start from the entropy of an ideal gas:

$$\sigma = N \ln \left(\frac{V}{N} \left(\frac{2\pi m \tau}{h^2} \right)^{3/2} \right) + \frac{5}{2}N.$$

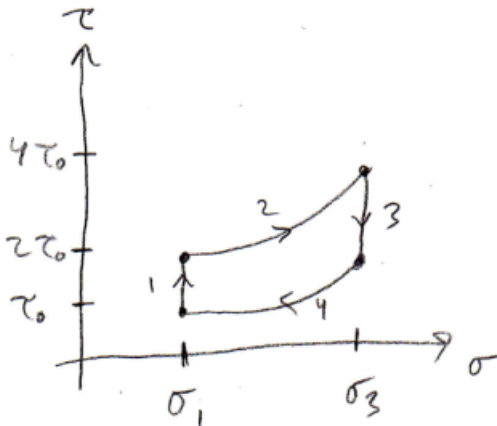
Solving for τ , we have

$$\begin{aligned} \frac{\sigma}{N} - \frac{5}{2} &= \ln \left(\frac{V}{N} \left(\frac{2\pi m \tau}{h^2} \right)^{3/2} \right) \\ \Rightarrow \frac{V}{N} \left(\frac{2\pi m \tau}{h^2} \right)^{3/2} &= e^{-\frac{5}{2}} e^{\frac{\sigma}{N}} \\ \Rightarrow \frac{2\pi m \tau}{h^2} &= \left(\frac{N}{V} \right)^{2/3} e^{-\frac{5}{3}} e^{\frac{2\sigma}{3N}} \\ \Rightarrow \tau &= \frac{h^2}{2\pi m} \left(\frac{N}{V} \right)^{2/3} e^{-\frac{5}{3}} e^{\frac{2\sigma}{3N}}. \end{aligned}$$

Thus $\tau = \alpha(N, V) \cdot e^{\frac{2\sigma}{3N}}$ with $\alpha(N, V) = \frac{h^2}{2\pi m} \left(\frac{N}{V} \right)^{2/3} e^{-\frac{5}{3}}$.

b. 7 points. 1 for curve shapes, 4 for curve labels, 1 for τ axis, 1 for σ axis.

The entropy-temperature diagram.



c. 6 points.

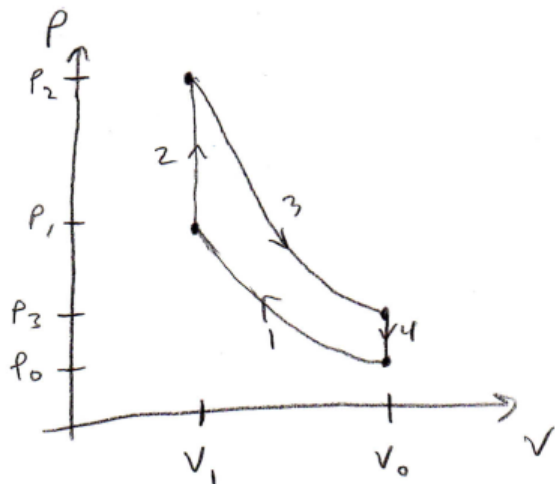
$$\begin{aligned}
 W &= \int_{\sigma_1}^{\sigma_3} \tau_2(\sigma) d\sigma + \int_{\sigma_3}^{\sigma_1} \tau_4(\sigma) d\sigma \\
 &= \int_{\sigma_1}^{\sigma_3} \alpha(N, V_1) \cdot e^{\frac{2\sigma}{3N}} d\sigma - \int_{\sigma_1}^{\sigma_3} \alpha(N, V_0) \cdot e^{\frac{2\sigma}{3N}} d\sigma \\
 &= \frac{3N}{2} \left[\alpha(N, V_1) \cdot e^{\frac{2\sigma}{3N}} \right]_{\sigma_1}^{\sigma_3} - \frac{3N}{2} \left[\alpha(N, V_0) \cdot e^{\frac{2\sigma}{3N}} \right]_{\sigma_1}^{\sigma_3} \\
 &= \frac{3N}{2} (4\tau_0 - 2\tau_0) - \frac{3N}{2} (2\tau_0 - \tau_0) \\
 &= \frac{3}{2} N \tau_0 = \frac{3}{2} P_0 V_0.
 \end{aligned}$$

d. 4 points.

$$\begin{aligned}
 W &= \frac{3}{2} \cdot (2 \cdot 10^{-3} \text{ m}^3) \cdot (10^5 \text{ Pa}) = 3 \cdot 10^2 \text{ J} \\
 P &= W / (1 \cdot 10^{-3} \text{ s}) = 3 \cdot 10^5 \text{ Watts}
 \end{aligned}$$

e. 8 points. 4 for curve labels, 3 for P axis, 1 for V axis

To draw the P-V diagram, we recall that the adiabats have the form $pV^\gamma = \text{constant}$, where $\gamma = C_p/C_V = 5/3$ for a monatomic ideal gas.



- $T_0 V_0^{2/3} = T_1 V_1^{2/3} = 2 T_0 V_1^{2/3}$
 $\Rightarrow V_1 = V_0 \cdot \frac{1}{2^{3/2}}$
- $P_1 V_1^{5/3} = P_0 V_0^{5/3} \Rightarrow P_1 = P_0 \cdot 2^{5/2}$
- $P_2 = 2 P_1 = P_0 \cdot 2^{7/2}$
- $P_3 = 2 P_0$

5. Microcanonical Entropy

a. 5 points.

We have $f > 0$ by Newton's third law. The velocity of the particle is

$$v = \sqrt{\frac{2E}{m}}.$$

The momentum exchanged with a wall during one round trip is

$$\Delta p = 2mv = 2\sqrt{2mE}$$

and the time of the round trip is

$$\Delta t = \frac{2L}{v} = 2L\sqrt{\frac{m}{2E}}$$

so the average force is

$$f = \frac{\Delta p}{\Delta t} = \frac{2E}{L}.$$

b. 2 points.

$$dE = \tau d\sigma - f dL$$

c. 2 points.

Setting $dE = 0$, we have

$$f dL = \tau d\sigma \implies f = \tau \left(\frac{\partial \sigma}{\partial L} \right)_E.$$

d. 7 points. 3 for correct density of states, 2 for a correct integral, 1 for a correct expansion in $\delta E/E$, 1 for answer.

$$\begin{aligned} g &= \int_{E-\delta E}^{E+\delta E} D(\epsilon) d\epsilon = \int_{E-\delta E}^{E+\delta E} \sqrt{\frac{mL^2}{2\hbar^2\pi^2}} \cdot \frac{1}{\epsilon^{1/2}} d\epsilon = \left[\sqrt{\frac{2mL^2}{\hbar^2\pi^2}} \epsilon^{1/2} \right]_{E-\delta E}^{E+\delta E} \\ &\approx \sqrt{\frac{2mL^2}{\hbar^2\pi^2}} E^{1/2} \cdot \frac{\delta E}{E} \\ &= \sqrt{\frac{2mL^2}{\hbar^2\pi^2}} \frac{\delta E}{E^{1/2}}. \end{aligned}$$

e. 4 points.

$$\begin{aligned} \sigma &= \ln g = \ln \left(\sqrt{\frac{2mL^2}{\hbar^2\pi^2}} \frac{\delta E}{E^{1/2}} \right) \\ \implies E(\sigma) &= \frac{2mL^2}{\hbar^2\pi^2} (\delta E)^2 e^{-2\sigma} \end{aligned}$$

f. 5 points. 2 for $\tau = \frac{\partial E}{\partial \sigma}$, 2 for answer, 1 for pointing out $\tau < 0$.

$$\tau = \left(\frac{\partial E}{\partial \sigma} \right)_L = -2 \cdot \frac{2mL^2}{\hbar^2\pi^2} (\delta E)^2 e^{-2\sigma} = -2E.$$

This result is unusual because $\tau < 0$.

g. 5 points. 3 for f (1 for consistency with previous sections, 1 for correct variables, 1 for answer), 2 for pointing out $f < 0$.

$$f = \tau \left(\frac{\partial \sigma}{\partial L} \right)_E = -2E \cdot \frac{1}{L} = \frac{-2E}{L}.$$

This result has the wrong sign.

h. +4 points. 1 for σ_G , 1 for $E(\sigma_G)$, 1 for τ_G , 1 for f .

If we use the Gibbs entropy, then we have

$$\sigma_G = \ln \sqrt{\frac{2mL^2}{\hbar^2 \pi^2}} E^{1/2} \implies E = \frac{\hbar^2 \pi^2}{2mL^2} e^{2\sigma_G}$$

and

$$\tau_G = 2 \frac{\hbar^2 \pi^2}{2mL^2} e^{2\sigma_G} = 2E.$$

Thus

$$f = \tau_G \left(\frac{\partial \sigma_G}{\partial L} \right)_E = 2E \cdot \frac{1}{L} = \frac{2E}{L}$$

which agrees with the classical result from part a, as we would expect.