MIDTERM EXAM

- You have 80 minutes
- You are permitted to have one letter-sized sheet of hand-written notes
- No other references or electronic devices are permitted
- It will be useful to recall:
  $e = 2.71828...$

1. *Equilibration of a trapped particle.* Consider a particle trapped in an isotropic three-dimensional harmonic potential. Its energy is given by

$$\varepsilon(n_x, n_y, n_z) = (n_x + n_y + n_z)\hbar\omega, \tag{1}$$

where we take the zero-point energy to be zero.

a. Calculate the entropy $\sigma(E)$ at total energy E for:
   i. $E = 0$
   ii. $E = \hbar\omega$
   iii. $E = 2\hbar\omega$
   iv. $E = 3\hbar\omega$

b. Suppose that the particle initially has energy $E = 2\hbar\omega$ and is then brought into thermal equilibrium with a bath at temperature $\tau = \hbar\omega$. Is the energy of the particle most likely to increase, decrease, or remain the same? Explain your answer using the definition of temperature and your results from a.

c. In the scenario described in b., does the entropy of the particle increase, decrease, or remain the same as it equilibrates with the bath? Is your result consistent with the second law of thermodynamics? Explain.

d. Calculate the average energy of the particle as a function of temperature τ and evaluate your result for $\tau = \hbar\omega$. Is the result consistent with your answer from b.? Explain.

2. *Elastic modulus.* A suspended rubber band may be modeled as a polymer chain consisting of N segments, each of which has length ℓ and can point either up or down. Consider a rubber band at temperature τ that is hanging from a peg and extended to a total length L by an adjustable weight w. Assume that the sole contribution to the energy of the system is the gravitational potential energy of the weight w, i.e., neglect the weights of the segments themselves, or any interaction between the segments.

a. Briefly explain how the weighted rubber band is equivalent to a paramagnet. Which parameter in the above description is analogous to the magnetization?
b. Write down the partition function and the Helmholtz free energy F for the weighted rubber band in terms of w and τ.

c. Starting from the fundamental thermodynamic relation $dE = -Ldw + \tau d\sigma$, show that the equilibrium length of the rubber band at temperature τ is given by

$$L = -\left(\frac{\partial F}{\partial w}\right)_\tau . \tag{2}$$

d. Calculate the equilibrium length L as a function of weight w and temperature τ.

e. The *elastic modulus* μ, defined as

$$\mu \equiv L\left(\frac{\partial w}{\partial L}\right)_\tau , \tag{3}$$

quantifies the difficulty of stretching the rubber band. Calculate the elastic modulus as a function of the tension w and temperature τ. *Hint:* it is easiest first to calculate μ^{-1}.

f. Sketch μ as a function of temperature τ at fixed tension w, and indicate the limiting value as $\tau \to \infty$.

g. Give a physical explanation for the behavior of μ in the limit as $\tau \to 0$.

3. *Particle in a 1D box.* Recall that the energy eigenstates of a particle of mass m in a one-dimensional box of length L are given by

$$\varepsilon_n = \frac{n^2\pi^2\hbar^2}{2mL^2}, \tag{4}$$

where $n = 1, 2, 3, \ldots$ is a positive integer.

a. What is the speed of the particle in the n^{th} energy eigenstate?

b. Write down an exact expression for the average speed $\langle |v| \rangle$ of the particle at temperature τ as a sum over states.

c. Give an approximate condition on the temperature for the particle to be in a classical regime. In this regime, evaluate $\langle |v| \rangle$ as a function of τ.

d. Determine the standard deviation of the particle's speed in the classical regime.

e. Suppose that the box expands from length L to length $2L$ at constant temperature τ. Evaluate:

   i. The Helmholtz free energy F before and after expansion. (You may express F in terms of the thermal de Broglie wavelength λ_τ.)

   ii. The work done by the particle.

   iii. The heat absorbed by the particle.

   Be sure to give the correct sign in each case!