Physics 170:
Statistical Mechanics and Thermodynamics

Lecture 10A

Administrative Notes

- **Final exam:** Wednesday, December 13
- You will be allowed 1 sheet of hand-written notes
- Additional review problems will be posted today
- **My office hours:** Monday, 12:15 - 1:30 pm

Today

- Wrap up Bose-Einstein Condensation

- What does it take to prepare and detect an ultra-cold quantum gases in the lab?

  Practice applying what you've learned in 170 outside the context of a textbook chapter.

Names:

EXERCISE 10A: QUANTUM GAS EXPERIMENTS

Objectives: Apply your broad knowledge of statistical mechanics to figure out...

- ...how to detect a Bose-Einstein condensate
- ...how we can study an ultracold quantum gas in a room-temperature apparatus

Useful past results:

- Thermal de Broglie wavelength: $\lambda_\tau = h/\sqrt{2\pi m\tau}$
- Condition for BEC in a 3D box: $n\lambda_\tau^3 > \zeta(3/2) \approx 2.6$

1. *Bose-Einstein condensation (BEC).* Last time, we derived the condition for obtaining a macroscopic occupation of the ground state in a gas of bosons with number density $n = N/V$ in a box of volume $V = L^3$.

   a. How does the critical temperature τ_E scale with the number of bosons, their mass m, and the dimension L of the box? (Don't worry about numerical factors.)

   $$\frac{N}{L^3}\cdot\frac{h^3}{(2\pi m\tau_E)^{3/2}} \sim 1 \Rightarrow \tau_E \sim \frac{N^{2/3}\cdot h^2}{mL^2}$$

   b. How does the critical temperature for Bose-Einstein condensation compare with the energy of the first excited state? Is the result surprising? Why or why not?

   $\tau_E \sim N^{2/3}\varepsilon_1$, where $\varepsilon_1 = \frac{(2^2-1^2)\pi^2\hbar^2}{2mL^2}$

   * τ_E much larger than ε_1 in the thermodynamic limit.

   * Evidently Bose-Einstein condensation is a collective effect, arising from interference of matter waves

c. Re-express the condition for Bose-Einstein condensation as a condition on the *phase space density* $n/(\Delta p)^3$, where Δp represents the spread of the momentum distribution at temperature τ. How do you interpret your result?

$$\tau \sim \frac{\langle p^2 \rangle}{2m} \sim \frac{(\Delta p)^2}{2m} \overset{!}{\lesssim} \frac{n^{2/3} \cdot h^2}{m}$$

$$\Rightarrow \frac{n^{2/3}}{(\Delta p)^2} \lesssim h^2 \Rightarrow \boxed{\frac{n}{(\Delta p)^3} \lesssim h^3}$$

d. For a given density n and temperature τ, what fraction of the atoms are in the ground state? ~~Calculate~~ First, calculate the ~~condensate~~ thermal fraction N_e/N at $\tau < \tau_E$

i. . . . as a function of density $n = N/V$ and thermal de Broglie wavelength λ_τ.

ii. . . . as a function of temperature τ and the critical temperature τ_E for Bose-Einstein condensation.

$$\frac{N_0}{N} = \frac{N - N_e}{N} = 1 - \frac{N_e}{N}$$

i) We showed last time: $(N_e/V)\lambda_\tau^3 = \zeta(3/2) \Rightarrow \boxed{\frac{N_e}{N} = \frac{\zeta(3/2)}{n\lambda_\tau^3}}$

for $\tau < \tau_E$, where excited orbitals are filled to the maximum ...

$$\Rightarrow \boxed{\frac{N_0}{N} = 1 - \frac{\zeta(3/2)}{n\lambda_\tau^3}}$$

ii) $\frac{N_e}{\tau^{3/2}} = \text{const}$, with $N_e = N$ @ $\tau = \tau_E$

$$\Rightarrow N_e = N(\tau/\tau_E)^{3/2} \Rightarrow \boxed{\frac{N_e}{N} = (\tau/\tau_E)^{3/2}}$$

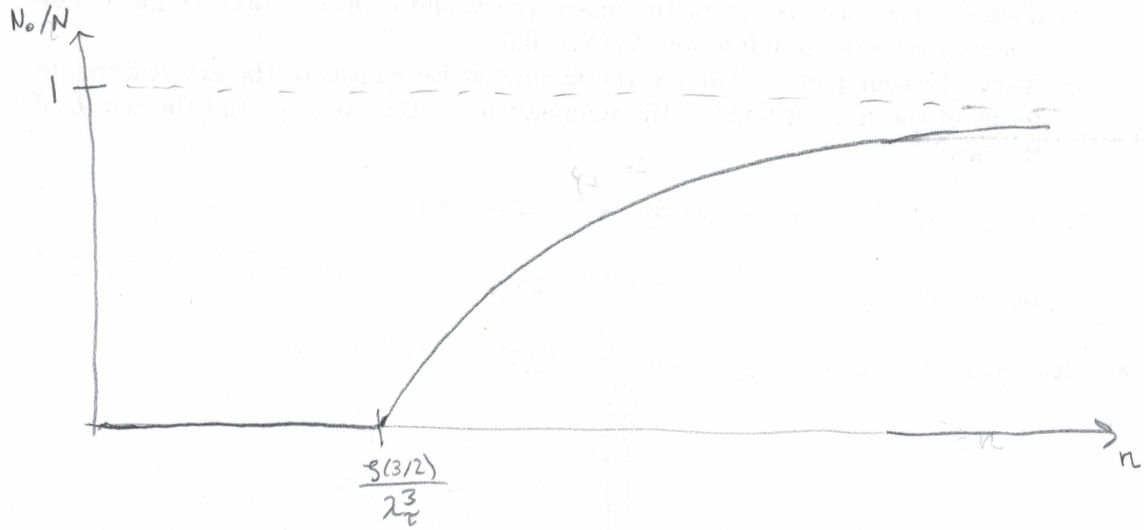
$$\boxed{\frac{N_0}{N} = 1 - (\tau/\tau_E)^{3/2}}$$

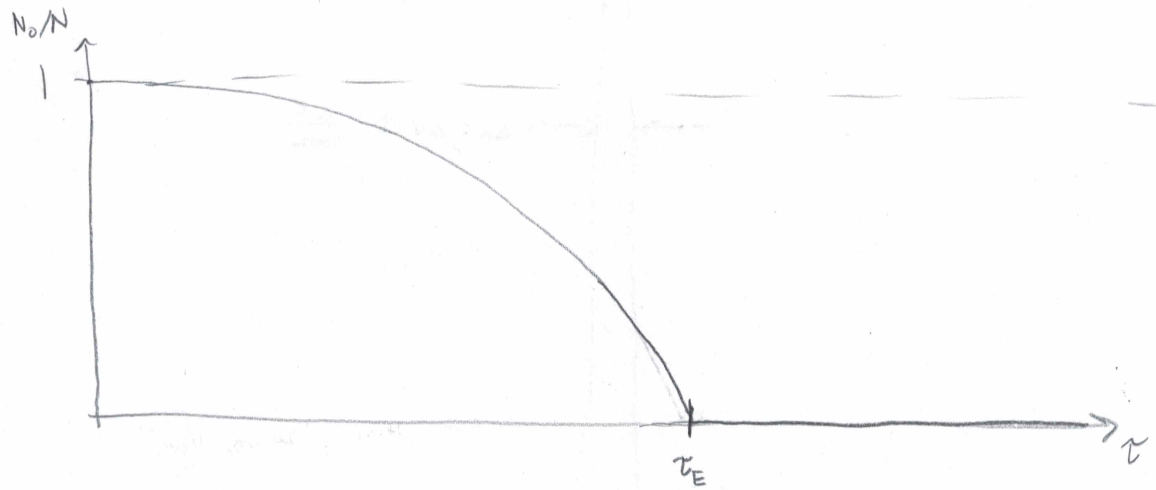
e. What is the thermal fraction for $\tau > \tau_E$? N_e/N_0 = 1

f. Sketch the condensate fraction N_0/N

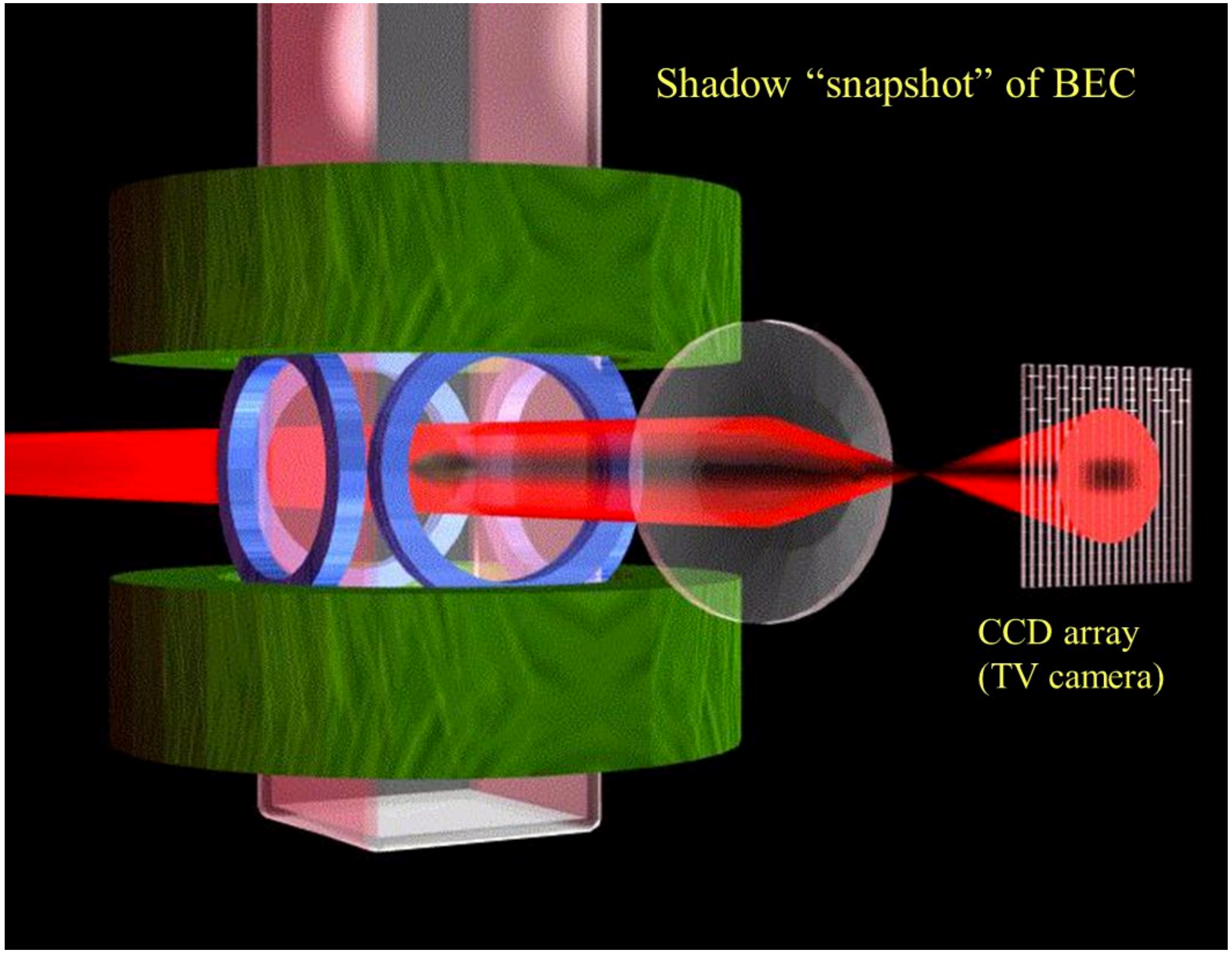
i. ...vs. density n at fixed temperature τ.

ii. ...vs. temperature τ at fixed density.





Detecting a BEC



What do you expect to observe?

2. To detect a Bose-Einstein condensate, atoms are suddenly released from an optical trap, allowed to expand for a fixed time t, and then imaged on a CCD camera. In the limit of a long time of flight t, an atom's position in the image is directly proportional to its velocity at the time that the trap was switched off.

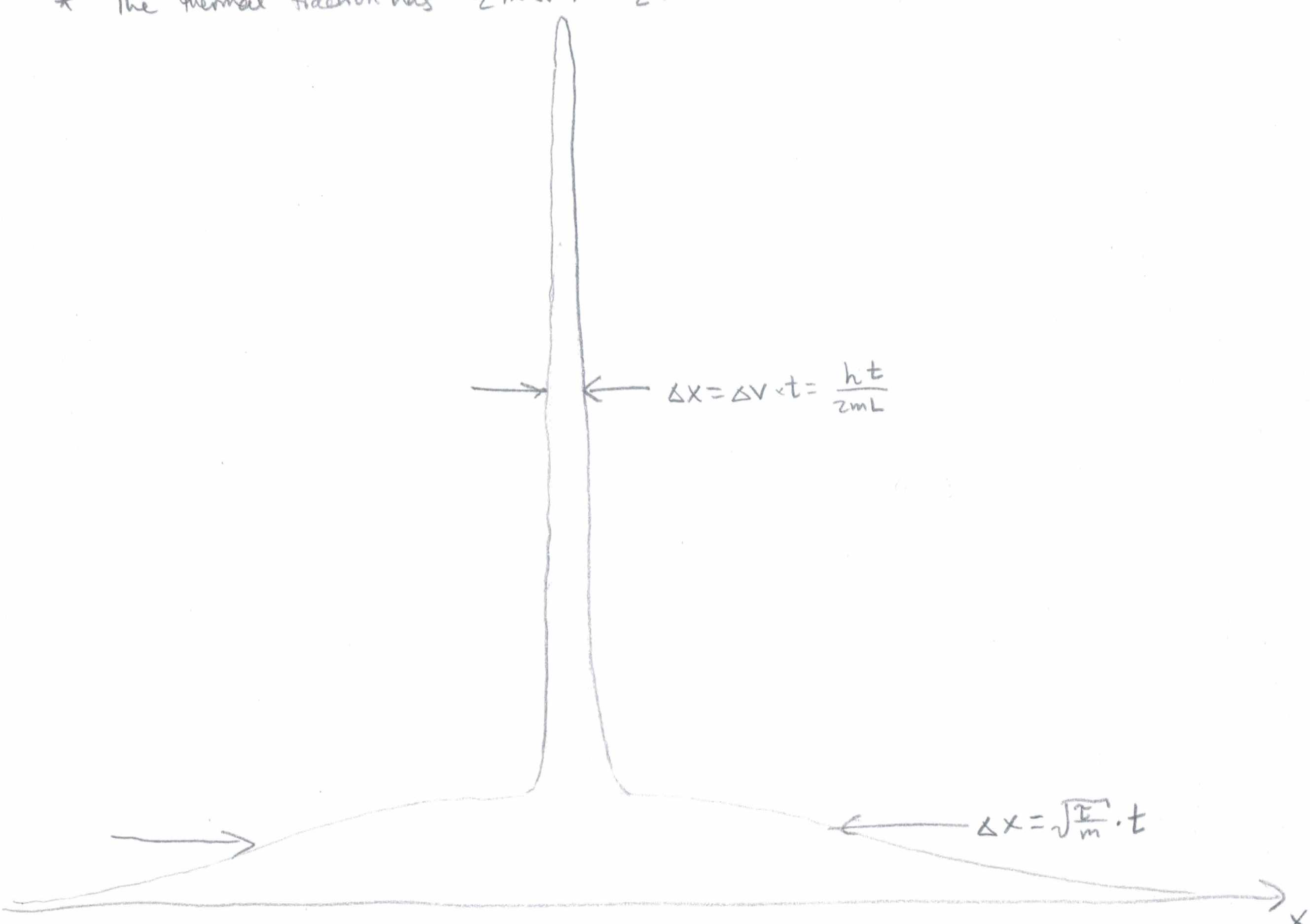
   a. Sketch a 1D cross-section of the image you would expect to observe for a Bose gas with a condensate fraction $N_0/N \sim 0.5$.

   b. Annotate your plot. What are the characteristic widths of the key features in terms of the time of flight t, the temperature τ, the mass m, and the size L of the box?

* In the ground state, the rms momentum Δp is given by

$$(\Delta p)^2 = \langle p^2 \rangle = \left(\frac{h}{2L}\right)^2 \Rightarrow \langle v^2 \rangle = \frac{h^2}{4m^2L^2} \qquad \left[\frac{1}{2}mv^2 = \frac{\pi^2\hbar^2}{2mL^2} \Rightarrow v^2 = \frac{h^2}{4m^2L^2} \checkmark\right]$$

* The thermal fraction has $\frac{1}{2}m\langle v^2\rangle \sim \frac{1}{2}\tau \Rightarrow \langle v^2 \rangle \sim \tau/m$

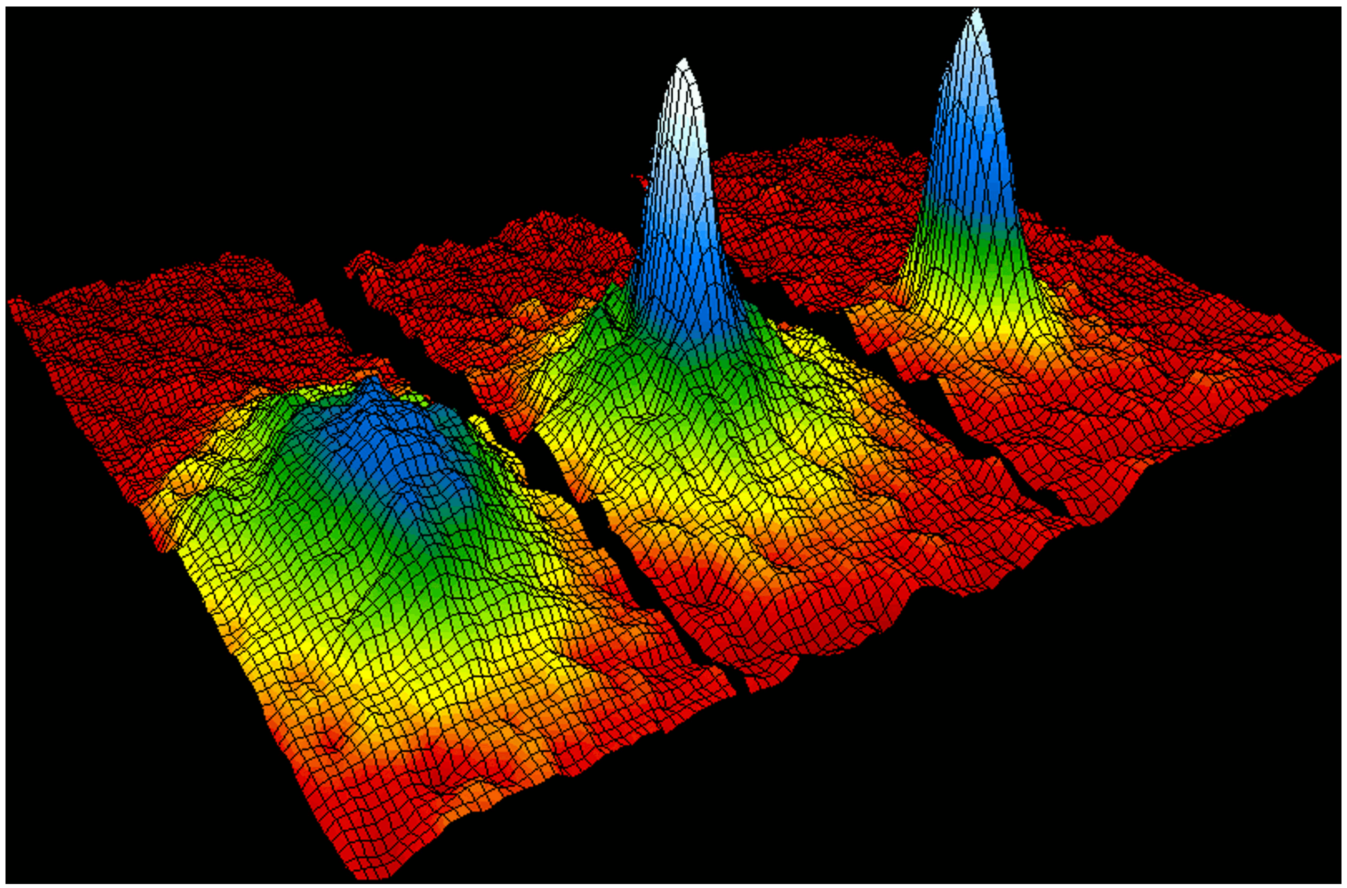


Note: central peak width $\Delta x \propto 1/L \Rightarrow$ inversion of aspect ratio @ long time of flight

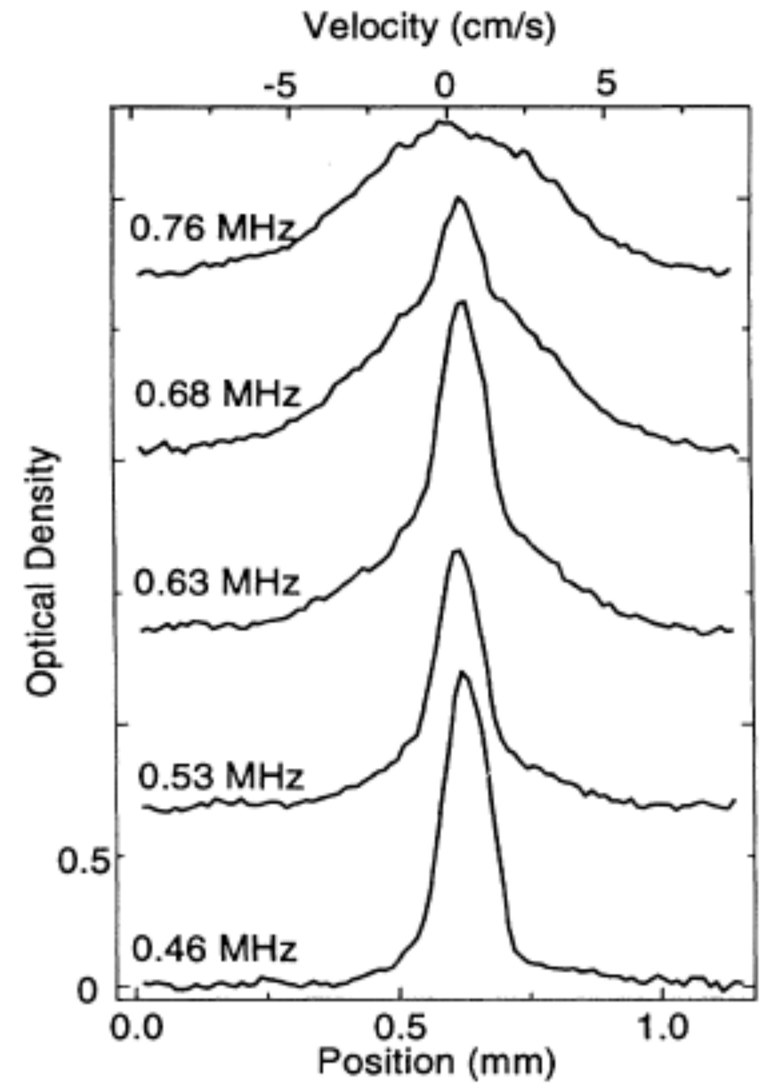
Real space

Momentum space

Bose-Einstein Condensation

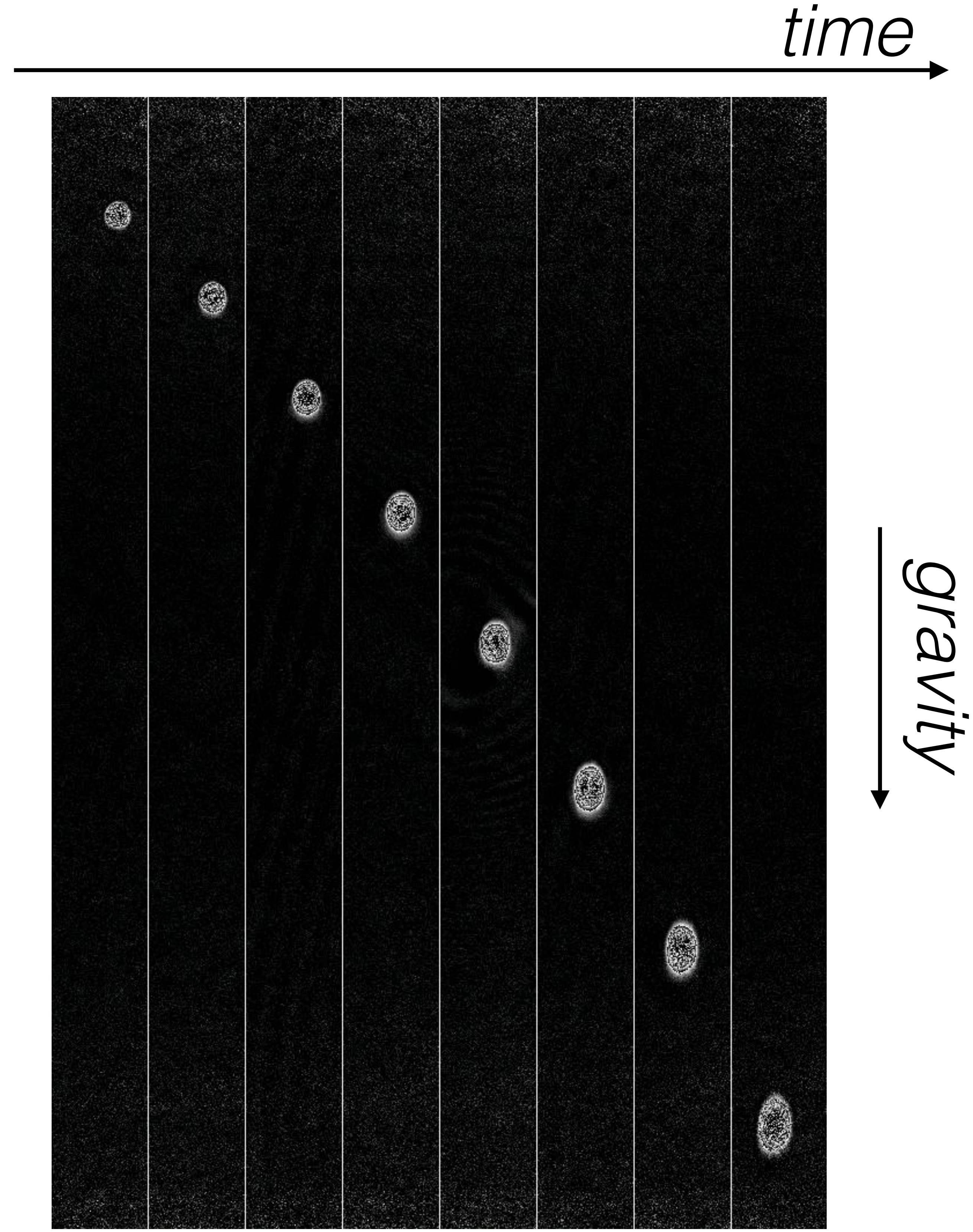


Anderson, …, Wieman & Cornell, *Science* (1995).



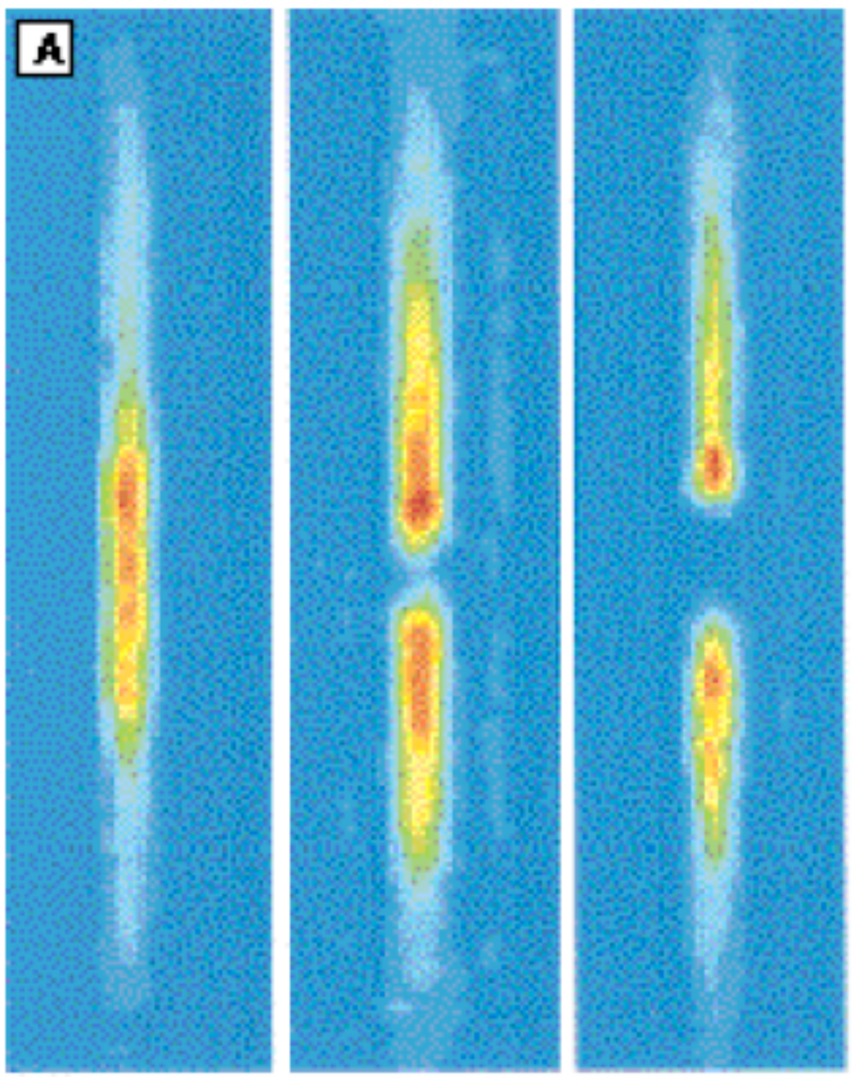
Davis, … & Ketterle, *PRL* (1995).

Time-of-flight images of a BEC

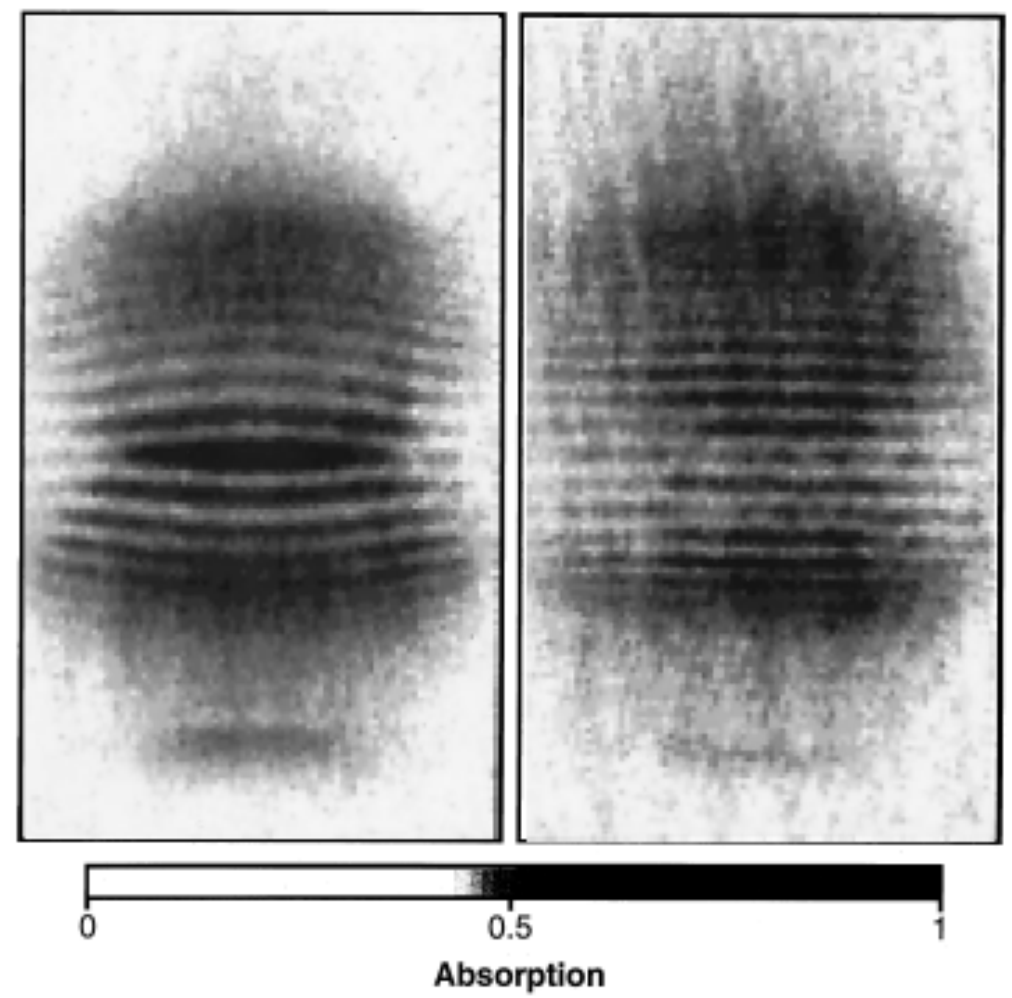


Matter-Wave Interference

Making two independent BECs…



…and watching them interfere



Andrews, …, & Ketterle, "Observation of interference between two Bose Condensates," *Science* (1997).