MIDTERM EXAM

- You have 80 minutes
- You are permitted to have one letter-sized sheet of hand-written notes
- No other references or electronic devices are permitted

1. *Quantum cantilever* (20 points total). In recent years, it has become possible to cool a mechanical cantilever to such low temperature that it is well described as a quantized harmonic oscillator, with energy levels

$$\varepsilon_n = n\hbar\omega. \tag{1}$$

(We have taken the zero-point energy to be zero.)

a. The cantilever is placed into a dilution refrigerator and allowed to equilibrate to a temperature τ_c. To precisely calibrate the temperature τ_c, you measure the ratio $\rho \equiv P_0/P_1$ of probabilities of finding the cantilever in its ground and first excited states. Express the following quantities in terms of the ratio ρ:

i. (2 points) …the temperature τ_c

ii. (3 points) …the free energy F

iii. (4 points) …the average energy E

b. (5 points) If the cantilever in part a. is twice as likely to be in the ground state as in the first excited state ($\rho = 2$), what is its entropy?

c. (6 points) *Three cantilevers.* Now consider a system of three cantilevers, each of frequency ω. Suppose that I give you only one of the following pieces of information:

i. The cantilevers are in thermal equilibrium at the temperature τ_c that you found in part a., with $\rho = 2$. (*Extra:* +1 point for considering general ρ).

—*or*—

ii. The total energy of the three cantilevers is $E = 3\hbar\omega$.

In which case have I given you more information about the microstate (n_1, n_2, n_3)? Explain. Quantify the difference in information content, if any.

2. *Electron vs proton spins* (25 points total). Consider two paramagnetic solids: solid $\mathcal{A}_e$ has N_e unpaired electron spins of magnetic moment μ_e, whereas solid $\mathcal{B}_p$ has N_p unpaired nuclear spins of magnetic moment μ_p. The ratio of the magnetic moments is $\mu_e/\mu_p = 662$. The solids are placed in a magnetic field B, where each spin has two states of energies 0 and $2\mu_i B$, measured relative to the ground state.
Answer the following questions using physical arguments, bolstered by simple equations and/or approximations.

   a. The two solids are in a magnetic field B and are in thermal equilibrium at a temperature $\tau = \mu_p B$.
      i. (4 points) Which solid has a higher energy per spin? Explain.
      ii. (4 points) Which solid has a higher entropy per spin (or are both entropies the same)? Explain.

   b. Now suppose that the two solids are prepared at different temperatures to achieve the same average energy per particle, $E_e/N_e = E_p/N_p = \mu_p B$. The solids are then isolated from their environments and brought into thermal contact with one another.
      i. (5 points) On average, does energy flow from $\mathcal{A}_e$ to $\mathcal{B}_p$, from $\mathcal{A}_e$ to $\mathcal{B}_p$, or neither?
      ii. (4 points) Does the entropy of solid $\mathcal{A}_e$ increase, decrease or remain the same?
      iii. (4 points) Does the entropy of solid $\mathcal{B}_p$ increase, decrease or remain the same?
      iv. (4 points) Are your results consistent with the 2nd Law of Thermodynamics? Why or why not?

3. *Non-ideal gas* (10 points total). For certain dense gases, a good approximation to the equation of state is

$$p = \frac{N\tau}{V - Nb} - \frac{a}{\sqrt{\tau}V(V+b)}, \tag{2}$$

where the coefficients a and b account for interactions among the gas molecules and the volume occupied by each molecule. Let's examine how the heat capacity of such a gas depends on volume V at fixed temperature τ.

   a. (2 points) Express the heat capacity in terms of a derivative of the entropy σ.
   b. (3 points) Derive a useful thermodynamic relation of the form:

$$\left(\frac{\partial \sigma}{\partial V}\right)_\tau = \left(\frac{\partial A}{\partial B}\right)_V \tag{3}$$

   What are A and B?
   c. Combine your results from above to find:
      i. (3 points) ...a general expression for $(\partial C_V/\partial V)_\tau$ in terms of state functions appearing in Eq. 2
      ii. (2 points) ...the derivative $(\partial C_V/\partial V)_\tau$ of the heat capacity with respect to volume for the gas of Eq. 2.

4. *Multi-species gas* (25 points total). A dilute mixture of several monatomic gases is contained in a volume $V = L^3$. There are N_α molecules of species α, each of molecular mass m_α. The mixture is in thermal equilibrium at temperature τ.

   a. (3 points) What is the total energy E of the mixture?

   b. Express the following in terms of the temperature τ:

      i. (2 points) The rms momentum of a single molecule of species α

      ii. (2 points) The rms velocity of a single molecule of species α

   c. (4 points) Find the rms momentum $\wp_{\rm rms} \equiv \sqrt{\langle \wp^2 \rangle}$ and rms velocity $v_{\rm rms} = \sqrt{\langle v^2 \rangle}$ for the mixture.

   d. (4 points) Simplify your results from c. for the case where there are two species with equal numbers of particles $N_1 = N_2 = N_{\rm tot}/2$ and with very unequal masses $m_1 \ll m_2$. Express $\wp_{\rm rms}$ and $v_{\rm rms}$ in terms of the temperature τ.

   e. Suppose that you perform an experiment to measure the velocity v_x of each molecule in the x direction. Sketch the distribution $P(v_x)$ that you would expect to measure in such an experiment for the case of $N_{\rm He} = N_{\rm Ar} \gg 1$ molecules of ^{4}He ($m_{\rm He} = 4$ amu) and ^{40}Ar ($m_{\rm Ar} = 40$ amu).

   f. (7 points) Now suppose that you performed many such experiments and, from each one, extracted the average velocity $\overline{v_x}$. Consider the probability distribution $P(\overline{v_x})$ for the *average* velocity $\overline{v_x}$.

      i. (3 points) Is the functional form of $P(\overline{v_x})$ the same or different from that of $P(v_x)$? Explain.

      ii. **Extra credit.** (+3 points) Find the standard deviation $\Delta\overline{v_x}$ in terms of τ, $N_{\rm He}$, $N_{\rm Ar}$, $m_{\rm He}$ and/or $m_{\rm Ar}$. Then simplify your result using the approximation $m_{\rm He} \ll m_{\rm Ar}$.