Problem Set 4 Solutions

Problem 1

a. The first vortex occupies an area πa^2, and the total area of the 2D superfluid is L^2, so there are $L^2/(\pi a^2)$ distinct locations where the first vortex could be located. The second vortex is located on the circumference of the circle with radius r centered on the first vortex. The circumference has length $2\pi r$, and the vortex has diameter $2a$, so there are $2\pi r/(2a)$ distinct locations for the second vortex.

Note that we have assumed $L \gg r$ so that we can ignore the case when the first vortex is located a distance less than r from the boundary. We have also assumed $r \gg a$ so that we can ignore geometric corrections to the fraction of the circumference occupied by the second vortex.

In these limits, the total number of allowed spatial configurations is

$$\frac{L^2}{\pi a^2} \cdot \frac{2\pi r}{2a} = \left(\frac{L}{a}\right)^2 \left(\frac{r}{a}\right)$$

and the entropy of one vortex pair of separation r is therefore

$$\sigma(r) = \ln\left(\frac{L}{a}\right)^2 \left(\frac{r}{a}\right) = 2\ln\frac{L}{a} + \ln\frac{r}{a}$$

b. The Helmholtz free energy is

$$F(r,\tau) = \epsilon_p(r) - \tau\sigma(r) = 2\pi J \ln\frac{r}{a} - \tau\ln\frac{r}{a} - 2\tau\ln\frac{r}{a} = (2\pi J - \tau)\ln\frac{r}{a} - 2\tau\ln\frac{L}{a}.$$

c. Note that

$$\frac{\partial F}{\partial r} = \frac{2\pi J - \tau}{r}.$$

Since r is positive, the sign of $\frac{\partial F}{\partial r}$ depends on the sign of $2\pi J - \tau$.

If $\tau < 2\pi J$, then $\frac{\partial F}{\partial r} > 0$, which means that the free energy of the vortex pair is minimized by reducing r toward a. In equilibrium, there will be no vortex pairs with radius $r \gg a$. Such pairs could reduce the free energy of the system by moving closer together, but we know that free energy must be minimized in equilibrium. So if there are any vortex pairs in the system, they must be separated by a distance $r \sim a$. These vortex pairs are called "bound" vortices.

However, if $\tau > 2\pi J$, then $\frac{\partial F}{\partial r} < 0$. In this case, the free energy of the vortex pair is minimized by increasing r toward L. In equilibrium, there will be no vortex pairs with $r \ll L$. Such pairs could reduce the free energy of the system by moving farther apart, but we know that free energy must be minimized in equilibrium. So if there are any vortex pairs in the system, they must be separated by a distance $r \sim L$, which scales with the system size. These vortex pairs are called "unbound" vortices.

From the above considerations, we see that the transition from bound to unbound vortices occurs at a critical temperature $\tau_c = 2\pi J$.

d. The two vortices are attracted to one another since $\frac{\partial \epsilon_p}{\partial r} > 0$. If the temperature is low enough, the vortices will form a bound state since that is the lowest energy configuration. However, each vortex is pushed around by thermal fluctuations. If the temperature is high enough, the thermal fluctuations are strong enough to rip the vortex pair apart. The temperature at which this occurs is the critical temperature.

Problem 2

a. The fundamental thermodynamic relation is

$$dE = f dL + \tau d\sigma.$$

Rearranging, we have

$$d\sigma = \frac{dE}{\tau} - \frac{f}{\tau} dL.$$

The fundamental relation for the free energy $F(L, \tau)$ is therefore

$$dF = dE - d(\tau\sigma) = dE - \tau d\sigma - \sigma d\tau = f dL - \sigma d\tau.$$

Taking two derivatives of the free energy to obtain Maxwell's relation, we have

$$\left(\frac{\partial}{\partial L}\right)_\tau \left(\frac{\partial}{\partial \tau}\right)_L F = -\left(\frac{\partial\sigma}{\partial L}\right)_\tau$$

and

$$\left(\frac{\partial}{\partial \tau}\right)_L \left(\frac{\partial}{\partial L}\right)_\tau F = \left(\frac{\partial f}{\partial \tau}\right)_L$$

so

$$\left(\frac{\partial\sigma}{\partial L}\right)_\tau = -\left(\frac{\partial f}{\partial \tau}\right)_L = -2a\tau(L - L_0).$$

b. To find $\sigma(\tau, L)$, we first integrate along a path of constant length, then along a path of constant temperature. So

$$\sigma(\tau, L) = \sigma(\tau_0, L_0) + \int_{\tau'=\tau_0}^\tau \frac{\partial\sigma(\tau', L_0)}{\partial\tau'} d\tau' + \int_{L'=L_0}^L \frac{\partial\sigma(\tau, L')}{\partial L'} dL'$$

$$= \sigma(\tau_0, L_0) + \int_{\tau'=\tau_0}^\tau \frac{C_L}{\tau} d\tau' + \int_{L'=L_0}^L -2a\tau(L' - L_0) dL'$$

$$= \sigma(\tau_0, L_0) + b(\tau - \tau_0) - a\tau(L - L_0)^2.$$

c. Since the process is adiabatic (the rod is thermally insulated) as well as quasistatic, the process is reversible. This means that $\Delta\sigma = 0$, i.e.

$$b(\tau_i - \tau_0) - a\tau_i(L_i - L_0)^2 = b(\tau_f - \tau_0) - a\tau_f(L_f - L_0)^2$$

or

$$\tau_f = \tau_i \cdot \frac{b - a(L_i - L_0)^2}{b - a(L_f - L_0)^2}.$$

If $L_f > L_i$, $\tau_f > \tau_i$, so the temperature increases.

d. To find the heat capacity $C_L(L)$ as a function of length, we need to find $\left(\frac{\partial C_L}{\partial L}\right)_\tau$ and integrate. Since $C_L = \tau \frac{\partial\sigma}{\partial\tau}$, we have

$$\left(\frac{\partial C_L}{\partial L}\right)_\tau = \left(\frac{\partial}{\partial L}\right)_\tau \tau \left(\frac{\partial}{\partial\tau}\right)_L \sigma = \tau \left(\frac{\partial}{\partial\tau}\right)_L \left(\frac{\partial}{\partial L}\right)_\tau \sigma$$

$$= -2a\tau(L - L_0).$$

Thus, the heat capacity at length L is

$$C_L(L) = C_L(L_0) + \int_{L_0}^L \left(\frac{\partial C_L(L')}{\partial L'}\right)_\tau dL' = b\tau - a\tau(L - L_0)^2.$$

e. We have

$$\sigma(L, \tau_0) - \sigma(L_0, \tau_0) = \int_{L_0}^{L} \left(\frac{\partial \sigma(L', \tau_0)}{\partial L'} \right)_\tau dL' = \int_{L_0}^{L} -2a\tau_0(L' - L_0)dL' = -a\tau_0(L - L_0)^2.$$

In addition, we have

$$\sigma(L, \tau) - \sigma(L, \tau_0) = \int_{\tau_0}^{\tau} \frac{C_L(L)}{\tau'} d\tau' = b(\tau - \tau_0) - a(L - L_0)^2(\tau - \tau_0).$$

Therefore,

$$\sigma(L, \tau) - \sigma(L_0, \tau_0) = b(\tau - \tau_0) - a\tau(L - L_0)^2,$$

which agrees with what we found in part b.

Problem 3

a. The fundamental thermodynamic relation for this system is

$$dE = 2sl\ dx + dQ.$$

Since we consider a quasi-static process, we have

$$dE = 2sl\ dx + \tau d\sigma.$$

b. From part a, we have

$$\left(\frac{\partial E}{\partial x}\right)_\tau = 2sl + \tau \left(\frac{\partial \sigma}{\partial x}\right)_\tau.$$

To find $\left(\frac{\partial \sigma}{\partial x}\right)_\tau$, we use Maxwell's relation for the free energy. Since

$$dF = 2sl\ dx - \sigma d\tau$$

we have

$$\frac{\partial}{\partial x}\frac{\partial}{\partial \tau}F = -\frac{\partial \sigma}{\partial x} = \frac{\partial}{\partial \tau}\frac{\partial}{\partial x}F = \frac{\partial}{\partial \tau}2sl = -2\alpha l.$$

Thus

$$\left(\frac{\partial \sigma}{\partial x}\right)_\tau = 2\alpha l,$$

the derivative of the mean energy with respect to x at constant temperature τ_0 is

$$\left(\frac{\partial E(x)}{\partial x}\right)_\tau = 2(s_0 - \alpha\tau_0)l + 2\alpha l\tau_0 = 2s_0 l$$

and the change in mean energy when stretching the film to length x is

$$E(x) - E(0) = \int_0^x \left(\frac{\partial E(x')}{\partial x'}\right)_\tau dx' = \int_0^x 2s_0 l\ dx' = 2s_0 lx.$$

c. The work done on the film in order to stretch it to length x is

$$W(0 \to x) = \int_0^x 2sl\ dx = 2slx.$$

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Problem 4

a. Expanding the potential to second order in x around its minimum at $x = 0$, we have

$$U(x) \approx -U_0 + \frac{2U_0}{a^2}x^2 + \dots$$

Ignoring the constant offset $-U_0$ as well as terms higher than second order in x, we can approximate the potential by a harmonic oscillator with frequency

$$\frac{1}{2}M\omega^2 = \frac{2U_0}{a^2} \implies \omega = \sqrt{\frac{4U_0}{Ma^2}}.$$

b. At constant temperature, the mean energy of the system is given by

$$E = \bar{n}\hbar\omega$$

where $\bar{n}$ is the mean vibrational quantum number. Differentiating, we have

$$dE = \bar{n}\hbar d\omega + \hbar\omega d\bar{n}.$$

We want to express this relation in the differentials $d\omega$ and $d\sigma$, so we need to relate σ to $\bar{n}$. To do this, we write the entropy as

$$\sigma = -\sum_{n=0}^{\infty} P(n)\ln P(n) = \sum_{n=0}^{\infty} \frac{e^{-\beta\hbar\omega n}}{Z}(\beta\hbar\omega n + \ln Z)$$

$$= \beta\hbar\omega \sum_{n=0}^{\infty} n\frac{e^{-\beta\hbar\omega n}}{Z} + \ln Z \sum_{n=0}^{\infty} \frac{e^{-\beta\hbar\omega n}}{Z}$$

$$= \beta\hbar\omega\bar{n} + \ln Z.$$

Thus

$$d\sigma = \frac{\hbar\omega}{\tau}d\bar{n} \implies d\bar{n} = \frac{\tau d\sigma}{\hbar\omega}$$

and we have

$$dE = \bar{n}\hbar d\omega + \tau d\sigma.$$

c. Computing the partition function, we have

$$Z = \sum_{n=0}^{\infty} e^{-\beta\hbar\omega n} = \frac{1}{1 - e^{-\beta\hbar\omega}}.$$

Thus, the mean vibrational quantum number is given by

$$\bar{n} = \frac{E}{\hbar\omega} = \frac{1}{\hbar\omega}\frac{\partial \ln Z}{\partial \beta} = \frac{1}{e^{\hbar\omega/\tau} - 1}.$$

d. Expanding the mean energy to first order in $\frac{\hbar\omega}{\tau}$, we have

$$E = \frac{\hbar\omega}{e^{\hbar\omega/\tau} - 1} \approx \frac{\hbar\omega}{1 + \frac{\hbar\omega}{\tau} - 1} = \tau.$$

Note that we also could have gotten this result from the equipartition theorem. For an isothermal process, it follows that $dE = 0$. The work done on the system is given by

$$\Delta W = \int_{\omega_1}^{\omega_2} \bar{n}\hbar d\omega = \tau \int_{\omega_1}^{\omega_2} \frac{1}{\omega}d\omega = \tau \ln \frac{\omega_2}{\omega_1}.$$

Thus, the heat absorbed by the system is

$$dQ = dW = -\tau \ln \frac{\omega_2}{\omega_1}.$$

e. In an adiabatic process, $dQ = 0$. Thus, the change in energy of the system is equal to the work done on the system. So we have

$$dE = \bar{n}\hbar d\omega = \frac{\tau}{\omega}d\omega$$

and since $E = \tau$ in the classical limit,

$$d\tau = \frac{\tau}{\omega}d\omega.$$

Integrating, we have

$$\ln\frac{\tau_2}{\tau_1} = \ln\frac{\omega_2}{\omega_1}$$

or

$$\Delta\tau = \tau_2 - \tau_1 = \left(\frac{\omega_2}{\omega_1} - 1\right)\tau_1.$$

If $\omega_2 > \omega_1$, then $\tau_2 > \tau_1$. This is reasonable because increasing the trap frequency adds energy to the system, and we generally expect that the temperature of a thermally isolated system will rise if we add energy to it.

Problem 5

a. The energy eigenstates for a single particle in a 1D box are

$$E_n = \frac{\hbar^2 \pi^2}{2ML^2} n^2.$$

The partition function for a single particle is therefore

$$z_1 = \sum_{n=1}^{\infty} e^{-\beta E_n} \approx \int_0^{\infty} e^{-\beta E_n} \, dn = \frac{L}{\lambda_\tau}$$

where

$$\lambda_\tau \equiv \sqrt{\frac{2\pi \hbar^2}{M\tau}}.$$

For N identical particles, we estimate that the N-particle partition function is

$$Z = \frac{z_1^N}{N!}.$$

The free energy is then

$$F = -\tau \ln Z = \tau \ln N! - \tau N \ln z_1$$

$$\approx \tau (N \ln N - N) - \tau N \ln \frac{L}{\lambda_\tau}$$

$$= \tau N \ln \left(\frac{N}{L} \lambda_\tau \right) - \tau N$$

$$= \tau N \left(\ln \left(\frac{N}{L} \lambda_\tau \right) - 1 \right).$$

The entropy is

$$\sigma = -\left(\frac{\partial F}{\partial \tau} \right)_L = N \left(\ln \left(\frac{L}{N \lambda_\tau} \right) + \frac{3}{2} \right).$$

Our estimate of the N-particle partition function is accurate only when $\frac{L}{N\lambda_\tau} \gg 1$, or in other words, when

$$L \gg N \lambda_\tau,$$

which obtains when the box is so large that the particle wavefunctions are very unlikely to overlap.

b. The heat capacity is

$$C_L = \left(\frac{\partial E}{\partial \tau} \right)_L = \tau \left(\frac{\partial \sigma}{\partial \tau} \right)_L = \frac{N}{2}.$$

The specific heat is then $c_L = 1/2$.

c. The molar heat capacity is

$$6.022 \cdot 10^{23} \cdot c_L \cdot 1.381 \cdot 10^{-23} \, \frac{J}{K} = 4.157 \, \frac{J}{K}.$$