

Physics 170 concept questions

1. Heat Capacity

Consider the following three systems, all with N particles of mass m at temperature τ_0 :

1. An ideal gas at constant volume with no internal degrees of freedom.
2. An ideal gas at constant pressure with no internal degrees of freedom.
3. An ideal gas at constant pressure with $f > 0$ internal degrees of freedom.

Which system is the best heat reservoir? That is, if each system was attached to a Carnot engine along with a cold reservoir at temperature $\tau < \tau_0$, which engine would perform the most work?

System 1 has the smallest heat capacity, followed by system 2, then system 3. More heat capacity means more energy is released or absorbed when changing the temperature. Thus system 3 is the best heat reservoir.

Which system would be easiest to refrigerate to temperature $\tau < \tau_0$?

System 1, since it has the lowest heat capacity and thus takes the least work to cool.

If systems 1 and 3 were each used as the working fluid of a Carnot engine, which engine would perform more work per cycle?

The engine using system 1 would perform more work per cycle because the adiabats are steeper for gases with fewer internal degrees of freedom. This means that the P-V diagram encloses more area.

2. Ensembles

Describe in words the physical situation that corresponds to each of the following:

- a. Fixed σ, V, N ; energy $E(\sigma, V, N)$.
- b. Fixed τ, V, N ; Helmholtz free energy $F(\tau, V, N)$.
- c. Fixed τ, V, μ ; grand-canonical free energy $\Omega(\tau, V, \mu)$.
- d. Fixed N, p, τ ; Gibbs free energy $G(\tau, p, N)$.

System (a) is an isolated system. Energy, volume, and particle number are all conserved. In equilibrium, entropy is conserved as well. We referred to this system as the microcanonical ensemble.

System (b) is a system that can exchange energy with a heat reservoir. The volume and particle number are still conserved, but the energy of the system is not conserved because energy can fluctuate into or out of the reservoir. Instead, the reservoir fixes the temperature of the system. We referred to this system as the canonical ensemble.

System (c) is a system that can exchange energy and particles with a reservoir. The average energy of the system is given by the temperature of the reservoir, and the average particle number is given by the chemical potential of the reservoir. We referred to this system as the grand canonical ensemble.

System (d) can exchange energy and volume with a reservoir. The temperature of the reservoir sets the average energy of the system, and the pressure of the reservoir sets the average volume of the system. We didn't give this ensemble a name in class.

Mathematically, how do we transform from one ensemble to another? Transform the microcanonical ensemble into the canonical ensemble.

We can transform between any of the pairs of variables (σ, τ) , (V, p) , and (N, μ) by using Legendre transformations. For example, to go from the microcanonical ensemble to the canonical ensemble, we want to use τ as an independent variable instead of σ , so we define a new energy function

$$F = E - \tau\sigma.$$

The differential element is then

$$dF = dE - \tau d\sigma - \sigma d\tau$$

and by substituting $dE = \tau d\sigma - p dV + \mu dN$, we obtain

$$dF = -\sigma d\tau - p dV + \mu dN$$

which looks very similar to dE except that the differential element is now τ instead of σ . So we have gone from $E(\sigma, V, N)$ to $F(\tau, V, N)$ as desired.

3. Temperature

Define temperature in terms of energy and entropy. Explain why the temperatures of two systems that can exchange energy will be equal in equilibrium.

The temperature is defined by

$$\frac{1}{\tau} = \frac{\partial \sigma}{\partial E}$$

holding all external parameters (V, N , etc.) fixed. In equilibrium, if energy ΔE fluctuates from system 2 to system 1, the total entropy of the two systems does not change. So we have

$$\begin{aligned}\Delta\sigma_1 + \Delta\sigma_2 &= 0 \\ \Rightarrow \frac{\partial\sigma_1}{\partial E}\Delta E + \frac{\partial\sigma_2}{\partial E}(-\Delta E) &= 0 \\ \Rightarrow \tau_1 &= \tau_2.\end{aligned}$$

4. Thermodynamic relations

- Define a generalized force f .
- Describe how to write down the fundamental thermodynamic relation for a system.

If a process can do work dW on a system by changing some external parameter X by a small amount, then the generalized force F associated with that process is defined by

$$dW = -FdX.$$

To write down the fundamental thermodynamic relation for a system, we include a term for each form of work along with a term for the heat $dQ = \tau d\sigma$.

For example, work can be done on an ideal gas by changing its volume, so at fixed particle number, the fundamental thermodynamic relation for an ideal gas is

$$dE = \tau d\sigma - p dV.$$

Here the generalized force is called pressure.

5. Ideal Gas

- a. One mole of ideal gas at 300K is allowed to double its volume by free expansion through a orifice. What is the change in Temperature? Pressure? Entropy?

There will be no change in temperature because there is no energy left the system and - by the equipartition theorem - the energy of an ideal gas is directly proportional to the energy and independent of volume.

The pressure will be cut in half, $pV = N\tau$.

Entropy will go up. Because each particle has twice as many states afterwards then before $\Delta g = 2^N$ so that $\Delta\sigma = N \ln(2)$. An alternative solution would be to integrate the change in entropy for the isothermal reversible process with the same starting and ending points

$$\Delta\sigma = \int \frac{dQ}{\tau}$$

Because in an isothermal process the heat flows must be negative of the work done, $dQ = -pdV$. Using this and the ideal gas law:

$$\Delta\sigma = \int \frac{N\tau dV}{V\tau} = N \ln\left(\frac{V_f}{V_o}\right) = N \ln(2)$$

- b. For an ideal gas undergoing an isentropic $pV^\gamma = \text{const.}$. How to you prove this?

An isentropic process has no heat flow. A small move along the isentropic P-V curve can be accomplished by a small change in V while holding p constant followed by a small change in p while holding V constant. The heat flow of these two processes must add up to zero. The heat flows are characterized by

$$C_V = \left(\frac{dQ}{dT}\right)_V, \quad C_p = \left(\frac{dQ}{dT}\right)_p$$

using the ideal gas law $T = pV/N$ and noting what is held constant in each case

$$C_V = \frac{N}{V} \left(\frac{dQ}{dP}\right)_V, \quad C_p = \frac{N}{P} \left(\frac{dQ}{dV}\right)_p$$

Now set $dQ_p = -dQ_V$ so that the overall heat flow is zero

$$C_V \frac{V}{N} dP = -C_p \frac{P}{N} dV$$

or

$$\frac{1}{P} dP = -\frac{C_p}{C_V} \frac{1}{V} dV$$

integrating

$$\ln(P) = -\gamma \ln(V) + \text{const.}$$

which is what we were trying to prove.

6. Chemical potential range of bosons

- a. What is the chemical potential of photons and phonons? Why?

The chemical potential of photons and phonons is zero because you don't need to get them from anywhere, you can just create them with energy $\hbar\omega$.

- b. Why must $\mu < 0$ for massive bosons?

We typically define 0 energy to the ground state. The chemical potential can't be above the ground state because this would cause an infinite flow of particles into the ground state. There are not an infinite number of massive bosons in any reservoir.

7. Energy variations

- a. Suppose a block of material is in thermal equilibrium with its surroundings. Suppose that this block can be divided into small pieces that are equivalent to each-other. Use this to show how the variation in energy ΔE scales with the size of the system N .

By the central limit theorem. The variance of energy of the entire block is the sum of the variances of the pieces $(\Delta E)^2 = \sum_i (\Delta E_i)^2$. The number of pieces of a particular size scales with N the have that $\Delta E \propto \sqrt{N}$

- b. When can a similar argument be made for other variables: ΔN , ΔP , ΔV , ΔT .

All of these scale with \sqrt{N} provided that the material is uniform and that it is in an ensemble where it is allowed to vary.

8. $\tilde{n} + 1$ state system

- a. Suppose a single particle that has $\tilde{n} + 1$ possible states. These are a \tilde{n} fold degenerate excited states and a single ground state that has an energy ϵ lower. Suppose there are many $\lim_{N \rightarrow \infty}$ such distinguishable particles (they can be made distinguishable by being in different containers or located at different points in a lattice). Find the ratio of the number of particles in the ground state to the number not in the ground state. Assume they are in contact with a reservoir with finite temperature τ .

We use the Boltzmann factors keeping the degeneracy of the excited state in mind we express the ration of the number in the ground state N_0 to the number in the excited states $N - N_0$.

$$\frac{N_0}{N - N_0} = \frac{1}{\tilde{n} \exp(-\beta\epsilon)}$$

Which is some finite number. Because this is independent of N the fraction is unchanged when $N \rightarrow \infty$.

- b. Instead suppose we have N massive bosons that we are putting into a similar system with $\tilde{n} + 1$ states. Now multiple bosons can share the same quantum state. Find the ratio of the number of particles in the ground state to the number not in the ground state in terms of ϵ and μ .

Now we use the occupation number for bosons $f = 1/(\exp(\beta(\epsilon - \mu)) - 1)$ and take the ratio of the two states. By convention we define the ground state as having zero energy.

$$\frac{N_0}{N - N_0} = \frac{e^{\beta(\epsilon - \mu)} - 1}{\tilde{n} (e^{-\beta\mu} - 1)}$$

- c. If $N \gg \tilde{n}$ show that $\frac{-\tau}{\mu} > \approx \frac{N}{\tilde{n} + 1}$. (This shows what chemical potential results when I put in a large number of particles.)

There are an average of $N/(\tilde{n} + 1)$ particles per state. Because the ground state has a lower energy it has even more particles per state so $N_0 > \frac{N}{\tilde{n} + 1} \gg 1$. Which means

$$\frac{1}{e^{-\beta\mu} - 1} > \frac{N}{\tilde{n} + 1} \gg 1$$

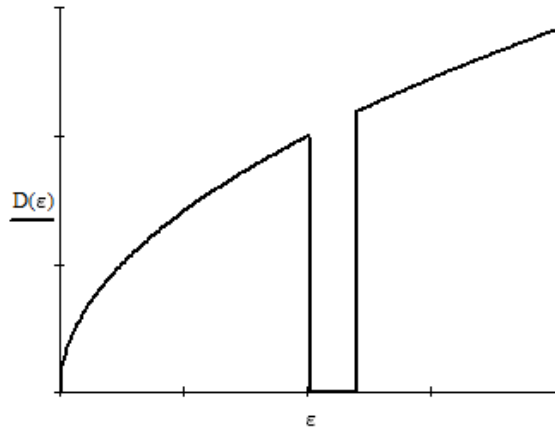
Because the exponential must be close to 1 we may Taylor expand it

$$\frac{1}{-\beta\mu} > \approx \frac{N}{\tilde{n} + 1} \gg 1$$

- d. Now what happens to this fraction in the limit $\lim_{N \rightarrow \infty}$. Why is this different that what you got in part a? In the limit that N gets large then $\exp(-\beta\mu) \rightarrow 1$ so the ratio $N_0/(N - N_0) \rightarrow \infty$. More specifically it will scale as N in the large N limit. This is quite different then the case in part a) because now we are working with bosons which interact quantum-mechanically with each-other.

9. Band Gap

Suppose you have a gas of electrons with a density of states which looks like



Also assume that $0 < \tau \ll \mu$.

- a. What will the heat capacity look like as a function of μ ? How will this change with the width of the gap?

At low temperatures the heat capacity is approximately proportional to the density of states. A plot of heat capacity vs μ will look like the plot of D except with a slight rounding of the corners. If the gap is small then temperature then the rounding of the corners will smooth out much of the gap.

- b. What will μ be if the number of electrons is slightly too large to fit below the gap?

The chemical potential will be inside the gap but near the upper end of the gap so that there are the tail of $f(\epsilon)$ is large enough above the gap to account for the particles which don't fit below the gap.