

Entropy - Definition: The entropy of a macro-state is the log of the number of micro-states in that macro-state (times k_B if you want it in historical units).

Inverse Temperature - Definition: The change in entropy for a change in energy of a system. $\Delta\sigma/\Delta E$

Density of States - Definition: The number of states between energy E and $E + \Delta E$ divided by ΔE . That is, binning micro-states into macro-state bins by their energy.

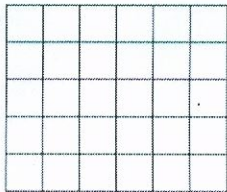
n	$\frac{20!}{n!(20-n)!}$	$\ln \binom{20}{n}$	$\Delta \ln (...)$
0	1	0.000	
1	20	2.996	2.996
2	190	5.247	2.251
3	1140	7.039	1.792
4	4845	8.486	1.447
5	15504	9.649	1.163
6	38760	10.565	0.916
7	77520	11.258	0.693
8	125970	11.744	0.486
9	167960	12.031	0.288
10	184756	12.127	0.095
11	167960	12.031	-0.095
12	125970	11.744	-0.288
13	77520	11.258	-0.486
14	38760	10.565	-0.693
15	15504	9.649	-0.916
16	4845	8.486	-1.163
17	1140	7.039	-1.447
18	190	5.247	-1.792
19	20	2.996	-2.251
20	1	0	-2.996



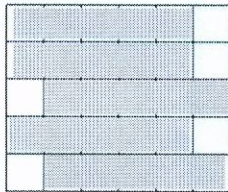
Entropy (not) as disorder

We often hear that entropy is disorder, this isn't always true. Suppose you are trying to arrange 5 identical tiles each of which is 1x5 grid spaces in size onto a 5x6 grid. How many ways are there to arrange these tiles in an "ordered" way where they are all parallel? How many ways are there to arrange them in a "disordered" way where they are not all parallel. Which of these two macro-states has more entropy?

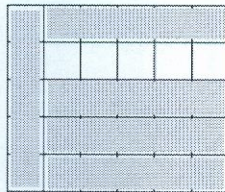
This is an extremely simplified example of a system of liquid crystals which are characterized by oblong molecules which tend to align because of their geometry.



Grid



Ordered



Disordered

Ordered

$$2^5 + 6 = 32 + 6 = 38$$

Disordered

$$2 \cdot 5 = 10$$

The ordered one has more entropy.

Temperature as an entropic price for energy

The second law of thermodynamics says that total entropy always increases. Energy tends to move from one system to another when doing so increases total entropy of the two systems.

As useful analogy to temperature is the price of various goods in two different countries. If a oil costs \$60/barrel in one country and \$40/barrel in another country, people will ship oil to the first country until the price equilibrates. Likewise, if system A can increase in entropy by 60 by gaining a unit of energy and system B only loses an entropy of 40 when it loses a unit of energy, then energy moves to system A.

1. Suppose we have a system of 20 spinners. Suppose further that for each spinner there is an energy ϵ for being up and 0 for being down. Suppose that the system starts with three spins in the up state 17 in the down state. If there is conservation of energy, how many micro-states are accessible? What is the entropy of this system. *The set of states with the same energy is referred to as the micro-canonical ensemble.*

$$g = \binom{20}{3} = 1140$$

$$\sigma = 7.039$$

2. How much would the entropy change if there were one more units ϵ of energy in the system?

$$1.447 \quad (\text{see table})$$

3. How much would the entropy change if there were one fewer units ϵ of energy?

$$-1.792 \quad (\text{see table})$$

4. Answer the same above three question for another identical system that starts with 6 units of energy.

$$g = 38760$$

$$\sigma = 10.565$$

$$\text{more } E \quad \Delta\sigma = 0.693$$

$$\text{less } E \quad \Delta\sigma = -0.916$$

5. If a unit of energy is to be transferred between the systems from 1. and 4., which direction must it go in order to increase entropy?

To the one with three spins because that will generate entropy

$$1.447 - 0.916 > 0$$

6. One of the systems lost entropy during the transfer. Why doesn't this violate the second law of thermodynamics which says that entropy always increases?

Total entropy must decrease, not the entropy of a particular system.

7. There is a non-zero change (you could calculate it if you wanted to using ratios of probabilities) that the combined system would go to a state of ^{higher} ~~lower~~ entropy. What does this say about the second law of thermodynamics?

It is only probably true for small systems.

For large systems it effectively always true.

8. What would a thermal reservoir be in the above economic example?

A large country that can buy and sell oil without the price changing.

9. How many units of energy would the first system most likely have if it were placed in contact with a system with inverse temperature $1/\tau = \beta = 1.1\epsilon^{-1}$

Five because $\Delta \ln\left(\frac{20}{5}\right) - \Delta \ln\left(\frac{20}{4}\right) = 1.163 \sim 1.1$

We will show that it isn't 6 in the next part.

10. For the first system at this temperature, what is the ratio of the probability of having 5 units of energy to probability of having 6 units of energy. That is $\frac{P(5)}{P(6)}$. (Hint: Remember that ratio of to micro-states goes as $e^{-\beta \Delta E}$. Now we are asking about macro-states.)

$$\begin{aligned} \frac{P(5)}{P(6)} &= \frac{e^{-\beta E(5)} g(5)}{e^{-\beta E(6)} g(6)} = e^{-\beta(E(5) - E(6)) + \sigma(5) - \sigma(6)} \\ &= e^{-\frac{1.1}{\epsilon}(-1\epsilon) - 0.916} = e^{0.184} > 1 \end{aligned}$$

11. What are some other aspects of the economic analogy that are shared by statistical mechanics? Where does the analogy break down?

Open Ended...

Density of states

The 3D harmonic oscillator. Consider a particle trapped in an isotropic three-dimensional harmonic potential. Its energy is given by

$$E(n_x, n_y, n_z) = (n_x + n_y + n_z) \hbar \omega$$

where we take the zero-point energy to be zero.

1. Calculate the entropy $\sigma(E)$ at a total energy E for:

(a) $E = 0$

(b) $E = \hbar \omega$

(c) $E = 2\hbar \omega$

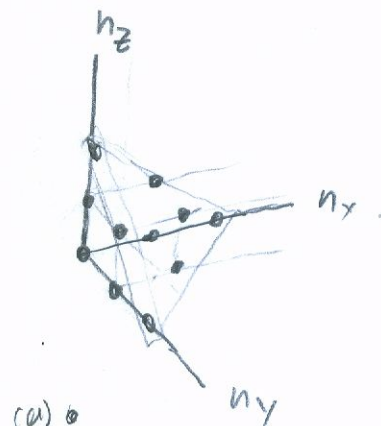
(d) $E = 3\hbar \omega$

(a) $g = 1$ $\sigma = \ln(1) = 0$

(b) $g = 3$ $\sigma = \ln(3)$

(c) $g = 6$ $\sigma = \ln(6)$

(d) $g = 10$ $\sigma = \ln(10)$



(a)

(b)

(c)

(d)

2. What is the multiplicity as a function of energy, $g(E)$? Write the answer as a function of E .

$$\begin{pmatrix} \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet \end{pmatrix} = \begin{pmatrix} \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet \end{pmatrix} \frac{1}{2} + \begin{pmatrix} \times & & \\ & \times & \\ & & \times \end{pmatrix} \frac{1}{2}$$

$$\frac{(n+1)^2}{2} + \frac{n+1}{2} = \frac{(n+1)(n+2)}{2}$$

$$g(E) = \frac{\left(\frac{E}{\hbar \omega} + 1\right) \left(\frac{E}{\hbar \omega} + 2\right)}{2}$$

3. What is the density of states $D(E) dE = \bar{g}(E)$?

$$g(E) = \frac{\left(\frac{E}{\hbar \omega} + 1\right) \left(\frac{E}{\hbar \omega} + 2\right)}{2 \hbar \omega}$$

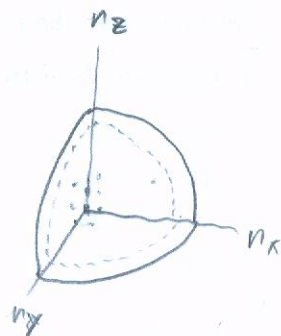
4. For an extreme relativistic gas particle in a box, the kinetic energy is given by

$$E = \hbar\omega (n_x^2 + n_y^2 + n_z^2)$$

What is the approximate density of states as a function of energy. (Hint: draw a picture).

First, let's find the density of states in n where $n^2 = n_x^2 + n_y^2 + n_z^2$. This can be approximated by the surface of a $\frac{1}{8}$ sphere shell.

$$DN = \frac{1}{8} \left(\frac{4\pi}{3} n^3 \right) \leftarrow \# \text{ inside sphere}$$



$$D(E) = \frac{dN}{dE} = \frac{dN}{dn} \frac{dn}{dE} = \frac{1}{8} (4\pi n^2) \left(\frac{1}{\sqrt{E\hbar\omega}} \right) = \frac{1}{8} \left(4\pi \frac{E}{\hbar\omega} \right) \left(\frac{1}{\sqrt{E\hbar\omega}} \right)$$

$$= \boxed{\frac{\pi E^{1/2}}{2(\hbar\omega)^{3/2}}}$$

Note $n = \sqrt{\frac{E}{\hbar\omega}}$ $\frac{dn}{dE} = \frac{1}{\sqrt{E\hbar\omega}}$

5. Write the approximate entropy as a function of energy? How about temperature as a function of energy?

$$\sigma(E) = \ln(D) + \text{const.}$$

$$= \frac{1}{2} \ln(E/\hbar\omega) + \text{const.}$$

\leftarrow small compared to large E

$$\frac{1}{T} = \frac{d\sigma}{dE} = \frac{1}{2} \frac{\hbar\omega}{E} + \text{const}$$

$$T \approx \frac{2E}{\hbar\omega}$$