Names:

EXERCISE 1A: RANDOM WALK

Objectives:

- Derive the binomial distribution describing the statistics of a random walk
- Calculate and visualize the mean and variance of the binomial distribution

Reference: Kittel & Kroemer, Ch. 1

1. A drunkard, initially standing under a lamppost, sets out on a random walk along a road, taking either a step to the right with probability p or a step to the left with probability $q = 1 - p$.

   a. After a total of N steps, what is the probability $P_N(X)$ to find the drunkard a distance of X steps to the right of the lamppost ($-N \leq X \leq N$)? Derive your answer based on the following considerations:

      i. How many different sequences of N steps are possible?

      ii. How many of these sequences include R steps to the right (and $N - R$ steps to the left)? Explain your reasoning.

iii. What is the probability of executing a specific one of the sequences in ii.?

iv. What is the probability $P_N(R)$ for the drunkard to take R steps to the right?

v. Verify that the distribution $P_N(R)$ is properly normalized.

vi. What is the probability $P_N(X)$ for the drunkard to end up a distance X from the lamppost? Write it out explicitly, including factorials.

b. Find the *mean* $\langle X \rangle$ and *standard deviation* ΔX of the drunkard's final position:

   i. Write down generic expressions for the mean $\langle Y \rangle$ and second moment $\langle Y^2 \rangle$ of a random variable Y described by a discrete probability distribution $P(Y)$.

   ii. How would you obtain the standard deviation ΔY from $\langle Y \rangle$ and $\langle Y^2 \rangle$?

   iii. For the case of the random walk, it will be convenient first to calculate the mean $\langle R \rangle$ and variance ΔR of the number of steps to the right. How are these quantities related to $\langle X \rangle$ and ΔX?

iv. Calculate the average number of steps $\langle R \rangle$ taken to the right using $P_N(R)$.

v. Calculate the standard deviation $\Delta R = \sqrt{\langle R^2 \rangle - \langle R \rangle^2}$ of the number of steps taken to the right.

vi. Based on iii.-v., determine $\langle X \rangle$ and ΔX. Check your results explicitly for the special cases $p = 0$, $p = 1/2$, $p = 1$. Do they make sense?

c. Compare the random walk of the human drunkard with a random walk undertaken by an ant, both tending towards the right with probability $p = 3/4$. The drunkard takes steps of length $\ell = 40$ cm, whereas the ant takes steps of length $\ell = 1$ mm. Each walks the same total distance $L = N\ell = 10$ m, but each with a different number of steps N. Sketch, on a single set of axes, probability distributions for the final positions $x = X\ell$ of the drunkard and the ant.

Two other commonly encountered probability distributions arise as limiting cases of the binomial distribution. The limit $p \ll 1$ yields the *Poisson distribution* (the "law of rare events"), which you will derive in the homework. We will next consider the large-N limit, where the size of each individual step is small (i.e., ant-like) compared to the total length of the walk. This limit will lead us to the *Gaussian distribution*, and to a profound theorem that explains why we encounter the Gaussian in so many different contexts.

For next time: think of other phenomena that can be modeled as random walks…