FINAL REVIEW SOLUTIONS

1. a. Assume there are n links pointing down, $N-n$ links pointing up, then The partition function is

$$
\begin{aligned}
Z = &\sum_{n=0}^{N} \binom{N}{n} e^{-\beta(-wl)(2n-N)} \\
= &e^{-\beta w N l}(1+e^{2\beta wl})^N \\
= &\left(e^{\beta wl} + e^{-\beta wl}\right)^N
\end{aligned}
$$

The free energy is

$$-\beta F = \log Z = N \log\left(e^{\beta wl} + e^{-\beta wl}\right)$$

Energy

$$E = -\frac{\partial}{\partial \beta} \log Z = -Nwl \tanh(\beta wl)$$

So the equilibrium length will be

$$L = -\frac{E}{w} = Nl \tanh(\beta wl)$$

b. From the expression of L, we see that as we heat up the band, the length will decrease, so the bucket will move up.
From an entropic point of view, as heat comes into the system, the entropy should increase, and we know that the entropy of the band increases when it gets shorter.

c. See Fig. 1 on next page.

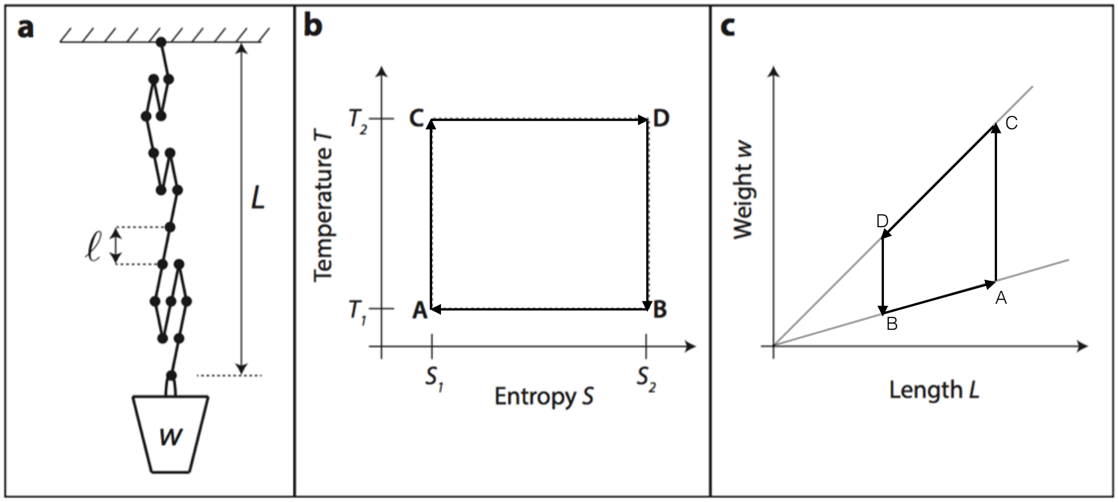

FIG. 1. Rubber band and Carnot cycle.

2. a. At zero temperature,

$$N = \frac{2A}{(2\pi\hbar)^2}\int_0^{p_F} dp 2\pi p = \frac{2A}{(2\pi\hbar)^2}2\pi m\int_0^{\epsilon_F} d\epsilon = \frac{A}{\pi\hbar^2}m\epsilon_F$$

So the Fermi energy is

$$\epsilon_F = \frac{n\pi\hbar^2}{m}$$

b. A spin-up particle with energy ϵ will have kinetic energy $\epsilon+\gamma B$, while a spin-down particle with energy ϵ will have kinetic energy $\epsilon - \gamma B$

$$N_+ = \frac{A}{2\pi\hbar^2}m(\epsilon_F + \gamma B)$$
$$N_- = \frac{A}{2\pi\hbar^2}m(\epsilon_F - \gamma B)$$

i. The magnetization is

$$M = \gamma(N_+ - N_-) = \frac{Am}{\pi\hbar^2}\gamma^2 B$$

ii. The magnetic susceptibility is

$$\chi = \frac{\partial M}{\partial B} = \frac{Am}{\pi\hbar^2}\gamma^2$$

c. At temperature T,

$$\begin{aligned}
N =& \frac{Am}{\pi\hbar^2}\int_0^\infty d\epsilon \frac{1}{e^{\beta(\epsilon-\mu)}+1} \\
=& \frac{Amk_BT}{\pi\hbar^2}\log\frac{e^{-\beta\mu}+1}{e^{-\beta\mu}} \\
=& \frac{Am\epsilon_F}{\pi\hbar^2} + \mathcal{O}(e^{-\frac{T_F}{T}}) \\
E =& \frac{Am}{\pi\hbar^2}\int_0^\infty d\epsilon \frac{\epsilon}{e^{\beta(\epsilon-\mu)}+1} \\
=& \frac{Am(k_BT)^2}{\pi\hbar^2}\left(\frac{\pi^2}{6}+\frac{1}{2}\left(\frac{T_F}{T}\right)^2 + \mathcal{O}(e^{-\frac{T_F}{T}})\right) \\
=& \frac{Am\epsilon_F^2}{2\pi\hbar^2} + \frac{\pi^2}{6}\frac{Am}{\pi\hbar^2}(k_BT)^2
\end{aligned}$$

The energy per particle is

$$\varepsilon = \frac{E}{N} = \frac{\epsilon_F}{2} + \frac{\pi^2}{6}\frac{(k_BT)^2}{k_BT_F}$$

The specific heat per particle is

$$C_V = \frac{\partial E}{\partial T} = k_B\frac{\pi^2}{3}\frac{T}{T_F}$$

For two dimensional ideal gas,

$$C_V = k_B$$

We see that the specific heat per electron is parametrically small by a factor of $\frac{T}{T_F}$.

4.10) HEAT CAPACITY OF INTERGALACTIC SPACE

For the atoms, $C_H/V = (3/2)nk_B$. For photons, we first re-express the proportionality factor in the expression (20) by inserting the Stefan-Boltzmann constant, $\sigma_B = \pi^2 k_B^4/60\hbar^3 c^2$:

$$U/V = 4\sigma_B T^4/c \quad , \quad C_{rad}/V = 16\sigma_B T^3/c \quad ;$$

$$C_H/C_{rad} = \frac{3ck_B}{32\sigma_B}\,\frac{N/V}{T^3} \cong 2.84\times10^{-10} \quad .$$

Comment. It usually simplifies numerical calculations if one expresses U/V in the form used here, involving σ_B/c and T, rather than the form (20) directly.

6.3) DISTRIBUTION FUNCTION FOR DOUBLE OCCUPANCY STATISTICS

(a)

$$\begin{array}{llccccc} \text{Occupancy:} & & 0 & & 1 & & 2 \\ \text{Energy} & : & 0 & & \varepsilon & & 2\varepsilon \\ \mathcal{Z}_{DO} & = & 1 & + & \lambda e^{-\varepsilon/\tau} & + & \lambda^2 e^{-2\varepsilon/\tau} \\ \langle N\rangle_{DO} & = \frac{1}{\mathcal{Z}}[& 0 & + & \lambda e^{-\varepsilon/\tau} & + & 2\lambda^2 e^{-2\varepsilon/\tau}] \quad . \end{array}$$

(b)

$$\begin{array}{llccccccc} \text{Occupancy:} & & 0,0 & & 0,1 & & 1,0 & & 1,1 \\ \text{Energy} & : & 0 & & \varepsilon & & \varepsilon & & 2\varepsilon \\ \mathcal{Z} & = & 1 & + & & 2\lambda e^{-\varepsilon/\tau} & & + & \lambda^2 e^{-2\varepsilon\tau} \\ & = & \multicolumn{7}{l}{[1+\lambda e^{-\varepsilon/\tau}]^2;} \\ \langle N\rangle & = \frac{1}{\mathcal{Z}}[& 0 & + & & 2\lambda e^{-\varepsilon/\tau} & & + & 2\lambda^2 e^{-2\varepsilon/\tau}] \quad . \end{array}$$

The last line may be written

$$\langle N\rangle \;=\; 2\lambda e^{-\varepsilon/\tau}\,\frac{1+\lambda e^{-\varepsilon/\tau}}{[1+\lambda e^{-\varepsilon/\tau}]^2} = \frac{2}{\exp[(\varepsilon-\mu)/\tau]+1} \quad .$$

The Gibbs sum is square of a single-orbital Gibbs sum, as one would expect for two independent single-orbital systems. See Problem 3.9 for comparison. The occupancy is just twice the Fermi-Dirac distribution function.

6.6) ENTROPY OF MIXING

For a single gas of N atoms with the concentration $n = N/V$, from (34):

$$\sigma = N[\log(n_Q/n) + 5/2] \quad .$$

For two distinguishable independent gases, each state of gas A may be combined with each state of gas B to generate a distinct state of the combined system. If g_A and g_B are the numbers of states accessible to each gas, $g = g_A g_B$ is the number of states accessible to the two-gas system, and the two-gas entropy is the sum of the one-gas entropies: $\sigma = \sigma_A + \sigma_B$.

This is true both before and after diffusive contact. Before contact, each of the two concentrations is $n = n_i = N/V$, hence

$$\sigma_i = N[\log(n_{QA}/n_i) + \log(n_{QB}/n_i) + 5] \quad ,$$

where n_{QA} and n_{QB} will in general be different. After diffusive equilibrium, $n = n_f = N/2V = n_i/2$; therefore

$$\begin{aligned} \sigma_f &= N[\log(n_{QA}/n_f) + \log(n_{QB}/n_f) + 5] \\ &= N[\log(2n_{QA}/n_i) + \log(2n_{QB}/n_i) + 5] \\ &= \sigma_i + 2N \log 2 \quad . \end{aligned}$$

The additional term $2N \log 2$ may be understood as follows. For every A-atom orbital in volume V_A there is an equivalent A-atom orbital in V_B, and for every B-atom orbital in V_B there is an equivalent B-atom orbital in V_A. States in which these equivalent orbitals are occupied are initially inaccessible, but they become accessible upon diffusive contact. Every accessible state of the two-gas system with diffusive contact may be viewed as having been generated from one of the initially accessible states by interchanging the occupancies of an arbitrary number (from 0 to 2N) of equivalent orbital pairs. For 2N occupied orbitals, there are 2^{2N} distinguishable combinations of interchanges, leading to 2^{2N} distinguishable accessible states for every initially accessible state. The entropy of mixing is the logarithm of this multiplicity of mixing: $\sigma_M = \log 2^{2N} = 2N \log 2$.

If the particles are indistinguishable, these occupancy interchanges do not lead to new distinguishable states but merely re-arrange the accessible states amongst each other. The two gases in diffusive contact are then simply a single gas of 2N atoms with the unchanged concentration $n_f = 2N/2V = n_i$, and with the entropy

$$\sigma_f = 2N[\log(n_Q/n_i) + 5/2] \quad .$$

For indistinguishable gases $n_{QA} = n_{QB}$; therefore $\sigma_f = \sigma_i$.

<u>Comment</u>. Students often ask whether particles can be "nearly" identical. The answer is: no! Particles differ either by a finite discrete amount or not at all. For example, the different isotopes of an element differ in the (discrete) number of neutrons in their nuclei.

7.2) ENERGY OF RELATIVISTIC FERMI GAS

(a) With the help of (6) -- $n_F = (3N/\pi)^{1/3}$, which remains valid -- we obtain, with $n = N/L^3$:

$$\varepsilon_F = p_F c = (\hbar\pi c/L)n_F = \hbar\pi c(3n/\pi)^{1/3}.$$

(b) With (9):

$$U(0) = 2\sum\varepsilon(n) = \pi\int_0^{n_F} n^2\varepsilon(n)dn = \pi(\hbar\pi c/L)\int_0^{n_F} n^3 dn$$

$$= \pi(\hbar\pi c/L)n_F^{\,4}/4 = \pi\varepsilon_F n_F^{\,3}/4 = (3N/4)\varepsilon_F \;.$$

7.10) RELATIVISTIC WHITE DWARF STARS

(a) We take over the result (88,89) for the energy of the extreme relativistic electron Fermi gas. If we write $n = N/V = 3N/4\pi R^3$, this result becomes

$$U_0 = \frac{3}{4}N\varepsilon_F = \frac{3}{4}\,N\hbar\pi c(9N/4\pi^2R^3)^{1/3}$$

$$= \frac{3}{4}\left(\frac{9\pi}{4}\right)^{1/3}\frac{\hbar c N^{4/3}}{R} \quad .$$

We equate this to the magnitude of the gravitational potential energy, assuming a uniform density:

$$\langle P.E.\rangle = \frac{3}{5}\,\frac{GM^2}{R} = \frac{3}{5}\,GM_H^2\,\frac{N^{6/3}}{R} \quad .$$

(See Problems 4.1 and 7.6). The radius cancels out and we obtain the relation for N

$$N^{2/3} = \frac{5}{4}\left(\frac{9\pi}{4}\right)^{1/3}\frac{\hbar c}{GM_H^2} \quad .$$

(b) $N = 8.4\times10^{57}$. If the Sun were all hydrogen: $N = M/M_H = 2\times10^{33}\,g/M_H = 1.3\times10^{57}$.