Problem Set 2 Solutions

Problem 1

a. Let n_i be a random variable that is equal to 1 if the ball switches urns on the ith time step and equal to 0 otherwise. Then

$$n = \sum_{i=1}^{M} n_i.$$

The n_i are a collection of independent random variables. We compute the expectation and variance of n from the expectation and variance of n_i:

$$\langle n \rangle = M \langle n_i \rangle = \frac{M}{N}$$

and

$$\Delta n^2 = M \Delta n_i^2 = M \left[1^2 \cdot \frac{1}{N} + 0^2 \cdot \left(1 - \frac{1}{N}\right) - \left(\frac{1}{N}\right)^2 \right] = \frac{M}{N^2}(N-1)$$

so

$$\Delta n = \sqrt{\frac{M(N-1)}{N^2}}.$$

b. As M/N approaches infinity, Δn diverges. This means that after $M \gg N$ steps, the ball is equally likely to have switched urns an odd or even number of times, i.e. it is equally likely to be located in urn A or urn B. Since this is true of all balls, each microstate is equally likely.

c. There are 2^{100} microstates, and by part (b), all are equally likely for a given $M \gg N$. There is only one microstate in which all of the balls are located in urn A, so we would expect it to occur once every 2^{100} steps. Since each step takes one millisecond, we would expect the initial configuration to be reached in $2^{100}/10^{-3}$ seconds or once every $4 \cdot 10^{19}$ years.

Problem 2

a. The only energy in the model is the gravitational potential energy of the weight. Defining l to be positive when the rubber band is extended downward, we have $E = -wl$.

b. We can write the length as $l = a(N_\uparrow - N_\downarrow)$, where $N_\uparrow$ is the number of segments parallel to the direction of gravity and $N_\downarrow$ is the number of anti-parallel segments. Defining $X \equiv N_\uparrow - N_\downarrow$, we have

$$g(X) = \frac{N!}{\left(\frac{N+X}{2}\right)!\left(\frac{N-X}{2}\right)!},$$

which is the same result as for the paramagnet. Now $E = -waX$, so

$$g(E) = \frac{N!}{\left(\frac{N-E/wa}{2}\right)!\left(\frac{N+E/wa}{2}\right)!}.$$

The entropy is

$$\sigma(E) = \ln g(E) = \ln N! - \ln\left(\frac{N-E/wa}{2}\right)! - \ln\left(\frac{N+E/wa}{2}\right)!.$$

c. In order to use Stirling's approximation, the argument of every factorial must be much larger than one, so we assume $N \gg 1$, $N - X \gg 1$, and $N + X \gg 1$. These three conditions imply that $-N \ll X \ll N$, i.e. the rubber band is not fully extended in either direction. With these assumptions, we have

$$\begin{aligned}\sigma(E) &\approx N\ln N - N - \left[\frac{N-E/wa}{2}\ln\frac{N-E/wa}{2} - \frac{N-E/wa}{2}\right] - \left[\frac{N+E/wa}{2}\ln\frac{N+E/wa}{2} - \frac{N+E/wa}{2}\right] \\ &= N\ln N - \frac{N-E/wa}{2}\ln\frac{N-E/wa}{2} - \frac{N+E/wa}{2}\ln\frac{N+E/wa}{2}.\end{aligned}$$

d. The temperature τ is given by

$$\frac{1}{\tau} = \frac{d\sigma}{dE} = \frac{1}{2wa}\ln\frac{N-E/wa}{N+E/wa}.$$

e. Solving (d) for $l = -E/w$, we have

$$l(\tau) = Na\left(\frac{1-e^{2wa/\tau}}{1+e^{2wa/\tau}}\right) = Na\tanh\frac{wa}{\tau}.$$

f. As $\tau \to 0$, $l(\tau) \to Na$, i.e. the rubber band is fully extended. That violates our assumption from (c) that $X \ll N$.

Problem 3

The energy of one spin with magnetic moment m in a magnetic field B is given by $U_i = \pm mB$, so for N spins with spin excess $X = 2s$, we have

$$U = -XmB = -2smB$$

and in particular

$$\langle U \rangle = -\langle X \rangle mB = -2 \langle s \rangle mB.$$

Note that in the canonical ensemble, i.e. when the system is in contact with a reservoir of fixed temperature, the energy of the system is not constant (it can exchange energy with the reservoir). This is why we consider the expectation values of U, X, and s in the above equation.

The entropy is then given by

$$\sigma(s) = \sigma_0 - \frac{2s^2}{N} = \sigma_0 - \frac{U^2}{2m^2B^2N}$$

and the relationship between temperature and the expected value of the energy by

$$\frac{1}{\tau} = \left.\frac{d\sigma}{dU}\right|_{U=\langle U \rangle} = -\frac{\langle U \rangle}{m^2B^2N}.$$

Substituting $\langle s \rangle$ and solving for the fractional magnetization, we have

$$\frac{2\langle s \rangle}{N} = \frac{mB}{\tau}.$$

a. The temperature of the paramagnet is negative when $\langle s \rangle < 0$, which means that more spins are aligned against the magnetic field than with the magnetic field. In this case, the paramagnet could gain entropy by losing energy (i.e. moving toward $\langle s \rangle = 0$), so $\frac{d\sigma}{dU}$ is negative.

b. When two systems are brought into thermal contact, energy will always flow in the direction that maximizes the total entropy. In this case, energy will flow from the paramagnet into the bath, since this will increase both the entropy of the paramagnet and the entropy of the bath (which implies that the total entropy will increase).

Problem 4

a. The multiplicity of the system of oscillators is, from (55),

$$g(n, N) = \frac{(N-1+n)!}{(n)!(N-1)!}.$$

The entropy is therefore

$$\sigma(n, N) = \ln g(N, n) = \ln(N-1+n)! - \ln(n)! - \ln(N-1)!$$

$$\approx \ln(N+n)! - \ln(n)! - \ln(N)!$$

where we have approximated $N-1$ by N in the arguments of the logarithms. Using Stirling's approximation, we have

$$\sigma \approx (N+n)\ln(N+n) - (N+n) - N\ln N + N - n\ln n + n$$
$$= (N+n)\ln(N+n) - N\ln N - n\ln n.$$

b. Substituting $U = n\hbar\omega$, we have

$$\sigma(U, N) = (N + \frac{U}{\hbar\omega})\ln(N + \frac{U}{\hbar\omega}) - N\ln N - \frac{U}{\hbar\omega}\ln\frac{U}{\hbar\omega}.$$

At temperature τ, the expectation value of the energy is given by

$$\frac{1}{\tau} = \left.\frac{d\sigma}{dU}\right|_{U=\langle U\rangle}$$

$$= \frac{1}{\hbar\omega}\ln(N + \frac{\langle U\rangle}{\hbar\omega}) + 1 - \frac{1}{\hbar\omega}\ln\frac{\langle U\rangle}{\hbar\omega} - 1$$
$$= \frac{1}{\hbar\omega}\ln\left(\frac{N\hbar\omega}{\langle U\rangle} + 1\right).$$

Solving for $\langle U\rangle$, we have

$$\langle U\rangle = \frac{N\hbar\omega}{e^{\frac{\hbar\omega}{\tau}} - 1}$$

as desired.

Problem 5

a. Applying the results of problem 3, we have

$$\frac{1}{\tau_1} = -\frac{\hat{E}_1}{N_1\mu_1^2 B^2}$$

and

$$\frac{1}{\tau_2} = -\frac{\hat{E}_2}{N_2\mu_2^2 B^2}$$

so in equilibrium, i.e. when $\tau_1 = \tau_2$, we have

$$\frac{\hat{E}_1}{N_1\mu_1^2} = \frac{\hat{E}_2}{N_2\mu_2^2}.$$

b. The total energy is $E = E_1 + E_2 = (b_1\mu_1 N_1 + b_2\mu_2 N_2)B$. Since energy is conserved, at equilibrium we have $E = \hat{E}_1 + \hat{E}_2$. Substituting for $\hat{E}_2$,

$$\hat{E}_1 = \frac{\mu_1^2 N_1}{\mu_1^2 N_1 + \mu_2^2 N_2} E = \frac{\mu_1^2 N_1}{\mu_1^2 N_1 + \mu_2^2 N_2}(b_1\mu_1 N_1 + b_2\mu_2 N_2)B.$$

c. The absorbed energy E_1^{abs} is the difference between the initial energy and the energy at equilibrium, so

$$E_1^{\text{abs}} = \hat{E}_1 - b_1\mu_1 N_1 B = \frac{\mu_1\mu_2 N_1 N_2 B(\mu_1 b_2 - \mu_2 b_1)}{\mu_1^2 N_1 + \mu_2^2 N_2}.$$

d. To find the variance ΔE_1^2, we write the multiplicity of the combined system as

$$g(E_1, E - E_1) = g_1(E_1)g_2(E - E_1).$$

Using the central limit theorem, we approximate each multiplicity function by a Gaussian distribution, so

$$g(E_1, E - E_1) \propto \exp\left(-\frac{E_1^2}{2\mu_1^2 B^2 N_1} - \frac{(E - E_1)^2}{2\mu_2^2 B^2 N_2}\right).$$

Now we need to rewrite the multiplicity in equilibrium as a Gaussian centered at $\hat{E}_1$. To do this, we must eliminate the total energy E. From part (b), we have

$$E = \frac{\mu_1^2 N_1 + \mu_2^2 N_2}{\mu_1^2 N_1}\hat{E}_1.$$

Substituting for E and factoring the argument of the exponential, we obtain

$$g(E_1, E - E_1) \propto \exp\left(-\frac{\mu_1^2 N_1 + \mu_2^2 N_2}{2\mu_1^2 N_1 \mu_2^2 N_2 B^2}\left(E_1^2 - 2E_1\hat{E}_1 + \hat{E}_1^2 + \hat{E}_1^2\frac{\mu_2^2 N_2}{\mu_1^2 N_1}\right)\right).$$

The fourth term is not a function of E_1, so it can be absorbed into the normalization constant. The remaining terms factor to produce

$$g(E_1, E - E_1) \propto \exp\left(-\frac{\mu_1^2 N_1 + \mu_2^2 N_2}{2\mu_1^2 N_1 \mu_2^2 N_2 B^2}\left(E_1 - \hat{E}_1\right)^2\right).$$

This is a Gaussian of the desired form

$$g \propto \exp\left(-\frac{1}{2\Delta E_1^2}\left(E_1 - \hat{E}_1\right)^2\right)$$

where we identify

$$\Delta E_1^2 = \frac{\mu_1^2 N_1 \mu_2^2 N_2 B^2}{\mu_1^2 N_1 + \mu_2^2 N_2}.$$

Note that this problem can also be solved by completing the square in the exponential, following the lecture notes.

e. When $N_2 \gg N_1$, we can Taylor expand ΔE_1 to zeroth order in the small parameter N_1/N_2 to obtain

$$\Delta E_1 = B\mu_1 \sqrt{\frac{\mu_2^2 N_1 N_2}{\mu_1^2 N_1 + \mu_2^2 N_2}} = B\mu_1 \sqrt{\frac{N_1}{1 + \left(\frac{\mu_1}{\mu_2}\right)^2 \frac{N_1}{N_2}}}$$

$$\approx B\mu_1 \sqrt{N_1}.$$

Similarly,

$$\hat{E}_1^2 = \frac{\mu_1^2 N_1}{\mu_1^2 N_1 + \mu_2^2 N_2}(b_1 \mu_1 N_1 + b_2 \mu_2 N_2) B \approx \frac{b_2 \mu_1^2 N_1 B}{\mu_2}.$$

Thus,

$$\frac{\Delta E_1}{\hat{E}_1} \approx \frac{\mu_2}{b_2 \mu_1 \sqrt{N_1}}.$$

Problem 6

a. Since $s = s_1 + s_2 = 0$ is constant, $s_1 = \hat{s}_1 + \delta$ requires that $s_2 = \hat{s}_2 - \delta$. Thus, we can use the Gaussian approximation of (17) to write

$$\frac{g_1 g_2}{(g_1 g_2)_{\max}} = \exp\left(-\frac{2\delta^2}{N_1} - \frac{2\delta^2}{N_2}\right).$$

Plugging in $N_1 = N_2 = 10^{22}$ and $\delta = 10^{11}$, we have

$$\frac{g_1 g_2}{(g_1 g_2)_{\max}} = \exp\left(-\frac{4 \cdot 10^{22}}{10^{22}}\right) = \frac{1}{e^4}.$$

b. First, note that since $N \equiv N_1 = N_2$ and $\hat{s}_1 = \hat{s}_2$,

$$\sum_{s_1} g_1(N_1, s_1) g_2(N_2, s - s_1) = g(2N, s) = g(2N, 2\hat{s}_1).$$

Using the Gaussian approximation,

$$g(2N, 2\hat{s}_1) = g(2N, 0) \exp\left(-\frac{2(2\hat{s}_1)^2}{2N}\right) = g(2N, 0) \exp\left(-\frac{4\hat{s}_1^2}{N}\right)$$

and since

$$(g_1 g_2)_{\max} = g_1(N, 0) \exp\left(-\frac{2\hat{s}_1^2}{N}\right) g_2(N, 0) \exp\left(-\frac{2\hat{s}_2^2}{N}\right) = g_1(N, 0) g_2(N, 0) \exp\left(-\frac{4\hat{s}_1^2}{N}\right)$$

we have

$$\frac{g(2N, 2\hat{s}_1)}{(g_1 g_2)_{\max}} = \frac{g(2N, 0)}{g_1(N, 0) g_2(N, 0)} = \frac{2^{2N}\sqrt{\frac{2}{2\pi N}}}{2^N \sqrt{\frac{2}{\pi N}}\, 2^N \sqrt{\frac{2}{\pi N}}} = \sqrt{\frac{\pi N}{4}}.$$

Since $N = 10^{22}$, this factor is about 10^{11}.

c. The fractional entropy error is

$$\frac{\Delta\sigma}{\sigma} = \frac{\ln g(2N, 2\hat{s}_1) - \ln(g_1 g_2)_{\max}}{\ln g(2N, 2\hat{s}_1)} \approx \frac{\ln 10^{11}}{2 \cdot 10^{22} \ln 2} \approx 10^{-21}.$$

Problem 7

a. Since all of the atoms have the same probability distribution and are (almost) independent, it is sufficient to calculate the temperature at which the probability that any *one* atom will align with the field is 75%. The probability that an atom is aligned with the field is given by

$$P(-\mu B) = \frac{e^{\mu B/kT}}{e^{\mu B/kT} + e^{-\mu B/kT}}$$

so the temperature as a function of the probability and magnetic field is

$$T = \frac{2\mu B}{k \ln \frac{P}{1-P}}.$$

Plugging in $P = 0.75$, $B = 10^4$ gauss, $\mu = h \cdot 1.4$ MHz/gauss, and $k = 1.38 \cdot 10^{-23}$ J/K, we have $T < 1.2$ K.

b. The same calculation with $\mu = h \cdot 2.1$ kHz/gauss yields $T < 1.8 \cdot 10^{-3}$ K.