

**EXERCISE 6C: BLACK BODY RADIATION AND DETAILED BALANCE***Objectives:*

- Finish deriving the radiant energy flux from the surface of a black body.
- Derive **Kirchhoff's Law** from the principle of detailed balance.
- Calculate the **Einstein coefficients** of a two-level system. (Supplemental)

*Useful results from last time:*

- Planck distribution for photon number  $s$  in a single mode:  $\langle s \rangle_\omega = \frac{1}{e^{\hbar\omega/\tau} - 1}$
- Planck spectrum:  $u_\omega = \frac{\hbar\omega^3}{\pi^2 c^3 (e^{\hbar\omega/\tau} - 1)}$
- Stefan-Boltzmann law:  $U/V = \frac{\pi^2}{15\hbar^3 c^3} \tau^4$
- Spectral density of radiant energy flux:  $j_\omega = cu_\omega/4$

1. Last time, we calculated the spectral density of radiant energy flux  $j_\omega$  from the surface of a black body.

- a. The total flux at the surface,  $J = \int j_\omega d\omega$ , is often expressed as

$$J = \sigma_B T^4, \quad (1)$$

where  $\sigma_B$  is called the **Stefan-Boltzmann constant**. What is the value of  $\sigma_B$ ?

$$\begin{aligned} J &= \int j_\omega d\omega = \frac{c}{4} \int u_\omega d\omega = \frac{c}{4} \cdot \frac{\pi^2}{15\hbar^3 c^3} \tau^4 \\ &= \frac{\pi^2}{60\hbar^3 c^2} (k_B T)^4 \\ \Rightarrow \sigma_B &= \frac{\pi^2 k_B^4}{60\hbar^3 c^2}. \end{aligned}$$

- b. In the homework, you will calculate the relationship between the temperature of the sun and the temperature of the earth. For now:

- i. How much higher is the radiant energy flux emitted from the surface of the sun ( $T \sim 6000$  K) than from the surface of the earth ( $T \sim 300$  K)?

$$\frac{J_{\text{sun}}}{J_{\text{earth}}} = \left( \frac{6000}{300} \right)^4 = 20^4 = 1.6 \times 10^5$$

Way more power!

- ii. What other factors affect how bright the sun and earth appear to an observer comparing the two from the same distance?

• Frequency sensitivity of detector (different spectra)  
 • Surface properties (Earth not a good black body).

2. *Kirchhoff's law*. The container in Fig. 1 is filled with hot water. Its left side is silvered on the outside, whereas its right side is blackened on the outside.

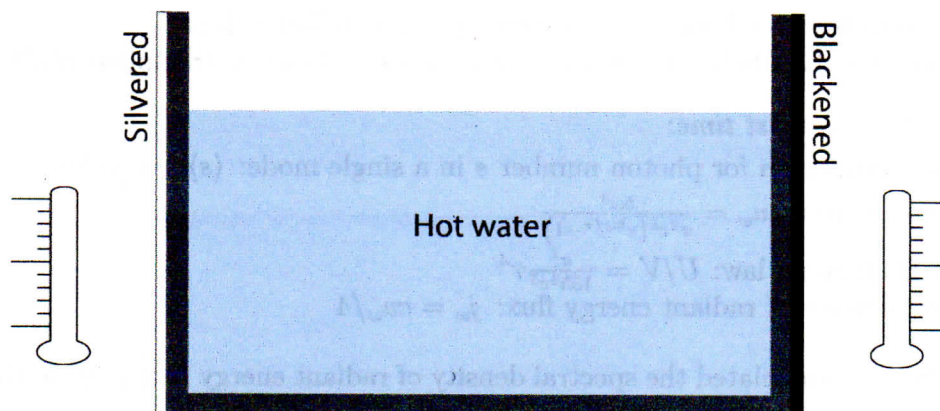


FIG. 1. Which thermometer reads a higher temperature?

- a. The principle of **detailed balance** states that in thermal equilibrium, every reversible microscopic process occurs at the same rate as its reverse process.

- i. What process is the time reverse of emitting a photon at frequency  $\omega$ ?

• Absorbing that photon



- ii. Suppose that at frequency  $\omega$ , the silvered wall has absorptivity  $a_s(\omega)$  and the blackened wall has absorptivity  $a_b(\omega)$ , with  $a_b(\omega) > a_s(\omega)$ . Write an expression for the emissivities  $e_s$  and  $e_b$  in terms of  $a_s$  and  $a_b$ .

$$e_s(\omega) = a_s(\omega) \quad \text{and} \quad e_b(\omega) = a_b(\omega).$$

- iii. For simplicity, assume  $e_s$  and  $e_b$  are constant as a function of frequency. Write an expression for the ratio of total power per unit area emitted by the silvered and blackened walls as a function of temperature in terms of  $a_s$  and  $a_b$ .

$$\frac{J_s}{J_b} = \frac{e_s \sigma_B T^4}{e_b \sigma_B T^4} = \frac{e_s}{e_b} = \frac{a_s}{a_b}.$$

b. We make the following assumptions:

- The tank of water is losing energy slowly enough that it is nearly in thermal equilibrium.
- The thermometers can exchange energy with the tank solely through the radiation field.
- The thermometers are nearly in thermal equilibrium with the radiation field.
- The two thermometers are equally far away from the tank.

With these assumptions, which thermometer reads a higher temperature? Why? Explain why each of the above assumptions was necessary.

• Silvered mirror:  $P_s = a_s \sigma_B T^4 \cdot A_{\text{wall}}$

• Blackened mirror:  $P_b = a_b \sigma_B T^4 \cdot A_{\text{wall}}$

• Left thermometer absorbs:  $P_s \cdot a_t \cdot g \quad ; \quad g \sim \frac{\Omega_t}{2\pi}$

• Right thermometer absorbs:  $P_b \cdot a_t \cdot g$

• Left thermometer emits:  $P_{st} = e_t \sigma_B T_{st}^4 \cdot A_t = a_t \sigma_B T_{st}^4 A_t$

• Right thermometer emits:  $P_{bt} = e_t \sigma_B T_{bt}^4 \cdot A_t = a_t \sigma_B T_{bt}^4 A_t$

• Equilibrium:  $P_{st} = P_s a_t g$  and  $P_{bt} = P_b a_t g$

$$\Rightarrow \frac{a_t \sigma_B T_{st}^4 A_t}{a_t \sigma_B T_{bt}^4 A_t} = \frac{a_s \sigma_B T^4 A_{\text{wall}}}{a_b \sigma_B T^4 A_{\text{wall}}} \Rightarrow \frac{T_{st}}{T_{bt}} = \left( \frac{a_s}{a_b} \right)^{1/4} < 1$$

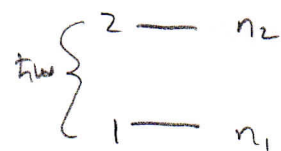
so  $T_{st} < T_{bt}$ .

c. How is your result consistent with the notion that if three systems are in thermal equilibrium with one another, they should all have the same temperature?

This system is not at thermal equilibrium.  
Energy is leaving the system.

3. *Einstein coefficients (supplemental)*. Consider a two-level atom with transition frequency  $\omega$  in thermal equilibrium with a black body radiation field at temperature  $\tau$ . Let  $n_1$  and  $n_2$  be the probability that the atom is in the ground and excited state, respectively.

- a. Spontaneous emission from the excited state to the ground state occurs at the rate  $A$ . Write an expression that relates  $A$  to  $n_1$ ,  $n_2$ , and their time derivatives.



$$\left(\frac{dn_1}{dt}\right)_{\text{spont}} = A n_2$$

- b. Stimulated absorption from the ground state to the excited state occurs at the rate  $B_{12}$  per unit spectral intensity  $u_\omega$ . Likewise, stimulated emission occurs at the rate  $B_{21}$  per unit spectral intensity. Write expressions that relate  $B_{12}$  and  $B_{21}$  to  $n_1$ ,  $n_2$ , and their time derivatives.

$$\left(\frac{dn_1}{dt}\right)_{\text{abs}} = -B_{12} n_1 u_\omega$$

$$\left(\frac{dn_1}{dt}\right)_{\text{stim}} = B_{21} n_2 u_\omega$$

- c. Using the results of parts (a) and (b) along with the principle of detailed balance, write an expression that relates  $A$ ,  $B_{12}$ , and  $B_{21}$ .

$$\frac{dn_1}{dt} = 0 = -B_{12} n_1 u_\omega + B_{21} n_2 u_\omega + A n_2.$$



- d. Simplify the expression from part (c) by using the equilibrium distributions of  $n_1$ ,  $n_2$ , and  $u_\omega$ .

$$[F \equiv \frac{\hbar \omega^3}{\pi^2 c^3}] \quad B_{12} \cdot \frac{1}{z} \cdot \frac{F}{e^{\hbar \omega / \tau} - 1} = B_{21} \cdot \frac{e^{-\hbar \omega / \tau}}{z} \cdot \frac{F}{e^{\hbar \omega / \tau} - 1} + A \cdot \frac{e^{-\hbar \omega / \tau}}{z}$$

$$\Rightarrow B_{12} \cdot F \cdot e^{\hbar \omega / \tau} = B_{21} \cdot F + A (e^{\hbar \omega / \tau} - 1).$$

- e. The equation in part (d) is true at all temperatures. By considering the  $\tau \rightarrow \infty$  and  $\tau \rightarrow 0$  limits, find two equations that relate  $A$ ,  $B_{12}$ , and  $B_{21}$ .

$$\tau \rightarrow \infty : e^{\hbar \omega / \tau} \rightarrow 1 ; \quad B_{12} F = B_{21} F$$

$$\Rightarrow B_{12} = B_{21} \equiv B.$$

$$\tau \rightarrow 0 : e^{\hbar \omega / \tau} \gg 1 ; \quad B \cdot F \cdot e^{\hbar \omega / \tau} = A e^{\hbar \omega / \tau}$$

$$\Rightarrow \frac{A}{B} = F = \frac{\hbar \omega^3}{\pi^2 c^3}.$$

- These coefficients are properties of the atoms, i.e. valid in all situations.
- To calculate them, it was sufficient to think about what happens in equilibrium.