Problem Set 1 Solutions

Problem 1

a.

$$(x+y)^N = \underbrace{(x+y)(x+y)...(x+y)}_{\text{N of them}}$$

There are product of N $(x+y)$'s. From each of them, we can choose either x or y. The coefficient of $x^n y^{N-n}$ is the number of ways of choosing n x's and $N-n$ y's. This is equivalent to counting the number of different ways for a coin to land heads n times and tails $N-n$ times in N tosses.

b.

$$\left(\frac{\partial}{\partial x}\right)^n (x+y)^N = N(N-1)...(N-n+1)(x+y)^{N-n}$$

So

$$\left(\frac{\partial}{\partial x}\right)^n (x+y)^N|_{x=0} = N(N-1)...(N-n+1)y^{N-n}$$

On the other hand,

$$\left(\frac{\partial}{\partial x}\right)^n (x+y)^N|_{x=0} = \sum_{i=0}^{N} a_i \left(\frac{\partial}{\partial x}\right)^n x^i y^{N-i} = a_n n! y^{N-n}$$

So

$$\begin{aligned} a_n n! &= N(N-1)...(N-n+1) = \frac{N!}{(N-n)!} \\ a_n &= \frac{N!}{n!(N-n)!} = \binom{N}{n} \end{aligned}$$

Problem 2

a.

$$\left(\frac{5}{6}\right)^N$$

b. The probability of being shot on the N^{th} trial is the probability of surviving $N-1$ multiplied by the probability of shooting oneself.

$$\left(\frac{5}{6}\right)^{N-1}\left(\frac{1}{6}\right)$$

c. We can use the derivative of the geometric series

$$\begin{aligned}
\sum_{N=1}^{\infty} N\left(\frac{5}{6}\right)^{N-1}\left(\frac{1}{6}\right) &= (1-x)\,\frac{d}{dx}\sum_{N=1}^{\infty} x^N\Bigg|_{x=\frac{5}{6}} \\
&= (1-x)\,\frac{d}{dx}\frac{1}{1-x}\Bigg|_{x=\frac{5}{6}} \\
&= \frac{1}{1-x}\Bigg|_{x=\frac{5}{6}} \\
&= 6
\end{aligned}$$

Problem 3

a. Recall that

$$P_N(n) = \frac{N!}{n!\,(N-n)!} p^n (1-p)^{N-n}$$

When $p \ll 1$ the we have $P \underset{\sim}{\propto} p^n$ which is much much smaller for large n then for small n. Meaning that n is almost always small. Typically we are interested in things that actually happen.
Note that this is only the case when N is large which it will be for most problems we are interested in in statistical mechanics.

b. Taking the natural log of both sides [In general it is not OK to take the natural logarithm of both sides of a $\approx$ sign. For example, one billion dollars is not approximately equal to two billion dollars yet their logarithms are. However, if A and B are less then one than $\ln(A) \approx \ln(B)$ is a more restrictive criterion and implies $A \approx B$. Because both sides are clearly less then one we are OK.]

$$\ln\left(\left(1 - p^{(N-n)}\right)\right) \overset{?}{\approx} \ln\left(e^{-Np}\right)$$

$$(N-n)\ln(1-p) \overset{?}{\approx} -Np$$

by assumption $n \ll N$ and $n \ll N$

$$N(-p) \approx -Np$$

c. Now we will make stretch our definition of approximately equal to even more.

$$\frac{N!}{(N-n)!} \overset{?}{\approx} N^n$$

In this part of the problem what we mean by approximately equal to is that the logarithms of both sides are approximately equal to [otherwise the statement is not true].

$$\begin{aligned}
\ln\left(\frac{N!}{(N-n)!}\right) &= \ln\left(N(N-1)\ldots(N-n+1)\right) \\
&= \underbrace{\ln(N) + \ln(N-1) + \ldots + \ln(N-n+1)}_{n \text{ terms}} \\
&= n\ln(N) + \ln(1) + \ln\left(1 - \frac{1}{N}\right) + \ldots + \ln\left(1 - \frac{n-1}{N}\right) \\
&\approx n\ln(N) - \frac{1}{N} - \ldots - \frac{n-1}{N} \\
&\approx n\ln(N) \\
&\approx \ln(N^n)
\end{aligned}$$

Which, as I said earlier, will be considered good enough to consider the statement shown.

d.

$$P_N(n) = \frac{N!}{n!\,(N-n)!} p^n (1-p)^{N-n}$$

and using $N!/(N-n)! \approx N^n$ and $(1-p)^{N-n} \approx e^{-Np}$ we have

$$P_N(n) \approx \frac{1}{n!}(Np)^n e^{-Np}$$

and we recognize $\lambda = Np$.

e. There are a lot of examples from photon counting by a detector to number of people in your neighborhood that are selected for jury duty.

Problem 4

a. Because the approximation will become exact when $N \gg 1$ we take the difference between the value at $N = \infty$

$$\begin{aligned}
\sum_{n=0}^{N} \frac{1}{n!} (Np)^n e^{-Np} &= \sum_{n=0}^{\infty} \frac{1}{n!} (Np)^n e^{-Np} - \sum_{n=N+1}^{\infty} \frac{1}{n!} (Np)^n e^{-Np} \\
&= e^{-Np} \left(\sum_{n=0}^{\infty} \frac{1}{n!} (Np)^n - \sum_{n=N+1}^{\infty} \frac{1}{n!} (Np)^n \right)
\end{aligned}$$

now we recognize the Taylor series expansion for the exponential.

$$\sum_{n=0}^{N} \frac{1}{n!} (Np)^n e^{-Np} = e^{-Np} \left(e^{Np} - \sum_{n=N+1}^{\infty} \frac{1}{n!} (Np)^n \right)$$

The second term is latter terms of the Taylor expansion which is small as long $\frac{e^{-Np}}{(N+1)!} (Np)^{N+1} \ll 1$ which is will for large N or small Np.

$$\sum_{n=0}^{N} \frac{1}{n!} (Np)^n e^{-Np} \approx 1$$

b.

$$\begin{aligned}
\langle n \rangle &= \sum_{n=0}^{\infty} n \frac{\lambda^n}{n!} e^{-\lambda} \\
&= \lambda e^{-\lambda} \frac{\partial}{\partial \lambda} \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} \\
&= \lambda e^{-\lambda} \frac{\partial}{\partial \lambda} e^{\lambda} \\
&= \lambda
\end{aligned}$$

alternative derivation:

$$\begin{aligned}
\langle n \rangle &= \sum_{n=0}^{\infty} n \frac{\lambda^n}{n!} e^{-\lambda} \\
&= \sum_{n=1}^{\infty} \frac{\lambda^n}{(n-1)!} e^{-\lambda} \\
&= e^{-\lambda} \lambda \sum_{n=1}^{\infty} \frac{\lambda^{n-1}}{(n-1)!} \\
&= e^{-\lambda} \lambda \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} \\
&= e^{-\lambda} \lambda e^{\lambda} \\
&= \lambda
\end{aligned}$$

c. First lets calculate the second moment

$$
\begin{aligned}
\left\langle n^2 \right\rangle &= \sum_{n=0}^{\infty} n^2 \frac{\lambda^n}{n!} e^{-\lambda} \\
&= e^{-\lambda} \lambda \frac{\partial}{\partial \lambda} \lambda \frac{\partial}{\partial \lambda} \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} \\
&= e^{-\lambda} \lambda \frac{\partial}{\partial \lambda} \lambda \frac{\partial}{\partial \lambda} e^{\lambda} \\
&= e^{-\lambda} \lambda \frac{\partial}{\partial \lambda} \lambda e^{\lambda} \\
&= e^{-\lambda} \lambda \left(e^{\lambda} + \lambda e^{\lambda} \right) \\
&= \lambda + \lambda^2
\end{aligned}
$$

this moment can be used to calculate the variance

$$
\begin{aligned}
\Delta n^2 &= \left\langle (n - \langle n \rangle)^2 \right\rangle \\
&= \left\langle n^2 - 2n \langle n \rangle + \langle n \rangle^2 \right\rangle \\
&= \left\langle n^2 \right\rangle - \langle n \rangle^2 \\
&= \lambda^2 + \lambda - \lambda^2 \\
&= \lambda
\end{aligned}
$$

Problem 5

a. Assuming each position is equally likely

$$\langle N\rangle = N_0 \frac{V}{V_0}$$

b. This should obey binomial statistics which each molecule is or isn't in V.

$$P_{N_0}(N) = \frac{N_0!}{N!\,(N_0 - N)!}\left(\frac{V}{V_0}\right)^N \left(1 - \frac{V}{V_0}\right)^{N-N_0}$$

We need to calculate the fluctuations $\Delta N = \sqrt{\langle N^2\rangle - \langle N\rangle^2}$ of this distribution. To do this we calculate $\langle N^2\rangle$

$$\langle N^2\rangle = \sum_{N=0}^{\infty} N^2 P_{N_0}(N)$$

We will calculate the variance the same way we did in class. Let $p = V/V_0$ and $q = 1 - V/V_0$

$$\begin{aligned}\sum_{N=0}^{\infty} P_{N_0}(N) &= \sum_{N=0}^{\infty} \frac{N_0!}{N!\,(N_0 - N)!} p^N q^{N-N_0} \\ &= (p+q)^{N_0}\end{aligned}$$

likewise

$$\begin{aligned}\sum_{N=0}^{\infty} N^2 P_{N_0}(N) &= p\frac{\partial}{\partial p} p\frac{\partial}{\partial p} \sum_{N=0}^{\infty} \frac{N_0!}{N!\,(N_0 - N)!} p^N q^{N-N_0} \\ &= p\frac{\partial}{\partial p} p\frac{\partial}{\partial p}(p+q)^{N_0} \\ &= p\frac{\partial}{\partial p} pN_0 (p+q)^{N_0-1} \\ &= p\left(N_0(N_0-1)\,p\,(p+q)^{N_0-2} + N_0(p+q)^{N_0-1}\right) \\ &= pN_0(N_0 p + q)(p+q)^{N_0-2} \\ &= \frac{V}{V_0} N_0 \left(N_0 \frac{V}{V_0} + 1 - \frac{V}{V_0}\right)(1)^{N_0-2} \\ &= \langle N\rangle \left(\langle N\rangle + 1 - \frac{V}{V_0}\right)\end{aligned}$$

Now we can calculate the fractional variation

$$\frac{\Delta N}{\langle N\rangle} = \frac{\sqrt{\langle N\rangle\left(\langle N\rangle + 1 - \frac{V}{V_0}\right) - \langle N\rangle^2}}{\langle N\rangle}$$

$$\boxed{\frac{\Delta N}{\langle N\rangle} = \sqrt{\frac{1}{\langle N\rangle}\left(1 - \frac{V}{V_0}\right)}}$$

c. When $V \ll V_0$ this simplifies to $1/\sqrt{\langle N\rangle}$. Which is what we would have from Poisson's statistics.

d. When $V \to V_0$ this quantity goes to zero because all of the particles are inside the volume so there is no variance.

Problem 6

Solution Option 1: From Binomial distribution

What is meant by $0 \ll V/V_0 \ll 1$ is that $1 \ll N_0 V$ and $V \ll V_0$. Because $1 \ll N_0 V$ there is usually a large number, N, of particles and because $V \ll V_0$ we have $N \ll N_0$. These where exactly the conditions needed in problem 3 for the Poisson distribution to apply.

$$P(N)\,dN \approx \frac{\langle N\rangle^N}{N!} e^{-\langle N\rangle} dN$$

To simplify we take the natural log

$$\ln(P(N)) \approx N\ln(\langle N\rangle) - \ln(N!) - \langle N\rangle$$

Because $1 \ll N$ in almost all cases that we care about we can use Stirling's approximation

$$\ln(P(N)) \approx N\ln(\langle N\rangle) - N\ln(N) + N - \frac{1}{2}\ln(2\pi N) - \langle N\rangle$$

$$\ln(P(N)) \approx N\ln\left(\frac{\langle N\rangle}{N}\right) + N - \frac{1}{2}\ln(2\pi N) - \langle N\rangle$$

Now $N \approx \langle N\rangle$ for large N because we have already shown that relative variation decreases with N. So we can Taylor expansion $\ln(1+x) \approx x - \frac{1}{2}x^2$ where $x = \frac{\langle N\rangle}{N} - 1$.

$$\begin{aligned}\ln\left(\frac{\langle N\rangle}{N}\right) &= \ln\left(1 + \left(\frac{\langle N\rangle}{N} - 1\right)\right) \\ &\approx \left(\frac{\langle N\rangle}{N} - 1 + \frac{1}{2}\left(\frac{\langle N\rangle}{N} - 1\right)^2\right)\end{aligned}$$

plugging this in above we have

$$\ln(P(N)) \approx N\left(\frac{\langle N\rangle}{N} - 1 - \frac{1}{2}\left(\frac{\langle N\rangle}{N} - 1\right)^2\right) + N - \frac{1}{2}\ln(2\pi N) - \langle N\rangle$$

$$\ln(P(N)) \approx -\frac{1}{2N}(\langle N\rangle - N)^2 - \frac{1}{2}\ln(2\pi N)$$

because $N \approx \langle N\rangle$ most of the time

$$\ln(P(N)) \approx -\frac{1}{2\langle N\rangle}(\langle N\rangle - N)^2 - \frac{1}{2}\ln(2\pi\langle N\rangle)$$

$$\boxed{P(N)\,dN = \frac{1}{\sqrt{2\pi\langle N\rangle}}\exp\left(-\frac{(\langle N\rangle - N)^2}{2\langle N\rangle}\right)dN}$$

Solution Option 2: Using central limit theorem

Alternatively we could have just appealed to the central limit theorem proven in the next problem. The central limit theorem applies because we have many independent random samples: whether each particle is or isn't in the volume.

$$x_i = \begin{cases} 1 & \text{with probability } \frac{V}{V_0} \\ 0 & \text{otherwise} \end{cases}$$

The average for a single draw particle is

$$\langle x_i\rangle = \frac{V}{V_0}$$

To get the variance we need the second moment

$$\left\langle x_i^2 \right\rangle = 1^2 \cdot \left(\frac{V}{V_0}\right) + 0^2 \cdot \left(1 - \frac{V}{V_0}\right)$$

so the variance is

$$\begin{aligned}
(\Delta x)^2 &= \left\langle x^2 \right\rangle - \langle x \rangle^2 \\
&= \left(\frac{V}{V_0}\right) - \left(\frac{V}{V_0}\right)^2
\end{aligned}$$

By the central limit theorem the distribution of the sum $N = \sum_{i=1}^{N_0} x_i$ is a Gaussian with mean and variance N_0 times as big

$$\langle N \rangle \approx N_0 \frac{V}{V_0}$$

$$\begin{aligned}
(\Delta N)^2 &\approx N_0 \left(\frac{V}{V_0}\right)\left(1 - \frac{V}{V_0}\right) \\
&\approx \langle N \rangle \left(1 - \frac{\langle N \rangle}{N_0}\right) \\
&\approx \langle N \rangle
\end{aligned}$$

where in the last line we use the assumption that $V/V_0 \ll 1$. Putting this into the Gaussian distribution we have

$$\boxed{P(N)\, dN = \frac{1}{\sqrt{2\pi \langle N \rangle}} \exp\left(-\frac{(\langle N \rangle - N)^2}{2 \langle N \rangle}\right) dN}$$

Problem 7

a. The delta function clicks when the sum adds up to 1. This effectively counts the ways of adding up the x_i times their respective probabilities.

b.

$$P(Y) = \int dx_1 ... \int dx_N \delta\left(Y - \sum_{i=1}^{N} x_i\right) \prod_{i=1}^{N} p(x_i)$$

using $2\pi\delta(u) = \int_{-\infty}^{\infty} dk e^{-iku}$

$$P(Y) = \frac{1}{2\pi} \int dx_1 ... \int dx_N \int_{-\infty}^{\infty} dk \exp\left[-ik\left(Y - \sum_{i=1}^{N} x_i\right)\right] \prod_{i=1}^{N} p(x_i)$$

Now we insert the reverse forier transform

$$p(x_i) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk_i e^{-ik_i x_i} Q(k_i)$$

$$P(Y) = (2\pi)^{-N-1} \int dx_1 ... \int dx_N \int_{-\infty}^{\infty} dk \int dk_1 ... \int dk_N \exp\left[-ik\left(Y - \sum_{i=1}^{N} x_i\right) - ik_i x_i\right] \prod_{i=1}^{N} Q(k_i)$$

Now we use

$$\int_{-\infty}^{\infty} dx_i e^{-ix_i(kY+k_i)} = 2\pi\delta(kY + k_i)$$

$$P(Y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk \int_{-\infty}^{\infty} dk_i \exp[-ikY] \prod_{i=1}^{N} \delta(kY + k_i) Q(k_i)$$

$$P(Y) = \frac{1}{2\pi} \int dk_1 ... \int dk_N \exp[-ikY] Q^N(-Yk)$$

c. Because we are interested in small k only we use can use the Taylor expansion $e^{ikx} \approx 1 + ikx - \frac{1}{2}k^2x^2$ in the definition of $Q(k)$.

$$\begin{aligned} Q(k) &= \int_{-\infty}^{\infty} dx \left(1 + ikx - \frac{1}{2}k^2x^2\right) p(x) \\ &= 1 + ik\langle x\rangle - \frac{1}{2}k^2\langle x^2\rangle \end{aligned}$$

d. We would like to approximate $Q^N(k)$

$$\ln\left(Q^N(k)\right) \approx N \ln\left(1 + ik\langle x\rangle - \frac{1}{2}k^2\langle x^2\rangle\right)$$

using $\ln(1+x) \approx x - \frac{1}{2}x^2$

$$\ln\left(Q^N(k)\right) = N\left(ik\langle x\rangle - k^2\frac{1}{2}\left(\langle x^2\rangle - \langle x\rangle^2\right) + O\left(k^3x^3\right)\right)$$

$$Q^N(k) \approx \exp\left[N\left(ik\langle x\rangle - k^2\frac{1}{2}\left(\langle x^2\rangle - \langle x\rangle^2\right)\right)\right]$$

e.

$$P(Y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk \exp\left[-ikY\right]\exp\left[N\left(ik\langle x\rangle - k^2\frac{1}{2}\left(\langle x^2\rangle - \langle x\rangle^2\right)\right)\right]$$

$$P(Y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk \exp\left[-\frac{N}{2}\left(\langle x^2\rangle - \langle x\rangle^2\right)k^2 + \left(iN\langle x\rangle - iY\right)k\right]$$

using the Gaussian integral $\int dx e^{-ax^2+bx+c} = \sqrt{\frac{\pi}{a}}e^{\frac{b^2}{4a}+c}$

$$P(Y) = \frac{1}{2\pi}\sqrt{\frac{\pi}{\frac{N}{2}\left(\langle x^2\rangle - \langle x\rangle^2\right)}}\exp\left(-\frac{\left(N\langle x\rangle - Y\right)^2}{4\frac{N}{2}\left(\langle x^2\rangle - \langle x\rangle^2\right)}\right)$$

$$P(Y) = \frac{1}{\sqrt{2\pi N\left(\langle x^2\rangle - \langle x\rangle^2\right)}}\exp\left(-\frac{\left(N\langle x\rangle - Y\right)^2}{2N\left(\langle x^2\rangle - \langle x\rangle^2\right)}\right)$$

f. To calculate $\langle Y\rangle$ we need to do the integral $\langle Y\rangle = \int YP(Y)\,dY$. This is made simpler if we make the substitution $Y' = Y - N\langle x\rangle$. The integral because

$$\langle Y\rangle = N\langle x\rangle \int P(Y)\,dY + \frac{1}{\sqrt{\cdots}}\int Y'\exp\left(\frac{Y'^2}{\cdots}\right)dY'$$

The second integral is an integral of an odd function times an even one so it is zero. The first integral is one.

$$\boxed{\langle Y\rangle = N\langle x\rangle}$$

Now we need to calculate$\Delta Y = \sqrt{\langle Y^2\rangle - \langle Y\rangle^2}$. For simplicity let $\alpha \equiv \frac{1}{2N\left(\langle x^2\rangle - \langle x\rangle^2\right)}$

$$\int Y^2 P(Y)\,dY = \sqrt{\frac{\alpha}{\pi}}\int Y^2 e^{-\alpha Y^2}dY$$

By integrating by parts where $\frac{\partial}{\partial Y}Y = 1$ and $\int Ye^{-\alpha Y^2}dY = \frac{-1}{2\alpha}e^{-\alpha Y^2}$ we have

$$\begin{aligned}\int Y^2 P(Y)\,dY &= \sqrt{\frac{1}{4\alpha\pi}}\int e^{-\alpha Y^2}dY \\ &= \frac{1}{2\alpha}\end{aligned}$$

Putting the definition of α back into the formula for ΔY we have

$$\Delta Y = \sqrt{N\left(\langle x^2\rangle - \frac{1}{2}\langle x\rangle^2\right)}$$

$$\boxed{\Delta Y = \sqrt{N\Delta x^2}}$$

Which makes the distribution

$$P(Y) = \frac{1}{\sqrt{2\pi\left(\Delta Y\right)^2}}\exp\left(-\frac{\left(Y - \langle Y\rangle\right)^2}{2\left(\Delta Y\right)^2}\right)$$

g. With the original distribution

$$p(x_i) = \begin{cases} 1 & \text{if } 0 \le x_i \le 1 \\ 0 & \text{otherwise} \end{cases}$$

i.

$$\begin{aligned} |Q_i(k)| &= \left| \int_0^1 dx e^{ikx} \right| \\ &= \left| \frac{e^{ik} - 1}{k} \right| \end{aligned}$$

Note that

$$\lim_{k \to 0} \frac{e^{ik} - 1}{k} = 1$$

To see how wide this is either plot it or see then next section.

iii.

$$\begin{aligned} \left|Q_i^N(k)\right| &= \left| \left(\int_0^1 dx e^{ikx} \right)^N \right| \\ &= \left| \frac{\left(e^{ik} - 1\right)^N}{k^N} \right| \end{aligned}$$

Now I'd like to find how wide this function is. To do this I use some approximations. Q^N will only be non-zero when $k \ll 1$ so

$$\begin{aligned} \left|Q_i^N(k)\right| &\approx \left| \frac{\left(ik - \frac{1}{2}k^2 ...\right)^N}{k^N} \right| \\ &\approx \left| \frac{k^N - N\frac{1}{2}k^{N+1}}{k^N} \right| \\ &\approx 1 - \frac{1}{2}Nk \end{aligned}$$

In order to get to half height we set $\left|Q^N\right| = \frac{1}{2}$ so that

$$\left| \frac{e^{ik} - 1}{k} \right| = \left(\frac{1}{2}\right)^{1/N}$$

$$\left| \frac{ik - \frac{1}{2}k^2 - i\frac{1}{6}k^3 ..}{k} \right| = \left(\frac{1}{2}\right)^{1/N}$$

$$\sqrt{\left(i - \frac{1}{2}k - i\frac{1}{6}k^2 + O\left(k^3\right)\right)\left(-i - \frac{1}{2}k + i\frac{1}{6}k^2 + O\left(k^3\right)\right)} = \left(\frac{1}{2}\right)^{1/N}$$

$$\sqrt{1 + \frac{1}{4}k^2 - \frac{2}{6}k^2 + O\left(k^3\right)} = \left(\frac{1}{2}\right)^{1/N}$$

$$1 - \frac{1}{12}k^2 + O\left(k^3\right) = \left(\frac{1}{2}\right)^{2/N}$$

$$\ln\left(1-\frac{1}{12}k^2\right)\approx\frac{2}{N}\ln\left(\frac{1}{2}\right)$$

using $\ln(1+x)=x-\frac{1}{2}x^2 ...$

$$-\frac{1}{12}k^2=\frac{2}{N}\ln\left(\frac{1}{2}\right)$$

$$k_{1/2}\approx\frac{1}{\sqrt{N}}\sqrt{24}\ln(2)$$

$$k_{1/2}\approx\frac{1}{\sqrt{N}}\sqrt{24}\ln(2)$$

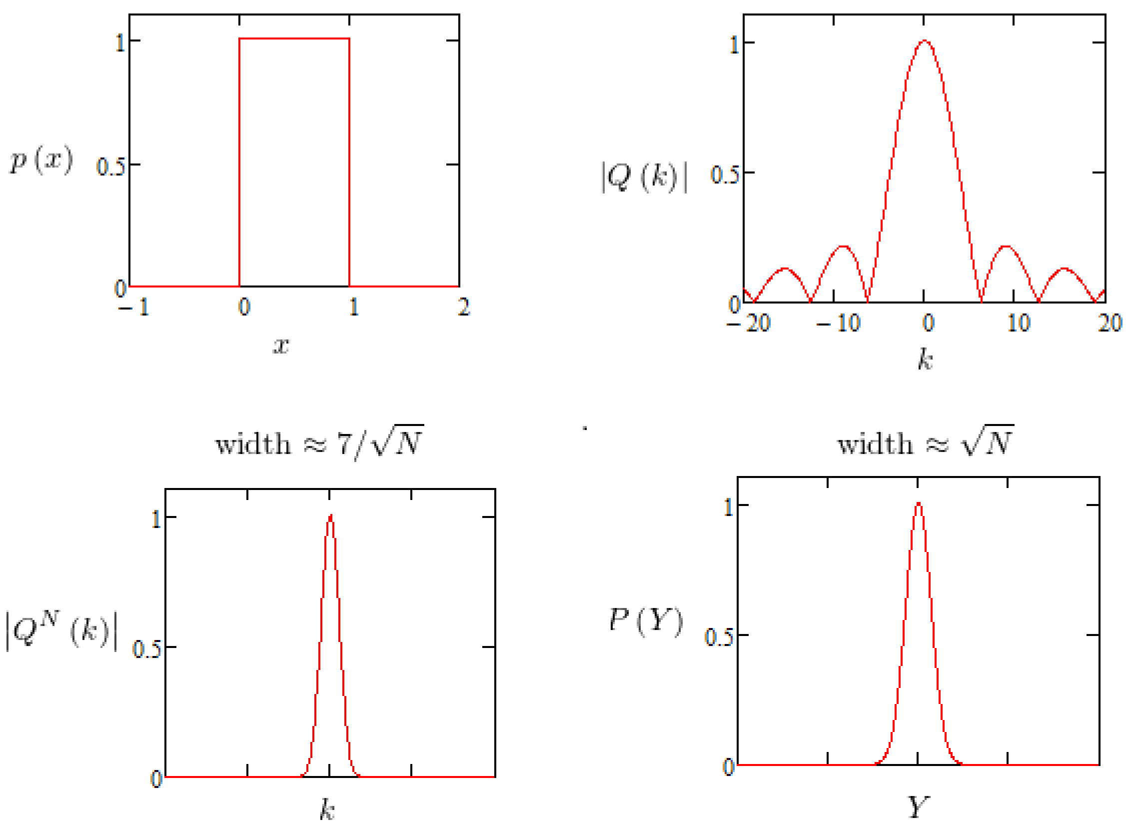
The value of $\sqrt{24}\ln(2)\approx 3.4$ which really isn't all that important. What is important is that it scales with $N^{-1/2}$.

$$\text{width}\approx 7/\sqrt{N}$$

iv.

To get the width of $P(Y)$ we have

$$\begin{aligned}\sigma&=\sqrt{N\left(\langle x^2\rangle-\frac{1}{2}\langle x\rangle^2\right)}\\&=\sqrt{N\left(\frac{1}{3}-\frac{1}{8}\right)}\end{aligned}$$



or $\sigma\approx 0.46\sqrt{N}$.