**PROBLEM SET 3**

1. a. From $P \propto e^{-\beta E}$, we get that

$$\frac{P_1}{P_0} = e^{-\beta\hbar\omega}$$

b. First we calculate the partition function only counting the first two states

$$Z = e^{-\beta\frac{1}{2}\hbar\omega} + e^{-\beta\frac{3}{2}\hbar\omega} = e^{-\beta\hbar\omega} 2\cosh(\frac{1}{2}\beta\hbar\omega)$$
$$\log Z = -\beta\hbar\omega + \log 2 + \log\cosh(\frac{1}{2}\beta\hbar\omega)$$

$$\begin{aligned}
\langle E\rangle &= -\frac{\partial}{\partial\beta}\log Z \\
&= \hbar\omega + \frac{1}{\cosh(\beta\hbar\omega/2)}\frac{\partial}{\partial\beta}\cosh(\frac{1}{2}\beta\hbar\omega) \\
&= \hbar\omega - \frac{1}{2}\hbar\omega\tanh(\frac{1}{2}\beta\hbar\omega) \\
&= \hbar\omega\left(1 - \frac{1}{2}\tanh\left(\frac{\hbar\omega}{2k_BT}\right)\right)
\end{aligned}$$

2. a. Because the spinners are independent the partition function factors into a product partition functions of single spinners.

$$Z = Z_1^N = (e^{-\beta\mu B} + e^{\beta\mu B})^N$$
$$\log Z = N\log(2\cosh(\beta\mu B))$$
$$\langle E\rangle = -\frac{\partial}{\partial\beta}\log Z = -N\mu B\tanh(\beta\mu B)$$

Because $\langle E\rangle = -MB$ were $M$ is the expectation value of the magnetization

$$M = N\mu\tanh(\beta\mu B) = N\mu\tanh\left(\frac{\mu B}{\tau}\right)$$
$$\chi \equiv \frac{dM}{dB} = \frac{N\mu^2\beta}{\cosh^2\left(\frac{\mu B}{\tau}\right)} = 4N\mu^2\beta\left(e^{-\mu\beta B} + e^{\mu\beta B}\right)^{-2}$$
$$\chi = \frac{4N\mu^2/\tau}{\left(e^{-\mu B/\tau} + e^{\mu B/\tau}\right)^2}$$

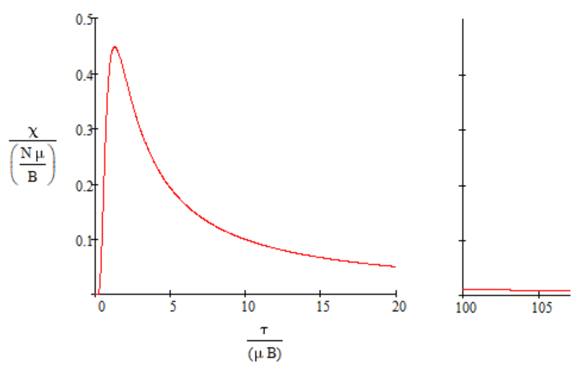
b. When $\tau \ll \mu B$ the second term in parentheses dominates,

$$\lim_{\mu B/\tau\to\infty}\chi = 4N\mu^2\beta\left(e^{-2\mu\beta B}\right)$$

which goes to zero because the exponential term outruns the linear term. When $\tau \gg \mu B$ we can Taylor expand the exponentials,

$$\chi = \frac{4N\mu^2/\tau}{\left(1 - \mu B/\tau + ... + 1 + \mu B/\tau + ...\right)^2}$$
$$\lim_{\mu B/\tau\to 0}\chi = 4N\mu^2\beta\,(2)^{-2} = N\mu^2\beta$$

c. For the plot I will nondimensionalize $\tau$ by $\mu B$ and $\chi$ by $N\mu/B$



3. (K&K 3.4) It is usually easier to work with $\beta$ rather than $\tau$.

$$\left(\frac{\partial U}{\partial \tau}\right)_V = \frac{\partial \beta}{\partial \tau}\frac{\partial U}{\partial \beta} = -\beta^2 \frac{\partial}{\partial \beta}(-\frac{\partial}{\partial \beta}\log Z) = \beta^2 \frac{\partial^2}{\partial \beta^2}\log Z$$

On the other hand,

$$\langle (E - \langle E\rangle)^2\rangle = \langle E^2\rangle - \langle E\rangle^2 = \frac{1}{Z}\frac{\partial^2}{\partial \beta^2}Z - \left(\frac{1}{Z}\frac{\partial Z}{\partial \beta}\right)^2 = \frac{\partial}{\partial \beta}\left(\frac{1}{Z}\frac{\partial Z}{\partial \beta}\right) = \frac{\partial^2}{\partial \beta^2}\log Z$$

So we see that

$$\langle (E - \langle E\rangle)^2\rangle = \tau^2\left(\frac{\partial U}{\partial \tau}\right)_V$$

4. (K&K 3.5) The Boltzmann factor is ratio of probabilities of two microstates of the system. Let's denote these states as $S$ with energy $\varepsilon$ and $S'$ with energy $\varepsilon'$.

$$\frac{P(S')}{P(S)} = \frac{g_R(S')}{g_R(S)} = e^{\sigma_R(S') - \sigma_R(S)}$$

First we write the entropy of the reservoir in terms of the energy of the reservoir, $\varepsilon_R$.

$$\begin{aligned}\sigma_R\left(S'\right) - \sigma_R\left(S\right) &= \sigma_R\left(\epsilon'_R\right) - \sigma_R\left(\epsilon_R\right)\\ &\approx \sigma_R\left(\epsilon_R\right) + \frac{d\sigma_R}{dE}\left(\epsilon'_R - \epsilon_R\right) - \sigma_R\left(\epsilon_R\right)\\ &= \beta\left(\epsilon'_R - \epsilon_R\right)\end{aligned}$$

Whenever energy flows into the system a net amount $1-\alpha$ flows out of the reservoir, that is,

$$\Delta\epsilon_R = (\alpha - 1)\,\Delta\epsilon$$

therefore

$$\sigma_R\left(S'\right) - \sigma_R\left(S\right) = -\beta\left(1-\alpha\right)\Delta\epsilon$$

$$\frac{P\left(S'\right)}{P\left(S\right)} = e^{-(1-\alpha)\Delta\epsilon\beta}$$

Therefore $P(S')$ is proportional to $e^{-(1-\alpha)\varepsilon\beta}$, where the proportionality constant depends on the baseline state we are comparing it to, in this case $P(S)$. Regardless, this constant is taken care of by the partition function.

5. (K&K 3.7)

   a. The energy of state $s$, where links 1 through $s$ are open, is $s\varepsilon$. Therefore the partition function is

   $$Z = \sum_{s=0}^{N} e^{-\beta s\varepsilon}$$

   which is just a geometric sum where $r = e^{-\beta\varepsilon}$, so

   $$Z = \frac{1 - r^{N+1}}{1 - r} = \frac{1 - e^{-\beta(N+1)\varepsilon}}{1 - e^{-\beta\varepsilon}}$$

   b.

   $$\begin{aligned}
   \langle s \rangle &= \frac{1}{Z}\sum_{s=0}^{N} s e^{-\beta s\varepsilon} = \frac{1}{Z}\left(-\frac{1}{\varepsilon}\right)\frac{\partial Z}{\partial \beta} \\
   &= \frac{-1}{Z\varepsilon}\left(\frac{e^{-\beta\varepsilon(N+1)}\varepsilon(N+1)}{1 - e^{-\beta\varepsilon}} + \frac{(1 - e^{-\beta\varepsilon(N+1)})(-\varepsilon)(e^{-\varepsilon\beta})}{(1 - e^{-\beta\varepsilon})^2}\right) \\
   &= \left(\frac{1}{1 - e^{-\beta(N+1)\varepsilon}}\right)\left(-e^{-\beta\varepsilon(N+1)}(N+1) + \frac{(1 - e^{-\beta\varepsilon(N+1)})(e^{-\varepsilon\beta})}{(1 - e^{-\beta\varepsilon})}\right)
   \end{aligned}$$

   When $\beta\varepsilon \gg 1$,

   $$\langle s \rangle \approx \frac{e^{-\varepsilon\beta}}{1 - e^{-\beta\varepsilon}} = \frac{1}{e^{\beta\varepsilon} - 1} \approx e^{-\beta\varepsilon}$$

6. (K&K 3.9) Let's say we have a combined state $s$, where the first system is in state $a$ and the second system is in state $b$. Since they are independent, $P(s) = P(a)P(b)$, and therefore $Z(1+2) = \sum_s P(a)P(b)$. In addition, every possible combination of $a$ and $b$ is possible (once again because they are independent). Therefore summing over $s$ is equivalent to summing over $a$ and over $b$ separately, so $Z(1+2) = \sum_a P(a) \sum_b P(b) = Z(1)Z(2)$.

7. (K&K 3.3)

   a. First we need to calculate the partition function, and then we can calculate the free energy.

$$Z = \sum_{s=0}^{s=\infty} e^{-s\hbar\omega\beta} = \frac{1}{1-e^{-\hbar\omega\beta}}$$

$$F = -\tau \ln Z = -\tau \ln\left[\frac{1}{1-e^{-\hbar\omega\beta}}\right] = \tau \ln\left[1-e^{-\hbar\omega\beta}\right]$$

   b.

$$\begin{aligned}\sigma = -\frac{\partial F}{\partial \tau} &= -\ln\left(1-e^{-\hbar\omega\beta}\right) - \frac{\tau}{1-e^{-\hbar\omega\beta}}(-e^{-\hbar\omega\beta})(-\hbar\omega)(-\tau^{-2}) \\ &= \frac{(e^{-\hbar\omega\beta})(\hbar\omega\beta)}{1-e^{-\hbar\omega\beta}} - \ln\left(1-e^{-\hbar\omega\beta}\right) \\ &= \frac{\hbar\omega\beta}{e^{\hbar\omega\beta}-1} - \ln\left(1-e^{-\hbar\omega\beta}\right)\end{aligned}$$