# Midterm Solutions

## Problem 1

a. i. (2 points)
Using Boltzmann's statistics

$$\rho \equiv \frac{P_0}{P_1} = \frac{e^{-\beta\epsilon_0}}{e^{-\beta\epsilon_1}} = e^{\hbar\omega/\tau_c}$$

$$\tau_c = \frac{\hbar\omega}{\ln(\rho)}$$

ii. (3 points)

$$F = -\tau \ln(Z)$$

Using the geometric series

$$Z = \sum_{n=0}^{\infty} e^{-\beta\hbar\omega} = \frac{1}{1-e^{-\beta\hbar\omega}}$$

$$\begin{aligned}
F &= -\tau_c \ln\left(\frac{1}{1-e^{-\beta\hbar\omega}}\right) \\
&= -\frac{\hbar\omega}{\ln(\rho)} \ln\left(\frac{1}{1-\rho^{-1}}\right) \\
&= -\frac{\hbar\omega}{\ln(\rho)} \ln\left(\frac{\rho}{\rho-1}\right) \\
&= \hbar\omega\left(\frac{\ln(\rho-1)}{\ln(\rho)} - 1\right)
\end{aligned}$$

iii. (4 points)

$$\begin{aligned}
\langle E\rangle &= -\frac{\partial}{\partial\beta}\ln(Z) \\
&= \frac{\partial}{\partial\beta}\ln\left(1-e^{-\beta\hbar\omega}\right) \\
&= \frac{\hbar\omega e^{-\beta\hbar\omega}}{1-e^{-\beta\hbar\omega}} \\
&= \frac{\hbar\omega\rho^{-1}}{1-\rho^{-1}} \\
&= \frac{\hbar\omega}{\rho-1}
\end{aligned}$$

b. (5 points)
First derive a general expression

$$F = E - \tau\sigma$$

$$\sigma = \frac{E-F}{\tau_c}$$

$$\begin{aligned}\sigma &= \frac{\frac{\hbar\omega}{\rho-1} - \hbar\omega\left(\frac{\ln(\rho-1)}{\ln(\rho)} - 1\right)}{\frac{\hbar\omega}{\ln(\rho)}}\\ &= \frac{\ln(\rho)}{\rho-1} - (\ln(\rho-1) - \ln(\rho))\\ &= \frac{\rho}{\rho-1}\ln(\rho) - \ln(\rho-1)\\ &= \frac{2}{2-1}\ln(2) - \ln(2-1)\\ &= 2\ln(2) - \ln(1)\\ &= \ln(4)\end{aligned}$$

c. (6 points)

In order to compare informational content we compare entropies. In case i) we have 3 cantilever's from above

$$\sigma_i = 3\left(\frac{\rho}{\rho-1}\ln(\rho) - \ln(\rho-1)\right)$$

In case ii) there are 10 micro states.

$$\begin{aligned}n_1 + n_2 + n_3 &= 3+0+0\\ &= (3+0+0)\times 3\text{permutations}\\ &= (2+1+0)\times 6\text{permutations}\\ &= (1+1+1)\times 6\text{permutations}\end{aligned}$$

For a total of 10 microstates which gives an entropy of $\sigma = \ln(10)$. Which case has less entropy, and therefor more information is know about it, depends on temperature. If $\tau_c$ is high $\sigma_i$ will be bigger than $\sigma_{ii}$. If we take $\rho = 2$ then $\sigma_i = 3\ln(4) = \ln(64) > \ln(10) = \sigma_{ii}$. So more information is know about $ii$. This difference can be quntified by $\sigma_i - \sigma_{ii}$=ln(64) − ln(10). If we want to measure this in bits we can write it as ln(6.4)/ ln(2).

## Problem 2

a. i. (4 points) System $A_e$ has higher energy per excited spin by a factor of $\mu_e/\mu_p$, but fewer excited spins by a factor of $\sim \exp(-2\mu_e/\mu_p)$, so the ratio of average energy per spin for the two systems scales like

$$\frac{\mu_e}{\mu_p} e^{-2\mu_e/\mu_p} \ll 1$$

i.e. system $B_p$ has higher average energy per spin.
Although not required for full credit, it is easy to calculate the average energy per spin of each system exactly. The partition function for a single spin in system $i$ is

$$Z_i = 1 + e^{-2\mu_i B/\tau},$$

and the average energy per spin is

$$E_i = -\frac{\partial}{\partial \beta} \ln Z_i = \frac{2\mu_i B e^{-2\mu_i B/\tau}}{1 + e^{-2\mu_i B/\tau}}.$$

Comparing the average energy per spin for the two systems, and setting $\tau = \mu_p B$, we have

$$\frac{E_e}{E_p} = \frac{2\mu_e B e^{-2\mu_e B/\mu_p B}}{2\mu_p B e^{-2\mu_p B/\mu_p B}} \cdot \frac{1 + e^{-2\mu_p B/\mu_p B}}{1 + e^{-2\mu_e B/\mu_p B}} = \frac{\mu_e}{\mu_p} e^{-2\mu_e/\mu_p + 2} \cdot \frac{1 + e^{-2}}{1 + e^{-2\mu_e/\mu_p}} \propto \frac{\mu_e}{\mu_p} e^{-2\mu_e/\mu_p}$$

which has the expected scaling.

ii. (4 points) Each spin in system $A_e$ has a probability of $\sim e^{-2\mu_e/\mu_p} = e^{-1324}$ of being in the excited state, but each spin in system $B_p$ has a probability of $\sim e^{-2}$ of being in the excited state. Thus, the state of each spin in $B_p$ is more uncertain, i.e. system $B_p$ has higher entropy per spin.

b. i. (5 points) In order to have average energy $\mu_p B = 1/2 \cdot 0 + 1/2 \cdot 2\mu_p B$, the spins in system $B_p$ must be equally likely to be in the ground state or the excited state. Thus, the temperature of system $B_p$ is infinite. Since $\mu_p B < \mu_e B$, the temperature of system $A_e$ is finite. To maximize entropy, energy will flow from the hotter system to the colder system, i.e. from system $B_p$ to system $A_e$.

ii. (4 points) The entropy of $A_e$ increases because it absorbs energy from system $B_p$ and because its temperature is positive, so $\frac{\Delta\sigma}{\Delta E} \cdot \Delta E = \Delta\sigma > 0$.

iii. (4 points) The entropy of $B_p$ decreasese because it loses energy to system $A_e$ and because its temperature is positive, so $\frac{\Delta\sigma}{\Delta E} \cdot \Delta E = \Delta\sigma < 0$

iv. (4 points) Our results are consistent with the second law of thermodynamics because the entropy of system $A_e$ increases by more than the entropy of system $B_p$ decreases when the two systems are brought into thermal contact. In other words, the total entropy of the combined system increases, as required by the second law.

## Problem 3

a. (2 points)

$$C_V \equiv \left(\frac{\partial E}{\partial \tau}\right)_V = \left(\frac{\partial E}{\partial \sigma}\right)_V \left(\frac{\partial \sigma}{\partial \tau}\right)_V = \tau \left(\frac{\partial \sigma}{\partial \tau}\right)_V.$$

b. (3 points)

$$\frac{\partial}{\partial V}\frac{\partial}{\partial \tau}F(\tau, V) = -\left(\frac{\partial \sigma}{\partial V}\right)_\tau = \frac{\partial}{\partial \tau}\frac{\partial}{\partial V}F(\tau, V) = -\left(\frac{\partial p}{\partial \tau}\right)_V$$

so

$$\left(\frac{\partial \sigma}{\partial V}\right)_\tau = \left(\frac{\partial p}{\partial \tau}\right)_V$$

i.e. $A = p$ and $B = \tau$.

c. i. (3 points)

$$\left(\frac{\partial C_V}{\partial V}\right)_\tau = \frac{\partial}{\partial V}\tau\frac{\partial \sigma}{\partial \tau} = \tau\frac{\partial}{\partial V}\frac{\partial \sigma}{\partial \tau} = \tau\frac{\partial}{\partial \tau}\frac{\partial \sigma}{\partial V} = \tau\left(\frac{\partial^2 p}{\partial \tau^2}\right)_V$$

where we have used the result of part (a) in the first step and part (b) in the last step.

ii. (2 points)

$$\left(\frac{\partial^2 p}{\partial \tau^2}\right)_V = -\frac{3}{4}\frac{a}{V(V+b)}\tau^{-5/2}$$

so

$$\left(\frac{\partial C_V}{\partial V}\right)_\tau = -\frac{3}{4}\frac{a}{V(V+b)}\tau^{-3/2}.$$

## Problem 4

a. (3 points)
The gas is dilute so we treat each species independently and as an ideal gas.

$$E_\alpha = \sum_\alpha \frac{3}{2}\tau N_\alpha = \frac{3}{2}\tau \sum_\alpha N_\alpha$$

b. i. (2 points) We want the rms momentum for species $\alpha$. Because all particles are identical, we find $p_{rms}$ of a single particle of $\alpha$. Note that $p_{rms}$ is an intensive quantity. For a single particle $E = p^2/2m$

$$\begin{aligned} p_{rms,\alpha} &= \sqrt{\langle p_\alpha^2 \rangle} \\ &= \sqrt{2m_\alpha \langle E \rangle} \\ &= \sqrt{3m_\alpha \tau} \end{aligned}$$

ii. (2 points) likewise for $v$. Note that $v = p/m$ so

$$\begin{aligned} v_{rms} &= \sqrt{\langle p_\alpha^2 / m_\alpha^2 \rangle} \\ &= \sqrt{3\tau / m_\alpha} \end{aligned}$$

c. (4 points)
RMS means root of the mean of the square. When combining many particles calculate the mean by summing the momentum/velocity and dividing by the total number

$$p_{rms} = \sqrt{\frac{\sum_\alpha \sum_{i=1}^{N_i} p_{\alpha,i}^2}{\sum_\alpha \sum_{i=1}^{N_i} 1}}$$

$$p_{rms} = \sqrt{\frac{\sum_\alpha N_\alpha p_{rms,\alpha}^2}{\sum_\alpha N_\alpha}}$$

$$p_{rms} = \sqrt{\frac{3\tau \sum_\alpha N_\alpha m_\alpha}{\sum_\alpha N_\alpha}}$$

likewise for momentum

$$v_{rms} = \sqrt{\frac{3\tau \sum_\alpha N_\alpha / m_\alpha}{\sum_\alpha N_\alpha}}$$

d. (4 points)
We have $N_1 = N_2 = N_{tot}/2$. For momentum only the heavy particles, $m_2$ will contribute

$$p_{rms} = \sqrt{\frac{3\tau \left(N_{tot}/2\right) m_2}{N_{tot}/2 + N_{tot}/2}}$$

$$p_{rms} = \sqrt{\frac{3\tau m_2}{2}}$$

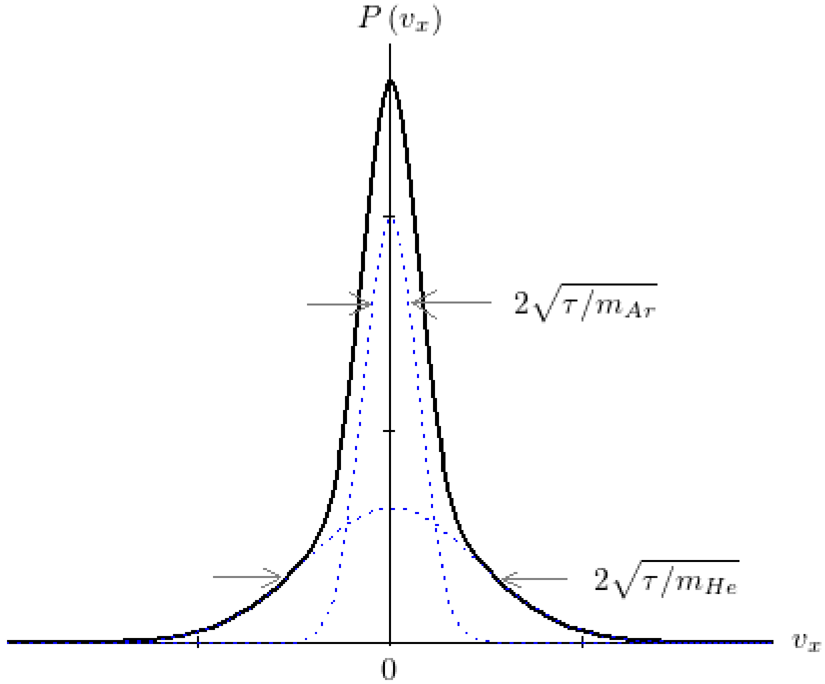
likewise for velocity except only the light particles will matter

$$v_{rms} = \sqrt{\frac{3\tau}{2m_1}}$$

e. (7 points)

$$P \propto e^{-\beta E_{kenetic}}$$

$$P(v_x) = \frac{N_{Ar}}{N_{tot}} \frac{1}{\sqrt{2\pi m_{Ar}}} e^{-v_x^2 m_{Ar}/2\tau} + \frac{N_{He}}{N_{tot}} \frac{1}{\sqrt{2\pi m_{Ar}}} e^{-v_x^2 m_{He}/2\tau}$$




Here we sketch $N_{Ar} \approx N_{He}$.

f. i. (3 points)

The function in part e) non-Gaussian. By the central limit theorem the distribution of the mean will be Gaussian (with a much smaller width). So they are different.

ii. (+3 extra points)

The average velocity is

$$\bar{v}_x = \frac{\sum_{i=1}^{N_{\mathrm{He}}} v_x^2 + \sum_{i=1}^{N_{\mathrm{Ar}}} v_x^2}{N_{\mathrm{He}} + N_{\mathrm{Ar}}}$$

The variance of a single molecule is $\sigma_i^2 = \tau/m_i$ and the sum of variances is the variance of a sum

$$(\Delta\bar{v}_x)^2 = \frac{1}{(N_{\mathrm{He}} + N_{\mathrm{Ar}})^2} \left(N_{\mathrm{He}}\sigma_{\mathrm{He}}^2 + N_{\mathrm{Ar}}\sigma_{\mathrm{Ar}}^2\right)$$

$$\Delta\bar{v}_x = \sqrt{\frac{1}{(N_{\mathrm{He}} + N_{\mathrm{Ar}})^2} \left(N_{\mathrm{He}}\frac{\tau}{m_{\mathrm{He}}} + N_{\mathrm{Ar}}\frac{\tau}{m_{\mathrm{Ar}}}\right)}$$

with $m_{\mathrm{He}} \ll m_{\mathrm{Ar}}$ we have

$$\Delta\bar{v}_x \approx \frac{\sqrt{N_{\mathrm{He}}\frac{\tau}{m_{\mathrm{He}}}}}{N_{\mathrm{He}} + N_{\mathrm{Ar}}}$$