FINAL EXAM

- You have 180 minutes
- You are permitted to have one letter-sized sheet of hand-written notes
- No other references or electronic devices are permitted
- There are 120 possible points on the exam. 100 points count as a perfect score. Anything beyond that is extra credit.

Each problem on this exam has multiple parts (a, b, c, ...). Some parts can be done out of sequence. In case you get stuck on one part, still look at the rest of the problem!

1. *Equation of state for relativistic fermions* (40 points). Consider a three-dimensional gas of N relativistic spin-1/2 particles in a box of volume $V = L^3$. The energy of each particle is given by $\varepsilon = \hbar c k$, where $\hbar k$ represents the particle's momentum. The objective of this problem is to sketch the relationship between pressure p and temperature τ at fixed density $n = N/V$.

   a. Properties of the relativistic Fermi gas:
      i. Calculate the Fermi energy as a function of density
      ii. Calculate the density of states $\mathcal{D}(\varepsilon)$ per unit energy
   b. Useful Maxwell relations:
      i. Derive a relationship between $\left(\frac{\partial p}{\partial N}\right)_{\tau,V}$ and $\left(\frac{\partial \mu}{\partial V}\right)_{\tau,N}$
      ii. Derive a relationship between $\left(\frac{\partial p}{\partial \tau}\right)_{V,N}$ and $\left(\frac{\partial \sigma}{\partial V}\right)_{\tau,N}$
   c. Low-temperature limit of the pressure:
      i. Determine the pressure at $\tau = 0$ as a function of density n. One approach is to start by obtaining $\left(\frac{\partial p}{\partial N}\right)_{\tau=0,V}$ via a Maxwell relation.
      ii. What is the initial slope of the pressure vs. temperature curve? Determine $\left(\frac{\partial p}{\partial \tau}\right)_{V,N}$ at $\tau = 0$ using a Maxwell relation from b. and a law of thermodynamics. State the relevant law.
   d. "Classical" limit of the pressure:
      i. Write down an integral expression for the total number N of particles in terms of the chemical potential μ, temperature τ, and density of states.
      ii. Give a condition on the fugacity $\phi = e^{\beta\mu}$ for the system to be "classical" in the sense that the quantum statistics are irrelevant. ("Classical" is in quotes because the gas is relativistic.)
      iii. Calculate the chemical potential $\mu(n, \tau)$ in the "classical" regime.
      iv. Re-express your condition for classicality in terms of...
         A. the temperature and density
         B. the temperature and the Fermi energy
      v. Calculate $p(n, \tau)$ in the classical regime
   e. Sketch the the pressure as a function of temperature at fixed density by interpolating smoothly between the low-temperature and classical limits. Label characteristic features on the plot in terms of their scaling with n and/or the Fermi energy ε_F.
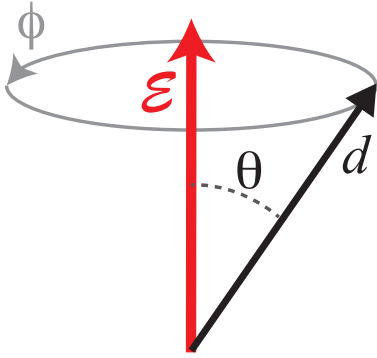
FIG. 1. The microstate of a classical dipole is specified by polar and azimuthal angles (θ, ϕ). The polar angle θ is measured relative to the direction of the electric field.

2. *Classical dipoles* (25 points). Consider a classical model of a system of N electric dipoles, each of which has a dipole moment d. The orientation of each dipole is specified in spherical coordinates by a pair of angles (θ, ϕ), where θ denotes the angle between the dipole moment and an applied electric field of strength $\mathcal{E}$. The energy of each dipole is thus $-d\mathcal{E}\cos\theta$. We assume that the system is sufficiently dilute that the dipoles do not interact, and that the kinetic energy is negligibly small.

   a. Write down the classical partition function Z for the system of N dipoles.
   b. Calculate the total energy $\langle E \rangle$ and the polarization $\mathcal{P}$ as a function of temperature τ. The polarization is defined such that $\langle E \rangle = -\mathcal{P}\mathcal{E}$.
   c. Find the following limiting values:
      i. The polarization $\mathcal{P}$ in the strong-field limit $\mathcal{E} \gg \tau/d$
      ii. The polarizability $\chi = \partial\mathcal{P}/\partial\mathcal{E}$ in the weak-field limit $\mathcal{E} \ll \tau/d$. (Here, it is convenient first to expand $\mathcal{P}$ in the small parameter $\mathcal{E}d/\tau$.)
   d. Sketch the polarization $\mathcal{P}$ as a function of electric field strength $\mathcal{E}$ for two different temperatures $\tau_1 < \tau_2$. Label the two curves corresponding to τ_1 and τ_2.
   e. Suppose that you wish to distinguish whether a system of dipoles is better described by (A) the classical model considered above or (B) a model in which each dipole has only two discrete energy states, either aligned or anti-aligned with the electric field. To this end, you measure the heat capacity as a function of temperature. Which model would predict a higher heat capacity for low temperatures (or strong fields) $\tau \ll d\mathcal{E}$? Explain.

3. *Entropy of gold* (12 points). Use arguments based on the heat capacity to answer the following questions without detailed calculations:

   a. The Debye temperature of gold is $\theta_D = 170$ K. Suppose that the temperature of a piece of gold is doubled from $T_1 = 2$ K to $T_2 = 4$ K.
      i. By what factor does the entropy of the lattice vibrations change?
      ii. By what factor does the entropy of the conduction electrons change?
   b. Suppose that the temperature of the gold were instead doubled from $T_1 = 300$ K to $T_2 = 600$ K. Which of your answers, i. or ii., would be significantly different? Give a revised estimate for the factor by which this component of the entropy changes. Approximate the relevant heat capacity by its behavior for $T_1, T_2 \gg \theta_D$.

4. *Surface adsorption* (25 points). Consider a surface consisting of N_s sites, each of which can bind at most one molecule from a surrounding ideal gas at temperature τ. Let N denote the number of molecules adsorbed to the surface, and let $\varepsilon < 0$ represent the energy of an occupied site. The energy of a vacant site is taken to be zero.

    a. Calculate the occupation fraction N/N_s as a function of the binding energy ε, the temperature τ, and the chemical potential μ of the gas.

    b. Let σ_s denote the entropy of the surface. Consider its dependence on N:

        i. Calculate the derivative $(\partial\sigma_s/\partial N)_\tau$ as a function of N.

        ii. Evaluate $(\partial\sigma_s/\partial N)_\tau$ at $N = 0$, $N = N_s/2$, and $N = N_s$.

    c. Sketch qualitatively:

        i. the derivative $(\partial\sigma_s/\partial N)_\tau$ as a function of N at fixed temperature τ

        ii. the entropy σ_s of the surface as a function N at fixed temperature τ

    d. Suppose that the gas initially occupies a volume V, and that more than half of the adsorption sites are initially occupied ($N > N_s/2$). The gas is then quasi-statically compressed from its initial volume V to a final volume $V' < V$, while remaining in thermal equilibrium with a reservoir at temperature τ.

        i. Does the entropy of the gas increase, decrease, or remain the same?

        ii. Does the entropy of the surface increase, decrease, or remain the same?

        iii. Does the entropy of the reservoir increase, decrease, or remain the same? *Justify your responses to questions i., ii., and iii. by physical arguments.*

5. *Two-dimensional Bose gas* (18 points). A two-dimensional gas of identical spinless bosons of mass m is trapped in a harmonic potential of the form

$$U(x,y) = \frac{1}{2}m\omega^2(x^2 + y^2). \tag{1}$$

The energy eigenvalues are of the form $\varepsilon = (n_x + n_y)\hbar\omega$, where $n_{x,y} = 0, 1, 2, \ldots$ and we have taken the zero-point energy to be zero.

    a. Sketch a graphical representation of the eigenstates as points in the (n_x, n_y) plane. Shade in the region corresponding to all eigenstates of energy $\leq \varepsilon$, and calculate its area $\mathcal{N}(\varepsilon)$.

    b. Calculate the density of states $\mathcal{D}(\varepsilon) = d\mathcal{N}/d\varepsilon$.

    c. Show that a Bose-Einstein condensate can form in this system, and calculate the critical temperature τ_E in terms of the total number N of bosons. Express your result in terms of a dimensionless integral $\mathcal{I}$, which you may leave unevaluated.

    d. Re-express the condition for Bose-Einstein condensation in the harmonic trap as a critical density, as follows:

        i. Classically, the average potential energy $\langle U \rangle$ can be related to the temperature using the equipartition theorem. In the thermodynamic limit, the temperature τ_E is sufficiently high that the equipartition result is still a good approximation. Use it to estimate the rms cloud size $x_{\rm rms} = y_{\rm rms}$ in terms of the temperature τ.

        ii. By estimating the density as $n \approx N/(x_{\rm rms}\, y_{\rm rms})$, find the critical density n_E for Bose-Einstein condensation at temperature τ.

        iii. How does the critical density scale with the thermal de Broglie wavelength? Explain.