PROBLEM SET 4

Reading: Kittel & Kroemer Ch. 3, pp. 63 - 81

When to start which problem?

- You can start on problems 1-4 immediately and problem 5 after class on Friday.
- For problems 2-4, be sure to read about *Maxwell relations* on pg. 71 of K&K.

1. *Topological defects.* The energy of a single vortex in a superfluid film of dimensions $L \times L$ is $J\pi \ln(L/a)$, where a is the radius of the vortex core and the constant J is determined by the properties of the superfluid. Since ε diverges as $L \to \infty$, it was long assumed that isolated vortices would be unstable at any temperature in the thermodynamic limit. By recognizing a flaw in this argument, Kosterlitz and Thouless [1] discovered a seminal example of a *topological phase transition*, for which they were awarded the 2016 Nobel Prize in Physics [2].

   The energy of a *pair* of vortices with opposite signs of the vorticity is

   $$\varepsilon_p = 2\pi J \ln(r/a), \tag{1}$$

   where r is the separation between the vortices.

   a. Calculate the entropy of a system containing a vortex pair of separation r by arguing that there are $(L/a)^2(r/a)$ possible spatial configurations.

   b. Calculate the Helmholtz free energy $F(r, \tau)$.

   c. Show that above a critical temperature τ_c, the Helmholtz free energy is minimized by allowing the vortex pair to "unbind." Explain what is meant by unbinding, and give the value of τ_c.

   d. Give a physical argument for why the transition to free (unbound) vortices occurs.

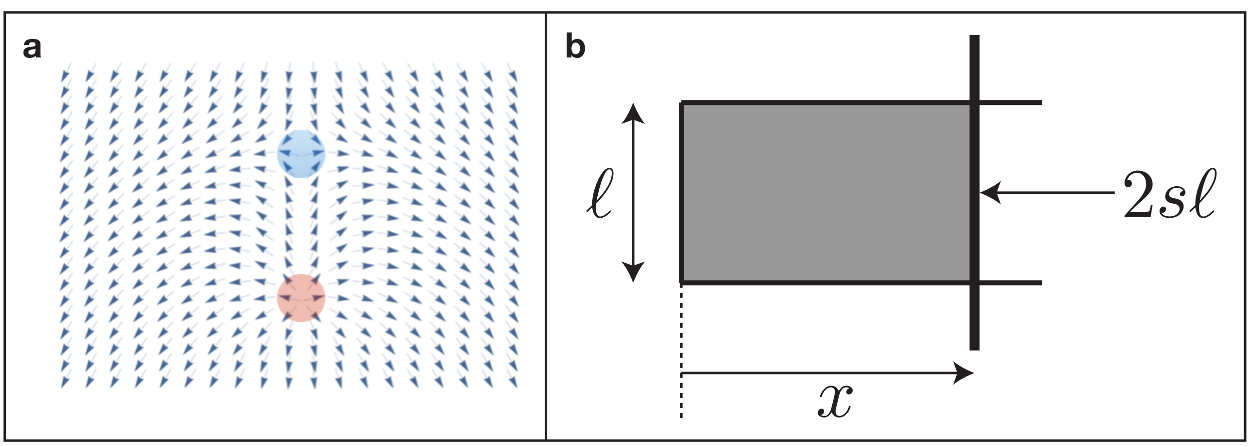

FIG. 1. **(a)—Problem 1.** Illustration of a vortex pair, from Ref. [2]. Arrows represent the phase of the wavefunction. **(b)—Problem 3.** A soap film (gray) supported by a wire frame.

2. In some range of the temperature τ, the tension force f of a stretched plastic rod is related to its length L by
$$f = a\tau^2(L - L_0), \tag{2}$$
where a and L_0 are positive constants, L_0 being the unstretched length of the rod. When $L = L_0$, the heat capacity C_L of the rod (measured at constant length) is given by the relation $C_L = b\tau$, where b is a constant.

   a. Write down the fundamental thermodynamic relation for this system, expressing $d\sigma$ in terms of dE and dL. Compute $(\partial\sigma/\partial L)_\tau$.

   b. Knowing $\sigma(\tau_0, L_0)$, compute the entropy $\sigma(\tau, L)$ at *any* other temperature τ and length L. (It is most convenient to calculate first the change of entropy with temperature at the length L_0 where the heat capacity is known.)

   c. If one starts at $\tau = \tau_i$ and $L = L_i$ and stretches the thermally insulated rod quasi-statically until it attains the length L_f, what is the final temperature τ_f? Is τ_f larger or smaller than τ_i?

   d. Calculate the heat capacity $C_L(\tau)$ of the rod when its length is L instead of L_0.

   e. Calculate $\sigma(\tau, L)$ by writing $\sigma(\tau, L) - \sigma(\tau_0, L_0) = [\sigma(\tau, L) - \sigma(\tau_0, L)] + [\sigma(\tau_0, L) - \sigma(\tau_0, L_0)]$ and using the result of part d. to compute the first term in the square brackets. Show that the final answer agrees with that found in b.

3. Fig. 1b illustrates a soap film (shown in gray) supported by a wire frame. Because of surface tension the film exerts a force $2s\ell$ on the cross wire. This force is in such a direction that it tends to move this wire so as to decrease the area of the film. The quantity s is called the "surface tension" of the film and the factor 2 occurs because the film has two surfaces. The temperature dependence of s is given by
$$s = s_0 - \alpha\tau \tag{3}$$
where s_0 and α are constants independent of τ or x.

   a. Suppose that the distance x (or equivalently the total film area $2\ell x$) is the only external parameter of significance in the problem. Write a relation expressing the change dE in mean energy of the film in terms of the heat $đQ$ absorbed by it and the work done by it in an infinitesimal quasi-static process in which the distance x is changed by an amount dx.

   b. Calculate the change in mean energy $\Delta E = E(x) - E(0)$ of the film when it is stretched at a constant temperature τ_0 from a length $x = 0$ to a length x.

   c. Calculate the work $\mathcal{W}(0 \to x)$ done on the film in order to stretch it at this constant temperature from a length $x = 0$ to a length x.

4. A small glass bead of mass M is held in an optical tweezer formed by a focused laser beam. Consider its motion in one dimension, for which the laser beam generates an attractive potential $U(x) = -U_0 e^{-2x^2/a^2}$ of depth U_0 proportional to the laser intensity.

   a. Assume that the bead has a temperature $\tau \ll U_0$, so that we may approximate it as a harmonic oscillator with energy spectrum
$$E_n = n\hbar\omega, \tag{4}$$

where we neglect the zero-point energy and an offset associated with the trap depth. Find the characteristic frequency ω.

b. Write down the fundamental thermodynamic relation describing the change in energy dE with respect to variations $d\sigma, d\omega$ in entropy and trap frequency. Describe the state of the bead by the mean vibrational quantum number $\overline{n}(\omega, \tau)$.

c. Calculate the mean vibrational quantum number $\overline{n}$ as a function of temperature τ and trap frequency ω.

d. Suppose that the trap frequency is changed from ω_1 to ω_2 at constant temperature τ. Calculate the work ΔW done on the bead and the heat ΔQ absorbed by the bead in the classical limit $\hbar\omega_{1,2} \ll \tau \ll U_0$.

e. Suppose that the trap frequency is changed adiabatically from ω_1 to ω_2. Determine the resulting change in temperature $\Delta\tau$ in the classical limit. Explain your result.

5. *One-dimensional gas* (adapted from K&K 3.11). Consider an ideal gas of N particles, each of mass M, confined to a one-dimensional line of length L. The particles have spin 0.

   a. Find the entropy at temperature τ. What is the minimum length L for your calculation of the entropy to be valid? Explain.

   b. Calculate the heat capacity C_L at constant length L and the *specific heat* $c_L \equiv C_L/N$.

   c. Give the *molar heat capacity*, i.e., the heat capacity of a mole (Avogadro's number) of particles, in units of Joules/Kelvin.

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[1] J M Kosterlitz and D J Thouless, "Long range order and metastability in two dimensional solids and superfluids. (application of dislocation theory)," J. Phys. C: Solid State Phys. (1972).

[2] "Scientific background on the Nobel prize in physics 2016: Topological phase transitions and topological phases of matter," Compiled by the Class for Physics of the Royal Swedish Academy of Sciences.