Problem 1) Subtle point about Bose gasses

In class on Friday we approximated the Bose gas a single lowest state plus a smooth density of states $\mathcal{D}(\epsilon) \propto \epsilon^{1/2}$. Show that this approximation is good and that the particles actually will condense into ground state.

Calculate the ratio of number of particles in the ground state N_0 and those in the first excited state N_1 which have energy ϵ_1. Give the answer in terms of ϵ_1 and μ. We may assume and that $N_0 \gg 1$.

$$N_0 = \frac{1}{e^{-\beta\mu} - 1}$$

By assumption $N_0 \gg 1$ so $|\beta\mu| \ll 1$. It is therefor possible to approximate the exponential with the first two terms of its Taylor series.

$$N_0 \approx \frac{1}{-\beta\mu}$$

$$N_1 = \frac{1}{e^{\beta(\epsilon_1 - \mu)} - 1}$$

$$\frac{N_1}{N_0} = \frac{e^{\frac{1}{N_0}} - 1}{e^{\beta\epsilon_1 + \frac{1}{N_0}} - 1}$$

Because $\epsilon_1 \ll \tau$ the we can Taylor expand that exponential as well

$$\frac{e^{-\beta\mu} - 1}{e^{\beta(\epsilon_1 - \mu)} - 1} \approx \frac{-\mu}{\epsilon_1 - \mu}$$

Under what condition on N_0 is $N_0 \gg N_1$? You can assume that $\epsilon_1 \ll \tau$. Give the result in terms of the size of the box L.

Recall that $\epsilon_{\vec{n}} = \frac{\hbar^2\left(n_x^2 + n_y^2 + n_z^2\right)\pi^2}{2mL^2}$ for massive bozons in a box.

The ground state is $(1, 1, 1)$ while the first excited state is $(2, 1, 1)$

$$\begin{aligned}\epsilon_1 &= \frac{\left(2^2 + 1^2 + 1^2 - 1^2 - 1^2 - 1^2\right)\pi^2\hbar^2}{2mL^2} \\ &= \frac{3\pi^2\hbar^2}{2mL^2}\end{aligned}$$

in order for the first excited state to have much less occupancy than the ground state

$$|\mu| \ll \epsilon_1$$

$$\frac{1}{\beta N_0} \ll \frac{3\pi^2\hbar^2}{2mL^2}$$

$$N_0 \gg \frac{2mL^2\tau}{3\pi^2\hbar^2}$$

When is this condition more restrictive then the condition we already derived for the BE condensate to occur?

Last Friday we found that the condition for the density of states to be overfilled so that particles would pile up in the ground state was

$$N > \frac{L^3}{4\pi\hbar^3}(2m\tau)^{3/2}\frac{\sqrt{\pi}}{2}\zeta\left(\frac{2}{3}\right)$$

Dividing the two conditions and assuming that N_0 is of the same order of magnitude as N (which it will be at the crossover) we have

$$1 \ll \frac{L\sqrt{m\tau}}{\hbar}$$

Under this condition the condition for the density of states to be overfilled is more restrictive. For $100nK$ and the mass of a few protons I get that the crossover is on the scale of microns. At system sizes much larger than this the condition we found from the density of states last Friday is more restrictive.

Problem 2) A more exact treatment of the heat capacity of a Fermi gas

Note that

$$\int_0^\infty \frac{x}{e^x+1}dx = \frac{\pi^2}{12} \approx 0.822467$$

a) Write an expression for the energy of a Fermi gas in terms of $\mathcal{D}(\epsilon)$.

$$\langle E \rangle = \int_0^\infty \frac{1}{e^{\beta(\epsilon-\mu)}+1}\mathcal{D}(\epsilon)\,\epsilon d\epsilon$$

b) What is $\mathcal{D}(\epsilon)$? See table below. Assuming that $T \ll T_F$ we can approximate the density of states near the Fermi level as a constant. What is this constant?

The density of states is $\alpha\epsilon^{1/2}$ where $\alpha = \frac{gV(2m)^{3/2}}{4\pi^2\hbar^3}$. At the fermi level, $\epsilon_F = (3N/2\alpha)^{2/3}$, the density of states is

$$\begin{aligned}\mathcal{D}(\epsilon_F) &= \alpha\left(\frac{3N}{2\alpha}\right)^{1/3} \\ &= \left(\frac{gV(2m)^{3/2}}{4\pi^2\hbar^3}\right)^{2/3}\left(\frac{3N}{2}\right)^{1/3}\end{aligned}$$

c) Find $\langle E \rangle - \langle E(\tau = 0) \rangle$ find under this approximation. (A plot may help.)

Because the fermi occupation function is anti-symmetric about $\epsilon = \mu$ and $P = 0.5$ then we can calculate the once sided integral and multiply by 2.

$$\langle E \rangle = 2\int_\mu^\infty \frac{1}{e^{\beta(\epsilon-\mu)}+1}\alpha\epsilon_F^{\frac{1}{2}}(\epsilon-\epsilon_F)\,d\epsilon$$

By symmetry we also have that $\epsilon_F \approx \mu$. We use $x \equiv \beta(\epsilon-\mu)$

$$\langle E \rangle = 2\alpha\epsilon_F^{\frac{1}{2}}\tau^2\int_0^\infty \frac{1}{e^x+1}xdx$$

$$\langle E \rangle = \frac{\pi^2}{6}\alpha\epsilon_F^{\frac{1}{2}}\tau^2$$

If we would like this in terms of N rather then ϵ_F then

$$N \approx \int_0^{\epsilon_F} \alpha\epsilon^{1/2}d\epsilon$$

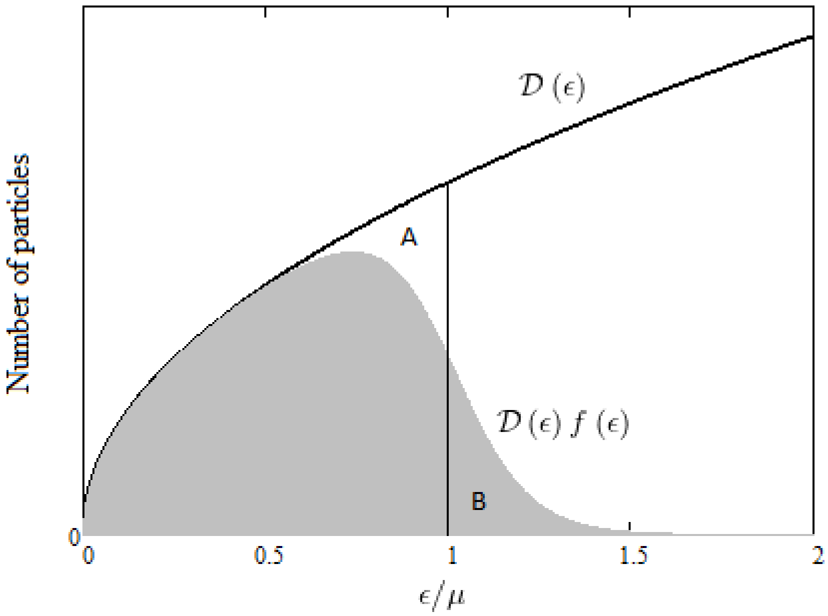
$$N \approx \frac{2}{3}\alpha\epsilon_F^{3/2}$$

$$\langle E \rangle = \frac{\pi^2}{6}\alpha^{2/3}\left(\frac{3N}{2}\right)^{1/3}\tau^2$$

recall that $\alpha = \frac{gV(2m)^{3/2}}{4\pi^2\hbar^3}$.

d) Sketch the distribution of particle energies at a finite temperature. Is our above calculation of $\langle E\rangle - \langle E(\tau=0)\rangle$ an overestimate or an underestimate?

My appolgies for the confusion about this problem as presented in class. To visulize the increase in energy when there is a finite tempertuare we plot $\mathcal{D}(\epsilon) f(\epsilon)$. Note that the energy is $\epsilon \times$ Area not just Area; more precicly the energy is the first moment this shape about 0. The increase in energy is visulaized by noting that particles need to be lifed out of lower energy states (A in plot) into high energy states (B in plot).



Because $\mathcal{D}(\epsilon)$ is increasing rather than being constant we see that increase in energy do the particles in B is larger then the decrease in energy due to particles in A. This means that our origional estimate was an **unerestimate**.

However, in most real systems of fermions we actually have a fixed N rather than a fixed μ. The above plot was made for a fixed μ. Because $B > A$ there was an increase in N. In order to keep N constant we would need to decrease the chemical potential which would reduce then energy counteracting the above effect. Judging which effect is larger is difficult to judge by looking at this polt.

Problem 3) Equivalence of ensembles

When calculated explored Bose/Fermi gases we assumed that we where working in the grand canonical ensemble. Suppose in reality we are working with a system that has a fixed number of particles, N. We may approximate this situation by working in the grand canonical ensemble and setting $\langle N_{\text{Grand con. ens.}} \rangle = N_{\text{con. ens.}}$.

How will this approximation affect the various observables ($\langle N \rangle, \Delta N, \langle E \rangle, \sigma, C_V ...$)? (Qualitatively, no need for mathematical expressions).

The value of $\langle N \rangle$ is the same by assumption. ΔN will be 0 in the canonical ensemble and $\propto 1/\sqrt{N}$ in the grand canonical. The values of $\langle E \rangle$ and C_V will be approximately the same though may have a correction of the order $1/\sqrt{N}$ that is situation specific. The value of σ will be larger (there are more possibilities for the system in the grand canonical ensemble) but the difference also shrinks as $1/\sqrt{N}$.

Under what conditions are the results for the two ensembles the same same?

They are the same in the thermodynamic limit when $1/\sqrt{N} \ll 1$.

Problem 4) Fill out the Table

Assume particles non-interacting and that there are not externally applied fields. Fill in as many as time permits.

	Fermi-Gas	Mass-less Bosons	Massive Bosons	"Boltzons"
examples	e^- in metal and starts.	photons/phonons	Integer spin atoms. BE condensate	Low pressure gas.
occupation	$\frac{1}{e^{\beta(\epsilon-\mu)}+1}$	$\frac{1}{e^{\beta(\epsilon-\mu)}-1}$	$\frac{1}{e^{\beta(\epsilon-\mu)}-1}$	$e^{-\beta(\epsilon-\mu)}$
$\epsilon(k)$	$\epsilon=\frac{\hbar^2k^2}{2m}$	$\epsilon=\hbar ck$	$\epsilon=\frac{\hbar^2k^2}{2m}$	$\epsilon=\frac{\hbar^2k^2}{2m}$
density of states	$\mathcal{D}(\epsilon)=\frac{Vgm^{3/2}}{\sqrt{2}\hbar\pi^3}\epsilon^{1/2}$	$\mathcal{D}(\epsilon)=\frac{g}{3}\left(\frac{L}{\pi\hbar c}\right)^3\epsilon^2$	$\mathcal{D}(\epsilon)=\frac{Vgm^{3/2}}{\sqrt{2}\hbar\pi^3}\epsilon^{1/2}$	$\mathcal{D}(\epsilon)=\frac{Vgm^{3/2}}{\sqrt{2}\hbar\pi^3}\epsilon^{1/2}$
$\langle E\rangle$	low τ... $\propto\tau^2$	Black body $\propto\tau^4$ high temp solid $\propto\tau$	$\propto\tau^{5/2}$	equi partiition $\propto N\tau$
Heat capacity	low τ... $\propto\tau$	photons $\propto\tau^3$ high temp solid $\propto\tau^0$	$\propto\tau^{3/2}$	$\propto N\tau^0$
$\langle n\rangle$	$\frac{N}{V}$	$\propto\tau^3$	$\frac{2}{3}\frac{g(2m)^{3/2}}{4\pi^2\hbar^3}\epsilon_F^{3/2}$	$\propto\tau^{3/2}e^{-\beta\mu}$
Equation of state	(Do after homework)	(Do after homework)	(Do after homework)	$pV=n\tau$

To get the density of states we recall the 1/8th sphere in reciprocal space

$$\mathcal{N}=\frac{g}{8}\frac{4}{3}\left(\frac{k}{\pi/L}\right)^3$$

for massive particles $k=\frac{1}{\hbar}\sqrt{2m\epsilon}$

$$\mathcal{N}=\frac{Vg}{6\hbar^3\pi^3}\left(2m\epsilon\right)^{3/2}$$

$$\mathcal{D}\left(\epsilon\right)=\frac{Vgm^{3/2}}{\sqrt{2}\hbar\pi^3}\epsilon^{1/2}$$

for mass-less particles. $\epsilon=\hbar\omega$ and $\omega=ck$ so $k=\epsilon/\hbar c$

$$\mathcal{D}\left(\epsilon\right)=\frac{g}{3}\left(\frac{L}{\pi\hbar c}\right)^3\epsilon^2$$

The energy is given by

$$\langle E\rangle=\int_0^\infty f\left(\epsilon\right)\mathcal{D}\left(\epsilon\right)\epsilon d\epsilon$$

The heat capacities are given by

$$C_V=\frac{d}{d\tau}\langle E\rangle$$

The number density is given by

$$\langle n\rangle=\frac{1}{V}\int_0^\infty f\left(\epsilon\right)\mathcal{D}\left(\epsilon\right)d\epsilon$$

For both the BE condensate you usually set N rather than μ so N/V is a what you usually care about. In the case of the Fermi gas I give the answer in terms of the Fermi level rather than the chemical potential (see problem 2).

Which ensemble did you assume when you filled out the table?

The grand canonical ensemble in most cases. For $\langle n\rangle$ for a BE condensate I switched back to the canonical. For

When filling out this table how would the answers change if particles where moving at relativistic speeds? (Qualitatively, you don't need to work them all out, though it would be a good practice problem).

The mass-less Bosons are already relativistic. For the massive particles the energy becomes $\epsilon^2=\left(mc^2\right)^2+\left(\hbar kc\right)^2$ which reduces to the mass-less limit in the ultra-relativistic limit. This will change all subsequent entries in the table.

Which cells would change if the particles where in a harmonic potential rather than a square well? (Once again, a good practice problem would be to work what the results would be.)

The density of states (and all following obervables) would change. The density of states of a HO is uniform.

How might these answers change if particles where interacting?

All results will change. However, if the interactions are week we can often include them as a correction to the non-interacting case. For example, a mean field approximation changes the energy of each state by the interaction with the average occupation of each of the other states.