Problem Set 5

November 9, 2017

Problem 1. Quadrupole Trap

(8 points)

a)

Because the N particles are non-interacting the density is N times higher than it would be for a single particle. For a single particle the density is proportional to the Boltzmann factor

$$n_0 dV = N \frac{1}{\int dV e^{-\beta U}} dV$$

$$n_0 = \frac{N}{\int dx \int dy \int dz \exp\left(-\beta\alpha\sqrt{x^2+y^2+4z^2}\right)}$$

Now we substitute $\tilde{z} \equiv 2z$ and $d\tilde{z} \equiv 2dz$ so that

$$n_0 = \frac{N}{\int dx \int dy \frac{1}{2} \int dz \exp\left(-\beta\alpha\sqrt{x^2+y^2+\tilde{z}^2}\right)}$$

Define $r = \sqrt{x^2+y^2+\tilde{z}^2}$ as in polar coordinates

$$n_0 = \frac{N}{4\pi\frac{1}{2}\int_0^\infty r^2 \exp\left(-\beta\alpha r\right) dr}$$

you can do this by integration by parts, or just look up the integral.

$$n_0 = \frac{N}{4\pi\frac{1}{2}\left(\frac{2}{\alpha^3\beta^3}\right)}$$

$$\boxed{n_0 = \frac{N\left(\alpha\beta\right)^3}{4\pi}}$$

Let's take a moment to understand this result. The units work because $\alpha\,[=]\,\frac{\text{Energy}}{\text{length}}$, $\beta\,[=]\,\frac{1}{\text{Energy}}$, and $n_0\,[=]\,\frac{1}{\text{Volume}}$. The density goes down with temperature and up with the steepness of the trap.

b)

We will calculate the Helmholtz free energy using $F = -\tau\ln(Z)$. Because the atoms are assumed to be dilute and independent we can take the single particle partition function to the N^{th} power. I will assume that the particles are indistinguishable. This is most likely a good assumtion because there are only a relitively small number of elements and from a practical point of view atom in a trap are likely of the same element.

$$Z \approx \frac{1}{N!}\left(Z_1\right)^N$$

The partition function of a single atom is the integral over its phase space divided by the granularity of phase space - planks constant.

$$Z_1 = \frac{1}{h^3}\int d\vec{r}\int d\vec{p} e^{-U(\vec{r})\beta} e^{-\vec{p}^2\beta/2m}$$

Doing the Gaussian integral over the momentums we have

$$Z_1 = \frac{(2\pi m/\beta)^{3/2}}{h^3} \int d\vec{r} e^{-U(\vec{r})\beta}$$

or simply

$$Z_1 = \frac{1}{\lambda_D^3} \int d\vec{r} e^{-U(\vec{r})\beta}$$

In other words, the granularity of position space is the thermal de Broglie wavelength.
Using the integral from part a. we have

$$Z = \frac{1}{N!} \left(\frac{1}{\lambda_\tau^3} \frac{4\pi}{(\alpha\beta)^3} \right)^N$$

Which makes the free energy

$$F \approx -\tau \left(-N \ln(N) + N + N \ln \left(\frac{4\pi}{(\alpha\beta\lambda_\tau)^3} \right) \right)$$

using $n_0 = \frac{N(\alpha\beta)^3}{4\pi}$ from above we have

$$F \approx \tau N \left(-1 + \ln \left(n_0 \lambda_\tau^3 \right) \right)$$

If the particles are sufficiently dilute, which it must be for this derivation to work, then $n_0 \lambda_\tau^3 \ll 1$ so

$$\boxed{F \approx \tau N \ln \left(n_0 \lambda_\tau^3 \right)}$$

To find the entropy we will use $F = E - \sigma\tau$. To use this we will need $\langle E \rangle = -\frac{\partial}{\partial\beta} \ln(Z)$. Noting that $\lambda_\tau = \frac{h\sqrt{\beta}}{\sqrt{2\pi m}}$ we have

$$\begin{aligned} Z &\propto \left(\frac{1}{\lambda_\tau^3 \beta^3} \right)^N \\ &= \beta^{-9N/2} \end{aligned}$$

so

$$\begin{aligned} \langle E \rangle &= -\frac{\partial}{\partial\beta} \left(-\frac{9N}{2} \ln(\beta) + \text{const.} \right) \\ &= \frac{9N}{2\beta} \\ &= \frac{9}{2} N\tau \end{aligned}$$

Note that this energy can be divided into $\frac{3}{2}N\tau$ of kinetic energy and $3N\tau$ of potential energy. To see this note that the kinetic energy obeys the equi-partition theorem because it it quadratic. If you don't take into account the β dependence of λ_τ you are only counting the potential energy.

To get the entropy use use

$$\sigma = \frac{E - F}{\tau}$$

$$\sigma = \frac{\frac{9}{2}N\tau - \tau N \left(\ln \left(n_0 \lambda_\tau^3 \right) - 1 \right)}{\tau}$$

$$\sigma = N \left(\frac{11}{2} - \ln \left(n_0 \lambda_\tau^3 \right) \right)$$

one again we work in the dilute limit

$$\boxed{\sigma \approx -N \ln \left(n_0 \lambda_\tau^3 \right)}$$

Alternatively you could calculate the entropy by

$$\sigma = -\left(\frac{\partial F}{\partial \tau}\right)_{\delta W=0} = N\left(\frac{11}{2} - \ln\left(n_0\lambda_\tau^3\right)\right)$$

where the work was held constant. Note that the work because the system expends it doesn't do work on an external body so a change in volume doesn't imply work done. This is why $\delta W = 0$ doesn't imply that $dV = 0$.

c)

The probability of two atoms trying to occupy the same state at the same time need to be negotiable. The most common state to occupy is the one at the center of the trap where n_0 is measured. The volume "occupied" by an atom is roughly the λ_τ^3 for out purposes so the assumption boils down to

$$n_0 \ll \frac{1}{\lambda_\tau^3}$$

The de Broglie wavelength is $\lambda_\tau = \frac{h}{\sqrt{2\pi m\tau}}$ so we may write this condition as

$$n_0\left(\frac{h}{\sqrt{2\pi m\tau}}\right)^3 \ll 1$$

alternatively

$$\frac{N}{4\pi}\left(\frac{\alpha h}{\sqrt{2\pi m\tau^3}}\right)^3 \ll 1$$

d)

i)

In order to hold entropy which we derived above constant we need $n_0\lambda_\tau^3$ to be constant. Hence

$$n_0\lambda_\tau^3 = \frac{N\left(\alpha\beta\right)^3}{4\pi}\left(\frac{h}{\sqrt{2\pi m\tau}}\right)^3 = \text{const.}$$

$$\frac{\alpha}{\tau^{3/2}} = \text{const.}$$

so $\alpha \propto \tau^{3/2}$.

ii)

Because $n_0\lambda_\tau^3$ is constant at constant entropy we can't make a system which initiallly has $n_0\lambda_\tau^3 \ll 1$ to a system which doesn't. This is because even though the temperature drops when you decrease α the density also drops to compensate.

Problem 3. (K&K 4.5) Surface temperature of the earth

(2 points)

This is a problem of energy ballance. The total energy emmited by the sun is

$$P_{sun} = 4\pi R_{sun}^2\sigma T_{sun}^4$$

The power recieved by earth is

$$P_{sun}\frac{\pi R_{earth}^2}{4\pi R_{orbit}^2}$$

Which must be equal to the energy emmit by the earth

$$4\pi R_{sun}^2\sigma T_{sun}^4\frac{\pi R_{earth}^2}{4\pi R_{orbit}^2} = 4\pi R_{earth}^2\sigma T_{earth}^4$$

$$R_{sun}^2 T_{sun}^4\frac{1}{4R_{orbit}^2} = T_{earth}^4$$

$$T_{earth} = T_{sun}\sqrt{\frac{R_{sun}}{2R_{orbit}}}$$

This works out to be about 280K on earth.

Problem 4. (K&K 4.6) Pressure of thermal radiation

(6 points)

a)

We would like to calculate pressure which can be expressed as

$$p = -\left(\frac{\partial \langle E\rangle}{\partial V}\right)_\sigma$$

To make this calculation easy let's consider a cubic box which we expand uniformally in each direction. This way the energy levels of the modes will all scale together in the same way (as will be shown in part b) making our calculation easy. Because the momenta carried by photons will be evenly distributed in all directions for a reasonably large box we expect the pressure measured this way will be the same as the pressure measured in other ways (say by moving one side of the box).

We may write the energy of photons in the box as

$$U = \sum_j s_j \hbar\omega_j$$

where the sum is carried out over all of the modes and s_j is the occupancy of the modes.

Suppose we expand the box as described above adiabatically. Because we do this expansion adiabatically the and the energy levels are all scaled uniformly maintaining a thermal distribution, the occupation values s_j will not change. Hence,

$$\left(\frac{\partial U}{\partial V}\right)_\sigma = \sum_j s_j\hbar\left(\frac{\partial\omega}{\partial V}\right)$$

b)

In order to have nodes on the walls the wavelenth λ needs if proportional to the box lenth L_{Box}. That is[1] $\lambda_j a_j = L_{Box}$. Noteing that $\frac{\lambda}{2\pi}\omega = c$ we have

$$\omega = \frac{2\pi c}{a_j L_{Box}}$$

$$\frac{d\omega}{dV} = \left(\frac{d\omega}{dL}\right)\left(\frac{dL}{dV}\right)$$

taking the derivative of the expression for ω above and $L_{Box} = V^{1/3}$

$$\frac{d\omega}{dV} = \left(-\frac{2\pi c}{a_j L_{Box}^2}\right)\left(\frac{1}{3}V^{-2/3}\right)$$

[1] Incedently the propartionality constant can be found in the following way. The wavelengths of modes must be such that there is a node at the walls

$$E_j = A_j \exp\left(i\frac{\pi}{L_{Box}}\left(n_{j,x}x + n_{j,y}y + n_{j,z}z\right) + i\omega t\right)$$

where the n values are integers. This solves the wave equation $\nabla^2 E = c^{-2}\partial_t^2 E$ when

$$\pi^2\frac{n_{j,x}^2 + n_{j,y}^2 + n_{j,z}^2}{L_{Box}} = \frac{\omega^2}{c^2}$$

recall that $\frac{\lambda}{2\pi}\omega = c$ so

$$\lambda = \frac{L_{Box}}{2\sqrt{n_{j,x}^2 + n_{j,y}^2 + n_{j,z}^2}}$$

so $a_j = 2\sqrt{n_{j,x}^2 + n_{j,y}^2 + n_{j,z}^2}$.

$$\frac{d\omega}{dV} = \left(-\frac{\omega}{L_{Box}}\right)\left(\frac{1}{3}\frac{L_{Box}}{V}\right)$$

$$\frac{d\omega}{dV} = -\frac{\omega}{3V}$$

c)

Putting together the above two results

$$\begin{aligned} p &= -\left(\frac{\partial U}{\partial V}\right)_\sigma \\ &= -\sum_j s_j \hbar \left(\frac{\partial \omega}{\partial V}\right) \\ &= -\sum_j s_j \hbar \left(-\frac{\omega}{3V}\right) \\ &= \frac{1}{3V}\sum_j s_j \hbar\omega \\ &= \frac{U}{3V} \end{aligned}$$

d)

To get the radiation pressure we take recall the energy density of photons u from class

$$p_{light} = \frac{U}{3V} = \frac{\pi^2}{45\hbar^3 c^3}(k_B T)^4$$

The kinetic pressure

$$p_{kinetic} = \frac{N}{V} k_B T$$

these are rougly equal when

$$\frac{N}{V} k_B T = \frac{\pi^2}{45\hbar^3 c^3}(k_B T)^4$$

$$T = \frac{1}{k_B}\left(\frac{45\hbar^3 c^3}{\pi^2}\frac{N}{V}\right)^{1/3}$$

$$T \approx 3.2 \times 10^7 \mathrm{K}$$

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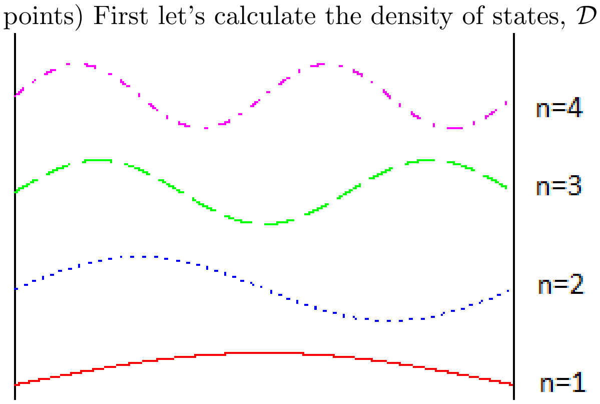
(4 points) First let's calculate the density of states, $\mathcal{D}(\omega)$. Consider the modes



For light moving with speed $v = \omega\lambda/2\pi$ and $n_1\lambda = 2L$. The number of modes (of one polarization) with frequencies at or less than or equal to $\omega$is

$$n_1 = \frac{2L}{(2\pi v/\omega)}$$

because there are 2 possible polarization the total number of modes with frequency less then or equal to ω is

$$n = \frac{2L\omega}{\pi v}$$

which gives a density of states of

$$\mathcal{D}(\omega) = \frac{\partial n}{\partial \omega} = \frac{2L}{\pi v}$$

The energy is given by the density of states times their occupation times the energy per photon

$$\langle E \rangle = \sum_{\omega} \mathcal{D}(\omega)\, s_\omega \hbar\omega$$

where then occupation is that of a quantum harmonic oscillator

$$\begin{aligned}
s_\omega &= \frac{\sum_{N=1}^{\infty} N e^{-\hbar\omega N/\tau}}{\sum_{N=1}^{\infty} e^{-\hbar\omega N/\tau}} \\
&= \frac{1}{\hbar\omega}\frac{\partial}{\partial\beta}\ln\left(\sum_{N=1}^{\infty} e^{-\hbar\omega N/\tau}\right) \\
&= \frac{1}{\hbar\omega}\frac{\partial}{\partial\beta}\ln\left(\frac{1}{1-e^{-\hbar\omega/\tau}}\right) \\
&= \frac{-e^{-\hbar\omega/\tau}}{1-e^{-\hbar\omega/\tau}} \\
&= \frac{1}{e^{\hbar\omega/\tau}-1}
\end{aligned}$$

assuming $\tau \gg \hbar\omega$.

$$\langle E \rangle \approx \int_0^{\infty} \frac{2L}{\pi v} \cdot \frac{\hbar\omega}{e^{\hbar\omega/\tau}-1} d\omega$$

$$\langle E \rangle \approx \int_0^{\infty} \frac{2L}{\pi v} \cdot \frac{u}{e^{u}-1}\frac{\tau^2}{\hbar} du$$

define $u = \hbar\omega/\tau$ and $du = \hbar d\omega/\tau$

$$\langle E \rangle \approx \frac{2L}{\pi v}\frac{\pi^2}{6}\frac{\tau^2}{\hbar}$$

$$\langle E \rangle \approx \frac{\pi L \tau^2}{3 v\hbar}$$

$$C = k_b \frac{\partial \langle E \rangle}{\partial \tau}$$

$$\boxed{C = \frac{2\pi L k_b \tau}{3 v \hbar}}$$

Note that if you assumed only one polarization your answer would be a factor of 2 smaller.