PROBLEM SET 8

Reading in Kittel & Kroemer:

- Ch. 8, pp. 227-232 (Heat and Work)
- Ch. 7 (Fermi and Bose Gases)

1. *Heat engine with finite reservoirs.* Two identical bodies, each characterized by a heat capacity at constant pressure C_p which is independent of temperature, are used as heat reservoirs for a heat engine (Fig. 1). The bodies remain at constant pressure. Initially, their temperatures are τ_1 and τ_2, respectively; finally, as a result of the operation of the heat engine, the bodies will attain a common final temperature τ_f.

   a. What is the total amount of work W done by the engine? Express the answer in terms of C_p, τ_1, τ_2, and τ_f.
      Hint: By definition, the heat engine operates in a cycle, i.e., at the end of the process, the heat engine is in the same macrostate in which it began.
   b. Use arguments based upon entropy considerations to derive an inequality relating τ_f to the initial temperatures τ_1 and τ_2.
   c. For given initial temperatures τ_1 and τ_2, what is the maximum amount of work obtainable from the engine?

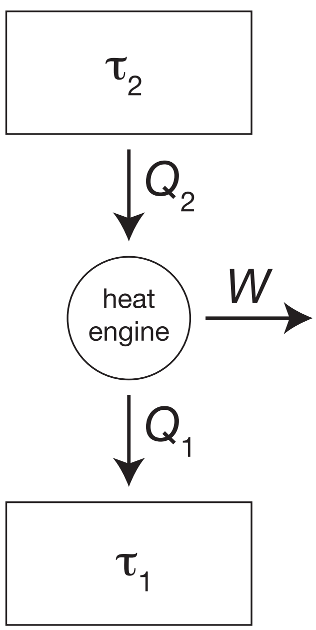

FIG. 1. Schematic of the heat engine and reservoirs in Problem 1.

2. *Pressure of a degenerate Fermi gas.* The equation of state of a strongly interacting Fermi gas, relevant to the study of neutron stars, is difficult to calculate from first principles but can be measured in experiments with ultracold atoms [1]. These experiments make use of the Gibbs-Duhem equation

$$dp = nd\mu + sd\tau \tag{1}$$

to determine the pressure p from measurements of the atomic density $n \equiv N/V$ as a function of chemical potential at fixed temperature τ; $s \equiv \sigma/V$ denotes the entropy density.

a. Derive the Gibbs-Duhem equation (Eq. 1).

b. Show that a *non-interacting* Fermi gas in the ground state exerts a pressure

$$p = \frac{(3\pi^2)^{2/3}}{5}\frac{\hbar^2}{m}\left(\frac{N}{V}\right)^{5/3}. \tag{2}$$

3. *Mass-radius relationship for white dwarfs* (K&K 7.6). Consider a white dwarf star of mass M and radius R. Let the electrons be degenerate but non-relativistic; the protons are nondegenerate.

   a. Show that the order of magnitude of the gravitational self-energy is $-GM^2/R$, where G is the gravitational constant. (If the mass density is constant within the sphere of radius R, the exact potential energy is $-3GM^2/5R$.)

   b. Show that the order of magnitude of the kinetic energy of the electrons in the ground state is
   $$\frac{\hbar^2 N^{5/3}}{mR^2} \approx \frac{\hbar^2 M^{5/3}}{m M_H^{5/3} R^2}, \tag{3}$$
   where m is the mass of an electron and M_H is the mass of a proton.

   c. Show that if the gravitational and kinetic energies are of the same order of magnitude (as required by the virial theorem of mechanics), $M^{1/3}R \approx 10^{20}\mathrm{g}^{1/3}$ cm.

   d. If the mass is equal to that of the Sun (2×10^{33} g), what is the density of the white dwarf?

   e. It is believed that pulsars are stars composed of a cold degenerate gas of neutrons. Show that for a neutron star $M^{1/3}R \approx 10^{17}\mathrm{g}^{1/3}$ cm. What is the value of the radius for a neutron star with a mass equal to that of the Sun? Express the result in km.

4. *Boson gas in one dimension* (from K&K 7.9). Calculate the integral representing the number N_e of atoms in excited states for a one-dimensional gas of non-interacting bosons. Show that the integral does not converge. This result suggests that an ideal boson gas in one dimension does not form a condensate. *Note:* take $\mu = 0$ for the calculation, and give a justification for this approximation.

5. *Energy, heat capacity, and entropy of a 3D Bose gas* (from K&K 7.8). Find expressions as a function of temperature in the region $\tau < \tau_E$ for the energy, heat capacity, and entropy of a gas of N non-interacting bosons of spin zero confined to a volume V. Put the definite integral in dimensionless form; it need not be evaluated.

6. *Fluctuations of Fermi and Bose gases* (from K&K 7.11-7.12).

   a. Show for a single orbital of a non-interacting fermion system that
   $$\langle(\Delta N)^2\rangle = \langle N\rangle\left(1 - \langle N\rangle\right), \tag{4}$$
   where $\langle N\rangle$ is the average number of fermions in that orbital. (By definition, $\Delta N = N - \langle N\rangle$.)

b. Show that for a single orbital of a non-interacting boson system,

$$\langle(\Delta N)^2\rangle = \langle N\rangle\,(1+\langle N\rangle)\,. \tag{5}$$

c. Sketch the variance $\langle(\Delta N)^2\rangle$ vs. $\langle N\rangle$ for fermions and bosons on a single plot, labeling each. Notice that for fermions, the fluctuation vanishes for orbitals with energies deep enough below the Fermi energy so that $\langle N\rangle = 1$. By contrast, for bosons, if the occupancy is large ($\langle N\rangle \gg 1$), the fractional fluctuations are of the order of unity: $\langle N^2\rangle/\langle N\rangle^2 \approx 1$, so that the actual fluctuations can be enormous. It has been said that "bosons travel in flocks."

7. *Optional review problems (more will soon be posted separately):*

- K&K 7.2
- K&K 7.10

[1] Immanuel Bloch, Jean Dalibard, and Sylvain Nascimbene, "Quantum simulations with ultracold quantum gases," Nature Physics **8**, 267–276 (2012).