

Problem Set 6 Solutions

Problem 1

The flux density between the plane at temperature T_u and the plane at temperature T_m is given by

$$J_{um} = \sigma_B (T_u^4 - T_m^4) .$$

Similarly, the flux density between the plane at temperature T_m and the plane at temperature T_l is given by

$$J_{ml} = \sigma_B (T_m^4 - T_l^4) .$$

In equilibrium, the flux of energy into the middle plane equals the flux of energy out, so $J_{um} = J_{ml}$, and

$$T_m = \left(\frac{T_u^4 + T_l^4}{2} \right)^{1/4} .$$

The flux density is then given by

$$J_{um} = J_{ml} = \frac{1}{2} \sigma_B (T_u^4 - T_l^4)$$

which is half of the flux density without the middle plane.

Problem 2

The plane at temperature T_u is emitting flux density $\sigma_B T_u^4$ toward the plane at temperature T_m . The plane at temperature T_m is emitting flux density $e\sigma_B T_m^4$ toward the plane at temperature T_u and also reflecting back a fraction r of the incident radiation. Thus, the flux density between the planes is

$$J_{um} = \sigma_B T_u^4 - e\sigma_B T_m^4 - r\sigma_B T_u^4.$$

Now $a = 1 - r$, and by Kirchhoff's law, $e = a$. Thus $e = 1 - r$, and we have

$$J_{um} = (1 - r)\sigma_B T_u^4 - (1 - r)\sigma_B T_m^4.$$

Similarly, the flux density between the plane at temperature T_m and the plane at temperature T_l is given by

$$J_{ml} = (1 - r)\sigma_B T_m^4 - (1 - r)\sigma_B T_l^4.$$

In equilibrium, $J_{um} = J_{ml}$. As before, we have

$$T_m = \left(\frac{T_u^4 + T_l^4}{2} \right)^{1/4},$$

but now the flux density is given by

$$J_{um} = J_{ml} = \frac{1 - r}{2} \sigma_B (T_u^4 - T_l^4).$$

Problem 3

The entropy of the black body radiation field is given by

$$\sigma = \frac{4\pi^2 V}{45} \left(\frac{\tau}{\hbar c} \right)^3 = \frac{4\pi^2}{45} \left(\frac{1}{\hbar c} \right)^3 (V\tau^3).$$

Once the black body radiation decoupled from matter, the number of photons in each mode became constant as a function of time. This means that the entropy remained constant, so the black body radiation expanded isentropically. Letting σ_i and σ_f be the entropies of the radiation before and after expansion, respectively, we have

$$\sigma_f = \sigma_i \implies (V_f \tau_f^3) = (V_i \tau_i^3)$$

or equivalently $\tau_f V_f^{1/3} = \tau_i V_i^{1/3}$.

- a. Since the radius R of the universe is given by $R \propto V^{1/3}$, we have

$$\tau_i R_i = \tau_f R_f.$$

With $\tau_f = 2.9\text{K}$ and $\tau_i = 3000\text{K}$, we have

$$\frac{R_i}{R_f} = \frac{2.9}{3000} \approx 10^{-3}.$$

Likewise, if the radius of the universe expanded linearly from zero, the decoupling occurred at fractional time

$$\frac{t_i}{t_f} \approx 10^{-3}.$$

- b. For isentropic expansion, $dQ = 0$, so the work done by the photons is equal to the inverse of the change in internal energy of the photons:

$$dW = -dU.$$

Now

$$U = \frac{\pi^2}{15\hbar^3 c^3} \tau^4 V = \frac{\pi^2}{15\hbar^3 c^3} (\tau_i^3 V_i) \tau$$

so

$$W = U_i - U_f = \frac{\pi^2}{15\hbar^3 c^3} (\tau_i^3 V_i) (\tau_i - \tau_f).$$

Problem 4

Since $1\text{K} \ll 100\text{K}$, the phonon heat capacity is given by

$$\frac{C_V^{\text{phonons}}}{V} = \frac{12\pi^4}{5} \frac{N}{V} k_B \left(\frac{1\text{K}}{100\text{K}} \right)^3 = 32.3 \frac{\text{J}}{\text{K m}^3}$$

The photon heat capacity is given by

$$\frac{C_V^{\text{photons}}}{V} = \frac{\partial}{\partial T} \left(\frac{U}{V} \right)_V = \frac{4\pi^2}{15\hbar^3 c^3} k_B^4 T^3 = \left(3.05 \cdot 10^{-15} \frac{\text{J}}{\text{K}^4 \text{m}^3} \right) \cdot T^3.$$

Setting the two heat capacities equal and solving for T , we have

$$T = 2.2 \cdot 10^5 \text{ K}.$$

Problem 5

- a. Since there are no transverse modes in the helium, the number of modes with frequency $\leq \omega$ is a factor of 3 smaller than what we calculated in class (for a solid with both longitudinal and transverse modes). So we have

$$N(\omega) = \frac{1}{8} \cdot \frac{4\pi}{3} n_w^3.$$

The helium still supports $3N$ total modes, so the Debye temperature is given by

$$\theta = \left(\frac{\hbar v}{k_B} \right) \left(\frac{18\pi^2 N}{V} \right)^{1/3}.$$

The density is given by

$$\frac{N}{V} = 0.145 \frac{\text{g}}{\text{cm}^3} \cdot 6.022 \cdot 10^{23} \frac{\text{atoms}}{\text{mol}} \cdot \frac{1 \text{ mol}}{4 \text{ g}} = 2.18 \cdot 10^{22} \frac{\text{atoms}}{\text{cm}^3}$$

and the velocity of sound in the helium is given by

$$v = 2.383 \cdot 10^4 \frac{\text{cm}}{\text{s}}$$

so the Debye temperature is

$$\theta = 28.5 \text{ K}.$$

- b. One gram of helium contains

$$N = 6.022 \cdot 10^{23} \frac{\text{atoms}}{\text{mol}} \cdot \frac{1 \text{ mol}}{4 \text{ g}} \cdot 1 \text{ g} = 1.51 \cdot 10^{23} \text{ atoms}$$

so the heat capacity per gram is

$$c_V = \frac{12\pi^4 N k_b}{5} \left(\frac{T}{\theta} \right)^3 = \left(2.11 \cdot 10^{-2} \frac{\text{J}}{\text{K g}} \right) T^3$$

This prediction of the heat capacity is within 10% of the experimentally measured value.

Problem 6

The difference in K^+ chemical potential between the sap and the pond water is given by

$$\mu_{\text{sap}} - \mu_{\text{pond}} = \tau \ln \left(\frac{n_{\text{sap}}}{n_{\text{pond}}} \right) = k_B T \ln 10^4 = 9.21 k_B T.$$

Substituting $k_B = 8.617 \cdot 10^{-5} \frac{\text{eV}}{\text{K}}$ and $T = 300 \text{ K}$, we have

$$\mu_{\text{sap}} - \mu_{\text{pond}} = 0.238 \text{ eV}.$$

Since each K^+ ion has one electron's worth of charge, the voltage difference is 0.238 V, as expected.

Problem 7

- a. The grand partition function is defined as

$$\mathcal{Z} = \sum_i e^{\beta\mu N_i - \beta E_i} = \sum_i \phi^{N_i} e^{-\beta E_i}$$

where the sum is taken over all possible microstates. In general, we have

$$\phi \frac{\partial \ln \mathcal{Z}}{\partial \phi} = \phi \cdot \frac{\sum_i N_i \phi^{N_i-1} e^{-\beta E_i}}{\mathcal{Z}} = \sum_i \frac{N_i \cdot \phi^{N_i} e^{-\beta E_i}}{\mathcal{Z}} = \langle N \rangle.$$

For this system, the grand partition function is

$$\mathcal{Z} = e^{\beta\delta/2} + \phi \left(e^{\beta\Delta/2} + e^{-\beta\Delta/2} \right) + \phi^2 e^{-\beta\delta/2}.$$

- b. For the average number of electrons per atom to be unity, each atom must be equally likely to be in the positive or negative ionization state. In other words,

$$e^{\beta\delta/2} = \phi^2 e^{-\beta\delta/2}$$

or

$$\phi = e^{\beta\delta/2}.$$

- c. Since $\phi = e^{\beta\mu}$, we have $\mu = \delta/2$.
- d. At finite temperature and fixed particle number, the system will preferentially occupy states of lower energy (in this case, the positive ionization state). However, at fixed temperature and positive chemical potential, the system will preferentially occupy states with more electrons (in this case, the negative ionization state). Thus, there is a competition between the energy difference (which drives atoms toward the positive ionization state) and the chemical potential (which drives atoms toward the negative ionization state). When $\mu = \delta/2$, these effects balance, and each ionization state is equally likely.

The value of the chemical potential at which $\langle N \rangle = 1$ is independent of Δ because both states of the neutral atom have $N_i = 1$. From the perspective of the chemical potential, which is sensitive to electron number differences, the two states of the neutral atom are indistinguishable.