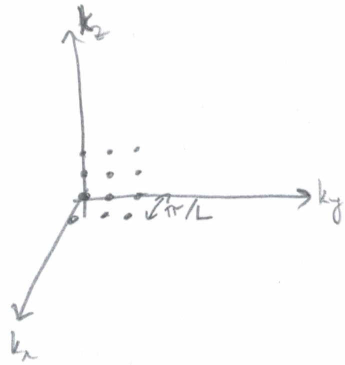
1. (a)



# of states of momentum $\le k$:

$$\mathcal{N}(k) = \frac{g}{8}\,\frac{4}{3}\pi k^3 / \left(\frac{\pi}{L}\right)^3 \quad \text{, where multiplicity } g=2$$

$$= \frac{Vk^3}{3\pi^2}$$

$$\varepsilon = \hbar c k \Rightarrow k = \varepsilon/(\hbar c) \Rightarrow \mathcal{N}(\varepsilon) = \frac{V\varepsilon^3}{3\pi^2(\hbar c)^3}$$

$$\therefore\ N = \frac{V\varepsilon_F^3}{3\pi^2(\hbar c)^3}$$

(i)

$$\varepsilon_F^3 = n\cdot 3\pi^2(\hbar c)^3 \Rightarrow \boxed{\varepsilon_F = (3\pi^2 n)^{1/3}\cdot \hbar c}$$

(ii)

$$\mathcal{D}(\varepsilon) = \frac{\partial \mathcal{N}}{\partial \varepsilon} = \frac{V\varepsilon^2}{\pi^2(\hbar c)^3}$$

(b) Maxwell relations for control parameters $\tau, V, N \Rightarrow$ start from $F(\tau, V, N)$

$$dF = -\sigma d\tau - p dV + \mu dN$$

(i)

$$\left.\frac{\partial^2 F}{\partial V \partial N}\right|_\tau = \left[\frac{\partial}{\partial V}\left(\frac{\partial F}{\partial N}\right)_{\tau,V}\right]_N = \left(\frac{\partial \mu}{\partial V}\right)_{\tau N}$$

$$= \left[\frac{\partial}{\partial N}\left(\frac{\partial F}{\partial V}\right)_{\tau,N}\right]_V = -\left(\frac{\partial p}{\partial N}\right)_{\tau,V}$$

$$\Rightarrow \left(\frac{\partial p}{\partial N}\right)_{\tau,V} = -\left(\frac{\partial \mu}{\partial V}\right)_{\tau,N}$$

(ii)

$$\left(\frac{\partial^2 F}{\partial \tau \partial V}\right)_N = -\left.\frac{\partial \sigma}{\partial V}\right|_{\tau,N} = -\left.\frac{\partial p}{\partial \tau}\right|_{V,N} \Rightarrow \left(\frac{\partial p}{\partial \tau}\right)_{V,N} = \left(\frac{\partial \sigma}{\partial V}\right)_{\tau,N}$$

(c) We know that at $\tau = 0$, $\mu = \varepsilon_F$. Thus,

$$\left(\frac{\partial \mu}{\partial V}\right)_{\tau=0,N} = \left(\frac{\partial \varepsilon_F}{\partial V}\right)_{\tau=0,N} = \frac{\partial}{\partial V}\left[\left(\frac{3\pi^2 N}{V}\right)^{1/3}\cdot \hbar c\right] = \frac{-1}{3V}\left(\frac{3\pi^2 N}{V}\right)^{1/3}\cdot \hbar c$$

By (b).i, we then have

$$\left(\frac{\partial p}{\partial N}\right)_{\tau,V} = \frac{1}{3V}\left(\frac{3\pi^2 N}{V}\right)^{1/3}\cdot \hbar c$$

Since $p=0$ @ $N=0$, we can integrate:

$$p = \int_0^N dN'\,\frac{\hbar c}{3V}\left(\frac{3\pi^2 N}{V}\right)^{1/3}$$

$$= \frac{3}{4}\,\frac{\hbar c}{3}\left[3\pi^2\left(\frac{N}{V}\right)^4\right]^{1/3}$$

$$p = \frac{\hbar c}{4}(3\pi^2)^{1/3} n^{4/3} \quad @\ \tau = 0$$

We can also write this as $\boxed{p = \frac{1}{4} n \varepsilon_F}$ @ $\tau = 0$

1 (c) ii.

$$\left(\frac{\partial p}{\partial \tau}\right)_{V,N} = \left(\frac{\partial \sigma}{\partial V}\right)_{\tau,N}$$

By the 3rd law, $\sigma(V)$ should be constant at $\tau = 0$.

(For the case of a Fermi gas, it is evident that $\sigma = 0$ at $\tau = 0$.)

Hence $\left(\frac{\partial p}{\partial \tau}\right)_{V,N} = 0$.

(d) (i) $N = \int_0^\infty d\varepsilon\, \mathcal{D}(\varepsilon) \cdot \frac{1}{e^{\beta(\varepsilon-\mu)}+1}$

(ii) Quantum statistics irrelevant if $e^{\beta(\varepsilon-\mu)} \gg 1$ for all ε

We always have $\mu < 0$, so having $\boxed{\frac{|\mu|}{\tau} \gg 1}$ will ensure $e^{\beta(\varepsilon-\mu)} \gg 1$ even at $\varepsilon = 0$

i.e., $|\mu| \gg \tau$, $\mu < 0$

[$e^{-\mu/\tau} \gg 1$ is also a sufficient condition, i.e., fugacity $e^{\beta\mu} \ll 1$.]

(iii) $N \approx \int_0^\infty d\varepsilon\, \mathcal{D}(\varepsilon) e^{\beta(\mu-\varepsilon)} = \frac{V}{\pi^2(\hbar c)^3} \int_0^\infty \varepsilon^2 e^{\beta(\mu-\varepsilon)} d\varepsilon$

$$= \frac{V e^{\beta\mu}}{\pi^2(\hbar c)^3} \frac{\partial^2}{\partial \beta^2} \underbrace{\int_0^\infty d\varepsilon\, e^{-\beta\varepsilon}}_{= \left[-\frac{1}{\beta}e^{-\beta\varepsilon}\right]_0^\infty = \frac{1}{\beta}} = \frac{2Ve^{\beta\mu}}{\pi^2(\hbar c)^3}\beta^{-3} = \frac{2Ve^{\beta\mu}\tau^3}{\pi^2(\hbar c)^3}$$

$$\therefore \quad \frac{\pi^2 n (\hbar c/\tau)^3}{2} = e^{\mu/\tau} \rightarrow \boxed{\mu = \tau \ln\left[\frac{\pi^2}{2}\left(\frac{\hbar c}{\tau}\right)^3 n\right]}$$

Can also express as: $\mu = \tau \ln\left[\frac{\pi^2}{2}\left(\frac{\hbar c}{\tau}\right)^3 \frac{\varepsilon_F^3}{3\pi^2(\hbar c)^3}\right]$

$$\therefore \quad \mu = \tau \ln\left[\frac{1}{6}\left(\frac{\tau_F}{\tau}\right)^3\right]$$

(iv) A: $n \ll \frac{2}{\pi^2}\left(\frac{\tau}{\hbar c}\right)^3$

B: $\tau \gg \tau_F/\sqrt[3]{6}$ ← to be exact...

(precise numerical factors aren't critical here)

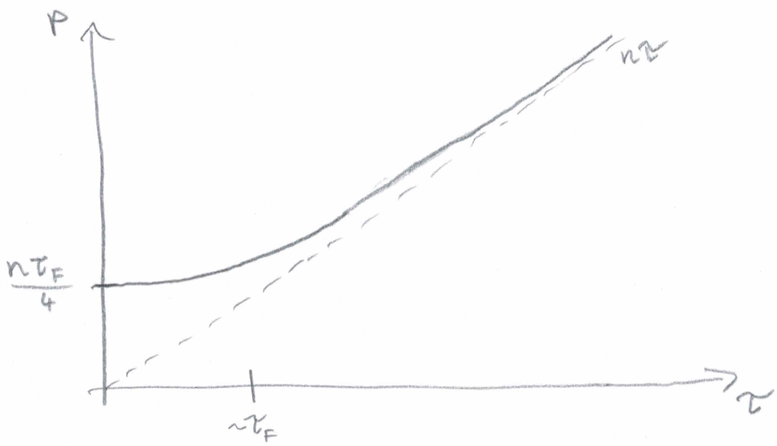
(cont'd)

(v) Again can use $-\left(\frac{\partial \mu}{\partial V}\right)_{\tau,N} = \left(\frac{\partial p}{\partial N}\right)_{\tau,V}$

$$\mu = -\tau \ln V + \text{const} \Rightarrow -\left(\frac{\partial \mu}{\partial V}\right)_{\tau,N} = +\frac{\tau}{V}$$

$$\therefore\ p = \int_0^N dN' \frac{\tau}{V} = \frac{N\tau}{V}$$

$$\boxed{pV = N\tau} \Rightarrow p = n\tau$$



2. $H = -d\mathcal{E}\cos\theta$

(a) $Z_1 = \int_0^{2\pi} d\varphi \int_0^{\pi} d\theta \cdot \sin\theta \, e^{\beta d \mathcal{E}\cos\theta}$

$$= -2\pi \int_0^{\pi} d(\cos\theta) e^{\beta d\mathcal{E}\cos\theta} = \frac{2\pi}{\beta d\mathcal{E}}\left[e^{\beta d\mathcal{E}} - e^{-\beta d\mathcal{E}}\right] = \frac{4\pi\tau}{\mathcal{E}d}\sinh(\mathcal{E}d/\tau)$$

$Z = Z_1^N$

(b) $$\boxed{\langle E\rangle = -\frac{\partial \ln Z}{\partial\beta} = -N\frac{\partial}{\partial\beta}\left(\ln[\sinh(\beta\mathcal{E}d)] - \ln\beta\right) = \left[\frac{1}{\beta} - \frac{\cosh(\beta\mathcal{E}d)}{\sinh(\beta\mathcal{E}d)}\cdot\mathcal{E}d\right]\cdot N}$$

$$= \boxed{N[\tau - \mathcal{E}d\coth(\mathcal{E}d/\tau)]}$$

$$\boxed{P = -\langle E\rangle/\mathcal{E} = -\frac{N\tau}{\mathcal{E}} + Nd\coth(\mathcal{E}d/\tau)}$$

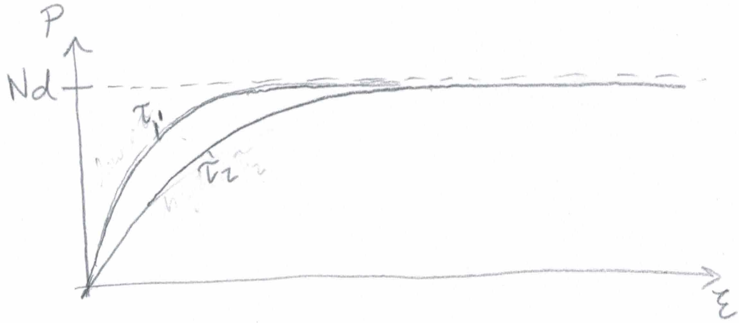
(c) $\chi = \partial P/\partial\mathcal{E}$

Let $x \equiv \mathcal{E}d/\tau$. Then $P/N = -d/x + d\cdot\frac{e^x + e^{-x}}{e^x - e^{-x}}$ $\xrightarrow{x \ll 1}$ $-\frac{d}{x} + d\cdot\frac{2+x^2}{2x} \approx \frac{dx}{2} = \mathcal{E}(d/\tau)^2$

$\xrightarrow{x \gg 1}$ $d - d/x \to d$

$$\chi = \frac{\partial P}{\partial x}\frac{\partial x}{\partial\mathcal{E}} = \frac{d}{\tau}\frac{\partial P}{\partial x} \xrightarrow{x\ll 1} \frac{Nd^2}{2\tau} \text{ — weak field}$$

$$\xrightarrow{x \gg 1} 0 \text{ — strong field}$$

(d) 

$\tau_1 < \tau_2$

(e) The classical model should have a higher heat capacity at low temperature, because it is possible to tilt the dipoles by an arbitrarily small angle, thereby increasing energy & entropy. By contrast, if there are only two discrete states, then excitations are exponentially suppressed at $\tau \ll d\mathcal{E}$. This is similar to the difference between the Debye model ($\exists$ low-energy excitations) and the Einstein model.

3. Entropy of gold

(a) All below Debye temperature τ_D

(i). Heat capacity $C_\ell \propto \tau^3$ for $\tau < \tau_D$

$$\tau \left(\frac{\partial \sigma}{\partial \tau}\right)_V = \alpha \tau^3 \Rightarrow \int_0^\sigma d\sigma' = \int_0^\tau \alpha \tau'^2 \, d\tau' \Rightarrow \sigma = \frac{1}{3}\alpha \tau^3$$

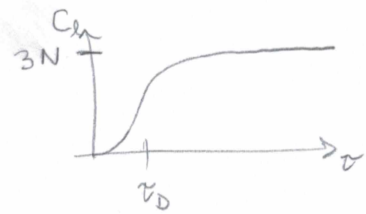
τ doubles $\Rightarrow$ σ increases by a factor of 8.

(ii.) Heat capacity $C_{el} \propto \tau$ since $\tau \ll \tau_F$ (metal at low temperature)

$$\tau \left(\frac{\partial \sigma}{\partial \tau}\right) \propto \tau \Rightarrow \frac{\partial \sigma}{\partial \tau} = \text{constant} \Rightarrow \sigma \propto \tau$$

$\Rightarrow$ σ increases by a factor of 2

(b) Now starting above τ_D — heat capacity $\approx$ const for lattice vibrations



So (i) is significantly different:

$$C_\ell \approx \text{const} = \tau \left(\frac{\partial \sigma}{\partial \tau}\right)_V \Rightarrow \int_{\sigma_1}^{\sigma_2} d\sigma = \int_{\tau_1}^{\tau_2} \frac{d\tau}{\tau} \Rightarrow \sigma_2 - \sigma_1 = \ln(\tau_2) - \ln(\tau_1)$$

$$\frac{\sigma_2}{\sigma_1} = 1 + \ln(\tau_2/\tau_1)$$

$$\boxed{\frac{\sigma_2}{\sigma_1} \approx 1 + \ln 2}$$

$\therefore$ The entropy of lattice vibrations increases by less than a factor of 2.

4. For each site, $\mathcal{Z} = 1 + e^{\beta(\mu-\varepsilon)}$ and the occupation is simply given by the Fermi-Dirac function. Explicitly $P_{occupied} = \dfrac{e^{\beta(\mu-\varepsilon)}}{1+e^{\beta(\mu-\varepsilon)}} = \dfrac{1}{e^{\beta(\varepsilon-\mu)}+1}$

(a) $$\rightarrow \frac{N}{N_s} = \frac{1}{e^{\beta(\varepsilon-\mu)}+1} = \frac{1}{e^{(\varepsilon-\mu)/\tau}+1}$$

(b) Entropy σ_s.

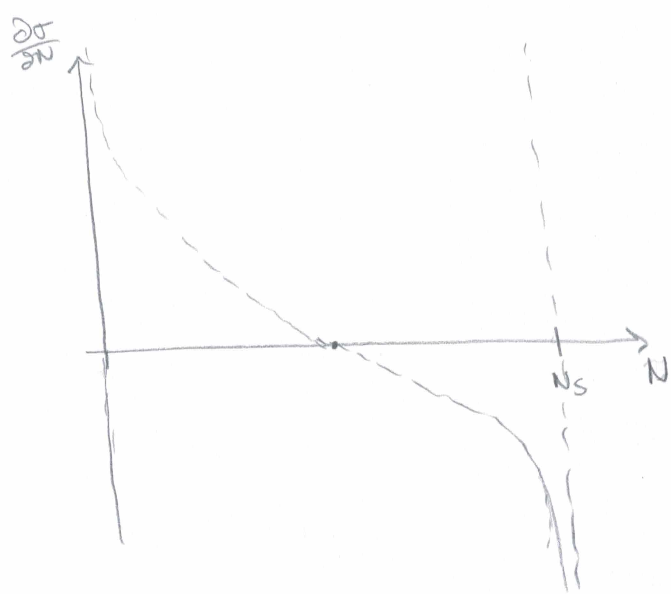
$$dF_s = -\sigma_s\, d\tau + \mu\, dN \Rightarrow \sigma_s = -\frac{\partial F_s}{\partial \tau}\Big|_N$$

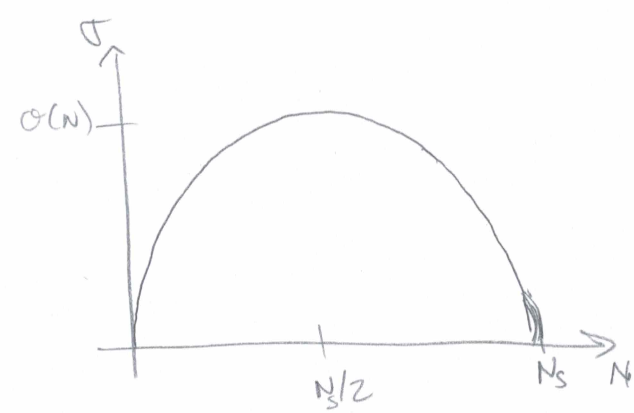
$$\left(\frac{\partial \sigma_s}{\partial N}\right)_\tau = -\frac{\partial^2 F_s}{\partial N \partial \tau} = -\left(\frac{\partial \mu}{\partial \tau}\right)_N$$

$$e^{(\varepsilon-\mu)/\tau} = \frac{N_s}{N} - 1 \Rightarrow \frac{\varepsilon-\mu}{\tau} = \ln\left(\frac{N_s}{N}-1\right) \Rightarrow \mu = \varepsilon - \tau \ln\left(\frac{N_s}{N}-1\right)$$

$$-\left(\frac{\partial \mu}{\partial \tau}\right)_N = \boxed{\ln\left(\frac{N_s}{N}-1\right) = \left(\frac{\partial \sigma_s}{\partial N}\right)_\tau}$$

(c)



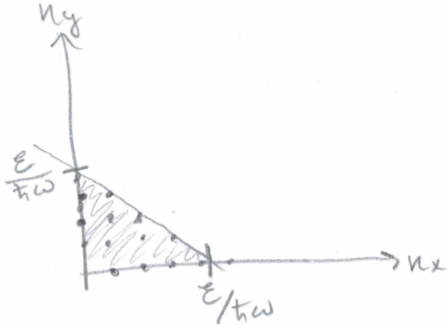


4. (d) (i) The entropy of the gas <u>decreases</u>, since it is now occupying the same amount of momentum space (set by τ) but less volume in real space.

(ii) The entropy of the surface <u>decreases</u>. This is because the chemical potential of the gas is given by $\mu = \tau \ln(n_g \lambda_\tau^3)$, where the compression has increased the density n_g of the gas. The increase in chemical potential increases the number N of molecules adsorbed, and since we already had $N > N_s/2$ this decreases the entropy according to our graphs in (c). Physically: we are getting closer to all sites being occupied, which is a low-entropy state.

(iii) The entropy of the reservoir must increase by the 2nd law of thermodynamics, since the entropy of the surface and gas decreased.

5.



(a) $\mathcal{N}(\varepsilon) = \frac{1}{2}\left(\frac{\varepsilon}{\hbar\omega}\right)^2$

(b) $\mathcal{D}(\varepsilon) = \frac{\partial \mathcal{N}}{\partial \varepsilon} = \frac{\varepsilon}{(\hbar\omega)^2}$

(c) $$N_e = \int_0^\infty d\varepsilon \, \frac{\varepsilon/(\hbar\omega)^2}{e^{\beta(\varepsilon-\mu)}-1} \xrightarrow[\mu\to 0]{} \left(\frac{\tau}{\hbar\omega}\right)^2 \underbrace{\int_0^\infty \frac{dx \cdot x^2}{e^x - 1}}_{\mathcal{I}}$$

As $x \to 0$, the integrand $\to x$

As $x \to \infty$, integrand $\to x^2 e^{-x}$

The integral converges

$\Rightarrow$ BEC for $N > \left(\frac{\tau}{\hbar\omega}\right)^2 \cdot \mathcal{I}$

$$\boxed{\tau_E = \hbar\omega\sqrt{N/\mathcal{I}}}$$

(d) (i). By equipartition, $\frac{1}{2} m\omega^2 \langle x^2 \rangle = \frac{1}{2}\tau$

$$\Rightarrow \langle x^2 \rangle = \langle y^2 \rangle = \frac{\tau}{m\omega^2}$$

$x_{rms} = y_{rms} = \frac{1}{\omega}\sqrt{\tau/m}$

(ii) $n \approx \frac{N\omega^2 m}{\tau} \Rightarrow$ BEC condition $N\omega^2 > (\tau/\hbar)^2 \mathcal{I}$, yields

$$\frac{n\tau}{m} > (\tau/\hbar)^2 \mathcal{I}$$

$$\Rightarrow \tau_E = \frac{\hbar^2 n}{m\mathcal{I}} \quad \text{or} \quad \boxed{n_E = \frac{m\tau}{\hbar^2}\mathcal{I}}$$

(iii) $\lambda_\tau = \frac{h}{\sqrt{2\pi m\tau}} \Rightarrow n_E \propto 1/\lambda_\tau^2$

As expected, need interparticle spacing $< \lambda_\tau$.