EXERCISE 6A: PLANCK LAW OF RADIATION

Objectives:

- Derive the Planck distribution for the thermal occupation of an electromagnetic mode
- Introduce the concept of a **black body**
- Derive the **Planck law of radiation** describing the black body spectrum

1. *Planck distribution.* The simple model of the harmonic oscillator provides the foundation for the quantum theory of light. A photon of frequency ω carries energy $\hbar\omega$, so that a system of s indistinguishable photons can be regarded as a harmonic oscillator with total energy

$$\varepsilon_s = s\hbar\omega. \tag{1}$$

   a. Under what circumstances are photons indistinguishable?

   b. Write down the partition function for the harmonic oscillator and simplify the geometric series.

   c. Calculate the average number of photons $\langle s \rangle$ in a mode of frequency ω at temperature τ. Your result is the **Planck distribution**.

d. Sketch the Planck distribution as a function of the reduced temperature $\tau/\hbar\omega$.

e. In the limit $\tau \gg \hbar\omega$, evaluate:

i. the average number of photons in the mode

ii. the average energy in the mode

Is your result in c.ii. consistent with the equipartition theorem? Why or why not?

2. *Black body radiation.* A **black body** is defined as a perfect absorber of radiation. For such an object to be in equilibrium with its surroundings, it must emit radiation at the same rate at which it absorbs it. A major shortcoming of classical physics, recognized at the turn of the 20th century, was its failure to fully describe the spectrum of black body radiation. Planck's successful derivation of the black body spectrum was an early triumph of quantum mechanics. We will reproduce it here.

   a. First, give a few examples of real physical entities that are well approximated as black bodies.

   b. Our starting point for analyzing black-body emission is to calculate the energy density of radiation in a volume V at temperature τ. Consider a conducting cavity of volume $V = L \times L \times L$, and let $\boldsymbol{\mathcal{E}}(x, y, z, t)$ denote the electric field in the cavity. The most general form of $\mathcal{E}$ consistent with the wave equation

$$\nabla^2 \boldsymbol{\mathcal{E}} = \frac{1}{c^2} \frac{\partial^2 \boldsymbol{\mathcal{E}}}{\partial t^2} \tag{2}$$

   and with the boundary conditions imposed by the cavity is

$$\begin{aligned} \mathcal{E}_x &= \mathcal{E}_{0x} \sin(\omega t + \phi) \cos(n_x \pi x/L) \sin(n_y \pi y/L) \sin(n_z \pi z/L), \\ \mathcal{E}_y &= \mathcal{E}_{0y} \sin(\omega t + \phi) \sin(n_x \pi x/L) \cos(n_y \pi y/L) \sin(n_z \pi z/L), \\ \mathcal{E}_z &= \mathcal{E}_{0z} \sin(\omega t + \phi) \sin(n_x \pi x/L) \sin(n_y \pi y/L) \cos(n_z \pi z/L). \end{aligned} \tag{3}$$

   Write down the following quantities for a mode labeled (n_x, n_y, n_z):

   i. the wavevector $\mathbf{k}$
   ii. the frequency ω
   iii. the photon energy E

c. The condition that the field be divergence-free ($\nabla \cdot \boldsymbol{\mathcal{E}} = 0$) implies

$$\mathcal{E}_{0x} n_x + \mathcal{E}_{0y} n_y + \mathcal{E}_{0z} n_z = 0. \tag{4}$$

For a given triplet (n_x, n_y, n_z), how many distinct polarization modes are supported by the cavity? Justify your answer with a sketch.

d. The **density of states** $\mathcal{D}(\omega)$ is defined as the number of modes per unit frequency. It can be calculated as the derivative $\mathcal{D}(\omega) = d\mathcal{N}/d\omega$ of the total number $\mathcal{N}(\omega)$ of modes with frequency $\leq \omega$.

   i. Sketch the modes supported by the box of volume $V = L^3$ as points in momentum space. Graphically, what determines the frequency ω? Use your sketch to determine $\mathcal{N}(\omega)$.

ii. Determine the density of states $\mathcal{D}(\omega)$.

e. The radiant energy per unit volume U/V can be found by integrating the **spectral density of radiation** u_ω over all frequencies:

$$U/V = \int d\omega u_\omega. \tag{5}$$

i. Give an expression for u_ω in terms of the density of states $\mathcal{D}(\omega)$ and the average number $\langle s \rangle_\omega$ of photons in the mode.

ii. Plug in your results for $\mathcal{D}(\omega)$ and $\langle s \rangle_\omega$ to obtain an explicit expression for the spectral density of radiation at temperature τ. The result is the **Planck radiation law**.

iii. Sketch the Planck spectrum u_ω. Which portion of this spectrum could have been correctly predicted by classical theory? Which range of parameters requires the quantum mechanical description?

iv. Explain physically the behavior of the black body spectrum in the low- and high-frequency limits.