

Fermi Degeneracy

	T_F	T/T_F
Electrons in Cu $T = 300 \text{ K}$	80,000 K	0.004
White dwarf (e⁻) $T = 10^7 \text{ K}$	$4 \times 10^9 \text{ K}$	0.002
White dwarf (p⁺) $T = 10^7 \text{ K}$	$2 \times 10^6 \text{ K}$	4
K-40 (1/μm^3) $T = 10 \text{ nK}$	90 nK	0.1

- f. Calculate the Fermi temperature $T_F = \varepsilon_F/k_B$ and the ratio T/T_F for the following systems:

- i. A gas of electrons in copper at room temperature; the density of conduction electrons is $n = (0.23 \text{ nm})^{-3}$.

$$T_F = \frac{\hbar^2}{2mk_B} (3\pi^2 n)^{2/3} = 8 \times 10^6 \text{ K}$$

$$T_{\text{room}} \approx 300 \text{ K} \Rightarrow \frac{T}{T_F} \approx 4 \times 10^{-3}$$

- ii. A white dwarf star composed of hydrogen atoms at a density $n = (1 \text{ pm})^{-3}$. At such a high density, the electrons are not bound to the protons; consider each in turn. The temperature of such a star is believed to be on the order of $T \sim 10^7 \text{ K}$.

$$T_F^e = 4.2 \times 10^9 \text{ K} \Rightarrow T/T_F^e \approx 0.2 \times 10^{-2} = 2 \times 10^{-3}$$

$$T_F^p = 2.3 \times 10^6 \text{ K} \Rightarrow T/T_F^p \approx 4 - \text{not degenerate}$$

- * Electrons are degenerate but protons are not.
- * Remarkably, electrons in copper are very similar to electrons in white dwarf star except for overall length scale (and, correspondingly, effects of gravity...)

spin-polarized

- iii. A gas of potassium-40 atoms at a density $n = (1 \mu\text{m})^{-3}$ and a temperature of 10 nK.

First, why fermionic? ^{40}K has 19 protons, hence 19 e^- , 21 neutrons
— all spin-1/2

Spin-polarized \Rightarrow multiplicity $g=1$ (all same internal state),

$$N(\varepsilon) = \frac{1}{2} N_{g=2}(\varepsilon) \Rightarrow \varepsilon_F = \frac{\hbar^2}{2m} (6\pi^2 n)^{2/3} \Rightarrow \boxed{T_F = 91 \text{ nK}}$$

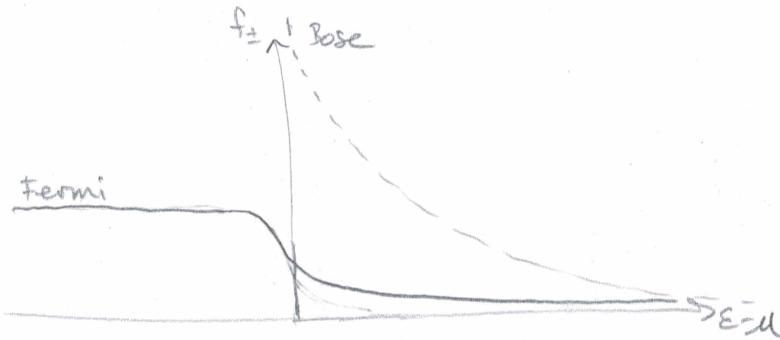
$\therefore T/T_F \approx 0.1 \Rightarrow$ degenerate

Not as deeply degenerate as electrons in a metal @ room temperature,
despite being way colder!

⑥ Useful past results:

$$D(\epsilon) = \frac{gV}{4\pi^2 h^3} (2m)^{3/2} \epsilon^{1/2} = \frac{3N(\epsilon)}{2\epsilon} \quad \text{total } \# N(\epsilon) = \int_0^\epsilon d\epsilon' D(\epsilon') \quad \text{up to energy } \epsilon$$

$$f_\pm(\epsilon) = \frac{1}{e^{\beta(\epsilon - \mu)} \pm 1}$$



Wrap up Ex. 9A (E, C_{el})

- * Degenerate = simple
- * C_{el} (c.f. phonons — how does e^- orbital contribution compare?)

4. Energy and heat capacity of an electron gas.

- a. Write down an integral expression for the kinetic energy E of an electron gas in terms of $\mathcal{D}(\varepsilon)$ and $f_+(\varepsilon)$.

$$E = \int_0^\infty d\varepsilon \mathcal{D}(\varepsilon) f_+(\varepsilon) \cdot \varepsilon$$

- b. Calculate the kinetic energy E_0 of an electron gas in the zero-temperature limit. Express it in terms of the Fermi energy ε_F and particle number N .
- The density of states is of the form $\mathcal{D}(\varepsilon) = \alpha \varepsilon^\zeta$. What is ζ ? To minimize algebra, leave α unevaluated.

$$\zeta = 1/2 \rightarrow \mathcal{D}(\varepsilon) = \alpha \varepsilon^{1/2}$$



- Find E_0 in terms of α and the Fermi energy ε_F .

$$E_0 = \int_0^{\varepsilon_F} \alpha \varepsilon^{3/2} d\varepsilon = \frac{2\alpha}{5} \varepsilon_F^{5/2}$$

- Use your result from 2.d. to express the particle number N in terms of α and the Fermi energy ε_F .

$$\mathcal{D}(\varepsilon) = \frac{3N(\varepsilon)}{2\varepsilon} \Rightarrow \mathcal{D}(\varepsilon_F) = \frac{3N}{2\varepsilon_F} = \alpha \varepsilon_F^{1/2}$$

$$\Rightarrow N = \frac{2}{3} \alpha \varepsilon_F^{3/2}$$

- Re-express E_0 in terms of only N , ε_F , and a numerical prefactor.

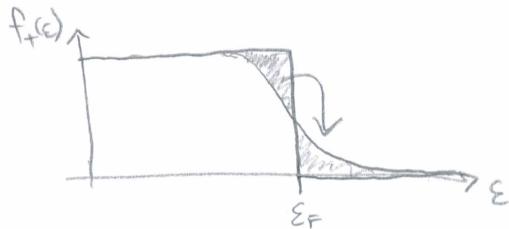
We have $N\varepsilon_F = \frac{2}{3} \alpha \varepsilon_F^{5/2} \Rightarrow \boxed{E_0 = \frac{3}{5} N\varepsilon_F}$

- c. In section, you will show that the heat capacity of an electron gas for $\tau \ll \tau_F$ is

$$C_{\text{el}} = \frac{\pi^2}{3} D(\varepsilon_F) \tau. \quad (2)$$

For now, let us find a physical explanation for the dependence of C_{el} on $D(\varepsilon_F)$ and τ . Don't worry about numerical factors.

- i. Sketch the Fermi-Dirac distribution at $\tau = 0$ and at a small but non-zero temperature $\tau \ll \tau_F$.



- ii. Based on your sketch, estimate the energy difference $\Delta E \equiv E_\tau - E_0$ between the finite-temperature and zero-temperature cases.

Roughly we remove an energy $\underbrace{\tau D(\varepsilon_F)}_{\# \text{ of states}} \times (\varepsilon_F - \tau)$

and add an energy $\tau D(\varepsilon_F) \times (\varepsilon_F + \tau)$

$$\Rightarrow \Delta E \sim \tau^2 D(\varepsilon_F)$$

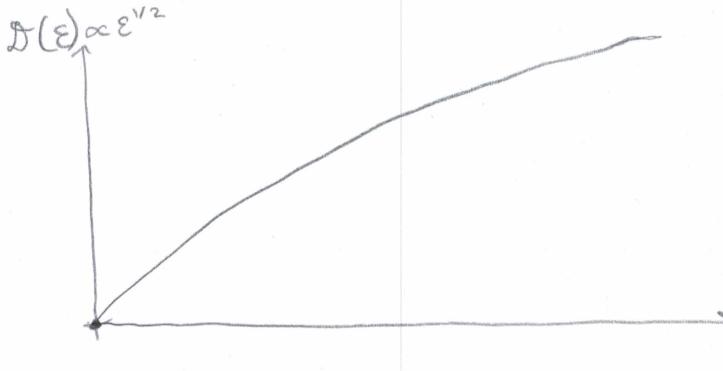
Here, we assume $D(\varepsilon_F)$ roughly constant over range $[\varepsilon_F - \tau, \varepsilon_F + \tau]$

- iii. Estimate the heat capacity C_{el} associated with the kinetic energy of the electron gas. Is the scaling with temperature consistent with Eq. 2?

$$C_{\text{el}} = \left(\frac{\partial E}{\partial T} \right)_{N,V} \quad \text{where } E = E_0 + \tau^2 D(\varepsilon_F)$$

$$\Rightarrow C_{\text{el}} \sim (2) \tau D(\varepsilon_F), \text{ scaling as in Eq. 2.}$$

Will do more precisely in section.



- * Fermions: vanishingly small fraction of the particles are @ $\epsilon \approx 0$... behavior dominated by particles near Fermi surface.
- * ... in stark contrast to bosons...

Bosons:

We know that the ground state is  all particles in lowest-energy orbital

How low must the temperature be for a macroscopic occupation ($\langle \hat{O}(N) \rangle$) in the ground state?

$$\begin{aligned} * \text{ Let's calculate: } N_0 \text{ vs. } \mu ? \\ & N_e \text{ vs. } \mu ? \end{aligned} \} \Rightarrow P_{\text{ground}} = \frac{N_0}{N_0 + N_e}$$

* We give the ground state a special treatment b/c $D(0)=0$ (continuum approx) and yet there is a state here that clearly matters...

EXERCISE 9B: BOSE-EINSTEIN CONDENSATION*Objective:*

- Calculate the critical temperature for **Bose-Einstein condensation**
- Understand what determines the **condensate fraction** and how it is measured

References: Kittel & Kroemer, Ch. 7, pp. 199-206*Useful past results:*

- Fermi-Dirac (+) and Bose-Einstein (-) distributions

$$f_{\pm}(\varepsilon) = \frac{1}{e^{\beta(\varepsilon-\mu)} \pm 1}$$

describing the mean occupation of a single orbital of energy ε .

- Density of states

$$D(\varepsilon) = \frac{gV}{4\pi^2\hbar^3} (2m)^{3/2} \varepsilon^{1/2}$$

for particles of mass m with multiplicity g of internal states, in a box of volume V .

1. *Bose-Einstein condensation.* For a gas of non-interacting bosons, the many-particle ground state consists of all particles occupying the orbital of lowest energy, which we shall take to be at $\varepsilon = 0$. How low must the temperature be in order to accrue a macroscopic occupation of the ground state?

- a. Let N_0 denote the number of bosons in the ground state and N_e the population of all excited states combined. Write down expressions for N_0 and N_e , the latter in terms of the density of states.

$$N_0 = f_-(0) = \frac{1}{e^{\beta\mu} - 1}$$

$$N_e = \int_0^\infty d\varepsilon D(\varepsilon) f_-(\varepsilon) = \int_0^\infty d\varepsilon \frac{D(\varepsilon)}{e^{\beta(\varepsilon-\mu)} - 1}$$

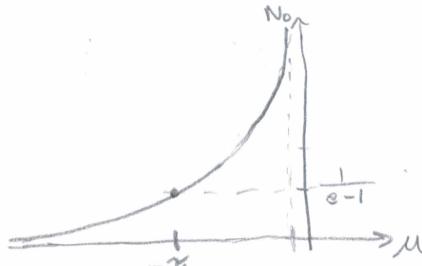
- b. What condition must be imposed on the chemical potential for your expressions in a. to be physically reasonable?

Need $\mu < 0$ s.t. $N_0 > 0$. This will also ensure $e^{\beta(\varepsilon-\mu)} > 1 \quad \forall \varepsilon > 0$, so both N_0 and N_e will be positive.

- c. Assume that the gas consists of sufficiently many atoms that $N_0 \gg 1$, without making any assumption about the fraction N_0/N_e of ground- to excited-state atoms.

- i. What does the condition $N_0 \gg 1$ imply about the absolute value of the chemical potential $|\mu|$? (It may help to sketch N_0 vs μ .)

$$N_0 \gg 1 \Rightarrow e^{-\beta\mu} - 1 \ll 1 \Rightarrow \frac{|\mu|}{\tau} \ll 1 \Rightarrow |\mu| \ll \tau$$



- ii. Use the assumption $N_0 \gg 1$ to give an approximate expression for N_e that is independent of the chemical potential μ .

$$N_e = \int_0^\infty dE \frac{D(E)}{e^{-\beta\mu} e^{\beta E} - 1} \approx \int_0^\infty dE \frac{D(E)}{e^{\beta E} - 1}$$

↑
 very close to 1
 for $N_0 \gg 1$

- d. Evaluate N_e as a function of temperature τ for bosons in a three-dimensional box of volume V . Your result will include a definite integral that you should express in dimensionless form and leave unevaluated.

$$N_e \approx \int_0^\infty dE \frac{gV}{4\pi^2\hbar^3} (2m)^{3/2} \epsilon^{1/2} / (e^{\beta E} - 1) \quad || \text{ spinless} \Rightarrow g=1$$

$$= \frac{V}{4\pi^2\hbar^3} (2m)^{3/2} \int_0^\infty dE \cdot \frac{\epsilon^{1/2}}{e^{\beta E} - 1}$$

$$\chi \equiv \epsilon/\tau \Rightarrow \tau d\chi = d\epsilon, \quad \epsilon^{1/2} = \tau^{1/2} \chi^{1/2}$$

$$\therefore N_e = \frac{V}{4\pi^2\hbar^3} (2m\tau)^{3/2} \underbrace{\int_0^\infty dx \cdot \frac{x^{1/2}}{e^x - 1}}_{\mathcal{I}}$$

III

e. Does the definite integral from part d. converge or diverge...

i. ... at $x = 0$?

ii. ... as $x \rightarrow \infty$?

$$\text{i)} \quad \text{For small } x: \frac{x^{1/2}}{e^x - 1} \approx \frac{x^{1/2}}{x} = x^{-1/2}$$

$$\int_0^a dx \cdot x^{-1/2} = 2a^{1/2}, \text{ so integral converges at } x=0$$

$$\text{ii)} \quad \text{For large } x: \frac{x^{1/2}}{e^x - 1} \approx \sqrt{x} e^{-x} \text{ converges due to exponential decay}$$

Note: $\mathcal{I} = \frac{\sqrt{\pi}}{2} \zeta(3/2)$

ζ Riemann zeta function
 ≈ 2.6

f. Physically, how do you interpret the convergence or divergence of the integral from part d.?

Even as $\beta\mu \rightarrow 0$ (making N_e as large as possible), the # of particles in all excited orbitals is bounded to a value that depends only on m, V, τ . Thus, for total # $N > N_e$, a macroscopic number $N_0 = N - N_e$ of bosons must occupy the ground state.



- g. For a given total atom number N , let τ_E denote the lowest temperature for which all atoms can be accommodated in excited states; this is the **critical temperature for Bose-Einstein condensation**. Below this temperature, there must be a macroscopic occupation N_0 of the ground state.
- Express the condition for Bose-Einstein condensation in terms of the density n and the thermal de Broglie wavelength $\lambda_\tau = h/\sqrt{2\pi m\tau}$.
 - How does the critical temperature τ_E scale with density n ?
 - Let Δp represent the spread of the momentum distribution at temperature τ . What do your results imply about the critical phase space density $n/(\Delta p)^3$ for Bose-Einstein condensation?

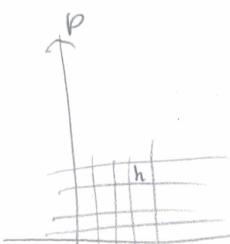
$$\begin{aligned} i) \quad N &= \frac{V}{4\pi^2 h^3} (2m\tau)^{3/2} \cdot \frac{\sqrt{\pi}}{2} \zeta(3/2) \\ &= \frac{2\pi V}{h^3} (2m\tau)^{3/2} \frac{\sqrt{\pi}}{2} \zeta(3/2) = \frac{(2\pi m\tau)^{3/2}}{h^3} \zeta(3/2) \\ \frac{N}{V} &= \frac{1}{\lambda_\tau^3} \zeta(3/2) \Rightarrow \boxed{n \lambda_\tau^3 = \zeta(3/2) \approx 2.6} \end{aligned}$$

\therefore interparticle spacing must be $\lesssim \lambda_\tau$ for BEC to occur
 \Rightarrow overlap of de Broglie waves

$$ii) \quad n \propto \tau_E^{3/2} \Rightarrow \tau_E \propto n^{2/3}$$

The higher the density, the higher the critical temperature.

$$iii) \quad \frac{(\Delta p)^2}{2m} \sim \tau \Rightarrow \Delta p \sim \sqrt{2m\tau} \sim \frac{h}{\lambda_\tau}$$



$$\frac{n}{(\Delta p)^3} \sim \frac{1}{\lambda_\tau^3} \cdot \frac{\lambda_\tau^3}{h^3} \sim \frac{1}{h^3}$$

\Rightarrow Need more than one boson per cell of volume h^3 in phase space
 Critical phase space density set only by Planck's constant.

- h. How does the critical temperature for Bose-Einstein condensation compare with the energy of the first excited state? Is the result surprising? Why or why not?

* τ_E can be made arbitrarily high by increasing N , without changing

energy $\frac{\hbar^2}{2mL^2}$ of 1st excited state

$$\text{for } N=1: \quad \varepsilon_1 = \frac{\pi^2 \hbar^2}{2mV^{2/3}} = \frac{n^{2/3} \pi^2 \hbar^2}{2m} \sim \underbrace{n^{2/3} \hbar^2}_{\substack{\text{ss} \\ 1}} \cdot \tau_E \approx \tau_E$$

... but for $N \gg 1$, $\tau_E \gg \varepsilon_1$

- i. Calculate the condensate fraction N_0/N as a function of density $n = N/V$ and temperature $\tau < \tau_E$. Then sketch it, as follows:
 i. Sketch N_0/N vs. temperature $\tau < \tau_E$ at fixed density.
 ii. Sketch N_0/N vs. density n at fixed temperature $\tau < \tau_E$.

Another way to say the above:

$$\tau_E \text{ given by } \lambda_{\tau_E} \sim n^{-1/3}$$

$$\sqrt{\frac{\hbar}{2\pi m \tau_E}} \sim n^{-1/3} \Rightarrow \tau_E \sim \frac{n^{2/3} \hbar^2}{m} = \frac{N^{2/3} \hbar^2}{m L^2}$$

By comparison:

$$\varepsilon_1 \equiv \varepsilon[(n_x, n_y, n_z) - \varepsilon[(1, 1, 1)] = \frac{(2^2 - 1^2)\pi^2 \hbar^2}{2mL^2} \sim \frac{\hbar^2}{mL^2} \Rightarrow \tau_E \sim N^{2/3} \varepsilon_1$$

$\uparrow \quad \uparrow$
1st exc. state ground state

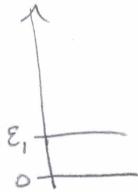
\therefore For $N=1$, $\tau_E \sim \varepsilon_1$, but for $N \gg 1$, $\tau_E \gg \varepsilon_1$.

\Rightarrow Macroscopic occupation of ground state even though τ is much higher than excitation energy. This is a collective effect that relies on constructive interference of matter waves.

(Next time: experimental signatures...).

Added Note:

As a function of N_0 , what is the ratio of occupations of the ground and first excited states?



$$N_0 = \frac{1}{e^{-\beta \mu} - 1} \gg 1$$

assume

$$\approx -\frac{\tau}{\mu} \Rightarrow -\beta \mu = \frac{1}{N_0} \Rightarrow e^{-\beta \mu} \approx e^{VN_0}$$

$$N_1 = \frac{1}{e^{-\beta \mu} e^{\beta E_1} - 1}$$

Plugging in $e^{-\beta \mu} = e^{VN_0}$, we have:

$$\begin{aligned} N_1 &\approx \frac{1}{e^{VN_0} e^{\beta E_1} - 1} \approx \frac{1}{(1 + \frac{1}{N_0}) e^{\beta E_1} - 1} \\ &= \frac{N_0}{N_0(e^{\beta E_1} - 1) + e^{\beta E_1}} \approx \frac{N_0}{N_0(E_1/\tau)} \approx N^{2/3} \text{ for } \tau \leq T_E \end{aligned}$$

For $\tau \approx T_E$, have: $E_1/\tau_E \approx N^{-2/3}$

$$\text{so } \frac{N_1}{N_0} \approx \frac{N^{2/3}}{N} \approx N^{-1/3} \ll 1 \text{ for } \tau \leq T_E$$

\Rightarrow In thermodynamic limit, $N_0 \gg N_1$.