Names:

EXERCISE 2A: ENTROPY AS MISSING INFORMATION

New concepts:

- **Entropy** as missing information
- The **fundamental assumption** of statistical mechanics
- The **First Law** and **Second Law of Thermodynamics**
- **Ensembles**; the **microcanonical ensemble**.

Reference: Kittel & Kroemer, Ch. 2

1. *Ehrenfest's Urns.* Watch the simulation of Ehrenfest's urns with $N = 3$ balls.

    a. Explain in words the microscopic laws governing the time evolution of the system.

    b. The plot labeled "percentage of balls" indicates the **macrostate** of the system. How will this plot look if we initialize the system with $N = 50$ balls in the left urn and allow it to evolve for a few hundred time steps? Sketch your prediction.

    c. Are the dynamics of Ehrenfest's urns reversible, i.e., would the simulation look the same if we ran it backwards? Why or why not?

2. Consider, quite generically, two physical systems $\mathcal{A}_1, \mathcal{A}_2$ with respectively g_1 and g_2 possible microstates.
If you are unsure about a. or b., you may start with part c..

   a. How many microstates are available to the composite system $\mathcal{A} = \mathcal{A}_1 + \mathcal{A}_2$?

   b. What is the number of bits $\mathcal{I}$ required to specify the state of the composite system?

   c. Explicitly check your answers to a.-b. for the case where each system $\mathcal{A}_i$ consists of N_i spin-1/2 magnetic moments in a solid.

   d. Based on your answers above, why might it be convenient to quantify the information content of physical systems in terms of a number of bits $\mathcal{I}$ rather than the multiplicity g?

   Up to a multiplicative constant, $\mathcal{I}$ is equivalent to the **entropy**, variously defined as $\sigma = \ln g$ or $S = k \ln g$. We further examine its physical significance below.

3. *Entropy of a paramagnet.* Let us return to the paramagnet of N spins of magnetic moment μ in a magnetic field B, described by a Hamiltonian

$$H = -\varepsilon \sum_{i=1}^{N} \sigma_i^z, \tag{1}$$

where $\varepsilon = \mu B$ and $\sigma_i^z = \pm 1$ describes the state of the i^{th} spin. Suppose that we know the total energy E of the system.

   a. Write down the **multiplicity** $g(E)$ of states with energy E. *Assume the thermodynamic limit (large N) and simplify the math by using the Gaussian approximation to the binomial distribution.*

   b. Given our knowledge of the total energy (which specifies a macrostate of the system), how much additional information $\sigma(E)$ would be required to specify the microstate?

c. Sketch the following over the full domain $-N\mu B \leq E \leq N\mu B$, using your physical intuition in the regime where the Gaussian approximation breaks down:

   i. $g(E)$

   ii. $\sigma(E)$

4. *Equilibration.* Suppose that we bring together two systems $\mathcal{A}_1, \mathcal{A}_2$ of N_1 and N_2 spins and allow energy to flow between them. The systems eventually equilibrate while conserving the total energy E.

   a. In **equilibrium**, what is the most probable value $\hat{E}_1$ of the energy of the subsystem of N_1 spins? Apply the following principles:

      - Conservation of energy, also known as the **First Law of Thermodynamics**.
      - The **fundamental assumption of thermodynamics**: all microstates consistent with our knowledge of the macroscopic properties of the system (e.g., total energy, particle number) are equally probable.

b. For an **ensemble** of many identically prepared systems of N spins with the same total energy E, determine the rms fluctuations ΔE_1 in the energy of a subsystem of N_1 spins.
The ensemble of many replicas of a system with the same total energy E and particle number N is called the **microcanonical ensemble**.

i. Write down the multiplicity $g(E_1, E_2 = E - E_1)$ of states of the composite system with energy E_1 in $\mathcal{A}_1$. We will only care about the width of the distribution, so don't worry about the overall prefactor.

ii. Show that your result from i. can be expressed in the form $g \propto e^{-\alpha(E_1 - \hat{E}_1)^2}$. Equivalently, show that

$$\ln g = -\alpha(E_1 - \hat{E}_1)^2 + \text{const.} \tag{2}$$

What is the value of α?

iii. Determine ΔE_1 from α.

c. Sketch the multiplicity function $g(E_1, E-E_1)$ illustrating the likelihood of finding the subsystem $\mathcal{A}_1$ to have energy E_1 for $N_1 = N_2$ and $E = (N_1 + N_2)\mu B/2$.

d. Does your ability to predict the fraction of the total energy in each subsystem improve or worsen in the thermodynamic limit?

e. Suppose that, prior to letting the two systems exchange energy, the energy of system $\mathcal{A}_1$ was known to be E_0 (and hence the energy of $\mathcal{A}_2$ was $E - E_0$). When the systems are allowed to equilibrate, does the amount of information σ missing from a full description of $\mathcal{A} = \mathcal{A}_1 + \mathcal{A}_2$ increase, decrease, or remain the same? Explain.

Your result in Ex. 3.e. is a manifestation of the **Second Law of Thermodynamics**: when a constraint internal to a closed system is removed, the entropy tends to ....