GATE ASSIGNMENT 1

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Download all python codes from

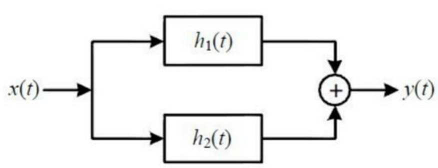
https://github.com/V−Gopireddy/EE3900/blob/main/GATE_Assignment1/codes/GATE_Assignment−1.py

and latex-tikz codes from

https://github.com/V−Gopireddy/EE3900/blob/main/GATE_Assignment1/GATE_Assignment−1.tex

1 GATE EC 2017 Q.35

Consider the parallel combination of two LTI systems shown in the figure.



The impulse responses of the systems are

$$h_1(t) = 2\delta(t+2) - 3\delta(t+1) \tag{1.0.1}$$
$$h_2(t) = \delta(t-2) \tag{1.0.2}$$

If the input $x(t)$ is a unit step signal, then find the energy of $y(t)$.

2 Solution

Definition 1 (Laplace Transform)**.** *It is an integral transform that converts a function of a real variable t to a function of a complex variable s. The Laplace transform of $f(t)$ is denoted by $\mathcal{L}\{f(t)\}$ or $F(s)$.*

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)dt \tag{2.0.1}$$

Definition 2 (Bilateral Laplace Transform)**.** *The Laplace transform can be alternatively defined as the bilateral Laplace transform, by extending the limits of integration to be the entire real axis The bilateral laplace transform of $f(t)$ is denoted by $\mathcal{L}_b\{f(t)\}$ or $F_b(s)$.*

$$F_b(s) = \mathcal{L}_b\{f(t)\} = \int_{-\infty}^\infty e^{-st} f(t)dt \tag{2.0.2}$$

Lemma 2.1. *Linearity of laplace transform*

$$\mathcal{L}\{af(t) + bg(t)\} = a\mathcal{L}\{f(t)\} + b\mathcal{L}\{g(t)\} \tag{2.0.3}$$

Proof.

$$\mathcal{L}\{af(t) + bg(t)\} \tag{2.0.4}$$
$$= \int_0^\infty e^{-st}\{af(t) + bg(t)\}dt \tag{2.0.5}$$
$$= a\int_0^\infty e^{-st} f(t)dt + b\int_0^\infty e^{-st} g(t)dt \tag{2.0.6}$$
$$= a\mathcal{L}\{f(t)\} + b\mathcal{L}\{g(t)\} \tag{2.0.7}$$

□

Lemma 2.2. *For any real number c,*

$$\mathcal{L}_b\{u(t-c)\} = \frac{e^{-cs}}{s}, s > 0 \tag{2.0.8}$$

Proof.

$$\mathcal{L}_b\{u(t-c)\} = \int_{-\infty}^\infty e^{-st} u(t-c)dt = \int_c^\infty e^{-st} dt \tag{2.0.9}$$
$$= \left[-\frac{e^{-st}}{s}\right]_c^\infty = \frac{e^{-cs}}{s}, s > 0 \tag{2.0.10}$$

□

Lemma 2.3. *For any real number c,*

$$\mathcal{L}_b\{\delta(t-c)\} = e^{-cs}, s > 0 \tag{2.0.11}$$

Proof. Let $f_k(t-a)$ be a function defined as,

$$f_k(t-c) = \begin{cases} \frac{1}{k}, & c \le t < c+k \\ 0, & \text{otherwise} \end{cases} \tag{2.0.12}$$
$$\implies \lim_{k\to 0} f_k(t-c) = \delta(t-c) \tag{2.0.13}$$

$$\mathcal{L}_b\left\{\delta(t-c)\right\} = \int_{-\infty}^{\infty} e^{-st}\delta(t-c)dt \tag{2.0.14}$$

$$= \lim_{k\to 0}\int_{-\infty}^{\infty} e^{-st} f_k(t-c)dt \tag{2.0.15}$$

$$= \lim_{k\to 0}\left[-\frac{e^{-st}}{ks}\right]_c^{c+k} \tag{2.0.16}$$

$$= \lim_{k\to 0}\frac{e^{-sc}-e^{-s(c+k)}}{sk} = e^{-cs}, s>0 \tag{2.0.17}$$

□

Definition 3 (Inverse Laplace Transform)**.** *It is the transformation of a Laplace transform into a function of time. If* $F(s) = \mathcal{L}\{f(t)\}$*, then the Inverse laplace transform of* $F(s)$ *is* $\mathcal{L}^{-1}\{F(s)\} = f(t)$*.*

Lemma 2.4. *Linearity of Inverse Laplace Transform*

$$\mathcal{L}^{-1}\{af(t)+bg(t)\} = a\mathcal{L}^{-1}\{f(t)\} + b\mathcal{L}^{-1}\{g(t)\} \tag{2.0.18}$$

Theorem 2.1 (Convolution Theorem)**.** *Let* $F(s)$ *and* $G(s)$ *be the Laplace-transform of two functions* $f(t)$ *and* $g(t)$ *respectively. Then*

$$\mathcal{L}\{f(t)*g(t)\} = F(s)G(s) \tag{2.0.19}$$

Given that,

$$x(t) = u(t) \tag{2.0.20}$$

And,

$$h_1(t) = 2\delta(t+2) - 3\delta(t+1) \tag{2.0.21}$$

$$h_2(t) = \delta(t-2) \tag{2.0.22}$$

Since the systems are in parallel,
Overall impulse response of the system is,

$$h(t) = h_1(t) + h_2(t) \tag{2.0.23}$$

$$= 2\delta(t+2) - 3\delta(t+1) + \delta(t-2) \tag{2.0.24}$$

From (2.0.11) and (2.0.3) we have

$$H(s) = \mathcal{L}\{h(t)\} = 2e^{2s} - 3e^{s} + e^{-2s} \tag{2.0.25}$$

From (2.0.20) and (2.0.8) we have,

$$X(s) = \mathcal{L}\{u(t)\} = \frac{1}{s} \tag{2.0.26}$$

The output signal $y(t)$ is given by,

$$y(t) = x(t) * h(t) \tag{2.0.27}$$

$$\implies \mathcal{L}\{y(t)\} = X(s)H(s) \tag{2.0.28}$$

$$= \frac{1}{s}\left[2e^{2s} - 3e^{s} + e^{-2s}\right] \tag{2.0.29}$$

Therefore we have,

$$y(t) = \mathcal{L}^{-1}\left\{\frac{2e^{2s}}{s} - \frac{3e^{s}}{s} + \frac{e^{-2s}}{s}\right\} \tag{2.0.30}$$

$$= 2\mathcal{L}^{-1}\left\{\frac{e^{2s}}{s}\right\} - 3\mathcal{L}^{-1}\left\{\frac{e^{s}}{s}\right\}\mathcal{L}^{-1} + \left\{\frac{e^{-2s}}{s}\right\} \tag{2.0.31}$$

$$= 2u(t+2) - 3u(t+1) + u(t-2) \tag{2.0.32}$$

Solving we get,

$$y(t) = \begin{cases} 2, & -2 \le t < -1 \\ -1, & -1 \le t < 2 \\ 0, & \text{otherwise} \end{cases} \tag{2.0.33}$$

Energy of the output signal E_y is given by,

$$E_y = \int_{-\infty}^{\infty} |y(t)|^2 dt \tag{2.0.34}$$

$$= \int_{-2}^{-1} 4dt + \int_{-1}^{2} dt \tag{2.0.35}$$

$$= 7 \tag{2.0.36}$$

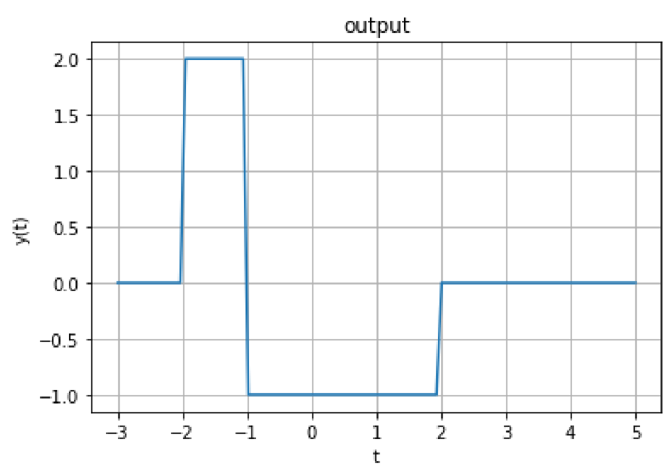


Fig. 0: Output signal $y(t)$