Assignment 1

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Download all python codes from

```
https://github.com/V−Gopireddy/EE3900/blob/
    main/Assignment3/codes/Assignment−3.py
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and latex-tikz codes from

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https://github.com/V−gopireddy/EE3900/blob/main
    /Assignment3/Assignment−3.tex
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1 Ramsey/4.4 Systems of circles/Q.2

Find the equation of a circle which cuts orthogonally the three circles

$$\mathbf{x}^T\mathbf{x} + \begin{pmatrix} 4 & -5 \end{pmatrix}\mathbf{x} + 6 = 0 \tag{1.0.1}$$

$$\mathbf{x}^T\mathbf{x} + \begin{pmatrix} 5 & -6 \end{pmatrix}\mathbf{x} + 7 = 0 \tag{1.0.2}$$

$$\mathbf{x}^T\mathbf{x} - \begin{pmatrix} 1 & 1 \end{pmatrix}\mathbf{x} - 1 = 0 \tag{1.0.3}$$

2 Solution

Lemma 2.1. *Tangent to a circle : Consider a circle*

$$\mathbf{x}^\top\mathbf{x} + 2\mathbf{c}^\top\mathbf{x} + f = 0 \tag{2.0.1}$$

Given a point $\mathbf{q}$ *on the circle, the tangent at that point is given as,*

$$(\mathbf{q} + \mathbf{c})^\top \mathbf{x} + \mathbf{c}^\top\mathbf{q} + f = 0 \tag{2.0.2}$$

Lemma 2.2. *Orthogonality of circles : Two circles are said to be orthogonal if the tangents at their points of intersection are perpendicular to each other.*
That implies, tangents to one circle at the points of contact are normals to the other circle. Given two circles,

$$\mathbf{x}^\top\mathbf{x} + 2\mathbf{c_1}^\top\mathbf{x} + f_1 = 0 \tag{2.0.3}$$

$$\mathbf{x}^\top\mathbf{x} + 2\mathbf{c_2}^\top\mathbf{x} + f_2 = 0 \tag{2.0.4}$$

They are orthogonal if

$$2\mathbf{c_1}^\top\mathbf{c_2} = f_1 + f_2 \tag{2.0.5}$$

Proof. Let the two circles (2.0.3) and (2.0.4) meet at a point $\mathbf{q}$ i.e $\mathbf{q}$ satisfies the equation of the circles

$$\mathbf{q}^\top\mathbf{q} + 2\mathbf{c_1}^\top\mathbf{q} + f_1 = 0 \tag{2.0.6}$$

$$\mathbf{q}^\top\mathbf{q} + 2\mathbf{c_2}^\top\mathbf{q} + f_2 = 0 \tag{2.0.7}$$

Eliminating quadratic term,

$$2\left(\mathbf{c_1}^T - \mathbf{c_2}^T\right)\mathbf{q} + f_1 - f_2 = 0 \tag{2.0.8}$$

Given the point of contact $\mathbf{q}$, the equation of tangent to circle (2.0.3) is

$$(\mathbf{q} + \mathbf{c_1})^\top \mathbf{x} + \mathbf{c_1}^\top\mathbf{q} + f_1 = 0 \tag{2.0.9}$$

As it is a normal to the second circle (2.0.4), it passes through the center of it

$$(\mathbf{q} + \mathbf{c_1})^\top (-\mathbf{c_2}) + \mathbf{c_1}^\top\mathbf{q} + f_1 = 0 \tag{2.0.10}$$

$$\implies \left(\mathbf{c_1}^T - \mathbf{c_2}^T\right)\mathbf{q} + f_1 - \mathbf{c_1}^T\mathbf{c_2} = 0 \tag{2.0.11}$$

$$\implies \left(\mathbf{c_1}^T - \mathbf{c_2}^T\right)\mathbf{q} = \mathbf{c_1}^T\mathbf{c_2} - f_1 \tag{2.0.12}$$

Substituting (2.0.12) in (2.0.8),

$$2\left(\mathbf{c_1}^T\mathbf{c_2} - f_1\right) + f_1 - f_2 = 0 \tag{2.0.13}$$

$$\implies 2\mathbf{c_1}^\top\mathbf{c_2} = f_1 + f_2 \tag{2.0.14}$$

□

Let the equation of the circle be

$$\mathbf{x}^\top\mathbf{x} + 2\mathbf{c}^\top\mathbf{x} + f = 0 \tag{2.0.15}$$

It is orthogonal to the circles (1.0.1), (1.0.2) and (1.0.3)

$$\begin{pmatrix} 2 & \frac{-5}{2} \end{pmatrix}\mathbf{c} - f = 6 \tag{2.0.16}$$

$$\begin{pmatrix} \frac{5}{2} & -3 \end{pmatrix}\mathbf{c} - f = 7 \tag{2.0.17}$$

$$\begin{pmatrix} \frac{-1}{2} & \frac{-1}{2} \end{pmatrix}\mathbf{c} - f = -1 \tag{2.0.18}$$

Expressing in the form of a matrix

$$\begin{pmatrix} 2 & -5/2 & -1 \\ 5/2 & -3 & -1 \\ -1/2 & -1/2 & -1 \end{pmatrix}\begin{pmatrix} \mathbf{c} \\ f \end{pmatrix} = \begin{pmatrix} 6 \\ 7 \\ -1 \end{pmatrix} \tag{2.0.19}$$

Row reducing the augumented matrix,

$$\begin{pmatrix} 2 & -5/2 & -1 & 6 \\ 5/2 & -3 & -1 & 7 \\ -1/2 & -1/2 & -1 & -1 \end{pmatrix} \tag{2.0.20}$$

$$\xleftrightarrow{R_1 \to 5R_1 - 4R_2 - R_3} \begin{pmatrix} 1/2 & 0 & 0 & 3 \\ 5/2 & -3 & -1 & 7 \\ -1/2 & -1/2 & -1 & -1 \end{pmatrix} \tag{2.0.21}$$

$$\xleftrightarrow[R_3 \to 6R_3 - R_2 + 11R_1]{R_2 \to R_2 - R_3 - 6R_1} \begin{pmatrix} 1/2 & 0 & 0 & 3 \\ 0 & -5/2 & 0 & -10 \\ 0 & 0 & -5 & 20 \end{pmatrix} \tag{2.0.22}$$

$$\mathbf{c} = \begin{pmatrix} 6 \\ 4 \end{pmatrix} \tag{2.0.23}$$

$$f = -4 \tag{2.0.24}$$

The required equation of circle,

$$S = \mathbf{x}^\top \mathbf{x} + 2\begin{pmatrix} 6 & 4 \end{pmatrix}\mathbf{x} - 4 = 0 \tag{2.0.25}$$

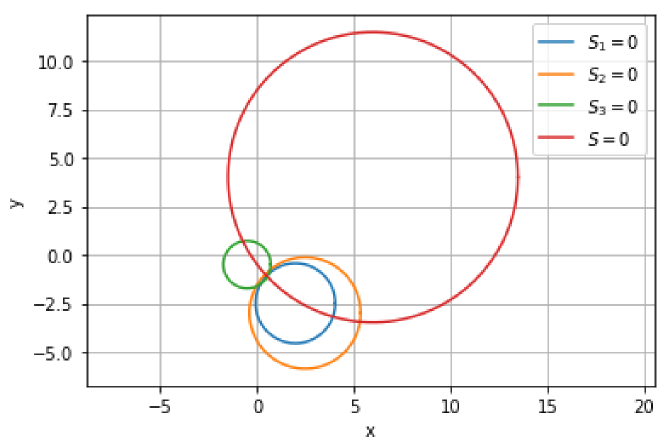

Fig. 0: Plot of circles