**Not-for-Publication Appendix to**

# Evaluating Direct Multi–Step Forecasts *

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Abstract

This not-for-publication appendix contains proofs of Theorems 3.1 - 3.4 as discussed in the text. It also contains lemmas used to prove the theorems.

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## 1. Introduction

This not-for-publication appendix contains proofs of Theorems 3.1 - 3.4 as discussed in the text. It also contains lemmas used to prove the theorems. Herein we focus on an environment in which the models are linear and either $0 < \pi < \infty$ or $\pi = 0$. The proofs provide results for each of the recursive, rolling and fixed schemes. Note that, for simplicity, the $P - \tau + 1$ terms that appear in the text formulas are replaced by P in the theoretical results below, without any consequence. Throughout, the null is maintained.

## 2. Notation

The following notation will be used. Forecasts of the scalar $y_{t+\tau}$, $t = R,\ldots,T-\tau$, $1 \leq \tau < \infty$ are generated using a $(k_1 + k_2 = k\times 1)$ vector of covariates $x_{2,t} = (x_{1,t}^{'}, x_{22,t}^{'})^{'}$ and two linear parametric models, $x_{i,t}^{'}\beta_i^*$, $i = 1,2$, each of which is estimated. Under the null, model 2 is unrestricted and nests the restricted model 1. We denote the $\tau$-step ahead forecast errors as $\hat{u}_{1,t+\tau} = y_{t+\tau} - x_{1,t}^{'}\hat{\beta}_{1,t}$ and $\hat{u}_{2,t+\tau} = y_{t+\tau} - x_{2,t}^{'}\hat{\beta}_{2,t}$ for models 1 and 2, respectively.

Let $h_{i,t+\tau}(\beta_i) = (y_{t+\tau} - x_{i,t}^{'}\beta_i)x_{i,t}$, $h_{i,t+\tau} = h_{i,t+\tau}(\beta_i^*)$, $q_{i,t} = x_{i,t}x_{i,t}^{'}$ and $B_i = (Eq_{i,t})^{-1}$. Let J denote the selection matrix $(I_{k_1\times k_1}, 0_{k_1\times k_2})^{'}$, $\sup_t$ denote $\sup_{R\leq t\leq T}$, and for matrices A and C defined in Lemma A4, $\tilde{h}_{2,t+\tau} = \sigma^{-1}A^{'}CB_2^{1/2}h_{2,t+\tau}$ and $\tilde{H}_2(t) = \sigma^{-1}A^{'}CB_2^{1/2}H_2(t)$. For any $(m\times n)$ matrix G with elements $g_{i,j}$ and column vectors $g_j$ let vec(G) denote the $(mn\times 1)$ vector $[g_1^{'}, g_2^{'}, \ldots, g_n^{'}]^{'}$ and let $|G|$ denote $\max_{i,j}|g_{i,j}|$. For the sequence $U_{t+\tau}$ defined in Assumption 2, U(t) is defined analogous to H(t) in Assumption 1. For $\hat{c}_{t+\tau} = \hat{u}_{1,t+\tau}^2 - \hat{u}_{1,t+\tau}\hat{u}_{2,t+\tau}$, $\hat{d}_{t+\tau} = \hat{u}_{1,t+\tau}^2 - \hat{u}_{2,t+\tau}^2$, $\overline{c} = (P-\tau+1)^{-1}\sum_{t=R+j}^{T-\tau}\hat{c}_{t+\tau}$, $\overline{d} = (P-\tau+1)^{-1}\sum_{t=R+j}^{T-\tau}\hat{d}_{t+\tau}$ and $0 \leq j \leq \overline{j} < \infty$, $\hat{\Gamma}_{cc}(j) = (P-\tau+1)^{-1}\sum_{t=R+j}^{T-\tau}(\hat{c}_{t+\tau} - \overline{c})(\hat{c}_{t+\tau-j} - \overline{c})$, $\hat{\Gamma}_{dd}(j) =$

$(P\text{-}\tau+1)^{-1}\sum_{t=R+j}^{T-\tau}(\hat{d}_{t+\tau}-\bar{d})(\hat{d}_{t+\tau-j}-\bar{d})$, $\hat{\Gamma}_{cc}(j) = \hat{\Gamma}_{cc}(-j)'$ and $\hat{\Gamma}_{dd}(j) = \hat{\Gamma}_{dd}(-j)'$. Throughout, $\sum_t z_t$ denotes $\sum_{t=R}^{T-\tau} z_t$ and $S_{zz}$ denotes $\lim Var(P^{-1/2}\sum_t z_t)$ for any sequence of variables $z_t$.

In the text we use the index s to denote the time argument of Brownian motion W(s). In this appendix we instead use ω. We do so in order to preserve s as a generic index on summations throughout the appendix. Having made that change, let W(ω; Ω) denote a vector Brownian motion with covariance kernel Ω, W(ω; I) = W(ω) and let ⇒ denote weak convergence.

## 3. Statistics

For ease of reference, the encompassing and forecast accuracy statistics are presented below.

$$\text{MSE-T} = (P\text{-}\tau+1)^{1/2} \times \frac{\bar{d}}{\sqrt{\sum_{j=-\bar{j}}^{\bar{j}} K(j/M)\hat{\Gamma}_{dd}(j)}}$$

$$= \frac{\sum_t (\hat{u}_{1,t+\tau}^2 - \hat{u}_{2,t+\tau}^2)}{\sqrt{\sum_{j=-\bar{j}}^{\bar{j}} K(j/M)[\sum_{t=R+j}^{T-\tau}(\hat{u}_{1,t+\tau}^2-\hat{u}_{2,t+\tau}^2-\bar{d})(\hat{u}_{1,t+\tau-j}^2-\hat{u}_{2,t+\tau-j}^2-\bar{d})]}}$$

$$\text{ENC-T} = (P\text{-}\tau+1)^{1/2} \times \frac{\bar{c}}{\sqrt{\sum_{j=-\bar{j}}^{\bar{j}} K(j/M)\hat{\Gamma}_{cc}(j)}}$$

$$= \frac{\sum_t (\hat{u}_{1,t+\tau}^2 - \hat{u}_{1,t+\tau}\hat{u}_{2,t+\tau})}{\sqrt{\sum_{j=-\bar{j}}^{\bar{j}} K(j/M)[\sum_{t=R+j}^{T-\tau}(\hat{u}_{1,t+\tau}^2-\hat{u}_{1,t+\tau}\hat{u}_{2,t+\tau}-\bar{c})(\hat{u}_{1,t+\tau-j}^2-\hat{u}_{1,t+\tau-j}\hat{u}_{2,t+\tau-j}-\bar{c})]}}$$

$$\text{MSE-F} = (P\text{-}\tau+1)\times\frac{\bar{d}}{MSE_2} = \frac{\sum_t(\hat{u}_{1,t+\tau}^2-\hat{u}_{2,t+\tau}^2)}{(P\text{-}\tau+1)^{-1}\sum_t \hat{u}_{2,t+\tau}^2} \qquad \left(\text{or } \left(R/(P\text{-}\tau+1)\right)^{1/2}\times(P\text{-}\tau+1)\times\frac{\bar{d}}{MSE_2}\right)$$

$$\text{ENC-NEW} = (P\text{-}\tau+1)\times\frac{\bar{c}}{MSE_2} = \frac{\sum_t(\hat{u}_{1,t+\tau}^2-\hat{u}_{1,t+\tau}\hat{u}_{2,t+\tau})}{(P\text{-}\tau+1)^{-1}\sum_t \hat{u}_{2,t+\tau}^2} \qquad \left(\text{or } \left(R/(P\text{-}\tau+1)\right)^{1/2}\times(P\text{-}\tau+1)\times\frac{\bar{c}}{MSE_2}\right).$$

## 4. Assumptions

Assumption 1: The parameter estimates $\hat{\beta}_{i,t}$, i = 1,2, t = R,…,T-τ, satisfy $\hat{\beta}_{i,t}$-$\beta_i^*$ = $B_i(t)H_i(t)$ where $B_i(t)H_i(t)$ equals $(t^{-1}\sum_{s=1}^{t-\tau}q_{i,s})^{-1}(t^{-1}\sum_{s=1}^{t-\tau}h_{i,s+\tau})$, $(R^{-1}\sum_{s=t-R+1}^{t-\tau}q_{i,s})^{-1}(R^{-1}\sum_{s=t-R+1}^{t-\tau}h_{i,s+\tau})$, $(R^{-1}\sum_{s=1}^{R-\tau}q_{i,s})^{-1}(R^{-1}\sum_{s=1}^{R-\tau}h_{i,s+\tau})$ for the recursive, rolling and fixed schemes respectively.

Assumption 2: (a) $U_{t+\tau} = [u_{t+\tau}, x'_{2,t}-Ex'_{2,t}, h'_{2,t+\tau}]'$ is covariance stationary, (b) $EU_{t+\tau} = 0$, (c) $Eq_{2,t} < \infty$ and is positive definite, (d) For some r > 8, $U_{t+\tau}$ is uniformly $L^r$ bounded, (e) For some r > d > 2, $U_{t+\tau}$ is strong mixing with coefficients of size –rd/(r – d), (f) With $\tilde{U}_{t+\tau}$ denoting the vector of nonredundant elements of $U_{t+\tau}$, $\lim_{T\to\infty}T^{-1}E(\sum_{s=1}^{T-\tau}\tilde{U}_{s+\tau})(\sum_{s=1}^{T-\tau}\tilde{U}_{s+\tau})' = \Omega < \infty$ is positive definite.

Assumption 3: (a) Let K(x) be a continuous kernel such that for all real scalars x, |K(x)| ≤ 1, K(x) = K(-x) and K(0) = 1, (b) For some bandwidth M and constant i ∈ (0, 0.5), M = O($P^i$), (c) For all j > τ – 1, $Eh_{2,t+\tau}h'_{2,t+\tau-j} = 0$, (d) The number of covariances $\bar{j}$, used to estimate the long-run covariances $S_{cc}$ and $S_{dd}$ defined in Section 3.1, satisfies $\tau - 1 \le \bar{j} < \infty$.

Assumption 4: $\lim_{P,R\to\infty}P/R = \pi \in (0, \infty)$, with $\lambda = (1+\pi)^{-1}$.

Assumption 4′: $\lim_{P,R\to\infty}P/R = 0$, with λ = 1.

5. Lemmas

**Lemma A1**: Let Assumptions 1, 2 and either 4 or 4′ hold. For each i = 1,2, (a) $\sup_t T^{1/2}|U(t)| = O_p(1)$, (b) $\sup_t T^{1/2}|vec[B_i(t)]-vec[B_i]| = O_p(1)$, (c) $\sup_t T^{1/2}|\hat{\beta}_{i,t}-\beta_i^*| = O_p(1)$.

**Proof of Lemma A1**: (a) We will show this for the recursive scheme. The fixed scheme follows immediately from the recursive and the rolling follows from a proof similar to that for the recursive. Note that by definition $T^{1/2}U(t) = (T/t)T^{-1/2}\sum_{s=1}^{t-\tau}U_{s+\tau}$. Recall that |.| denotes the max norm and hence for $\tilde{U}_{t+\tau}$ defined in Assumption 2, $|U_{t+\tau}| = |\tilde{U}_{t+\tau}|$. Therefore $\sup_t T^{1/2}|U(t)| \leq (T/R)\sup_t|T^{-1/2}\sum_{s=1}^{t-\tau}\tilde{U}_{s+\tau}|$. Both Assumption 4 and 4′ imply that T/R is bounded. Given Assumption 2 and Corollary 29.19 of Davidson (1994) we know that $T^{-1/2}\sum_{s=1}^{t-\tau}\tilde{U}_{s+\tau} \Rightarrow \Omega^{1/2}W(\omega)$. Following arguments similar to those in Lemma 2.1 of Corradi, Swanson and Olivetti (2001) we obtain $\sup_t|T^{-1/2}\sum_{s=1}^{t-\tau}\tilde{U}_{s+\tau}| \to_d \sup_{\lambda\leq\omega\leq 1}|\Omega^{1/2}W(\omega)| = O_p(1)$ and the proof is complete.

(b) We will show this for the recursive scheme. The fixed scheme follows immediately and the rolling follows from a proof similar to that for the recursive.

First note that $f(vec[B_i^{-1}(t)])-f(vec[B_i^{-1}]) = vec[B_i(t)]-vec[B_i]$ for a continuously differentiable function f(.) with $\partial f_v(\nu)/\partial f(\nu) \equiv f_v(\nu)$. Second note that there exists an open neighborhood $N(vec[B_i^{-1}])$ of $vec[B_i^{-1}] = vec[Eq_{i,t}]$ and a finite positive constant D such that $\sup_{\nu\in N(vec[B_i^{-1}])}|f_v(\nu)| < D$. Taking a first order Taylor expansion of f(.) around $vec[B_i^{-1}]$ we obtain

$$\begin{aligned}
&\sup_t T^{1/2}|vec[B_i(t)]-vec[B_i]| = \sup_t T^{1/2}|f_v(\tilde{\nu}_t)'(vec[B_i^{-1}(t)]-vec[B_i^{-1}])| \\
&\quad = \sup_t|f_v(\tilde{\nu}_t)'((T/t)T^{-1/2}\textstyle\sum_{s=1}^{t-\tau}vec[q_{i,s}-Eq_{i,s}])| \\
&\quad \leq k^2(T/R)(\sup_t|f_v(\tilde{\nu}_t)|)(\sup_t|T^{-1/2}\textstyle\sum_{s=1}^{t-\tau}vec[q_{i,s}-Eq_{i,s}]|)
\end{aligned}$$

for some $\tilde{v}_t$ with elements that lie on the line between the corresponding elements of $\text{vec}[B_i^{-1}(t)]$ and $\text{vec}[B_i^{-1}]$. Both Assumption 4 and 4′ imply that T/R is bounded. That $\sup_t|T^{-1/2}\sum_{s=1}^{t-\tau}\text{vec}[q_{i,s}\text{-}Eq_{i,s}]| = O_p(1)$ follows from Lemma A1 (a). The result will follow if $\sup_t|f_v(\tilde{v}_t)| = O_p(1)$. By Lemma A1 (a) we know that $\text{vec}[B_i^{-1}(t)] \rightarrow_{a.s.} \text{vec}[B_i^{-1}]$ and hence for all $\varepsilon > 0$ there exists R sufficiently large that $\text{Prob}(\tilde{v}_t \in N(\text{vec}[B_i^{-1}])) > 1 - \varepsilon$. This in turn implies that for large enough R, $\sup_t|f_v(\tilde{v}_t)| \le \sup_{v\in N(\text{vec}[B_i^{-1}])}|f_v(v)| < D$ with probability greater than $1 - \varepsilon$ and the proof is complete.

(c) If we add and subtract $B_i$ and apply the triangle inequality we obtain

$$\sup_t T^{1/2}|B_i(t)H_i(t)| \le k(\sup_t|B_i(t)\text{-}B_i|)(\sup_t T^{1/2}|H_i(t)|) + k|B_i|(\sup_t T^{1/2}|H_i(t)|).$$

Lemma A1 (b) implies that $\sup_t|B_i(t)\text{-}B_i| = o_p(1)$. Since Lemma A1 (a) implies that $\sup_t T^{1/2}|H_i(t)| = O_p(1)$ the proof is complete.

**Lemma A2**: Let Assumptions 1, 2 and 4 hold. For i = 1,2, $\sum_t h_{i,t+\tau}'B_i(t)H_i(t) = \sum_t h_{i,t+\tau}'B_iH_i(t) + o_p(1)$.

**Proof of Lemma A2**: Add and subtract $B_i$ to obtain

$$\text{(A1)} \quad \sum_t h_{i,t+\tau}'B_i(t)H_i(t) = \sum_t h_{i,t+\tau}'B_iH_i(t) + \sum_t h_{i,t+\tau}'(B_i(t)\text{-}B_i)H_i(t).$$

We must then show that the last right-hand side term in (A1) is $o_p(1)$. We do so for the recursive scheme; the arguments are similar for the other schemes. Note that

$$\sum_t h_{i,t+\tau}'(B_i(t)\text{-}B_i)H_i(t) = T^{-1/2}\sum_t (T/t)\text{vec}[T^{1/2}(B_i(t)\text{-}B_i)]'[T^{-1/2}h_{i,t+\tau} \otimes (T^{-1/2}\sum_{s=1}^{t-\tau} h_{i,s+\tau})].$$

Given Assumption 2, Lemmas A1 (a)-(b) and Corollary 29.19 of Davidson (1994), that $\sum_t (T/t)\text{vec}[T^{1/2}(B_i(t)-B_i)]'[T^{-1/2}h_{i,t+\tau} \otimes (T^{-1/2}\sum_{s=1}^{t-\tau} h_{i,s+\tau})]$ is $O_p(1)$ follows from Theorem 4.1 of Hansen (1992). Since $T^{-1/2} = o(1)$ the proof is complete.

**Lemma A3**: Let Assumptions 1, 2 and 4 hold. For i,j = 1,2 and $0 \le m \le \bar{j}$

(a) $\sum_{t=R+m}^{T-\tau} H_i'(t)B_i(t)h_{i,t+\tau}h_{j,t+\tau-m}'B_j(t-m)H_j(t-m) = \sum_t H_i'(t)B_iE(h_{i,t+\tau}h_{j,t+\tau-m}')B_jH_j(t) + o_p(1)$.

(b) $\sum_t H_i'(t)B_i(t)x_{i,t}x_{j,t}'B_j(t)H_j(t) = \sum_t H_i'(t)B_iE(x_{i,t}x_{j,t}')B_jH_j(t) + o_p(1)$.

**Proof of Lemma A3**: We will show the two results for the recursive scheme. Proofs for the rolling and fixed schemes are similar. Since m and the arguments of the summations are finite, we immediately dispense with the summation $\sum_{t=R+m}^{T-\tau}(.)$ and replace it with $\sum_t(.)$. (a) The proof is conducted in two stages. The first stage consists of showing that

$$\sum_t H_i'(t)B_i(t)h_{i,t+\tau}h_{j,t+\tau-m}'B_j(t-m)H_j(t-m) = \sum_t H_i'(t)B_ih_{i,t+\tau}h_{j,t+\tau-m}'B_jH_j(t) + o_p(1).$$

In this proof only let $a_1 = B_i$, $a_2 = B_i(t) - B_i$, $b_1 = B_j$, $b_2 = B_j(t-m) - B_j$, $c_1 = H_j(t)$ and $c_2 = H_j(t-m) - H_j(t)$. Using this notation, if we add and subtract $B_i$, $B_j$ and $H_j(t)$ we obtain the identity

$$\text{(A2)} \quad \sum_t H_i'(t)B_i(t)h_{i,t+\tau}h_{j,t+\tau-m}'B_j(t-m)H_j(t-m) = \sum_{1\le v,w,x\le 2}[\sum_t H_i'(t)a_vh_{i,t+\tau}h_{j,t+\tau-m}'b_wc_x].$$

Note that the outer summation in the right-hand side of (A2) contains 8 terms corresponding to different combinations of (v, w, x). When v = w = x = 1 the argument takes the value $\sum_t H_i'(t)B_ih_{i,t+\tau}h_{j,t+\tau-m}'B_jH_j(t)$. To obtain the result we must show that the remaining seven pieces in (A2) are each $o_p(1)$. The proof of each is very similar. Here we only show that the term $\sum_t H_i'(t)a_2h_{i,t+\tau}h_{j,t+\tau-m}'b_2c_2 = \sum_t H_i'(t)(B_i(t)-B_i)h_{i,t+\tau}h_{j,t+\tau-m}'(B_j(t-m)-B_j)(H_j(t-m)-H_j(t))$ is $o_p(1)$.

Taking absolute values we immediately have

$$|\sum_t H_i'(t)(B_i(t)\text{-}B_i)h_{i,t+\tau}h'_{j,t+\tau\text{-}m}(B_j(t\text{-}m)\text{-}B_j)(H_j(t\text{-}m)\text{-}H_j(t))|$$
$$\leq k^4(P/T)(P^{-1}\sum_t|h_{i,t+\tau}h'_{j,t+\tau-m}|)(\sup_t|T^{1/4}H_i(t)|)(\sup_t|T^{1/4}H_j(t\text{-}m)|+\sup_t|T^{1/4}H_j(t)|)\times$$
$$(\sup_t T^{1/4}|B_i(t)\text{-}B_i|)(\sup_t T^{1/4}|B_j(t\text{-}m)\text{-}B_j|)\,.$$

Given Assumption 2, that $P^{-1}\sum_t|h_{i,t+\tau}h'_{j,t+\tau\text{-}m}| = O_p(1)$ follows from Markov's inequality. That $\sup_t T^{1/4}|H_i(t)|$, $\sup_t T^{1/4}|H_j(t)|$, $\sup_t T^{1/4}|H_j(t-m)|$, $\sup_t T^{1/4}|B_i(t)\text{-}B_i|$ and $\sup_t T^{1/4}|B_j(t\text{-}m)\text{-}B_j|$ are $o_p(1)$ follows from Lemma A1 (a)-(b) and the fact that m is finite.

The second stage of the proof consists of showing that

$$\sum_t H_i'(t)B_ih_{i,t+\tau}h'_{j,t+\tau\text{-}m}B_jH_j(t) = \sum_t H_i'(t)B_iE(h_{i,t+\tau}h'_{j,t+\tau\text{-}m})B_jH_j(t) + o_p(1).$$

To do so add and subtract $E(h_{i,t+\tau}h'_{j,t+\tau\text{-}m})$ to obtain

$$\text{(A3)}\quad \sum_t H_i'(t)B_ih_{i,t+\tau}h'_{j,t+\tau\text{-}m}B_jH_j(t)$$
$$= \sum_t H_i'(t)B_iE(h_{i,t+\tau}h'_{j,t+\tau\text{-}m})B_jH_j(t) + \sum_t H_i'(t)B_i(h_{i,t+\tau}h'_{j,t+\tau\text{-}m}\text{-}E(h_{i,t+\tau}h'_{j,t+\tau\text{-}m}))B_jH_j(t)\,.$$

It then suffices to show that the second right-hand side term in (A3) is $o_p(1)$. Rearranging terms we obtain

$$\sum_t H_i'(t)B_i(h_{i,t+\tau}h'_{j,t+\tau\text{-}m}\text{-}E(h_{i,t+\tau}h'_{j,t+\tau\text{-}m}))B_jH_j(t)$$
$$= T^{-1/2}\sum_t(T/t)^2[T^{-1/2}\sum_{s=1}^{t-\tau}h'_{j,s+\tau}B_j\otimes T^{-1/2}\sum_{s=1}^{t-\tau}h'_{i,s+\tau}B_i)]\text{vec}[T^{-1/2}(h_{i,t+\tau}h'_{j,t+\tau\text{-}m}\text{-}E(h_{i,t+\tau}h'_{j,t+\tau\text{-}m}))]\,.$$

That $\sum_t(T/t)^2[T^{-1/2}\sum_{s=1}^{t-\tau}h'_{j,s+\tau}B_j\otimes T^{-1/2}\sum_{s=1}^{t-\tau}h'_{i,s+\tau}B_i)]\text{vec}[T^{-1/2}(h_{i,t+\tau}h'_{j,t+\tau\text{-}m}\text{-}E(h_{i,t+\tau}h'_{j,t+\tau\text{-}m}))]$ is $O_p(1)$ follows from Assumption 2, Corollary 29.19 of Davidson (1994) and Theorem 4.1 of Hansen (1992). Since $T^{-1/2}$ is o(1) the proof is complete.

(b) The proof is conducted in two stages. The first stage consists of showing that

$$\sum_t H_i'(t)B_i(t)x_{i,t}x_{j,t}'B_j(t)H_j(t) = \sum_t H_i'(t)B_i x_{i,t}x_{j,t}'B_j H_j(t) + o_p(1).$$

In this proof only let $a_1 = B_i$, $a_2 = B_i(t) - B_i$, $b_1 = B_j$ and $b_2 = B_j(t) - B_j$. Using this notation, if we add and subtract $B_i$ and $B_j$ we obtain the identity

$$\text{(A4)} \quad \sum_t H_i'(t)B_i(t)x_{i,t}x_{j,t}'B_j(t)H_j(t) = \sum_{1\le v,w\le 2}[\sum_t H_i'(t)a_v x_{i,t}x_{j,t}'b_w H_j(t)].$$

Note that the outer summation in the right-hand side of (A4) contains four terms corresponding to different combinations of (v, w). When v = w = 1 the argument takes the value $\sum_t H_i'(t)B_i x_{i,t}x_{j,t}'B_j H_j(t)$. To obtain the result we must show that the remaining three pieces in (A4) are each $o_p(1)$. The proof of each is very similar. Here we only show that the term $\sum_t H_i'(t)a_2 x_{i,t}x_{j,t}'b_2 H_j(t) = \sum_t H_i'(t)(B_i(t)\text{-}B_i)x_{i,t}x_{j,t}'(B_j(t)\text{-}B_j)H_j(t)$ is $o_p(1)$. Taking absolute values we immediately have

$$\begin{aligned} &|\sum_t H_i'(t)(B_i(t)\text{-}B_i)x_{i,t}x_{j,t}'(B_j(t)\text{-}B_j)H_j(t)| \\ &\quad \le k^4(P/T)(P^{-1}\sum_t|x_{i,t}x_{j,t}'|)(\sup_t|T^{1/4}H_i(t)|)(\sup_t|T^{1/4}H_j(t)|) \times \\ &\qquad\qquad (\sup_t T^{1/4}|B_i(t)\text{-}B_i|)(\sup_t T^{1/4}|B_j(t)\text{-}B_j|). \end{aligned}$$

Given Assumption 2, that $P^{-1}\sum_t|x_{i,t}x_{j,t}'| = O_p(1)$ follows from Markov's inequality. That $\sup_t T^{1/4}|H_i(t)|$, $\sup_t T^{1/4}|H_j(t)|$, $\sup_t T^{1/4}|B_i(t)\text{-}B_i|$ and $\sup_t T^{1/4}|B_j(t)\text{-}B_j|$ are $o_p(1)$ follows from Lemma A1 (a)-(b).

The second stage of the proof consists of showing that

$$\sum_t H_i'(t)B_i x_{i,t}x_{j,t}'B_j H_j(t) = \sum_t H_i'(t)B_i E(x_{i,t}x_{j,t}')B_j H_j(t) + o_p(1).$$

To do so add and subtract $E(x_{i,t}x'_{j,t})$ to obtain

$$\text{(A5)} \quad \sum_t H_i'(t)B_i x_{i,t}x'_{j,t}B_jH_j(t)$$
$$= \sum_t H_i'(t)B_iE(x_{i,t}x'_{j,t})B_jH_j(t) + \sum_t H_i'(t)B_i(x_{i,t}x'_{j,t}-E(x_{i,t}x'_{j,t}))B_jH_j(t).$$

It then suffices to show that the second right-hand side term in (A5) is $o_p(1)$. Rearranging terms we obtain

$$\sum_t H_i'(t)B_i(x_{i,t}x'_{j,t}-E(x_{i,t}x'_{j,t}))B_jH_j(t)$$
$$= T^{-1/2}\sum_t \{(T/t)^2[T^{-1/2}\sum_{s=1}^{t-\tau} h'_{j,s+\tau}B_j \otimes T^{-1/2}\sum_{s=1}^{t-\tau} h'_{i,s+\tau}B_i)]\text{vec}[T^{-1/2}(x_{i,t}x'_{j,t}-E(x_{i,t}x'_{j,t}))]\}.$$

That $\sum_t \{(T/t)^2[T^{-1/2}\sum_{s=1}^{t-\tau} h'_{j,s+\tau}B_j \otimes T^{-1/2}\sum_{s=1}^{t-\tau} h'_{i,s+\tau}B_i)]\text{vec}[T^{-1/2}(x_{i,t}x'_{j,t}-E(x_{i,t}x'_{j,t}))]\}$ is $O_p(1)$ follows from Assumption 2, Corollary 29.19 of Davidson (1994) and Theorem 4.1 of Hansen (1992). Since $T^{-1/2}$ is o(1) the proof is complete.

**Lemma A4**: Let Assumptions 1-2 hold. (a) Let $-J'B_1J+B_2 = D$ and $B_2^{-1/2}DB_2^{-1/2} = Q$, then Q is idempotent. (b) Let A be a $(k\times k_2)$ matrix with $I_{k_2\times k_2}$ on the upper diagonal $(k_2\times k_2)$ block and zeroes elsewhere. There exists a symmetric orthonormal matrix C such that $Q = CAA'C$.

**Proof of Lemma A4**: (a) For i,j = 1,2, let $q_{2,t}^{(i,j)}$ denote the $(k_i\times k_j)$ ij-block of the matrix $q_{2,t}$. Since

$$B_2 = \begin{pmatrix} B_1(I+[Eq_{2,t}^{(1,2)}]N_2[Eq_{2,t}^{(2,1)}]B_1) & -B_1[Eq_{2,t}^{(1,2)}]N_2 \\ -N_2[Eq_{2,t}^{(2,1)}]B_1 & N_2 \end{pmatrix},$$

$N_2 = [Eq_{2,t}^{(2,2)}-Eq_{2,t}^{(2,1)}B_1Eq_{2,t}^{(1,2)}]^{-1}$ and

$$D = \begin{pmatrix} B_1[Eq_{2,t}^{(1,2)}]N_2[Eq_{2,t}^{(2,1)}]B_1 & -B_1[Eq_{2,t}^{(1,2)}]N_2 \\ -N_2[Eq_{2,t}^{(2,1)}]B_1 & N_2 \end{pmatrix}$$

the result follows from straightforward algebra. (b) Since Q is idempotent the result follows from Schur's Decomposition Theorem (Magnus and Neudecker, 1988, p.16).

**Lemma A5**: Let Assumptions 1, 2 and 4 hold. For $\omega \in [\lambda, 1]$, (a) $T^{-1/2}\sum_{s=1}^{t-\tau}\tilde{h}_{2,s+\tau} \Rightarrow S_{\tilde{h}\tilde{h}}^{1/2}W(\omega)$,

(b) $(T/t)T^{-1/2}\sum_{s=1}^{t-\tau}\tilde{h}_{2,s+\tau} \Rightarrow \omega^{-1}S_{\tilde{h}\tilde{h}}^{1/2}W(\omega)$, (c) $(T/R)T^{-1/2}\sum_{s=t-R+1}^{t-\tau}\tilde{h}_{2,s+\tau} \Rightarrow \lambda^{-1}S_{\tilde{h}\tilde{h}}^{1/2}\{W(\omega)\text{-}W(\omega\text{-}\lambda)\}$.

**Proof of Lemma A5**: (a) Given Assumption 2 and the fact that $T^{-1}E(\sum_{s=1}^{T}\tilde{h}_{2,s+\tau})(\sum_{s=1}^{T}\tilde{h}'_{2,s+\tau}) \rightarrow S_{\tilde{h}\tilde{h}}$ $< \infty$ the result follows from Corollary 29.19 of Davidson (1994). (b) Given (a) the result follows from the Continuous Mapping Theorem. (c) That $T/R \rightarrow \lambda^{-1}$ is immediate. Write

$T^{-1/2}\sum_{s=t-R+1}^{t-\tau}\tilde{h}_{2,s+\tau}$ as $T^{-1/2}\sum_{s=1}^{t-\tau}\tilde{h}_{2,s+\tau}\text{-}T^{-1/2}\sum_{s=1}^{t-R}\tilde{h}_{2,s+\tau}$. That $T^{-1/2}\sum_{s=1}^{t-\tau}\tilde{h}_{2,s+\tau} \Rightarrow S_{\tilde{h}\tilde{h}}^{1/2}W(\omega)$ follows from (a). For the second piece, if we define $\omega' = \omega - \lambda$ then Corollary 29.19 of Davidson (1994) implies $T^{-1/2}\sum_{s=1}^{t-R}\tilde{h}_{2,s+\tau} \Rightarrow S_{\tilde{h}\tilde{h}}^{1/2}W(\omega')$.

**Lemma A6**: Let Assumptions 1, 2 and 4 hold. $\sum_t \tilde{H}'_2(t)\tilde{h}_{2,t+\tau} \rightarrow_d \Gamma_1$ where $\Gamma_1$ equals

$\int_\lambda^1 \omega^{-1}W'(\omega)S_{\tilde{h}\tilde{h}}dW(\omega)$, $\lambda^{-1}\{W(1)\text{-}W(\lambda)\}'S_{\tilde{h}\tilde{h}}W(\lambda)$ and $\lambda^{-1}\int_\lambda^1\{W(\omega)\text{-}W(\omega\text{-}\lambda)\}'S_{\tilde{h}\tilde{h}}dW(\omega)$ for the recursive, fixed and rolling schemes respectively.

**Proof of Lemma A6**: The result is simple for the fixed. We will show the result for the recursive scheme. For the rolling scheme if we note that $\sum_t \tilde{H}'_2(t)\tilde{h}_{2,t+\tau}$ can be rewritten as

$(T/R)\sum_t(T^{-1/2}\sum_{s=1}^{t-\tau}\tilde{h}_{2,s+\tau})'(T^{-1/2}\tilde{h}_{2,t+\tau}) - (T/R)\sum_t(T^{-1/2}\sum_{s=1}^{t-R}\tilde{h}_{2,s+\tau})'(T^{-1/2}\tilde{h}_{2,t+\tau})$ then the result will follow from an argument similar to that for the recursive but repeated for each of the two components. Throughout assume that $\tau > 1$. If $\tau = 1$ the argument is made simpler since many of the terms do not exist.

The results are modifications of those in Hansen (1992). As such we will use his notation throughout. Let the operator $E_iX$ denote $E(X|\Im_i)$ where $\Im_t \equiv \sigma(T^{-1/2}\sum_{s=1}^{i}\tilde{h}_{2,s}, \tilde{h}_{2,i}; i \leq t, T \geq 1)$ is the smallest sigma-field containing the past history of $\{T^{-1/2}\sum_{s=1}^{t}\tilde{h}_{2,s}, \tilde{h}_{2,t}\}$ for all T. Define $\varepsilon_{t+\tau} = \sum_{i=0}^{\infty}(E_i\tilde{h}_{2,t+\tau+i} - E_{i-1}\tilde{h}_{2,t+\tau+i})$ and $z_{t+\tau} = \sum_{i=1}^{\infty}E_i\tilde{h}_{2,t+\tau+i}$. Then $\tilde{h}_{2,t+\tau} = \varepsilon_{t+\tau} + z_{t+\tau-1} - z_{t+\tau}$.

In the above notation

$$
\begin{aligned}
&\sum_t (T/t)(T^{-1/2}\sum_{s=1}^{t-\tau}\tilde{h}_{2,s+\tau})'(T^{-1/2}\tilde{h}_{2,t+\tau}) \\
&\quad = \sum_t (T/t)(T^{-1/2}\sum_{s=\tau+1}^{t+\tau-1}\tilde{h}_{2,s})'(T^{-1/2}\tilde{h}_{2,t+\tau}) - \sum_t (1/t)(\sum_{j=1}^{\tau-1}\tilde{h}_{2,t+j})'\tilde{h}_{2,t+\tau} \\
&\quad = \sum_t (T/t)(T^{-1/2}\sum_{s=\tau+1}^{t+\tau-1}\tilde{h}_{2,s})'(T^{-1/2}\varepsilon_{t+\tau}) + \sum_t (1/t)(\sum_{s=\tau+1}^{t+\tau-1}\tilde{h}_{2,s})'(z_{t+\tau-1} - z_{t+\tau}) \\
&\qquad\qquad - \sum_t (1/t)(\sum_{j=1}^{\tau-1}\tilde{h}_{2,t+j})'\tilde{h}_{2,t+\tau} \\
&\quad = \sum_t (T/t)(T^{-1/2}\sum_{s=1}^{t-\tau}\tilde{h}_{2,s+\tau})'(T^{-1/2}\varepsilon_{t+\tau}) + R^{-1}(\sum_{s=1+\tau}^{R+\tau-2}\tilde{h}_{2,s}')z_{R+\tau-1} - T^{-1}(\sum_{s=1+\tau}^{T+\tau-1}\tilde{h}_{2,s}')z_{T+\tau} \\
&\qquad\qquad - \sum_{t=R}^{T-1}(t^2+t)^{-1}(\sum_{s=1+\tau}^{t+\tau-1}\tilde{h}_{2,s})'z_{t+\tau} + \sum_{t=R-1}^{T-1}(1/t)\tilde{h}_{2,t+\tau}'z_{t+\tau} - \sum_t (1/t)(\sum_{j=1}^{\tau-1}\tilde{h}_{2,t+j})'\tilde{h}_{2,t+\tau}\,.
\end{aligned}
$$

That $\sum_t (T/t)(T^{-1/2}\sum_{s=1}^{t-\tau}\tilde{h}_{2,s+\tau})'(T^{-1/2}\varepsilon_{t+\tau}) \to_d \int_\lambda^1 \omega^{-1}W'(\omega)S_{\tilde{h}\tilde{h}}dW(\omega)$ follows from Theorem 4.1 of Hansen (1992). The result will follow if the sum of the remaining five terms is $o_p(1)$.

Consider the second and third right-hand side terms. If we take the absolute value of each we obtain both $|R^{-1}(\sum_{s=1+\tau}^{R+\tau-2}\tilde{h}_{2,s}')z_{R+\tau-1}| \leq (T/R)k_2|T^{-1/2}\sum_{s=1+\tau}^{R+\tau-2}\tilde{h}_{2,s}||T^{-1/2}z_{R+\tau-1}|$ and $|R^{-1}(\sum_{s=1+\tau}^{T+\tau-1}\tilde{h}_{2,s}')z_{T+\tau}| \leq (T/R)k_2|T^{-1/2}\sum_{s=1+\tau}^{T+\tau-1}\tilde{h}_{2,s}||T^{-1/2}z_{T+\tau}|$. Assumption 4 implies that (T/R) is bounded while Lemma A1 (a) implies that both $|T^{-1/2}\sum_{s=1+\tau}^{R+\tau-2}\tilde{h}_{2,s}|$ and $|T^{-1/2}\sum_{s=1+\tau}^{T+\tau-1}\tilde{h}_{2,s}|$ are $O_p(1)$. That the second and third right-hand side terms are $o_p(1)$ follows from (A.3) of Hansen (1992) wherein he shows that both $|T^{-1/2}z_{R+\tau-1}|$ and $|T^{-1/2}z_{T+\tau}|$ are $o_p(1)$.

Consider the fourth right-hand side term. If we take its absolute value we obtain $|\sum_{t=R}^{T-1}(t^2+t)^{-1}(\sum_{s=1+\tau}^{t+\tau-1}\tilde{h}_{2,s})'z_{t+\tau}| \le [(T-1-R)/(R^2+R)]k_2(\sup_t|T^{-1/2}\sum_{s=1+\tau}^{t+\tau-1}\tilde{h}_{2,s}|)(\sup_{t\le T}|T^{-1/2}z_{t+\tau}|)$. Lemma A1 (a) implies that $(\sup_t|T^{-1/2}\sum_{s=1+\tau}^{t+\tau-1}\tilde{h}_{2,s}|)$ is $O_p(1)$. That $(\sup_{t\le T}|T^{-1/2}z_{t+\tau}|) = o_p(1)$ follows from (A.3) of Hansen (1992). The result follows since by Assumption 4, $(T-1-R)/(R^2+R)$ is o(1).

Consider the fifth right-hand side term. We show that it converges in probability to $-\ln(\lambda)\sum_{j=1}^{\tau-1}E\tilde{h}_{2,t+j}'\tilde{h}_{2,t+\tau}$. To do so, add and subtract $\sum_{t=R-1}^{T-1}(1/t)E(\tilde{h}_{2,t+\tau}'z_{t+\tau})$ and obtain $\sum_{t=R-1}^{T-1}(1/t)(\tilde{h}_{2,t+\tau}'z_{t+\tau}-E\tilde{h}_{2,t+\tau}'z_{t+\tau}) + \sum_{t=R-1}^{T-1}(1/t)(E\tilde{h}_{2,t+\tau}'z_{t+\tau})$. Since $E\tilde{h}_{2,t+\tau}\tilde{h}_{2,t+\tau-j}' = 0$ all $j \ge \tau$ it is clear that $\sum_{t=R-1}^{T-1}(1/t)(E\tilde{h}_{2,t+\tau}'z_{t+\tau}) = \sum_{t=R-1}^{T-1}(1/t)(E\tilde{h}_{2,t+\tau}'[\sum_{i=1}^{\infty}E_i\tilde{h}_{2,t+\tau+i}]) = (T^{-1}\sum_{t=R-1}^{T-1}(T/t))(\sum_{j=1}^{\tau-1}E\tilde{h}_{2,t+j}'\tilde{h}_{2,t+\tau})$. Since for large enough T, $T^{-1}\sum_{t=R}^{T}(T/t) \sim \int_{\lambda}^{1}\omega^{-1}d\omega = -\ln(\lambda)$, the result will follow if $\sum_{t=R-1}^{T-1}(1/t)(\tilde{h}_{2,t+\tau}'z_{t+\tau}-E\tilde{h}_{2,t+\tau}'z_{t+\tau}) = T^{-1}\sum_{t=R-1}^{T-1}(T/t)(\tilde{h}_{2,t+\tau}'z_{t+\tau}-E\tilde{h}_{2,t+\tau}'z_{t+\tau}) = o_p(1)$. If we define $U_{Tt} \equiv (T/t)$ and $e_t \equiv \tilde{h}_{2,t+\tau}'z_{t+\tau}-E\tilde{h}_{2,t+\tau}'z_{t+\tau}$ then the result follows from Theorem 3.2 of Hansen (1992).

Because of the minus sign, the proof will be complete if the final right-hand side term converges in probability to $-\ln(\lambda)\sum_{j=1}^{\tau-1}E\tilde{h}_{2,t+j}'\tilde{h}_{2,t+\tau}$. To show this add and subtract $\sum_{j=1}^{\tau-1}E\tilde{h}_{2,t+j}'\tilde{h}_{2,t+\tau}$ to obtain $\sum_t(1/t)(\sum_{j=1}^{\tau-1}\tilde{h}_{2,t+j})'\tilde{h}_{2,t+\tau} = T^{-1/2}\sum_{j=1}^{\tau-1}T^{-1/2}\sum_t(T/t)(\tilde{h}_{2,t+j}'\tilde{h}_{2,t+\tau}-E\tilde{h}_{2,t+j}'\tilde{h}_{2,t+\tau}) + (T^{-1}\sum_t(T/t))(\sum_{j=1}^{\tau-1}E\tilde{h}_{2,t+j}'\tilde{h}_{2,t+\tau})$. Given Assumption 2, Corollary 29.11 of Davidson (1994) implies that $\sum_{j=1}^{\tau-1}T^{-1/2}\sum_t(T/t)(\tilde{h}_{2,t+j}'\tilde{h}_{2,t+\tau}-E\tilde{h}_{2,t+j}'\tilde{h}_{2,t+\tau}) = O_p(1)$. Since $T^{-1/2} = o(1)$ the result is obtained because $(T^{-1}\sum_t(T/t))(\sum_{j=1}^{\tau-1}E\tilde{h}_{2,t+j}'\tilde{h}_{2,t+\tau}) = -\ln(\lambda)\sum_{j=1}^{\tau-1}E\tilde{h}_{2,t+j}'\tilde{h}_{2,t+\tau} + o(1)$ was established in the preceding paragraph.

**Lemma A7**: Let Assumptions 1, 2 and 4 hold. $\sum_t \tilde{H}_2'(t)\tilde{H}_2(t) \to_d \Gamma_2$ where $\Gamma_2$ equals

$\int_\lambda^1 \omega^{-2} W'(\omega) S_{\tilde{h}\tilde{h}} W(\omega) d\omega$, $\lambda^{-2} \int_\lambda^1 \{W(\omega)-W(\omega-\lambda)\}' S_{\tilde{h}\tilde{h}} \{W(\omega)-W(\omega-\lambda)\} d\omega$ and $\pi\lambda^{-1} W'(\lambda) S_{\tilde{h}\tilde{h}} W(\lambda)$ for the recursive, rolling and fixed schemes respectively.

**Proof of Lemma A7**: The result is immediate for the fixed. Given Lemma A5 the results for the recursive and rolling follow from the Continuous Mapping Theorem.

**Lemma A8:** Let Assumptions 1, 2 and 4′ hold. For i = 1,2, $\sup_t R^{1/2}|\hat{\beta}_{i,t} - \hat{\beta}_{i,R}| = o_p(1)$.

**Proof of Lemma A8:** We will show the result for the recursive scheme. The fixed is immediate and the rolling follows from a proof similar to that for the recursive. First note that

$$\begin{aligned}
\hat{\beta}_{i,t} - \hat{\beta}_{i,R} &= (\hat{\beta}_{i,t} - \beta_i^*) - (\hat{\beta}_{i,R} - \beta_i^*) = (\textstyle\sum_{s=1}^{t-\tau} q_{i,s})^{-1}(\sum_{s=1}^{t-\tau} h_{i,s+\tau}) - B_i(R)H_i(R) \\
&= (B_i^{-1}(R) + R^{-1}\textstyle\sum_{s=R-\tau+1}^{t-\tau} q_{i,s})^{-1}(H_i(R) + R^{-1}\sum_{s=R-\tau+1}^{t-\tau} h_{i,s+\tau}) - B_i(R)H_i(R) \\
&= B_i(R)(R^{-1}\textstyle\sum_{s=R-\tau+1}^{t-\tau} h_{i,s+\tau}) - B_i(R)(R^{-1}\sum_{s=R-\tau+1}^{t-\tau} q_{i,s})[I + B_i(R)(R^{-1}\sum_{s=R-\tau+1}^{t-\tau} q_{i,s})]^{-1} B_i(R)H_i(R) \\
&\quad - B_i(R)(R^{-1}\textstyle\sum_{s=R-\tau+1}^{t-\tau} q_{i,s})[I + B_i(R)(R^{-1}\sum_{s=R-\tau+1}^{t-\tau} q_{i,s})]^{-1} B_i(R)(R^{-1}\sum_{s=R-\tau+1}^{t-\tau} h_{i,s+\tau})
\end{aligned}$$

where the final equality follows from the formula for the inverse of a sum of matrices as presented in Greene (2000, p. 32). Rearranging terms and using the triangle inequality we obtain

$$\begin{aligned}
\text{(A6)} \quad \sup_t R^{1/2}|\hat{\beta}_{i,t} - \hat{\beta}_{i,R}| &= (P/R)^{1/2} \sup_t |B_i(R)(P^{-1/2}\textstyle\sum_{s=R-\tau+1}^{t-\tau} h_{i,s+\tau})| \\
&+ (P/R)\sup_t |B_i(R)(P^{-1}\textstyle\sum_{s=R-\tau+1}^{t-\tau} q_{i,s}) [I + (P/R)B_i(R)(P^{-1}\sum_{s=R-\tau+1}^{t-\tau} q_{i,s})]^{-1} B_i(R)(R^{-1/2}\sum_{s=1}^{R-\tau} h_{i,s+\tau})| \\
&+ (P/R)^{3/2}\sup_t |B_i(R)(P^{-1}\textstyle\sum_{s=R-\tau+1}^{t-\tau} q_{i,s}) \times \\
&\qquad [I + (P/R)B_i(R)(P^{-1}\textstyle\sum_{s=R-\tau+1}^{t-\tau} q_{i,s})]^{-1} B_i(R)(P^{-1/2}\sum_{s=R-\tau+1}^{t-\tau} h_{i,s+\tau})|.
\end{aligned}$$

We must then show that the three right-hand side terms in (A6) are each $o_p(1)$. First note that both $B_i(R)$ and $R^{-1/2}\sum_{s=1}^{R-\tau} h_{i,s+\tau}$ are invariant to the index t. Given Lemmas A1 (a) and (b), each of these terms can be treated as $O_p(1)$ constants with respect to the operator $\sup_t$.

Consider the first right-hand side term in (A6). Given Assumption 2 and Corollary 29.19 of Davidson (1994) we know that $P^{-1/2}\sum_{s=R-\tau+1}^{t-\tau} h_{i,s+\tau} \Rightarrow S_{hh}^{1/2} W(\omega)$. Then following arguments similar to those in Lemma A1 (a) we obtain $\sup_t |B_i(R)(P^{-1/2}\sum_{s=R-\tau+1}^{t-\tau} h_{i,s+\tau})| \to_d \sup_{0\le\omega\le1} |B_i S_{hh}^{1/2} W(\omega)| = O_p(1)$. Since $(P/R)^{1/2} = o(1)$ the desired result follows.

Consider the second right-hand side term in (A6). In particular rewrite $P^{-1}\sum_{s=R-\tau+1}^{t-\tau} q_{i,s}$ as $P^{-1/2}[P^{-1/2}\sum_{s=R-\tau+1}^{t-\tau}(q_{i,s}-Eq_{i,s})] + Eq_{i,s}(t-R-1)/P$. Given Assumption 2 and Corollary 29.19 of Davidson (1994) we know that $P^{-1/2}\sum_{s=R-\tau+1}^{t-\tau} \mathrm{vec}(q_{i,s}-Eq_{i,s}) \Rightarrow S_{qq}^{1/2} W(\omega)$. The Continuous Mapping Theorem then implies that $P^{-1/2}[P^{-1/2}\sum_{s=R-\tau+1}^{t-\tau}(q_{i,s}-Eq_{i,s})] + Eq_{i,s}(t-R-1)/P \Rightarrow \omega Eq_{i,s} = \omega B_i^{-1}$. Given Assumption 2 and Corollary 29.19 of Davidson (1994) we know that $R^{-1/2}\sum_{s=1}^{R-\tau} h_{i,s+\tau}$ $\to_d S_{hh}^{1/2} V$, V a standard normal vector. Then following arguments similar to those in Lemma A1 (a) we obtain $\sup_t |B_i(R)(P^{-1}\sum_{s=R-\tau+1}^{t-\tau} q_{i,s}) \times [I+(P/R)B_i(R)(P^{-1}\sum_{s=R-\tau+1}^{t-\tau} q_{i,s})]^{-1} B_i(R)(R^{-1/2}\sum_{s=1}^{R-\tau} h_{i,s+\tau})| \to_d$ $\sup_{0\le\omega\le1} |\omega B_i S_{hh}^{1/2} V| = |B_i S_{hh}^{1/2} V| = O_p(1)$. Since $P/R = o(1)$ the desired result follows.

Consider the third right-hand side term in (A6). Arguments as for the first two pieces imply $\sup_t |B_i(R)(P^{-1}\sum_{s=R-\tau+1}^{t-\tau} q_{i,s})\,[I+(P/R)B_i(R)(P^{-1}\sum_{s=R-\tau+1}^{t-\tau} q_{i,s})]^{-1} B_i(R)(P^{-1/2}\sum_{s=R-\tau+1}^{t-\tau} h_{i,s+\tau})| \to_d$ $\sup_{0\le\omega\le1} |\omega B_i S_{hh}^{1/2} W(\omega)| = O_p(1)$. Since $(P/R)^{3/2} = o(1)$ the proof is complete.

**Lemma A9:** Let Assumptions 1, 2 and 4′ hold, let i,j,m,n = 1,2 and let $0 \le r,s \le \bar{j}$. (a) Each of the following are $o_p((P/R)^{1/2})$:

(i) $\sum_t h_{i,t+\tau}'(\hat{\beta}_{i,t}-\hat{\beta}_{i,R})$ and (ii) $\sum_t (\hat{\beta}_{i,R}-\beta_i^*)' x_{i,t} x_{j,t}' (\hat{\beta}_{j,R}-\beta_j^*)$.

(b) Each of the following are $o_p(P/R)$ (and hence also $o_p((P/R)^{1/2})$):

(iii) $\sum_t (\hat{\beta}_{i,R} - \beta_i^*)' x_{i,t} x_{j,t}' (\hat{\beta}_{j,t} - \hat{\beta}_{j,R})$,

(iv) $\sum_t (\hat{\beta}_{i,t} - \hat{\beta}_{i,R})' x_{i,t} x_{j,t}' (\hat{\beta}_{j,t} - \hat{\beta}_{j,R})$,

(v) $\sum_t (\hat{\beta}_{i,R} - \beta_i^*)' h_{i,t+\tau-r} h_{j,t+\tau-s}' (\hat{\beta}_{j,t-s} - \hat{\beta}_{j,R})$,

(vi) $\sum_t (\hat{\beta}_{i,t-r} - \hat{\beta}_{i,R})' h_{i,t+\tau-r} h_{j,t+\tau-s}' (\hat{\beta}_{j,t-s} - \hat{\beta}_{j,R})$,

(vii) $\sum_t (\hat{\beta}_{i,R} - \beta_i^*)' h_{i,t+\tau-r} vec(x_{m,t-s} x_{j,t-s}')' ((\hat{\beta}_{j,R} - \beta_j^*) \otimes (\hat{\beta}_{m,R} - \beta_m^*))$,

(viii) $\sum_t (\hat{\beta}_{i,R} - \beta_i^*)' h_{i,t+\tau-r} vec(x_{m,t-s} x_{j,t-s}')' ((\hat{\beta}_{j,t-s} - \hat{\beta}_{j,R}) \otimes (\hat{\beta}_{m,R} - \beta_m^*))$,

(ix) $\sum_t (\hat{\beta}_{i,R} - \beta_i^*)' h_{i,t+\tau-r} vec(x_{m,t-s} x_{j,t-s}')' ((\hat{\beta}_{j,t-s} - \hat{\beta}_{j,R}) \otimes (\hat{\beta}_{m,t-s} - \hat{\beta}_{m,R}))$,

(x) $\sum_t (\hat{\beta}_{i,t-r} - \hat{\beta}_{i,R})' h_{i,t+\tau-r} vec(x_{m,t-s} x_{j,t-s}')' ((\hat{\beta}_{j,R} - \beta_j^*) \otimes (\hat{\beta}_{m,R} - \beta_m^*))$,

(xi) $\sum_t (\hat{\beta}_{i,t-r} - \hat{\beta}_{i,R})' h_{i,t+\tau-r} vec(x_{m,t-s} x_{j,t-s}')' ((\hat{\beta}_{j,t-s} - \hat{\beta}_{j,R}) \otimes (\hat{\beta}_{m,R} - \beta_m^*))$,

(xii) $\sum_t (\hat{\beta}_{i,t-r} - \hat{\beta}_{i,R})' h_{i,t+\tau-r} vec(x_{m,t-s} x_{j,t-s}')' ((\hat{\beta}_{j,t-s} - \hat{\beta}_{j,R}) \otimes (\hat{\beta}_{m,t-s} - \hat{\beta}_{m,R}))$,

(xiii) $\sum_t ((\hat{\beta}_{i,R} - \beta_i^*)' \otimes (\hat{\beta}_{j,R} - \beta_j^*)') vec(x_{i,t-r} x_{j,t-r}') vec(x_{n,t-s} x_{m,t-s}')' ((\hat{\beta}_{m,R} - \beta_m^*) \otimes (\hat{\beta}_{n,R} - \beta_n^*))$,

(xiv) $\sum_t ((\hat{\beta}_{i,R} - \beta_i^*)' \otimes (\hat{\beta}_{j,R} - \beta_j^*)') vec(x_{i,t-r} x_{j,t-r}') vec(x_{n,t-s} x_{m,t-s}')' ((\hat{\beta}_{m,t-s} - \hat{\beta}_{m,R}) \otimes (\hat{\beta}_{n,R} - \beta_n^*))$,

(xv) $\sum_t ((\hat{\beta}_{i,R} - \beta_i^*)' \otimes (\hat{\beta}_{j,R} - \beta_j^*)') vec(x_{i,t-r} x_{j,t-r}') vec(x_{n,t-s} x_{m,t-s}')' ((\hat{\beta}_{m,t-s} - \hat{\beta}_{m,R}) \otimes (\hat{\beta}_{n,t-s} - \hat{\beta}_{n,R}))$,

(xvi) $\sum_t ((\hat{\beta}_{i,t-r} - \hat{\beta}_{i,R})' \otimes (\hat{\beta}_{j,R} - \beta_j^*)') vec(x_{i,t-r} x_{j,t-r}') vec(x_{n,t-s} x_{m,t-s}')' ((\hat{\beta}_{m,t-s} - \hat{\beta}_{m,R}) \otimes (\hat{\beta}_{n,R} - \beta_n^*)$,

(xvii) $\sum_t ((\hat{\beta}_{i,t-r} - \hat{\beta}_{i,R})' \otimes (\hat{\beta}_{j,R} - \beta_j^*)') vec(x_{i,t-r} x_{j,t-r}') vec(x_{n,t-s} x_{m,t-s}')' ((\hat{\beta}_{m,t-s} - \hat{\beta}_{m,R}) \otimes (\hat{\beta}_{n,t-s} - \hat{\beta}_{n,R}))$,

(xviii) $\sum_t ((\hat{\beta}_{i,t-r} - \hat{\beta}_{i,R})' \otimes (\hat{\beta}_{j,t-r} - \hat{\beta}_{j,R})') vec(x_{i,t-r} x_{j,t-r}') vec(x_{n,t-s} x_{m,t-s}')' ((\hat{\beta}_{m,t-s} - \hat{\beta}_{m,R}) \otimes (\hat{\beta}_{n,t-s} - \hat{\beta}_{n,R}))$.

**Proof of Lemma A9:** (a) It suffices to show that $(R/P)^{1/2}$ times each of the terms is $o_p(1)$. Consider (i), $(R/P)^{1/2} \sum_t h_{i,t+\tau}' (\hat{\beta}_{i,t} - \hat{\beta}_{i,R}) = (P/R)^{1/2} \sum_t (P^{-1/2} h_{i,t+\tau}')[(R/P)^{1/2} R^{1/2} (\hat{\beta}_{i,t} - \hat{\beta}_{i,R})]$. The second paragraph of the proof of Lemma A8 implies that $(R/P)^{1/2} R^{1/2} (\hat{\beta}_{i,t} - \hat{\beta}_{i,R}) \Rightarrow B_i S_{hh}^{1/2} W(\omega)$. Given

Assumption 2, Corollary 4.1 of Hansen (1992) then implies that

$\sum_t (P^{-1/2}h'_{i,t+\tau})[(R/P)^{1/2}R^{1/2}(\hat{\beta}_{i,t} - \hat{\beta}_{i,R})] = O_p(1)$. Since $(P/R)^{1/2} = o(1)$ the result follows.

Now consider the remaining term. Taking absolute values, we obtain the inequality

$$\text{(ii)}\ |(R/P)^{1/2}\sum_t (\hat{\beta}_{i,R} - \beta^*_i)'x_{i,t}x'_{j,t}(\hat{\beta}_{j,R} - \beta^*_j)|$$
$$\leq k^2(P/R)^{1/2}(R^{1/2}|\hat{\beta}_{i,R} - \beta^*_i|)(R^{1/2}|\hat{\beta}_{j,R} - \beta^*_j|)(P^{-1}\sum_t|x_{i,t}x'_{j,t}|),$$

Given Assumption 2, Markov's inequality implies that $P^{-1}\sum_t|x_{i,t}x'_{j,t}| = O_p(1)$. That $R^{1/2}|\hat{\beta}_{i,R} - \beta^*_i| = O_p(1)$ follows from Lemma A1 (c). Since $\sup_t R^{1/2}|\hat{\beta}_{i,t} - \hat{\beta}_{i,R}| = o_p(1)$ by Lemma A8 and $P/R = o(1)$ the proof is complete.

(b) It suffices to show that (R/P) times each of the terms is $o_p(1)$. Taking absolute values, we obtain the inequalities

$$\text{(iii)}\ |(R/P)\sum_t (\hat{\beta}_{i,R} - \beta^*_i)'x_{i,t}x'_{j,t}(\hat{\beta}_{j,t} - \hat{\beta}_{j,R})| \leq k^2(R^{1/2}|\hat{\beta}_{i,R} - \beta^*_i|)(\sup_t R^{1/2}|\hat{\beta}_{j,t} - \hat{\beta}_{j,R}|)(P^{-1}\sum_t|x_{i,t}x'_{j,t}|),$$

$$\text{(iv)}\ |(R/P)\sum_t (\hat{\beta}_{i,t} - \hat{\beta}_{i,R})'x_{i,t}x'_{j,t}(\hat{\beta}_{j,t} - \hat{\beta}_{j,R})|$$
$$\leq k^2(\sup_t R^{1/2}|\hat{\beta}_{i,t} - \hat{\beta}_{i,R}|)(\sup_t R^{1/2}|\hat{\beta}_{j,t} - \hat{\beta}_{j,R}|)(P^{-1}\sum_t|x_{i,t}x'_{j,t}|).$$

$$\text{(v)}\ |(R/P)\sum_t (\hat{\beta}_{i,R} - \beta^*_i)'h_{i,t+\tau-r}h'_{j,t+\tau-s}(\hat{\beta}_{j,t-s} - \hat{\beta}_{j,R})|$$
$$\leq k^2(R^{1/2}|\hat{\beta}_{i,R} - \beta^*_i|)(\sup_t R^{1/2}|\hat{\beta}_{j,t-s} - \hat{\beta}_{j,R}|)(P^{-1}\sum_t|h_{i,t+\tau-r}h'_{j,t+\tau-s}|),$$

$$\text{(vi)}\ |(R/P)\sum_t (\hat{\beta}_{i,t-r} - \hat{\beta}_{i,R})'h_{i,t+\tau-r}h'_{j,t+\tau-s}(\hat{\beta}_{j,t-s} - \hat{\beta}_{j,R})|$$
$$\leq k^2(\sup_t R^{1/2}|\hat{\beta}_{i,t-r} - \hat{\beta}_{i,R}|)(\sup_t R^{1/2}|\hat{\beta}_{j,t-s} - \hat{\beta}_{j,R}|)(P^{-1}\sum_t|h_{i,t+\tau-r}h'_{j,t+\tau-s}|),$$

$$\text{(vii)}\ |(R/P)\sum_t (\hat{\beta}_{i,R} - \beta^*_i)'h_{i,t+\tau-r}\text{vec}(x_{m,t-s}x'_{j,t-s})'((\hat{\beta}_{j,R} - \beta^*_j)\otimes(\hat{\beta}_{m,R} - \beta^*_m))|$$
$$\leq R^{-1/2}k^3(R^{1/2}|\hat{\beta}_{i,R} - \beta^*_i|)(R^{1/2}|\hat{\beta}_{j,R} - \beta^*_j|)\ (R^{1/2}|\hat{\beta}_{m,R} - \beta^*_m|)(P^{-1}\sum_t|h_{i,t+\tau-r}\text{vec}(x_{m,t-s}x'_{j,t-s})'|),$$

(viii) $|(R/P)\sum_t (\hat{\beta}_{i,R} - \beta_i^*)' h_{i,t+\tau-r} \mathrm{vec}(x_{m,t-s} x'_{j,t-s})' ((\hat{\beta}_{j,t-s} - \hat{\beta}_{j,R}) \otimes (\hat{\beta}_{m,R} - \beta_m^*))|$

$\leq R^{-1/2} k^3 (R^{1/2}|\hat{\beta}_{i,R} - \beta_i^*|)(\sup_t R^{1/2}|\hat{\beta}_{j,t-s} - \hat{\beta}_{j,R}|)\ (R^{1/2}|\hat{\beta}_{m,R} - \beta_m^*|)(P^{-1}\sum_t |h_{i,t+\tau-r} \mathrm{vec}(x_{m,t-s} x'_{j,t-s})'|)$,

(ix) $|(R/P)\sum_t (\hat{\beta}_{i,R} - \beta_i^*)' h_{i,t+\tau-r} \mathrm{vec}(x_{m,t-s} x'_{j,t-s})' ((\hat{\beta}_{j,t-s} - \hat{\beta}_{j,R}) \otimes (\hat{\beta}_{m,t-s} - \hat{\beta}_{m,R}))|$

$\leq R^{-1/2} k^3 (R^{1/2}|\hat{\beta}_{i,R} - \beta_i^*|)(\sup_t R^{1/2}|\hat{\beta}_{j,t-s} - \hat{\beta}_{j,R}|) \times$

$(\sup_t R^{1/2}|\hat{\beta}_{m,t-s} - \hat{\beta}_{m,R}|)(P^{-1}\sum_t |h_{i,t+\tau-r} \mathrm{vec}(x_{m,t-s} x'_{j,t-s})'|)$,

(x) $|(R/P)\sum_t (\hat{\beta}_{i,t-r} - \hat{\beta}_{i,R})' h_{i,t+\tau-r} \mathrm{vec}(x_{m,t-s} x'_{j,t-s})' ((\hat{\beta}_{j,R} - \beta_j^*) \otimes (\hat{\beta}_{m,R} - \beta_m^*))|$

$\leq R^{-1/2} k^3 (\sup_t R^{1/2}|\hat{\beta}_{i,t-r} - \hat{\beta}_{i,R}|)(R^{1/2}|\hat{\beta}_{j,R} - \beta_j^*|)\ (R^{1/2}|\hat{\beta}_{m,R} - \beta_m^*|)(P^{-1}\sum_t |h_{i,t+\tau-r} \mathrm{vec}(x_{m,t-s} x'_{j,t-s})'|)$,

(xi) $|(R/P)\sum_t (\hat{\beta}_{i,t-r} - \hat{\beta}_{i,R})' h_{i,t+\tau-r} \mathrm{vec}(x_{m,t-s} x'_{j,t-s})' ((\hat{\beta}_{j,t-s} - \hat{\beta}_{j,R}) \otimes (\hat{\beta}_{m,R} - \beta_m^*))|$

$\leq R^{-1/2} k^3 (\sup_t R^{1/2}|\hat{\beta}_{i,t-r} - \hat{\beta}_{i,R}|)(\sup_t R^{1/2}|\hat{\beta}_{j,t-s} - \hat{\beta}_{j,R}|) \times$

$(R^{1/2}|\hat{\beta}_{m,R} - \beta_m^*|)(P^{-1}\sum_t |h_{i,t+\tau-r} \mathrm{vec}(x_{m,t-s} x'_{j,t-s})'|)$,

(xii) $|(R/P)\sum_t (\hat{\beta}_{i,t-r} - \hat{\beta}_{i,R})' h_{i,t+\tau-r} \mathrm{vec}(x_{m,t-s} x'_{j,t-s})' ((\hat{\beta}_{j,t-s} - \hat{\beta}_{j,R}) \otimes (\hat{\beta}_{m,t-s} - \hat{\beta}_{m,R}))|$

$\leq R^{-1/2} k^3 (\sup_t R^{1/2}|\hat{\beta}_{i,t-r} - \hat{\beta}_{i,R}|)(\sup_t R^{1/2}|\hat{\beta}_{j,t-s} - \hat{\beta}_{j,R}|) \times$

$(\sup_t R^{1/2}|\hat{\beta}_{m,t-s} - \hat{\beta}_{m,R}|)(P^{-1}\sum_t |h_{i,t+\tau-r} \mathrm{vec}(x_{m,t-s} x'_{j,t-s})'|)$,

(xiii) $|(R/P)\sum_t ((\hat{\beta}_{i,R} - \beta_i^*)' \otimes (\hat{\beta}_{j,R} - \beta_j^*)') \mathrm{vec}(x_{i,t-r} x'_{j,t-r})\ \mathrm{vec}(x_{n,t-s} x'_{m,t-s})' ((\hat{\beta}_{m,R} - \beta_m^*) \otimes (\hat{\beta}_{n,R} - \beta_n^*))|$

$\leq R^{-1} k^4 (R^{1/2}|\hat{\beta}_{i,R} - \beta_i^*|)(R^{1/2}|\hat{\beta}_{j,R} - \beta_j^*|)(R^{1/2}|\hat{\beta}_{m,R} - \beta_m^*|) \times$

$(R^{1/2}|\hat{\beta}_{n,R} - \beta_n^*|)(P^{-1}\sum_t |\mathrm{vec}(x_{i,t-r} x'_{j,t-r}) \mathrm{vec}(x_{n,t-s} x'_{m,t-s})'|)$,

(xiv) $|(R/P)\sum_t ((\hat{\beta}_{i,R} - \beta_i^*)' \otimes (\hat{\beta}_{j,R} - \beta_j^*)') \mathrm{vec}(x_{i,t-r} x'_{j,t-r}) \mathrm{vec}(x_{n,t-s} x'_{m,t-s})' ((\hat{\beta}_{m,t-s} - \hat{\beta}_{m,R}) \otimes (\hat{\beta}_{n,R} - \beta_n^*))|$

$\leq R^{-1} k^4 (R^{1/2}|\hat{\beta}_{i,R} - \beta_i^*|)(R^{1/2}|\hat{\beta}_{j,R} - \beta_j^*|)(\sup_t R^{1/2}|\hat{\beta}_{m,t-s} - \hat{\beta}_{m,R}|) \times$

$$(R^{1/2}|\hat{\beta}_{n,R}-\beta^*_n|)(P^{-1}\textstyle\sum_t|vec(x_{i,t-r}x'_{j,t-r})vec(x_{n,t-s}x'_{m,t-s})'|),$$

(xv) $|(R/P)\sum_t((\hat{\beta}_{i,R}-\beta^*_i)'\otimes(\hat{\beta}_{j,R}-\beta^*_j)')vec(x_{i,t-r}x'_{j,t-r})vec(x_{n,t-s}x'_{m,t-s})'((\hat{\beta}_{m,t-s}-\hat{\beta}_{m,R})\otimes(\hat{\beta}_{n,t-s}-\hat{\beta}_{n,R}))|$

$$\leq R^{-1}k^4(R^{1/2}|\hat{\beta}_{i,R}-\beta^*_i|)(R^{1/2}|\hat{\beta}_{j,R}-\beta^*_j|)(\sup_t R^{1/2}|\hat{\beta}_{m,t-s}-\hat{\beta}_{m,R}|)\times$$

$$(\sup_t R^{1/2}|\hat{\beta}_{n,t-s}-\hat{\beta}_{n,R}|)(P^{-1}\textstyle\sum_t|vec(x_{i,t-r}x'_{j,t-r})vec(x_{n,t-s}x'_{m,t-s})'|),$$

(xvi) $(R/P)\sum_t((\hat{\beta}_{i,t-r}-\hat{\beta}_{i,R})'\otimes(\hat{\beta}_{j,R}-\beta^*_j)')vec(x_{i,t-r}x'_{j,t-r})\ vec(x_{n,t-s}x'_{m,t-s})'((\hat{\beta}_{m,t-s}-\hat{\beta}_{m,R})\otimes(\hat{\beta}_{n,R}-\beta^*_n)$

$$\leq R^{-1}k^4(\sup_t R^{1/2}|\hat{\beta}_{i,t-r}-\hat{\beta}_{i,R}|)(R^{1/2}|\hat{\beta}_{j,R}-\beta^*_j|)(\sup_t R^{1/2}|\hat{\beta}_{m,t-s}-\hat{\beta}_{m,R}|)\times$$

$$(R^{1/2}|\hat{\beta}_{n,R}-\beta^*_n|)(P^{-1}\textstyle\sum_t|vec(x_{i,t-r}x'_{j,t-r})vec(x_{n,t-s}x'_{m,t-s})'|),$$

(xvii) $(R/P)\sum_t((\hat{\beta}_{i,t-r}-\hat{\beta}_{i,R})'\otimes(\hat{\beta}_{j,R}-\beta^*_j)')vec(x_{i,t-r}x'_{j,t-r})\times$

$$vec(x_{n,t-s}x'_{m,t-s})'((\hat{\beta}_{m,t-s}-\hat{\beta}_{m,R})\otimes(\hat{\beta}_{n,t-s}-\hat{\beta}_{n,R}))$$

$$\leq R^{-1}k^4(\sup_t R^{1/2}|\hat{\beta}_{i,t-r}-\hat{\beta}_{i,R}|)(R^{1/2}|\hat{\beta}_{j,R}-\beta^*_j|)(\sup_t R^{1/2}|\hat{\beta}_{m,t-s}-\hat{\beta}_{m,R}|)\times$$

$$(\sup_t R^{1/2}|\hat{\beta}_{n,t-s}-\hat{\beta}_{n,R}|)(P^{-1}\textstyle\sum_t|vec(x_{i,t-r}x'_{j,t-r})vec(x_{n,t-s}x'_{m,t-s})'|),$$

(xviii) $(R/P)\sum_t((\hat{\beta}_{i,t-r}-\hat{\beta}_{i,R})'\otimes(\hat{\beta}_{j,t-r}-\hat{\beta}_{j,R})')vec(x_{i,t-r}x'_{j,t-r})\times$

$$vec(x_{n,t-s}x'_{m,t-s})'((\hat{\beta}_{m,t-s}-\hat{\beta}_{m,R})\otimes(\hat{\beta}_{n,t-s}-\hat{\beta}_{n,R}))$$

$$\leq R^{-1}k^4(\sup_t R^{1/2}|\hat{\beta}_{i,t-r}-\hat{\beta}_{i,R}|)(\sup_t R^{1/2}|\hat{\beta}_{j,t-r}-\hat{\beta}_{j,R}|)(\sup_t R^{1/2}|\hat{\beta}_{m,t-s}-\hat{\beta}_{m,R}|)\times$$

$$(\sup_t R^{1/2}|\hat{\beta}_{n,t-s}-\hat{\beta}_{n,R}|)(P^{-1}\textstyle\sum_t|vec(x_{i,t-r}x'_{j,t-r})vec(x_{n,t-s}x'_{m,t-s})'|).$$

Given Assumption 2, Markov's inequality implies that $P^{-1}\sum_t|vec(x_{i,t-r}x'_{j,t-r})vec(x_{n,t-s}x'_{m,t-s})'|$ and $P^{-1}\sum_t|h_{i,t+\tau-r}vec(x_{m,t-s}x'_{j,t-s})'|$ $P^{-1}\sum_t|h_{i,t+\tau-r}h'_{j,t+\tau-s}|$ are $O_p(1)$. That $R^{1/2}|\hat{\beta}_{i,R}-\beta^*_i|$ = $O_p(1)$ follows from Lemma A1 (c). Since $\sup_t R^{1/2}|\hat{\beta}_{i,t-r}-\hat{\beta}_{i,R}|$ = $o_p(1)$ by Lemma A8 and both $R^{-1/2}$ and $R^{-1}$ are o(1) the proof is complete.

**Lemma A10:** (a) Let Assumptions 1, 2 and 4 hold. $\sum_t (\hat{u}^2_{1,t+\tau} - \hat{u}_{1,t+\tau}\hat{u}_{2,t+\tau}) = \sigma^2 \sum_t \tilde{H}_2'(t)\tilde{h}_{2,t+\tau}$ + $o_p(1)$. (b) Let Assumptions 1, 2 and 4′ hold. $\sum_t (\hat{u}^2_{1,t+\tau} - \hat{u}_{1,t+\tau}\hat{u}_{2,t+\tau}) =$ $(P/R)^{1/2}\sigma^2[R^{1/2}\tilde{H}_2'(R)][P^{-1/2}\sum_t \tilde{h}_{2,t+\tau}] + o_p((P/R)^{1/2})$.

**Proof of Lemma A10:** (a) If we note that $\hat{u}_{1,t+\tau} = u_{t+\tau} - x_{1,t}'(\hat{\beta}_{1,t} - \beta_1^*)$ and $\hat{u}_{2,t+\tau} = u_{t+\tau} - x_{2,t}'(\hat{\beta}_{2,t} - \beta_2^*)$ we obtain

$$
\text{(A7)}\quad \sum_t (\hat{u}^2_{1,t+\tau} - \hat{u}_{1,t+\tau}\hat{u}_{2,t+\tau}) = \sum_t \{-h_{1,t+\tau}'B_1(t)H_1(t) + h_{2,t+\tau}'B_2(t)H_2(t)\}
$$
$$
- \sum_t \{-H_1'(t)B_1(t)q_{1,t}B_1(t)H_1(t) + H_1'(t)B_1(t)x_{1,t}x_{2,t}'B_2(t)H_2(t)\}.
$$

Consider the first bracketed right-hand side term in (A7). If we note that $h_{1,t+\tau} = Jh_{2,t+\tau}$ and apply the definition of $\tilde{h}_{2,t+\tau}$, by Lemmas A2 and A4 we obtain

$$
\sum_t \{-h_{1,t+\tau}'B_1(t)H_1(t) + h_{2,t+\tau}'B_2(t)H_2(t)\} = \sigma^2 \sum_t \tilde{H}_2'(t)\tilde{h}_{2,t+\tau} + o_p(1).
$$

We now need only show that the second bracketed right-hand side term in (A7) is $o_p(1)$. Since $Ex_{1,t}x_{2,t}' = JEq_{2,t} = JB_2^{-1}$ the result follows by Lemma A3 (b).

(b) If we note that $\hat{u}_{1,t+\tau} = u_{t+\tau} - x_{1,t}'(\hat{\beta}_{1,R} - \beta_1^*) - x_{1,t}'(\hat{\beta}_{1,t} - \hat{\beta}_{1,R})$ and $\hat{u}_{2,t+\tau} = u_{t+\tau} - x_{2,t}'(\hat{\beta}_{2,R} - \beta_2^*) -$ $x_{2,t}'(\hat{\beta}_{2,t} - \hat{\beta}_{2,R})$, we obtain

$$
\sum_t (\hat{u}^2_{1,t+\tau} - \hat{u}_{1,t+\tau}\hat{u}_{2,t+\tau}) = \sum_t \{r_{0,t} + r_{1,t}\} + \sum_t \{\textstyle\sum_{j=2}^{10} r_{j,t}\}
$$
$$
\equiv \sum_t \{-h_{1,t+\tau}'B_1(R)H_1(R) + h_{2,t+\tau}'B_2(R)H_2(R)\}
$$
$$
+ \sum_t \{-h_{1,t+\tau}'(\hat{\beta}_{1,t} - \hat{\beta}_{1,R}) + h_{2,t+\tau}'(\hat{\beta}_{2,t} - \hat{\beta}_{2,R}) + (\hat{\beta}_{1,R} - \beta_1^*)'q_{1,t}(\hat{\beta}_{1,R} - \beta_1^*)
$$

$$+ (\hat\beta_{1,R}-\beta_1^*)'x_{1,t}x_{2,t}'(\hat\beta_{2,R}-\beta_2^*) + 2(\hat\beta_{1,R}-\beta_1^*)'q_{1,t}(\hat\beta_{1,t}-\hat\beta_{1,R}) - (\hat\beta_{1,R}-\beta_1^*)'x_{1,t}x_{2,t}'(\hat\beta_{2,t}-\hat\beta_{2,R})$$

$$- (\hat\beta_{2,R}-\beta_2^*)'x_{2,t}x_{1,t}'(\hat\beta_{1,t}-\hat\beta_{1,R}) + (\hat\beta_{1,t}-\hat\beta_{1,R})'q_{1,t}(\hat\beta_{1,t}-\hat\beta_{1,R}) - (\hat\beta_{1,t}-\hat\beta_{1,R})'x_{1,t}x_{2,t}'(\hat\beta_{2,t}-\hat\beta_{2,R})\}$$

$$= (P/R)^{1/2}[P^{-1/2}\sum_t h_{2,t+\tau}'][-JB_1(R)J'+B_2(R)][R^{1/2}H_2(R)] + \sum_t\{\sum_{j=2}^{10} r_{j,T}\}.$$

Lemma A1 (b) implies that for i = 1,2, $B_i(R) \rightarrow_{a.s.} B_i$. Given Assumption 2, Corollary 29.19 of Davidson (1994) implies that both $R^{1/2}H_2(R)$ and $P^{-1/2}\sum_t h_{2,t+\tau}$ are $O_p(1)$. That $\sum_t\{r_{0,t}+r_{1,t}\}$ = $(P/R)^{1/2}\sigma^2[R^{1/2}\tilde H_2'(R)][P^{-1/2}\sum_t \tilde h_{2,t+\tau}]$ + $o_p((P/R)^{1/2})$ follows from Lemma A4 and the definition of $\tilde h_{2,t+\tau}$. The result follows since by Lemma A9 (a) - (b), $\sum_t\{\sum_{j=2}^{10} r_{j,t}\}$ = $o_p((P/R)^{1/2})$.

**Lemma A11**: (a) Let Assumptions 1, 2 and 4 hold. For $0 \le j \le \bar j$ $\sum_{t=R+j}^{T-\tau}(\hat u_{1,t+\tau}^2-\hat u_{1,t+\tau}\hat u_{2,t+\tau}-\bar c)(\hat u_{1,t+\tau-j}^2-\hat u_{1,t+\tau-j}\hat u_{2,t+\tau-j}-\bar c) = \sigma^4\sum_t \tilde H_2'(t)[E\tilde h_{2,t+\tau}\tilde h_{2,t+\tau-j}']\tilde H_2(t)$ + $o_p(1)$. (b) Let Assumptions 1, 2 and 4′ hold. For $0 \le j \le \bar j$ $\sum_{t=R+j}^{T-\tau}(\hat u_{1,t+\tau}^2-\hat u_{1,t+\tau}\hat u_{2,t+\tau}-\bar c)(\hat u_{1,t+\tau-j}^2-\hat u_{1,t+\tau-j}\hat u_{2,t+\tau-j}-\bar c)$ = $(P/R)\sigma^4[R^{1/2}\tilde H_2'(R)][E\tilde h_{2,t+\tau}\tilde h_{2,t+\tau-j}'][R^{1/2}\tilde H_2(R)]$ + $o_p(P/R)$.

**Proof of Lemma A11**: (a) Given Lemma A10 (a), Lemma A6 implies that $\bar c$ is $O_p(P^{-1})$ and hence $(P-j-\tau)\bar c^2$ = $o_p(1)$. Since j is finite this also implies that both $\bar c\sum_{t=R+j}^{T-\tau}(\hat u_{1,t+\tau-j}^2-\hat u_{1,t+\tau-j}\hat u_{2,t+\tau-j})$ and $\bar c\sum_{t=R+j}^{T-\tau}(\hat u_{1,t+\tau}^2-\hat u_{1,t+\tau}\hat u_{2,t+\tau})$ are $o_p(1)$. It then suffices to show that

$$\sum_{t=R+j}^{T-\tau}(\hat u_{1,t+\tau}^2-\hat u_{1,t+\tau}\hat u_{2,t+\tau})(\hat u_{1,t+\tau-j}^2-\hat u_{1,t+\tau-j}\hat u_{2,t+\tau-j}) = \sigma^4\sum_t \tilde H_2'(t)[E\tilde h_{2,t+\tau}\tilde h_{2,t+\tau-j}']\tilde H_2(t) + o_p(1).$$

If we note that $\hat u_{1,t+\tau} = u_{t+\tau} - x_{1,t}'(\hat\beta_{1,t}-\beta_1^*)$ and $\hat u_{2,t+\tau} = u_{t+\tau} - x_{2,t}'(\hat\beta_{2,t}-\beta_2^*)$, we have

$$\text{(A8)}\quad \sum_{t=R+j}^{T-\tau}(\hat u_{1,t+\tau}^2-\hat u_{1,t+\tau}\hat u_{2,t+\tau})(\hat u_{1,t+\tau-j}^2-\hat u_{1,t+\tau-j}\hat u_{2,t+\tau-j})$$

$$= [\sum_{t=R+j}^{T-\tau}H_1'(t)B_1(t)h_{1,t+\tau}h_{1,t+\tau-j}'B_1(t-j)H_1(t-j) - \sum_{t=R+j}^{T-\tau}H_1'(t)B_1(t)h_{1,t+\tau}h_{2,t+\tau-j}'B_2(t-j)H_2(t-j)$$

$$- \sum_{t=R+j}^{T-\tau}H_2'(t)B_2(t)h_{2,t+\tau}h_{1,t+\tau-j}'B_1(t-j)H_1(t-j) + \sum_{t=R+j}^{T-\tau}H_2'(t)B_2(t)h_{2,t+\tau}h_{2,t+\tau-j}'B_2(t-j)H_2(t-j)]$$

$$
\begin{aligned}
&+ [-\textstyle\sum_{t=R+j}^{T-\tau}(\hat{\beta}_{1,t}-\beta_1^*)'h_{1,t+\tau}\mathrm{vec}(x_{1,t-j}x_{1,t-j}^{'})'((\hat{\beta}_{1,t-j}-\beta_1^*)\otimes(\hat{\beta}_{1,t-j}-\beta_1^*)) \\
&\quad + \textstyle\sum_{t=R+j}^{T-\tau}(\hat{\beta}_{1,t}-\beta_1^*)'h_{1,t+\tau}\mathrm{vec}(x_{1,t-j}x_{2,t-j}^{'})'((\hat{\beta}_{1,t-j}-\beta_1^*)\otimes(\hat{\beta}_{2,t-j}-\beta_2^*)) \\
&\quad + \textstyle\sum_{t=R+j}^{T-\tau}(\hat{\beta}_{2,t}-\beta_2^*)'h_{2,t+\tau}\mathrm{vec}(x_{1,t-j}x_{1,t-j}^{'})'((\hat{\beta}_{1,t-j}-\beta_1^*)\otimes(\hat{\beta}_{1,t-j}-\beta_1^*)) \\
&\quad - \textstyle\sum_{t=R+j}^{T-\tau}(\hat{\beta}_{2,t}-\beta_2^*)'h_{2,t+\tau}\mathrm{vec}(x_{1,t-j}x_{2,t-j}^{'})'((\hat{\beta}_{1,t-j}-\beta_1^*)\otimes(\hat{\beta}_{2,t-j}-\beta_2^*)) \\
&\quad - \textstyle\sum_{t=R+j}^{T-\tau}(\hat{\beta}_{1,t-j}-\beta_1^*)'h_{1,t+\tau-j}\mathrm{vec}(x_{1,t}x_{1,t}^{'})'((\hat{\beta}_{1,t}-\beta_1^*)\otimes(\hat{\beta}_{1,t}-\beta_1^*)) \\
&\quad + \textstyle\sum_{t=R+j}^{T-\tau}(\hat{\beta}_{1,t-j}-\beta_1^*)'h_{1,t+\tau-j}\mathrm{vec}(x_{1,t}x_{2,t}^{'})'((\hat{\beta}_{1,t}-\beta_1^*)\otimes(\hat{\beta}_{2,t}-\beta_2^*)) \\
&\quad + \textstyle\sum_{t=R+j}^{T-\tau}(\hat{\beta}_{2,t-j}-\beta_2^*)'h_{2,t+\tau-j}\mathrm{vec}(x_{1,t}x_{1,t}^{'})'((\hat{\beta}_{1,t}-\beta_1^*)\otimes(\hat{\beta}_{1,t}-\beta_1^*)) \\
&\quad - \textstyle\sum_{t=R+j}^{T-\tau}(\hat{\beta}_{2,t-j}-\beta_2^*)'h_{2,t+\tau-j}\mathrm{vec}(x_{1,t}x_{2,t}^{'})'((\hat{\beta}_{1,t}-\beta_1^*)\otimes(\hat{\beta}_{2,t}-\beta_2^*)) \\
&\quad + \textstyle\sum_{t=R+j}^{T-\tau}((\hat{\beta}_{1,t}-\beta_1^*)'\otimes(\hat{\beta}_{1,t}-\beta_1^*)')\mathrm{vec}(x_{1,t}x_{1,t}^{'})\mathrm{vec}(x_{1,t-j}x_{1,t-j}^{'})'((\hat{\beta}_{1,t-j}-\beta_1^*)\otimes(\hat{\beta}_{1,t-j}-\beta_1^*)) \\
&\quad - \textstyle\sum_{t=R+j}^{T-\tau}((\hat{\beta}_{1,t}-\beta_1^*)'\otimes(\hat{\beta}_{1,t}-\beta_1^*)')\mathrm{vec}(x_{1,t}x_{1,t}^{'})\mathrm{vec}(x_{1,t-j}x_{2,t-j}^{'})'((\hat{\beta}_{1,t-j}-\beta_1^*)\otimes(\hat{\beta}_{2,t-j}-\beta_2^*)) \\
&\quad - \textstyle\sum_{t=R+j}^{T-\tau}((\hat{\beta}_{1,t}-\beta_1^*)'\otimes(\hat{\beta}_{2,t}-\beta_2^*)')\mathrm{vec}(x_{1,t}x_{2,t}^{'})\mathrm{vec}(x_{1,t-j}x_{1,t-j}^{'})'((\hat{\beta}_{1,t-j}-\beta_1^*)\otimes(\hat{\beta}_{1,t-j}-\beta_1^*)) \\
&\quad + \textstyle\sum_{t=R+j}^{T-\tau}((\hat{\beta}_{1,t}-\beta_1^*)'\otimes(\hat{\beta}_{2,t}-\beta_2^*)')\mathrm{vec}(x_{1,t}x_{2,t}^{'})\mathrm{vec}(x_{1,t-j}x_{2,t-j}^{'})'((\hat{\beta}_{1,t-j}-\beta_1^*)\otimes(\hat{\beta}_{2,t-j}-\beta_2^*))\,].
\end{aligned}
$$

Consider the first bracketed right-hand side term in (A8). If we apply the definition of $\tilde{h}_{2,t+\tau}$ then by Lemmas A3 (a) and A4 we obtain

$$
\begin{aligned}
&[\textstyle\sum_{t=R+j}^{T-\tau}H_1^{'}(t)B_1(t)h_{1,t+\tau}h_{1,t+\tau-j}^{'}B_1(t\text{-}j)H_1(t\text{-}j) - \sum_{t=R+j}^{T-\tau}H_1^{'}(t)B_1(t)h_{1,t+\tau}h_{2,t+\tau-j}^{'}B_2(t\text{-}j)H_2(t\text{-}j) \\
&\quad - \textstyle\sum_{t=R+j}^{T-\tau}H_2^{'}(t)B_2(t)h_{2,t+\tau}h_{1,t+\tau-j}^{'}B_1(t\text{-}j)H_1(t\text{-}j) + \sum_{t=R+j}^{T-\tau}H_2^{'}(t)B_2(t)h_{2,t+\tau}h_{2,t+\tau-j}^{'}B_2(t\text{-}j)H_2(t\text{-}j)\,] \\
&= \sigma^4\textstyle\sum_t\tilde{H}_2^{'}(t)[E\tilde{h}_{2,t+\tau}\tilde{h}_{2,t+\tau-j}^{'}]\tilde{H}_2(t) + o_p(1).
\end{aligned}
$$

We now need only show that the second bracketed term on the right-hand side of (A8) is $o_p(1)$. To do so we show that $\sum_{t=R+j}^{T-\tau}(\hat{\beta}_{1,t}-\beta_1^*)'h_{1,t+\tau}\mathrm{vec}(x_{1,t-j}x_{1,t-j}^{'})'((\hat{\beta}_{1,t-j}-\beta_1^*)\otimes(\hat{\beta}_{1,t-j}-\beta_1^*))$ is $o_p(1)$. The remaining terms follow similar arguments. Taking the absolute value we obtain

$$|\sum_{t=R+j}^{T-\tau}(\hat{\beta}_{1,t}-\beta_1^*)'h_{1,t+\tau}vec(x_{1,t-j}x_{1,t-j}^{'})'((\hat{\beta}_{1,t-j}-\beta_1^*)\otimes(\hat{\beta}_{1,t-j}-\beta_1^*))|$$

$$\leq k^3(P/T)(\sup_t T^{1/3}|\hat{\beta}_{1,t}-\beta_1^*|)(P^{-1}\sum_t|h_{1,t+\tau}vec(x_{1,t-j}x_{1,t-j}^{'})'|)(\sup_t T^{1/3}|\hat{\beta}_{1,t-j}-\beta_1^*|)^2.$$

Assumption 4 implies that P/T is bounded. Given Assumption 2, that $P^{-1}\sum_t|h_{1,t+\tau}vec(x_{1,t-j}x_{1,t-j}^{'})'|$ is $O_p(1)$ follows from Markov's inequality. Since j is finite, that $(\sup_t T^{1/3}|\hat{\beta}_{1,t}-\beta_1^*|)$ and $(\sup_t T^{1/3}|\hat{\beta}_{1,t-j}-\beta_1^*|)^2$ are $o_p(1)$ follows from Lemma A1 (c) and the proof is complete.

(b) Given Lemma A10 (b), Lemma A6 implies that $\overline{c}$ is $O_p(P^{-1})$ and hence $(P\text{-}j)\overline{c}^2 = o_p(1)$. Since j is finite this also implies that both $\overline{c}\sum_{t=R+j}^{T-\tau}(\hat{u}_{1,t+\tau-j}^2\text{-}\hat{u}_{1,t+\tau-j}\hat{u}_{2,t+\tau-j})$ and $\overline{c}\sum_{t=R+j}^{T-\tau}(\hat{u}_{1,t+\tau}^2\text{-}\hat{u}_{1,t+\tau}\hat{u}_{2,t+\tau})$ are $o_p(1)$. It then suffices to show that $\sum_{t=R+j}^{T-\tau}(\hat{u}_{1,t+\tau}^2\text{-}\hat{u}_{1,t+\tau}\hat{u}_{2,t+\tau})(\hat{u}_{1,t+\tau-j}^2\text{-}\hat{u}_{1,t+\tau-j}\hat{u}_{2,t+\tau-j}) =$ $(P/R)\sigma^4[R^{1/2}\tilde{H}_2^{'}(R)][E\tilde{h}_{2,t+\tau}\tilde{h}_{2,t+\tau-j}^{'}][R^{1/2}\tilde{H}_2(R)] + o_p(P/R)$.

First note that, in the notation of Lemma A10 (b), $\sum_{t=R+j}^{T-\tau}(\hat{u}_{1,t+\tau}^2\text{-}\hat{u}_{1,t+\tau}\hat{u}_{2,t+\tau})(\hat{u}_{1,t+\tau-j}^2\text{-}\hat{u}_{1,t+\tau-j}\hat{u}_{2,t+\tau-j}) =$ $\sum_{t=R+j}^{T-\tau}\{r_{0,t}+r_{1,t}+\sum_{i=2}^{10}r_{i,t}\}\{r_{0,t-j}+r_{1,t-j}+\sum_{i=2}^{10}r_{i,t-j}\}$. If we take the product within the argument we obtain 121 distinct crossproduct terms. Rather than list each explicitly, we will simply note that the important terms are $z_{0,t}=r_{0,t}r_{0,t-j}$, $z_{1,t}=r_{0,t}r_{1,t-j}$, $z_{2,t}=r_{1,t}r_{0,t-j}$ and $z_{3,t}=r_{1,t}r_{1,t-j}$. The remaining terms are each of the form in Lemma A9 (b) for some choice of indices $i,j,m,n \in \{1, 2\}$ and $0 \leq r,s \leq \overline{j}$. Using this notation we obtain

$$\sum_{t=R+j}^{T-\tau}(\hat{u}_{1,t+\tau}^2\text{-}\hat{u}_{1,t+\tau}\hat{u}_{2,t+\tau})(\hat{u}_{1,t+\tau-j}^2\text{-}\hat{u}_{1,t+\tau-j}\hat{u}_{2,t+\tau-j}) = \sum_{t=R+j}^{T-\tau}\{z_{0,t}+z_{1,t}+z_{2,t}+z_{3,t}\} + \sum_{t=R+j}^{T-\tau}\{\sum_{i=4}^{121}z_{i,t}\}$$

$$\equiv \sum_{t=R+j}^{T-\tau}\{(\hat{\beta}_{1,R}\text{-}\beta_1^*)'h_{1,t+\tau}h_{1,t+\tau-j}^{'}(\hat{\beta}_{1,R}\text{-}\beta_1^*) - (\hat{\beta}_{1,R}\text{-}\beta_1^*)'h_{1,t+\tau}h_{2,t+\tau-j}^{'}(\hat{\beta}_{2,R}\text{-}\beta_2^*)$$

$$- (\hat{\beta}_{2,R}\text{-}\beta_2^*)'h_{2,t+\tau}h_{1,t+\tau-j}^{'}(\hat{\beta}_{1,R}\text{-}\beta_1^*)\} + (\hat{\beta}_{2,R}\text{-}\beta_2^*)'h_{2,t+\tau}h_{2,t+\tau-j}^{'}(\hat{\beta}_{2,R}\text{-}\beta_2^*)\} + \sum_{t=R+j}^{T-\tau}\{\sum_{i=4}^{121}z_{i,t}\}$$

$$= (P/R)[R^{1/2}H_2^{'}(R)][JB_1(R)(P^{-1}\sum_{t=R+j}^{T-\tau}h_{1,t+\tau}h_{1,t+\tau-j}^{'})B_1(R)J'\text{-}JB_1(R)(P^{-1}\sum_{t=R+j}^{T-\tau}h_{1,t+\tau}h_{2,t+\tau-j}^{'})B_2(R)$$

$+\ B_2(R)(P^{-1}\sum_{t=R+j}^{T-\tau} h_{2,t+\tau}h'_{1,t+\tau-j})B_1(R)J' + B_2(R)(P^{-1}\sum_{t=R+j}^{T-\tau} h_{2,t+\tau}h'_{2,t+\tau-j})B_2(R)][R^{1/2}H_2(R)]$

$+\ \sum_{t=R+j}^{T-\tau}\{\sum_{i=4}^{121} z_{i,t}\}$.

Lemma A1 (b) implies that both $B_1(R) \rightarrow_{a.s.} B_1$ and $B_2(R) \rightarrow_{a.s.} B_2$. Given Assumption 2, Corollary 29.19 of Davidson (1994) implies that $R^{1/2}H_2'(R)$ is $O_p(1)$. Assumption 2 also suffices for $P^{-1}\sum_{t=R+j}^{T-\tau} h_{1,t+\tau}h'_{1,t+\tau-j} \rightarrow_{a.s.} Eh_{1,t+\tau}h'_{1,t+\tau-j}$, $P^{-1}\sum_{t=R+j}^{T-\tau} h_{1,t+\tau}h'_{2,t+\tau-j} \rightarrow_{a.s.} Eh_{1,t+\tau}h'_{2,t+\tau-j}$, $P^{-1}\sum_{t=R+j}^{T-\tau} h_{2,t+\tau}h'_{1,t+\tau-j} \rightarrow_{a.s.} Eh_{2,t+\tau}h'_{1,t+\tau-j}$ and $P^{-1}\sum_{t=R+j}^{T-\tau} h_{2,t+\tau}h'_{2,t+\tau-j} \rightarrow_{a.s.} Eh_{2,t+\tau}h'_{2,t+\tau-j}$. That $\sum_{t=R+j}^{T-\tau}\{z_{0,t}+z_{1,t}+z_{2,t}+z_{3,t}\} = (P/R)\sigma^4[R^{1/2}\tilde{H}_2'(R)][E\tilde{h}_{2,t+\tau}\tilde{h}'_{2,t+\tau-j}][R^{1/2}\tilde{H}_2(R)] + o_p(P/R)$ then follows from Lemma A4 and the definition of $\tilde{h}_{2,t+\tau}$. The result follows since by Lemma A9 (b), $\sum_{t=R+j}^{T-\tau}\{\sum_{i=4}^{121} z_{i,t}\} = o_p(P/R)$.

**Lemma A12:** (a) Let Assumptions 1, 2 and 4 hold. $\sum_t(\hat{u}_{1,t+\tau}^2-\hat{u}_{2,t+\tau}^2) = 2\sigma^2\sum_t \tilde{H}_2'(t)\tilde{h}_{2,t+\tau}$ $-$ $\sigma^2\sum_t \tilde{H}_2'(t)\tilde{H}_2'(t)$ $+ o_p(1)$. (b) Let Assumptions 1, 2 and 4′ hold. $\sum_t(\hat{u}_{1,t+\tau}^2-\hat{u}_{2,t+\tau}^2) =$ $2(P/R)^{1/2}\sigma^2[R^{1/2}\tilde{H}_2'(R)][P^{-1/2}\sum_t \tilde{h}_{2,t+\tau}] + o_p((P/R)^{1/2})$.

**Proof of Lemma A12:** (a) If we note that $\hat{u}_{1,t+\tau} = u_{t+\tau} - x'_{1,t}(\hat{\beta}_{1,t}-\beta_1^*)$ and $\hat{u}_{2,t+\tau} = u_{t+\tau} - x'_{2,t}(\hat{\beta}_{2,t}-\beta_2^*)$ we obtain

$$\text{(A9)}\quad \sum_t(\hat{u}_{1,t+\tau}^2-\hat{u}_{2,t+\tau}^2) = 2\sum_t\{-h'_{1,t+\tau}B_1(t)H_1(t)+h'_{2,t+\tau}B_2(t)H_2(t)\}$$
$$-\ \sum_t\{-H_1'(t)B_1(t)q_{1,t}B_1(t)H_1(t) + H_1'(t)B_1(t)q_{2,t}B_2(t)H_2(t)\}.$$

If we note that $h_{1,t+\tau} = Jh_{2,t+\tau}$, $Eq_{i,t} = B_i^{-1}$ and apply the definition of $\tilde{h}_{2,t+\tau}$, the result follows from Lemmas A2, A3 and A4.

(b) If we note that $\hat{u}_{1,t+\tau} = u_{t+\tau} - x'_{1,t}(\hat{\beta}_{1,R}-\beta_1^*) - x'_{1,t}(\hat{\beta}_{1,t}-\hat{\beta}_{1,R})$ and $\hat{u}_{2,t+\tau} = u_{t+\tau} - x'_{2,t}(\hat{\beta}_{2,R}-\beta_2^*) - x'_{2,t}(\hat{\beta}_{2,t}-\hat{\beta}_{2,R})$, we obtain

$$
\begin{aligned}
\textstyle\sum_t(\hat{u}^2_{1,t+\tau}-\hat{u}^2_{2,t+\tau}) &= 2\textstyle\sum_t\{r_{0,t}+r_{1,t}\} + \textstyle\sum_t\{2r_{2,t}+2r_{3,t}+r_{4,t}+r_{6,t}+r_{9,t}+\textstyle\sum_{j=11}^{13}r_{j,t}\} \\
&\equiv 2\textstyle\sum_t\{-h'_{1,t+\tau}B_1(R)H_1(R)+h'_{2,t+\tau}B_2(R)H_2(R)\} \\
&\quad + \textstyle\sum_t\{-2h'_{1,t+\tau}(\hat{\beta}_{1,t}-\hat{\beta}_{1,R}) + 2h'_{2,t+\tau}(\hat{\beta}_{2,t}-\hat{\beta}_{2,R}) + (\hat{\beta}_{1,R}-\beta_1^*)'q_{1,t}(\hat{\beta}_{1,R}-\beta_1^*) \\
&\quad + 2(\hat{\beta}_{1,R}-\beta_1^*)'q_{1,t}(\hat{\beta}_{1,t}-\hat{\beta}_{1,R}) + (\hat{\beta}_{1,t}-\hat{\beta}_{1,R})'q_{1,t}(\hat{\beta}_{1,t}-\hat{\beta}_{1,R}) - (\hat{\beta}_{2,R}-\beta_2^*)'q_{2,t}(\hat{\beta}_{2,R}-\beta_2^*) \\
&\quad - 2(\hat{\beta}_{2,R}-\beta_2^*)'q_{2,t}(\hat{\beta}_{2,t}-\hat{\beta}_{2,R}) - (\hat{\beta}_{2,t}-\hat{\beta}_{2,R})'q_{2,t}(\hat{\beta}_{2,t}-\hat{\beta}_{2,R})\} \\
&= 2(P/R)^{1/2}[P^{-1/2}\textstyle\sum_t h'_{2,t+\tau}][-JB_1(R)J'+B_2(R)][R^{1/2}H_2(R)] \\
&\quad + \textstyle\sum_t\{2r_{2,T}+2r_{3,T}+r_{4,T}+r_{6,T}+r_{9,T}+\textstyle\sum_{j=11}^{13}r_{j,T}\}\,.
\end{aligned}
$$

Lemma A1 (b) implies that for i = 1,2, $B_i(R) \to_{a.s.} B_i$. Assumption 2 and Corollary 29.19 of Davidson (1994) imply that both $R^{1/2}H_2(R)$ and $P^{-1/2}\sum_t h_{2,t+\tau}$ are $O_p(1)$. That $2\sum_t\{r_{0,T}+r_{1,T}\} = 2(P/R)^{1/2}\sigma^2[R^{1/2}\tilde{H}'_2(R)][P^{-1/2}\sum_t \tilde{h}_{2,t+\tau}] + o_p((P/R)^{1/2})$ follows by Lemma A4 and the definition of $\tilde{h}_{2,t+\tau}$. The result follows since by Lemma A9 (a)-(b), $\sum_t\{2r_{2,T}+2r_{3,T}+r_{4,T}+r_{6,T}+r_{9,T}+\sum_{j=11}^{13}r_{j,T}\} = o_p((P/R)^{1/2})$.

**Lemma A13**: (a) Let Assumptions 1, 2 and 4 hold. $\sum_{t=R+j}^{T-\tau}(\hat{u}^2_{1,t+\tau}-\hat{u}^2_{2,t+\tau}-\bar{d})(\hat{u}^2_{1,t+\tau-j}-\hat{u}^2_{2,t+\tau-j}-\bar{d}) = 4\sigma^4\sum_t\tilde{H}'_2(t)[E\tilde{h}_{2,t+\tau}\tilde{h}'_{2,t+\tau-j}]\tilde{H}_2(t) + o_p(1)$. (b) Let Assumptions 1, 2 and 4′ hold. $\sum_{t=R+j}^{T-\tau}(\hat{u}^2_{1,t+\tau}-\hat{u}^2_{2,t+\tau}-\bar{d})(\hat{u}^2_{1,t+\tau-j}-\hat{u}^2_{2,t+\tau-j}-\bar{d}) = 4(P/R)\sigma^4[R^{1/2}\tilde{H}'_2(R)][E\tilde{h}_{2,t+\tau}\tilde{h}'_{2,t+\tau-j}][R^{1/2}\tilde{H}_2(R)] + o_p(P/R)$.

**Proof of Lemma A13**: (a) Given Lemma A12 (a), Lemmas A6 and A7 imply that $\bar{d}$ is $O_p(P^{-1})$ and hence $(P-\tau-j)\bar{d}^2 = o_p(1)$. Since j is finite this also implies that both $\bar{d}\sum_{t=R+j}^{T-\tau}(\hat{u}^2_{1,t+\tau-j}-\hat{u}^2_{2,t+\tau-j})$ and

$\bar{d}\sum_{t=R+j}^{T-\tau}(\hat{u}_{1,t+\tau}^2-\hat{u}_{2,t+\tau}^2)$ are $o_p(1)$. It then suffices to show that $\sum_{t=R+j}^{T-\tau}(\hat{u}_{1,t+\tau}^2-\hat{u}_{2,t+\tau}^2)(\hat{u}_{1,t+\tau-j}^2-\hat{u}_{2,t+\tau-j}^2) =$

$4\sigma^4\sum_t \tilde{H}_2'(t)[E\tilde{h}_{2,t+\tau}\tilde{h}_{2,t+\tau-j}']\tilde{H}_2(t) + o_p(1)$.

If we note that $\hat{u}_{1,t+\tau} = u_{t+\tau} - x_{1,t}'(\hat{\beta}_{1,t}-\beta_1^*)$ and $\hat{u}_{2,t+\tau} = u_{t+\tau} - x_{2,t}'(\hat{\beta}_{2,t}-\beta_2^*)$, we have

$$
\begin{aligned}
\text{(A10)}\quad & \sum_{t=R+j}^{T-\tau}(\hat{u}_{1,t+\tau}^2-\hat{u}_{2,t+\tau}^2)(\hat{u}_{1,t+\tau-j}^2-\hat{u}_{2,t+\tau-j}^2) \\
&= 4[\sum_{t=R+j}^{T-\tau} H_1'(t)B_1(t)h_{1,t+\tau}h_{1,t+\tau-j}'B_1(t\text{-}j)H_1(t\text{-}j) - \sum_{t=R+j}^{T-\tau} H_1'(t)B_1(t)h_{1,t+\tau}h_{2,t+\tau-j}'B_2(t\text{-}j)H_2(t\text{-}j) \\
&\quad - \sum_{t=R+j}^{T-\tau} H_2'(t)B_2(t)h_{2,t+\tau}h_{1,t+\tau-j}'B_1(t\text{-}j)H_1(t\text{-}j) + \sum_{t=R+j}^{T-\tau} H_2'(t)B_2(t)h_{2,t+\tau}h_{2,t+\tau-j}'B_2(t\text{-}j)H_2(t\text{-}j)\,] \\
&+ [-2\sum_{t=R+j}^{T-\tau}(\hat{\beta}_{1,t}-\beta_1^*)'h_{1,t+\tau}\text{vec}(x_{1,t-j}x_{1,t-j}')'((\hat{\beta}_{1,t-j}-\beta_1^*)\otimes(\hat{\beta}_{1,t-j}-\beta_1^*)) \\
&\quad + 2\sum_{t=R+j}^{T-\tau}(\hat{\beta}_{1,t}-\beta_1^*)'h_{1,t+\tau}\text{vec}(x_{2,t-j}x_{2,t-j}')'((\hat{\beta}_{2,t-j}-\beta_1^*)\otimes(\hat{\beta}_{2,t-j}-\beta_2^*)) \\
&\quad + 2\sum_{t=R+j}^{T-\tau}(\hat{\beta}_{2,t}-\beta_2^*)'h_{2,t+\tau}\text{vec}(x_{1,t-j}x_{1,t-j}')'((\hat{\beta}_{1,t-j}-\beta_1^*)\otimes(\hat{\beta}_{1,t-j}-\beta_1^*)) \\
&\quad - 2\sum_{t=R+j}^{T-\tau}(\hat{\beta}_{2,t}-\beta_2^*)'h_{2,t+\tau}\text{vec}(x_{2,t-j}x_{2,t-j}')'((\hat{\beta}_{2,t-j}-\beta_1^*)\otimes(\hat{\beta}_{2,t-j}-\beta_2^*)) \\
&\quad - 2\sum_{t=R+j}^{T-\tau}(\hat{\beta}_{1,t-j}-\beta_1^*)'h_{1,t+\tau-j}\text{vec}(x_{1,t}x_{1,t}')'((\hat{\beta}_{1,t}-\beta_1^*)\otimes(\hat{\beta}_{1,t}-\beta_1^*)) \\
&\quad + 2\sum_{t=R+j}^{T-\tau}(\hat{\beta}_{1,t-j}-\beta_1^*)'h_{1,t+\tau-j}\text{vec}(x_{2,t}x_{2,t}')'((\hat{\beta}_{2,t}-\beta_2^*)\otimes(\hat{\beta}_{2,t}-\beta_2^*)) \\
&\quad + 2\sum_{t=R+j}^{T-\tau}(\hat{\beta}_{2,t-j}-\beta_2^*)'h_{2,t+\tau-j}\text{vec}(x_{1,t}x_{1,t}')'((\hat{\beta}_{1,t}-\beta_1^*)\otimes(\hat{\beta}_{1,t}-\beta_1^*)) \\
&\quad - 2\sum_{t=R+j}^{T-\tau}(\hat{\beta}_{2,t-j}-\beta_2^*)'h_{2,t+\tau-j}\text{vec}(x_{2,t}x_{2,t}')'((\hat{\beta}_{2,t}-\beta_2^*)\otimes(\hat{\beta}_{2,t}-\beta_2^*)) \\
&\quad + \sum_{t=R+j}^{T-\tau}((\hat{\beta}_{1,t}-\beta_1^*)'\otimes(\hat{\beta}_{1,t}-\beta_1^*)')\text{vec}(x_{1,t}x_{1,t}')\text{vec}(x_{1,t-j}x_{1,t-j}')'((\hat{\beta}_{1,t-j}-\beta_1^*)\otimes(\hat{\beta}_{1,t-j}-\beta_1^*)) \\
&\quad - \sum_{t=R+j}^{T-\tau}((\hat{\beta}_{1,t}-\beta_1^*)'\otimes(\hat{\beta}_{1,t}-\beta_1^*)')\text{vec}(x_{1,t}x_{1,t}')\text{vec}(x_{2,t-j}x_{2,t-j}')'((\hat{\beta}_{2,t-j}-\beta_2^*)\otimes(\hat{\beta}_{2,t-j}-\beta_2^*)) \\
&\quad - \sum_{t=R+j}^{T-\tau}((\hat{\beta}_{2,t}-\beta_2^*)'\otimes(\hat{\beta}_{2,t}-\beta_2^*)')\text{vec}(x_{2,t}x_{2,t}')\text{vec}(x_{1,t-j}x_{1,t-j}')'((\hat{\beta}_{1,t-j}-\beta_1^*)\otimes(\hat{\beta}_{1,t-j}-\beta_1^*)) \\
&\quad + \sum_{t=R+j}^{T-\tau}((\hat{\beta}_{2,t}-\beta_2^*)'\otimes(\hat{\beta}_{2,t}-\beta_2^*)')\text{vec}(x_{2,t}x_{2,t}')\text{vec}(x_{2,t-j}x_{2,t-j}')'((\hat{\beta}_{2,t-j}-\beta_2^*)\otimes(\hat{\beta}_{2,t-j}-\beta_2^*))\,].
\end{aligned}
$$

Consider the first bracketed right-hand side term in (A10). If we apply the definition of $\tilde{h}_{2,t+\tau}$ then by Lemmas A3 (a) and A4 we obtain

$$4[\sum_{t=R+j}^{T-\tau} H_1'(t)B_1(t)h_{1,t+\tau}h_{1,t+\tau-j}'B_1(t\text{-}j)H_1(t\text{-}j) - \sum_{t=R+j}^{T-\tau} H_1'(t)B_1(t)h_{1,t+\tau}h_{2,t+\tau-j}'B_2(t\text{-}j)H_2(t\text{-}j)$$

$$- \sum_{t=R+j}^{T-\tau} H_2'(t)B_2(t)h_{2,t+\tau}h_{1,t+\tau-j}'B_1(t\text{-}j)H_1(t\text{-}j) + \sum_{t=R+j}^{T-\tau} H_2'(t)B_2(t)h_{2,t+\tau}h_{2,t+\tau-j}'B_2(t\text{-}j)H_2(t\text{-}j) ]$$

$$= 4\sigma^4 \sum_t \tilde{H}_2'(t)[E\tilde{h}_{2,t+\tau}\tilde{h}_{2,t+\tau-j}']\tilde{H}_2(t) + o_p(1).$$

We now need only show that the second bracketed term on the right-hand side of (A10) is $o_p(1)$. To do so we show that $\sum_{t=R+j}^{T-\tau}(\hat{\beta}_{1,t}\text{-}\beta_1^*)'h_{1,t+\tau}vec(x_{1,t\text{-}j}x_{1,t\text{-}j}')'((\hat{\beta}_{1,t\text{-}j}\text{-}\beta_1^*)\otimes(\hat{\beta}_{1,t\text{-}j}\text{-}\beta_1^*))$ is $o_p(1)$. The remaining terms follow similar arguments. Taking the absolute value we obtain

$$| \sum_{t=R+j}^{T-\tau}(\hat{\beta}_{1,t}\text{-}\beta_1^*)'h_{1,t+\tau}vec(x_{1,t\text{-}j}x_{1,t\text{-}j}')'((\hat{\beta}_{1,t\text{-}j}\text{-}\beta_1^*)\otimes(\hat{\beta}_{1,t\text{-}j}\text{-}\beta_1^*)) |$$
$$\leq k^3(P/T)(\sup_t T^{1/3}|\hat{\beta}_{1,t}\text{-}\beta_1^*|)(P^{-1}\sum_t|h_{1,t+\tau}vec(x_{1,t\text{-}j}x_{1,t\text{-}j}')'|)(\sup_t T^{1/3}|\hat{\beta}_{1,t\text{-}j}\text{-}\beta_1^*|)^2 .$$

Assumption 4 implies that P/T is bounded. Given Assumption 2, that $P^{-1}\sum_t|h_{1,t+\tau}vec(x_{1,t\text{-}j}x_{1,t\text{-}j}')'|$ is $O_p(1)$ follows from Markov's inequality. Since j is finite, that $(\sup_t T^{1/3}|\hat{\beta}_{1,t}\text{-}\beta_1^*|)$ and $(\sup_t T^{1/3}|\hat{\beta}_{1,t\text{-}j}\text{-}\beta_1^*|)^2$ are $o_p(1)$ follows from Lemma A1 (c) and the proof is complete.

(b) Given Lemma A12 (b), Lemmas A6 and A7 imply that $\overline{d}$ is $O_p(P^{-1})$ and hence $(P\text{-}\tau\text{-}j)\overline{d}^2$ = $o_p(1)$. Since j is finite this also implies that both $\overline{d}\sum_{t=R+j}^{T-\tau}(\hat{u}_{1,t+\tau\text{-}j}^2\text{-}\hat{u}_{2,t+\tau\text{-}j}^2)$ and $\overline{d}\sum_{t=R+j}^{T-\tau}(\hat{u}_{1,t+\tau}^2\text{-}\hat{u}_{2,t+\tau}^2)$ are $o_p(1)$. It then suffices to show that $\sum_{t=R+j}^{T-\tau}(\hat{u}_{1,t+\tau}^2\text{-}\hat{u}_{2,t+\tau}^2)(\hat{u}_{1,t+\tau\text{-}j}^2\text{-}\hat{u}_{2,t+\tau\text{-}j}^2)$ =

$4(P/R)\sigma^4[R^{1/2}\tilde{H}_2'(R)][E\tilde{h}_{2,t+\tau}\tilde{h}_{2,t+\tau\text{-}j}'][R^{1/2}\tilde{H}_2(R)] + o_p(1)$.

First note that, in the notation of Lemma A12 (b), $\sum_{t=R+j}^{T-\tau}(\hat{u}_{1,t+\tau}^2\text{-}\hat{u}_{2,t+\tau}^2)(\hat{u}_{1,t+\tau\text{-}j}^2\text{-}\hat{u}_{2,t+\tau\text{-}j}^2)$ =

$\sum_{t=R+j}^{T-\tau}\{2r_{0,t}+2r_{1,t}+2r_{2,t}+2r_{3,t}+r_{4,t}+r_{6,t}+r_{9,t}+\sum_{j=11}^{13}r_{j,t}\}\{2r_{0,t\text{-}j}+2r_{1,t\text{-}j}+2r_{2,t\text{-}j}+2r_{3,t\text{-}j}+r_{4,t\text{-}j}+r_{6,t\text{-}j}+r_{9,t\text{-}j}+\sum_{j=11}^{13}r_{j,t\text{-}j}\}$.

If we take the product within the argument we obtain 100 distinct crossproduct terms. Rather than list each explicitly, we will simply note that the important terms are $w_{0,t} \equiv 4z_{0,t} = 4\,r_{0,t}r_{0,t-j}$, $w_{1,t} \equiv 4z_{1,t} = 4\,r_{0,t}r_{1,t-j}$, $w_{2,t} \equiv 4z_{2,t} = 4\,r_{1,t}r_{0,t-j}$ and $w_{3,t} \equiv 4z_{3,t} = 4\,r_{1,t}r_{1,t-j}$. The remaining terms are each of the form in Lemma A9 (b) for some choice of index i,j,m,n $\in$ {1, 2} and $0 \le r,s \le \bar{j}$.

Using this notation we obtain

$$\begin{aligned}
&\textstyle\sum_{t=R+j}^{T-\tau}(\hat{u}_{1,t+\tau}^2-\hat{u}_{2,t+\tau}^2)(\hat{u}_{1,t+\tau-j}^2-\hat{u}_{2,t+\tau-j}^2) = \sum_{t=R+j}^{T-\tau}\{w_{0,t}+w_{1,t}+w_{2,t}+w_{3,t}\} + \sum_{t=R+j}^{T-\tau}\{\sum_{i=4}^{99} w_{i,t}\} \\
&\equiv 4\textstyle\sum_{t=R+j}^{T-\tau}\{(\hat{\beta}_{1,R}-\beta_1^*)'h_{1,t+\tau}h_{1,t+\tau-j}'(\hat{\beta}_{1,R}-\beta_1^*) - (\hat{\beta}_{1,R}-\beta_1^*)'h_{1,t+\tau}h_{2,t+\tau-j}'(\hat{\beta}_{2,R}-\beta_2^*) \\
&\quad - (\hat{\beta}_{2,R}-\beta_2^*)'h_{2,t+\tau}h_{1,t+\tau-j}'(\hat{\beta}_{1,R}-\beta_1^*)\} + (\hat{\beta}_{2,R}-\beta_2^*)'h_{2,t+\tau}h_{2,t+\tau-j}'(\hat{\beta}_{2,R}-\beta_2^*)\} + \textstyle\sum_{t=R+j}^{T-\tau}\{\sum_{i=4}^{99} w_{i,t}\} \\
&= 4(P/R)[R^{1/2}H_2'(R)][JB_1(R)(P^{-1}\textstyle\sum_{t=R+j}^{T-\tau}h_{1,t+\tau}h_{1,t+\tau-j}')B_1(R)J' - JB_1(R)(P^{-1}\sum_{t=R+j}^{T-\tau}h_{1,t+\tau}h_{2,t+\tau-j}')B_2(R) \\
&\quad + B_2(R)(P^{-1}\textstyle\sum_{t=R+j}^{T-\tau}h_{2,t+\tau}h_{1,t+\tau-j}')B_1(R)J' + B_2(R)(P^{-1}\sum_{t=R+j}^{T-\tau}h_{2,t+\tau}h_{2,t+\tau-j}')B_2(R)][R^{1/2}H_2(R)] \\
&\quad + \textstyle\sum_{t=R+j}^{T-\tau}\{\sum_{i=4}^{99} w_{i,t}\}.
\end{aligned}$$

Since $w_{i,t} = 4z_{i,t}$ $i = 0,\ldots,3$, the proof of Lemma A11 (b) implies that $\sum_{t=R+j}^{T-\tau}\{w_{0,t}+w_{1,t}+w_{2,t}+w_{3,t}\}$ $= 4(P/R)\sigma^4[R^{1/2}\tilde{H}_2'(R)][E\tilde{h}_{2,t+\tau}\tilde{h}_{2,t+\tau-j}'][R^{1/2}\tilde{H}_2(R)] + o_p(P/R)$. The result follows since by Lemma A9 (b), $\sum_{t=R+j}^{T-\tau}\{\sum_{i=4}^{99} w_{i,t}\} = o_p(P/R)$.

6. Theorems

**Theorem 3.1:** (a) Let Assumptions 1-4 hold. For the recursive, fixed and rolling schemes, let $\Gamma_1$ equal $\int_\lambda^1 \omega^{-1}W'(\omega)S_{\tilde{h}\tilde{h}}dW(\omega)$, $\lambda^{-1}\{W(1)\text{-}W(\lambda)\}'S_{\tilde{h}\tilde{h}}W(\lambda)$ and $\lambda^{-1}\int_\lambda^1\{W(\omega)\text{-}W(\omega\text{-}\lambda)\}'S_{\tilde{h}\tilde{h}}dW(\omega)$ respectively. Similarly, for the recursive, fixed and rolling schemes let $\Gamma_3$ denote $\int_\lambda^1 \omega^{-2}W'(\omega)S_{\tilde{h}\tilde{h}}^2W(\omega)d\omega$, $\pi\lambda^{-1}W'(\lambda)S_{\tilde{h}\tilde{h}}^2W(\lambda)$ and $\lambda^{-2}\int_\lambda^1\{W(\omega)\text{-}W(\omega\text{-}\lambda)\}'S_{\tilde{h}\tilde{h}}^2\{W(\omega)\text{-}W(\omega\text{-}\lambda)\}d\omega$ respectively. ENC-T $\to_d$ $\Gamma_1/\Gamma_3^{1/2}$. (b) Let Assumptions 1-3 and 4′ hold and let $V_0$ and $V_1$ denote $(k_2\times 1)$ independent standard normal vectors. ENC-T $\to_d$ $V_0'S_{\tilde{h}\tilde{h}}V_1/[V_0'S_{\tilde{h}\tilde{h}}^2V_0]^{1/2}$ ~ N(0,1).

**Proof of Theorem 3.1**: We will prove this for the recursive scheme. The fixed follows from the recursive while the rolling follows from a proof similar to that for the recursive (a) Given Theorem 3.4 and the Continuous Mapping Theorem it suffices to show that $P\sum_{j=-\bar{j}}^{\bar{j}}K(j/M)\hat{\Gamma}_{cc}(j)$ $\to_d$ $\sigma^4\Gamma_3$. That

$$\begin{aligned}P\sum_{j=-\bar{j}}^{\bar{j}}K(j/M)\hat{\Gamma}_{cc}(j) &= \sigma^4\sum_{j=-\bar{j}}^{\bar{j}}K(j/M)[\sum_t\tilde{H}_2'(t)[E\tilde{h}_{2,t+\tau}\tilde{h}_{2,t+\tau-j}']\tilde{H}_2(t)] + o_p(1)\\ &= \sigma^4\sum_t\tilde{H}_2'(t)[\sum_{j=-\bar{j}}^{\bar{j}}K(j/M)(E\tilde{h}_{2,t+\tau}\tilde{h}_{2,t+\tau-j}')]\tilde{H}_2(t) + o_p(1)\\ &= \sigma^4(\sum_t[\tilde{H}_2'(t)\otimes\tilde{H}_2'(t)])\text{vec}[\sum_{j=-\bar{j}}^{\bar{j}}K(j/M)(E\tilde{h}_{2,t+\tau}\tilde{h}_{2,t+\tau-j}')] + o_p(1)\end{aligned}$$

follows from Lemma A11 (a) and the fact that $\bar{j}$ is finite. Given Assumption 3, we know that $\sum_{j=-\bar{j}}^{\bar{j}}K(j/M)(E\tilde{h}_{2,t+\tau}\tilde{h}_{2,t+\tau-j}') \to S_{\tilde{h}\tilde{h}}$. Note that by Lemma A6 and the Continuous Mapping Theorem $\sum_t\tilde{H}_2'(t)\otimes\tilde{H}_2'(t)$ $\to_d$ $\int_\lambda^1\omega^{-2}[W'(\omega)S_{\tilde{h}\tilde{h}}^{1/2}\otimes W'(\omega)S_{\tilde{h}\tilde{h}}^{1/2}]d\omega$. The result follows since $(\int_\lambda^1\omega^{-2}[W'(\omega)S_{\tilde{h}\tilde{h}}^{1/2}\otimes W'(\omega)S_{\tilde{h}\tilde{h}}^{1/2}]d\omega)\text{vec}[S_{\tilde{h}\tilde{h}}] = \Gamma_3$.

(b) Given Lemmas A10 (b) and A11 (b) we can write ENC-T as

$$\text{ENC-T} = \frac{(P/R)^{1/2}\sigma^2[R^{1/2}\tilde{H}_2'(R)][P^{-1/2}\sum_t \tilde{h}_{2,t+\tau}]+o_p((P/R)^{1/2})}{[(P/R)\sigma^4[R^{1/2}\tilde{H}_2'(R)][\sum_{j=-\bar{j}}^{\bar{j}} K(j/M)(E\tilde{h}_{t+\tau}\tilde{h}_{t+\tau-j}')][R^{1/2}\tilde{H}_2(R)]+o_p(P/R)]^{1/2}}$$

$$= \frac{[R^{1/2}\tilde{H}_2'(R)][P^{-1/2}\sum_t \tilde{h}_{2,t+\tau}]+o_p(1)}{[[R^{1/2}\tilde{H}_2'(R)][\sum_{j=-\bar{j}}^{\bar{j}} K(j/M)(E\tilde{h}_{t+\tau}\tilde{h}_{t+\tau-j}')][R^{1/2}\tilde{H}_2(R)]+o_p(1)]^{1/2}}$$

$$= \frac{[R^{1/2}\tilde{H}_2'(R)][P^{-1/2}\sum_t \tilde{h}_{2,t+\tau}]}{[[R^{1/2}\tilde{H}_2'(R)][\sum_{j=-\bar{j}}^{\bar{j}} K(j/M)(E\tilde{h}_{t+\tau}\tilde{h}_{t+\tau-j}')][R^{1/2}\tilde{H}_2(R)]]^{1/2}} + o_p(1).$$

Given Assumption 2, Corollary 29.19 of Davidson (1994) suffices for $(P^{-1/2}\sum_t \tilde{h}_{2,t+\tau}', R^{1/2}\tilde{H}_2'(R))'$ $\rightarrow_d (V'_1 S_{\tilde{h}\tilde{h}}^{1/2}, V'_0 S_{\tilde{h}\tilde{h}}^{1/2})'$ for independent $(k\times 1)$ standard normal vectors $V_0$ and $V_1$. Given Assumption 3, we know that $\sum_{j=-\bar{j}}^{\bar{j}} K(j/M)(E\tilde{h}_{2,t+\tau}\tilde{h}_{2,t+\tau-j}') \rightarrow S_{\tilde{h}\tilde{h}}$. The result follows immediately from the Continuous Mapping Theorem.

**Theorem 3.2:** (a) Let Assumptions 1-4 hold. For the recursive, fixed and rolling schemes let $\Gamma_2$ denote $\int_\lambda^1 \omega^{-2}W'(\omega)S_{\tilde{h}\tilde{h}}W(\omega)d\omega$, $\pi\lambda^{-1}W'(\lambda)S_{\tilde{h}\tilde{h}}W(\lambda)$ and $\lambda^{-2}\int_\lambda^1 \{W(\omega)-W(\omega-\lambda)\}'S_{\tilde{h}\tilde{h}}\{W(\omega)-W(\omega-\lambda)\}d\omega$ respectively. MSE-T $\rightarrow_d$ $(\Gamma_1-(0.5)\Gamma_2)/\Gamma_3^{1/2}$. (b) Let Assumptions 1-3 and 4′ hold. MSE-T – ENC-T = $o_p(1)$.

**Proof of Theorem 3.2**: We will prove this for the recursive scheme. The fixed follows from the recursive while the rolling follows from a proof similar to that for the recursive (a) Given Theorem 3.3 and the Continuous Mapping Theorem it suffices to show that $\sum_{j=-\bar{j}}^{\bar{j}} K(j/M)\hat{\Gamma}_{dd}(j)$ $\rightarrow_d 4\sigma^4\Gamma_3$. That

$$P\sum_{j=-\bar{j}}^{\bar{j}} K(j/M)\hat{\Gamma}_{dd}(j) = 4\sigma^4\sum_{j=-\bar{j}}^{\bar{j}} K(j/M)[\sum_t \tilde{H}_2'(t)[E\tilde{h}_{2,t+\tau}\tilde{h}_{2,t+\tau-j}']\tilde{H}_2(t)] + o_p(1)$$

$$= 4\sigma^4\sum_t \tilde{H}_2'(t)[\sum_{j=-\bar{j}}^{\bar{j}} K(j/M)(E\tilde{h}_{2,t+\tau}\tilde{h}_{2,t+\tau-j}')]\tilde{H}_2(t) + o_p(1)$$

$$= 4\sigma^4(\sum_t[\tilde{H}_2'(t)\otimes\tilde{H}_2'(t)])\mathrm{vec}[\sum_{j=-\bar{j}}^{\bar{j}}K(j/M)(E\tilde{h}_{2,t+\tau}\tilde{h}_{2,t+\tau-j}')] + o_p(1)$$

follows from Lemma A13 (a) and the fact that $\bar{j}$ is finite. Given Assumption 3, we know that $\sum_{j=-\bar{j}}^{\bar{j}}K(j/M)(E\tilde{h}_{2,t+\tau}\tilde{h}_{2,t+\tau-j}') \rightarrow S_{\tilde{h}\tilde{h}}$. Note that by Lemma A6 and the Continuous Mapping Theorem $\sum_t \tilde{H}_2'(t)\otimes\tilde{H}_2'(t) \rightarrow_d \int_\lambda^1 \omega^{-2}[W'(\omega)S_{\tilde{h}\tilde{h}}^{1/2}\otimes W'(\omega)S_{\tilde{h}\tilde{h}}^{1/2}]d\omega$. The result follows since $(\int_\lambda^1 \omega^{-2}[W'(\omega)S_{\tilde{h}\tilde{h}}^{1/2}\otimes W'(\omega)S_{\tilde{h}\tilde{h}}^{1/2}]d\omega)\mathrm{vec}[S_{\tilde{h}\tilde{h}}] = \Gamma_3$.

(b) Given Lemmas A12 (b) and A13 (b) we can write MSE-T as

$$\text{MSE-T} = \frac{2(P/R)^{1/2}\sigma^2[R^{1/2}\tilde{H}_2'(R)][P^{-1/2}\sum_t\tilde{h}_{2,t+1}]+o_p((P/R)^{1/2})}{[4(P/R)\sigma^4[R^{1/2}\tilde{H}_2'(R)][\sum_{j=-\bar{j}}^{\bar{j}}K(j/M)(E\tilde{h}_{t+\tau}\tilde{h}_{t+\tau-j}')][R^{1/2}\tilde{H}_2(R)]+o_p(P/R)]^{1/2}}$$

$$= \frac{[R^{1/2}\tilde{H}_2'(R)][P^{-1/2}\sum_t\tilde{h}_{2,t+1}]+o_p(1)}{[[R^{1/2}\tilde{H}_2'(R)][\sum_{j=-\bar{j}}^{\bar{j}}K(j/M)(E\tilde{h}_{t+\tau}\tilde{h}_{t+\tau-j}')][R^{1/2}\tilde{H}_2(R)]+o_p(1)]^{1/2}}$$

$$= \frac{[R^{1/2}\tilde{H}_2'(R)][P^{-1/2}\sum_t\tilde{h}_{2,t+1}]}{[[R^{1/2}\tilde{H}_2'(R)][\sum_{j=-\bar{j}}^{\bar{j}}K(j/M)(E\tilde{h}_{t+\tau}\tilde{h}_{t+\tau-j}')][R^{1/2}\tilde{H}_2(R)]]^{1/2}} + o_p(1).$$

The result follows immediately from the proof of Theorem 3.1 (b).

**Theorem 3.3:** (a) Let Assumptions 1, 2 and 4 hold. MSE-F $\rightarrow_d 2\Gamma_1-\Gamma_2$. (b) Let Assumptions 1, 2 and 4′ hold. For the $(k_2\times 1)$ independent standard normal vectors $V_0$ and $V_1$ defined in Theorem 3.1 (b), $(R/P)^{1/2}$MSE-F $\rightarrow_d 2V_0'S_{\tilde{h}\tilde{h}}V_1$.

**Proof of Theorem 3.3**: (a) That $P^{-1}\sum_t\hat{u}_{2,t+\tau}^2 \rightarrow_p \sigma^2$ follows from Theorem 4.1 of West (1996). Given Lemma A12 (a) the result then follows from Lemmas A6 and A7.

(b) That $P^{-1}\sum_t\hat{u}_{2,t+\tau}^2 \rightarrow_p \sigma^2$ follows from Theorem 4.1 of West (1996). Note that Lemma A12 (b) implies that $\sum_t(\hat{u}_{1,t+\tau}^2-\hat{u}_{2,t+\tau}^2) = 2\sigma^2(P/R)^{1/2}[R^{1/2}\tilde{H}_2'(R)][P^{-1/2}\sum_t\tilde{h}_{2,t+\tau}] + o_p((P/R)^{1/2})$. Given

Assumption 2, Corollary 29.19 of Davidson (1994) suffices for $(P^{-1/2}\sum_t \tilde{h}'_{2,t+\tau}, R^{1/2}\tilde{H}'_2(R))' \to_d$ $(V'_1 S^{1/2}_{\tilde{h}\tilde{h}}, V'_0 S^{1/2}_{\tilde{h}\tilde{h}})'$ for the independent $(k_2\times 1)$ standard normal vectors $V_0$ and $V_1$ from Theorem 3.1. Scaling by $(R/P)^{1/2}$ provides the desired result.

**Theorem 3.4:** (a) Let Assumptions 1, 2 and 4 hold. ENC-NEW $\to_d \Gamma_1$. (b) Let Assumptions 1, 2 and 4′ hold. $2(R/P)^{1/2}$ENC-NEW – $(R/P)^{1/2}$MSE-F = $o_p(1)$.

**Proof of Theorem 3.4**: (a) That $P^{-1}\sum_t \hat{u}^2_{2,t+\tau} \to_p \sigma^2$ follows from Theorem 4.1 of West (1996). Given Lemma A10 (a) the result then follows from Lemma A6.

(b) That $P^{-1}\sum_t \hat{u}^2_{2,t+\tau} \to_p \sigma^2$ follows from Theorem 4.1 of West (1996). Note that Lemma A10 (b) implies that $\sum_t(\hat{u}^2_{1,t+\tau}-\hat{u}_{1,t+\tau}\hat{u}_{2,t+\tau}) = \sigma^2(P/R)^{1/2}[R^{1/2}\tilde{H}'_2(R)][P^{-1/2}\sum_t \tilde{h}_{2,t+\tau}] + o_p((P/R)^{1/2})$. Other than the factor 2, this is identical to the expansion in the proof of Theorem 3.3 (b). Scaling by $(R/P)^{1/2}$ provides the desired result.

## 7. References


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## 8. Referenced Corollaries and Theorems

This section contains six of the primary technical references used throughout the appendix: Corollaries 29.11 and 29.19 of Davidson (1994), Theorems 3.1, 3.3 and 4.1 of Hansen (1992) and Theorem 4.1 of West (1996). For ease of reference, the notation used within this paper is adopted as much as possible. To insure understanding of the issues, we recommend reading the original sources.

**Corollary 29.11 of Davidson (1994)**: Let the conditions (a), (b) and (c) of 29.4 hold, and instead of condition 29.4(d) assume (d′) $E(X_n(t)) \to 0$ and $E(X_n(t)^2) \to \eta(t)$ as $n \to \infty$, each $t \in [0,1]$. Then $X_n \to_d B_\eta$.

**Corollary 29.19 of Davidson (1994)**: Let $\{\tilde{U}_{t+\tau}\}$ be a zero-mean, uniformly $L_r$-bounded m-vector sequence with each element $L_2$-NED of size -1/2 on an $\alpha$-mixing process of size -r/(r-2); and assume $T^{-1}E(\sum_{t=1}^{T-\tau}\tilde{U}_{t+\tau})(\sum_{t=1}^{T-\tau}\tilde{U}_{t+\tau})' \to \Omega$. If $X_T(s) = T^{-1/2}\sum_{t=1}^{[sT]}\tilde{U}_{t+\tau}$, then $X_T \Rightarrow W(s; \Omega)$.

**Theorem 3.1 of Hansen (1992)**: If Assumption 1 (from the Hansen article) holds and $(T^{-1/2}\sum_{t=1}^{[sT]}\tilde{U}_t, T^{-1/2}\sum_{t=1}^{[sT]}\tilde{V}_t) \Rightarrow (U, V)$ in $D_{M^{km}\times\Re^m}[0,1]$ then $\sum_{i=1}^{[rT]}(T^{-1/2}\sum_{t=1}^{i}\tilde{U}_t)(T^{-1/2}\tilde{V}_{i+1}) \to^d \int_0^r U^- dV$ with $V(s) = W(s; \Omega)$, and $\Omega$ being defined in Assumption 1 (from the Hansen article).

**Theorem 3.3 of Hansen (1992)**: Suppose $U_n \Rightarrow U$ in $D_{M^{km}}[0,1]$ and U(.) is almost surely continuous. For a random sequence $\{e_i\}$ and a sequence of nondecreasing sigma fields $\{\Im_i^e\}$ to which $\{e_i\}$ is adapted, assume that $\sup_i E|E(e_i|\Im_{i-m}^r)| \to 0$ as $m \to \infty$. Then $\sup_{0\le s\le 1}|n^{-1}\sum_{i=1}^{[ns]}U_{ni}e_i| \to_p 0$.

**Theorem 4.1 of Hansen (1992)**: If Assumption 1 (from the Hansen article) holds, then

$\sum_{i=1}^{[rT]}(T^{-1/2}\sum_{t=1}^{i}\tilde{U}_{t+\tau})(T^{-1/2}\tilde{U}_{i+\tau}^{'}) \to^{d} \int_0^r BdB^{'}+r\Lambda$ as $T \to \infty$, where $B \equiv W(s;\Omega)$, and $\Lambda$ being defined in Assumption 1 (from the Hansen article).

**Theorem 4.1 of West (1996)**: (a) If $\pi = 0$ or $F \equiv E\partial f_{t+1}(\beta)/\partial\beta = 0$, $P^{1/2}(P^{-1}\sum_{t=R}^{T} f_{t+1}(\hat{\beta}_t)\text{-}Ef_{t+1}(\beta^*))$ $\to^{d} N(0,\Omega)$ where $\Omega = S_{ff} \equiv \lim_{T\to\infty}E[P^{1/2}(P^{-1}\sum_{t=R}^{T} f_{t+1}(\beta^*)\text{-}Ef_{t+1}(\beta^*))]^2$. (b) If $S = \begin{pmatrix} S_{ff} & S_{fh}B^{'} \\ BS_{fh}^{'} & BS_{hh}B^{'} \end{pmatrix}$ $\equiv \begin{pmatrix} S_{ff} & S_{fh}B^{'} \\ BS_{fh}^{'} & V_{\beta} \end{pmatrix}$ is p.d., $P^{1/2}(P^{-1}\sum_{t=R}^{T} f_{t+1}(\hat{\beta}_t)\text{-}Ef_{t+1}(\beta^*)) \to^{d} N(0,\Omega)$ where

$\Omega = S_{ff} + (1 - \pi^{-1}\ln(1+\pi))(FBS_{fh}^{'} + S_{fh}B^{'}F^{'}) + 2(1-\pi^{-1}\ln(1+\pi))FV_{\beta}F^{'}$.

## 9. Summary Table of Limiting Distributions

This section contains 4 panels, one each for the MSE-T, ENC-T, MSE-F, and ENC-NEW statistics. Each panel contains the limiting distributions of the statistics for a given forecasting scheme (recursive, rolling or fixed) and value of $\lim_{P,R\to\infty} P/R = \pi$ ($0 < \pi < \infty$ or $\pi = 0$).

**Panel 1: MSE-T**

| Scheme\π | $0 < \pi < \infty$ |
|---|---|
| Recursive | $\dfrac{\int_\lambda^1 \omega^{-1}W'(\omega)S_{\tilde{h}\tilde{h}}dW(\omega)-(0.5)\int_\lambda^1 \omega^{-2}W'(\omega)S_{\tilde{h}\tilde{h}}W(\omega)d\omega}{[\int_\lambda^1 \omega^{-2}W'(\omega)S^2_{\tilde{h}\tilde{h}}W(\omega)d\omega]^{1/2}}$ |
| Rolling | $\dfrac{\lambda^{-1}\int_\lambda^1 \{W(\omega)-W(\omega-\lambda)\}'S_{\tilde{h}\tilde{h}}dW(\omega)-(0.5)\lambda^{-2}\int_\lambda^1 \{W(\omega)-W(\omega-\lambda)\}'S_{\tilde{h}\tilde{h}}\{W(\omega)-W(\omega-\lambda)\}d\omega}{[\lambda^{-2}\int_\lambda^1 \{W(\omega)-W(\omega-\lambda)\}'S^2_{\tilde{h}\tilde{h}}\{W(\omega)-W(\omega-\lambda)\}d\omega]^{1/2}}$ |
| Fixed | $\dfrac{\lambda^{-1}\{W(1)-W(\lambda)\}'S_{\tilde{h}\tilde{h}}W(\lambda)-(0.5)\pi\lambda^{-1}W'(\lambda)S_{\tilde{h}\tilde{h}}W(\lambda)}{[\pi\lambda^{-1}W'(\lambda)S^2_{\tilde{h}\tilde{h}}W(\lambda)]^{1/2}}$ |

Note: If $\pi = 0$ MSE-T − ENC-T = $o_p(1)$ so the MSE-T distribution can be inferred from Panel 2.

**Panel 2: ENC-T**

| Scheme\π | $\pi = 0$ | $0 < \pi < \infty$ |
|---|---|---|
| Recursive | N(0, 1) | $\dfrac{\int_\lambda^1 \omega^{-1}W'(\omega)S_{\tilde{h}\tilde{h}}dW(\omega)}{[\int_\lambda^1 \omega^{-2}W'(\omega)S^2_{\tilde{h}\tilde{h}}W(\omega)d\omega]^{1/2}}$ |
| Rolling | N(0, 1) | $\dfrac{\lambda^{-1}\int_\lambda^1 \{W(\omega)-W(\omega-\lambda)\}'S_{\tilde{h}\tilde{h}}dW(\omega)}{[\lambda^{-2}\int_\lambda^1 \{W(\omega)-W(\omega-\lambda)\}'S^2_{\tilde{h}\tilde{h}}\{W(\omega)-W(\omega-\lambda)\}d\omega]^{1/2}}$ |
| Fixed | N(0, 1) | $\dfrac{\lambda^{-1}\{W(1)-W(\lambda)\}'S_{\tilde{h}\tilde{h}}W(\lambda)}{[\pi\lambda^{-1}W'(\lambda)S^2_{\tilde{h}\tilde{h}}W(\lambda)]^{1/2}}$ |

**Panel 3: MSE-F**

| Scheme\π | $\pi = 0$ | $0 < \pi < \infty$ |
| --- | --- | --- |
| Recursive | $2V_0^{'}S_{\tilde{h}\tilde{h}}V_1$ | $2\int_\lambda^1 \omega^{-1}W^{'}(\omega)S_{\tilde{h}\tilde{h}}dW(\omega)-\int_\lambda^1 \omega^{-2}W^{'}(\omega)S_{\tilde{h}\tilde{h}}W(\omega)d\omega$ |
| Rolling | $2V_0^{'}S_{\tilde{h}\tilde{h}}V_1$ | $2\lambda^{-1}\int_\lambda^1 \{W(\omega)-W(\omega-\lambda)\}^{'}S_{\tilde{h}\tilde{h}}dW(\omega)$ $-\lambda^{-2}\int_\lambda^1 \{W(\omega)-W(\omega-\lambda)\}^{'}S_{\tilde{h}\tilde{h}}\{W(\omega)-W(\omega-\lambda)\}d\omega$ |
| Fixed | $2V_0^{'}S_{\tilde{h}\tilde{h}}V_1$ | $2\lambda^{-1}\{W(1)-W(\lambda)\}^{'}S_{\tilde{h}\tilde{h}}W(\lambda)-\pi\lambda^{-1}W^{'}(\lambda)S_{\tilde{h}\tilde{h}}W(\lambda)$ |

Note: When $\pi = 0$ the definition of the MSE-F changes. It is rescaled by $(R/P)^{1/2}$.

**Panel 4: ENC-NEW**

| Scheme\π | $\pi = 0$ | $0 < \pi < \infty$ |
| --- | --- | --- |
| Recursive | $V_0^{'}S_{\tilde{h}\tilde{h}}V_1$ | $\int_\lambda^1 \omega^{-1}W^{'}(\omega)S_{\tilde{h}\tilde{h}}dW(\omega)$ |
| Rolling | $V_0^{'}S_{\tilde{h}\tilde{h}}V_1$ | $\lambda^{-1}\int_\lambda^1 \{W(\omega)-W(\omega-\lambda)\}^{'}S_{\tilde{h}\tilde{h}}dW(\omega)$ |
| Fixed | $V_0^{'}S_{\tilde{h}\tilde{h}}V_1$ | $\lambda^{-1}\{W(1)-W(\lambda)\}^{'}S_{\tilde{h}\tilde{h}}W(\lambda)$ |

Note: When $\pi = 0$ the definition of the ENC-F changes. It is rescaled by $(R/P)^{1/2}$.