$$\underbrace{\mathbb{P}\{\Theta \mid X\}}_{\text{апост.}} = \frac{\overbrace{\mathbb{P}\{X=x \mid \Theta\}}^{\text{функц. пр.}}\;\overbrace{\mathbb{P}\{\Theta\}}^{\text{априор.}}}{\underbrace{\sum_{\Theta} \mathbb{P}\{X=x \mid \Theta\}\,\mathbb{P}\{\Theta\}}_{\mathbb{P}\{X=x\}}}$$

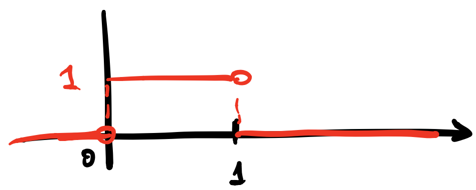
$$\underbrace{f(\Theta \mid x)}_{\text{апост.}} = \frac{\overbrace{f(x \mid \Theta)}^{\text{правд.}}\;\overbrace{f(\Theta)}^{\text{апр.}}}{\int f(x \mid \Theta)\, f(\Theta)\, d\Theta}$$

② $$\underbrace{p(\Theta \mid x)}_{\text{апост.}} \propto \underbrace{p(x \mid \Theta)}_{\text{правд.}} \cdot \underbrace{p(\Theta)}_{\text{априор.}}$$

Точечные оц-и: моду апост. распр., среднее

№1 $X_1 \dots X_n \sim Bern(p)$

$f(p) = 1$ (flat)



а) $$f(p \mid x) \propto f(x \mid p) \cdot \underset{1}{\underset{\shortparallel}{f(p)}} = \underbrace{p^{\sum_i x_i} (1-p)^{n-\sum_i x_i}}$$

б) $\sum_i x_i = s$

$$f(p \mid x) \propto \underbrace{p^s (1-p)^{n-s}} = p^{\overset{\alpha}{\boxed{(s+1)}}-1} (1-p)^{\overset{\beta}{\boxed{(n-s+1)}}-1}$$

$$\alpha = s+1$$
$$\beta = n-s+1$$

c) $$C = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\,\Gamma(\beta)} = \frac{\Gamma(n+2)}{\Gamma(s+1)\,\Gamma(n-s+1)}$$

Итого:
$$f(p|x) = \frac{\Gamma(n+2)}{\Gamma(s+1)\,\Gamma(n-s+1)}\, p^{(s+1)-1}(1-p)^{(n-s+1)-1}$$

$$p|x \sim Beta(s+1,\, n-s+1)$$

d) $Y \sim Beta(\alpha, \beta)$
$$\mathbb{E}(Y) = \frac{\alpha}{\alpha+\beta}$$
$$\boxed{\bar{p} = \frac{s+1}{n+2}}$$

$$s := \sum x_i$$
$$\hat{p}_{ML} = \frac{\sum x_i}{n}$$

e) $$\bar{p} = \underbrace{\frac{n}{n+2}}_{\lambda} \cdot \underbrace{\frac{s}{n}}_{\hat{p}_{ML}} + \underbrace{\left(1 - \frac{n}{n+2}\right)}_{1-\lambda} \underbrace{\frac{1}{2}}_{\text{мат. ожид. априор. распр.}}$$

f) $$\int_a^b f(p|x)\,dp = 0.95$$

численно $\Rightarrow a, b$

g) $\underline{f(p) = \mathrm{Beta}(\gamma, \xi)}$

$$f(p|x) \propto f(x|p)\cdot f(p) =$$

$$= p^{s}(1-p)^{n-s}\overbrace{\frac{\Gamma(\gamma+\xi)}{\Gamma(\gamma)\Gamma(\xi)}}^{c}p^{\gamma-1}(1-p)^{\xi-1} =$$

$$= c\, p^{\overbrace{\boxed{s+\gamma}}^{\alpha}-1}(1-p)^{\overbrace{\boxed{n-s+\xi}}^{\beta}-1}$$

$\alpha = s+\gamma$

$\beta = n-s+\xi$

$p|x \sim \mathrm{Beta}(s+\gamma; n-s+\xi)$

$$\underline{f(p|x) = \frac{\Gamma(\gamma+n+\xi)}{\Gamma(s+\gamma)\Gamma(n-s+\xi)}p^{s+\gamma-1}(1-p)^{n-s+\xi-1}}$$

Априор. и апостер. ∈ к одному сем-у распр., то априор. называв. ~~сопрят~~ сопряж. (conjugate prior)

№2

$X_1 \dots X_2 \sim N(\mu, \sigma^2)$

$\mu \sim N(a, b^2)$

$X_1 := x$

$$f(\mu \mid x) \propto f(x \mid \mu) \cdot f(\mu) =$$

$$= \underbrace{\frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}}_{\text{правдоп.}} \cdot \underbrace{\frac{1}{\sqrt{2\pi b^2}} e^{-\frac{1}{2}\left(\frac{\mu-a}{b}\right)^2}}_{\text{априор.}} =$$

$$= C e^{-\frac{1}{2}\left[\frac{b^2(x^2 - 2\mu x + \mu^2) + \sigma^2(\mu^2 - 2a\mu + a^2)}{\sigma^2 b^2}\right]} =$$

$$= C e^{-\frac{1}{2}\left[\frac{(b^2+\sigma^2)\mu^2 - 2\mu(b^2 x + \sigma^2 a) + b^2 x^2 + \sigma^2 a^2}{\sigma^2 b^2}\right]}$$

$$= C e^{-\frac{1}{2} \frac{\left[\mu^2 - 2\mu \left(\frac{b^2 x + \sigma^2 a}{b^2 + \sigma^2} \right) + \frac{b^2 x^2 + \sigma^2 a^2}{b^2 + \sigma^2} \right]}{\frac{\sigma^2 b^2}{b^2 + \sigma^2}}} =$$

$$= C e^{-\frac{1}{2} \frac{\left(\mu - \frac{b^2 x + \sigma^2 a}{b^2 + \sigma^2} \right)^2 + C_1}{C_2}}$$

$$\mu | X \sim N(\dots, \dots)$$

б)

$$\bar{\mu} = \lambda \underbrace{\bar{X}}_{\hat{\mu}_{ML}} + (1-\lambda) \underbrace{a}_{\text{мат. ож. априор.}}$$

$$S = Var(\bar{X}) = \frac{\sigma}{\sqrt{n}}$$

$$\lambda = \frac{\frac{1}{S^2}}{\frac{1}{S^2} + \frac{1}{b^2}} \xrightarrow{n \to \infty} 1$$

$$\bar{\mu} \to \bar{X}$$

$$\frac{W}{S} = \frac{\sqrt{\frac{1}{\frac{1}{S^2} + \frac{1}{b^2}}}}{S} \to 1$$

$$W \to S$$

$$\frac{1}{W^2} = \frac{1}{S^2} + \frac{1}{b^2}$$

$$W = \sqrt{\frac{1}{\frac{1}{S^2} + \frac{1}{b^2}}}$$

$$\mu | X \overset{\text{асимп.}}{\sim} \mathcal{N}(\bar{X}, s^2)$$

$$\mu | X \sim \mathcal{N}(\bar{\mu}, w^2)$$

---

c) Найти (c, d):

$$\mathbb{P}\{\mu \in (c, d) | X\} = 0.95$$

$\Leftrightarrow$

$$\mathbb{P}\{\mu < c \,|\, X\} = 0.025$$

$$\mathbb{P}\{\mu > d \,|\, X\} = 0.025$$

Байес.: credible intervals

Частот: confidence intervals

$$\mathbb{P}\{\mu < c | X\} = \mathbb{P}\Big\{\underbrace{\frac{\mu - \bar{\mu}}{w}}_{Z} < \underbrace{\frac{c - \bar{\mu}}{w}}_{1.96} \,\Big|\, X\Big\} = 0.025$$

Байес: $[\bar{\mu} - 1.96w; \bar{\mu} + 1.96w]$

Асимп: $[\bar{X} - 1.96s; \bar{X} + 1.96s]$

N3 Θ – апр., апост. $\oplus$ правд.

$$G = g(\Theta) - F_G,\ f_G$$

$$F_G(g \mid X) = \mathbb{P}\{G \le g \mid X\} = \int\limits_{G \le g} f(\Theta \mid X)\, d\Theta$$

$$f_G(g \mid X) = F_G'(g \mid X)$$

$$G = \log\left(\frac{p}{1-p}\right)$$

$$F_G(g \mid X) = \mathbb{P}\{G \le g \mid X\} = \mathbb{P}\{\overbrace{\log\left(\frac{p}{1-p}\right)}^{e} \le \overbrace{g}^{e} \mid X\} =$$

$$= \mathbb{P}\{p \le \frac{e^g}{1+e^g} \mid X\} = \int\limits_0^{e^g/(1+e^g)} \underbrace{f(p \mid X)}\, dp =$$

$$= \frac{\Gamma(n+2)}{\Gamma(s+1)\Gamma(n-s+1)} \int\limits_0^{e^g/(1+e^g)} p^{(s+1)-1} (1-p)^{(n-s+1)-1}\, dp$$

$$h'(g \mid x) = \frac{\Gamma(n+2)}{\Gamma(s+1)\Gamma(n-s+1)} \left(\frac{e^g}{1+e^g}\right)^{s+1-1} \cdot$$

$$\cdot \left(1 - \frac{e^g}{1+e^g}\right)^{(n-s+1)-1} \cdot \left(\frac{\partial\left(\frac{e^g}{1+e^g}\right)}{\partial g}\right).$$

$$\left(\frac{e^g}{1+e^g}\right)' = \left(\frac{1}{e^g+1}\right)^2$$

№4 Общая идея:

$$\Theta = (\Theta_1 \dots \Theta_p)$$

$$f(\Theta \mid X) \propto f(X \mid \Theta) \cdot f(\Theta)$$

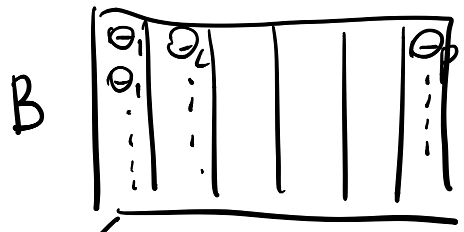
Как найти апост. для Θ_1?

$$f(\Theta_1 \mid X) = \iiint \dots \int f(\Theta_1 \dots \Theta_p \mid X)\, d\Theta_2 \dots d\Theta_p$$

$\|$

Симуляции! $\Theta^1 \dots \Theta^B \sim f(\Theta \mid X)$

$$\Theta^j = \underbrace{(\Theta_1^j \dots \Theta_p^j)}$$



$\theta_1^1 \dots \theta_1^B$ – выборка из $f(\theta_1|X)$

a) $$f(p_1,p_2|x_1,x_2) \propto \underbrace{\underbrace{p_1^{x_1}(1-p_1)^{n_1-x_1}}_{f(p_1|x_1)} \cdot \underbrace{p_2^{x_2}(1-p_2)^{n_2-x_2}}_{f(p_2|x_2)}}_{\text{правдоп.}}$$

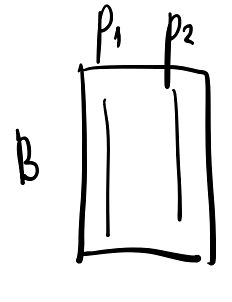
$$p_1|x_1 \perp\!\!\!\perp p_2|x_2$$

c) $p_1|x_1 \sim \mathrm{Beta}(x_1+1,\ n_1-x_1+1) \to$

$p_{11}, \dots, p_{1B}$

$p_2|x_2 \sim \mathrm{Beta}(x_2+1,\ n_2-x_2+1) \to$

$p_{21} \dots p_{2B}$



$\tau = p_{2i} - p_{1i}$, $i = 1 \dots B$ – выборка из $f(p_2 - p_1|x)$