$$y_i = \beta_0 + \beta_1 x_{1i} + \beta_2 x_{2i} + u_i$$

$$u \sim N(0, \sigma^2 I)$$

$\hat{y} = X\hat{\beta}$; $\hat{\beta}_{\text{мнк}}$

ТГМ выполн.

$n = 5$, $k = 3$

$$X = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} \quad y = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{pmatrix}$$

$$(X^T X)^{-1} = \begin{pmatrix} 1/3 & -1/3 & 0 \\ -1/3 & 4/3 & -1 \\ 0 & -1 & 2 \end{pmatrix}$$

(а) $\hat{\beta}_0, \hat{\beta}_1, \hat{\beta}_2$

$$\hat{\beta} = (X^T X)^{-1} X^T y = \begin{pmatrix} 1/3 & -1/3 & 0 \\ -1/3 & 4/3 & -1 \\ 0 & -1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{pmatrix} =$$

$$= \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} \hat{\beta}_0 \\ \hat{\beta}_1 \\ \hat{\beta}_2 \end{pmatrix}$$

(б) $$\hat{y} = X\hat{\beta} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 2 \\ 4 \\ 5 \end{pmatrix}$$

(в) TSS, ESS, RSS, R^2 $\qquad TSS = ESS + RSS$

$$TSS = \sum_i (y_i - \bar{y})^2 = 2^2 + 1^2 + 1^2 + 2^2 = 10$$

$$\bar{y} = \frac{15}{5} = 3 \quad ; \quad \bar{y} = \bar{\hat{y}}$$

$$ESS = \sum_i (\hat{y}_i - \bar{\hat{y}})^2 = \underbrace{1^2 + 1^2 + 1^2 + 1^2}_{4} + \underbrace{2^2}_{4} = 8$$

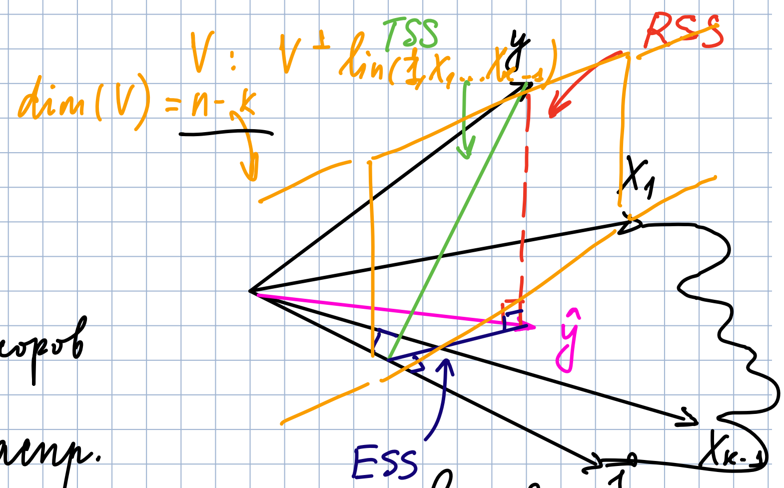
$$RSS = \sum_i (y_i - \hat{y}_i)^2 = 1^2 + 1^2 = 2$$

$$R^2 = \frac{ESS}{TSS} = \frac{8}{10} = 0.8$$

(г) $\hat{\sigma}^2$ – ?

$$\hat{\sigma}^2 = \frac{RSS}{n-k}$$

число набл. — n; число регрессоров — k



$\text{lin}(\vec{1}, X_1 \ldots, X_{k-1})$

$\dim(\text{lin}(\vec{1}, X_1 \ldots X_{k-1})) =$
$= k$

Геом. опред. χ^2_m – распр.

$$\underset{m\times 1}{u} \sim N(0, I)$$

I — един. м-ца

$$\|u\|_2^2 \sim \chi^2_m$$

Спроец. u на подпр-о разм-и p:

v – проекция u на подпр-о разм. p

$$\Rightarrow \|v\|_2^2 \sim \chi^2_p$$

$$RSS \sim \chi^2_{n-k} \cdot \text{const}$$

$y \sim N(\ldots, \ldots)$

$y = X\beta + u$

$u \sim N(0, \sigma^2 I)$

$$\underbrace{y \sim N(X\beta, \sigma^2 I)}$$

$$\frac{RSS}{\sigma^2} \sim \chi^2_{n-k}$$

$$\boxed{\hat{\sigma}^2 = \frac{RSS}{n-k}}$$

$$\mathbb{E}\left(\frac{\cancel{RSS}}{\cancel{RSS}/n-k}\right) = \underbrace{\mathbb{E}(\chi^2_{n-k})}_{= n-k}$$

$$n-k \equiv n-k$$

$$\hat{\sigma}^2 = \frac{2}{5-3} = 1$$

(g) $\hat{Var}(\hat{\beta})$

$$Var(\hat{\beta}) = Var\left((X^TX)^{-1}X^Ty\right) = (X^TX)^{-1}X^T \underset{\sigma^2 I}{\underbrace{Var(y)}} X(X^TX)^{-1} =$$

$$= \sigma^2 (X^TX)^{-1} \cancel{(X^TX)} \cancel{(X^TX)^{-1}} = \sigma^2 (X^TX)^{-1}.$$

$$\hat{Var}(\hat{\beta}) = \underset{1}{\underset{\parallel}{\hat{\sigma}^2}} (X^TX)^{-1} = (X^TX)^{-1} = \begin{pmatrix} 1/3 & -1/3 & 0 \\ -1/3 & 4/3 & -1 \\ 0 & -1 & 2 \end{pmatrix}$$

$\hat{cov}(\hat{\beta}_0, \hat{\beta}_1)$

$\hat{Var}(\hat{\beta}_0)$

$\hat{cov}(\hat{\beta}_2, \hat{\beta}_0)$ $\hat{Var}(\hat{\beta}_1)$

(e) 5%

$$\begin{cases} H_0: \beta_1 = 1, \\ H_1: \beta_1 \neq 1. \end{cases}$$

Гип-а об одном пар-е → z-test, t-test

Гип-а о неск. пар-х → F-test

$$Z = \frac{\hat{\beta}_1 - \beta_1}{\sqrt{Var(\hat{\beta}_1)}} \overset{H_0}{\sim} N(0, 1)$$

→ известна $Var(\hat{\beta}_1)$

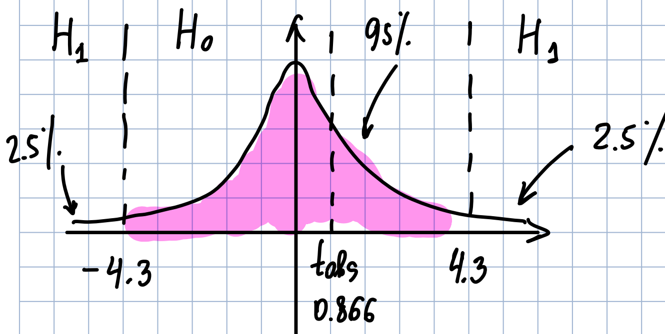
→ $n \to \infty$

$$t = \frac{\hat{\beta}_1 - \beta_1}{\sqrt{\hat{Var}(\hat{\beta}_1)}} \sim t_{n-k}$$

число набл. — число регрессоров

$$t_{obs} = \frac{2 - 1}{\sqrt{4/3}} = 0.866 \overset{H_0}{\sim} t_{5-3} = t_2$$

$t_{2,\,0.975} = 4.3$

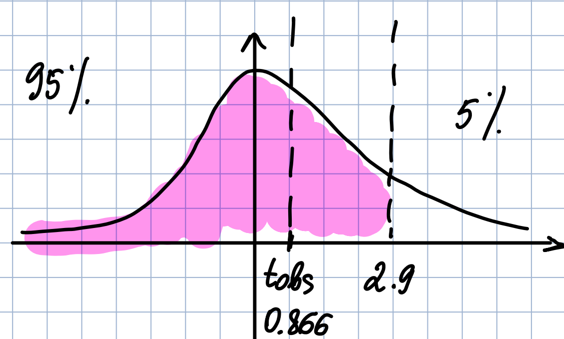


$\Rightarrow H_0$ не отверг.

(ё) $\begin{cases} H_0 : \beta_1 = 1 \\ H_1 : \beta_1 > 1 \end{cases}$

$t_{2,\,0.95} = 2.9$

$t_{obs} = 0.866$



$\Rightarrow H_0$ не отверг.

$\begin{cases} H_0 : \beta_i = 0 \\ H_1 : \beta_i \neq 0 \end{cases}$

Гипотеза о значим. коэф-а

H_0 не отверг. $\Rightarrow$ коэф. не значим (ж) Проверьте регр. на значим. в целом

$$\begin{cases} H_0: \beta_1 = \beta_2 = \ldots = \beta_{k-1} = 0 \\ H_1: \beta_1^2 + \beta_2^2 + \ldots + \beta_{k-1}^2 > 0 \end{cases}$$

$$UR: y_i = \beta_0 + \beta_1 x_{1i} + \beta_2 x_{2i} + u_i$$

$$\begin{cases} H_0: \beta_1 = \beta_2 = 0 \\ H_1: \beta_1^2 + \beta_2^2 > 0 \end{cases}$$

$$R: \quad y_i = \beta_0 + u_i$$

F-тест:

число рег. в UR м.

число рег. в R мод.

$$F = \frac{(RSS_R - RSS_{UR}) / (K_{UR} - K_R)}{RSS_{UR} / (n - K_{UR})} \overset{H_0}{\sim} F_{(K_{UR} - K_R,\ n - K_{UR})}$$

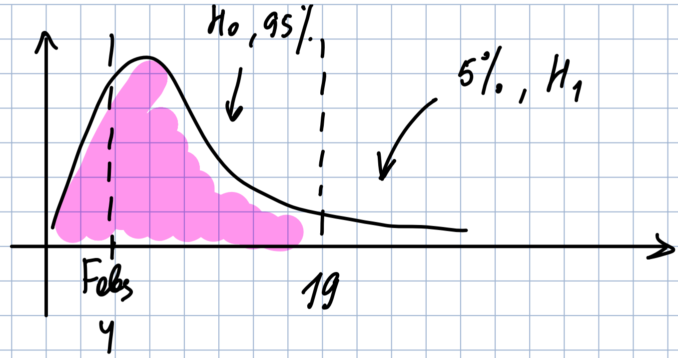
$$R: \quad y_i = \beta_0 + u_i$$

$$\hat{\beta}_{0,R} = \bar{y} \Rightarrow \hat{y}_{i,R} = \bar{y} \qquad \hat{y}_R = \begin{pmatrix} 3 \\ 3 \\ 3 \\ 3 \\ 3 \end{pmatrix}$$

$$RSS_R = \underbrace{\sum_i (y_i - \hat{y}_{R,i})^2}_{} = 10$$

$$RSS_{UR} = 2$$

$$F = \frac{(10 - 2) / (3 - 1)}{2 / (5 - 3)} = \frac{8/2}{2/2} = 4 \overset{H_0}{\sim} F_{2,2}$$



$F_{2,2,\,0.95} = 19$

$\Rightarrow H_0$ не отверг.

$\Rightarrow$ регрессия не значима в целом

(з) $\begin{cases} H_0: \beta_1 = \beta_2 \\ H_1: \beta_1 \neq \beta_2 \end{cases}$

$UR: \; y_i = \beta_0 + \beta_1 x_{1i} + \beta_2 x_{2i} + u_i$

$R: \; y_i = \beta_0 + \beta_1 \underbrace{(x_{1i} + x_{2i})} + u_i$

$$X^R = \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{pmatrix}$$

$$\hat{\beta}_R = \left((X^R)^T X^R \right)^{-1} (X^R)^T y = \begin{pmatrix} 2 \\ 1.6 \end{pmatrix}$$

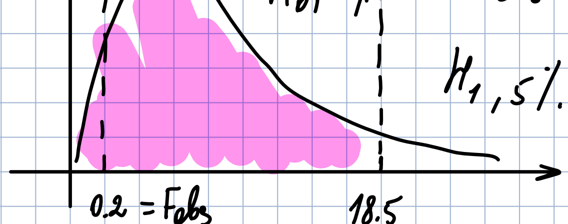
$$\hat{y}_R = X^R \cdot \hat{\beta}_R = \begin{pmatrix} 2 \\ 2 \\ 2 \\ 3.6 \\ 5.2 \end{pmatrix}$$

$RSS_R = 2.2$

$$F = \frac{(2.2 - 2) / (3 - 2)}{2 / (5 - 3)} \overset{H_0}{\sim} F_{1,2}$$

$F_{1,2,\,0.95} = 18.5$

$\shortparallel$



$F_{obs} = 0.2$

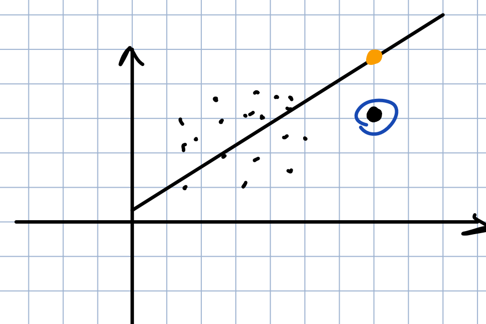
$\Rightarrow H_0$ не отверг.

(и) 95% CI для β_1

$$\beta_1 \in \left[\hat{\beta}_1 - t / \underset{n \to \infty}{Z} \cdot \sqrt{\widehat{Var}(\hat{\beta}_1)} \; ; \; \hat{\beta}_1 + t/Z \cdot \sqrt{\widehat{Var}(\hat{\beta}_1)}\right]$$

$$\beta_1 \in \left[2 - 4.3\sqrt{\frac{4}{3}} \; ; \; 2 + 4.3\sqrt{\frac{4}{3}}\right]$$

(й) $\hat{y}_6 \mid x_{1,6} = 10, \; x_{2,6} = 7$



$$\hat{y}_i = \hat{\beta}_0 + \hat{\beta}_1 x_{1i} + \hat{\beta}_2 x_{2i}$$

$$\hat{y}_6 = 2 + 2 \cdot 10 + 1 \cdot 7 = 29$$