скал　　　　　　　　матр

$I(\theta)$:　　　$Var(s(\theta))$

$\mathbb{E}(s^2)$　　　　　　$\mathbb{E}(s \cdot s^T)$

$\mathbb{E}(-H)$

резул.　$\boxed{Var(\hat{\theta}) \cdot I(\theta) \xrightarrow{n\to\infty} Id}$

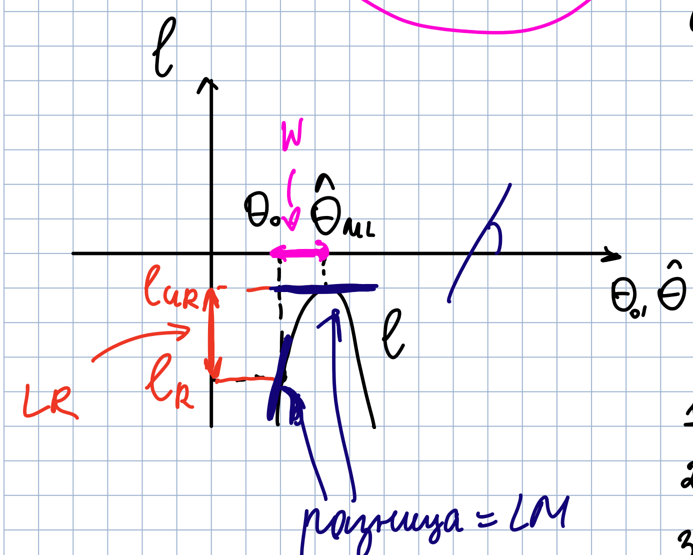
LR :　$2(l_{UR} - l_R) \xrightarrow[(dist)]{n\to\infty} \chi^2_k$

↑ число огранич.

W :　$\dfrac{(\hat{\theta}_{ML} - \theta_0)^2}{Var(\hat{\theta})}$　　$(\hat{\gamma}_{ML} - \gamma_0)^T \, Var(\hat{\gamma}_{ML})^{-1} (\hat{\gamma}_{ML} - \gamma_0)$

γ — подвектор парам-в, о к. пров. гип-у

Θ ; $\begin{cases} H_0 : \gamma = \gamma_0 \\ H_A : \gamma \neq \gamma_0 \end{cases}$



$\gamma = \Theta \Rightarrow Var(\hat{\gamma}_{ML})^{-1} = I$

(*) Иначе : $\gamma \neq \Theta$

1. $\hat{I}(\hat{\theta})$
2. $\hat{Var}(\hat{\theta}_{ML}) = \hat{I}^{-1}$
3. Вырезаем подматрицу $\hat{Var}(\hat{\gamma})$ из $\hat{Var}(\hat{\theta})$
4. ↳ $Var(\hat{\gamma}_{ML})^{-1}$.

скал $\overset{\text{"}0}{}$ матр

LM $\dfrac{\left(\ell'(\theta_0) - \ell'(\hat{\theta}_{ML})\right)^2}{Var(\ell'(\theta_0))}$ $S_R^{T} \cdot Var(s)^{-1} \cdot S_R$

$$= \frac{[\ell'(\theta_0)]^2}{Var(\ell'(\theta_0))}$$

$$S = \begin{bmatrix} \ell'_{\theta_1} \\ \vdots \\ \ell'_{\theta_k} \end{bmatrix} \qquad I = Var(s)$$

№1 $X_1 \dots X_{50} \overset{iid}{\sim} exp(\lambda)$ $\begin{cases} H_0: \lambda = 1 \\ H_1: \lambda \neq 1 \end{cases}$

$\bar{X} = 1.5$

5% при пом. LR, LM, W

$$f_X(x) \begin{cases} \lambda e^{-\lambda x}, & x > 0, \\ 0, & иначе \end{cases}$$

$$L = \prod_{i=1}^{n} \lambda e^{-\lambda x} = \lambda^n e^{-\lambda \sum_{i=1}^{n} x_i}$$

$$\ell = n \ln \lambda - \lambda \sum_{i=1}^{n} x_i \to \max_{\lambda}$$

$$S = \ell'_\lambda = \frac{n}{\lambda} - \sum x_i$$

$$\frac{n}{\hat\lambda} = \sum x_i \Rightarrow \hat\lambda_{ML} = \frac{n}{\sum x_i}\Bigg|_X = \frac{1}{\bar X} = \frac{2}{3}. \qquad \boxed{\hat\lambda_{ML} = \frac{2}{3}}$$

$$\ell''_\lambda = -\frac{n}{\lambda^2} \qquad I = \mathbb{E}(-\ell''_\lambda) = \frac{n}{\lambda^2}$$

$$\hat I = I\big|_{\hat\lambda} = \frac{n}{\hat\lambda^2} = \frac{50 \cdot 9}{4} = \frac{450}{4} = 112.5$$

$$\longrightarrow \hat{Var}(\hat\lambda_{ML}) = \frac{1}{\hat I} = \frac{1}{112.5} = \frac{4}{450}$$

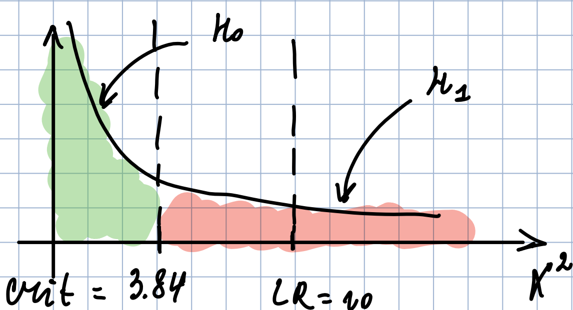
(1) $LR = 2(\ell_{UR} - \ell_R)$ $\qquad \ell = n \ln\lambda - \lambda \sum_{i=1}^n x_i \longrightarrow \max_\lambda$

$$\ell_{UR} = \ell\big|_{\hat\Theta_{UR}} = \ell\big|_{\hat\Theta_{ML}} = 50 \ln\frac{2}{3} - \frac{2}{3} \cdot 50 \cdot \frac{3}{2} =$$

$\hat\lambda_{UR} = \frac{2}{3}$ $$= 50 \ln\frac{2}{3} - 50$$

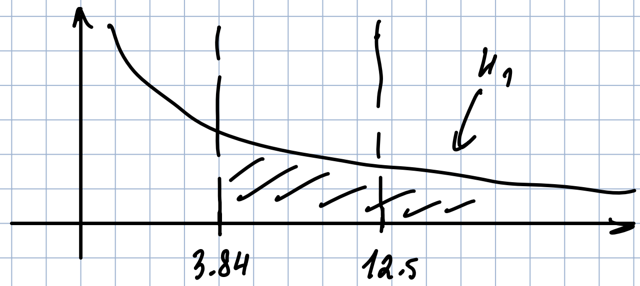
$$\ell_R = \ell\big|_{\Theta_0} = \ell\big|_{\lambda=1} = -50 \cdot \frac{3}{2} = -75$$

$$LR = 2\left(50 \ln\frac{2}{3} - 50 + 75\right) = 2\left(25 + 50 \ln\frac{2}{3}\right) \approx 10 \sim \chi^2_1$$



$\Rightarrow$ H_0 отверг. на ур. зн. 5%

$$W = \frac{(\hat{\theta}_{UR} - \theta_0)^2}{\hat{Var}(\hat{\theta}_{UR})} = \frac{\left(\frac{2}{3} - 1\right)^2}{4} \cdot 450 = \frac{450}{36} = 12.5$$



H_0 отверг.

$$LM = \frac{(l'(\theta_0))^2}{Var(l'(\theta_0))} \qquad \lambda_R = 1$$

$$S = l'_\lambda = \frac{n}{\lambda} - \sum x_i = 50 - 50 \cdot \frac{3}{2}$$

$$\boxed{I = \mathbb{E}(-l''_\lambda) = \frac{n}{\lambda^2}\Big|_{\lambda=1} = 50}$$

$$LM = \frac{625}{50} = 12.5 \Rightarrow H_0 \text{ отверг.}$$

Лин. модель ⊕ шп. с лин. огранич.

$$y_i = \beta_0 + \beta_1 x_i + \varepsilon_i$$

$$\varepsilon_i \overset{iid}{\sim} N(0, 1)$$

$$\begin{cases} H_0: f(\beta) = 3 \quad \text{лин.} \\ H_1: \beta_1 \neq 3 \end{cases} \Rightarrow \boxed{\begin{array}{l} LM \le LR \le W \\ \text{(на конеч. выборках)} \end{array}}$$

№2 $X_1 \dots X_{100} \sim N(\mu, \sigma^2)$

$$\sum_{i=1}^{100} X_i = 20, \quad \sum_{i=1}^{100} (X_i - \bar{X})^2 = 400$$

$$\begin{cases} H_0: \begin{pmatrix} \mu \\ \sigma \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ H_1: \begin{pmatrix} \mu \\ \sigma \end{pmatrix} \neq \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{cases}$$

5%, LR, LM, W

$$\hat{\mu} = \bar{X} = 0.2 \qquad \hat{\Theta}_{UR} = \begin{bmatrix} \hat{\mu}_{UR} \\ \hat{\sigma}_{UR} \end{bmatrix}$$

$$\hat{\sigma}^2 = \frac{\sum_{i=1}^{100} (X_i - \bar{X})^2}{n} = 4 \ ; \ \hat{\sigma} = 2$$

$$LR = 2(l_{UR} - l_R)$$

$$l = \cancel{-\frac{n}{2} \ln 2\pi} - n \ln \sigma - \frac{1}{2\sigma^2} \sum_{i=1}^{n} (x_i - \mu)^2$$

$$l_{UR} = -100 \ln 2 - \frac{1}{8} \cdot 400$$

$$l_R = -\frac{1}{2} \sum_{i=1}^{n} x_i^2 = -\frac{404}{2} = -202$$

$$400 = \sum_i (x_i - \bar{x})^2 = \sum x_i^2 - 2n \overbrace{\frac{\sum x_i}{n}}^{\bar{x}} \bar{x} + (\bar{x})^2 \cdot n =$$

$$= \sum x_i^2 - n(\bar{x})^2 = 400$$

$$\sum x_i^2 = 400 + 100 \cdot 0.04 = 404$$

$$LR = 2\left(-100 \ln 2 - \frac{1}{8} \cdot 400 + 202\right) = 82 \overset{H_0}{\sim} \chi^2_2$$

$\chi^2_{crit\ 95\%,2} = 5.99 \Rightarrow H_0$ отверг.

$$W = [\hat{\Theta}_{UR} - \Theta_0]^T \cdot \hat{Var}(\hat{\theta})^{-1} \cdot [\hat{\Theta}_{UR} - \Theta_0]$$

$$I(\Theta) = \begin{pmatrix} \frac{n}{\sigma^2} & 0 \\ 0 & \frac{2n}{\sigma^2} \end{pmatrix}$$

$\hat{\mu} = 0.2$

$\hat{\sigma} = 2$

$$W = \begin{bmatrix} 0.2 - 0 \\ 2 - 1 \end{bmatrix}^T \begin{pmatrix} \frac{100}{4} & 0 \\ 0 & \frac{200}{4} \end{pmatrix} \begin{bmatrix} 0.2 - 0 \\ 2 - 1 \end{bmatrix} = \dots$$

$$LM = S_R^T \cdot \hat{Var}(S)^{-1} \cdot S_R = \dots \sim \chi^2_2$$

$$S = \begin{bmatrix} l'_\mu \\ l'_\sigma \end{bmatrix} = \begin{bmatrix} -\frac{1}{\sigma^2} \sum_i (x_i - \mu) \\ -\frac{n}{\sigma} + \frac{1}{\sigma^3} \sum_i (x_i - \mu)^2 \end{bmatrix} \Bigg|_{R:\ \mu = 0,\ \sigma = 1} = \begin{bmatrix} \dots \\ \dots \end{bmatrix}$$

$$I\Big|_R = \begin{pmatrix} 100 & 0 \\ 0 & 200 \end{pmatrix}$$

$$I_R^{-1} = \begin{pmatrix} \frac{1}{100} & 0 \\ 0 & \frac{1}{200} \end{pmatrix}$$

N3 Палочки

вишня, дуб, вяз

$n = 150$

$n_в = 75$ вишня: p_1

$n_д = 30$ дуб: p_2

$n_{вяз} = 45$ вяз: $1 - p_1 - p_2$

a) $p = \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}$ $\hat{p}_{ML}$

$k_1 \quad k_2 \quad \dots \quad k_n$

$p_1 \quad p_2 \quad \dots \quad p_n \qquad \sum_{i=1}^{n} p_i = 1$

X_1 – число раз, когда вып. k_1

X_2 – – " – k_2

$\vdots$

X_n

$(X_1 \dots X_n) \sim Multinomial$

$\mathbb{P}\{X_1 = x_1 \dots X_n = x_n\} = C_n^{x_1} C_{n-x_1}^{x_2} C_{n-x_1-x_3}^{x_3} \dots C_{n-\dots}^{x_n} \cdot$
$\cdot p_1^{x_1} p_2^{x_2} \cdot \dots \cdot p_n^{x_n}$

$L = C_{150}^{75} C_{75}^{30} C_{45}^{45} p_1^{75} p_2^{30} (1-p_1-p_2)^{45}$

$n = 150$

$n_б = 75$

$n_g = 30$

$n_{ву} = 45$

$\ell = C + \underline{75 \ln p_1 + 30 \ln p_2 + 45 \ln(1-p_1-p_2)} \to \max_{p_1, p_2}$

$\hat{p}_1 = \frac{75}{150}$ (честно)

$\hat{p}_2 = \frac{30}{150}$

б) $\begin{cases} H_0: p_1 = 0.7 \\ H_1: p_1 \neq 0.7 \end{cases}$

$LR = 2(\ell_{UR} - \ell_R)$

ℓ_{UR} – понятно

$\ell_R = 75 \ln 0.7 + 30 \ln p_2 + 45 \ln(0.3 - p_2) \to \max_{p_2}$

$$\frac{30}{\hat{p}_2} - \frac{45}{0.3-\hat{p}_2} = 0$$

$$9 - 30\hat{p}_2 = 45\hat{p}_2$$

$$\hat{p}_{2_R} = \frac{9}{75}$$

$$\ell(a, b) = \ldots$$

$$\begin{cases} H_0: a = 2b \\ H_1: a \neq 2b \end{cases} \Rightarrow \begin{matrix} H_0: a - 2b = 0 \\ H_1: a - 2b \neq 0 \end{matrix}$$