Модели бинарного выбора

$$y_i = \begin{cases} 1, & p_i \\ 0, & 1-p_i \end{cases}$$

Идея 1: $y_i = \underset{k\times 1}{(x_i)}^T \beta + u_i$ Лин. вероятн. модель

Проблемы: $\mathbb{E}(y_i) = p_i \longrightarrow \underline{p_i = x_i^T\beta}$

Пусть $\mathbb{E}(u_i) = 0 \Rightarrow \mathbb{E}(y_i) = \underbrace{x_i^T\beta}$

1. $\hat{p}_i = x_i^T \hat{\beta} \notin [0;1]$
2. Гск по постр.

$$\mathrm{Var}(y_i) = p_i(1-p_i) = \mathrm{Var}(u_i)$$

$$\mathrm{Var}(u_i) = (x_i^T\beta)(1 - x_i^T\beta)$$

3. $$u_i = \begin{cases} 1 - x_i^T\beta, & p_i = x_i^T\beta \\ -x_i^T\beta, & 1-p_i = 1 - x_i^T\beta \end{cases}$$

$\Rightarrow u_i \sim N(\ldots, \ldots) \Rightarrow$ нельзя станд. тесты ($t, Z, F, \chi^2, \ldots$)

Идея 2: Введём лат. перем. y_i^*

$$y_i^* = x_i^T\beta + u_i$$

$$y_i = \begin{cases} 1, & y_i^* \geq 0 \\ 0, & y_i^* < 0 \end{cases}$$

$p(y_i = 1) = p(y_i^* \geq 0) = p(x_i^T\beta + u_i \geq 0) =$

$= p_i(u_i \geq -x_i^T\beta) = \left\{ \begin{array}{l} u_i \text{ имеет} \\ \text{сим. распр.} \end{array} \right\} =$

$= p_i(u_i \leq x_i^T\beta) = \underbrace{F_{u_i}(x_i^T\beta)}$

$u_i \sim logit$ $\qquad\qquad$ $u_i \sim N(0,1)$

$$F(t) = \Lambda(t) = \frac{e^t}{1+e^t} \qquad\qquad F(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{t} e^{-\frac{z^2}{2}} dz$$

(логит) $\qquad\qquad$ (пробит)

Задача:

$$\begin{cases} y_i = \begin{cases} 1, & p_i, \ y_i^* \geq 0 \\ 0, & 1-p_i, \ y_i^* < 0 \end{cases} \\ y_i^* = x_i^T \beta + u_i \\ p(y_i = 1) = F_{u_i}(x_i^T \beta) \\ u_i \sim logit \end{cases}$$

$$y_i \sim Bern\left(p_i = F_{u_i}(x_i^T \beta)\right) \qquad \underline{\mathbb{P}\{y_i = 1\} = p_i^{y_i}(1-p_i)^{1-y_i}}$$

$$L(y_i, x_i | \beta) = \prod_{i=1}^{n} F_{u_i}(x_i^T\beta)^{y_i} \left(1 - F_{u_i}(x_i^T \beta)\right)^{1-y_i}$$

$$l(y_i, x_i | \beta) = \sum_i y_i \underbrace{\ln F_{u_i}(x_i^T \beta)} + (1-y_i) \underbrace{\ln (1 - F_{u_i}(x_i^T\beta))}$$

$\to \max\limits_{\beta}$

№1

y_i	x_i
0	20
1	20
1	30
0	30

$y_i^* = \beta_0 + \beta_1 x_i + u_i$

$u_i \sim logit$

$p(y_i = 1) = \Lambda(x_i^T \beta)$

а) $\hat{\beta}_0, \hat{\beta}_1$

б) $\hat{I}$

$$l = \ln p(y_1=0) + \ln p(y_2=1) + \ln p(y_3=1) +$$
$$+ \ln p(y_4=0) = \ln(1-\Lambda(\beta_0+20\beta_1)) +$$
$$+ \ln \Lambda(\beta_0+20\beta_1) + \ln \Lambda(\beta_0+30\beta_1) +$$
$$+ \ln(1-\Lambda(\beta_0+30\beta_1))$$

$$\Lambda(t) = \frac{e^t}{1+e^t} \qquad \boxed{\Lambda'_t = \underbrace{\Lambda(t)}\cdot\underbrace{(1-\Lambda(t))}}$$

$$[\ln \Lambda(t)]'_t = \frac{\Lambda(t)'}{\Lambda(t)} = \frac{\cancel{\Lambda(t)}(1-\Lambda(t))}{\cancel{\Lambda(t)}} = \underbrace{1-\Lambda(t)}$$

$$[\ln(1-\Lambda(t))]'_t = \frac{-\Lambda'_t}{1-\Lambda(t)} = -\Lambda(t)$$

$$\frac{\partial l}{\partial \beta_0} = -\hat{\Lambda}(\beta_0+20\beta_1) + 1 - \hat{\Lambda}(\beta_0+20\beta_1) +$$
$$+ 1 - \hat{\Lambda}(\beta_0+30\beta_1) - \hat{\Lambda}(\beta_0+30\beta_1) = 0$$

$$\frac{\partial l}{\partial \beta_1} = -\Lambda(\beta_0+20\beta_1)\cdot 20 + \ldots \qquad + \ldots = 0$$

$$\longrightarrow \hat{\beta}_0, \hat{\beta}_1 \rightarrow \text{град. спуск}$$
$$\Lambda' = \Lambda(1-\Lambda) \qquad \hookrightarrow (*) \text{ явно}$$

б)
$$H = \begin{pmatrix} -\sum_i \Lambda(1-\Lambda) & -\sum_i \Lambda(1-\Lambda)x_i \\ -\text{''}- & -\sum_i \Lambda(1-\Lambda)x_i^2 \end{pmatrix}$$

$\hat{I} = -H$

$\widehat{Var}(\hat\beta) = (\hat{I})^{-1}$

$\hat\beta \equiv \hat\beta_{ММП}$

(+) LR, LM, W

(+) Z-test

1) инв. к гл. преобр.
2) асимпт. несм.
3) ас. сост.
4) ас. эф.
5) $\hat\beta \sim N(\beta, I^{-1})$

$$\hat p(y_i = 1) = F_{u_i}(x_i^T \hat\beta)$$

№2 $n = 10^7$; $\hat p(y_i = 1) = \Lambda(2.3 - 1.7x_i + 2.5z_i)$

$$\widehat{Var}(\hat\beta) = (\hat I)^{-1} = \begin{pmatrix} 2.56 & 0.3 & -0.2 \\ 0.3 & 1.69 & 0.1 \\ -0.2 & 0.1 & 1.44 \end{pmatrix}$$

а) $\begin{cases} H_0: \beta_x = 0 \\ H_A: \beta_x \neq 0 \end{cases}$

$$W = (-1.7 - 0) \cdot (1.69)^{-1} \cdot (-1.7 - 0) \sim \chi^2_1$$

б) $H_0: \begin{pmatrix} \beta_0 = 0 \\ \beta_z = 0 \end{pmatrix}$

$H_1: \begin{pmatrix} \beta_0 \neq 0 \\ \beta_z \neq 0 \end{pmatrix}$

$$W = \begin{pmatrix} 2.3 & 2.5 \end{pmatrix} \begin{pmatrix} 2.56 & -0.2 \\ -0.2 & 1.44 \end{pmatrix}^{-1} \begin{pmatrix} 2.3 \\ 2.5 \end{pmatrix} \sim \chi^2_2$$

$$p(y_i = 1) = F_{u_i}(x_i^T \beta)$$

Предельные эффекты:

в кажд. точке i свой пред. эф-т

$$\frac{\partial p(y_i = 1)}{\partial x_j} = \underbrace{f(x_i^T \beta)}_{>0} \cdot \underbrace{\beta_j}_{\text{знак пред. эф.} = \text{знак коэф.}}$$