Теория ($Var(u_i)$ - известно)

(N1) $y_i = \beta_0 + \beta_1 x_i + u_i \quad : \sqrt{f(x_i)}$

$$Var(u_i) = \frac{\sigma^2}{(3+x_i)^2} \qquad Var(u_i) = \sigma^2 \cdot f(x_i)$$

$$f(x_i) = \frac{1}{(3+x_i)^2}$$

$$\underbrace{(3+x_i)\,y_i}_{} = \underbrace{(3+x_i)\,\beta_0}_{} + \underbrace{\beta_1 x_i\,(3+x_i)}_{} + \underbrace{u_i \cdot (3+x_i)}_{}$$

$$\tilde{y}_i = \beta_0 \cdot \tilde{x}_{1i} + \beta_1 \cdot \tilde{x}_{2i} + \tilde{u}_i$$

$$Var(\tilde{u}_i) = Var(u_i(3+x_i)) = \frac{\cancel{(3+x_i)^2} \cdot \sigma^2}{\cancel{(3+x_i)^2}} = \sigma^2$$

$\Rightarrow$ в модели $\hat{y}_i = \beta_0 \cdot \hat{x}_{1i} + \beta_1 \cdot \hat{x}_{2i} + \hat{u}_i$ г/ск нет.

$$\hat{\beta}^{eff} = (\tilde{X}^T \tilde{X})^{-1} \tilde{X}^T \tilde{y}$$

(N2) $y = X\beta + u$ $\qquad$ A - известна

$Var(u) = \sigma^2 \cdot A$

м-ца соб. век.

$$A = P \Lambda P^{-1}$$

м-ца соб. чисел

$$\underbrace{A^{-1/2} y}_{} = \underbrace{A^{-1/2} X\beta}_{} + \underbrace{A^{-1/2} u}_{}$$

$$A^{-1/2} = P \begin{pmatrix} \frac{1}{\sqrt{\lambda_1}} & & \\ & \ddots & \\ & & \frac{1}{\sqrt{\lambda_n}} \end{pmatrix} P^{-1}$$

$$\tilde{y} = \tilde{X}\beta + \tilde{u}$$

$$\hat{\beta}^{eff} = (\tilde{X}^T\tilde{X})^{-1}\tilde{X}^T\tilde{y} = (\underbrace{X^T A^{-1/2}}_{\tilde{X}^T}\underbrace{A^{-1/2}X}_{\tilde{X}})^{-1}\underbrace{X^T A^{-1/2}}_{\tilde{X}^T}\underbrace{A^{-1/2}y}_{\tilde{y}}$$

$$= (X^T A^{-1} X)^{-1} X^T A^{-1} y$$

(б) $\mathrm{Var}(\hat{\beta}^{eff}) = \mathrm{Var}((X^T A^{-1} X)^{-1} X^T A^{-1} y) =$

$$= (X^T A^{-1} X)^{-1} X^T A^{-1} \underbrace{\mathrm{Var}(y)}_{=A\sigma^2} A^{-1} X (X^T A^{-1} X)^{-1} =$$

$$= \sigma^2 \cancel{(X^T A^{-1} X)^{-1} X^T A^{-1} A A^{-1} X} (X^T A^{-1} X)^{-1} =$$

$$= \sigma^2 (X^T \underbrace{A^{-1}}_{=A^{-1/2}\cdot A^{-1/2}} X)^{-1} = \sigma^2 (\tilde{X}^T\tilde{X})^{-1}$$

(в) $\mathrm{Var}(\hat{\beta}^{мнк}) = \mathrm{Var}((X^T X)^{-1} X^T y) =$

$$= \sigma^2 (X^T X)^{-1} X^T A X (X^T X)^{-1}$$

Практика.

$$\hat{Var}(\hat{\beta}^{мнк})_{HC} = (X^T X)^{-1} X^T \underbrace{\hat{Var}(u)_{HC}} X (X^T X)^{-1}$$

$$HC_0: \quad \hat{Var}(u)_{HC_0} = \begin{pmatrix} \hat{u}_1^2 & & 0 \\ & \ddots & \\ 0 & & \hat{u}_n^2 \end{pmatrix}$$

$$\hat{u}_i = y_i - \hat{y}_i$$

$$HC_3: \quad Var(u)_{HC_3} = \begin{pmatrix} \hat{u}_{cv,1}^2 & & 0 \\ & \ddots & \\ 0 & & \hat{u}_{cv,n}^2 \end{pmatrix}$$

N3 $\quad y_i = \beta x_i + u_i$

y_i	x_i
2	1
2	1
2	4

(a) $$\hat{\beta}_{мнк} = \frac{\sum_i x_i y_i}{\sum_i x_i^2} = \frac{2+2+8}{1^2+1^2+4^2} = \frac{12}{18} = \frac{2}{3}$$

$$x^T x = \sum_i x_i^2 = 18$$

(б) HC_0:

y_i	x_i	$\hat{y}_i = x_i \cdot \hat{\beta}$	$\hat{u}_i = y_i - \hat{y}_i$
2	1	2/3	4/3
2	1	2/3	4/3
2	4	8/3	-2/3

$$\hat{Var}(\hat{\beta}^{мнк})_{HC} = (X^T X)^{-1} X^T \hat{Var}(u)_{HC_0} X (X^T X)^{-1}$$

$$\hat{Var}(\hat{\beta}_{мнк})_{HC_0} = \frac{1}{18} \begin{pmatrix} 1 & 1 & 4 \end{pmatrix} \begin{pmatrix} (4/3)^2 & & 0 \\ & (4/3)^2 & \\ 0 & & (-\frac{2}{3})^2 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 4 \end{pmatrix} \cdot \frac{1}{18}$$

HC_3:

y_i	x_i
2	1
2	1
2	4

$\hat{\beta}_{cv}$

$\frac{2 \cdot 1 + 2 \cdot 4}{1^2 + 4^2} = \frac{10}{17}$

$\frac{2 \cdot 1 + 2 \cdot 4}{1^2 + 4^2} = 10/17$

$\frac{2 + 2}{1 + 1} = 2$

$\hat{y}_{i\,cv} = \hat{\beta}_{cv} \cdot x_i$

$10/17$

$10/17$

8

$\hat{u}_{cv} = y_i - \hat{y}_{i\,cv}$

$\vdots$