LR: $y_i = \beta_0 \cdot 1 + \beta_1 x_{1i} + \beta_2 x_{2i} + \dots + \beta_k x_{ki} + \varepsilon_i$

регрессоры

завис. перем.

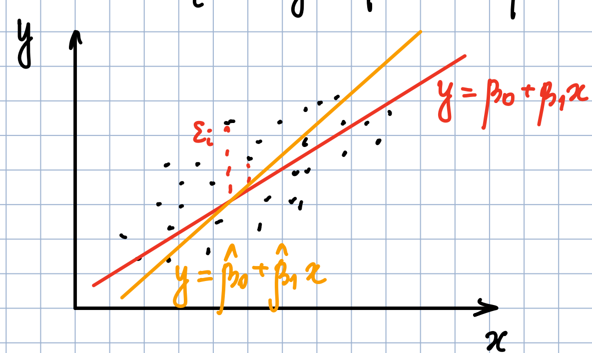
коэф-ты

случайная ошибка

(k+1) регрессор

β_i – consts

x_i – детерминир.



парная

$$\boxed{y_i = \beta_0 + \beta_1 x_i + \varepsilon_i}$$

$$\underbrace{y_i = \beta_0 + \beta_1 x_i}_{\text{истинная лин. регр.}}$$

множ-я

$$\underset{n\times 1}{y} = \underset{n\times(k+1)}{X}\,\underset{(k+1)\times 1}{\beta} + \underset{n\times 1}{\varepsilon}$$

β – не знаем

$$\hat\beta \Rightarrow \hat y_i = \hat\beta_0 + \hat\beta_1 x_i$$

(1) МНК $\Rightarrow \hat\beta_{МНК}$

RSS (residual sum of squares) $= \sum_i (y_i - \hat y_i)^2 \to \min\limits_{\hat\beta_0,\dots,\hat\beta_k}$

Пример

$y_i = \beta_1 x_i + \varepsilon_i$

$\hat\beta_{МНК}$ – ? $\quad \boxed{\hat y = \hat\beta_1 x_i}$

x_i	y_i
1	1
2	2
2	4

$$RSS = \sum_i (y_i - \hat\beta_1 x_i)^2 = (1-\hat\beta_1)^2 + (2-2\hat\beta_1)^2 + (4-2\hat\beta_1)^2 \to \min_{\hat\beta_1}$$

$$RSS'_{\hat\beta_1} = -2\sum_i x_i (y_i - \hat\beta_1 x_i)$$

$$\sum_i x_i y_i = \sum_i \hat\beta_1 x_i^2 \Rightarrow \boxed{\hat\beta_{1,мнк} = \frac{\sum_i x_i y_i}{\sum_i x_i^2}} =$$

$$= \frac{1+4+8}{1+4+4}.$$

$$y_i = \beta_0 + \beta_1 x_i + \varepsilon_i$$

$$\hat\beta_1 = \frac{\sum_i (x_i - \bar x)(y_i - \bar y)}{\sum_i (x_i - \bar x)^2}$$

$$\hat\beta_0 = \bar y - \hat\beta_1 \bar x$$

Пример 2

x_i	y_i
1	1
2	2
2	4

$$y_i = \beta_0 + \varepsilon_i$$

$$\hat\beta_{0,\text{мнк}} - ?$$

$$\hat y_i = \hat\beta_0$$

$$RSS = \sum_i (y_i - \hat\beta_0)^2 \to \min_{\hat\beta_0}$$

$$RSS'_{\hat\beta_0} = -2 \sum_i (y_i - \hat\beta_0)$$

$$\sum_i y_i = \sum_i \hat\beta_0 = n \cdot \hat\beta_0 \Rightarrow \hat\beta_{0,\text{мнк}} = \frac{\sum_i y_i}{n} = \bar y$$

Множ. регр.

$$y_i = \beta_0 + \beta_1 x_{1i} + \dots + \beta_k x_{ki} + \varepsilon_i$$

$$y = X\beta + \varepsilon$$

$$\hat\beta_{\text{мнк}}$$

аналитика

$$\sum_i (y_i - \hat y_i)^2 \to \min_{\hat\beta_0 \dots \hat\beta_1}$$

матрич.

$$(y - \hat y)^T (y - \hat y) \to \min_{\hat\beta}$$

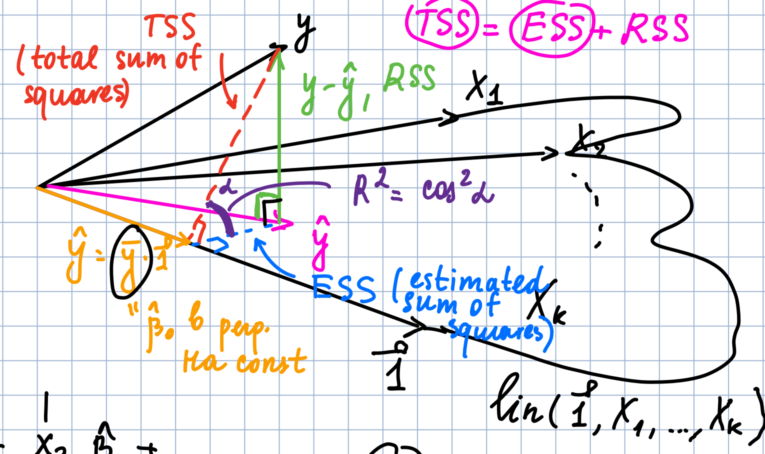
геом

$y = X\beta + \varepsilon$

$\underbrace{\hat{y} = X\hat{\beta}}$

$\hat{\beta}_{\text{мнк}}$
$(k+1)\times 1$

$\text{TSS} = \text{ESS} + \text{RSS}$



$\hat{y} \in \text{lin}$ (столбцов X)

$\hat{y} = X\hat{\beta} = x_1\hat{\beta}_0 + x_2\hat{\beta}_1 + \ldots$

(1)

$$X = \begin{bmatrix} | & | & | & & | \\ 1 & x_1 & x_2 & \ldots & x_k \\ | & | & | & & | \end{bmatrix}$$
$n\times(k+1)$

$$\text{RSS} = \sum_i (y_i - \hat{y}_i)^2 = \|y - \hat{y}\|^2 \to \min_{\hat{\beta}}$$

$(y - \hat{y}) \perp X$

$X^T(y - X\hat{\beta}) = 0$

$X^T y = X^T X\hat{\beta}$

$$\boxed{\hat{\beta}_{\text{мнк}} = \underbrace{(X^T X)^{-1} X^T}\, y}$$

случ. вел.

$\text{TSS} = \text{ESS} + \text{RSS}$

$$\underbrace{\sum_i (y_i - \bar{y}_i)^2}_{\text{TSS}} = \underbrace{\sum_i (\hat{y}_i - \bar{y})^2}_{\text{ESS}} + \underbrace{\sum_i (y_i - \hat{y}_i)^2}_{\text{RSS}}$$

$$\boxed{R^2 := \frac{ESS}{TSS}} = 1 - \frac{RSS}{TSS} = \frac{\sum_i (\hat{y}_i - \bar{y})^2/n}{\sum_i (y_i - \bar{y})^2/n} = \frac{\|\hat{y} - \bar{y}\cdot \vec{1}\|}{\|y - \bar{y}\cdot \vec{1}\|}$$

$\in [0; 1]$

В регр. без const: $TSS \neq ESS + RSS \Rightarrow R^2$ не интерпр.

Упр $R^2 = scorr^2(y, \hat{y})$

выбор. коррел.

$$scorr(x, y) = \frac{\sum_i (x_i - \bar{x})(y_i - \bar{y})}{\sqrt{\sum_i (x_i - \bar{x})^2} \sqrt{\sum_i (y_i - \bar{y})^2}}$$

$$\hat{\beta} = (X^T X)^{-1} X^T y$$

$$\mathbb{E}(\hat{\beta}) = \mathbb{E}(\underbrace{(X^T X)^{-1} X^T}y) =$$

$$= (X^T X)^{-1} X^T \mathbb{E}(y) =$$

$$= [y = X\beta + \varepsilon] =$$

$$= (X^T X)^{-1} X^T (X\beta + \varepsilon) =$$

$$= \cancel{(X^T X)^{-1}} \cancel{X^T X} \beta + (X^T X)^{-1} X^T \mathbb{E}(\varepsilon) =$$

$$= \beta + (X^T X)^{-1} X^T \cdot \mathbb{E}(\varepsilon)$$

Правила

$\underset{n\times 1}{u}, v$ – случ. вектора

$\underset{n\times n}{A}, B$ – не случ. матр.

$$\mathbb{E}(u) = \begin{pmatrix} \mathbb{E}(u_1) \\ \vdots \\ \mathbb{E}(u_n) \end{pmatrix}$$

$$\mathbb{E}(Au) = A\,\mathbb{E}(u)$$

$$Var(u) = \begin{pmatrix} Var(u_1) & cov(u_1, u_2) & \dots \\ \vdots & \ddots & \vdots \\ cov(u_1, u_n) & \dots & Var(u_n) \end{pmatrix}$$

$$Var(Au) = A\,Var(u)\,A^T$$

$$cov(Au, Bv) = A\,cov(u, v)\,B^T$$

$$cov(\underset{n\times 1}{u}, \underset{k\times 1}{v}) = \underset{n\times k}{\begin{pmatrix} cov(u_1, v_1) & \dots & cov(u_1, v_k) \\ \vdots & & \\ cov(u_n, v_1) & \dots & cov(u_n, v_k) \end{pmatrix}}$$

$$Var(\hat{\beta}) = Var(\underbrace{(X^TX)^{-1}X^T}y) =$$

$$= (X^TX)^{-1}X^T \, Var(y) \, X \, (X^TX)^{-1} =$$

$$= (X^TX)^{-1}X^T \, Var(X\beta + \varepsilon) \, X \, (X^TX)^{-1} =$$

$$= (X^TX)^{-1}X^T \, Var(\varepsilon) \, X \, (X^TX)^{-1}$$

Теорема Гаусса-Маркова

Если:

"Модель прав-о специфицир."

1. $y = X\beta + \varepsilon$
2. $\hat{y} = X\hat{\beta}$ при пом. МНК
3. β – const
4. X – детермин., X – полной ранг
5. $n \gg K$
6. $\mathbb{E}(\varepsilon_i) = 0 \quad \forall i$

$Var(\varepsilon_i) = \sigma^2$

$cov(\varepsilon_i, \varepsilon_j) = 0 \quad \forall i,j$

$\Big\} \; \underbrace{Var(\varepsilon) = \sigma^2 \cdot I_{n\times n}}_{\text{гомоскедастичн.}}$

То:

1. $\hat{\beta}_{МНК}$ сущ-т и единств.
2. $\hat{\beta}_{МНК}$ линейны по y $\left(\hat{\beta} = \underbrace{(X^TX)^{-1}X^T}y\right)$
3. $\mathbb{E}(\hat{\beta}_{МНК}) = \beta$

4. $Var(\hat\beta_{мнк}) \le Var(\hat\beta')$, $\hat\beta'$ — люб. др. лин. по y, несм. оц-а

Мегаматрица

(ТГМ вштол)

$y = X\beta + \varepsilon$

$\hat y = X\hat\beta$

	y	ε	$\hat y$	$\hat\beta$	$y-\hat y$
y	$Var(y)$	$\cdot$		$cov(y, \hat\beta)$	
ε	$cov(\varepsilon, y)$	$Var(\varepsilon)$			
$\hat y$	$\cdot$				
$\hat\beta$				$\sigma^2 (X^T X)^{-1}$	
$y-\hat y$					
$\mathbb{E}(\cdot)$	$X\beta$	0		β	

$$\mathbb{E}(y) = \mathbb{E}(X\beta + \varepsilon) = X\beta$$

$$Var(\hat\beta) = (X^T X)^{-1} X^T \underbrace{Var(\varepsilon)}_{\sigma^2 \cdot I_{n\times n}} X (X^T X)^{-1} =$$

$$= \sigma^2 \cdot \cancel{(X^T X)^{-1}} \cancel{X^T X} (X^T X)^{-1} = \sigma^2 (X^T X)^{-1}$$