University of Missouri

CS 8001 Quantum Computer Science

Quantum Game Theory

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Review of Relevant Concepts from Classical Game Theory

Normal-form Game

Normal-form Game [6]

A game in normal form is a tuple (P, S, u) where

- $P = \{1, 2, \ldots, n\}$ is a finite set of players numbered with natural numbers.
- $S_i = (s_1, \ldots, s_k)$ is a set of (pure) strategies for player i.
  Here we assume S_i is finite.
- $S = \{(s_1, \ldots, s_n) | s_1 \in S_1, \ldots, s_n \in S_n\}$ is the set of strategy profiles.
- $u_i : S \to \mathbb{R}$ is a payoff function for player i.
  $u : S \to \mathbb{R}^n = (u_1, \ldots, u_n)$ is a payoff vector.

Example: Bach or Stravinsky

Two players want to attend a concert and have to choose between two composers. Player One would prefer to go to Bach, and Player Two would prefer Starvinksy. Both would prefer to go anywhere together rather than going alone to the personally preferred event.

$$
\begin{array}{cc|c|c|}
 & & \multicolumn{2}{c}{P_2} \\
 & & \multicolumn{1}{c}{B} & \multicolumn{1}{c}{S} \\
\cline{3-4}
\multirow{2}{*}{P_1} & B & (3,5) & (0,0) \\
\cline{3-4}
 & S & (0,0) & (4,2) \\
\cline{3-4}
\end{array}
$$

Here $P = \{P_1, P_2\}$, $S_{P_1} = S_{P_2} = \{B, S\}$, and $S = \{(B,B), (B,S), (S,B), (S,S)\}$.

Nash Equilibrium in Pure Strategies

Nash Equilibrium in Pure Strategies

Let $G = (P, S, u)$ be a normal-form game. Then $s^* \in S$ is a Nash equilibrium in pure strategies if $\forall i$ and $\forall s_i' \in S$

$$u_i(s_i^*, s_{-i}^*) \geq u_i(s_i', s_{-i}^*)$$

P_2

		B	S
P_1	B	$(3, 5)$	$(0, 0)$
	S	$(0, 0)$	$(4, 2)$

- (B, B), (S, S) are two Nash equilibria in pure strategies.
- Intuitively, no player wants to change their strategy if other players' strategies are fixed.

Nash Equilibrium in Mixed Strategies

- Assume a player has access to an RNG that decides which strategy to play.
- Probability distribution $Pr_i = (pr_1, \ldots, pr_k)$ on the set of pure strategies of player i is called a **mixed strategy**.
- Other players know probabilities but are unaware of the final strategy choice.

Nash Equilibrium in Mixed Strategies

Let $G = (P, S, u)$ be a normal-form game. Let us call a game $G' = (P, \prod_i Pr_i, u)$ the **mixed extension** of G. The Nash equilibrium in mixed strategies of G is the Nash equilibrium of G'.

Nash Equilibrium in Mixed Strategies

$$
\begin{array}{cc|c|c|}
 & & \multicolumn{2}{c}{P_2} \\
 & & B & S \\
\cline{3-4}
\multirow{2}{*}{P_1} & B & (3,5) & (0,0) \\
\cline{3-4}
 & S & (0,0) & (4,2) \\
\cline{3-4}
\end{array}
$$

Let P_2 play B with probability p and S with probability $1-p$. Nash equilibrium is achieved when P_1 is indifferent between their pure strategies.

$$3p + 0(1-p) = 0p + 4(1-p) \Rightarrow p = 4/7.$$

Nash Equilibrium in Mixed Strategies

$$
\begin{array}{cc|c|c|}
 & & \multicolumn{2}{c}{P_2} \\
 & & B & S \\
\cline{3-4}
\multirow{2}{*}{P_1} & B & (3, 5) & (0, 0) \\
\cline{3-4}
 & S & (0, 0) & (4, 2) \\
\cline{3-4}
\end{array}
$$

In the same way we can find

$$5q + 0(1-q) = 0q + 2(1-q) \Rightarrow q = 2/7$$

Thus, the pair

$$(2/7; 5/7);\ (4/7, 3/7)$$

is the Nash equilibrium in mixed strategies for this game.

Nash Equilibrium in Mixed Strategies

		P_2	
		B	S
P_1	B	$(3,5)$	$(0,0)$
	S	$(0,0)$	$(4,2)$

- The same result can be obtained with

$$u_A(q|p) = 3pq + 4(1-p)(1-q) \to \max_q$$

$$u_B(p|q) = 5pq + 2(1-p)(1-q) \to \max_p$$

Existence of Nash Equilibrium

- "Matching Pennies" game: two players secretly turn the penny to heads or tails. If the pennies match, P_1 wins and P_2 loses and vice versa.

		P_2	
		H	T
P_1	H	(1, -1)	(-1, 1)
	T	(-1, 1)	(1, -1)

- Nash equilibrium in pure strategies does not exist. However, we can show that there is a Nash equilibrium in mixed strategies: $(1/2, 1/2);\ (1/2; 1/2)$.

Theorem (Nash, 1950)

Every finite-state game in normal form has at least one Nash equilibrium in mixed strategies.

Prisoner's Dilemma

Each of two players, Alice and Bob, must independently decide whether they choose to defect (D) or cooperate (C). [1], [3]

$$
\begin{array}{cc|c|c|}
 & & \multicolumn{2}{c}{B} \\
 & & C & D \\
\cline{3-4}
\multirow{2}{*}{A} & C & (3,3) & (0,5) \\
\cline{3-4}
 & D & (5,0) & (1,1) \\
\cline{3-4}
\end{array}
$$

(D, D) is the only Nash equilibrium in pure (and mixed) strategies.

Pareto-Optimality

Pareto-Optimality

Let $G = (P, S, u)$ be a normal-form game. Then a strategy profile $s \in S$ is Pareto-optimal, if for all $i \in P$ there is no such s' that

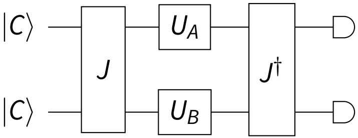
$$u_i(s') \geq u_i(s)$$

and at least for one i the inequality is "greater".

- Intuitively, we cannot make one player better off without harming somebody else.
- For the Prisoner's Dilemma, (D, D) is the only profile which is not Pareto-optimal.

Quantum Formulation of the Prisoner’s Dilemma

Game Circuit [3]



- Consider a system of two qubits $|C\rangle$, $|D\rangle$, corresponding to classical **outcomes** C and D. The system starts in $|CC\rangle$.
- The game's intial state vector is $|\psi_0\rangle = J\,|CC\rangle$.
- The game's final state which is measured is

$$|\psi_f\rangle = J^\dagger (U_A \otimes U_B) J\,|CC\rangle\,.$$

Strategies

- U_A and U_B are unitary operators which represent strategic moves.
- It is sufficient to restrict

$$U(\theta, \phi) = \begin{bmatrix} e^{i\phi}\cos(\frac{\theta}{2}) & \sin(\frac{\theta}{2}) \\ -\sin(\frac{\theta}{2}) & e^{-i\phi}\cos(\frac{\theta}{2}) \end{bmatrix}$$

  with $0 \leq \theta \leq \pi$ and $0 \leq \phi \leq \frac{\pi}{2}$.
- Specifically,

$$\mathcal{C} = U(0,0) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \mathcal{D} = U(\pi, 0) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

Payoffs

		B: C	B: D
A	C	$(3,3)$	$(0,5)$
	D	$(5,0)$	$(1,1)$

- The expected payoffs are given by

$$u_A = 3Pr_{CC} + 0Pr_{CD} + 5Pr_{DC} + 1Pr_{DD}$$

$$u_B = 3Pr_{CC} + 5Pr_{CD} + 0Pr_{DC} + 1Pr_{DD}$$

with $P_{ss'} = |\langle ss'|\psi_f\rangle|^2$.

Entanglement

- Players are allowed to entangle their qubits. The entangling operator is [5]

$$J = \cos\left(\frac{\gamma}{2}\right) I \otimes I + i \sin\left(\frac{\gamma}{2}\right) D \otimes D.$$

  where $\gamma \in [0, \frac{\pi}{2}]$ is entanglement strength.
- To guarantee faithful representation of the classical game, we require

$$J[U(\theta, 0) \otimes U(\theta', 0)] = [U(\theta, 0) \otimes U(\theta', 0)]J$$

  for all $\theta, \theta' \in [0, \pi]$.

Game Simulation Procedure

- The game can be solved explicitly by fixing (γ) and optimizing for $(\theta_A, \phi_A, \theta_B, \phi_B)$.
- Exact solutions are messy, e.g., for $\gamma = 0$

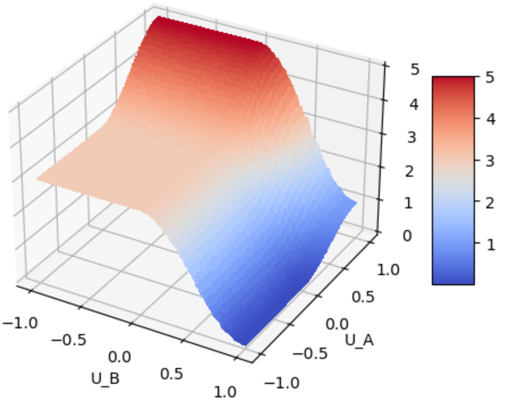
$$P_{CC} = \left| \cos\left(\phi_A + \phi_B\right) \cos\left(\frac{\theta_A}{2}\right) \cos\left(\frac{\theta_B}{2}\right) \right|^2$$

- So, we will simulate the game (1000 sims) and compute expected payoffs using `Qiskit`.
- Strategies are parameterized by $-1 \leq t \leq 1$,

$$U(t) = \begin{cases} U(0, -t\frac{\pi}{2}), & -1 \leq t < 0, \\ U(t\pi, 0), & 0 \leq t \leq 1 \end{cases}, \ \mathcal{D} = U(1),\ \mathcal{C} = U(0).$$

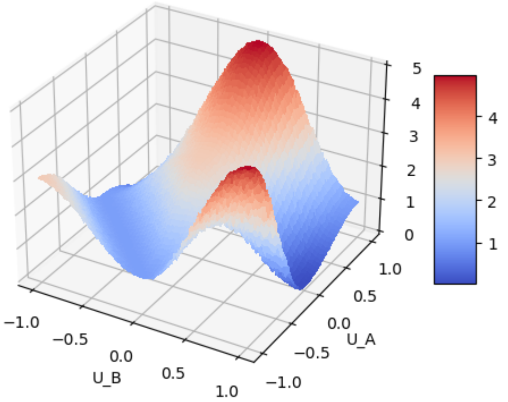
Classical Setting ($\gamma = 0$)

For any $U_B(t)$, $\arg\max_t U_A(t) = 1$, i.e., $U_A = \mathcal{D}$. Bob is symmetric, so $(\mathcal{D}, \mathcal{D})$ is the only Nash equilibrium.



Maximum Entanglement ($\gamma = \frac{\pi}{2}$)

- If Bob plays $\mathcal{C} = U(0)$, Alice should play $\mathcal{D} = U(1)$ as in the classical setting.
- However, if Bob plays $\mathcal{D} = U(1)$, Alice should play $\mathcal{Q} = U(-1)$.



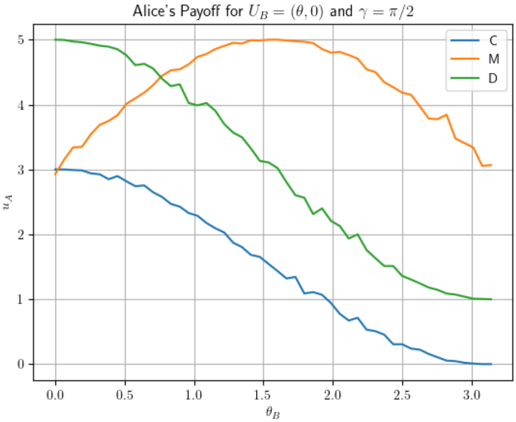
Analyzing $\mathcal{Q}$

$$\mathcal{Q} = U(-1) = U\left(0, \frac{\pi}{2}\right) = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}.$$

- Surprisingly, $\mathcal{D} \otimes \mathcal{D}$ is no longer a Nash equilibrium. The new equilibrium is $\mathcal{Q} \otimes \mathcal{Q}$. **Proof:** we can show $u_A(\mathcal{Q}, \mathcal{Q}) = 3$ and $u_A(U(\theta, \phi), \mathcal{Q}) \leq 3$ for all allowed (θ, ϕ). Bob is symmetric, so no player will deviate from Q if the other one plays Q.
- Moreover, $\mathcal{Q} \otimes \mathcal{Q}$ is Pareto-optimal. Thus, quantum strategies allow players to escape the dilemma.
- $\gamma_{\text{min}} \approx 0.685$. [2]

Unfair Game

- Assume Alice is allowed to use quantum strategy, but Bob is limited to apply only classical mixed strategies.
- Alice has the strategy $M = U\left(\frac{\pi}{2}, \frac{\pi}{2}\right)$ which ensures her payoff is always at least 3.
- Advantage monotonically depends on γ.



Quantum Formulation of Matching Pennies

Zero-sum Game

Zero-sum Game

A game $G = (P, S, u)$ is called a zero-sum game if for all $s \in S$

$$\sum_{p \in P} u_p(s) = 0.$$

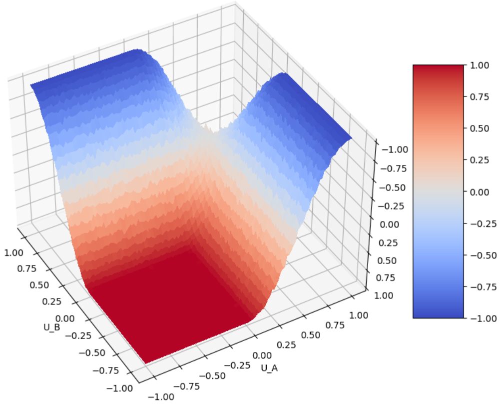
If $|P| = 2$, such a game is also called antagonistic.

Example: Matching Pennies.

$$\begin{array}{cc|c|c|} & & \multicolumn{2}{c}{P_2} \\ & & H & T \\ \hline P_1 & H & (1, -1) & (-1, 1) \\ \hline & T & (-1, 1) & (1, -1) \\ \hline \end{array}$$

Equilibria Analysis

- Payoff space does not change with different entanglement values.
- In the classical region, there are no equilibria in pure strategies and one equilibrium, $(0.5, 0.5), (0.5, 0.5)$, in mixed strategies.
- All pairs with one player utilizing classical mixture and another player utilizing quantum mixture are equilibria.
- Quantum advantage still holds. [4]



Conclusions

- Quantum strategies allow for new types of equilibria which solve classical paradoxes.
- NE in mixed strategies $\subset$ NE in quantum strategies?
- It is interesting to look at dynamic games.

Literature I

[1] Robert Axelrod and William D Hamilton. “The evolution of cooperation”. In: *science* 211.4489 (1981), pp. 1390–1396.

[2] Jiangfeng Du et al. “Experimental realization of quantum games on a quantum computer”. In: *Physical Review Letters* 88.13 (2002), p. 137902.

[3] Jens Eisert, Martin Wilkens, and Maciej Lewenstein. “Quantum games and quantum strategies”. In: *Physical Review Letters* 83.15 (1999), p. 3077.

[4] Edward W Piotrowski and Jan Sładkowski. “An invitation to quantum game theory”. In: *International Journal of Theoretical Physics* 42 (2003), pp. 1089–1099.

[5] Marek Szopa. “Efficiency of classical and quantum games equilibria”. In: *Entropy* 23.5 (2021), p. 506.

Literature II

[6] Alexei Zaharov. *Game Theory in Social Sciences*. Litres, 2022.

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