Computational Practicum Report

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1 Exact Solution of Differential Equation

Given Initial Value Problem (IVP):

$$\begin{cases} y' = f(x, y) \\ y(x_0) = y_0 \\ x \in (x_0; X) \end{cases}$$

In my case I obtain the following IVP:

$$\begin{cases} y' = \frac{y^2 - y}{x} \\ y(1) = \frac{1}{2} \\ x \in (1; 9) \end{cases}$$

1.1 Find General Solution

1.1.1 $y' = \frac{y^2-y}{x}$ - non-linear first order differential equation.

This equation is equivalent to differential equation in separable form.
Since there is a division by $x \Rightarrow x \neq 0$

1.1.2 Let's consider border cases: $1)\ y = 0,\ 2)\ y = 1$ on $\mathbf{R}/\{0\}$

1) $y = 0 \Rightarrow y' = 0$
   Check: $0 = \frac{0}{x} \Rightarrow 0 = 0$; Therefore $y = 0$ is a trivial solution on $\mathrm{R}/\{0\}$
2) $y = 1 \Rightarrow y' = 0$
   Check: $0 = \frac{1-1}{x} \Rightarrow 0 = 0$; Therefore $y = 1$ is a trivial solution on $\mathrm{R}/\{0\}$

1.1.3 Assume $y \neq 0$ and $y \neq 1$:

$$\frac{y'}{y^2 - y} = \frac{1}{x}$$

Let's bring differential equation to differential form:

$$\frac{dy}{y^2 - y} = \frac{dx}{x}$$

Let's integrate both parts:

$$\int \frac{dy}{y^2 - y} = \int \frac{dx}{x}$$

$$\int \left(\frac{1}{y-1} - \frac{1}{y} \right) dy = \ln|x| + \ln|C|, \;\; C - const \in R/\{0\}$$

$$\ln|y-1| + \ln|y^{-1}| = \ln|xC|$$

$$(y-1)y^{-1} = xC$$

Let's represent equation in explicit form:

$$y = \frac{1}{1 - Cx}, \; x \neq \frac{1}{C}$$

1.2 Let's move on to the solution of initial value problem:

$$\begin{cases} y = \frac{1}{1-Cx} \\ y(1) = \frac{1}{2} \end{cases} \Rightarrow \frac{1}{2} = \frac{1}{1-C} \Rightarrow C = -1, \; x \neq -1$$

1.3 Answer: $y = \frac{1}{x+1}$ on $R/\{-1, 0\}$

2 Analysis of points of discontinuity

The equation contains two points of discontinuity:

1) $x = -1$:

$$L^+ = \lim_{x \to -1^+} \frac{1}{x+1} = \infty$$

$$L^- = \lim_{x \to -1^-} \frac{1}{x+1} = -\infty$$

Both L^- and L^+ don't exist so $x = -1$ is point of infinite discontinuities

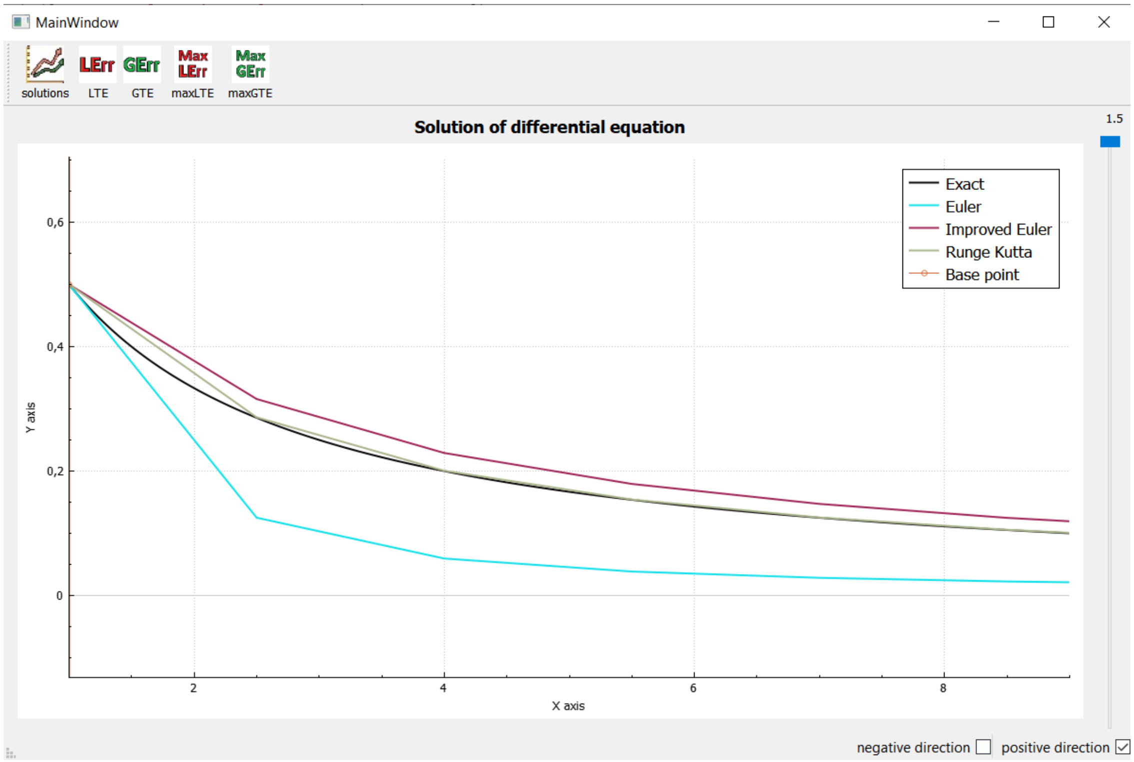
2) $x = 0$:

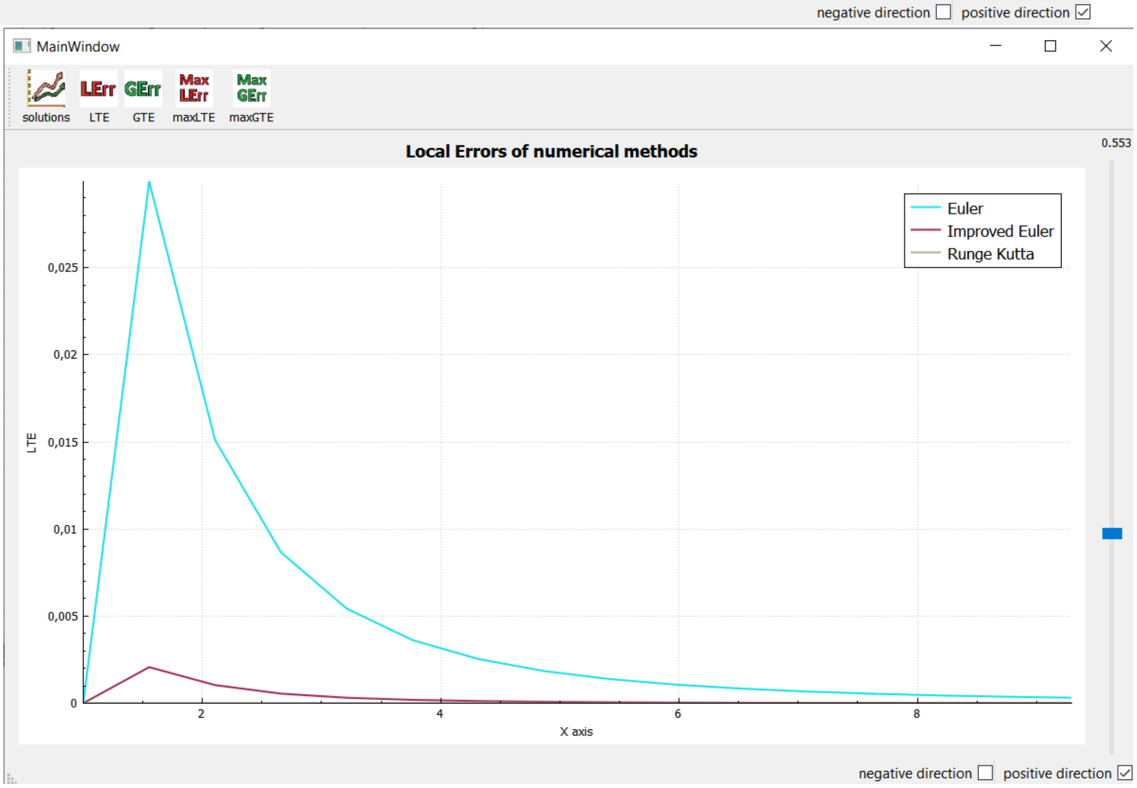
$$L^+ = \lim_{x \to 0^+} \frac{1}{x+1} = 1$$

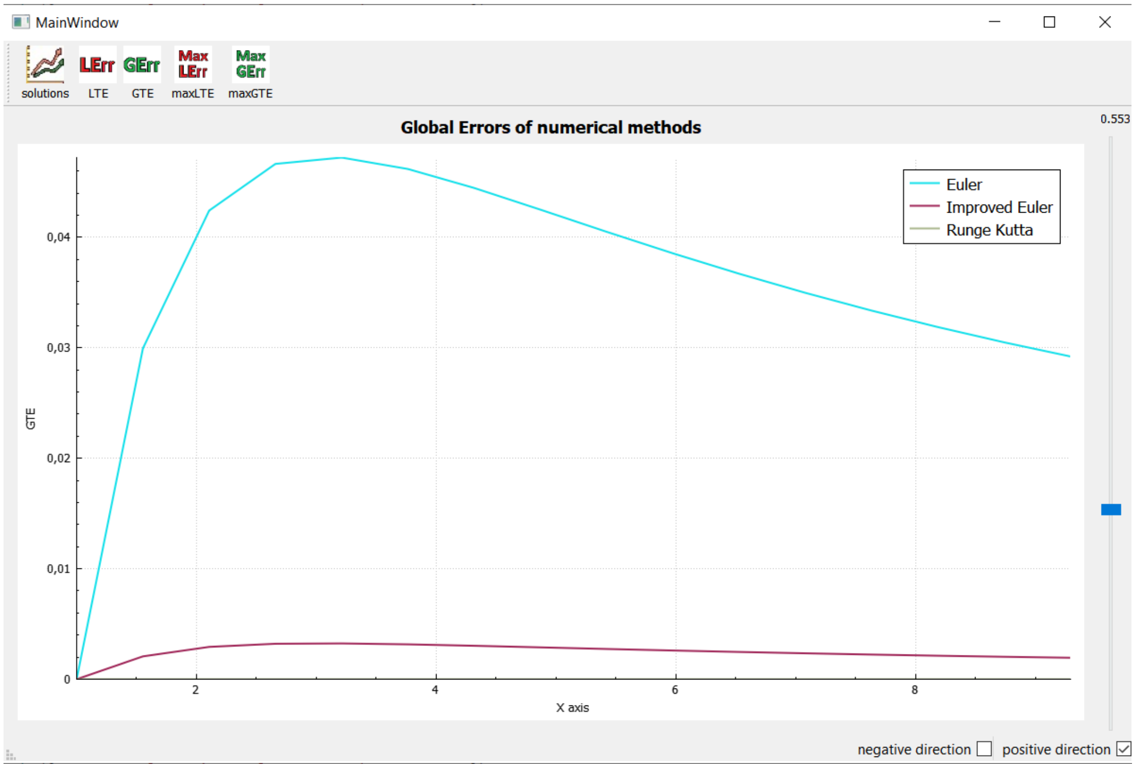
$$L^- = \lim_{x \to -1^-} \frac{1}{x+1} = 1$$

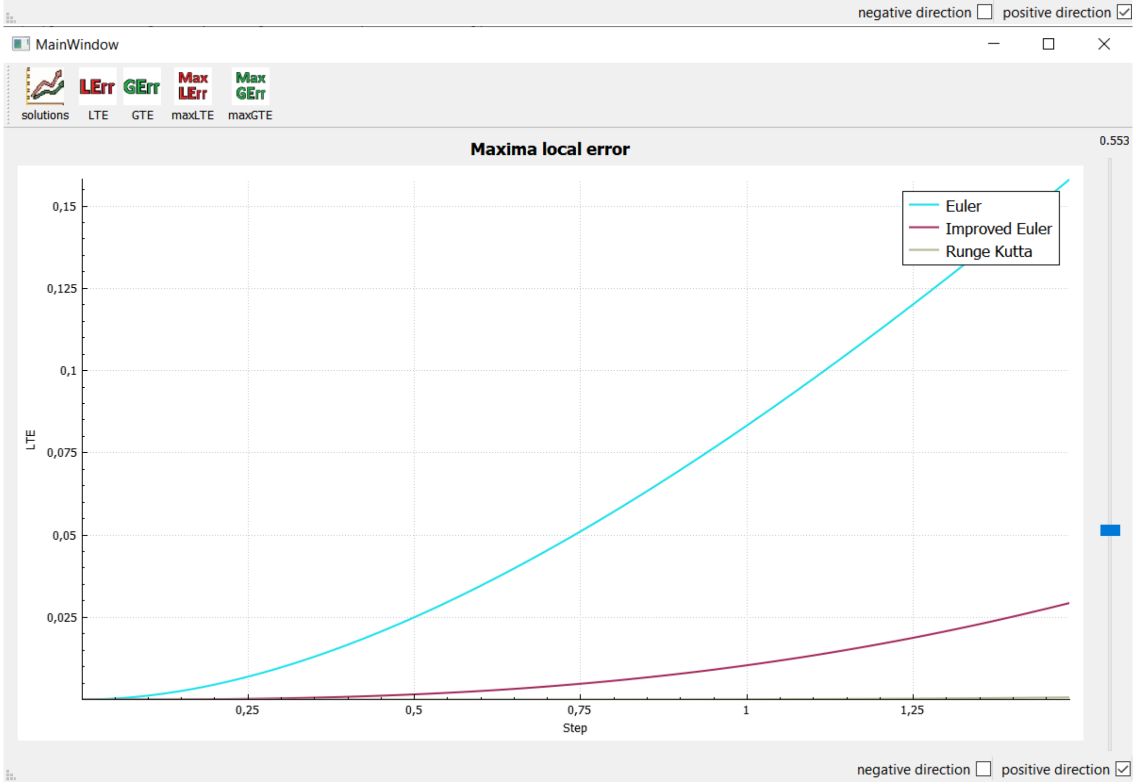
Both L^- and L^+ exist and equal so $x = 0$ is point of removable discontinuities

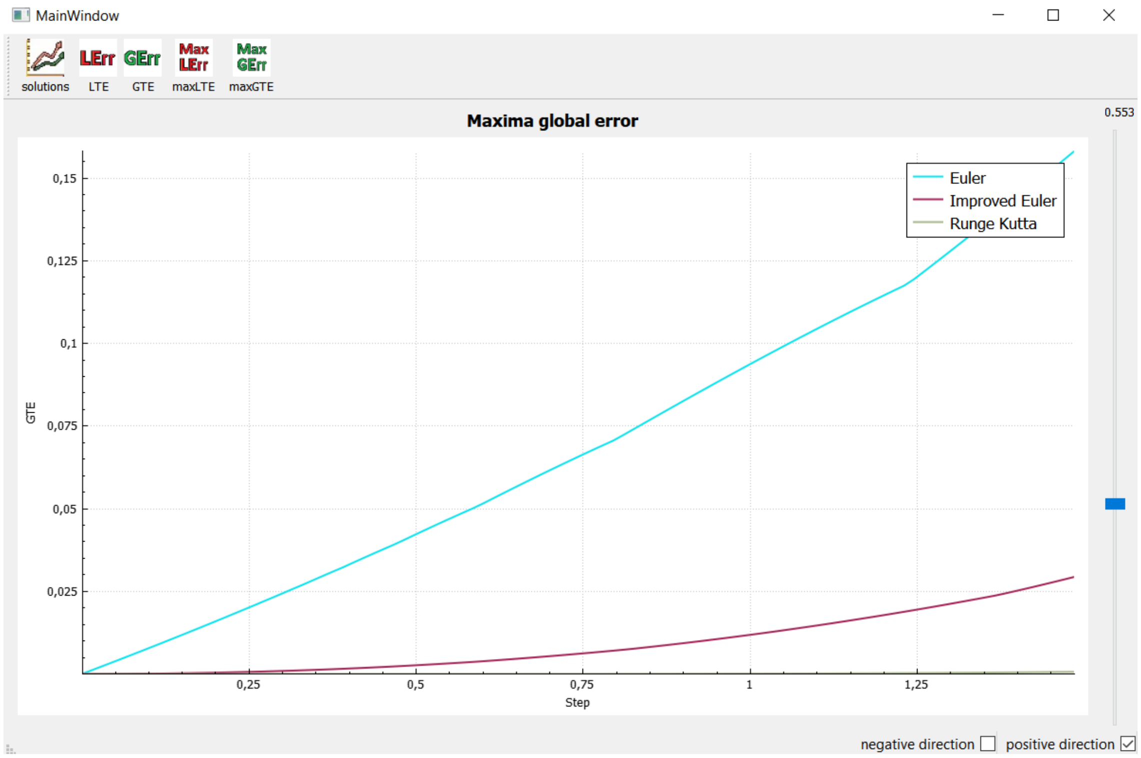
3 Computed Solution of Initial Value Problem (Graphs)





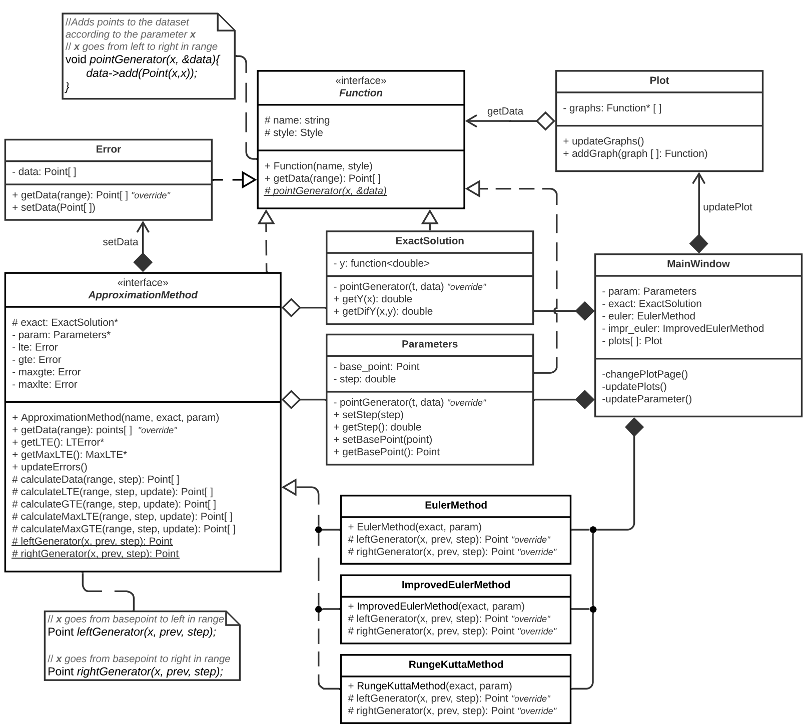

MainWindow
solutions
LErr
LTE
GErr
GTE
Max LErr
maxLTE
Max GErr
maxGTE
Global Errors of numerical methods
0.553
Euler
Improved Euler
Runge Kutta
GTE
0,04
0,03
0,02
0,01
0
2
4
6
8
X axis
negative direction
positive direction


MainWindow
solutions
LErr
LTE
GErr
GTE
Max LErr
maxLTE
Max GErr
maxGTE
Maxima local error
0.553
Euler
Improved Euler
Runge Kutta
LTE
0,15
0,125
0,1
0,075
0,05
0,025
0,25
0,5
0,75
1
1,25
Step
negative direction
positive direction

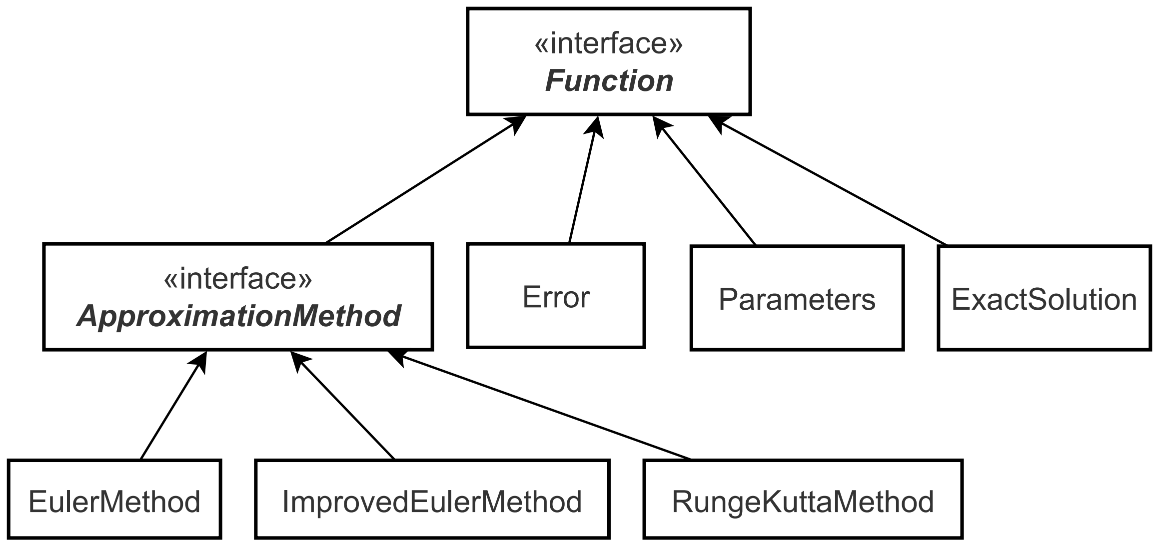


4 Code Comments

4.1 UML class diagram



4.2 Class Hierarchy



4.3 Used libraries

Qt libraries were used to display the window. In addition, the QCustomPlot library was used to build the graphs.

4.4 Class explanation

Interface layer:

- MainWindow: main class, it configures the relationships of objects and processes user actions.

Drawer layer:

- Plot: responsible for displaying functions.
- Function: provides an interface for creating graphs. It is enough to define the pointGenerator function.
- ApproximationMethod: provides an interface for creating numerical approximation methods. It is enough to define the functions leftGenerator and rightGenerator.

Graphs layer:

- ExactSolution: stores information about the function y.
- Parameters: stores information for numerical methods.
- Error: auxiliary class to display numerical method errors.
- Methods(3): define the way a differential equation is approximated.

4.5 Expanding and changing functionality

4.5.1 Replacing a estimated function

To replace the function to be evaluated, it is enough to change the formulas for the derivative and its solution, and to specify the points of discontinuity.

```
ExactSolution::ExactSolution()
    :Function("Exact", QColor(10,10,10))
{
    y    = [](double x)        { return 1./(x+1); };
    difY = [](double x, double y){ return y*(y-1)/x;};
    infDisc  = {-1}; // Infinite discontinuities points
    otherDisc = {0}; // all others discontinuities points
}
```

4.5.2 Adding a new numerical method

To add a new numerical method, you need to inherit ApproximationMethod and define two methods: leftGenerator and rightGenerator

```
QCPGraphData EulerMethod::leftGenerator(double t, const QCPGraphData& prev, double step) const {
    double k1 = eq->getDifY(prev);
    return {t, prev.value - step*k1};
}
QCPGraphData EulerMethod::rightGenerator(double t, const QCPGraphData& prev, double step) const {
    double k1 = eq->getDifY(prev);
    return {t, prev.value + step*k1};
}
```