NAME OF CANDIDATE: ....

STUDENT NUMBER: ....

THE UNIVERSITY OF NEW SOUTH WALES

SCHOOL OF MATHEMATICS AND STATISTICS

Midsession Examination, 2018

MATH2901
HIGHER THEORY OF STATISTICS

(1) TIME ALLOWED – 50 minutes

(2) TOTAL NUMBER OF QUESTIONS – 4

(3) ANSWER ALL 4 QUESTIONS

(4) THE QUESTIONS ARE OF EQUAL VALUE

(5) THIS PAPER MAY **NOT** BE RETAINED BY THE CANDIDATE

(6) **ONLY** CALCULATORS WITH AN AFFIXED “UNSW APPROVED” STICKER MAY BE USED

All answers must be written in ink. Except where they are expressly required pencils may only be used for drawing, sketching or graphical work.

1. [**10 marks**] Let X and Y be continuous random variables with joint density function:

$$f_{X,Y}(x,y) = \frac{1}{8}(x+y) \quad \text{for} \quad 0 < x < 2 \quad \text{and} \quad 0 < y < 2.$$

a) [**2 marks**] Determine the marginal density $f_Y(y)$.

b) [**3 marks**] Determine the conditional density function $f_{X|Y}(x|y)$.

c) [**2 marks**] Are X and Y independent? (Explain your answer.)

d) [**3 marks**] Determine the expected value $\mathbb{E}(X|Y=1)$.

2. **[10 marks]** A random point (X, Y) is uniformly distributed on the square with vertices $(1, 1)$, $(1, -1)$, $(-1, 1)$ and $(-1, -1)$.

a) **[1 marks]** Explain why the joint probability density function of (X, Y) is given by $f_{X,Y}(x, y) = \frac{1}{4}$, for $x \in (-1, 1)$ and $y \in (-1, 1)$.

b) **[9 marks]** By drawing the corresponding region in $\mathbb{R}^2$, determine the probabilities of the following events

i) **[3 marks]** $X^2 + Y^2 < 1$

ii) **[3 marks]** $2X - Y > 0$

iii) **[3 marks]** $|X + Y| < 2$.

3. [**10 marks**]

a) [**4 marks**] Let X and Y be two independent binomial random variables, more specifically, $X \sim \text{Bin}(3, 0.4)$ and $Y \sim \text{Bin}(3, 0.4)$.

   i) [**1 marks**] Compute the probability $\mathbb{P}(X = 0)$.

   ii) [**3 marks**] Compute the probability $\mathbb{P}(XY = 0)$.

b) [**6 marks**] A continuous random variable X is said to be *symmetric* if X and $-X$ has the same cumulative distribution function. That is $F_X(x) = F_{-X}(x)$ for all $x \in \mathbb{R}$.

   On the other hand, a density function f is called *symmetric* if $f(x) = f(-x)$ for all $x \in \mathbb{R}$.

   i) [**2 marks**] Show that $F_{-X}(x) = 1 - F_X(-x)$.

   ii) [**4 marks**] Hence or otherwise deduce that a random variable X is *symmetric* if and only if the density of X given by f_X is *symmetric*.

Please see over . . .

4. [10 marks]

a) **[3 marks]** Let $X \sim \text{Uniform}(0,1)$, show that $-\log(X) \sim \exp(1)$.

b) **[2 marks]** Let $X_1 \sim \exp(\lambda)$ and $X_2 \sim \exp(\lambda)$, show that $X_1 + X_2 \sim \text{Gamma}(2, \lambda)$.

c) **[5 marks]** Given a sequence of i.i.d. (independent identically distributed) Uniform$(0,1)$ random variables $(X_1, \ldots, X_n)$. Show that the probability density function of $Y := \prod_{i=1}^{n} X_i$ is given by

$$f_Y(y) = \frac{(-\log(y))^{n-1}}{(n-1)!}, \quad y \in (0,1)$$

Table of some common distributions

X	mass/density function	domain	mean	variance	mgf $m_X(u) = E(e^{uX})$
Bernoulli	$P(X=1)=p$ $P(X=0)=q=1-p$	$\{0,1\}$	p	pq	$q+pe^u$
Binomial $Bin(n,p)$	$\binom{n}{k}p^k(1-p)^{n-k}$	$k=0,1,\dots,n$	np	$np(1-p)$	$(1-p+pe^u)^n$
Geometric	$p(1-p)^{k-1}$	$k=1,2,\dots$	$\frac{1}{p}$	$\frac{(1-p)}{p^2}$	$\frac{p}{e^{-u}-1+p}$
Poisson	$e^{-\lambda}\lambda^k/k!$	$k=0,1,\dots$	λ	λ	$\exp\{\lambda(e^u-1)\}$
Uniform	$(b-a)^{-1}$	$x\in(a,b)$	$\frac{1}{2}(a+b)$	$\frac{1}{12}(b-a)^2$	$\frac{e^{bu}-e^{au}}{u(b-a)}$
Exponential	$\frac{1}{\beta}e^{-\frac{1}{\beta}x}$	$x\in(0,\infty)$	β	β^2	$\frac{1}{1-\beta u}$
Normal $N(\mu,\sigma^2)$	$\frac{1}{\sqrt{2\pi\,\sigma^2}}\exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\}$	$x\in\{-\infty,\infty\}$	μ	σ^2	$e^{\mu u+\frac{1}{2}\sigma^2u^2}$
Gamma (α,β)	$\frac{e^{-x/\beta}x^{\alpha-1}}{\Gamma(\alpha)\beta^\alpha}$	$x\in(0,\infty)$	$\alpha\beta$	$\alpha\beta^2$	$\left(\frac{1}{1-\beta u}\right)^\alpha$
Beta $\beta(\alpha,\beta)$	$\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}x^{\alpha-1}(1-x)^{\beta-1}$	$x\in[0,1]$	$\frac{\alpha}{\alpha+\beta}$	$\frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$	

Useful Fact

$$\lim_{n\to\infty}(1+\frac{x}{n})^n = e^x$$

The Delta Method

Let $\hat\theta_1,\hat\theta_2,\dots$ be a sequence of estimators of θ such that

$$\frac{\hat\theta_n-\theta}{\sigma/\sqrt{n}}\longrightarrow\mathcal{N}(0,1)\quad\text{in distribution.}$$

Suppose the function g is differentiable at θ and $g'(\theta)\neq 0$. Then

$$\frac{g(\hat\theta_n)-g(\theta)}{g'(\theta)\sigma/\sqrt{n}}\longrightarrow\mathcal{N}(0,1)\quad\text{in distribution.}$$