# ICS 2022 Problem Sheet #2

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## Problem 2.1:

- Let $x$ and $y$ be real numbers
- $\forall x, y \in \mathbb{R}$
  if $y^3+yx^2 \leq x^3+xy^2$, then $y \leq x$

we prove this by contrapositive:

- Lets start with $y > x$ and try to get
  $(y^3+yx^2 > x^3+xy^2)$

$(x^2+y^2 \in \mathbb{R}^+)$

$$y > x \implies y(x^2+y^2) > x(x^2+y^2)$$

$$\implies y^3+yx^2 > x^3+xy^2$$

Since $y > x \implies y^3+yx^2 > x^3+xy^2$

then $\forall x, y \in \mathbb{R}$:

by contrapositive $y^3+yx^2 \leq x^3+xy^2 \implies y \leq x$

## Problem 2.2:

For all $n \in \mathbb{N}^*$, $\sum_{K=1}^{n}(2K-1)^2 = \frac{2n(2n-1)(2n+1)}{6}$

- we prove the following by induction

- for $n=1$:

$$1 = \frac{2(2-1)(2+1)}{6} \implies 1 = \frac{6}{6} \implies 1=1$$

therefore, the equation is true for $n=1$

- for some random number $n \in \mathbb{N}^*$
  we assume $(P_n)$ is true
- Lets verify for $(P_{n+1})$

$$(P_{n+1}) = \frac{2(n+1)(2(n+1)-1)(2(n+1)+1)}{6}$$

$$= \frac{(2n+2)(2n+1)(2n+3)}{6} = \frac{8n^3+24n^2+22n+6}{6}$$

$$= \frac{2n(2n-1)(2n+1)}{6} + 2n+1 = \sum_{K=1}^{n+1}(2K-1)^2$$

therefore, the equation is true for $n+1$

$(\forall n \in \mathbb{N}^*)$ by induction:

$$\sum_{K=1}^{n}(2K-1)^2 = \frac{2n(2n-1)(2n+1)}{6}$$

Problem 2.3:

- $f_1(n) = \frac{1}{2} n \log n$ ; $f_2(n) = n^2$ ; $f_3(n) = \sqrt{n^3}$

$f_4(n) = n^n$ ; $f_5(n) = 100n^2 + 10n^3$ ; $f_6(n) = 2\log n$

$f_7(n) = (n^2)^2 = n^4$ ; $f_8(n) = \log\log n$

a-) The increasing order concerning big-O membership of the above fct is: $f_8 < f_6 < f_1 < f_3 < f_2 < f_5 < f_7 < f_4$

b-) Lets first write out the two statments $f \in O(g)$ and $f \in O(h)$ means according to the definitions we have:

- $\exists (K_f \in \mathbb{N}$ and $n_f \in \mathbb{N})$ s.t $\forall n > n_f$ $f(n) \leq K_f \cdot g(n)$
- $\exists K_g, n_g \in \mathbb{N}. \forall n > n_g : g(n) \leq K_g \cdot h(n)$

We have shown that:

$\exists K_t, n_t \in \mathbb{N}. \forall n > n_t : f(n) \leq K_t \cdot h(n)$

We choose $K_t = K_f \cdot K_g$ and $n_t = \max\{n_g, n_f\}$

$\forall n > n_t$, the following holds:

$f(n) \leq K_f \cdot g(n)$ — definition of $f \in O(g)$ with $n \geq n_f$

$\leq K_f \cdot (K_g \cdot h(n))$ — definition of $g \in O(h)$ with $n \geq n_g$

$\leq (K_f \cdot K_g) \cdot h(n)$ — Commutativity of multiplication

$(K_f \cdot K_g = K_t)$ $\leq K_t \cdot h(n)$