Mathematical morphology

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Slides inspired from the ones of Isabelle Bloch

Mathematical morphology?

- Study of the image geometry (way the shapes are combined to form the images)

- To do that, we ask the question of the algebraic structure of the *set of images* :

    - We will here consider that this space is a lattice (treillis) : its elements are combined through the « supremum » et « infimum » operations

    - Why don't simply consider that this set is a vector space?

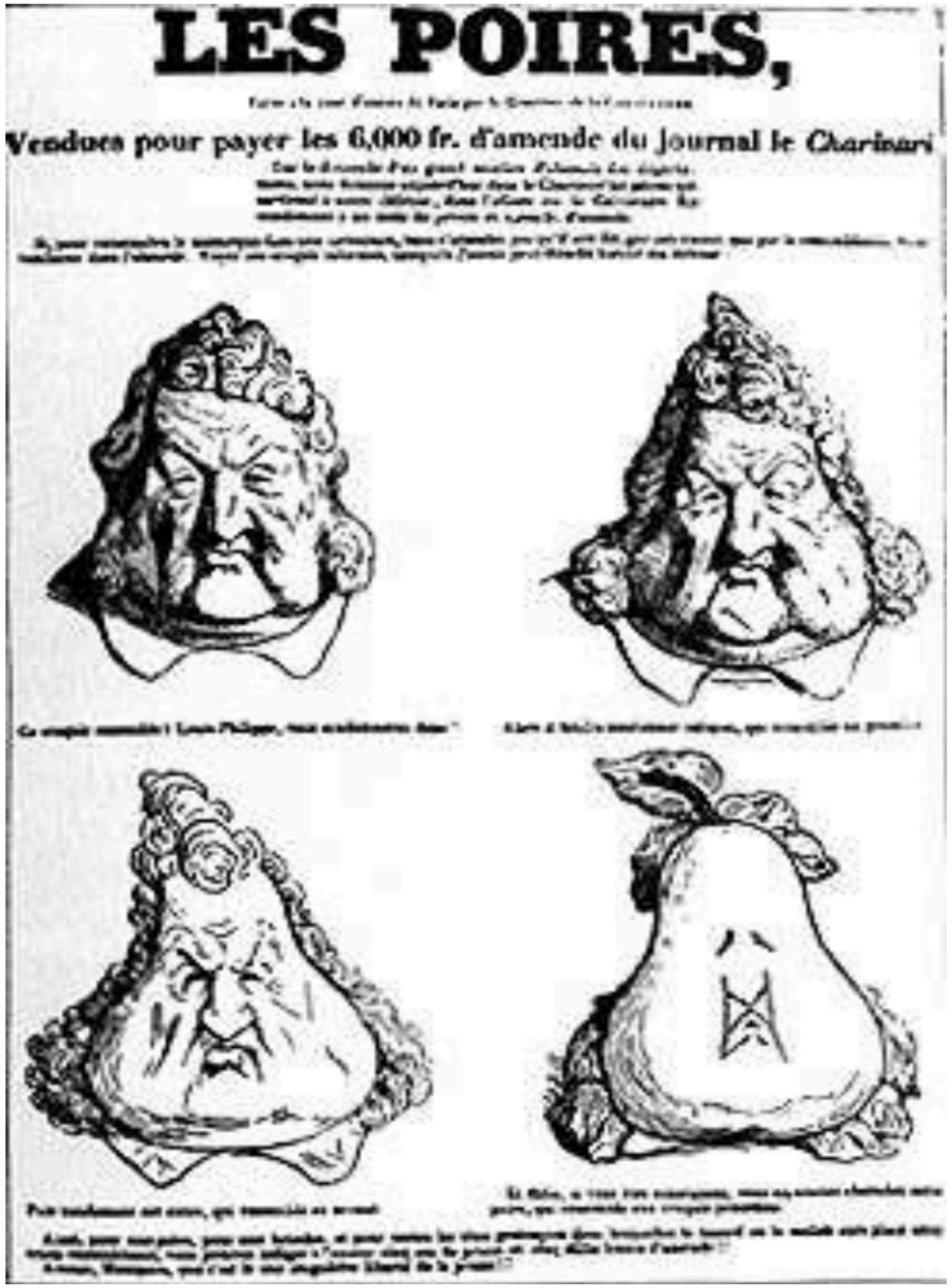
Mathematical morphology?

- Why not simply consider the set of images as a vector space?

- The images are generally correspond to 3-dimensional objects, which are projected into a 2D plane
- The luminances located along the perspective lines do not add to each other: they occlude each other => the linearity does not hold

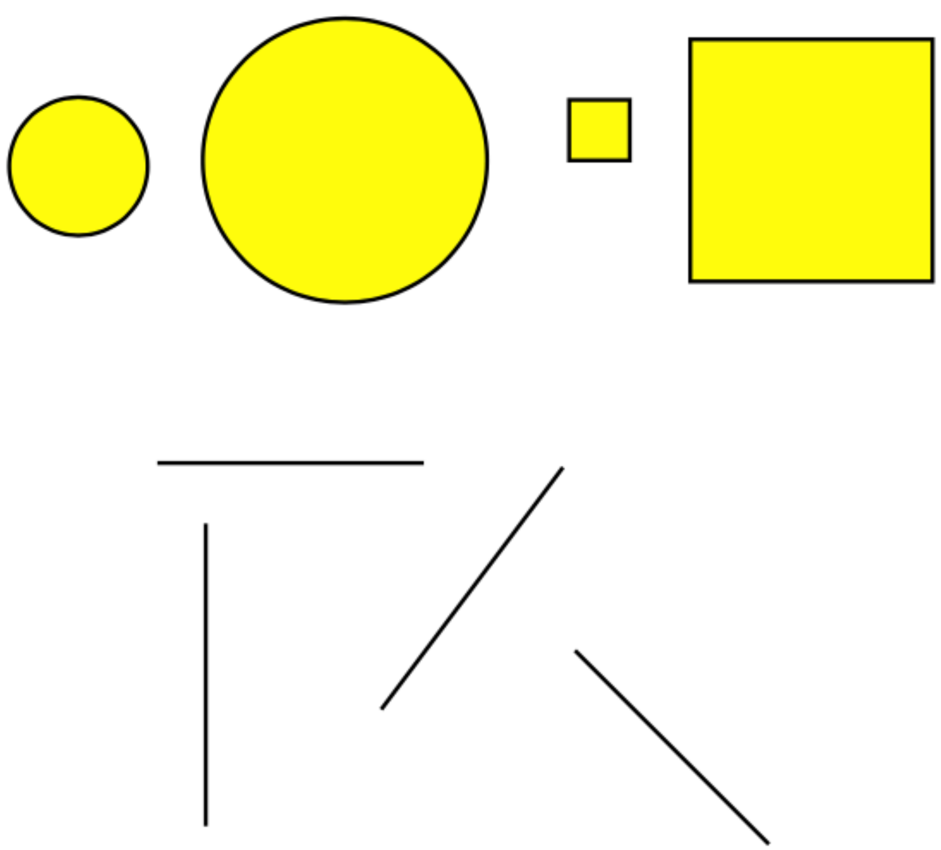
Interest of mathematical morphology?

- Simplify the images to retain only the relevant information

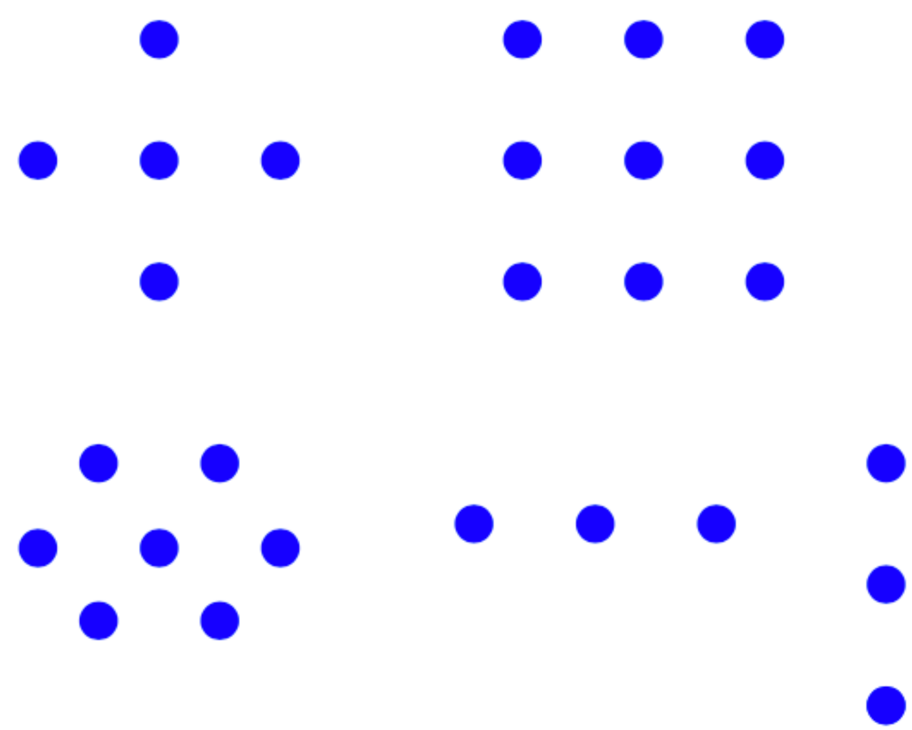


Structuring element

- Shape
- Size
- Origin (not necessarily in *B*)



- Continuous



- Discrete

Binary dilation

- Minkowski addition

$$X \oplus Y = \{x + y \mid x \in X, y \in Y\}$$

- Binary dilation

$$\begin{aligned} D(X,B) &= X \oplus B = \{x + y \mid x \in X, y \in B\} \text{ (ou } X \oplus \check{B} \text{ historiquement)} \\ &= \bigcup_{x \in X} B_x = \{x \in \mathbb{R}^n \mid \check{B}_x \cap X \neq \emptyset\} \end{aligned}$$

$\check{B}$ = symetric of *B*

Properties of binary dilation

- Extensive $(X \subseteq D(X,B))$ ssi $O \in B$
- Increasing $(X \subseteq Y \Rightarrow D(X,B) \subseteq D(Y,B))$

  $B \subseteq B' \Rightarrow D(X,B) \subseteq D(X,B')$

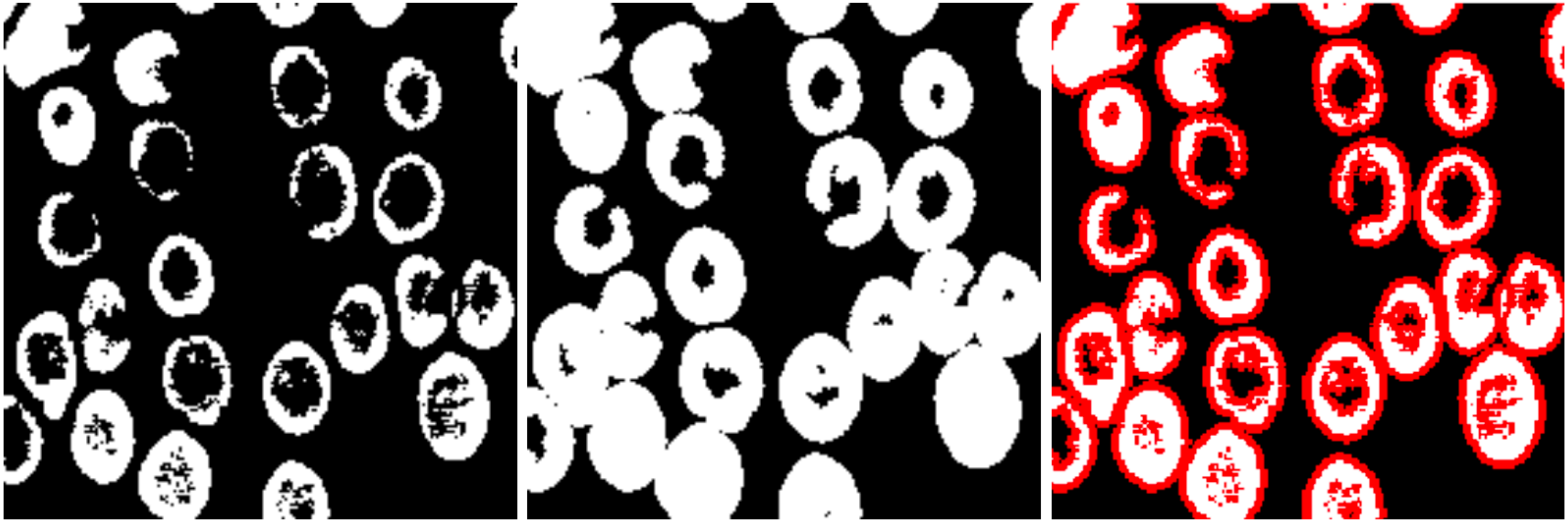
- Commute with union but not intersection

$$D(X \cup Y, B) = D(X,B) \cup D(Y,B), \quad D(X \cap Y, B) \subseteq D(X,B) \cap D(Y,B)$$

- Iteration property

$$D[D(X,B),B'] = D(X, B \oplus B').$$

Binary dilation example



Binary erosion

$$
\begin{aligned}
E(X,B) &= \{x \in \mathbb{R}^n \mid B_x \subseteq X\} \\
&= \{x \in \mathbb{R}^n \mid \forall y \in B, x+y \in X\} = X \ominus \check{B}
\end{aligned}
$$

Properties:

- Erosion and dilation duality w.r.t. complementation:

$$D[E(X,B), B'] \subseteq E[D(X,B'), B].$$

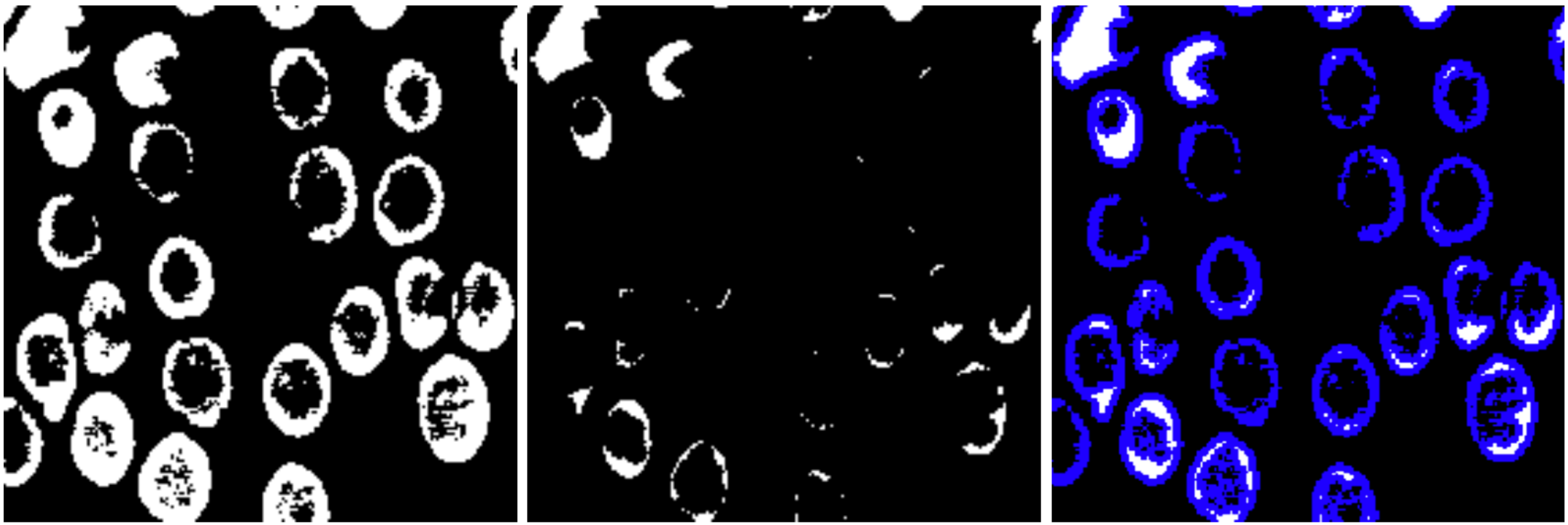
- **Anti**-extensive: $(E(X,B) \subseteq X)$ ssi $O \in B$
- Increasing: $(X \subseteq Y \Rightarrow E(X,B) \subseteq E(Y,B))$

  $B \subseteq B' \Rightarrow E(X,B') \subseteq E(X,B)$
- Commute with intersection but not union

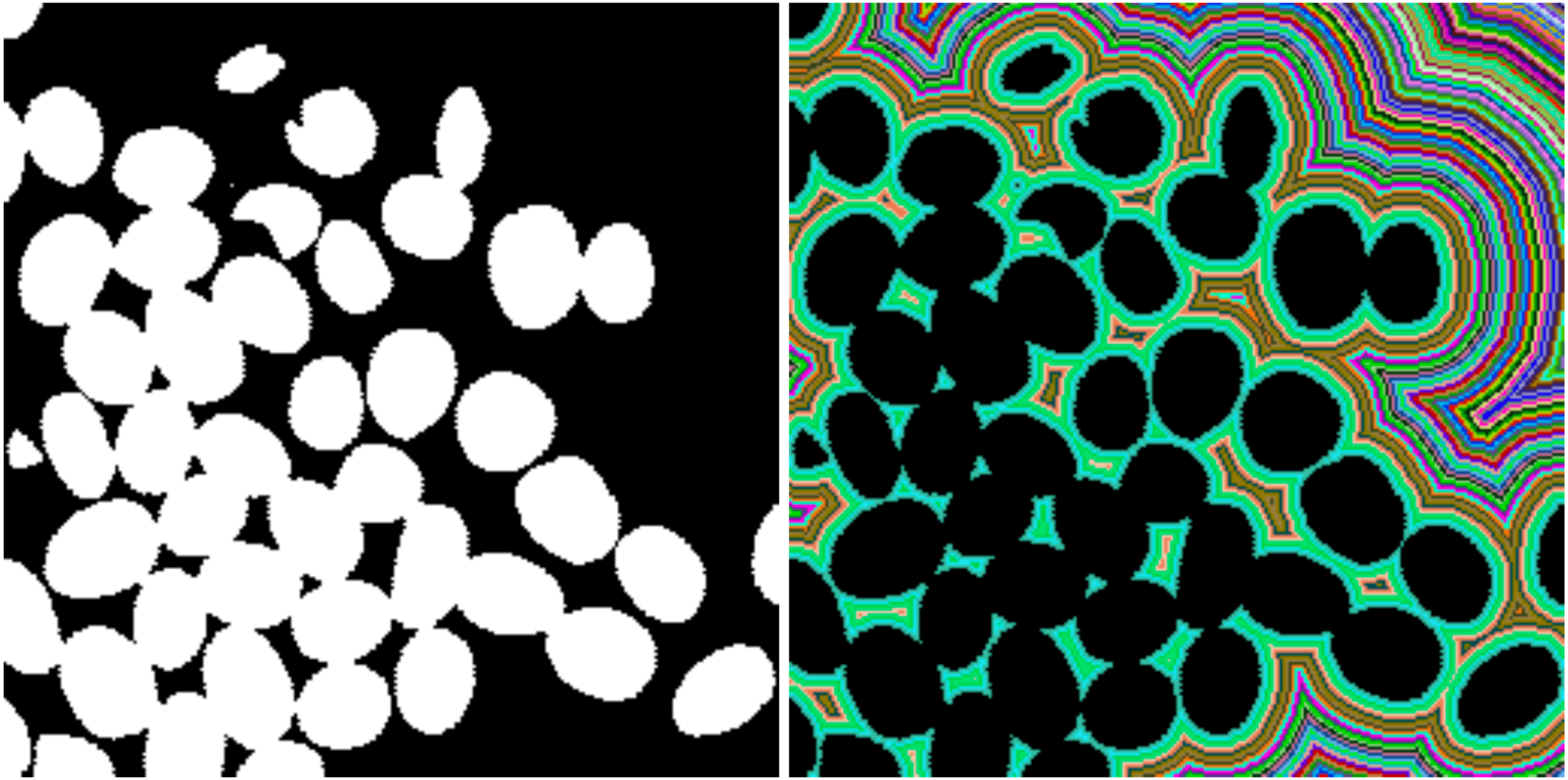
$$E[(X \cap Y), B] = E(X,B) \cap E(Y,B), \quad E[(X \cup Y), B] \supseteq E(X,B) \cup E(Y,B)$$

- Iteration property $E[E(X,B), B'] = E(X, B \oplus B').$
- $E(X,B) = [D(X^C, B)]^C$

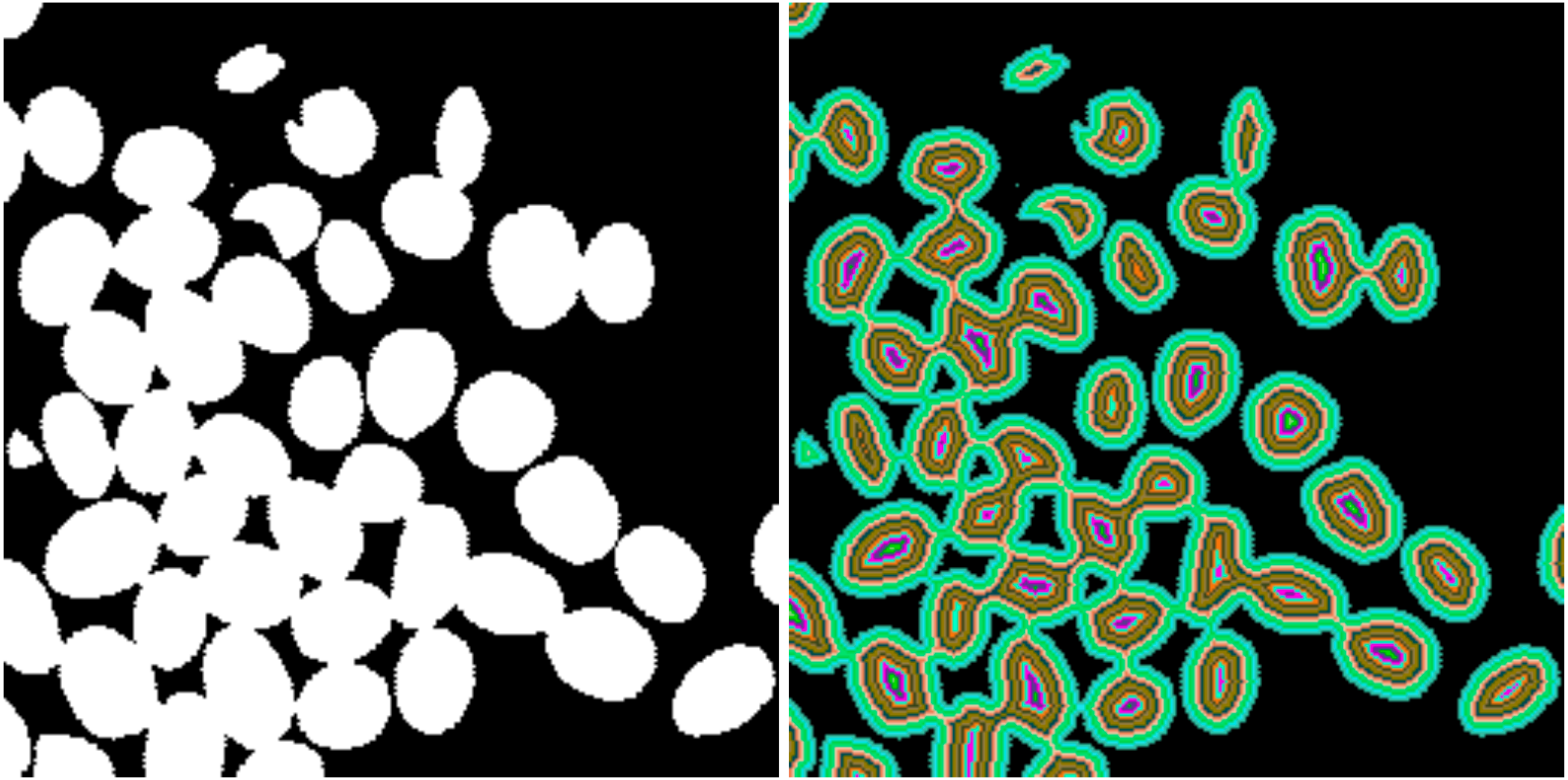
Example of binary erosion



Link with distances



Link with distances



Binary opening

$$\boxed{X_B = D[E(X,B),B]}$$

Properties:

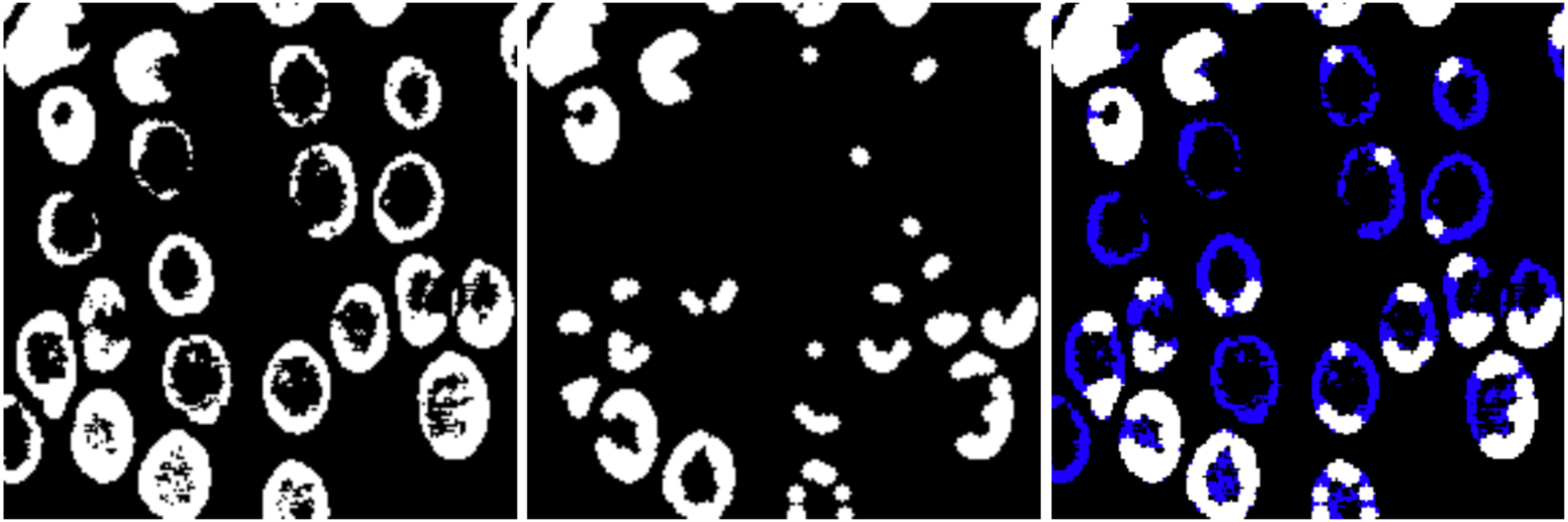
- **Anti**-extensive: $(X \supseteq X_B)$
- Increasing: $(X \subseteq Y \Rightarrow X_B \subseteq Y_B)$
- Idempotent: $((X_B)_B = X_B)$

=> Makes the binary aperture a morphological filter

$B \subseteq B' \Rightarrow X_{B'} \subseteq X_B$;

$(X_n)_{n'} = (X_{n'})_n = X_{\max(n,n')}$ with X_n the opening with a structuring element of size n

Example of aperture



Binary closing

$$\boxed{X^B = E[D(X,B),B]}$$

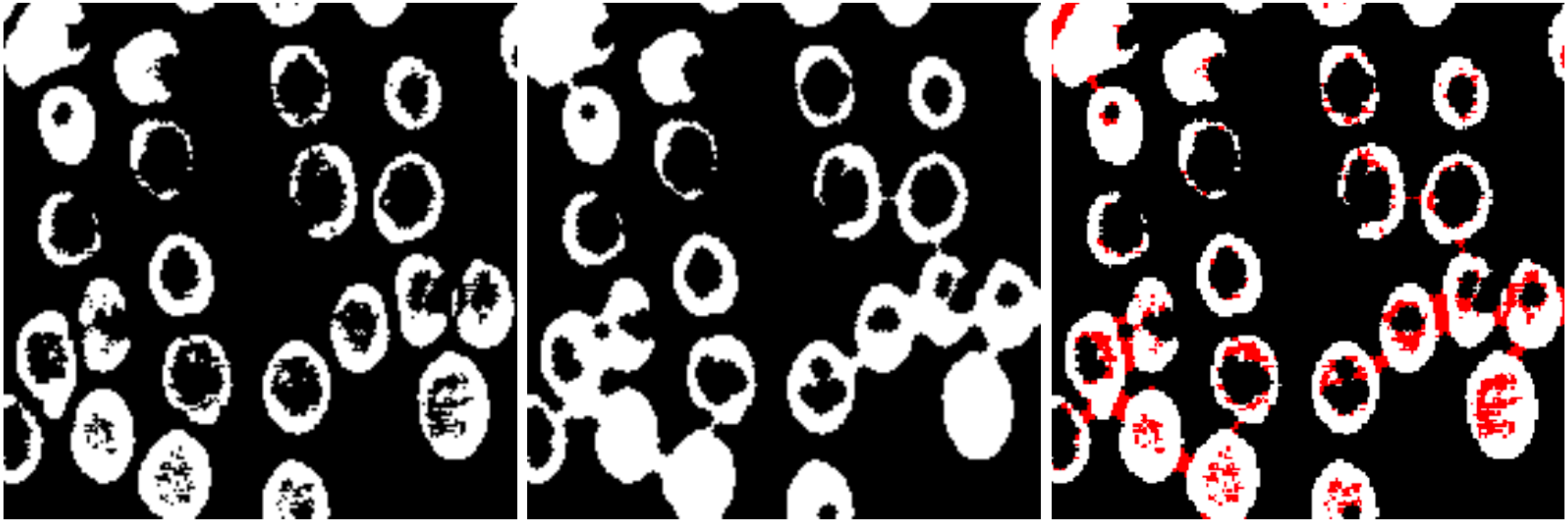
Properties:

- Extensive: $(X \subseteq X^B)$
- Increasing: $(X \subseteq Y \Rightarrow X^B \subseteq Y^B)$
- Idempotent: $((X^B)^B = X^B)$

=> Morphological filter

- $B \subseteq B' \Rightarrow X^B \subseteq X^{B'}$;
- $(X^n)^{n'} = (X^{n'})^n = X^{\max(n,n')}$;
- $X^B = [(X^C)_B]^C$.

Example of binary closing



From sets to functions

Two possibilities :

- **Use the subgraph of the function** : the sub-graph of a function of $\mathbb{R}^n$ is a subset of $\mathbb{R}^{n+1}$

=> this is often computationally expensive

- **Or use the functional counterparts of set operations :**

$$\begin{array}{ccl} \cup & \rightarrow & \sup / \vee \\ \cap & \rightarrow & \inf / \wedge \\ \subseteq & \rightarrow & \leq \\ \supseteq & \rightarrow & \geq \end{array}$$

Function (grayscale) dilation

$$\boxed{\forall x \in \mathbb{R}^n, \quad D(f,B)(x) = \sup\{f(y) \mid y \in \check{B}_x\}}$$

Properties:

- Extensive iff $O \in B$
- Increasing
- $D(f \vee g, B) = D(f,B) \vee D(g,B)$;
- $D(f \wedge g, B) \leq D(f,B) \wedge D(g,B)$;
- Iteration property

Example of function dilation

Function erosion

$$\boxed{\forall x \in \mathbb{R}^n, \quad E(f,B)(x) = \inf\{f(y) \mid y \in B_x\}}$$

Properties:

- Erosion and dilatation are dual operations
- Anti-extensive iff $O \in B$
- Increasing
- $E(f \vee g, B) \geq E(f,B) \vee E(g,B)$;
- $E(f \wedge g, B) = E(f,B) \wedge E(g,B)$
- Iteration property

Example of function erosion

Function opening

$$\boxed{f_B = D[E(f, B), B]}$$

Properties:

- Anti-extensive
- Increasing
- Idempotent

=> **morphological filter**

Example of function opening

Function closure

$$\boxed{f^B = E[D(f,B),B]}$$

Properties:

- Extensive
- Increasing
- Idempotent

=> **morphological filter**

- Duality between opening and closure

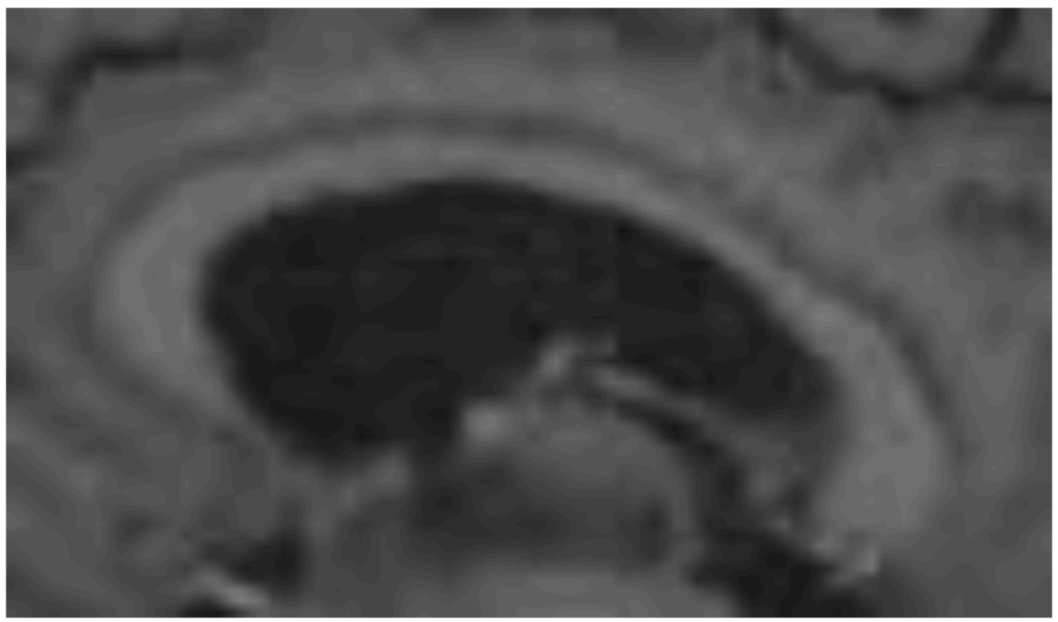
Example of function closure

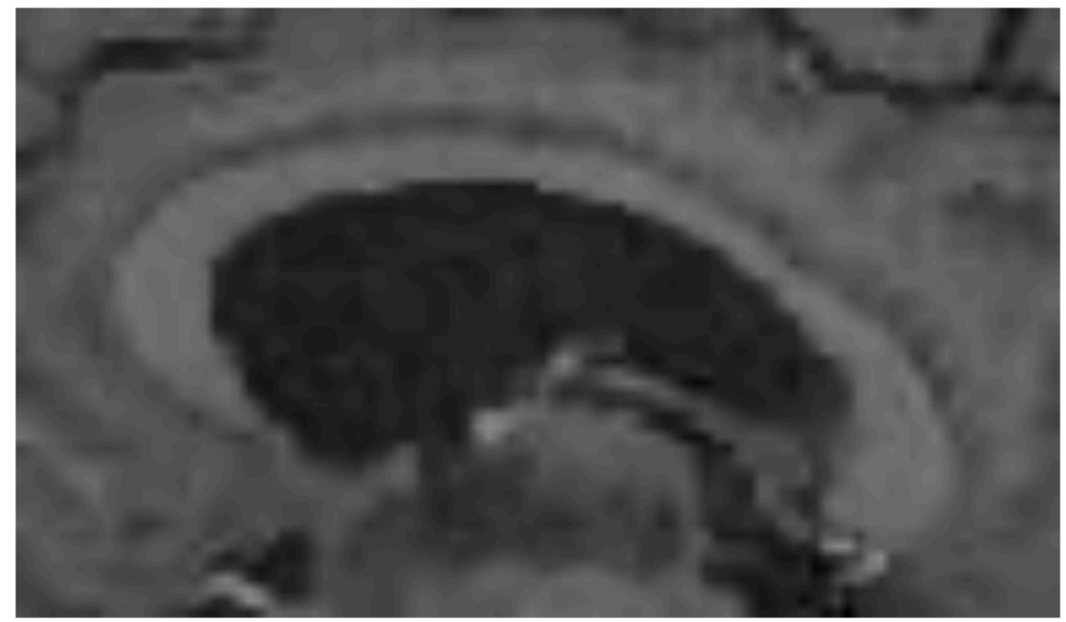
Applications of dilation and erosion

- Contrast enhancement

Applications of dilation and erosion

- Contrast enhancement





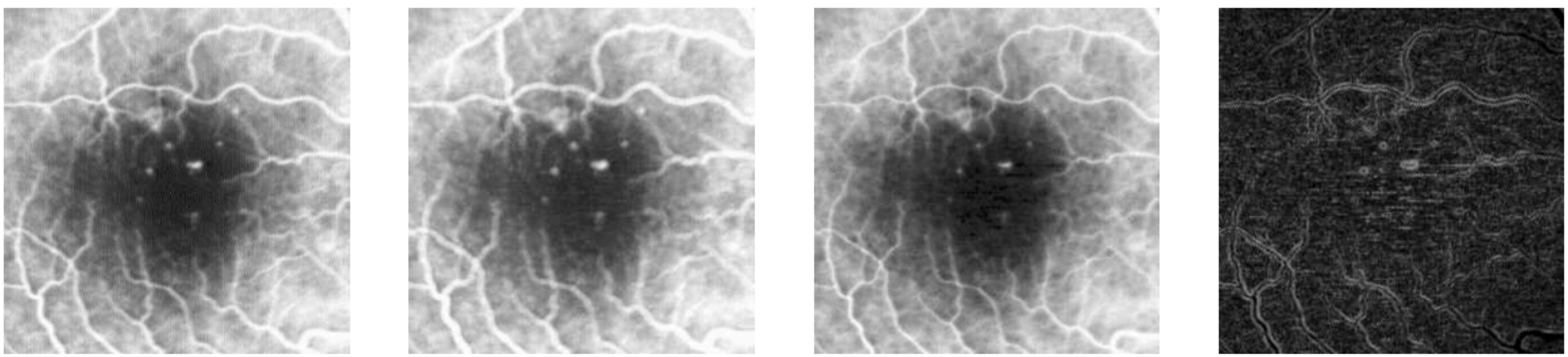
Applications of dilation and erosion

- Morphological gradient

$$\boxed{D_B(x) - E_B(x)}$$

Applications of dilation and erosion

- Morphological gradient



$$\boxed{D_B(x) - E_B(x)}$$

Sequential alternated filters

$$\boxed{(\ldots(((f_{B_1})^{B_1})_{B_2})^{B_2})\ldots_{B_n})^{B_n}}$$

Sequential alternated filters

$$\boxed{(\ldots(((f_{B_1})^{B_1})_{B_2})^{B_2})\ldots_{B_n})^{B_n}}$$

Sequential alternated filters

$$\boxed{(\ldots(((f_{B_1})^{B_1})_{B_2})^{B_2})\ldots_{B_n})^{B_n}}$$

Top-hat

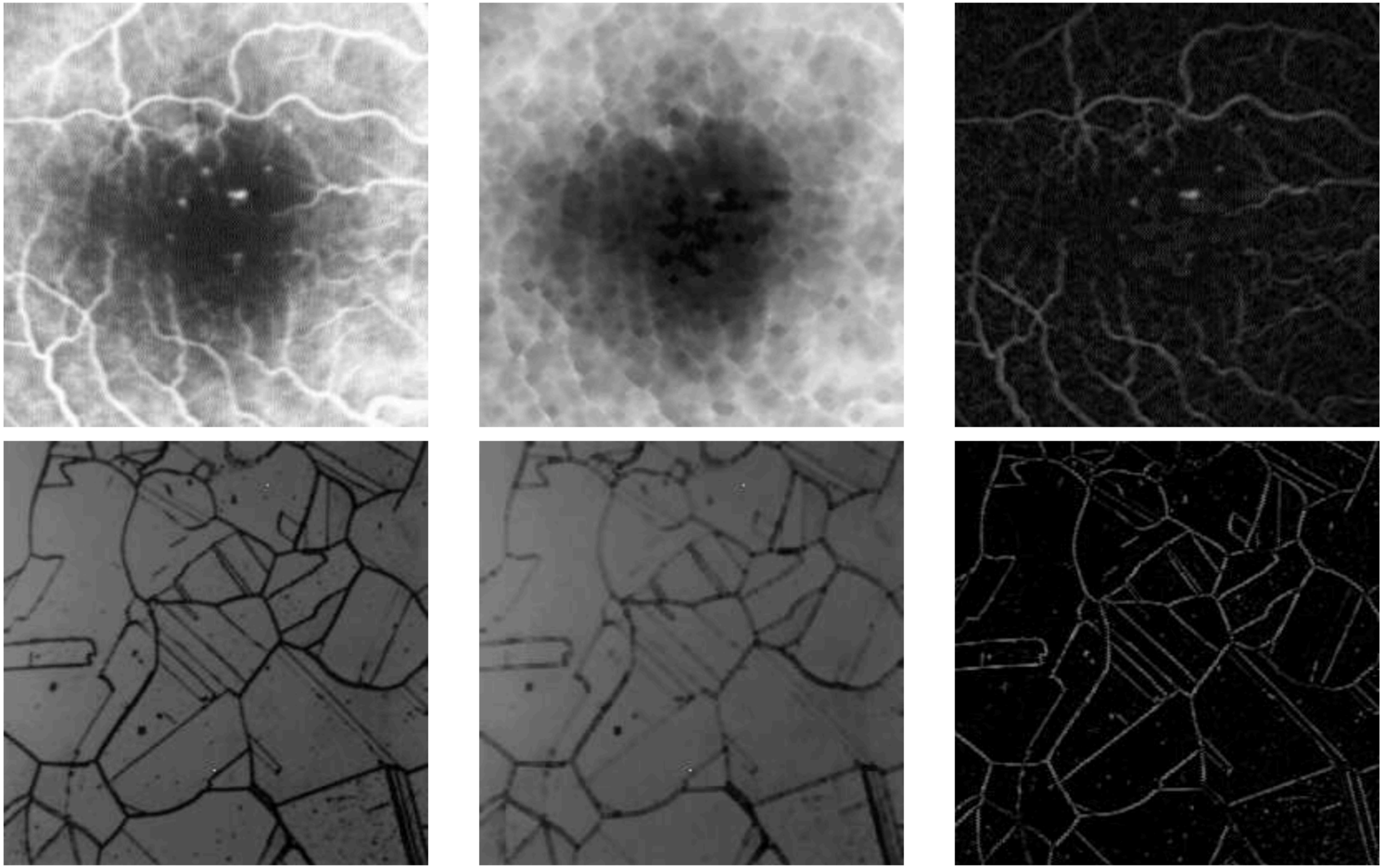
$$\boxed{f - f_B}$$

Top hat

$$\boxed{f - f_B}$$

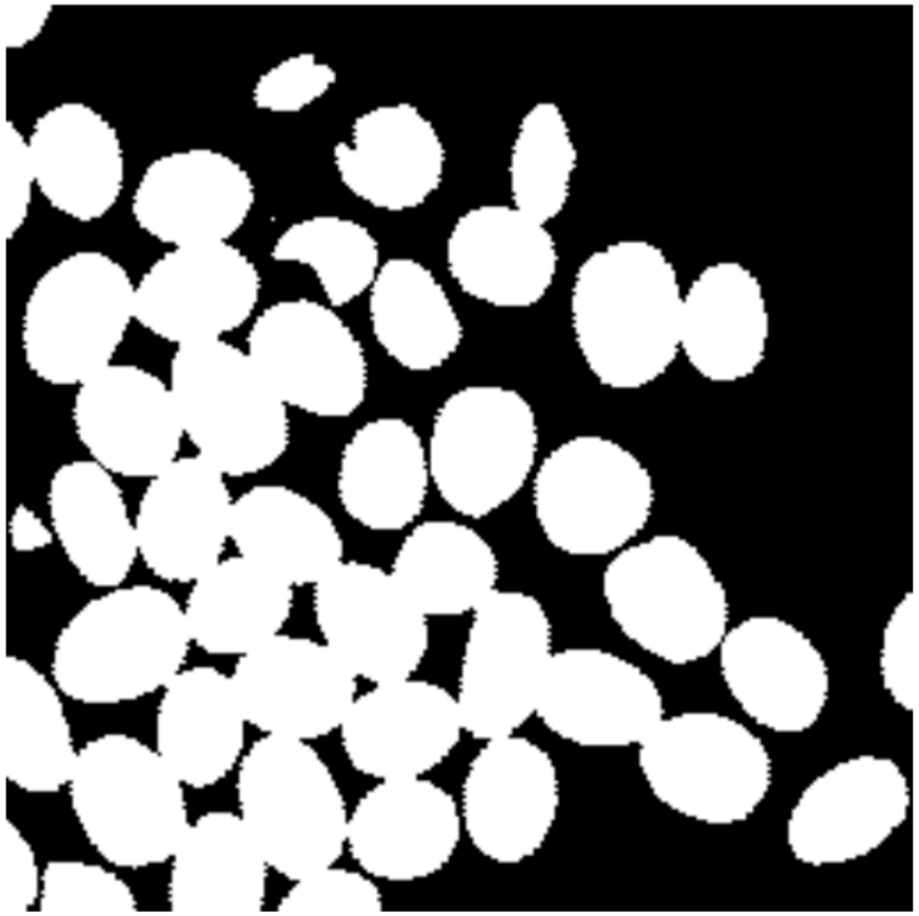
Top hat

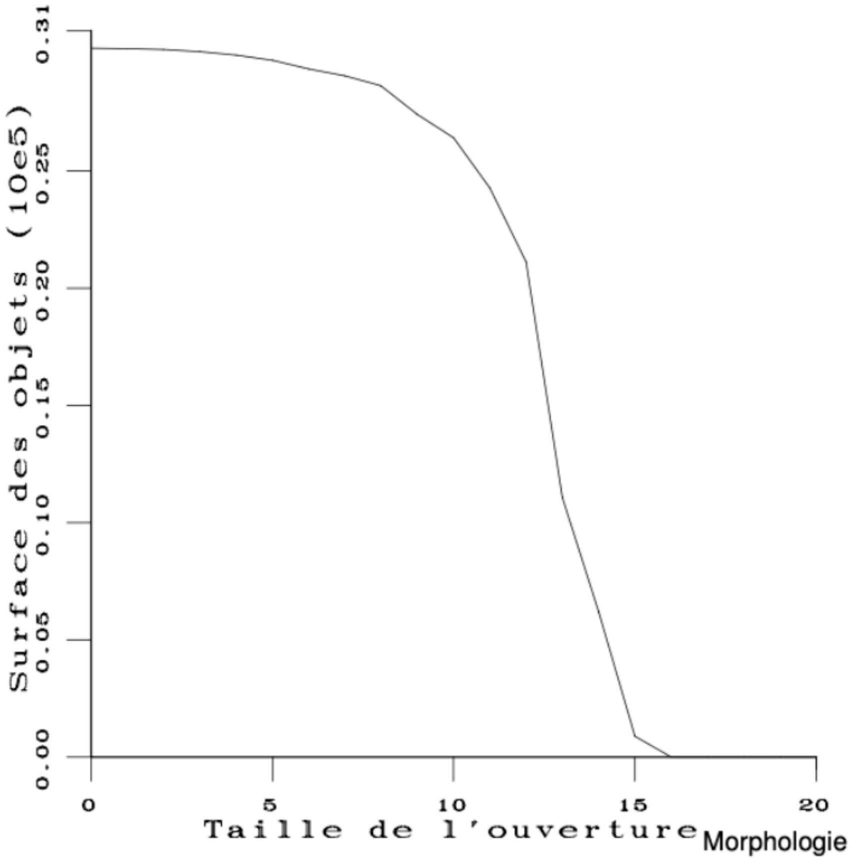
$$\boxed{f - f_B}$$



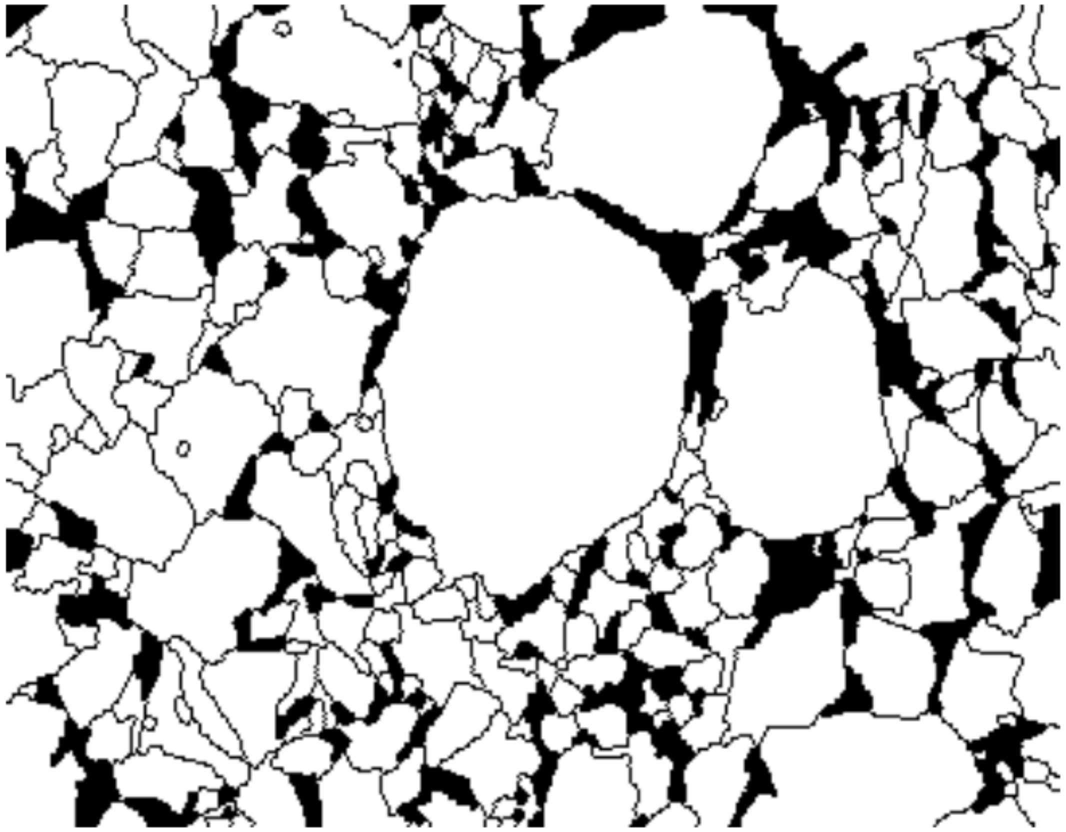
Granulometry

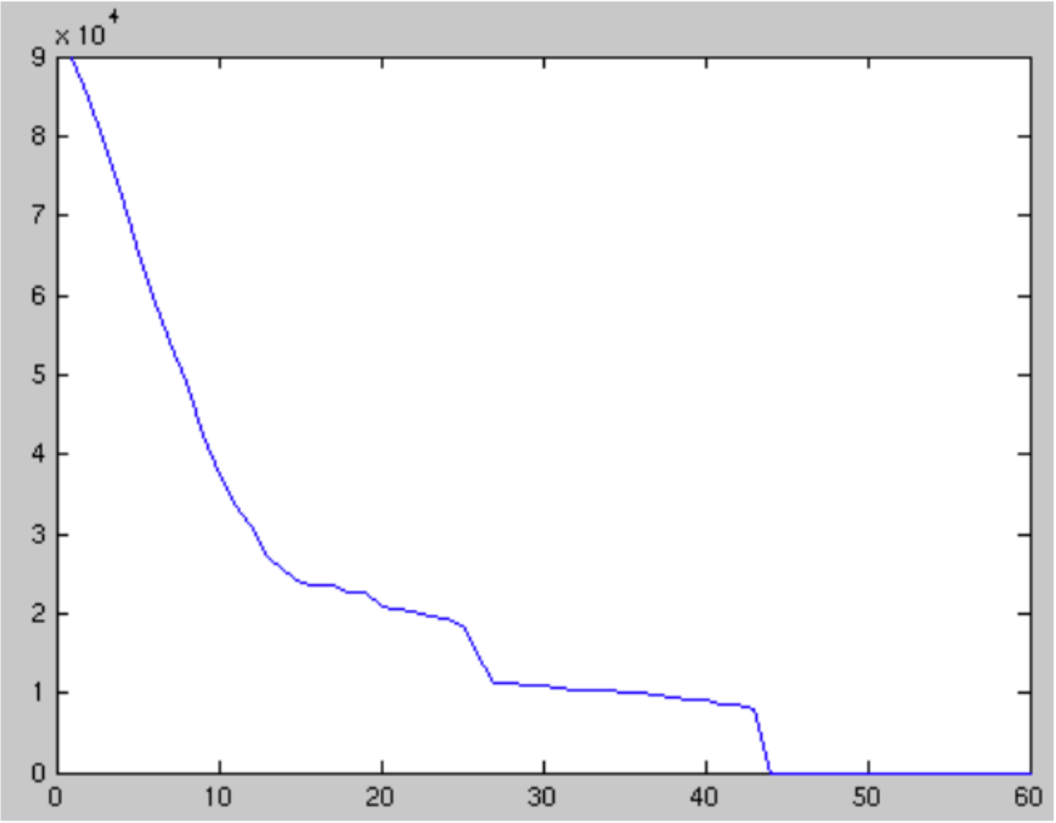
- $\forall X \in \mathcal{A}, \forall \lambda > 0, \quad \phi_\lambda(X) \subseteq X$ (ϕ_λ anti-extensive)
- $\forall (X,Y) \in \mathcal{A}^2, \forall \lambda > 0, \quad X \subseteq Y \Rightarrow \phi_\lambda(X) \subseteq \phi_\lambda(Y)$ (ϕ_λ increasing)
- $\forall X \in \mathcal{A}, \forall \lambda > 0, \forall \mu > 0 \quad \lambda \geq \mu \Rightarrow \phi_\lambda(X) \subseteq \phi_\mu(X)$ (ϕ_λ increasing w.r.t. parameter)
- $\forall \lambda > 0, \forall \mu > 0, \quad \phi_\lambda \circ \phi_\mu = \phi_\mu \circ \phi_\lambda = \phi_{\max(\lambda,\mu)}$





Granulometry

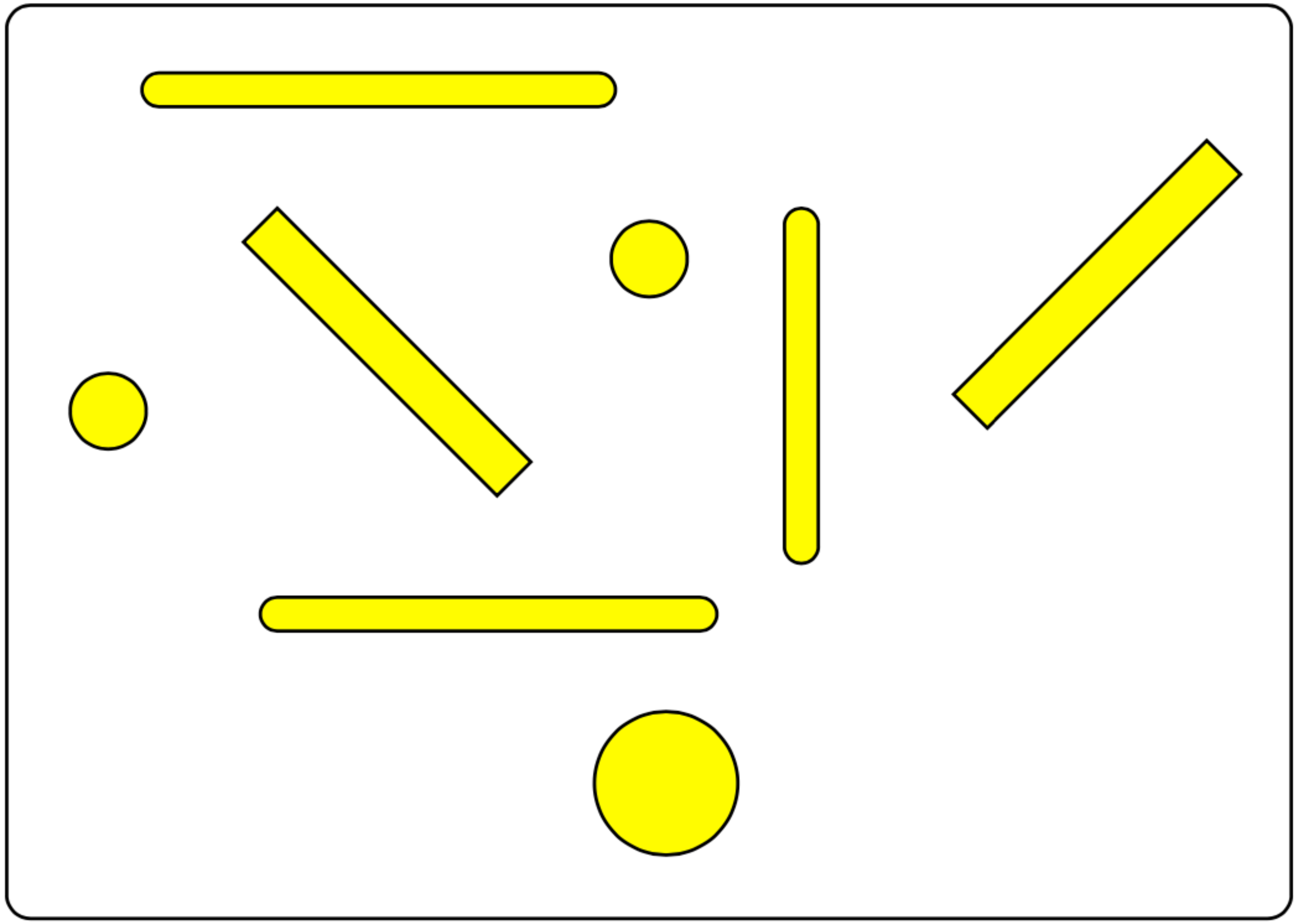




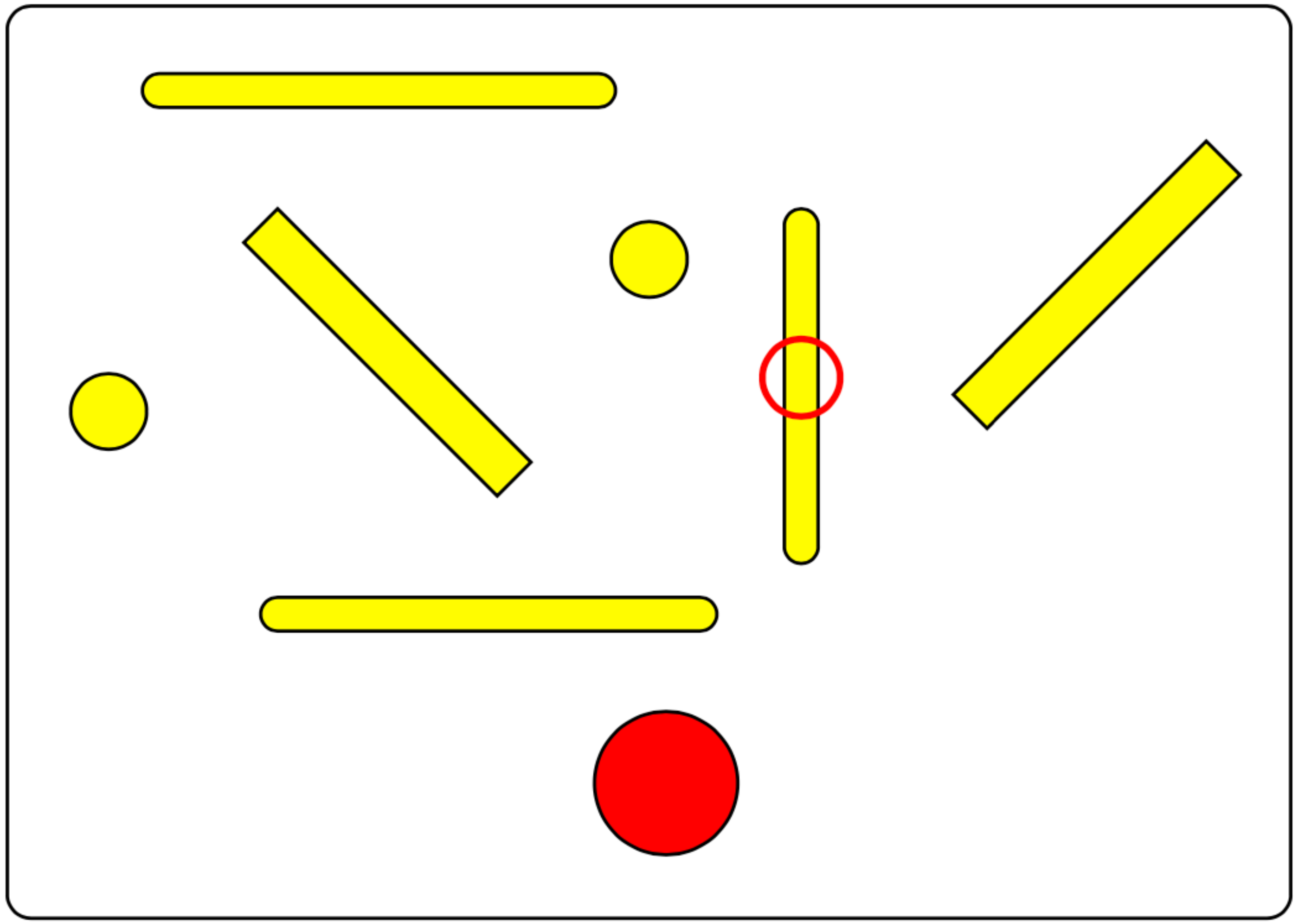
source de l'image : http://www.mamba-image.org/examples.html

Choice of the structuring element

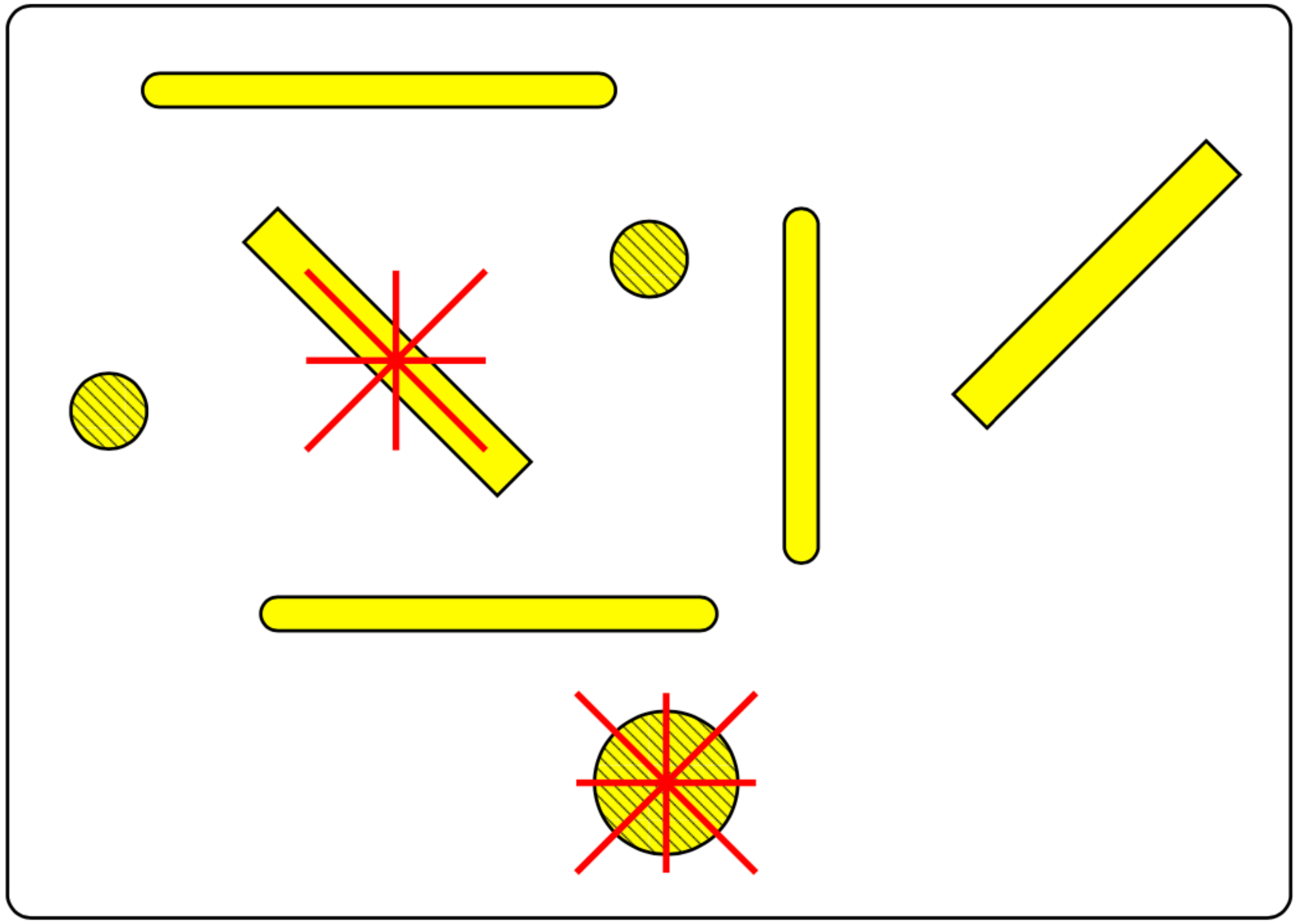
- Depends on what we want to keep
- Shape
- Size
- Example: opening through a disk or segment?



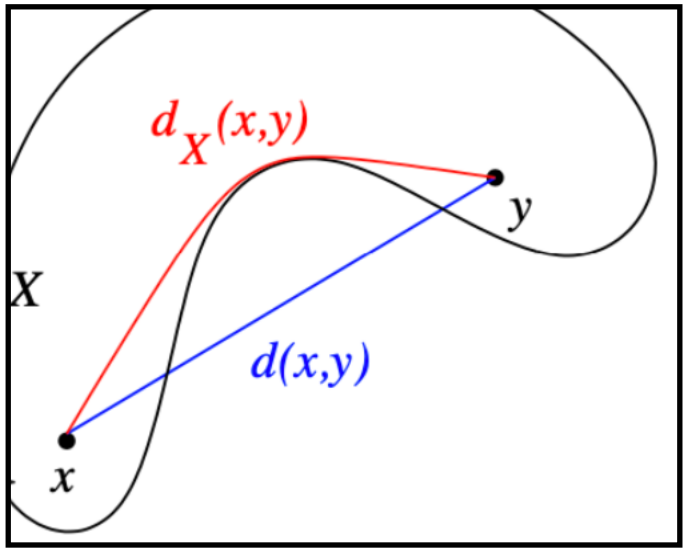
Choice of the structuring element



Choice of the structuring element



Geodesic operators and reconstruction



- Geodesic distance conditionally to X : d_X
- Geodesic ball:

$$B_X(x,r) = \{y \in X \mid d_X(x,y) \leq r\}$$

- Geodesic dilatation:

$$D_X(Y,B_r) = \{x \in \mathbb{R}^n \mid B_X(x,r) \cap Y \neq \emptyset\} = \{x \in \mathbb{R}^n \mid d_X(x,Y) \leq r\}$$

- Discret case:

$$D_X(Y,B_r) = [D(Y,B_1) \cap X]^r$$

- Geodesic erosion:

$$E_X(Y,B_r) = \{x \in \mathbb{R}^n \mid B_X(x,r) \subseteq Y\} = X \setminus D_X(X \setminus Y, B_r)$$

- Geodesic reconstruction:

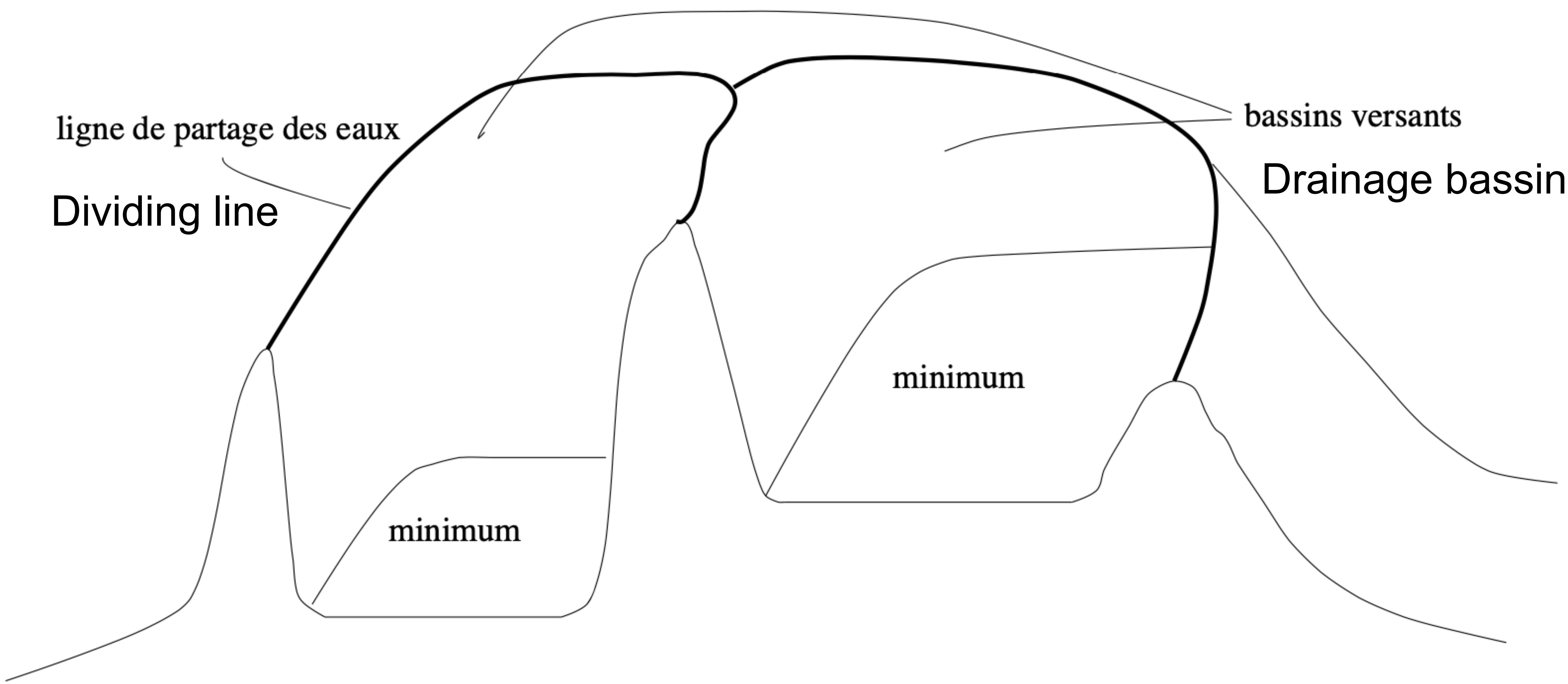
$[D(Y,B_1) \cap X]^\infty$ = connexe components of X that have an intersection with Y

- Function extension:

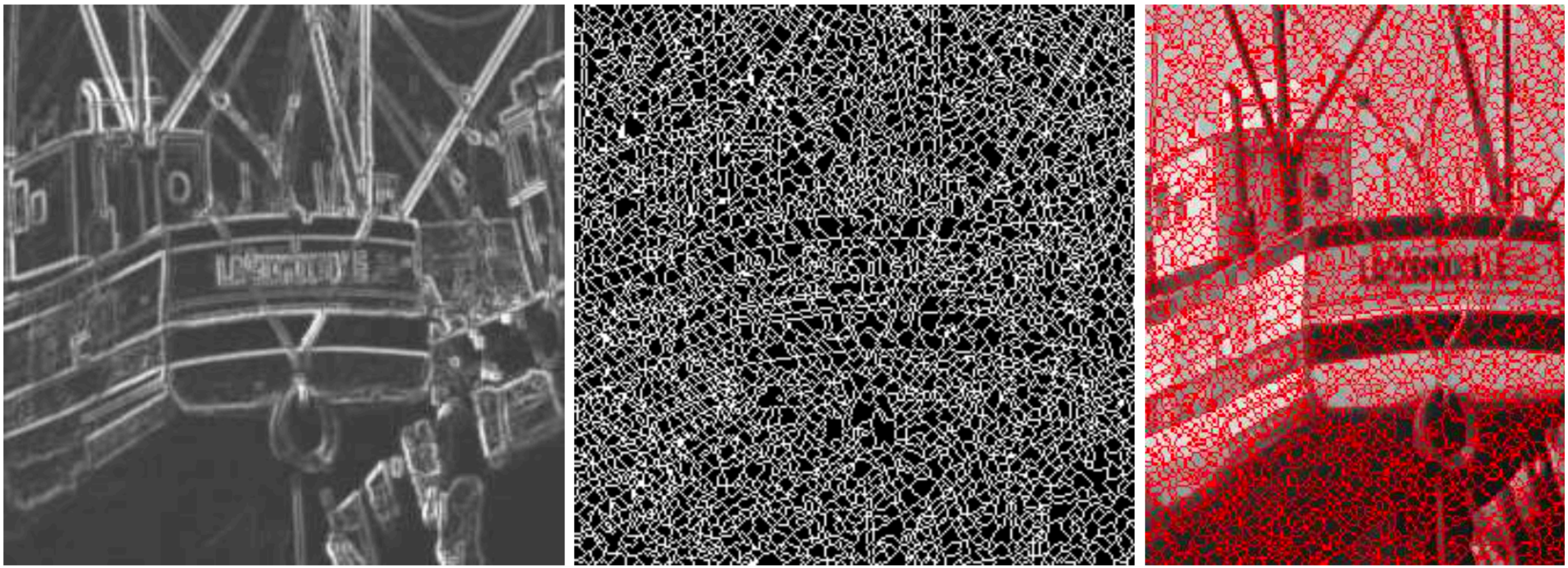
$$D_g(f,B_r) = [D(f,B_1) \wedge g]^r \quad E_g(f,B_r) = [E(f,B_1) \vee g]^r$$

Watershed dividing line

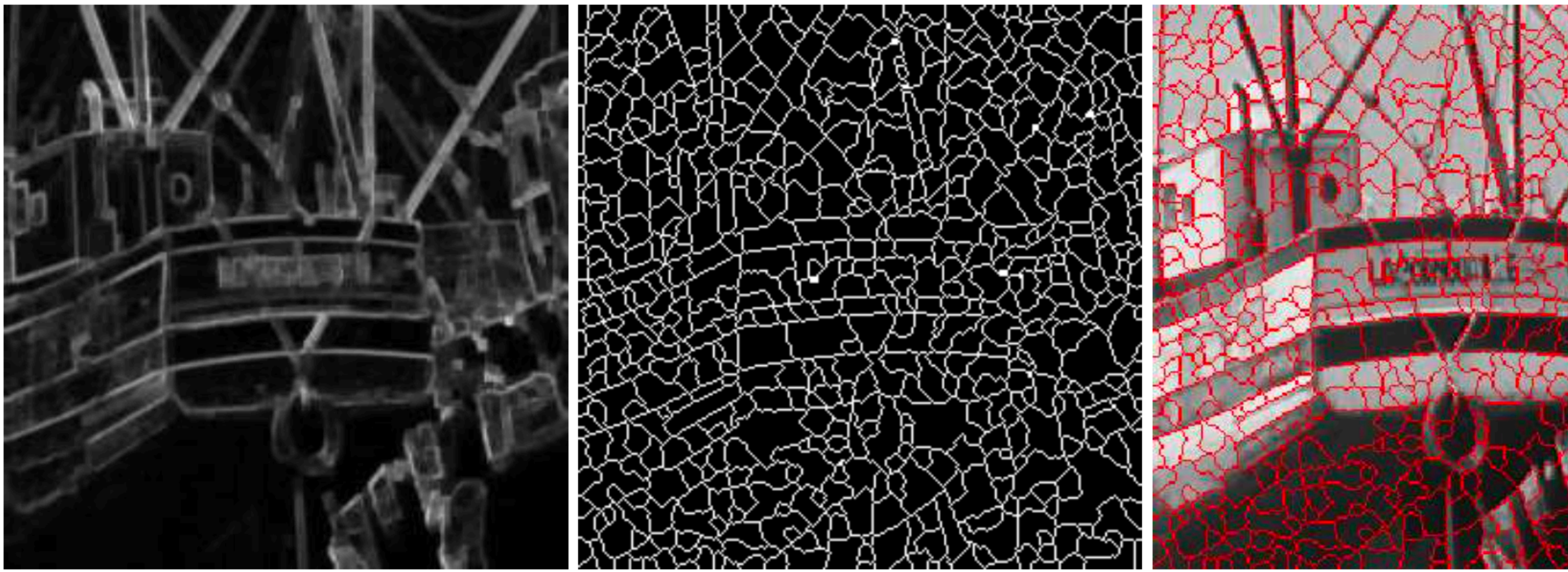
Watershed dividing line



Watersheding and oversegmentation

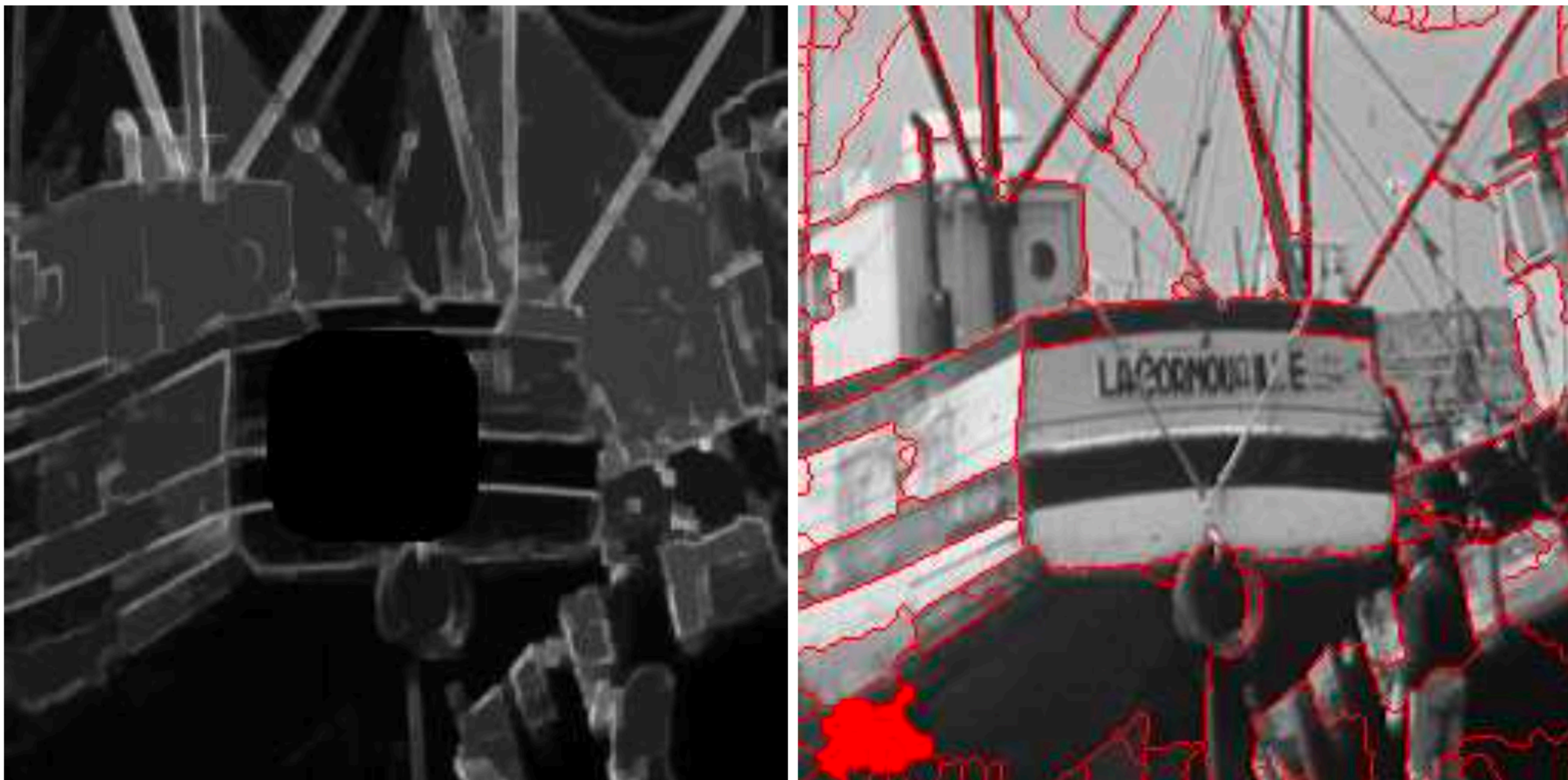


Watersheding and oversegmentation

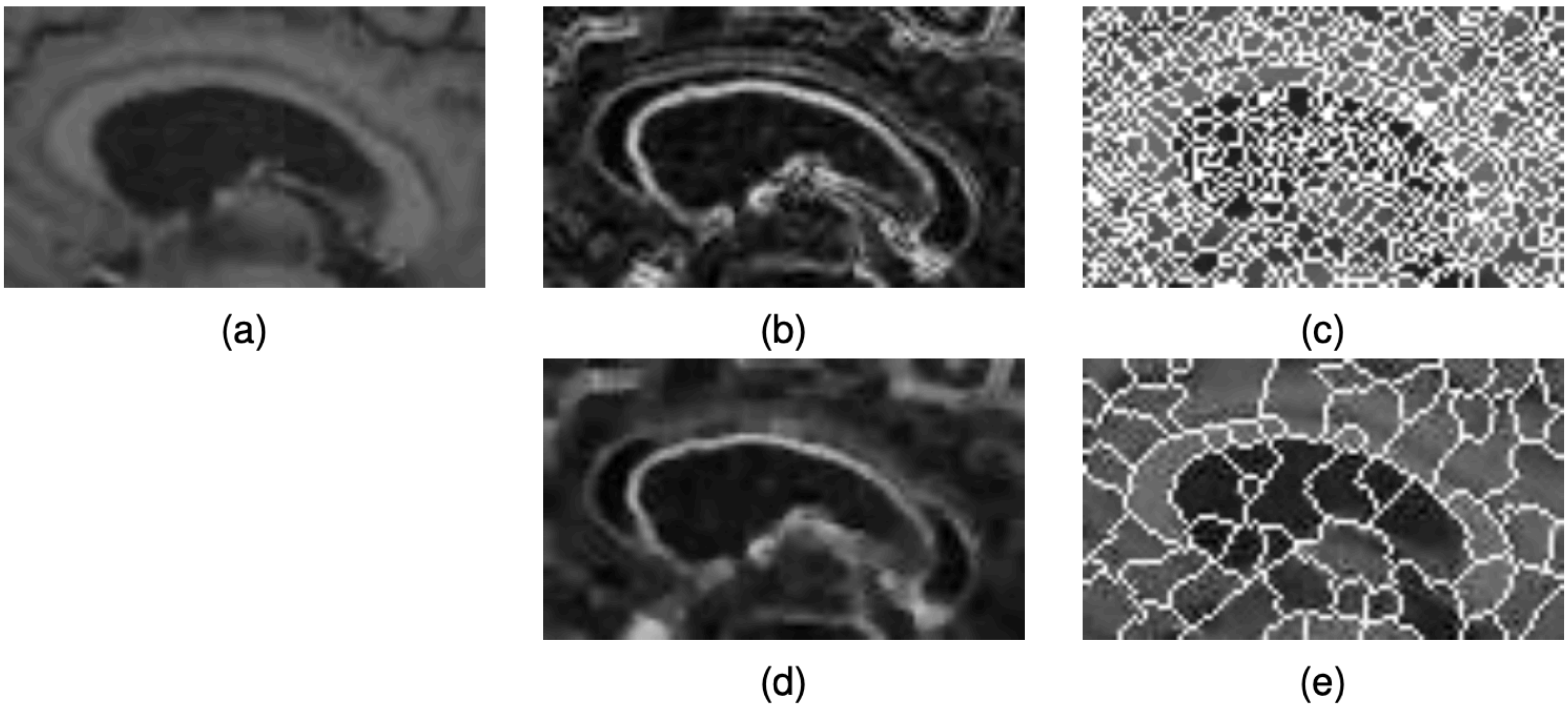


Watersheding constrained by markers

- f : function over which we want to find the dividing lines
- g : marker function, which select local minima
- Reconstruction $E_{f \wedge g}(g, B_\infty)$

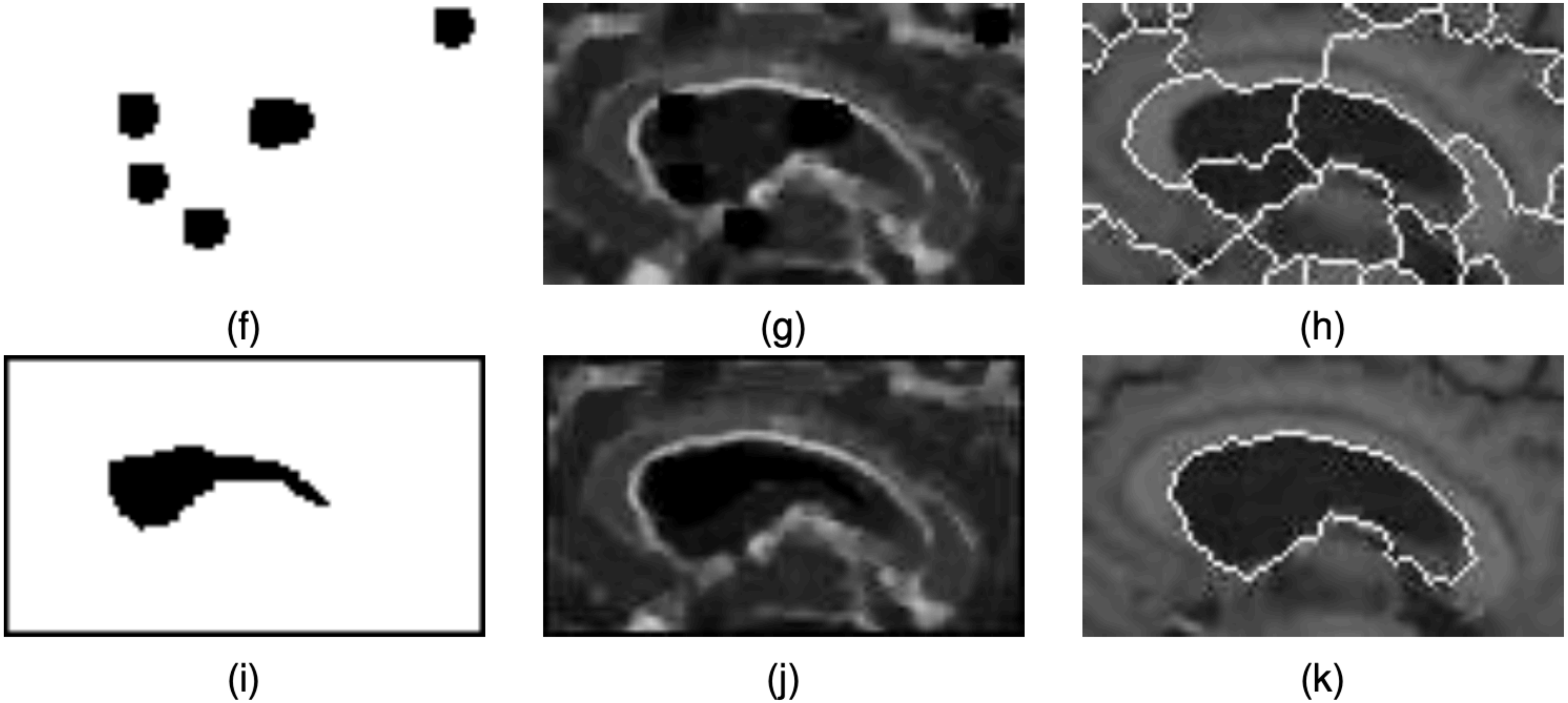


Example



(a) Original MRI, (b) Morphological gradient, (c) dividing lines, (d) closing of size 1 of the gradient image, (e) dividing lines of the closed gradient

Example



(f) Markers from the local minima, (g) Gradient reconstruction, (h) dividing lines, (i) ventricules markers and in the image boundary, (j) reconstructed gradients, (k) dividing lines

For more informations…

- M. Schmitt et J. Mattioli, *Morphologie mathématique*, Masson, Paris, 1994.
- J. Serra, *Image Analysis and Mathematical Morphology*, Academic Press, New-York, 1982.
- J. Serra (Ed.), *Image Analysis and Mathematical Morphology, Part II: Theoretical Advances*, Academic Press, London, 1988.
- P. Soille, *Morphological Image Analysis*, Springer-Verlag, Berlin, 1999.
- L. Najman, H. Talbot (Eds.), *Mathematical Morphology: From Theory to Applications*, ISTE-Wiley, 2010.