IMA 207

Statistics of coherent imaging

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Outline

Outline

Objective of the course

Noise models in image processing

- Most usual noise model: additive white gaussian noise
- Low light conditions: shot noise or Poisson noise (when the number of collected photons is small such as in fluorescence microscopy or astronomy) ; extension to Poisson-Gaussian noise when shot noise and thermal noise are mixed.
- Coherent imaging : speckle noise.

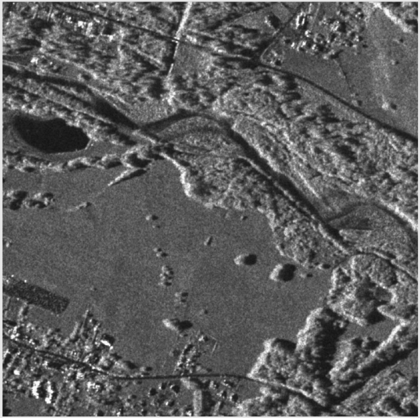
Main objective of the course

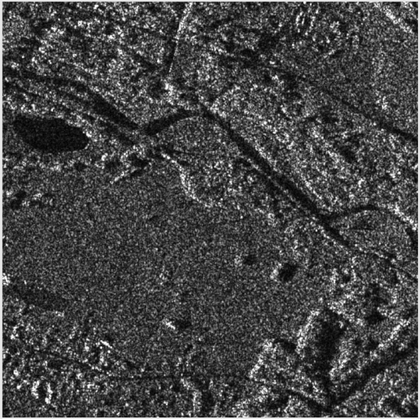
- To understand the reason of the speckle in coherent imaging
- To learn some characteristics of this noise and useful parameters
- To see some strategies to process the speckle noise

The notions are illustrated using SAR imaging but are general for coherent imaging.

Speckle in SAR images (TerraSAR-X images)

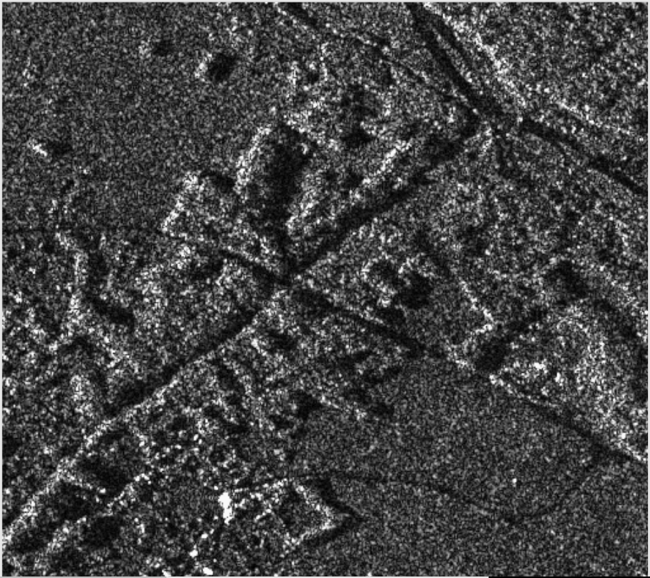
Speckle effect (amplitude image $A = |z|$, $\boldsymbol{\mu} + 3\sigma$ display)



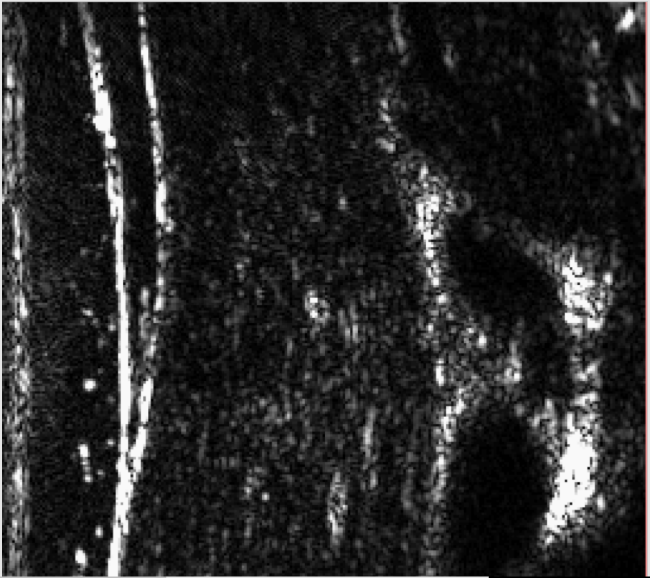


SAR image without speckle (temporal multi-looking) and original SAR image

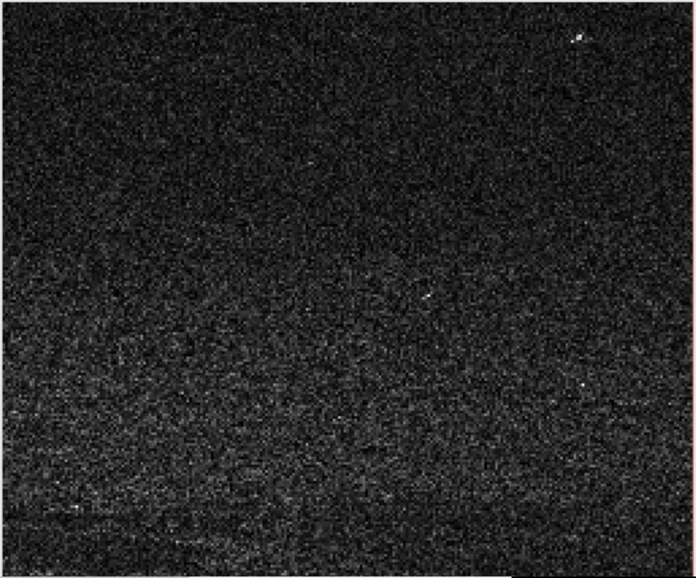
Speckle effect



Speckle effect

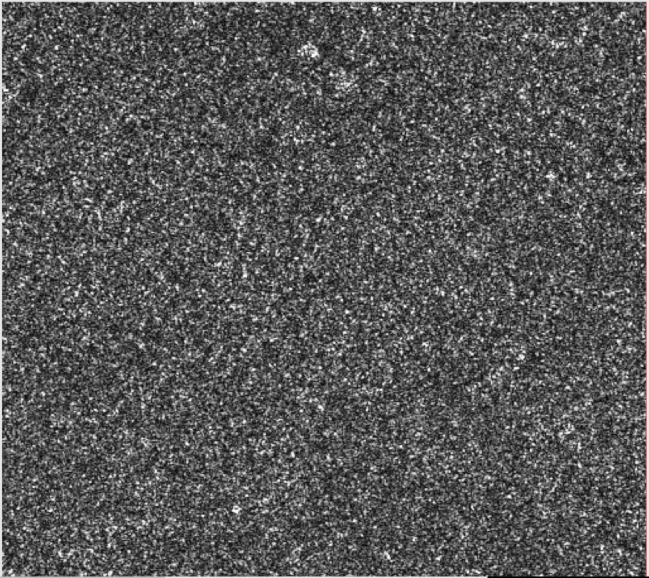


Speckle effect



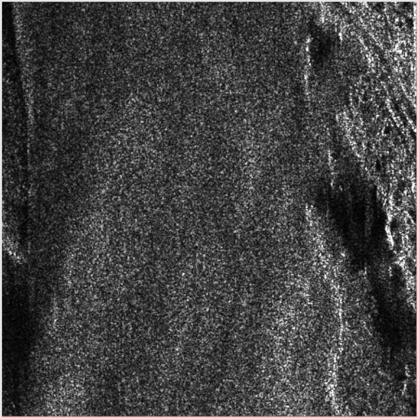
Back to SAR ...(desert in Australia)

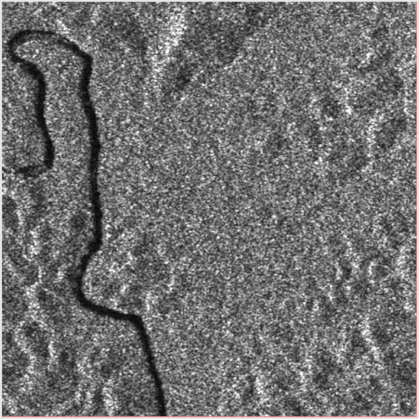
Speckle effect



Speckle in SAR images (TerraSAR-X / ERS images)

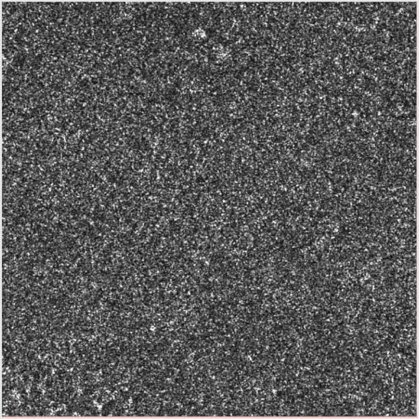
Speckle effect (amplitude image, $\mu + 3\sigma$ display)

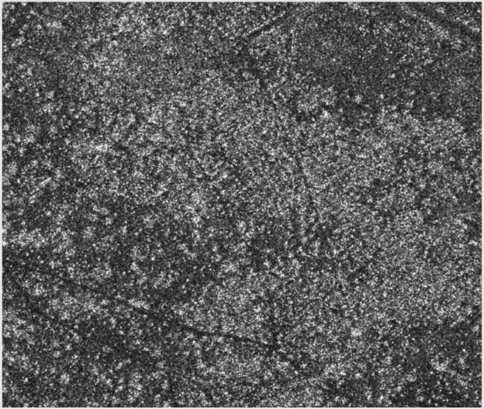




Speckle is present whatever the nature of the ground and the sensor

Experimental study





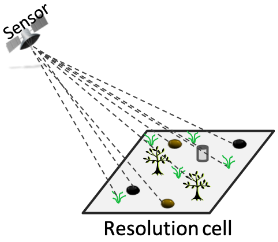
Experimental study

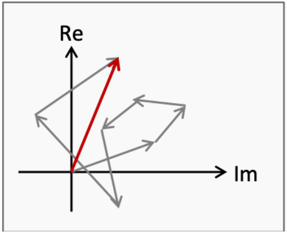
Even for different physical classes, the histogram is mono-modal...

Outline

Speckle phenomenon:

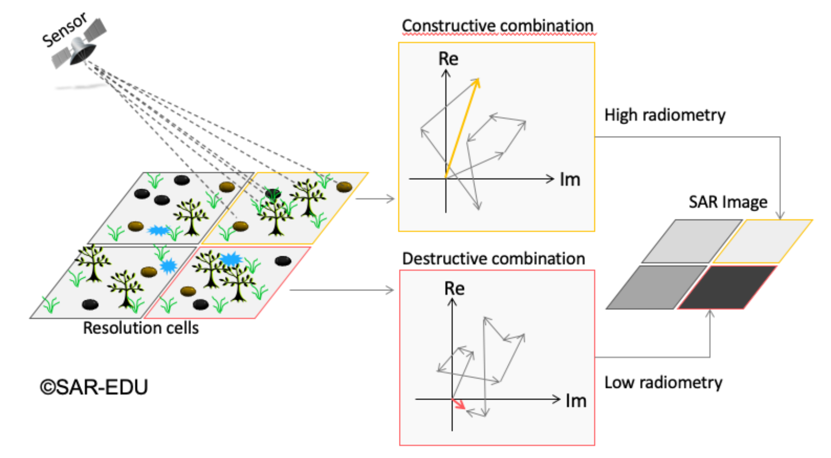
- Resolution cell much bigger than the wavelength: $\rightarrow$ many elementary scatterers inside a resolution cell
- coherent sum of the waves:
  - each scatterer backscatters the e.m. wave
  - vectorial addition in the complex plane of the backscattered waved
  - interference phenomenon





Speckle phenomenon - physical origin

Wave interferences



Speckle modeling

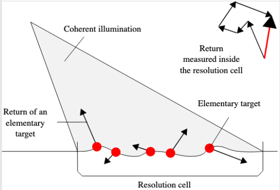
Statistical models

- No access to the organization of the elementary scatterers inside the resolution cell (even if it is deterministic!)
- Modeling with random variables at the pixel level
- Why developing models for the electro-magnetic field ?
  - Prediction of the performances of image processing methods (choice of the thresholds for false alarm rates, detection probability, etc.)
  - Adaptation of the processing to take into account the statistical models

Random variables that will be considered

- Active sensor: emits a wave and measures its echoes
- SAR: At each pixel: complex amplitude of the echo $z = Ae^{j\varphi}$
- Amplitude: $A = |z|$
- Intensity: $I = A^2 = |z|^2$
- Phase: $\varphi = \arg z$

Origins of speckle in SAR / coherent imaging systems / interferences



Goodman model

Coherent summation of N punctual echos

$$z = \frac{1}{\sqrt{N}} \sum_{i}^{N} z_i, \quad z_i \in \mathbb{C},\ z_i = a_i e^{j\phi_i}.$$

1. amplitude $|z_i|$ and phase $\arg z_i$ independent for each scatterer.
2. amplitude $|z_i|$ and phase $\arg z_i$ iid
3. phase $\arg z_i$ uniformly distributed on $[-\pi, \pi]$

Last hypothesis implies that the surface is **rough** compared to the wavelength ($\Delta h > \frac{\lambda}{8\cos(\boldsymbol{\theta})}$)

Goodman model - physically homogeneous area

- Real and imaginary parts

$$\mathbb{E}[\Re(z)] = E(\frac{1}{\sqrt{N}}\sum_{i}^{N} a_i \cos(\phi_i)) = 0$$

$$\mathbb{E}[\Im(z)] = E(\frac{1}{\sqrt{N}}\sum_{i}^{N} a_i \sin(\phi_i)) = 0$$

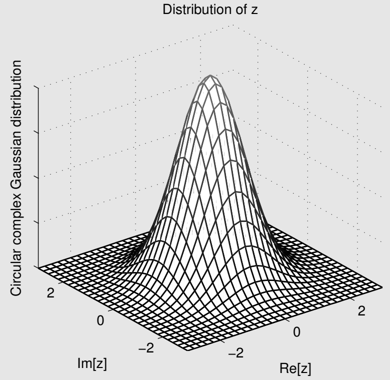
$$\mathbb{E}[\Re(z)^2] = \mathbb{E}[\Im(z)^2] = \sigma^2$$

σ^2 is linked to the physical properties of the surface (characteristic of it)

- By the law of large numbers wrt N

$$p(z|\sigma) \triangleq p(\Re(z), \Im(z)|\sigma) = \frac{1}{2\pi\sigma^2}\exp\left(-\frac{|z|^2}{2\sigma^2}\right)$$

Gaussian distribution of z



Statistical modeling of speckle - Goodman model

Distributions of phase and intensity

$z = Ae^{j\varphi}$ is distributed according to a complex circular Gaussian, thus

- $\varphi = \arg(z)$ uniformly distributed in $[-\pi, \pi]$, $\Rightarrow$ phase is non-informative.

$$p(\phi|R) = \frac{1}{2\pi}.$$

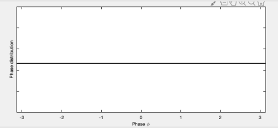
- $I = |z|^2$ exponentially distributed: $p(I\,|\sigma) = \frac{1}{2\sigma^2}\exp\left(-\frac{I}{2\sigma^2}\right).$

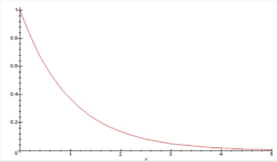
  $\mathbb{E}[I] = 2\sigma^2$.
  The parameter $R = 2\sigma^2$ is introduced and represents the reflectivity of the considered area. It is proportional to the backscattering coefficient of the surface and is the physical parameter of interest.

$$p(I\,|R) = \frac{1}{R}\exp\left(-\frac{I}{R}\right).$$

Statistical modeling of speckle - Goodman model

Distributions of phase and intensity





Mean and standard deviation for an area of reflectivity R

- Mean value : $\mathbb{E}[I] = R$
- Variance : $\text{Var}[I] = R^2$

The higher the reflectivity R, the higher the fluctuations (the variance is not stationary and increases with the mean value). The noise is said multiplicative.

Statistical modeling of speckle - Goodman model

Distribution of amplitude

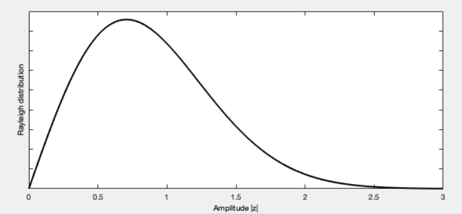
$A = |z|$ is distributed according to a Rayleigh distribution

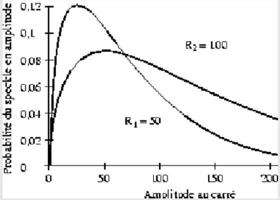
$$p(A \mid R) = \frac{2A}{R} \exp\left(-\frac{A^2}{R}\right).$$

$$\mathbb{E}[A] = \sqrt{\frac{\pi}{4} R}$$

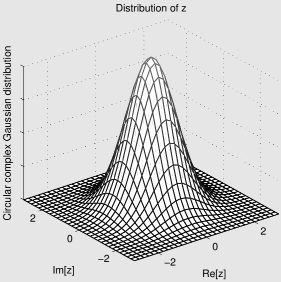
- The mode does not correspond to the mean
- The distribution has a heavy tail
- Mean and standard deviation are proportional

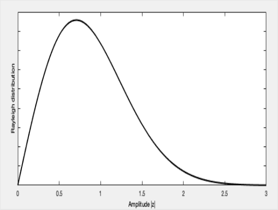
Distributions of amplitude

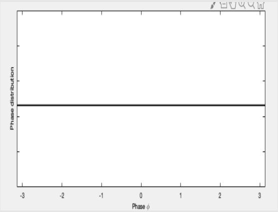




Statistics of the circular complex Gaussian distribution - summary







$z = Ae^{j\varphi}$ is distributed according to a complex circular Gaussian, thus

- $\varphi = \arg(z)$ uniformly distributed in $[-\pi, \pi]$, $\Rightarrow$ phase is non-informative.
- $I = |z|^2$ exponentially distributed: $$p(I \mid R) = \frac{1}{R} \exp\left(-\frac{I}{R}\right).$$
- $A = |z|$ Rayleigh distributed: $$p(A \mid R) = \frac{2A}{R} \exp\left(-\frac{A^2}{R}\right).$$ $\Rightarrow$ heavy right tail.
- I or A are sufficient statistics for R: $\mathbb{E}[I] = \mathbb{E}[A^2] = R$. $\Rightarrow$ many SAR applications focus only on $|z|$.

Measuring the homogeneity / heterogeneity of the scene in an image

Statistical modeling of speckle in SAR imaging

Measuring the heterogeneity of an area

- Mean and standard deviation are proportional for a physically homogeneous area (both for intensity and amplitude data)
- The standard deviation no longer measures the local heterogenity
- Standard deviation has to be normalized by the mean: it defines the coefficient of variation
  - *For intensity data* on an homogeneous area (R constant):

$$\gamma_I = \frac{\sigma_I}{\mathbb{E}(I)} = 1$$

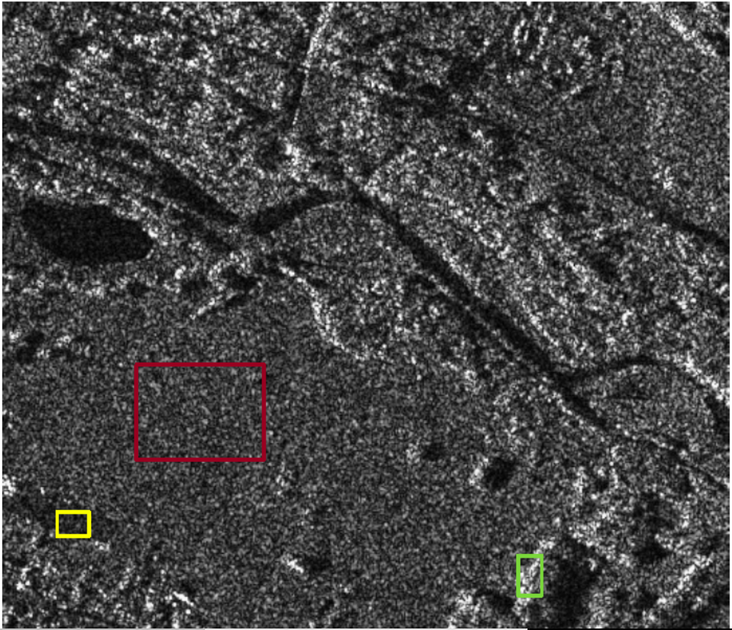
  - *For amplitude data* on an homogeneous area (R constant):

$$\gamma_A = \frac{\sigma_A}{\mathbb{E}(A)} = \sqrt{\frac{4}{\pi} - 1} \approx 0.523$$

Measuring the heterogeneity on an image

- Mean and standard deviation are computed locally on a window.
- For an homogeneous area (constant reflectivity R of the scene), it should be around 1 for intensity data, and 0.523 for amplitude data.

Measuring heterogeneity: coefficient of variation



Outline

1. Introduction
2. Speckle and Goodman model
3. Multi-look processing
4. Multiplicative noise model
5. Extension to vectorial data

Multi-looking

Motivations:

- The level of noise limits the understanding of data of coherent imaging
- It is preferable to reduce the spatial resolution to improve the radiometric resolution

Principle:

- Averaging L i.i.d samples divides the variance by L
- $X_1, X_2, ..., X_L$ N samples i.i.d then :

$$X_{ML} = \frac{1}{L}\sum_{i=1}^{L} X_i$$

$$\mathbb{E}[X_{ML}] = \mathbb{E}(X_i), \ \text{Var}[X_{ML}] = \frac{\text{Var}(X_i)}{L}$$

Data choice

- Complex data ? $z_1, z_2, ..., z_L$
- Intensity data ? $I_1, I_2, ..., I_L$ ($I_i = |z_i|^2$)
- Amplitude data ? $A_1, A_2, ..., A_L$ ($A_i = |z_i|$)

Effect of multi-looking

Multi-looking

- The averaging is done in intensity (should be an incoherent summation !)

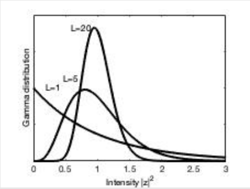
$$I = I_{\text{ML}} = \frac{1}{L}\sum_{i=1}^{L}|z_i|^2 = \frac{1}{L}\sum_{i=1}^{L} I_i$$

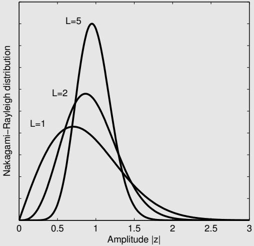
- Summation of random variables induces convolution of their pdf
- As I_i are iid and exponentially distributed, I is Gamma distributed (for a given reflectivity R of the surface)

$$p(I \mid R, L) = \mathop{\bigstar}_{i=1}^{L} p(I_i \mid R) = \frac{L^L}{\Gamma(L)}\frac{I^{L-1}}{R^L}\exp\left(-\frac{LI}{R}\right),$$

- $A = A_{\text{ML}} = \sqrt{I_{\text{ML}}}$ is Nakagami-Rayleigh distributed.

$$p(A \mid R, L) = \frac{2L^L}{\Gamma(L)}\frac{A^{2L-1}}{R^L}\exp\left(-\frac{LA^2}{R}\right),$$





Multi-looking and Equivalent Number of Looks (ENL)

Values of the coefficient of variation for multi-look data

- Multi-look intensity data:

$$\gamma_{I_{\text{ML}}} = \frac{1}{\sqrt{L}}$$

- Multi-look amplitude data

$$\gamma_{A_{\text{ML}}} = \frac{0.523}{\sqrt{L}}$$

Real data

- In practice, the samples to do the multi-looking are not i.i.d. In this case, the "number of looks" will be less than announced and no more an integer value.
- This real value is called the Equivalent Number of Looks (ENL). It should be used in the pdf.
- The data provider usually gives the theoretical number of looks.
- To compute the ENL, a physically homogeneous area is selected and the previous formula is inverted :

$$ENL = \frac{1}{\hat{\gamma}^2_{I_{\text{ML}}}}$$

$\hat{\gamma}_{I_{\text{ML}}}$ is obtained empirically using the empirical mean and variance.

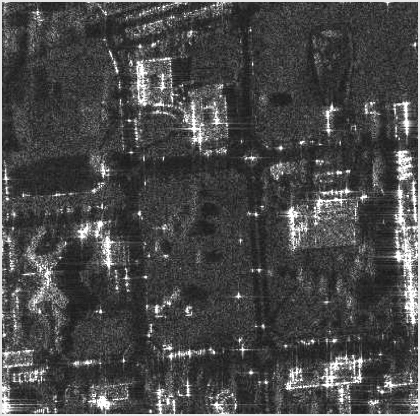
How choosing the samples to multi-look ?

Multi-looking in practice

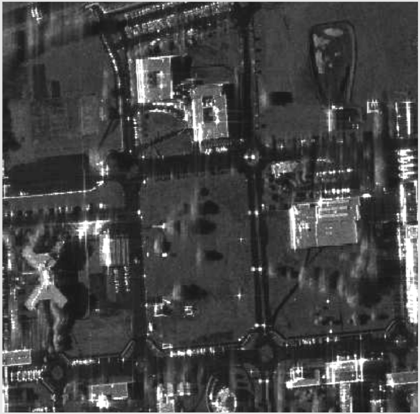
How choosing the samples to be averaged ?

- Spatial multi-looking
  - local averaging and subsampling
  - drawback: loss of resolution
  - advantage: the number of vertical and horizontal samples is chosen to obtain square pixels (example ERS: ground range resolution: 12m, azimuth resolution 3m $\Rightarrow 1 \times 4$ multi-looking)
  - alternative: mean filter with moving window (no explicit loss in resolution)
- Temporal multi-looking (remote sensing context)
  - use of the new acquisitions of the same area after a cycle of the satellite
  - drawback: cost of acquisitions, temporal changes
  - advantage: no loss in resolution
  - very good solution for stable features in the image

Very High Resolution SAR (© CNES) - spatial multi-looking



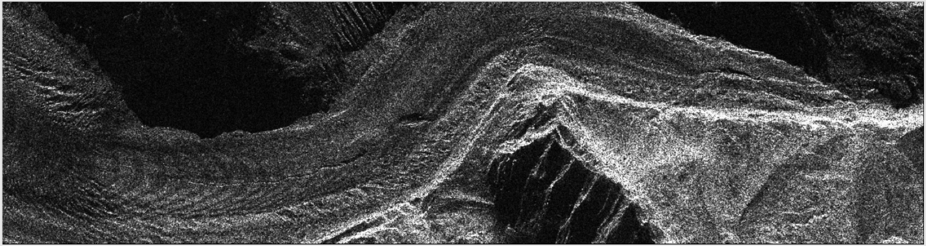
(a) Single look

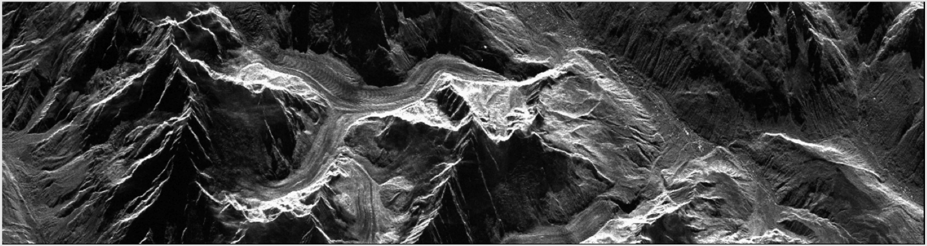


(b) 100-looks

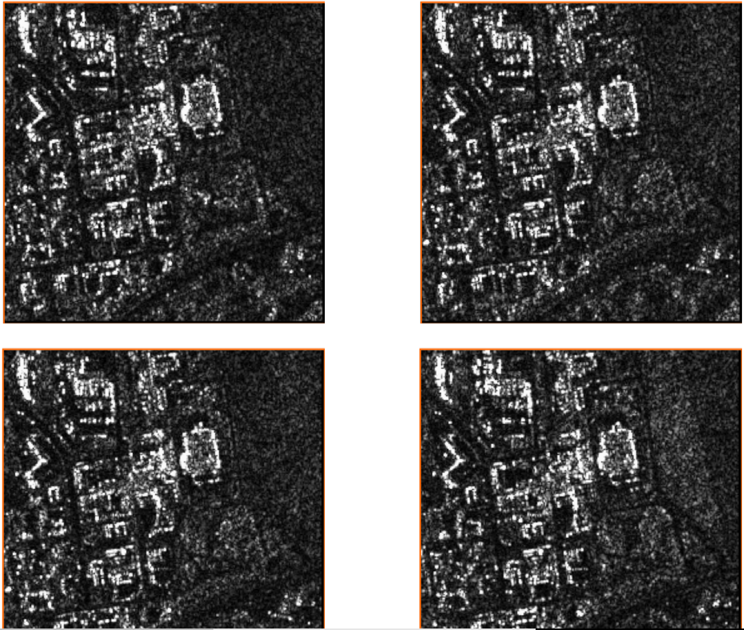
Example of spatial multi-looking

TerraSAR-X (© DLR





Example of multi-looking



Example of temporal multi-looking

TerraSAR-X image (© DLR)

(a) Temporally multi-looked image

(b) photo

Outline

Multiplicative noise modeling

Natural images and optical imaging

- Model of noise: AWGN (Additive White Gaussian Noise)
- Implicilty used in all approaches (NL-means, energy formulation with quadratic cost, etc.)

$$y = x + n, \quad n \sim \mathcal{N}(0, \sigma)$$

y is the observed value for a pixel and x the underlying scene

- The local standard deviation is compared to σ to evaluate the heterogeneity of the scene

Coherent imaging and speckle noise

- Model of noise: multiplicative
- should be used in all approaches to derive the adapted expressions (NL-means, energy formulation, etc.)

$$I = R \times S, \quad S \sim \mathcal{G}(1, L)$$

I observed intensity value, R underlying scene (reflectivity), S normalized speckle

- $\mathcal{G}(1, L)$ Gamma distribution of expectation 1 and number of looks L: $\mathbb{E}[S] = 1$, Var$[S] = \frac{1}{L}$

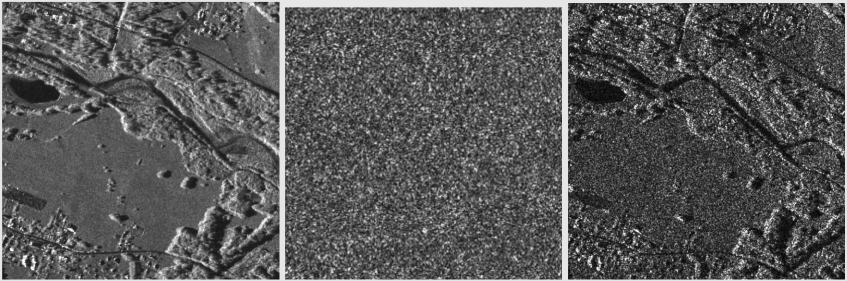
$$p(S \mid L) = \frac{L^L}{\Gamma(L)} S^{L-1} \exp\left(-LS\right),$$

Equivalence of modelings

Multiplicative model and equivalence with the bayesian formulation

- Pdf of a product

$$
\begin{aligned}
p(I = R.S \mid L) &= \int p(R) p(S = \frac{I}{R}) \frac{1}{R} dR \\
&= \int p(R) \frac{L^L}{\Gamma(L)} (\frac{I}{R})^{L-1} \exp\left(-LS\right) \frac{1}{R} dR \\
&= \int p(R) \frac{L^L}{\Gamma(L)} \frac{I^{L-1}}{R^L} \exp\left(-\frac{LI}{R}\right) dR \qquad = \int p(R) p(I \mid R, L) dR
\end{aligned}
$$



Multiplicative modeling

Interest of multiplicative noise model

- Generation of noisy images by pure noise simulation (evaluation of methods, training of deep learning approaches,...)
- Useful for some computations, like the coefficient of variation of the scene γ_R (measures the heterogeneity of the scene):

$$\gamma_R^2 = \frac{\gamma_I^2 - \gamma_S^2}{1 - \gamma_S^2}$$

  $\gamma_S^2 = \frac{1}{L}$ for a L-look image. The local variation coefficient of the scene is 0 if it is a constant reflectivity $R \times$ speckle and increases with the scene fluctuations (texture, strong scatterers, etc.).
- Lee filter (practical work):

$$\hat{R} = \overline{I} + k(I - \overline{I})$$

  $\overline{I}$ is the local empirical mean (computed on a local window)
  k is a coefficient controlling the balance between I and $\overline{I}$.

$$k = 1 - \frac{\gamma_S^2}{\gamma_I^2}$$

  γ_I is computed locally and γ_S is defined by L.

Taking into account the speckle distribution

- In many approaches the neg log-likelihood is used :

$$p(y|x) = \frac{1}{\sqrt{2\pi}\sigma}\exp(-\frac{(y-x)^2}{2\sigma^2})$$
$$-\log(p(y|x)) = \frac{(y-x)^2}{2\sigma^2} + ct$$

quadratic term between the solution and the observation appearing in many algorithms ($\Rightarrow$ implicit hypothesis of a white gaussian noise)

$$p(I|R,L) = \frac{L^L}{\Gamma(L)}\frac{I^{L-1}}{R^L}\exp\left(-\frac{LI}{R}\right)$$
$$-\log(p(I|R,L)) = \frac{LI}{R} + L\log(R) - (L-1)\log(I) + ct$$

- To get rid of the multiplicative noise: take the logarithm !

$$I = R.S \Longleftrightarrow \log(I) = \log(R) + \log(S)$$

This approach is called an homomorphic approach
But is $\log(S)$ a zero mean Gaussian noise ?

Statistical model of log-transformed speckle

Goodman model of fully developed speckle

SAR intensity is distributed according to a Gamma distribution:

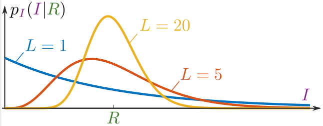
$$p_I(I|R) = \frac{L^L I^{L-1}}{\Gamma(L) R^L} \exp\left(-L\frac{I}{R}\right) \quad \text{with } R \text{ the radar reflectivity.}$$

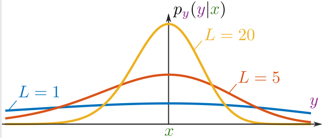
$$\rightarrow \ \mathbb{E}[I] = R$$
$$\rightarrow \ \text{Var}[I] = R^2/L$$

The log of the intensity y follows a Fisher-Tippett distribution:

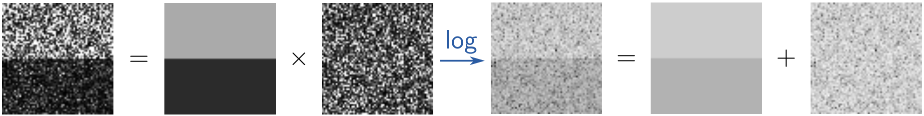
$$p_y(y|x) = \frac{L^L}{\Gamma(L)} e^{L(y-x)} \exp\left(-Le^{y-x}\right)$$

$$\rightarrow \ \mathbb{E}[y] = x - \log L + \Psi(L)$$
$$\rightarrow \ \text{Var}[y] = \Psi(1, L) \qquad (\Psi : \text{polygamma})$$





noisy image | noiseless image | exponential noise | noisy image | noiseless image | stationary noise



multiplicative speckle noise | additive stationary noise

Statistical model of log-transformed speckle

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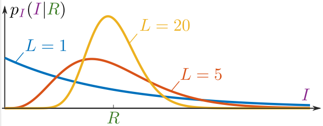
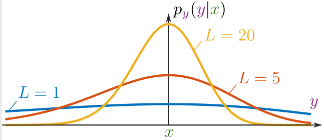
$$\rightarrow \ \text{Var}[I] = R^2/L$$

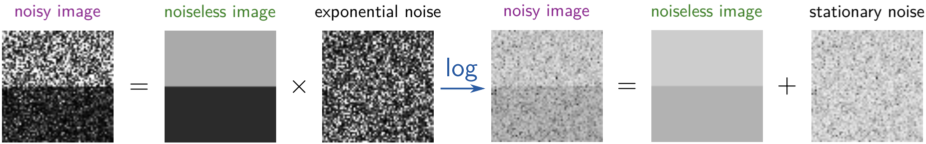


The log of the intensity y follows a Fisher-Tippett distribution:

$$p_y(y|x) = \frac{L^L}{\Gamma(L)} e^{L(y-x)} \exp\left(-Le^{y-x}\right)$$

$$\rightarrow \ \mathbb{E}[y] = x - \log L + \Psi(L)$$
$$\rightarrow \ \text{Var}[y] = \Psi(1, L) \qquad (\Psi : \text{polygamma})$$





multiplicative speckle noise

additive stationary noise

Statistical model of log-transformed speckle

Goodman model of fully developed speckle

SAR intensity is distributed according to a Gamma distribution:

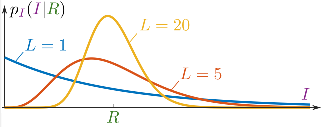
$$p_I(I|R) = \frac{L^L I^{L-1}}{\Gamma(L) R^L} \exp\left(-L\frac{I}{R}\right) \quad \text{with } R \text{ the radar reflectivity.}$$

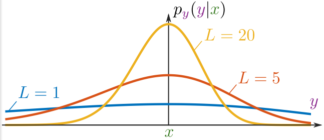
$$\rightarrow \mathbb{E}[I] = R$$
$$\rightarrow \text{Var}[I] = R^2/L$$

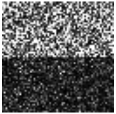
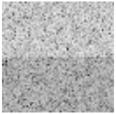
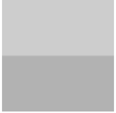
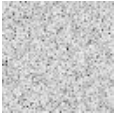
The log of the intensity y follows a Fisher-Tippett distribution:

$$p_y(y|x) = \frac{L^L}{\Gamma(L)} e^{L(y-x)} \exp\left(-L e^{y-x}\right)$$

$$\rightarrow \mathbb{E}[y] = x - \log L + \Psi(L)$$
$$\rightarrow \text{Var}[y] = \Psi(1, L) \qquad (\Psi : \text{polygamma})$$



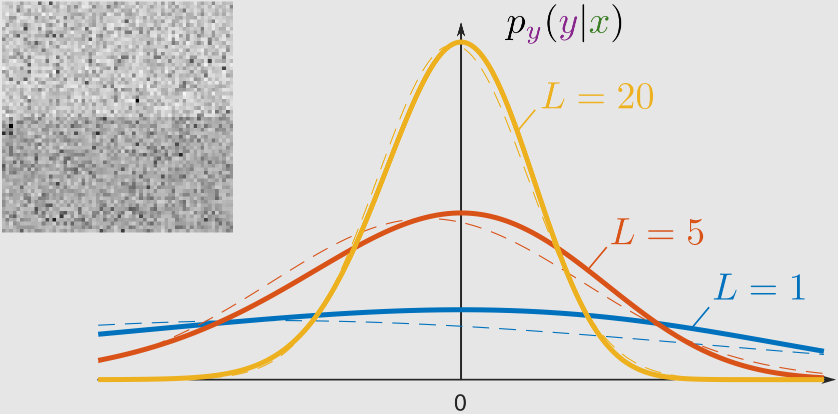


noisy image		noiseless image		exponential noise		noisy image		noiseless image		stationary noise
	=		×		log →		=		+	

multiplicative speckle noise — additive stationary noise

Statistical model of log-transformed speckle

Gaussian approximation of log-transformed speckle

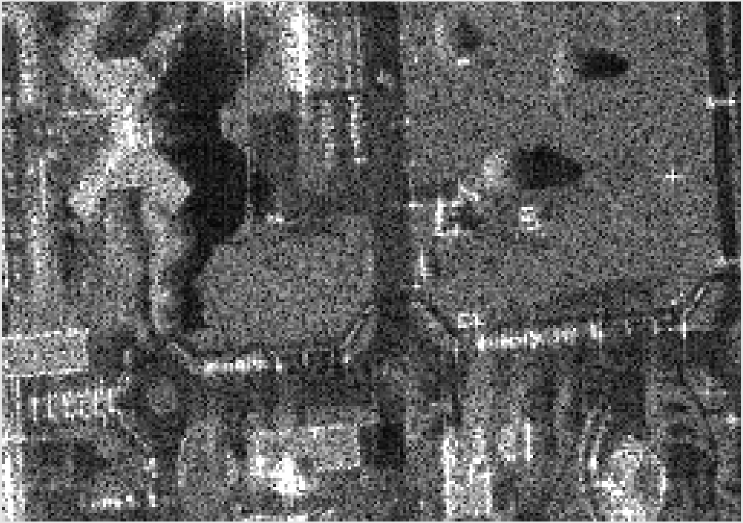


approximate log-transformed speckle as additive white Gaussian noise
$\rightarrow$ not very good for small L: *asymmetry* towards lower values
$\rightarrow$ not centered (a *debiaising step* is needed)

Denoising SAR intensity images with homomorphic approaches

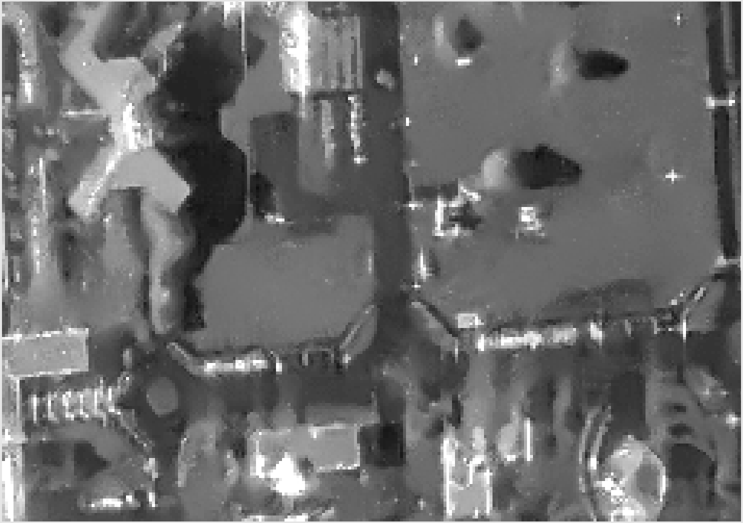
“speckle-free” reference image (100 looks, ©ONERA/CNES)

Denoising SAR intensity images with homomorphic approaches



simulated noisy image (1 look)

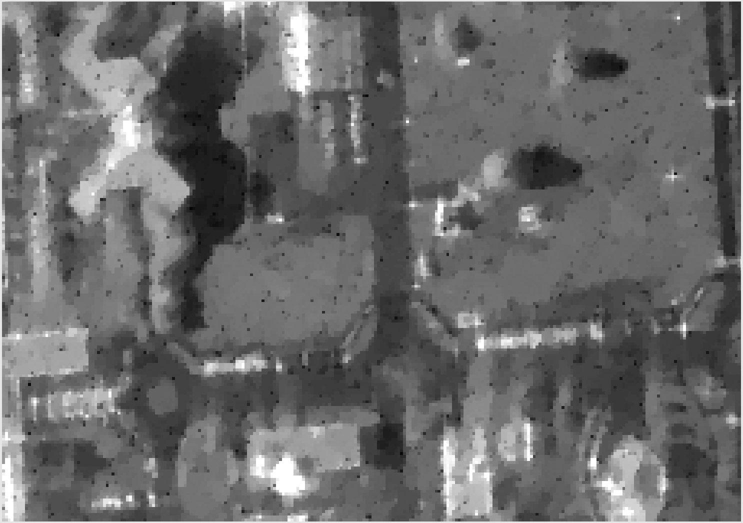
Denoising SAR intensity images with homomorphic approaches



state-of-the-art speckle-reduction (NL-SAR)

Homomorphic approaches

Denoising SAR intensity images with homomorphic approaches



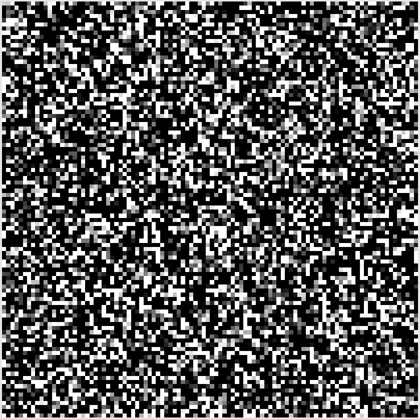
homomorphic approach with TV denoising

Denoising SAR intensity images with homomorphic approaches

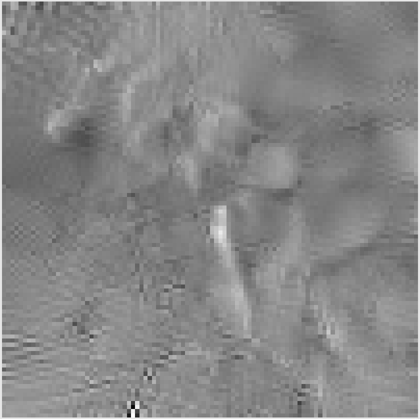
homomorphic approach with BM3D denoising

Denoising SAR intensity images with homomorphic approaches

Because of the Gaussian approximation, isolated dark pixels appear.



Gaussian noise
before BM3D filtering

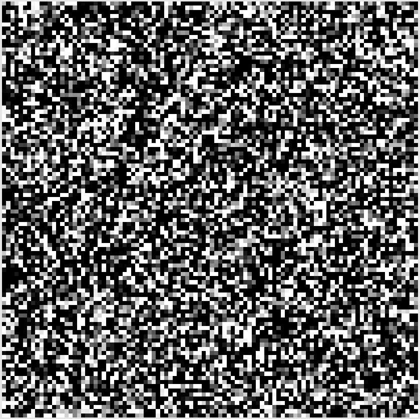


Gaussian noise
after BM3D filtering (enhanced contrast)

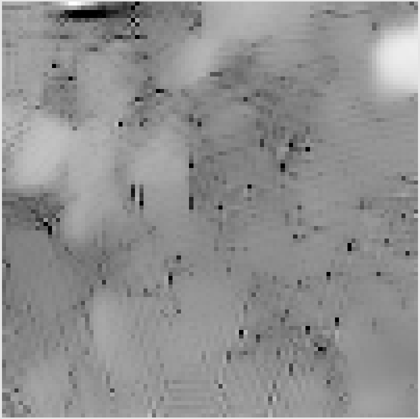
Homomorphic approaches

Denoising SAR intensity images with homomorphic approaches

Because of the Gaussian approximation, isolated dark pixels appear.



log-transformed speckle
before BM3D filtering

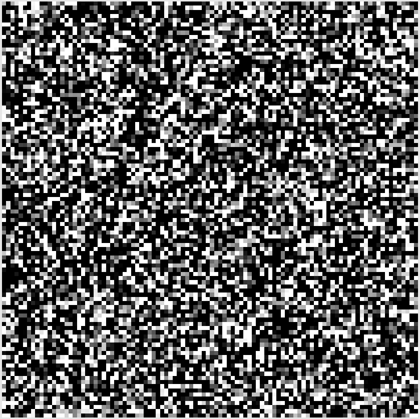


log-transformed speckle
after BM3D filtering (enhanced contrast)

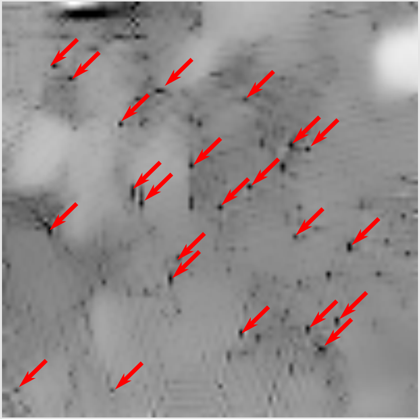
Homomorphic approaches

Denoising SAR intensity images with homomorphic approaches

Because of the Gaussian approximation, isolated dark pixels appear.



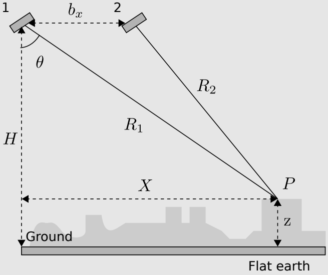
log-transformed speckle
before BM3D filtering



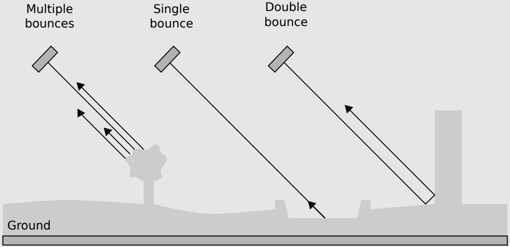
log-transformed speckle
after BM3D filtering

Outline

Motivations: InSAR, PolSAR, PolInSAR



(a) InSAR



(b) PolSAR

Multivariate SAR imagery

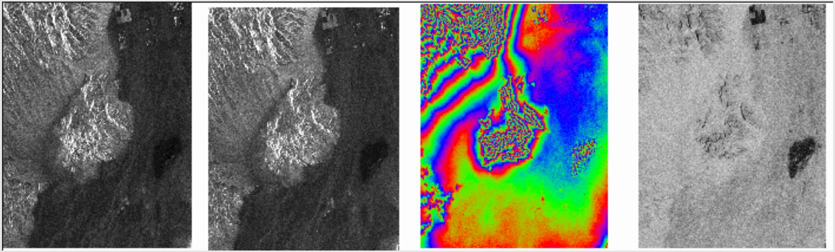
- Active sensor(s): measure echoes at different time, incidence angle, or polarization.
- Interferometry: 2 SAR images

  phase difference → **elevation**, ...
- Polarimetry: 3 SAR images

  complex correlation → **geophysical properties**

Example of interferometric data

Interferometric data



Interferometric data

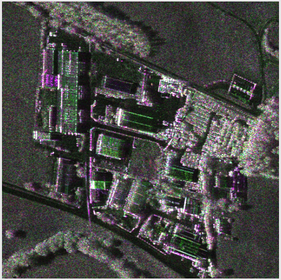
- Consider the scattering vector $\boldsymbol{k} = (z_1 \; z_2)^t$ of the 2 complex interferometric images
- The covariance matrix $\boldsymbol{\Sigma} = \mathbb{E}[\boldsymbol{k}.\boldsymbol{k}^\dagger]$ contains the following elements:

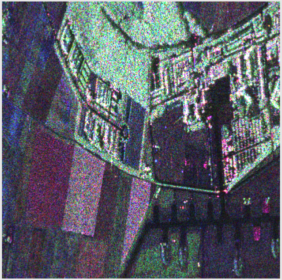
$$\boldsymbol{\Sigma} = \begin{pmatrix} R_1 & \sqrt{R_1 R_2}\rho_{1,2} \\ \sqrt{R_2 R_1}\rho^*_{1,2} & R_2 \end{pmatrix}$$

  - $R_k = \mathbb{E}[|z_k|^2]$ reflectivity for the channel k
  - $\rho_{1,2} = \frac{\mathbb{E}[z_1 z_2^*]}{\sqrt{\mathbb{E}[|z_1|^2]\mathbb{E}[|z_2|^2]}} = D_{1,2}e^{j\phi_{1,2}}$ coherence between the two acquisitions and interferometric phase $\phi_{1,2} = \phi_1 - \phi_2$ linked to topography

Example of polarimetric data

Polarimetric data





Polarimetric data

- Consider the scattering vector $\boldsymbol{k} = (z_{HH}\ z_{HV}\ z_{VV})^t$ of the 3 combinations of polarizations
- The covariance matrix $\boldsymbol{\Sigma} = \mathbb{E}[\boldsymbol{k}.\boldsymbol{k}^\dagger]$ contains the following elements:

$$\boldsymbol{\Sigma} = \begin{pmatrix} R_1 & \sqrt{R_1 R_2}\rho_{1,2} & \sqrt{R_1 R_3}\rho_{1,3} \\ \sqrt{R_1 R_2}\rho^*_{1,2} & R_2 & \sqrt{R_2 R_3}\rho_{2,3} \\ \sqrt{R_1 R_3}\rho^*_{1,3} & \sqrt{R_2 R_3}\rho^*_{2,3} & R_3 \end{pmatrix}$$

  - $R_k = \mathbb{E}[|z_k|^2]$ reflectivity for the channel k
  - $\rho_{1,2} = \frac{\mathbb{E}[z_1 z_2^*]}{\sqrt{\mathbb{E}[|z_1|^2]\mathbb{E}[|z_2|^2]}} = D_{1,2}e^{j\phi_{1,2}}$ physical information on the bacjscattering mechanisms

Distribution of multivariate Single Look Complex (SLC) SAR data

- Consider the scattering vector $\boldsymbol{k}$ whose elements form the sequence of complex echoes

$$\boldsymbol{k} = \begin{pmatrix} z_1 & z_2 & \dots & z_K \end{pmatrix}^t$$

- Under Goodman's model, $\boldsymbol{k}$ has a multivariate circular complex Gaussian distribution given by

$$p(\boldsymbol{k}|\boldsymbol{\Sigma}) = \frac{1}{\pi^K |\boldsymbol{\Sigma}|} \exp\left(-\boldsymbol{k}^\dagger \boldsymbol{\Sigma}^{-1} \boldsymbol{k}\right)$$

where $\boldsymbol{\Sigma}$ is an **unknown** complex covariance matrix (Hermitian positive definite)

$$\boldsymbol{\Sigma} = \begin{pmatrix} R_1 & \cdots & \sqrt{R_1 R_k}\rho_{1,k} & \cdots & \sqrt{R_1 R_K}\rho_{1,K} \\ \vdots & \ddots & \vdots & & \vdots \\ \sqrt{R_k R_1}\rho^*_{1,k} & \cdots & R_k & \cdots & \sqrt{R_k R_K}\rho_{k,K} \\ \vdots & & \vdots & \ddots & \vdots \\ \sqrt{R_K R_1}\rho^*_{1,K} & \cdots & \sqrt{R_K R_k}\rho^*_{k,K} & \cdots & R_K \end{pmatrix}$$

containing all **unknown** quantities of interest:

- $R_k = \mathbb{E}[|z_k|^2]$ reflectivity for the channel k → roughness
- $\rho_{k,l} = \frac{\mathbb{E}[z_k z_l^*]}{\sqrt{\mathbb{E}[|z_k|^2]\mathbb{E}[|z_l|^2]}} = D_{k,l} e^{j\beta_{k,l}}$ complex correlation between k and l → geophysical properties
- $D_{k,l} \leq 1$ true coherency → agreement between acquisitions k and l

Statistical modeling of multivariate SAR data

Multi Look Complex (MLC) SAR data

- As for the intensity, the SNR can be increased with spatial/temporal average

$$\boldsymbol{C} = \boldsymbol{C}_{\mathsf{ML}} = \arg\max_{\boldsymbol{\Sigma}} \sum_{t=1}^{L} \log p(\boldsymbol{C}_t|\boldsymbol{\Sigma}, L) = \frac{1}{L}\sum_{t=1}^{L} \boldsymbol{C}_t = \frac{1}{L}\sum_{t=1}^{L} \boldsymbol{k}_t \boldsymbol{k}_t^\dagger$$

- When $L \geq K$, $\boldsymbol{C}$ becomes full-rank and invertible and follows a complex Wishart distribution

$$p(\boldsymbol{C}|\boldsymbol{\Sigma}, L) = \frac{L^{LK}|\boldsymbol{C}|^{L-K}}{\Gamma_K(L)|\boldsymbol{\Sigma}|^L} \exp\left(-L\, \mathsf{Tr}(\boldsymbol{\Sigma}^{-1}\boldsymbol{C})\right)$$

Matrices Σ and C encode all information of interest

- Intensities

$$I_k = A_k^2 = \frac{1}{L}\sum_t |z_{k,t}|^2 = C_{k,k} \qquad \Rightarrow \text{statistic for the reflectivity } R_k = \boldsymbol{\Sigma}_{k,k}$$

- Empirical complex coherency

$$d_{k,l}e^{j\phi_{k,l}} = \frac{\sum_t z_{k,t} z_{l,t}^*}{\sqrt{\sum_t |z_{k,t}|^2 \sum_t |z_{l,t}|^2}} = \frac{C_{k,l}}{\sqrt{C_{k,k}C_{l,l}}}$$

$\Rightarrow$ $\phi_{k,l}$ statistic for the true phase shift $\beta_{k,l} = \arg \rho_{k,l}$

$\Rightarrow$ $d_{k,l}$ statistic for the true coherency $D_{k,l} = |\rho_{k,l}|$

Statistical modeling of coherent imaging- Take-home messages

Speckle phenomenon

- All data acquired in coherent imaging (single channel or multi-channel) are affected by the speckle leading to strong variations in the measured quantities
- Statistics of coherent data are well modeled by the Goodman model
- This model is verified for *rough* surfaces vs the wavelength

Distributions under Goodman model

phase	uniform	intensity	exponential (L=1), Gamma (L)
real and im parts	gaussian	amplitude	Rayleigh (L=1), Nakagami (L)

Important points

- Use coefficient of variation ($\gamma = \frac{\sqrt{\mathrm{Var}[X]}}{\mathbb{E}[X]}$) instead of standard deviation to characterize homogeneity of areas
- Adapt the image processing tools by taking into account the speckle phenomenon (use the statistics of coherent imaging to modify the criteria implicitly supposing AWGN: neg-log likelihood, adapted distances, thresholds computation,...)

Statistical modeling of coherent imaging- practical work

Speckle phenomenon

- Empirical checking of the presented distributions on real images
- Computation of equivalent number of looks on real images

SAR image processing

- Computation of a mean filter
- Computation of the local standard deviation using a moving window
- Computation of the local corfficient of variation using a moving window
- Implementation and analysis of the Lee filter