Q-1) Specify the following simple regression models & provide an interpretation for the slope & intercept parameters in each case.

1. $\boxed{\text{Level - Level:}}$

$$y = \beta_0 + \beta_1 x + u$$

$\beta_1 = dy/dx$

= measures the impact of a marginal change in x on the expected value of y.

β_0 = expected value of y when $x = 0$.

2. $\boxed{\text{Log - Level}}$:

$$\log y = \beta_0 + \beta_1 x + u.$$

β_0 = Log of the expected value of y when $x = 0$.

e.g. $\text{Log}(Y) = 3.03 - 0.2x$.

then $\text{Log}(Y \text{ when } x = 0) = 3.03$.

$$\beta_1 = \frac{1}{y}\frac{dy}{dx} = \frac{dy/y}{dx}.$$

$$\beta_1 = \frac{dy/y}{dx}.$$

(1 unit)
$\Rightarrow$ a marginal change in x changes the expected value of y by $100\beta\%$.

Clearly, the impact of x on y also depends on the ~~initial value~~ level of y.

3. $\boxed{\text{Level - Log:}}$

$$y = \beta_0 + \beta_1 \log(x) + u.$$

β_0 = expected value of y when ~~[illegible]~~ $x = 1$.

$$\beta_1 = x \cdot \frac{dy}{dx} = \frac{dy}{dx/x}.$$

$\Rightarrow$ a 1% (percent) change in x leads to a change in the expected value of y by $\beta/100$ units. (i.e; 1% of β units)

Clearly, the impact of x on y also depends on the ~~initial value~~ level of x.

4 Log - Log:

$$\log y = \beta_0 + \beta_1 \log(x) + u$$

$$\frac{1}{y}\frac{dy}{dx} = \beta_1 \frac{1}{x}$$

$$\Rightarrow \boxed{\beta_1 = \frac{dy/y}{dx/x}}$$

⇒ a 1 % (percent) change in x; changes the expected value of y by β_1 %.

β_0 = expected value of $\log(y)$ when $x = 1$.

Q-2) Show for a simple linear regression model : $\boxed{Y_i = \beta_0 + \beta_1 x_i + u_i}$ where $i = 1, 2 \ldots$

Ans. that OLS estimator is given as :

$$\hat{\beta}_1 = \frac{\Sigma_i (x_i - \bar{x})(y_i - \bar{y})}{\Sigma_i (x_i - \bar{x})^2}$$

$$\hat{u}_i = Y_i - \hat{Y}_i$$

$$\hat{u}_i = Y_i - (\hat{\beta}_0 - \hat{\beta}_1 x_i)$$

$$\hat{u}_i^2 = \left[Y_i - (\hat{\beta}_0 - \hat{\beta}_1 x_i)\right]^2$$

$$\Sigma_i \hat{u}_i^2 = \Sigma_i \left[Y_i - (\hat{\beta}_0 - \hat{\beta}_1 x_i)\right]^2$$

(2)

$$\min_{\hat\beta_0, \hat\beta_1} \Sigma_i \hat{u}_i^2 = \Sigma_i (y_i - \hat\beta_0 - \hat\beta_1 x_i)^2$$ ③

FOC w.r.t $\hat\beta_0$:

$$\frac{\partial \Sigma_i \hat{u}_i^2}{\partial \hat\beta_0} = 2 \Sigma_i (y_i - \hat\beta_0 - \hat\beta_1 x_i)(-1) = 0$$

$$\Rightarrow \Sigma_i y_i - n\hat\beta_0 - \hat\beta_1 \Sigma_i x_i = 0$$

Dividing by n:

$$\bar{y} - \hat\beta_0 - \hat\beta_1 \bar{x} = 0$$

$$\boxed{\hat\beta_0 = \bar{y} - \hat\beta_1 \bar{x}} \rightarrow ①$$

$\bar{x}\bar{y}$.

FOC w.r.t $\hat\beta_1$:

$$\frac{\partial \Sigma_i \hat{u}_i^2}{\partial \hat\beta_1} = 2 \Sigma_i (y_i - \hat\beta_0 - \hat\beta_1 x_i)(-x_i) = 0$$

$$\Sigma_i (y_i - \hat\beta_0 - \hat\beta_1 x_i)(x_i) = 0$$

$$\boxed{\Sigma_i x_i y_i - \hat\beta_0 \Sigma_i x_i - \hat\beta_1 \Sigma_i x_i^2 = 0} \rightarrow ②$$

subs ① in ②:

$$\Sigma_i x_i y_i - (\bar{y} - \hat\beta_1 \bar{x}) \Sigma_i x_i - \hat\beta_1 \Sigma_i x_i^2 = 0$$

$$\Sigma_i x_i y_i - (\bar{y} \cancel{+} \hat\beta_1 \bar{x}) \Sigma_i x_i - \hat\beta_1 \Sigma_i x_i^2 = 0$$

$$\Sigma_i x_i y_i - \bar{y} \Sigma_i x_i + \hat\beta_1 \bar{x} \Sigma_i x_i - \hat\beta_1 \Sigma_i x_i^2 = 0$$

④

$$\Sigma_i x_i y_i - \bar{y}\,\Sigma_i x_i - \hat{\beta}_1 (\Sigma_i x_i^2 - n\bar{x}^2) = 0$$

$$\hat{\beta}_1 (\Sigma_i x_i^2 - n\bar{x}^2) = \Sigma_i x_i y_i - \bar{y}\,\Sigma_i x_i$$

$$\boxed{\hat{\beta}_1 = \frac{\Sigma_i x_i y_i - n\bar{x}\bar{y}}{\Sigma_i x_i^2 - n\bar{x}^2}} \rightarrow (3)$$

given: $\hat{\beta}_1 = \dfrac{\Sigma_i (x_i - \bar{x})(y_i - \bar{y})}{\Sigma_i (x_i - \bar{x})^2}$

NR:

(2) $\Sigma_i (x_i - \bar{x})(y_i - \bar{y}) = \Sigma_i [x_i y_i - \bar{x} y_i - \bar{y} x_i + \bar{x}\bar{y}]$

$$= \Sigma_i x_i y_i - \bar{x}\,\Sigma_i y_i - \bar{y}\,\Sigma_i x_i + n\bar{x}\bar{y}$$

$$= \Sigma_i x_i y_i - \bar{x}(n\bar{y}) - \cancel{\bar{y}(n\bar{x})} + \cancel{n\bar{x}\bar{y}}$$

$$= \Sigma_i x_i y_i - n\bar{x}\bar{y}$$

DR:

$\Sigma_i (x_i - \bar{x})^2 = \Sigma_i [x_i^2 + \bar{x}^2 - 2x_i\bar{x}]$

(3) $= \Sigma_i x_i^2 + n\bar{x}^2 - 2\bar{x}\,\Sigma_i x_i$

$$= \Sigma_i x_i^2 + n\bar{x}^2 - 2\bar{x}(n\bar{x})$$

$$= \Sigma_i x_i^2 + n\bar{x}^2 - 2n\bar{x}^2 = \Sigma_i x_i^2 - n\bar{x}^2$$

$\Rightarrow \hat{\beta}_1 = NR / DR.$

$$\boxed{\hat{\beta}_1 = \frac{\sum_i x_i y_i - n\bar{x}\bar{y}}{\sum_i (x_i - \bar{x})^2} = \frac{\sum_i (x_i - \bar{x})(y_i - \bar{y})}{\sum_i (x_i - \bar{x})^2}.}$$

$y \rightarrow$ test scores.

$x \rightarrow$ no. of hours invested by the student in studying maths.

Show how $\hat{\beta}_0$ & $\hat{\beta}_1$ will change if an alternate regressor is included in the model.

$$\boxed{\hat{y}_i = \hat{\beta}_0 + \hat{\beta}_1 x_i}$$

Scanned by CamScanner

2) Hours to Minutes.

⑤

$$x_i^{NEW} = 60\, x_i^{OLD}.$$

$$\hat{\beta}_1^{NEW} = \frac{\sum_i (60x_i - 60\bar{x})(y_i - \bar{y})}{\sum_i (60x_i - 60\bar{x})^2}$$

$$= \frac{60 \sum_i (x_i - \bar{x})(y_i - \bar{y})}{60^2 \sum_i (x_i - \bar{x})^2}$$

$$\boxed{\hat{\beta}_1^{NEW} = \frac{1}{60} \hat{\beta}_1^{OLD}}$$

$$\hat{\beta}_0^{NEW} = \bar{y} - \hat{\beta}_1^{NEW} [60(\bar{x})]$$

$$= \bar{y} - \frac{1}{60} \hat{\beta}_1^{OLD} (60\bar{x})$$

$$\boxed{\hat{\beta}_0^{NEW} = \bar{y} - \hat{\beta}_1^{OLD} \bar{x} = \hat{\beta}_0^{OLD}}$$

2 $\hat{\beta}_1^{NEW} = \dfrac{\Sigma_i (3600x_i - 3600\bar{x})(y_i - \bar{y})}{\Sigma_i (3600x_i - 3600\bar{x})^2}$

$= \dfrac{3600\, \Sigma_i (x_i - \bar{x})(y_i - \bar{y})}{3600^2\, \Sigma_i (x_i - \bar{x})^2}$

$$\boxed{\hat{\beta}_1^{NEW} = \frac{1}{3600} \hat{\beta}_1^{OLD}}$$

$\hat{\beta}_0^{NEW} = \bar{y} - \hat{\beta}_1^{NEW} (3600x_i)$

$= \bar{y} - \dfrac{1}{\cancel{3600}} \hat{\beta}_1^{OLD} (\cancel{3600} x_i)$

$$\boxed{\hat{\beta}_0^{NEW} = \bar{y} - \hat{\beta}_1^{OLD} x_i = \hat{\beta}_0^{OLD}}$$

③ $$\hat{\beta}_1^{NEW} = \frac{\sum_i \left(\ln x_i - \frac{\sum_i \ln x_i}{n}\right)\left(y_i - \bar{y}\right)}{\sum_i \left(\ln x_i - \frac{\sum_i \ln x_i}{n}\right)^2}$$

↳ ②

$$\hat{\beta}_0^{NEW} = \bar{y} - \hat{\beta}_1^{NEW} (\ln x_i)$$

$$\boxed{\hat{\beta}_0^{NEW} = \bar{y} - \left[\frac{\left(\ln x_i - \frac{\sum_i \ln x_i}{n}\right)\left(y_i - \bar{y}\right)}{\sum_i \left(\ln x_i - \frac{\sum_i \ln x_i}{n}\right)^2}\right] \cdot (\ln x_i)}$$

↳ ②.