MT234P – MULTIVARIABLE CALCULUS – 2022

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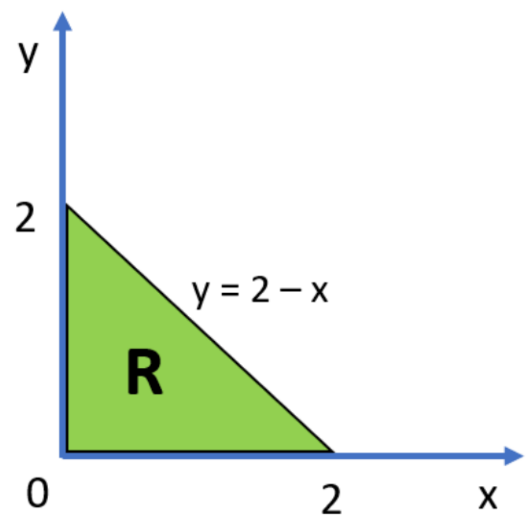
Lecture 20

Example 14 continued.

We have that $z = 2 - x - y$ and so we let $f(x, y) = 2 - x - y$ and $g(x, y, z) = xy + z$. Hence

$$\int_S\!\!\int (xy + z)\, dS = \int_R\!\!\int g(x, y, f(x, y))\sqrt{1 + f_x^2 + f_y^2}\, dA \quad \text{by theorem 6} \qquad (*)$$

where R is the triangle in the picture below.



So, $R = \{(x, y) : 0 \le x \le 2,\ 0 \le y \le 2 - x\}$

Now, $f_x = -1$ and $f_y = -1$ and so $(*)$ is

$$= \int_R\!\!\int (xy + 2 - x - y)\sqrt{1 + (-1)^2 + (-1)^2}\, dA$$

$$= \sqrt{3}\int_R\!\!\int (xy + 2 - x - y)\, dA$$

$$= \sqrt{3}\int_0^2\int_0^{2-x} (xy + 2 - x - y)\, dy dx \quad \text{by Fubini's theorem}$$

$$= \sqrt{3}\int_0^2 \left[\frac{xy^2}{2} + 2y - xy - \frac{y^2}{2}\right]_0^{2-x} dx$$

$$= \sqrt{3}\int_0^2 \left(\frac{x(2-x)^2}{2} + 2(2-x) - x(2-x) - \frac{(2-x)^2}{2}\right) dx$$

$$= \sqrt{3} \int_0^2 \left(\frac{x(2-x)^2}{2} + \frac{(2-x)^2}{2} \right) dx$$

$$= \sqrt{3} \int_0^2 \left(\frac{4x - 4x^2 + x^3}{2} + \frac{4 - 4x + x^2}{2} \right) dx$$

$$= \sqrt{3} \int_0^2 \left(\frac{4 - 3x^2 + x^3}{2} \right) dx$$

$$= \sqrt{3} \left[2x - \frac{x^3}{2} + \frac{x^4}{8} \right]_0^2$$

$$= \sqrt{3}(4 - 4 + 2)$$

$$= 2\sqrt{3}$$

Example 15.

Suppose a flat sheet of metal has the shape given by the surface $z = 1 + x + 2y$ that lies above the rectangle R given by $0 \le x \le 4$, $0 \le y \le 2$. If the density of the sheet is given by $g(x, y, z) = x^2yz$, then find the mass M of the sheet, where M is given by the surface integral

$$M = \int_S \int g(x, y, z)\, dS$$

where S is the surface given by $z = 1 + x + 2y$.

Solution.

Let $f(x, y) = 1 + x + 2y$. Then,

$$M = \int_S \int g(x, y, z)\, dS$$

$$= \int_R \int g(x, y, 1 + x + 2y) \sqrt{1 + f_x^2 + f_y^2}\, dA, \quad \text{by theorem 6}$$

$$= \int_0^4 \int_0^2 x^2 y(1 + x + 2y) \sqrt{1 + 1 + 4}\, dy dx, \quad \text{by Fubini's theorem}$$

$$= \sqrt{6} \int_0^4 \int_0^2 (x^2y + x^3y + 2x^2y^2)\, dy dx$$

$$= \sqrt{6} \int_0^4 \left[\frac{x^2y^2}{2} + \frac{x^3y^2}{2} + \frac{2x^2y^3}{3} \right]_0^2 dx$$

$$= \sqrt{6}\int_0^4 \left(2x^2 + 2x^3 + \frac{16x^2}{3}\right)\, dx$$

$$= \sqrt{6}\int_0^4 \left(\frac{22x^2}{3} + 2x^3\right)\, dx$$

$$= \sqrt{6}\left[\frac{22x^3}{9} + \frac{x^4}{2}\right]_0^4$$

$$= \sqrt{6}\left(\frac{1408}{9} + 128\right)$$

$$= \frac{2560\sqrt{6}}{9}$$

Section 4.5 – Triple Integrals.

Remark 14.

In definition 1 we defined the double integral of a function of two variables, $f(x, y)$, over a set in $\mathbb{R}^2$. In this section we will discuss (what will be called) the triple integral of a function of three variables, $g(x, y, z)$, over a set in $\mathbb{R}^3$. Recall how useful Fubini's theorem was for finding double integrals by performing two single integrals (one after the other). Well, we will also have a Fubini's theorem for triple integrals which will show how a triple integral can be evaluated by performing three single integrals (one after the other).

We will now motivate the definition of a triple integral. Suppose, $g(x, y, z)$ is a function defined on a closed bounded set in T in $\mathbb{R}^3$. We partition a rectangular region about T into n rectangular cells by planes parallel to the coordinate planes (i.e. the xy-plane, xz-plane and yz-plane). Suppose the k^{th} cell has volume ΔV_k. Pick a point (x_k, y_k, z_k) in the k^{th} cell and consider the sum

$$W = \sum_{k=1}^{n} g(x_k, y_k, z_k)\Delta V_k \qquad (*)$$

Then, the triple integral of g over T is denoted by

$$\int\int_T\int g(x, y, z)\, dV$$

and is defined to be the limit of $(*)$ above as the length of the longest diagonal of the n cells goes to zero. We say that g is integrable over T if this limit exists.

Remark 15.

If g is the constant function 1 in $(*)$ above, then $(*)$ will approximate the volume of T. This approximation will get better as the cells get smaller. So, we can define the volume of T to be the limit above and hence

$$\text{Volume of } T = \iiint_T dV$$

Remark 16 – Some properties of triple integrals.

Suppose all the integrals below exist. Then

$$\iiint_W kf\,dV = k\iiint_W f\,dV, \quad \text{for all} \quad k \in \mathbb{R}$$

$$\iiint_W (f+g)\,dV = \iiint_W f\,dV + \iiint_W g\,dV$$

$$\iiint_W (f-g)\,dV = \iiint_W f\,dV - \iiint_W g\,dV$$

$$\iiint_W f\,dV \geq 0 \quad \text{if} \quad f \geq 0 \quad \text{on} \quad W$$

$$\iiint_W f\,dV \geq \iiint_W g\,dV \quad \text{if} \quad f \geq g \quad \text{on} \quad W$$

$$\iiint_W f\,dV = \iiint_{A_1} f\,dV + \iiint_{A_2} f\,dV + \cdots \iiint_{A_n} f\,dV$$

where W is the union of the pairwise disjoint sets $\{A_1, A_2, \ldots, A_n\}$

Theorem 7 – Fubini's theorem for triple integrals.

Suppose g is continuous on a set W in $\mathbb{R}^3$.

(i) If $W = \{(x,y,z) \in \mathbb{R}^3 : a \leq x \leq b,\ f_1(x) \leq y \leq f_2(x),\ h_1(x,y) \leq z \leq h_2(x,y)\}$, where f_1, f_2, h_1, h_2 are continuous functions, then

$$\iiint_W g(x,y,z)\,dV = \int_a^b \int_{f_1(x)}^{f_2(x)} \int_{h_1(x,y)}^{h_2(x,y)} g(x,y,z)\,dzdydx$$

(ii) If $W = \{(x,y,z) \in \mathbb{R}^3 : a \leq x \leq b,\ f_1(x) \leq z \leq f_2(x),\ h_1(x,z) \leq y \leq h_2(x,z)\}$, where f_1, f_2, h_1, h_2 are continuous functions, then

$$\iiint_W g(x,y,z)\,dV = \int_a^b \int_{f_1(x)}^{f_2(x)} \int_{h_1(x,z)}^{h_2(x,z)} g(x,y,z)\,dydzdx$$

(iii) If $W = \{(x, y, z) \in \mathbb{R}^3 : c \leq y \leq d,\ k_1(y) \leq x \leq k_2(y),\ h_1(x, y) \leq z \leq h_2(x, y)\}$, where k_1, k_2, h_1, h_2 are continuous functions, then

$$\int\int_W\int g(x, y, z)\, dV = \int_c^d \int_{k_1(y)}^{k_2(y)} \int_{h_1(x,y)}^{h_2(x,y)} g(x, y, z)\, dzdxdy$$

(iv) If $W = \{(x, y, z) \in \mathbb{R}^3 : c \leq y \leq d,\ k_1(y) \leq z \leq k_2(y),\ h_1(y, z) \leq x \leq h_2(y, z)\}$, where k_1, k_2, h_1, h_2 are continuous functions, then

$$\int\int_W\int g(x, y, z)\, dV = \int_c^d \int_{k_1(y)}^{k_2(y)} \int_{h_1(y,z)}^{h_2(y,z)} g(x, y, z)\, dxdzdy$$

(v) If $W = \{(x, y, z) \in \mathbb{R}^3 : a \leq z \leq b,\ k_1(z) \leq x \leq k_2(z),\ s_1(x, z) \leq y \leq s_2(x, z)\}$, where k_1, k_2, s_1, s_2 are continuous functions, then

$$\int\int_W\int g(x, y, z)\, dV = \int_a^b \int_{k_1(z)}^{k_2(z)} \int_{s_1(x,z)}^{s_2(x,z)} g(x, y, z)\, dydxdz$$

(vi) If $W = \{(x, y, z) \in \mathbb{R}^3 : a \leq z \leq b,\ k_1(z) \leq y \leq k_2(z),\ s_1(y, z) \leq x \leq s_2(y, z)\}$, where k_1, k_2, s_1, s_2 are continuous functions, then

$$\int\int_W\int g(x, y, z)\, dV = \int_a^b \int_{k_1(z)}^{k_2(z)} \int_{s_1(y,z)}^{s_2(y,z)} g(x, y, z)\, dxdydz$$

Remark 17.

In Fubini's theorem (i) and (ii) we say W is simple in the x–direction. In these cases we integrate w.r.t. x last. In Fubini's theorem (iii) and (iv) we say W is simple in the y–direction. In these cases we integrate w.r.t. y last. In Fubini's theorem (v) and (vi) we say W is simple in the z–direction. In these cases we integrate w.r.t. z last. In relation to what variables we integrate w.r.t. first and second one should look at which of the six statements (in theorem 7) apply. All the information one needs is in the six statements.