MT251P – Lecture 4

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Implication and Conditional Statements.

Conditional statements of the form 'If P, then Q' are constructed using the implication operation $\Rightarrow$

Note that P implies Q (i.e. $P \Rightarrow Q$) is false if and only if P is true and Q is false.

The truth table for $P \Rightarrow Q$ is

P	Q	$P \Rightarrow Q$
T	T	T
T	F	F
F	T	T
F	F	T

Check that $P \Rightarrow Q$ is equivalent to $(\neg P) \vee Q$.

Remark 9.

The 'if and only if' statement is denoted by $P \iff Q$ and can now be defined as

$$P \iff Q = (P \Rightarrow Q) \wedge (Q \Rightarrow P)$$

Remark 10.

The operation $\neg$ has precedence over $\wedge$ which has precedence over $\vee$. For example

$$\neg P \wedge Q \quad \text{means} \quad (\neg P) \wedge Q \quad \text{and doesn't mean} \quad \neg(P \wedge Q)$$

Check that $(\neg P) \wedge Q$ is not equivalent to $\neg(P \wedge Q)$ by showing they don't have identical truth tables.

Tautology.

A formula is a tautology if it takes the value T for all possible values of its variables.

Example 14.

$P \vee \neg P$ is a tautology because its truth table is:

P	$\neg P$	$P \vee \neg P$
T	F	T
F	T	T

Example 15.

Suppose P, Q, R are three propositions. Then

$$(P \wedge Q) \wedge R \sim P \wedge (Q \wedge R) \qquad (*)$$

Proof:

P	Q	R	$P \wedge Q$	$(P \wedge Q) \wedge R$
T	T	T	T	T
T	T	F	T	F
T	F	T	F	F
T	F	F	F	F
F	T	T	F	F
F	T	F	F	F
F	F	T	F	F
F	F	F	F	F

P	Q	R	$Q \wedge R$	$P \wedge (Q \wedge R)$
T	T	T	T	T
T	T	F	F	F
T	F	T	F	F
T	F	F	F	F
F	T	T	T	F
F	T	F	F	F
F	F	T	F	F
F	F	F	F	F

Remark 11.

One can show that the following equivalences hold:

Idempotency: $P \vee P \sim P, \quad P \wedge P \sim P$

Associativity: $P \vee (Q \vee R) \sim (P \vee Q) \vee R, \quad P \wedge (Q \wedge R) \sim (P \wedge Q) \wedge R,$

Commutativity: $P \vee Q \sim Q \vee P, \quad P \wedge Q \sim Q \wedge P$

Distributivity: $P \wedge (Q \vee R) \sim (P \wedge Q) \vee (P \wedge R), \quad P \vee (Q \wedge R) \sim (P \vee Q) \wedge (P \vee R)$

Identity: Here we consider T to denote a formula which always has the value T and we consider F to denote a formula which always has the value F. Then, we have $P \vee F \sim P$, $P \vee T \sim T, \quad P \wedge F \sim F, \quad P \wedge T \sim P$

Complements: $P \vee \neg P \sim T, \quad P \wedge \neg P \sim F$

Involution: $\neg(\neg P) \sim P$

De Morgan's Laws: $\neg(P \vee Q) \sim \neg P \wedge \neg Q, \quad \neg(P \wedge Q) \sim \neg P \vee \neg Q$

Remark 12.

The equivalences in remark 11 can be used to prove other equivalences and to simplify propositional expressions by using the following two rules:

(i) If we replace a variable in two equivalent formulas by the same arbitrary formula in both, then we will end up with two equivalent formulas again.

(ii) A subformula within a formula can be replaced by an equivalent formula.