Fiacre Ó Cairbre

Lecture 18

Definition 5.

We will now define the length of a curve. Suppose C is a curve in the xy-plane given by the parametric equations $x = x(t),\ y = y(t)$ for $a \leq t \leq b$. Suppose

$$\frac{dx}{dt} \quad \text{and} \quad \frac{dy}{dt}$$

are continuous on $[a, b]$. If C is traced once as t moves from a to b, then the length of C is

$$\int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

Suppose B is a curve in $\mathbb{R}^3$ given by the parametric equations $x = x(t),\ y = y(t),\ z = z(t)$ for $a \leq t \leq b$. Suppose

$$\frac{dx}{dt}, \quad \frac{dy}{dt}, \quad \frac{dz}{dt}$$

are continuous on $[a, b]$. If B is traced once as t moves from a to b, then the length of B is

$$\int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt$$

Example 10.

Suppose B is the helix given by

$$x = \cos t,\ y = \sin t,\ z = t, \quad \text{for} \quad 0 \leq t \leq 2\pi$$

The length of B is

$$\int_0^{2\pi} \sqrt{(-\sin t)^2 + (\cos t)^2 + (1)^2}\, dt$$

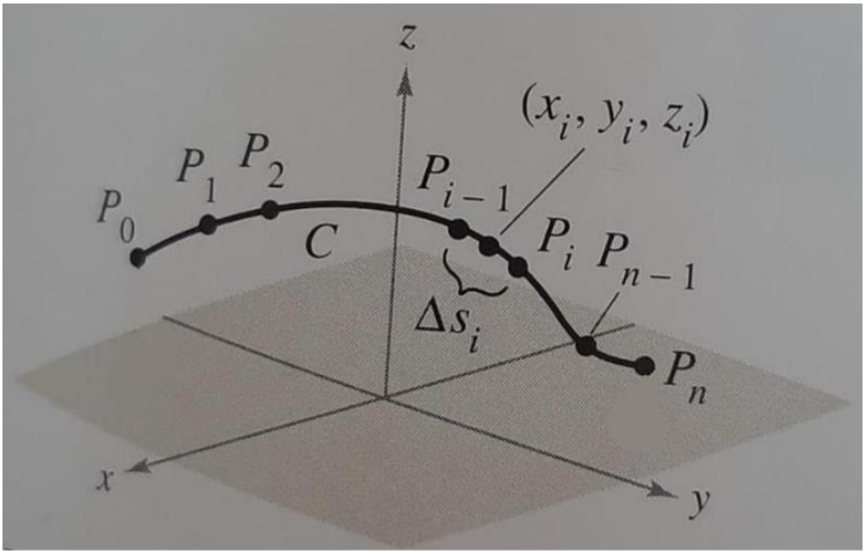
$$= \int_0^{2\pi} \sqrt{2}\, dt$$

$$= 2\pi\sqrt{2}$$

Remark 10.

We previously defined single integrals and double integrals. We will now define the so called line integral. Recall that we motivated the definition of a single integral by looking at the notion of area and we motivated the definition of a double integral by looking at volume. Here we will motivate the definition of a line integral by looking at the mass of a wire of finite length given by a curve C in $\mathbb{R}^3$.

Suppose the density (i.e. mass per unit length) of the wire at the point (x, y, z) is given by $f(x, y, z)$. Subdivide the curve C by the points $P_0, P_1, \ldots, P_n$ giving n subarcs as in the picture below.



Denote the length of the i^{th} subarc by Δs_i and let $||\Delta||$ denote the length of the longest subarc. Now, select a point (x_i, y_i, z_i) in the i^{th} subarc. We say that the sum

$$\sum_{i=1}^{n} f(x_i, y_i, z_i)\Delta s_i$$

approximates the total mass of the wire. Note that this approximation may not necessarily be a good approximation if $||\Delta||$ is big. However, we expect the approximation to improve as $||\Delta||$ approaches zero. With this as motivation, we define the mass of the wire to be

$$\lim_{||\Delta|| \to 0} \sum_{i=1}^{n} f(x_i, y_i, z_i)\Delta s_i$$

where this limit is defined in a similar way as (**) in remark 1

With the above as motivation (and using similar notation as above), we are now ready to define the so called line integral of f along C as follows:

Definition 6.

Suppose f is defined in a region containing a smooth curve C of finite length. The line integral of f along C is defined as

$$\int_C f(x,y,z)\ ds = \lim_{||\Delta||\to 0} \sum_{i=1}^{n} f(x_i, y_i, z_i)\Delta s_i$$

provided this limit exists.

We can similarly define the line integral for a function of two variables. So, suppose $g(x,y)$ is defined in a region containing a smooth curve C of finite length. The line integral of g along C is defined as

$$\int_C g(x,y)\ ds = \lim_{||\Delta||\to 0} \sum_{i=1}^{n} g(x_i, y_i)\Delta s_i$$

provided this limit exists.

Remark 11.

The following theorem shows how to calculate line integrals by using single integrals.

Theorem 3.

(i) Suppose g is continuous on a set containing a smooth curve C in $\mathbb{R}^2$. Suppose C has parametric equations

$$x = x(t),\ \ y = y(t) \quad \text{for} \quad t \in [a,b]$$

Then

$$\int_C g(x,y)\, ds = \int_a^b g(x(t), y(t))\sqrt{(x'(t))^2 + (y'(t))^2}\, dt$$

Note that here $x'(t) = \dfrac{dx}{dt}$ and $y'(t) = \dfrac{dy}{dt}$

(ii) Suppose h is continuous on a set containing a smooth curve C in $\mathbb{R}^3$. Suppose C has parametric equations

$$x = x(t),\ \ y = y(t), z = z(t) \quad \text{for} \quad t \in [a,b]$$

Then

$$\int_C h(x,y,z)\, ds = \int_a^b h(x(t), y(t), z(t))\sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}\, dt$$

Note that here $z'(t) = \dfrac{dz}{dt}$

Remark 12.

In the special case where $g(x, y) = 1$, for all (x, y) in the domain of g above, then we get that the line integral

$$\int_C g(x, y)\, ds$$

is the length of the curve C because

$$\int_C g(x, y)\, ds = \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2}\, dt$$

Example 11.

There are many applications of line integrals. For example, line integrals can be used to calculate masses. A coil spring lies along the helix C given by

$$x = \cos 4t,\ y = \sin 4t,\ z = t, \quad \text{for} \quad t \in [0, 2\pi]$$

The spring's density is the constant function $f(x, y, z) = 1$. Find the mass of the spring where the mass M of the spring is given by the line integral

$$M = \int_C f(x, y, z)\, ds$$

Solution.

$$M = \int_C f(x, y, z)\, ds$$

$$= \int_0^{2\pi} f(x(t), y(t), z(t))\sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}\, dt \qquad (*)$$

Now

$$x'(t) = -4\sin 4t,\ y'(t) = 4\cos 4t,\ z'(t) = 1$$

.

Hence

$$(*) = \int_0^{2\pi} \sqrt{17}\, dt$$

$$= 2\pi\sqrt{17}$$

So, the mass of the spring is $2\pi\sqrt{17}$.