MT251P – Lecture 15

Continuation of example 10.

Replace R_2 with $R_2 - 2R_1$ and replace R_3 with $R_3 - 3R_1$ to get

$$\begin{pmatrix} 1 & 6 & 0 & 3 & 2 \\ 0 & 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & 0 & -4 \end{pmatrix}$$

Replace R_3 with $R_3 - R_2$ to get

$$C = \begin{pmatrix} 1 & 6 & 0 & 3 & 2 \\ 0 & 0 & 1 & 0 & -4 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

which is in RREF.

By remark 4, we are in the free variable case. x_1 and x_3 are leading variables and x_2 and x_4 are free variables.

The system of linear equations V corresponding to C is:

$$x_1 + 6x_2 + 3x_4 = 2 \qquad (i)$$

$$x_3 = -4 \qquad (ii)$$

By remark 4, we say $x_2 = s$ and $x_4 = t$, where s can be any real number and t can be any real number. We write the leading variables, $x_1,\ x_3$ in terms of the free variables $x_2,\ x_4$ as follows:

$$x_1 = 2 - 6x_2 - 3x_4 = 2 - 6s - 3t \qquad (iii)$$

$$x_3 = -4 \qquad (ii)$$

So, the required solution set for W is

$$\{(2 - 6s - 3t, s, -4, t) : s, t \in \mathbb{R}\}$$

Section 4.4 – Matrix Operations.

Definition 9.

(i) If C is a $k \times n$ matrix, we say that $k \times n$ is the size of C.

(ii) Suppose A and B are two matrices. Then the sum, $A + B$ and the difference $A - B$ are only defined if A and B have the same size. In this case we add or subtract the matrices by adding or subtracting the corresponding entries.

(iii) If A is a matrix and t is a scalar (i.e. t is a real number), then tA is the matrix obtained by multiplying each entry of A by t.

Example 11.

$$\begin{pmatrix} 2 & -1 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 1 & 0 \end{pmatrix}$$

$$\text{If } \ A = \begin{pmatrix} 2 & -3 & 4 \\ 9 & 0 & 2 \end{pmatrix} \quad \text{then } \ 3A = \begin{pmatrix} 6 & -9 & 12 \\ 27 & 0 & 6 \end{pmatrix}$$

Definition 10 – Matrix multiplication.

Suppose $A = [a_{ij}]$ is a $k \times n$ matrix and $B = [b_{ij}]$ is an $n \times p$ matrix. Then, the (matrix) product AB is a $k \times p$ matrix and is defined as $AB = [c_{ij}]$ where

$$c_{ij} = \sum_{t=1}^{n} a_{it} b_{tj} = a_{i1} b_{1j} + a_{i2} b_{2j} + \cdots + a_{in} b_{nj}$$

Example 12.

$$A = \begin{pmatrix} 2 & 3 \\ -1 & 0 \\ 2 & 4 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & -2 \\ 3 & 2 \end{pmatrix}$$

$$AB = \begin{pmatrix} 2(1) + 3(3) & 2(-2) + 3(2) \\ -1(1) + 0(3) & -1(-2) + 0(2) \\ 2(1) + 4(3) & 2(-2) + 4(2) \end{pmatrix} = \begin{pmatrix} 11 & 2 \\ -1 & 2 \\ 14 & 4 \end{pmatrix}$$

Definition 11.

(i) A square matrix is a matrix with the same number of rows as columns.

(ii) If $A = [a_{ij}]$ is square matrix of size $n \times n$, then the entries $a_{11},\ a_{22},\ a_{33}, \ldots, a_{nn}$ are called the entries along the main diagonal of A.

(iii) The trace of a square matrix A is the sum of the entries along the main diagonal of A and is denoted by $tr(A)$, i.e.

$$tr(A) = \sum_{i=1}^{n} a_{ii}, \quad \text{if} \quad A = [a_{ij}] \quad \text{is of size } \ n \times n$$

Example 13.

$$A = \begin{pmatrix} 2 & -3 & 0 \\ 4 & 1 & 2 \\ 9 & -3 & -1 \end{pmatrix}$$

$tr(A) = 2$

Definition 12.

If A is a $k \times n$ matrix, then the transpose of A is denoted by A^T and is the $n \times k$ matrix we get by swapping the rows and columns of A, i.e. the i^{th} row of A^T is the i^{th} column of A and the j^{th} column of A^T is the j^{th} row of A.

Example 14.

If $A = \begin{pmatrix} -1 & 0 & 3 \\ 4 & 2 & 1 \end{pmatrix}$ then $A^T = \begin{pmatrix} -1 & 4 \\ 0 & 2 \\ 3 & 1 \end{pmatrix}$

Remark 5.

If A is a $k \times n$ matrix and B is an $n \times p$ matrix, then

(i) $(A^T)^T = A$

(ii) $(AB)^T = B^T A^T$.

Remark 6 – Properties of Matrix operations.

Suppose we have matrices $A,\ B,\ C$ and scalars $r,\ t$ and suppose that everything below is defined. Then,

1. $A + B = B + A$
2. $A + (B + C) = (A + B) + C$
3. $A(BC) = (AB)C$
4. $A(B + C) = AB + AC$
5. $(A + B)C = AC + BC$
6. $r(A + B) = rA + rB$
7. $(r + t)A = rA + tA$
8. $(rt)A = r(tA)$
9. $r(AB) = (rA)B = A(rB)$