MT251P - Foundations of Euclidean Geometry

Assignment #4

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Question 1

In each case below, state whether the statement is true or false. Justify your answer in each case.

Part A

There are infinitely many 4×4 matrices that are not invertible.

Solution

Since a square matrix is invertible if and only if its determinant is non-zero. Since there are infinitely many possible values for the elements of a 4×4 matrix, there are also infinitely many matrices that have a determinant of zero ie there are infinite amount of matrices with a zero row and any matrix with a zero row has det = 0 and therefore not invertible. Therefore the statement is True

Part B

There is a 4×4 invertible matrix A such that $A^3 = 2A^2$ and $\det A = 2$.

Solution

$$A^3 = 2A^2$$

$$A(AA) = There2(AA)$$

Since A is invertible, then A^{-1} exists. So:

$$A(AA)(A^{-1}A^{-1}) = 2(AA)(A^{-1}A^{-1})$$

Since $AA^{-1} = I_4 = A^{-1}A$, then:

$$A(AA^{-1})(AA^{-1}) = 2(AA^{-1})(AA^{-1})$$

We also know that $AA^{-1} = I_4$ since A is a 4×4 matrix. So:

$$A(I_4)(I_4) = 2(I_4)(I_4)$$

$$A = 2(I_4)$$

$$\det(A) = \det(2(I_4))$$

$$\det(A) = 16$$

Since $16 \neq 2$, then the statement is False.

Part C

Question 1 continued on next page. . .

There is a 4×4 matrix A such that $A^2 = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

Solution

$$\det\left(A^2\right) = \det(A)\det(A) = \det(A)^2$$

Since $\det(A) \in \mathbb{R}$, then $\det\left(A^2\right) \geq 0$
However,

$$\begin{aligned} \det\left(A^2\right) &= -1 \det \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ &= -1 \det \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ &= -1 \end{aligned}$$

This gives a contradiction. Therefore the statement is False.

Question 2

Part A

Prove that $\det\left(A^{-1}BA\right) = \det(B)$, for all n × n matrices A, B, where A is invertible and n > 1.

Solution

Let A be an invertible $n \times n$ matrix, and let B be any $n \times n$ matrix. We have $\det\left(A^{-1}BA\right) = \det\left(A^{-1}\right)\det(B)\det(A) = \frac{1}{\det(A)}\det(B)\det(A) = \det(B)$, so $\det\left(A^{-1}BA\right) = \det(B)$

Part B

Suppose $\underline{a} = i + 2j - k$, $\underline{b} = i + 3j + k$ and $\underline{c} = 3i + 8j + 4k$. Find $||\underline{w}||^2$ if $\underline{w} \in \mathbb{R}^3$ such that $\underline{w}.\underline{a} = 3$, $\underline{w}.\underline{b} = 5$ and $\underline{w}.\underline{c} = 17$.

Solution

Since $\underline{w}$ is in $\mathbb{R}^3$, it must be represented in the form of $\underline{w} = xi + yj + zk$ for some $x, y, z \in \mathbb{R}$.
We have the following:

$$\underline{w}.\underline{a} = (xi + yj + zk).(i + 2j - k) = x + 2y - z = 3$$

$$\underline{w}.\underline{b} = (xi + yj + zk).(i + 3j + k) = x + 3y + z = 5$$

Question 2 continued on next page. . .

$$\underline{w}.\underline{c} = (xi + yj + zk).(3i + 8j + 4k) = 3x + 8y + 4z = 17$$

Now we can solve for x, y, z using a matrix:

$$\begin{pmatrix} 1 & 2 & -1 & 3 \\ 1 & 3 & 1 & 5 \\ 3 & 8 & 4 & 17 \end{pmatrix}$$

Replace R_2 with $R_2 - R_1$

$$\begin{pmatrix} 1 & 2 & -1 & 3 \\ 0 & 1 & 2 & 2 \\ 3 & 8 & 4 & 17 \end{pmatrix}$$

Replace R_3 with $R_3 - 3R_1$

$$\begin{pmatrix} 1 & 2 & -1 & 3 \\ 0 & 1 & 2 & 2 \\ 0 & 2 & 7 & 8 \end{pmatrix}$$

Replace R_3 with $R_3 - 2R_2$

$$\begin{pmatrix} 1 & 2 & -1 & 3 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 3 & 4 \end{pmatrix}$$

Replace R_1 with $R_1 - 2R_2$

$$\begin{pmatrix} 1 & 0 & -5 & -1 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 3 & 4 \end{pmatrix}$$

Replace R_3 with $\frac{1}{3}R_3$

$$\begin{pmatrix} 1 & 0 & -5 & -1 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 1 & \frac{4}{3} \end{pmatrix}$$

Replace R_2 with $R_2 - 2R_3$

$$\begin{pmatrix} 1 & 0 & -5 & -1 \\ 0 & 1 & 0 & -\frac{2}{3} \\ 0 & 0 & 1 & \frac{4}{3} \end{pmatrix}$$

Replace R_1 with $R_1 - (-5)R_3$

$$\begin{pmatrix} 1 & 0 & 0 & \frac{17}{3} \\ 0 & 1 & 0 & -\frac{2}{3} \\ 0 & 0 & 1 & \frac{4}{3} \end{pmatrix}$$

Then: $x = \frac{17}{3}, y = -\frac{2}{3}, z = \frac{4}{3}$

Therefore $||\underline{w}|| = \sqrt{x^2 + y^2 + z^2} = \sqrt{\frac{17}{3}^2 - \frac{2}{3}^2 + \frac{4}{3}^2} = \frac{\sqrt{309}}{3}$

Then: $||\underline{w}||^2 = (\frac{\sqrt{309}}{3})^2 = \frac{103}{3}$

Question 3

Find the solution set of the following system of linear equations:

$$x_1 - 4x_2 + 3x_3 = 0$$

$$2x_1 - 6x_2 + 10x_3 = 6$$

$$x_1 - 2x_2 + 7x_3 = 5$$

Solution

We can solve for x_1, x_2, x_3 using a matrix:

$$\begin{pmatrix} 1 & -4 & 3 & 0 \\ 2 & -6 & 10 & 6 \\ 1 & -2 & 7 & 5 \end{pmatrix}$$

Replace R_2 with $R_2 - 2R_1$

$$\begin{pmatrix} 1 & -4 & 3 & 0 \\ 0 & 2 & 4 & 6 \\ 1 & -2 & 7 & 5 \end{pmatrix}$$

Replace R_3 with $R_3 - R_1$

$$\begin{pmatrix} 1 & -4 & 3 & 0 \\ 0 & 2 & 4 & 6 \\ 0 & 2 & 4 & 5 \end{pmatrix}$$

Replace R_3 with $R_3 - R_2$

$$\begin{pmatrix} 1 & -4 & 3 & 0 \\ 0 & 2 & 4 & 6 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Notice how the last row states: $0 = -1$, therefore the system has no solutions.

Question 4

Find the solution set of the following system of linear equations:

$$4x - 6y + 8z = 8$$

$$x + 2y - 5z = 2$$

$$y + 4x - 6z = 8$$

Solution

We can solve for x, y, z using a matrix:

$$\begin{pmatrix} 4 & -6 & 8 & 8 \\ 1 & 2 & -5 & 2 \\ 4 & 1 & -6 & 8 \end{pmatrix}$$

Replace R_1 with R_2

$$\begin{pmatrix} 1 & 2 & -5 & 2 \\ 4 & -6 & 8 & 8 \\ 4 & 1 & -6 & 8 \end{pmatrix}$$

Replace R_2 with $R_2 - 4R_1$

$$\begin{pmatrix} 1 & 2 & -5 & 2 \\ 0 & -14 & 28 & 0 \\ 4 & 1 & -6 & 8 \end{pmatrix}$$

Replace R_3 with $R_3 - 4R_1$

$$\begin{pmatrix} 1 & 2 & -5 & 2 \\ 0 & -14 & 28 & 0 \\ 0 & -7 & 14 & 0 \end{pmatrix}$$

Question 4 continued on next page. . .

Replace R_2 with $\frac{1}{2}R_2$

$$\begin{pmatrix} 1 & 2 & -5 & 2 \\ 0 & -7 & 14 & 0 \\ 0 & -7 & 14 & 0 \end{pmatrix}$$

Replace R_3 with $R_3 - R_2$

$$\begin{pmatrix} 1 & 2 & -5 & 2 \\ 0 & -7 & 14 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Replace R_2 with $-\frac{1}{7}R_2$

$$\begin{pmatrix} 1 & 2 & -5 & 2 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Replace R_1 with $R_1 - 2R_2$

$$\begin{pmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Notice how the matrix is in RREF and the last row states $0 = 0$, therefore the system has infinitely many solutions. Then the system of linear equations is:

$$x - z = 2$$

$$y - 2z = 0$$

By Remark 4, x, y are leading variables, and z is a free variable. We say that z = s, where s can be any real number. Then we can represent x, y in terms of z:

$$x = 2 + z = 2 + s$$

$$y = 2z = 2s$$

Therefore the solution set of the system of equations is:

$$\{(2 + s, 2s, s) : s \in \mathbb{R}\}$$

Question 5

Find A^{-1} if $A = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 4 & -3 & 8 \end{pmatrix}$

Solution

$$A = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 4 & -3 & 8 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I_3$$

Replace R_1 with R_2 in both matrices

$$\begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 4 & -3 & 8 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Question 5 continued on next page. . .

Replace R_3 with $R_3 - 4R_1$ in both matrices

$$\begin{pmatrix}1 & 0 & 3\\0 & 1 & 2\\0 & -3 & -4\end{pmatrix}\begin{pmatrix}0 & 1 & 0\\1 & 0 & 0\\0 & -4 & 1\end{pmatrix}$$

Replace R_3 with $R_3 - (-3)R_2$ in both matrices

$$\begin{pmatrix}1 & 0 & 3\\0 & 1 & 2\\0 & 0 & 2\end{pmatrix}\begin{pmatrix}0 & 1 & 0\\1 & 0 & 0\\3 & -4 & 1\end{pmatrix}$$

Replace R_3 with $\frac{1}{2}R_3$ in both matrices

$$\begin{pmatrix}1 & 0 & 3\\0 & 1 & 2\\0 & 0 & 1\end{pmatrix}\begin{pmatrix}0 & 1 & 0\\1 & 0 & 0\\\frac{3}{2} & -2 & \frac{1}{2}\end{pmatrix}$$

Replace R_2 with $R_2 - 2R_3$ in both matrices

$$\begin{pmatrix}1 & 0 & 3\\0 & 1 & 0\\0 & 0 & 1\end{pmatrix}\begin{pmatrix}0 & 1 & 0\\-2 & 4 & -1\\\frac{3}{2} & -2 & \frac{1}{2}\end{pmatrix}$$

Replace R_1 with $R_1 - 3R_3$ in both matrices

$$\begin{pmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{pmatrix}\begin{pmatrix}-\frac{9}{2} & 7 & -\frac{3}{2}\\-2 & 4 & -1\\\frac{3}{2} & -2 & \frac{1}{2}\end{pmatrix}$$

Since A is in RREF, then:

$$A^{-1} = \begin{pmatrix}-\frac{9}{2} & 7 & -\frac{3}{2}\\-2 & 4 & -1\\\frac{3}{2} & -2 & \frac{1}{2}\end{pmatrix}$$