MT251P – Lecture 18

Example 4.

Suppose $A = \begin{pmatrix} 2 & 9 & 3 \\ 0 & 1 & 4 \\ 0 & 4 & -\frac{1}{2} \end{pmatrix}$

Find $\det A$.

Solution.

Use (i) in Theorem 5 above and expand along the first column of A to get

$$\det(A) = a_{11}C_{11} + a_{21}C_{21} + a_{31}C_{31}$$

$$= a_{11}C_{11}$$

$$= 2\det \begin{pmatrix} 1 & 4 \\ 4 & -\frac{1}{2} \end{pmatrix}$$

$$= 2(-\frac{33}{2})$$

$$= -33$$

Theorem 6

Suppose A is an $n \times n$ matrix. Then, A is invertible $\iff$ $\det A \neq 0$.

Proof

We will prove $\Rightarrow$.

$$AA^{-1} = I_n$$

$$\Rightarrow \det(AA^{-1}) = \det I_n = 1$$

$$\Rightarrow \det A \det(A^{-1}) = 1$$

$$\Rightarrow \det A \neq 0$$

Example 5.

Suppose $A = \begin{pmatrix} 2 & 0 & 9 & 3 \\ 9 & 4 & 4 & 4 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 4 & -\frac{1}{2} \end{pmatrix}$

Find $\det A$ and use it to show A is invertible.

Solution.

Expand along the second column of A to get

$$\det A = 4 \det \begin{pmatrix} 2 & 9 & 3 \\ 0 & 1 & 4 \\ 0 & 4 & -\frac{1}{2} \end{pmatrix}$$

$$= 4(-33) \quad \text{by example 4}$$

$$= -132$$

So, $\det A \neq 0$ and hence A is invertible.

Example 6.

Suppose $B = \begin{pmatrix} 5 & 2 & 3 & 0 \\ 9 & -2 & 4 & 0 \\ 1 & 4 & 5 & -2 \\ 0 & 3 & 0 & 0 \end{pmatrix}$

Find $\det(B^2)$.

Solution.

Expand along the fourth row of B to get

$$\det B = 3 \det \begin{pmatrix} 5 & 3 & 0 \\ 9 & 4 & 0 \\ 1 & 5 & -2 \end{pmatrix}$$

$$= 3 \left(-2 \det \begin{pmatrix} 5 & 3 \\ 9 & 4 \end{pmatrix} \right)$$

$$= -6(-7) = 42$$

So, by (x) in Theorem 5, we get $\det(B^2) = (\det B)^2 = (42)^2 = 1764$.

Remark 4.

If A is an invertible $n \times n$ matrix, then the proof in Theorem 6 shows that

$$\det(A^{-1}) = \frac{1}{\det A}$$

Example 7.

Suppose $A = \begin{pmatrix} 3 & 0 & 0 \\ 4 & 2 & 0 \\ 5 & 5 & 1 \end{pmatrix} \quad B = \begin{pmatrix} 2 & 0 & 8 \\ 2 & 0 & 2 \\ 2 & 3 & 2 \end{pmatrix}$

Find $\det(A^{-1}B)$.

Solution.

$$\det(A^{-1}B) = \det(A^{-1}) \det B$$

$$\frac{\det B}{\det A}$$

Now, A is lower triangular and so $\det A = 3(2)(1) = 6$. Also, we can find $\det B$ by expanding along the second column to get

$$\det B = -3 \det \begin{pmatrix} 2 & 8 \\ 2 & 2 \end{pmatrix}$$

$$= -3(4 - 16) = 36$$

So,

$$\det(A^{-1}B) = \frac{36}{6} = 6$$

Theorem 7.

Suppose A is an $n \times n$ matrix and $b_1, b_2, \dots, b_n$ are real numbers. Then, the system of linear equations

$$A \begin{pmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \cdot \\ \cdot \\ \cdot \\ b_n \end{pmatrix}$$

has a unique solution $\iff$ $\det A \neq 0$.

Example 8.

Find the values of t for which

$$A_t = \begin{pmatrix} t-1 & 3 & -1 \\ 0 & t-2 & 4 \\ 0 & 0 & t+2 \end{pmatrix}$$

is not invertible.

Solution.

$\det A_t = (t-1)(t-2)(t+2)$ because A_t is upper triangular.

So, $\det A_t = 0 \iff t = -2, 1, 2$. Hence, A_t is not invertible exactly when $t = -2, 1, 2$.