2 Sequences

2.1 Introduction to Sequences

Definition 2.1. A sequence is a function whose domain is $\mathbb{N}$.

We usually write the sequence as $\{a_n\}_{n=1}^{\infty}$ or $\{a_n\}_{n\in\mathbb{N}}$. Often we will just write $\{a_n\}$.
Sometimes we have a formula which describes a_n for example if $a_n = \frac{1}{n}$ for all $n \in \mathbb{N}$ then $\{a_n\}_{n\in\mathbb{N}} = \{\frac{1}{n}\}_{n\in\mathbb{N}}$.
We will be interested in the behavior of the sequence $\{a_n\}_{n\in\mathbb{N}}$ as $n \to \infty$. What do we mean by saying that $\lim_{n\to\infty} a_n = a$? We mean that the distance between the a_n's and a can be made as small as we like after a certain point in our sequence. The distance between a_n and a is $|a_n - a|$. So given any small positive number (let's call it ϵ) we need to be able to find N (which will probably depend on ϵ) such that the distance $|a_n - a|$ is smaller than ϵ for all $n \geq N$.

We have the following definition:

Definition 2.2. The sequence $\{a_n\}_{n\in\mathbb{N}}$ *converges* to a if for all $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $|a_n - a| < \epsilon$ for all $n \geq N$. We write $\lim_{n\to\infty} a_n = a$.

Example 2.3. Show that $\{1 - \frac{1}{n}\}_{n=1}^{\infty}$ converges to 1.

Solution Let $\epsilon > 0$ be given. We need to show that we can find $N \in \mathbb{N}$ such that $|(1 - \frac{1}{n}) - 1| < \epsilon$ for all $n \geq N$.
Notice that $|(1 - \frac{1}{n}) - 1| = |\frac{1}{n}| = \frac{1}{n}$. So we need to find N such that $\frac{1}{n} < \epsilon$ for all $n \geq N$.
Let's choose N to be the smallest natural number bigger than $\frac{1}{\epsilon}$.
So if $n \geq N$ and $N > \frac{1}{\epsilon}$, we have $n > \frac{1}{\epsilon}$. Then $\frac{1}{n} < \epsilon$ as required.
So we have found $N \in \mathbb{N}$ such that $|(1 - \frac{1}{n}) - 1| < \epsilon$ for all $n \geq N$. That is $\lim_{n\to\infty} 1 - \frac{1}{n} = 1$.

Let's try another example:

Example 2.4. Show that $\{\frac{n-1}{n+1}\}_{n=1}^{\infty}$ converges to 1.

Solution Let $\epsilon > 0$ be given. We need to show that we can find $N \in \mathbb{N}$ such that $|\frac{n-1}{n+1} - 1| < \epsilon$ for all $n \geq N$. Notice that $|\frac{n-1}{n+1} - 1| = |\frac{-2}{n+1}| = \frac{2}{n+1}$. So we need to find N such that $\frac{2}{n+1} < \epsilon$ for all $n \geq N$.
If $\frac{2}{n+1} < \epsilon$ then $\frac{n+1}{2} > \frac{1}{\epsilon}$ or $n > \frac{2}{\epsilon} - 1$.
Let's choose N to be the smallest natural number bigger than $\frac{2}{\epsilon} - 1$.
Does this choice of N work? Well if $n > N$ then $n > N > \frac{2}{\epsilon} - 1$ so $n + 1 > \frac{2}{\epsilon}$ or $\frac{n+1}{2} > \frac{1}{\epsilon}$.
So $\frac{2}{n+1} < \epsilon$ which means that $|\frac{n-1}{n+1} - 1| < \epsilon$, as required.

Exercise 2.5. *Your turn:*

1. Prove that $\lim_{n\to\infty} \frac{1}{n} = 0$.

2. Prove that $\lim_{n\to\infty} \frac{(-1)^n}{n} = 0$.

3. Prove that $\lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$.

Let's try one more example:

Example 2.6. Show that $\{\frac{3n+1}{2n+5}\}_{n=1}^{\infty}$ converges to $\frac{3}{2}$.

Solution Let $\epsilon > 0$ be given. We need to show that we can find $N \in \mathbb{N}$ such that $|\frac{3n+1}{2n+5} - \frac{3}{2}| < \epsilon$ for all $n \geq N$.
Let's consider $|\frac{3n+1}{2n+5} - \frac{3}{2}| < \epsilon$. We can see that this is equivalent to
$|\frac{6n+2-(6n+15)}{4n+10}| < \epsilon$ or $|\frac{-13}{4n+10}| < \epsilon$. That is $\frac{13}{4n+10} < \epsilon$.
We need to find $N \in \mathbb{N}$ so that if $n > N$ we have $\frac{13}{4n+10} < \epsilon$.
We need to find $N \in \mathbb{N}$ so that if $n > N$ we have $\frac{4n+10}{13} > \frac{1}{\epsilon}$.
That is $4n + 10 > \frac{13}{\epsilon}$ or $n > \frac{1}{4}(\frac{13}{\epsilon} - 10)$.
Let's choose N to be the smallest natural number bigger than $\frac{1}{4}(\frac{13}{\epsilon} - 10)$.
Check that this choice of N works.

Divergent Sequences

Consider the sequence $\{(-1)^n\}_{n=1}^{\infty} = -1, 1, -1, 1, ...$ It seems obvious that this sequence does not converge to 0, or to 1 or to -1. But what would we need to do to prove this? For example what would we need to do to show that the sequence does not converge to 0?

Well, recall that to show $\lim_{n\to\infty} a_n = a$ we need to demonstrate that for all $\epsilon > 0$ we can find $N \in \mathbb{N}$ such that

$$|a_n - a| < \epsilon \qquad \forall n \geq N.$$

In order to show that $\lim_{n\to\infty} a_n \neq a$ we need to demonstrate that there exists at least one $\epsilon > 0$ such that for all $N \in \mathbb{N}$

$$|a_n - a| > \epsilon \text{ for some } n \geq N.$$

[*Note*: To negate a statement of the form $\forall P \;\; \exists Q$ you need to show that for at least one P no such Q is possible. For example, to disprove the statement 'Every Irish town has a post office', we just need to find one Irish town with no post office.]
So order to show that $\lim_{n\to\infty}(-1)^n \neq 0$ we need to demonstrate that there exists at least one $\epsilon > 0$ such that for all $N \in \mathbb{N}$

$$|(-1)^n - 0| > \epsilon \text{ for some } n \geq N.$$

Now $|(-1)^n - 0| > \epsilon$ means $|(-1)^n| > \epsilon$ or $1 > \epsilon$. So let's choose $\epsilon = 0.5$. Then $|(-1)^n - 0| > \epsilon$ for all $n \in \mathbb{N}$. So $\lim_{n\to\infty}(-1)^n \neq 0$. [*Think of* ϵ *as the Irish town here.*]

Exercise 2.7. *Your turn:*

1. Prove that $\lim_{n\to\infty}(-1)^n \neq 1$.

2. Prove that $\lim_{n\to\infty}(-1)^n \neq -1$.

We will soon be able to show that the sequence $\{(-1)^n\}$ does not converge to any real number. We will call such sequences divergent.

Definition 2.8. If the sequence $\{a_n\}_{n\in\mathbb{N}}$ does not converge to any real number a we say that it *diverges.*

2.2 Properties of Convergent Sequences

First we need a lemma:

Lemma 2.9. *Suppose $x \in \mathbb{R}$ has the property that $0 \leq x < \delta$ for all $\delta > 0$. Then $x = 0$.*

Proof: Suppose that we have a value X such that $0 < x < \delta$ for all $\delta > 0$. Then $x > 0$ means that $\frac{x}{2} > 0$. So let $\delta = \frac{x}{2} > 0$. Then by our assumption $0 < x < \delta = \frac{x}{2}$. But that is impossible, so the only possibility is that $x = 0$.

Next we consider the question: Is it possible for a sequence to converge to two different limits? The answer is no, as the next theorem shows.

Theorem 2.10. *Let $\{a_n\}$ be a sequence of real numbers with $\lim_{n\to\infty} a_n = L_1$ and $\lim_{n\to\infty} a_n = L_2$. Then $L_1 = L_2$.*

Proof: Let $\epsilon > 0$ be given. Since $\lim_{n\to\infty} a_n = L_1$ we know that there exists $N_1 \in \mathbb{N}$ such that

$$|a_n - L_1| < \frac{\epsilon}{2} \text{ for all } n \geq N_1.$$

And $\lim_{n\to\infty} a_n = L_2$ means that there exists $N_2 \in \mathbb{N}$ such that

$$|a_n - L_2| < \frac{\epsilon}{2} \text{ for all } n \geq N_2.$$

Let $N = \text{Max}(N_1, N_2)$. Then for $n \geq N$

$$\begin{aligned} |L_1 - L_2| &= |L_1 - a_n + a_n - L_2| \\ &\leq |L_1 - a_n| + |a_n - L_2| \\ &< \epsilon \end{aligned}$$

Thus $|L_1 - L_2| < \epsilon$ for all $\epsilon > 0$. By our lemma this means that $L_1 = L_2$.

Definition 2.11. We say that a sequence of real numbers $\{a_n\}$ is *bounded* if there exists $M > 0$ such that $|a_n| \leq M$ for all $n \in \mathbb{N}$.

Note that if a sequence is bounded by M it means that all terms lie in the interval $[-M, M]$.

Example 2.12.

1. The sequence $\{\frac{2}{n}\}$ is bounded by $M = 2$.
2. The sequence $\{(-1)^n\}$ is bounded by $M = 1$.
3. The sequence $\{n\}$ is not bounded.

Exercise 2.13. *Your turn:* Find examples of two bounded and two unbounded sequences (different from the ones above).

Since the sequence $\{(-1)^n\}$ is bounded but not convergent, it is clearly not true that all bounded sequences are convergent however it is true that all convergent sequences are bounded.

Theorem 2.14. *Every convergent sequence of real numbers is bounded.*

Proof: Let $\{a_n\}$ be a convergent sequence and suppose $\lim_{n\to\infty} a_n = a$.
Thus given any $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $|a_n - a| < \epsilon$ for all $n \geq N$.
Let $\epsilon = 1$ then there exists $N_1 \in \mathbb{N}$ such that $|a_n - a| < 1$ for all $n \geq N_1$.
Now $|a_n| = |a_n - a + a| \leq |a_n - a| + |a| < 1 + |a|$, for all $n \geq N_1$.
Now let $M = \text{Max}\{|a_1|,\ |a_2|,\ ..., |a_{N_1-1}|,\ 1 + |a|\}$.
Then $|a_n| \leq M$ for all $n \in \mathbb{N}$ and so $\{a_n\}$ is a bounded sequence.

We can now prove a result about combinations of sequences:

Theorem 2.15. *Let $\{a_n\}$ and $\{b_n\}$ be sequences of real numbers with $\lim\limits_{n\to\infty} a_n = a$ and $\lim\limits_{n\to\infty} b_n = b$. Then*

1. $\lim\limits_{n\to\infty} ca_n = ca$ *for all* $c \in \mathbb{R}$.
2. $\lim\limits_{n\to\infty} (a_n + b_n) = a + b$
3. $\lim\limits_{n\to\infty} a_n b_n = ab$
4. *If* $b \neq 0$ *then* $\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = \dfrac{a}{b}$.

Proof:
1. If $c = 0$ we are done so assume $c \neq 0$. Given $\epsilon > 0$ we need to find $N \in \mathbb{N}$ such that $|ca_n - ca| < \epsilon$ for all $n \geq N$.

But $|ca_n - ca| = |c||a_n - a|$ and since $\lim_{n\to\infty} a_n = a$, we can make $|a_n - a|$ as small as we like by choosing n large enough. So choose N such that

$$|a_n - a| < \frac{\epsilon}{|c|}, \text{ for all } n \geq N.$$

Thus

$$\begin{aligned}|ca_n - ca| &= |c||a_n - a| \\ &< |c|\frac{\epsilon}{|c|} \text{ for all } n \geq N \\ &= \epsilon \text{ for all } n \geq N\end{aligned}$$

2. Given $\epsilon > 0$ we need to find $N \in \mathbb{N}$ such that $|(a_n + b_n) - (a + b)| < \epsilon$ for all $n \geq N$. Since $\lim_{n\to\infty} a_n = a$ and $\lim_{n\to\infty} b_n = b$ we know that given $\epsilon > 0$ we can find N_1 and N_2 such that $|a_n - a| < \frac{\epsilon}{2}$ if $n \geq N_1$ and $|b_n - b| < \frac{\epsilon}{2}$ if $n \geq N_2$. So if we let $N = \text{Max}\{N_1, N_2\}$ then

$$\begin{aligned}|(a_n + b_n) - (a + b)| &= |a_n - a + b_n - b| \\ &\leq |a_n - a| + |b_n - b| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \text{ for all } n \geq N \\ &= \epsilon \text{ for all } n \geq N\end{aligned}$$

3. Given $\epsilon > 0$ we need to find $N \in \mathbb{N}$ such that $|a_n b_n - ab| < \epsilon$ for all $n \geq N$.
Let's look at $|a_n b_n - ab|$ more closely. We would like to bound this quantity by something involving $|a_n - a|$ and $|b_n - b|$ since we can control the size of both of these. Now
$|a_n b_n - ab| = |a_n b_n - ab_n + ab_n - ab| = |b_n(a_n - a) + a(b_n - b)| \leq |b_n||a_n - a| + |a||b_n - b|$. (*)
Since $\lim_{n\to\infty} b_n = b$ we know that given $\epsilon > 0$ we can find N_1 such that

$$|b_n - b| < \frac{\epsilon}{2|a|} \text{ for all } n \geq N_1$$

and so $|a||b_n - b| < | < \frac{\epsilon}{2}$ for all $n \geq N_1$.

Now since $\{b_n\}$ is a convergent sequence we know (by Thm2.14) that it is a bounded sequence so there exists $M > 0$ such that $|b_n| \leq M$ for all $n \in \mathbb{N}$. Now choose N_2 such that

$$|a_n - a| < \frac{\epsilon}{2M} \text{ for all } n \geq N_2$$

and so $|b_n||a_n - a| < \frac{\epsilon}{2}$ for all $n \geq N_2$. Let $N = \text{Max}\{N_1, N_2\}$, then from (*) we get

$$\begin{aligned}|a_n b_n - ab| &\leq |b_n||a_n - a| + |a||b_n - b| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \text{ for all } n \geq N \\ &= \epsilon \text{ for all } n \geq N\end{aligned}$$

4. Assume $b \neq 0$. We need to show that given any $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $|\frac{a_n}{b_n} - \frac{a}{b}| < \epsilon$ for all $n \geq N$. Consider

$$\begin{aligned}|\frac{a_n}{b_n} - \frac{a}{b}| &= |\frac{a_n b - b_n a}{b_n b}| \\ &= \frac{1}{|b_n b|}|a_n b - ab + ab - b_n a| \\ &\leq \frac{1}{|b_n b|}|b||a_n - a| + \frac{1}{|b_n b|}|a||b_n - b| \\ &= \frac{1}{|b_n|}|a_n - a| + \frac{|a|}{|b_n b|}|b_n - b|\end{aligned}$$

Let $\epsilon > 0$ be given. Since $\lim_{n\to\infty} b_n = b$ we know that there exists N_1 such that $|b_n - b| < \frac{|b|}{2}$ for all $n \geq N_1$.
Since $|b_n| = |b + (b_n - b)| \geq |b| - |b_n - b|$, we have $|b_n| > \frac{|b|}{2}$ so $\frac{1}{|b_n|} < \frac{2}{|b|}$.
Choose N_2 such that $|a_n - a| < \frac{\epsilon|b|}{4}$ for all $n \geq N_2$. Thus for all $n \geq N_2$

$$\frac{1}{|b_n|}|a_n - a| < \frac{1}{|b_n|}\frac{\epsilon|b|}{4} < \frac{2}{|b|}\frac{\epsilon|b|}{4} = \frac{\epsilon}{2}.$$

Now choose $N_3 \geq N_2$ and such that $|b_n - b| < \dfrac{\epsilon|b|^2}{4|a|}$ for all $n \geq N_3$.

Thus for all $n \geq N_3$

$$\frac{|a|}{|b_n b|}|b_n - b| < \frac{|a|}{|b_n b|}\frac{\epsilon|b|^2}{4|a|} = \frac{\epsilon|b|}{4|b_n|} < \frac{2\epsilon|b|}{4|b|} = \frac{\epsilon}{2}.$$

Choose $N = Max\{N_1, N_3\}$ so for all $n \geq N$ we have

$$|\frac{a_n}{b_n} - \frac{a}{b}| < \frac{1}{|b_n|}|a_n - a| + \frac{|a|}{|b_n b|}|b_n - b| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

as required.

We can also prove results about limits and order:

Theorem 2.16. *Let $\{a_n\}$ and $\{b_n\}$ be sequences of real numbers with $\lim\limits_{n\to\infty} a_n = a$ and $\lim\limits_{n\to\infty} b_n = b$. Then*

1. *If $a_n \geq 0$ for all $n \in \mathbb{N}$ then $a \geq 0$.*
2. *If $a_n \leq b_n$ for all $n \in \mathbb{N}$ then $a \leq b$.*

Proof:

1. Suppose $a < 0$ (and look for a contradiction). Let $\epsilon = |a| = -a$ (which is positive since $a < 0$). Then there exists $N \in \mathbb{N}$ such that $|a_n - a| < \epsilon = -a$ for all $n \geq N$.

Let $n = N + 1$ so $|a_{N+1} - a| < -a$ or $-(-a) < a_{N+1} - a < -a$. This means that $2a < a_{N+1} < 0$, that is $a_{N+1} < 0$ but this is impossible. Thus $a \geq 0$.

2. By the previous theorem we know that $\lim\limits_{n\to\infty} b_n - a_n = b - a$. By part 1 of this theorem we can deduce that because $a_n \leq b_n$ for all $n \in \mathbb{N}$ (or $b_n - a_n \geq 0$ for all $n \in \mathbb{N}$) we have $b - a \geq 0$.

2.3 Monotone Sequences

Definition 2.17. A sequence $\{a_n\}$ is *increasing* if $a_n \leq a_{n+1}$ for all $n \in \mathbb{N}$.

A sequence $\{a_n\}$ is *decreasing* if $a_n \geq a_{n+1}$ for all $n \in \mathbb{N}$.

A sequence is *monotone* if it is either an increasing sequence or a decreasing sequence.

Example 2.18.

1. $\{n\}$ and $\{1 - \frac{1}{n}\}$ are increasing sequences.
2. $\{-n^2\}$ and $\{\frac{1}{n}\}$ are decreasing sequences.
3. $\{(-1)^n\}$ and $\{\frac{(-1)^n}{n}\}$ are neither increasing nor decreasing sequences.
4. The sequences in 1. and 2. are monotone sequences and the examples in 3. are not.

These examples show us that some monotone sequences converge (for example $\{1 - \frac{1}{n}\}$ and $\{\frac{1}{n}\}$) and some do not (for example $\{n\}$ and $\{-n^2\}$). You might have noticed that our examples of convergent monotone sequences are bounded.

In fact we can prove:

Theorem 2.19 (The Monotone Convergence Theorem)**.** *If a sequence is monotone and bounded then it is convergent.*

Proof Let $\{a_n\}$ be a monotone and bounded sequence.

Case 1 Assume that $\{a_n\}$ is increasing. Consider the set of points $A = \{a_n | \ n \in \mathbb{N}\}$. By assumption this set is bounded so by the Axiom of Completeness it has a least upper bound. Let $s = lub(A)$. We claim that $\lim_{n\to\infty} a_n = s$.

Let $\epsilon > 0$. Since $s = lub(A)$ we know that $s - \epsilon$ is not an upper bound for A so there exists N such that $s - \epsilon < a_N$.

Since $\{a_n\}$ is an increasing sequence we have that $a_N \leq a_n$ for all $n > N$. Thus $s - \epsilon < a_N \leq a_n$ for all $n > N$. (*)

Since $s = lub(A)$ we know that $a_n \leq s < s + \epsilon$ for all $n \in \mathbb{N}$. (**)

Putting the information from (*) and (**) together we get

$$s - \epsilon < a_n < s + \epsilon \text{ for all } n \geq N.$$

In other words, given $\epsilon > 0$ we have found N such that $|a_n - s| < \epsilon$ for all $n \geq N$. Thus

$$\lim_{n\to\infty} a_n = s.$$

Case 2 Exercise: Modify the proof above to deal with the case of a decreasing sequence.

Note: The proof of the Monotone Convergence theorem tells us that if $\{a_n\}$ is increasing and bounded above then it converges to $lub\{a_n\}$. If $\{a_n\}$ is decreasing and bounded below then it converges to $glb\{a_n\}$.

Example 2.20. Show that the sequence $\sqrt{2},\ \sqrt{2\sqrt{2}},\ \sqrt{2\sqrt{2\sqrt{2}}}, ...$ is convergent and find its limit.

Notice that here $a_{n+1} = \sqrt{2a_n}$. If we can prove that $\{a_n\}$ is bounded and increasing then the MCT will tell us that the sequence converges.
Claim 1. We claim that $a_n < 2$ for all $n \in \mathbb{N}$.
We can prove this by induction: If $n = 1$ then $a_n = a_1 = \sqrt{2} < 2$.
If the statement is true for $n = k$ then $a_k < 2$ but $a_{k+1} = \sqrt{2a_k} < \sqrt{2 \times 2} = 2$.
Claim 2. We claim that $\{a_n\}$ is monotone increasing.
As $a_n < 2$ and each $a_n > 0$ we have $(a_n)^2 < 2a_n$ or $a_n < \sqrt{2a_n} = a_{n+1}$. So our sequence is increasing.

From Claim 1 and Claim 2 we see that $\{a_n\}$ is bounded and increasing and therefore converges to a limit L. To find L notice that $a_{n+1} = \sqrt{2a_n}$ so $(a_{n+1})^2 = 2a_n$. Now by Theorem 2.15 (3) we see that $\lim_{n\to\infty}(a_{n+1})^2 = L^2$ and by Theorem 2.15 (1) we see that $\lim_{n\to\infty} 2(a_n) = 2L$ and so $L^2 = 2L$ ie $L = 0$ or $L = 2$. It is clear that $L = 0$ is not possible so our series converges to 2.

Example 2.21. Define $e = \lim_{n\to\infty}(1 + \frac{1}{n})^n$. Show that $2 < e < 3$.

Let's define $a_n = (1 + \frac{1}{n})^n$. We want to show that $\{a_n\}$ is a convergent sequence. If we can show that it is bounded and monotone then the MCT will show that our sequence is convergent.
Claim 1: $\{a_n\}$ is increasing.
We want to show that $a_n \leq a_{n+1}$. The Binomial Theorem gives us:

$$\begin{aligned}
a_n &= (1 + \frac{1}{n})^n \\
&= 1 + n(\frac{1}{n}) + \frac{n(n-1)}{2}\frac{1}{n^2} + \frac{n(n-1)(n-2)}{3!}\frac{1}{n^3} + ... + \frac{n(n-1)...1}{n!}\frac{1}{n^n} \\
&= 1 + 1 + \frac{1}{2!}(1 - \frac{1}{n}) + \frac{1}{3!}(1 - \frac{1}{n})(1 - \frac{2}{n}) + ... + \frac{1}{n!}(1 - \frac{1}{n})(1 - \frac{2}{n})...(1 - \frac{n-1}{n}) \\
a_{n+1} &= (1 + \frac{1}{n+1})^{n+1} \\
&= 1 + 1 + \frac{1}{2!}(1 - \frac{1}{n+1}) + \frac{1}{3!}(1 - \frac{1}{n+1})(1 - \frac{2}{n+1}) + ... + \frac{1}{(n+1)!}(1 - \frac{1}{n+1})...(1 - \frac{n}{n+1})
\end{aligned}$$

Notice that $1-\frac{1}{n}<1-\frac{1}{n+1}$, $1-\frac{2}{n}<1-\frac{2}{n+1}$ etc. and a_{n+1} has an extra term so $a_n \leq a_{n+1}$.
Claim 2: $\{a_n\}$ is bounded.
It is clear from our analysis above that $2 \leq a_1 \leq a_2 \leq ... \leq a_n \leq ....$ So $\{a_n\}$ is bounded below by 2. To show that it is bounded above:
Note that if $p = 1, 2, ..., n$ then $1-\frac{p}{n} \leq 1$ and $2^{p-1} \leq p!$ [*you could use induction to show this*]. Thus $\frac{1}{p!} \leq \frac{1}{2^{p-1}}$.
So for $n > 2$ we have

$$\begin{aligned}
2 &< a_n \\
&= 1+1+\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+...+\frac{1}{2^{n-1}} \\
&= 1+1+\frac{1}{2}(1+\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+...+\frac{1}{2^{n-2}}) \\
&= 1+1+\frac{1}{2}(\frac{1-\frac{1}{2^{n-1}}}{1-\frac{1}{2}}) \\
&= 1+1+1-\frac{1}{2^{n-1}} \\
&< 3
\end{aligned}$$

Thus the sequence is bounded above by 3. The Monotone Convergence Theorem tells us that

$$2 < \lim_{n\to\infty}(1+\frac{1}{n})^n < 3$$

or $2 < e < 3$ as required.

Exercise 2.22. *Your Turn:* Decide if the following statements are true or false. If the statement is true give a reason, if it is false give a counterexample.

1. Every convergent sequence is bounded.
2. Every bounded sequence is convergent.
3. Every monotone sequence is convergent.
4. Every convergent sequence is monotone.
5. Every bounded monotone sequence is convergent.

2.4 Subsequences

Definition 2.23. Let $\{a_n\}$ be a sequence of real numbers and let $n_1 < n_2 < n_3 < ...$ be an increasing sequence of natural numbers. Then $\{a_{n_k}\} = a_{n_1}, a_{n_2}, ...$ is called a *subsequence* of $\{a_n\}$.

Example 2.24. Let $\{a_n\} = \{\frac{1}{n}\}$ and consider the sequence of natural numbers $2 < 4 < 6 < ... < 2k < ....$ Note that here $n_k = 2k$. Then $\{a_{n_k}\} = \{\frac{1}{2k}\} = \frac{1}{2}, \frac{1}{4}, \frac{1}{6}, ...$

If we consider the sequence of natural numbers $\{3k\}$ we get the subsequence $\{a_{n_k}\} = \{\frac{1}{3k}\} = \frac{1}{3}, \frac{1}{6}, \frac{1}{9}, ...$

Note that $\frac{1}{3}, \frac{1}{5}, \frac{1}{4}, ...$ is **not** a subsequence of $\{a_n\}$ since the sequence of natural numbers $3, 5, 4, ...$ is not an increasing sequence.

Example 2.25. Let $\{a_n\} = \{\frac{(-1)^n}{n}\}$ and consider the sequence of natural numbers $2 < 4 < 6 < ... < 2k < ...$ as above. Then $\{a_{n_k}\} = \{\frac{(-1)^{2k}}{2k}\} = \frac{1}{2}, \frac{1}{4}, \frac{1}{6}, ...$

If we consider the sequence of natural numbers $\{3k\}$ we get the subsequence $\{a_{n_k}\} = \{\frac{(-1)^{3k}}{3k}\} = -\frac{1}{3}, \frac{1}{6}, -\frac{1}{9}, ...$

Note that the subsequence $\{a_{n_k}\} = \{\frac{(-1)^{3k}}{3k}\}$ is **not** an increasing sequence but it is still a subsequence of $\{\frac{(-1)^n}{n}\}$ since $\{n_k\} = 3k$ is an increasing sequence of natural numbers.

Exercise 2.26. *Your turn*: Let $\{a_n\} = \{n^2\}$. Write down the first five elements of the subsequence $\{a_{n_k}\}$ if $n_k = 2k - 1$. What if $n_k = k^2$?

What does it mean to say that the subsequence $\{a_{n_k}\}$ converges to a? Well we need to show that for all $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $|a_{n_k} - a| < \epsilon$ if $k \geq N$. We can prove the following:

Theorem 2.27. *Let $\{a_n\}$ be a sequence which converges to a. Let $\{a_{n_k}\}$ be a subsequence of $\{a_n\}$. Then $\{a_{n_k}\}$ converges to a also.*

Proof: Let $\epsilon > 0$ be given. Then there exists $N \in \mathbb{N}$ such that $|a_n - a| < \epsilon$ if $n \geq N$.

Then $|a_{n_k} - a| < \epsilon$ if $n_k \geq N$. Note that from the definition of subsequences we always have $n_k \geq k$ so if $k \geq N$ we have $n_k \geq k \geq N$.

Exercise 2.28. *Your turn:* Is it true that every subsequence of a divergent sequence diverges?

The following two examples show how the previous theorem could be used:

Example 2.29. The sequence $\{b^n\}$ converges to 0 if $0 < b < 1$.

It is clear that if $0 < b < 1$ the $b > b^2 > b^3 > ...$ so the sequence is decreasing and since $b^n > 0$ for all $n \in \mathbb{N}$ we can see that the sequence is bounded below. By the Monotone Convergence Theorem we see that $\{b^n\}$ converges to some $L \geq 0$.

By the last theorem any subsequence of $\{b^n\}$ also converges to L.

Thus $\{b^{2n}\}$ converges to L.

But $b^{2n} = b^n \times b^n$ and so by Theorem 2.15 (3) we see that $L = L \times L = L^2$.

Thus $L = 0$ or $L = 1$, but it is clear that $L \neq 1$ therefore $L = 0$.

Example 2.30. The sequence $\{(-1)^n\}$ diverges.

We saw previously that this sequence does not converge to 0, to 1 or to -1. Now we can show that it does not converge to any $a \in \mathbb{R}$.

Suppose that the sequence converges to some a. Then the last theorem shows that any subsequence would also converge to a. But consider the subsequence $\{(-1)^{2k}\} = 1, 1, 1, 1, ...$ This clearly converges to 1 so $a = 1$. But if we consider the subsequence $\{(-1)^{2k+1}\} = -1, -1, -1, -1, ..,$ this clearly converges to -1 so $a = -1$. But it is impossible for a to be equal to 1 and -1, therefore no such a exists. That is $\{(-1)^n\}$ diverges.

Note: In the last example we saw that divergent sequences could have convergent subsequences. The next theorem says that this is always the case for bounded sequences.

Theorem 2.31 (The Bolzano Weierstass Theorem)**.** *Every bounded sequence has a convergent subsequence.*

Proof: Let $\{a_n\}$ be a bounded sequence, that is there exists $M > 0$ st $|a_n| \leq M$ for all $n \in \mathbb{N}$. In other words, all of the terms a_n lie in the interval $[-M, M]$. Call this interval I_1 and choose $a_{n_1} \in I_1$.

Now divide this interval I_1 into two equal subintervals. Now one of these intervals must contain infinitely many of the a_n's. Call this interval I_2 and choose n_2 such that $n_2 > n_1$ and $a_{n_1} \in I_1$. [Note that we can do this because there are infinitely many of the terms a_n in I_2.]

Continue in this way: Construct the interval I_k from I_{k-1} and choose $a_{n_k} \in I_k$ with $n_k > n_{k-1}$.

We have $n_1 < n_2 < n_3 < ... < n_{k-1} < n_k < ...$ so $\{a_{n_k}\}$ is a subsequence of $\{a_n\}$. We will show that this subsequence converges.

Notice that the intervals that we have constructed form a nested sequence of closed intervals: $I_1 \supset I_2 \supset I_3 \supset ....$ So by the Nested Intervals Theorem (Thm 1.16) there exists at least one element x in the intersection of these intervals i.e. $x \in I_k$ for all $k \in \mathbb{N}$.

We will show that $\lim_{k\to\infty} a_{n_k} = x$.

Let $\epsilon > 0$. We can see that the length of I_k is $2M(\frac{1}{2})^{k-1}$ (check!). We know from Example 2.29 above that $\lim_{k\to\infty} 2M(\frac{1}{2})^{k-1} = 0$ so we can find $N \in \mathbb{N}$ such that $2M(\frac{1}{2})^{N-1} < \epsilon$.

Since x and a_{n_k} both lie in I_k we have that $|a_{n_k} - x| <$ length of $I_k = 2M(\frac{1}{2})^{k-1} < \epsilon$ if $k > N$. Therefore

$$\lim_{k\to\infty} a_{n_k} = x.$$

And so $\{a_n\}$ has a convergent subsequence.

Exercise 2.32. *Your turn:* Is it possible for unbounded sequences to have convergent subsequences? If so, do all unbounded sequences have convergent subsequences?

2.5 Cauchy Sequences

You will have noticed that to prove that a sequence $\{a_n\}$ converges we depend on the $\epsilon - N$ definition which means we need to know the value of the limit a. We will see that there is an alternative to this and it involves a special class of sequences called Cauchy sequences.

Definition 2.33. A sequence $\{a_n\}$ is called a *Cauchy sequence* if for all $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that whenever $m, n \geq N$ we have $|a_n - a_m| < \epsilon$.

Example 2.34. The sequence $\{\frac{1}{n^2}\}$ is a Cauchy sequence.

To see this, let's look at the definition. We need to show that for all $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that whenever $m, n \geq N$ we have $|\frac{1}{n^2} - \frac{1}{m^2}| < \epsilon$.

But $|\frac{1}{n^2} - \frac{1}{m^2}| \leq \frac{1}{n^2} + \frac{1}{m^2}$. If $n \geq N$ then $\frac{1}{n} \leq \frac{1}{N}$ and so $\frac{1}{n^2} \leq \frac{1}{N^2}$.

Similarly if $m \geq N$ then $\frac{1}{m^2} \leq \frac{1}{N^2}$. Thus if $m, n \geq N$

$$|\frac{1}{n^2} - \frac{1}{m^2}| \leq \frac{1}{n^2} + \frac{1}{m^2} \leq \frac{2}{N^2}.$$

Let $\epsilon > 0$ be given.

We need to find N such that $\dfrac{2}{N^2} < \epsilon$, that is $N > \sqrt{\frac{2}{\epsilon}}$.

Let's choose N to be the smallest natural number bigger than $\sqrt{\frac{2}{\epsilon}}$.

Then if $m, n \geq N$ we have

$$|\frac{1}{n^2} - \frac{1}{m^2}| \leq \frac{2}{N^2} < \epsilon.$$

And so $\{\frac{1}{n^2}\}$ is a Cauchy sequence.

Example 2.35. The sequence $\{n\}$ is **not** a Cauchy sequence.

In order to prove that a sequence $\{a_n\}$ is not a Cauchy sequence we need to show that there exists $\epsilon > 0$ such that for all $N \in \mathbb{N}$ $\;|a_n - a_m| > \epsilon$ for some $m, n \geq N$.

So here we need to show that there exists $\epsilon > 0$ such that for all $N \in \mathbb{N}$ $\;|n - m| > \epsilon$ for some $m, n \geq N$.

If $n > m$ then $|n - m| = n - m$ and since $n, m \in \mathbb{N}$ $n - m \geq 1$ (unless $n = m$).

Let $\epsilon = 0.5$.

Then for all $N \in \mathbb{N}$ there exist $m, n \geq N$ such that $|n - m| > \epsilon$. (For example take $n = N + 3$ and $m = N + 1$ then $|n - m| = 2 > 0.5$.)

And so $\{n\}$ is not a Cauchy sequence.

Exercise 2.36. *Your turn:*

1. Can you explain the definition of a Cauchy sequence geometrically?

2. Before you read on think about the relationship between convergent sequences and Cauchy sequences. Can you make any conjectures?

It turns out that if $\{a_n\}$ is a sequence of real numbers then it converges if and only if it is a Cauchy sequence. One part of this is easy to prove:

Theorem 2.37. *Every convergent sequence of real numbers is a Cauchy sequence.*

Proof: Assume $\{a_n\}$ converges to a. We know then that given any $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $|a_n - a| < \frac{\epsilon}{2}$ if $n \geq N$.
If $n, m \geq N$ then we have

$$\begin{aligned} |a_n - a_m| &= |a_n - a + a - a_m| \\ &\leq |a_n - a| + |a - a_m| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \end{aligned}$$

Therefore $\{a_n\}$ is a Cauchy sequence.

Now we would like to show that every Cauchy sequence of real numbers converges. First we will show that Cauchy sequences are bounded. The proof is very similar to that of Theorem 2.14.

Theorem 2.38. *Every Cauchy sequence of real numbers is bounded.*

Proof: Let $\{a_n\}$ be a Cauchy sequence.
Thus given any $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $|a_n - a_m| < \epsilon$ for all $n, m \geq N$.
Let $\epsilon = 1$ then there exists $N \in \mathbb{N}$ such that $|a_n - a_m| < 1$ for all $n, m \geq N$.
Let $m = N$, then we have $|a_n - a_N| < 1$ for all $n \geq N$.
Now $|a_n| = |a_n - a_N + a_N| \leq |a_n - a_N| + |a_N| < 1 + |a_N|$, for all $n \geq N$.
Now let $M = \text{Max}\{|a_1|,\ |a_2|,\ ..., |a_{N-1}|,\ 1 + |a_N|\}$.
Then $|a_n| \leq M$ for all $n \in \mathbb{N}$ and so $\{a_n\}$ is a bounded sequence.

Theorem 2.39 (The Cauchy Convergence Criterion)**.** *A sequence of real numbers converges if and only if it is a Cauchy sequence.*

Proof: Theorem 2.37 tells us that every convergent sequence of real numbers is a Cauchy sequence.
We need to show that every Cauchy sequence of real numbers is convergent.
Let $\{a_n\}$ be a Cauchy sequence. Then Theorem 2.38 tells us that $\{a_n\}$ is a bounded sequence.
The Bolzano-Weierstrass Theorem tells us that every bounded sequence has a convergent subsequence.
Therefore there exists a subsequence $\{a_{n_k}\}$ of $\{a_n\}$ which converges to some real number a. That is $\lim\limits_{k\to\infty} a_{n_k} = a$.
We will prove that $\lim\limits_{n\to\infty} a_n = a$ also.

Let $\epsilon > 0$ be given.
Since $\{a_n\}$ is a Cauchy sequence there exists $N \in \mathbb{N}$ such that

$$|a_n - a_m| < \frac{\epsilon}{2} \text{ if } n, m \geq N.$$

Since $\lim_{k\to\infty} a_{n_k} = a$ there exists $n_J \in \mathbb{N}$ such that

$$|a_{n_J} - a| < \frac{\epsilon}{2} \text{ and } n_J \geq N.$$

If $n \geq N$ then we have

$$\begin{aligned} |a_n - a| &= |a_n - a_{n_J} + a_{n_J} - a| \\ &\leq |a_n - a_{n_J}| + |a_{n_J} - a| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \end{aligned}$$

Therefore $\{a_n\}$ is a convergent sequence.

2.6 Infinite Limits of Sequences

Consider the sequence $\{n^2\}$. This sequence is not bounded and therefore not convergent. However, it seems natural to say that $\lim_{n\to\infty} n^2 = \infty$. We need to define what we mean by this:

Definition 2.40. Let $\{a_n\}$ be a sequence of real numbers and suppose that for all $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $a_n > \epsilon$ if $n \geq N$ then we say $\lim_{n\to\infty} a_n = \infty$. If for all $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $a_n < -\epsilon$ if $n \geq N$ then we say $\lim_{n\to\infty} a_n = -\infty$.

Example 2.41. We can use this definition to see that if $\{a_n\} = \{n^2\}$ then $\lim_{n\to\infty} a_n = \infty$. Given $\epsilon > 0$, choose N to be the smallest natural number bigger than $\sqrt{\epsilon}$. Then if $n \geq N$ we have $n^2 \geq N^2 > \epsilon$, as required. Similarly we could show that $\lim_{n\to\infty} -n^2 = -\infty$. (Try it!)

Example 2.42. Prove that if $\{a_n\} = \{(-1)^n n\}$ then $\lim_{n\to\infty} a_n \neq \infty$. Note that the sequence looks like $-1, 2, -3, 4, -5, ...$
We need to show that there exists an $\epsilon > 0$ such that for all $N \in \mathbb{N}$ we have $a_n \leq \epsilon$ for some $n \geq N$.
Let $\epsilon = 1$, then there is no value of N such that $(-1)^n n > 1$ for all $n \geq N$ since for any odd value of n we have $(-1)^n n < 0 < 1 = \epsilon$.
Similarly we could show that $\lim_{n\to\infty} (-1)^n n \neq -\infty$. (Try it!)

Exercise 2.43. *Your turn:*
Let $\{a_n\} = \{n^{(-1)^n}\}$. Is $\lim_{n\to\infty} a_n = \infty$? If so, prove it. If not, explain why not.

2.7 Superior and Inferior Limits of Sequences

Let $\{a_n\}$ be a sequence of real numbers which is bounded above. Let

$$U_n = lub\{a_k |\ k \geq n\}.$$

Let's have a look at some examples:

Example 2.44. Suppose $\{a_n\} = \{\frac{1}{n}\}$. Then $U_n = lub\{\frac{1}{n}, \frac{1}{n+1}, \frac{1}{n+2}, ...\} = \frac{1}{n}$ and $U_{n+1} = lub\{\frac{1}{n+1}, \frac{1}{n+2}, \frac{1}{n+3}, ...\} = \frac{1}{n+1}$. So $U_{n+1} = \frac{1}{n+1} < \frac{1}{n} = U_n$.

If $\{b_n\} = \{1 - \frac{1}{n}\}$. Then $U_n = lub\{1 - \frac{1}{n}, 1 - \frac{1}{n+1}, 1 - \frac{1}{n+2}, ...\} = 1$ and $U_{n+1} = lub\{1 - \frac{1}{n+1}, 1 - \frac{1}{n+2}, 1 - \frac{1}{n+3}, ...\} = 1$. So $U_{n+1} = 1 = U_n$.

In our both of our examples above, we had $U_{n+1} \leq U_n$. This will be the case in general.

Suppose that $a_k \leq a_n$ for all $k \geq n$ then $U_{n+1} \leq a_n = U_n$. If, however, $a_k > a_n$ for some $k > n$ we have $U_{n+1} = U_n$. Thus, in any case, the sequence $\{U_n\}$ is decreasing.

Define $U = \lim\limits_{n\to\infty} U_n$.

Are we sure that this makes sense? Well, if $\{U_n\}$ is bounded below, then by the Monotone convergence theorem it will converge. If $\{U_n\}$ is not bounded below then it is a decreasing, unbounded sequence and so $\lim\limits_{n\to\infty} U_n = -\infty$. We will call U the limit superior of the sequence $\{a_n\}$.

Definition 2.45. If the sequence $\{a_n\}$ is bounded above then the *limit superior* of $\{a_n\}$ is

$$\limsup a_n = \lim_{n\to\infty} U_n = \lim_{n\to\infty} (lub\{a_k |\ k \geq n\}).$$

If the sequence $\{a_n\}$ is not bounded above define $\limsup a_n = \infty$.

Exercise 2.46. *Your turn:* Find $\limsup a_n$ and $\limsup b_n$ in the example above. Can you make a conjecture based on your answers?

Example 2.47. Let $\{a_n\} = \{(-1)^n\}$. Find $\limsup a_n$.

Let's compute U_n. If n is even then $U_n = lub\{1, -1, 1, -1, ...\} = 1$, and if n is odd $U_n = lub\{-1, 1, -1, 1, ...\} = 1$. So $U_n = 1$ for all $n \in \mathbb{N}$ and $\limsup a_n = \lim_{n\to\infty} U_n = 1$.

Note that even though $\{(-1)^n\}$ does not converge, $\limsup(-1)^n$ exists.

Example 2.48. Consider the sequence $\{a_n\} = \{(-1)^n(1 + \frac{1}{n})\}$. It is not hard to see that this sequence diverges. (Prove it!) Let's find $\limsup a_n$.

What is U_n here? If n is even then $U_n = lub\{1+\frac{1}{n}, -(1+\frac{1}{n+1}), 1+\frac{1}{n+2}, -(1+\frac{1}{n+3}), ...\} = 1+\frac{1}{n}$.

If n is odd then $U_n = lub\{-(1 + \frac{1}{n}), 1 + \frac{1}{n+1}, -(1 + \frac{1}{n+2}), 1 + \frac{1}{n+3}, ...\} = 1 + \frac{1}{n+1}$.

Thus $\limsup(-1)^n(1 + \dfrac{1}{n}) = \lim\limits_{n\to\infty} U_n = 1$.

So we have seen that even if a sequence diverges, we can compute the $\limsup$. In the case of a convergent sequence we have:

Theorem 2.49. *If $\{a_n\}$ converges to a then* $\limsup a_n = a$.

Proof: Let $\epsilon > 0$ be given. Choose $N \in \mathbb{N}$ such that $|a_n - a| < \frac{\epsilon}{2}$ if $n \geq N$. Thus

$$-\frac{\epsilon}{2} < a_n - a < \frac{\epsilon}{2} \text{ for all } n \geq N$$

or

$$a - \frac{\epsilon}{2} < a_n < a + \frac{\epsilon}{2} \text{ for all } n \geq N.$$

Thus $a + \frac{\epsilon}{2}$ is an upper bound for $\{a_n, a_{n+1}, a_{n+2}, ...\}$ and $a - \frac{\epsilon}{2}$ is not and so

$$a - \epsilon < a - \frac{\epsilon}{2} < U_n \leq a + \frac{\epsilon}{2} < a + \epsilon \text{ for all } n \geq N.$$

This implies that $|U_n - a| < \epsilon$ if $n \geq N$. That is $\limsup a_n = \lim\limits_{n\to\infty} U_n = a$.

Note: If $\lim_{n\to\infty} a_n = \infty$ then $\{a_n\}$ is not bounded above and so $\limsup a_n = \infty$.
If $\lim_{n\to\infty} a_n = -\infty$ then given any $A > 0$ we can choose $N \in \mathbb{N}$ such that $a_n < -A$ for all $n \geq N$ and so $U_n \leq -A$ for all $n \geq N$. Therefore $\limsup a_n = \lim_{n\to\infty} U_n = -\infty$.

Definition 2.50. If the sequence $\{a_n\}$ is bounded below then the *limit inferior* of $\{a_n\}$ is

$$\liminf a_n = \lim_{n\to\infty} L_n = \lim_{n\to\infty} (glb\{a_k|\ k \geq n\}).$$

If the sequence $\{a_n\}$ is not bounded below define $\liminf a_n = -\infty$.

Exercise 2.51. *Your turn:*

1. Check that $\{L_n\}$ is an increasing sequence.

2. Compute the lim inf of the sequences in Examples 2.44, 2.47 and 2.48 above.

How are $\liminf(a_n)$ and $\limsup(a_n)$ related?

Theorem 2.52. *Let $\{a_n\}$ be a sequence of real numbers. Then*

$$\liminf(a_n) = -\limsup(-a_n).$$

Proof: Let $b_n = -a_n$. Then if $\{a_n\}$ is not bounded below, we have that $\{b_n\}$ is not bounded above so

$$\liminf(a_n) = -\infty = -\limsup(b_n) = -\limsup(-a_n).$$

If $\{a_n\}$ is bounded below, let $L_n = glb\{a_k|\ k \geq n\}$. Thus $a_k \geq L_n$ for all $k \geq n$ and so $b_k = -a_k \leq -L_n$ for all $k \geq n$. So $-L_n$ is an upper bound for $\{b_k|\ k \geq n\}$. Thus $-L_n \geq lub\{b_k|\ k \geq n\}$. We can use a similar argument to show that $-lub\{b_k|\ k \geq n\}$ is a lower bound for $\{a_k|\ k \geq n\}$ and so $lub\{b_k|\ k \geq n\} \geq -L_n$. So we have that $lub\{b_k|\ k \geq n\} = -L_n$. Thus

$$\begin{aligned}\liminf(a_n) = \lim_{n\to\infty} L_n &= \lim_{n\to\infty} -lub\{b_k|\ k \geq n\} \\ &= -\lim_{n\to\infty} lub\{b_k|\ k \geq n\} = -\limsup(b_n)\end{aligned}$$

We will use this result to prove the following theorem:

Theorem 2.53. *Let $\{a_n\}$ be a sequence of real numbers. Then $\lim_{n\to\infty} a_n = a$ if and only if*

$$\liminf(a_n) = \limsup(a_n) = a.$$

Proof: Assume that $\lim_{n\to\infty} a_n = a$ then by Theorem 2.49 we have $\limsup(a_n) = a$ also. Let $b_n = -a_n$ then $\lim_{n\to\infty} b_n = -a$ and $\limsup(b_n) = -a$. By Theorem 2.52 we get

$$\begin{aligned} \liminf(a_n) &= -\limsup(-a_n) \\ &= -\limsup(b_n) = a \end{aligned}$$

Thus if $\lim_{n\to\infty} a_n = a$ then $\liminf(a_n) = \limsup(a_n) = a$.

Suppose now that $\liminf(a_n) = \limsup(a_n) = a$ and that a is a finite number. We must show that $\lim_{n\to\infty} a_n = a$.

Let $\epsilon > 0$ be given and choose $N_1 \in \mathbb{N}$ such that $|U_n - a| < \epsilon$ for all $n \geq N_1$. That gives us that $U_n < a + \epsilon$ for all $n \geq N_1$. That is $a_n \leq U_n < a + \epsilon$ for all $n \geq N_1$.

Now choose $N_2 \in \mathbb{N}$ such that $|L_n - a| < \epsilon$ for all $n \geq N_2$. Thus $L_n > a - \epsilon$ for all $n \geq N_2$. That is $a_n \geq L_n > a - \epsilon$ for all $n \geq N_2$.

Let $N = Max(N_1, N_2)$. If $n \geq N$ we have

$$a - \epsilon < a_n < a + \epsilon \text{ for all } n \geq N$$

Thus $\lim_{n\to\infty} a_n = a$.