MT232P - Analysis

Assignment #4

Viktor Ilchev
22337763

Question 1

Let $x_1 = 1$, and set $x_{n+1} = \sqrt{2 + x_n}$ for all $n \in \mathbb{N}$. Use the Monotone Convergence Theorem to show that $\{x_n\}_1^\infty$ converges, and find $\lim_{n\to\infty} x_n$.

Solution

First, we show that the sequence $\{x_n\}$ is increasing. Since $x_1 = 1$, we have $x_{n+1} = \sqrt{2 + x_n} > \sqrt{2 + 1} = \sqrt{3} > 1 = x_1$ for all $n \in \mathbb{N}$. Thus, the sequence is increasing. $\{x_n\}$ Next, we need to show that the sequence is bounded above. Note that $x_{n+1} = \sqrt{2 + x_n} \leq \sqrt{2 + 2} = \sqrt{4} = 2$ for all $n \in \mathbb{N}$. Thus, the sequence is bounded above.

Since the sequence is increasing and bounded above, by the Monotone Convergence Theorem, the sequence $\{x_n\}$ converges. To find the limit, we take the limit of both sides of the equation $x_{n+1} = \sqrt{2 + x_n}$ as $n \to \infty$:

$$\lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} \sqrt{2 + x_n}$$

Since the sequence $\{x_n\}$ converges, the limit on the right-hand side exists and is equal to the limit of the sequence, say L. We can then write:

$$L = \lim_{n\to\infty} \sqrt{2 + x_n} = \sqrt{2 + L}$$

Squaring both sides, we get $L^2 = 2 + L$, so $L^2 - L - 2 = 0$. This quadratic equation has two solutions, $L = \frac{1\pm\sqrt{5}}{2}$. However, since L is the limit of an increasing sequence, it must be greater than or equal to the first term of the sequence, which is 1. This implies that $L = \frac{1+\sqrt{5}}{2}$.

Therefore, the limit of the sequence $\{x_n\}$ is $\frac{1+\sqrt{5}}{2}$.

Question 2

Show that if every subsequence of $\{a_n\}_1^\infty$ has itself a subsequence which converges to 0, then $\{a_n\}_1^\infty$ converges to 0

Solution

Question 3

Assume $\{a_n\}_1^\infty$ and $\{b_n\}_1^\infty$ are Cauchy sequences. Use a triangle inequality argument to prove directly from the definition of a Cauchy sequence that $\{c_n\}_1^\infty$, where $c_n = |an - bn|$,is also a Cauchy sequence.

Solution

Question 4

What is the value of $\lim_{x \to 1} \frac{1}{1+\sqrt{x}}$? Prove your assertion using an $\epsilon - \delta$ argument.

Solution