MT251P – Lecture 11

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Remark 10. continued.

This means that we can define the angle θ between the two non–zero vectors $\underline{x}, \underline{y}$ as the unique θ satisfying

$$\cos\theta = \frac{\underline{x}.\underline{y}}{||\underline{x}||||\underline{y}||}$$

and this agrees with theorem 3(e).

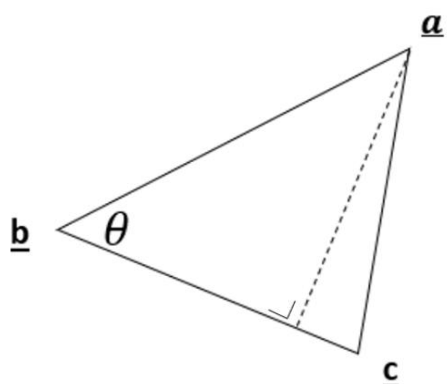
Example 9.

If θ is the angle between the vectors $\underline{x} = (2, 0, 1, 0)$ and $\underline{y} = (1, 3, 0, 0)$ in $\mathbb{R}^4$, then

$$\cos\theta = \frac{\underline{x}.\underline{y}}{||\underline{x}||||\underline{y}||} = \frac{2}{\sqrt{5}\sqrt{10}} = \frac{2}{\sqrt{50}}$$

Area of a triangle.

Three non–collinear vectors $\underline{a}$, $\underline{b}$, $\underline{c}$ in $\mathbb{R}^n$ define a triangle.



We will now derive a formula for the area A of the triangle in terms of the three vectors $\underline{a}$, $\underline{b}$, $\underline{c}$. Well, A is half the base by the perpendicular height.

Now, after translating $\underline{b}$ to the origin, the base can be described by the position vector $\underline{c} - \underline{b}$. The perpendicular height is $||\underline{a} - \underline{b}|||\sin\theta|$ where θ is the angle between $\underline{a} - \underline{b}$ and $\underline{c} - \underline{b}$.

So,

$$A = \frac{1}{2}||\underline{c} - \underline{b}||||\underline{a} - \underline{b}||| \sin\theta|$$

$$\Rightarrow A^2 = \frac{1}{4}||\underline{c} - \underline{b}||^2||\underline{a} - \underline{b}||^2\sin^2\theta$$

$$= \frac{1}{4}(||\underline{c} - \underline{b}||^2||\underline{a} - \underline{b}||^2 - ||\underline{c} - \underline{b}||^2||\underline{a} - \underline{b}||^2\cos^2\theta)$$

$$= \frac{1}{4}(||\underline{c} - \underline{b}||^2||\underline{a} - \underline{b}||^2 - ((\underline{c} - \underline{b}).(\underline{a} - \underline{b}))^2)$$

$$\Rightarrow A = \frac{1}{2}\sqrt{||\underline{c} - \underline{b}||^2||\underline{a} - \underline{b}||^2 - ((\underline{c} - \underline{b}).(\underline{a} - \underline{b}))^2}$$

Example 10.

Find the area A of the tiangle formed by the three vectors $\underline{a} = (1,1,0)$, $\underline{b} = (2,3,1)$, $\underline{c} = (0,2,2)$.

Solution.

$$||\underline{c} - \underline{b}||^2 = 6, \quad ||\underline{a} - \underline{b}||^2 = 6, \quad (\underline{c} - \underline{b}).(\underline{a} - \underline{b}) = 3$$

and so

$$A = \frac{1}{2}\sqrt{36 - 9} = \frac{1}{2}\sqrt{27}$$

Example 11.

Find the parametric equation of the line L which contains the point $(-1, 4, 3)$ and is parallel to the line K which has parametric equation

$$K = \{(9, 0, -1) + t(-1, 4, 2) : t \in \mathbb{R}\}$$

Also, find another point on L different from $(-1, 4, 3)$.

Solution.

K is parallel to the vector $\underline{z} = (-1, 4, 2)$ and so L is parallel to $\underline{z}$. Hence, the parametric equation of L is

$$L = \{(-1, 4, 3) + t(-1, 4, 2) : t \in \mathbb{R}\} \qquad (*)$$

To find another point on L different from $(-1, 4, 3)$, we let t be any non–zero real number in $(*)$. For example, let $t = 1$ to get the point

$$(-1, 4, 3) + (-1, 4, 2) = (-2, 8, 5)$$

and so $(-2, 8, 5)$ is another point on L different from $(-1, 4, 3)$.

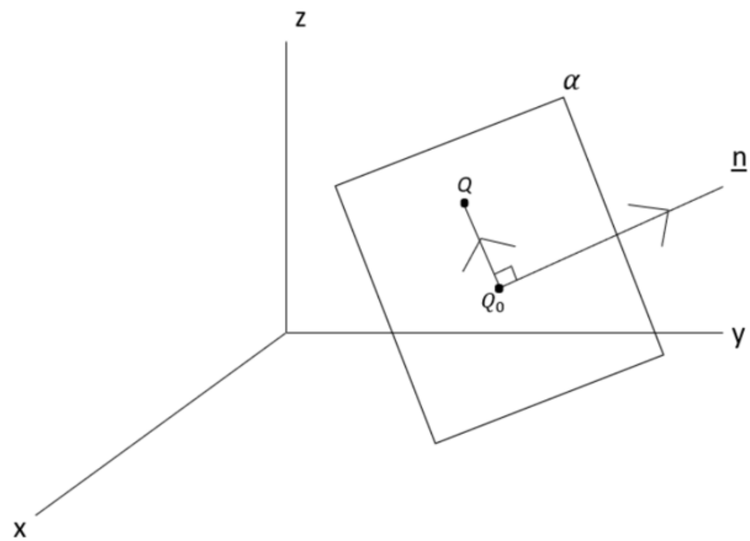
Section 3.3 – Planes in $\mathbb{R}^3$.

Definition 6.

Consider the plane α which contains the point $Q_0 = (x_0, y_0, z_0)$ and suppose the non–zero vector $\underline{n} = ai + bj + ck$ is perpendicular to α. Then, the equation of α is

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

Remark 11.



The motivation for the above definition is that if $Q = (x, y, z)$ is any point in α, then the vector $\vec{Q_0Q}$ is perpendicular to $\underline{n}$ and so the dot product

$$\underline{n}.\vec{Q_0Q} = 0$$

which means

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

Remark 12.

$\underline{n}$ is called a normal vector to α in definition 6 above.

Example 12.

Find the equation of the plane α which contains the point $(1, 2, -4)$ and has normal vector $\underline{n} = 3i - j + 2k$.

Solution.

The equation of α is

$$3(x-1)-(y-2)+2(z+4)=0$$

$$\Rightarrow 3x-y+2z+7=0$$

Example 13.

Find the equation of the plane α which contains the point $(1,-1,3)$ and is perpendicular to the line L with parametric equation

$$L=\{(2,1,-2)+t(-3,2,4):t\in\mathbb{R}\}$$

Solution.

L is parallel to $\underline{n}=-3i+2j+4k$ and so α is perpendicular to $-3i+2j+4k$. Hence, the equation of α is

$$-3(x-1)+2(y+1)+4(z-3)=0$$

$$\Rightarrow -3x+2y+4z=7$$

$$\Rightarrow 3x-2y-4z+7=0$$

Example 14.

Find the equation of the plane α containing the three points $(1,0,0)$, $(1,2,1)$, $(2,1,0)$.