MT251P – Lecture 20

Example 6.

(i) $(4, -4, 0)$ is a linear combination of $(2, 0, 0), (\frac{1}{2}, -1, 0)$ because

$$(4, -4, 0) = (2, 0, 0) + 4(\frac{1}{2}, -1, 0)$$

(ii) $(4, -2, 1)$ is not a linear combination of $(2, 0, 0), (\frac{1}{2}, -1, 0)$. because there are no $\alpha_1, \alpha_2 \in \mathbb{R}$ such that

$$(4, -2, 1) = \alpha_1(2, 0, 0) + \alpha_2(\frac{1}{2}, -1, 0)$$

Example 7.

$S = \{(a, b, 0) : a, b \in \mathbb{R}\}$ is subspace of $\mathbb{R}^3$.

Definition 6.

Suppose $\underline{u_1}, \underline{u_2}, \ldots, \underline{u_k}$ are vectors in $\mathbb{R}^n$. The set

$$S(\underline{u_1}, \underline{u_2}, \ldots, \underline{u_k}) = \{\alpha_1\underline{u_1} + \alpha_2\underline{u_2} + \cdots + \alpha_k\underline{u_k},\ \alpha_i \in \mathbb{R},\quad 1 \le i \le k\}$$

consisting of all linear combinations of $\underline{u_1}, \underline{u_2}, \ldots, \underline{u_k}$ is called the subspace generated by $\underline{u_1}, \underline{u_2}, \ldots, \underline{u_k}$.

Remark 1.

One can check that $S(\underline{u_1}, \underline{u_2}, \ldots, \underline{u_k})$ is a subspace of $\mathbb{R}^n$.

Definition 7.

Suppose S is a subspace of $\mathbb{R}^n$. The set $\{\underline{u_1}, \underline{u_2}, \ldots, \underline{u_k}\}$ is a basis for S if the following two conditions are satisfied:

(i) $\underline{u_1}, \underline{u_2}, \ldots, \underline{u_k}$ are linearly independent.

(ii) $\underline{u_1}, \underline{u_2}, \ldots, \underline{u_k}$ span S.

Example 8.

Suppose $\underline{e_i}$ is the vector in $\mathbb{R}^4$ with zero in every coordinate except the i^{th} coordinate where there is a one. Then $B = \{\underline{e_1}, \underline{e_2}, \underline{e_3}, \underline{e_4}\}$ is a basis for $\mathbb{R}^4$.

Proof.

We first prove that B is linearly independent. Suppose $\alpha_i \in \mathbb{R},\ 1 \le i \le 4$ and

$$\alpha_1\underline{e_1} + \alpha_2\underline{e_2} + \alpha_3\underline{e_3} + \alpha_4\underline{e_4} = (0, 0, 0, 0)$$

Thus,

$$\alpha_1(1,0,0,0)+\alpha_2(0,1,0,0)+\alpha_3(0,0,1,0)+\alpha_4(0,0,0,1)=(0,0,0,0)$$

$$\iff (\alpha_1,\alpha_2,\alpha_3,\alpha_4)=(0,0,0,0)$$

$$\iff \alpha_1=\alpha_2=\alpha_3=\alpha_4=0$$

and so B is linearly independent.

We next prove that B spans $\mathbb{R}^4$. So, suppose $\underline{x}=(x_1,x_2,x_3,x_4)\in\mathbb{R}^4$. Then, we can write

$$\underline{x}=x_1\underline{e}_1+x_2\underline{e}_2+x_3\underline{e}_3+x_4\underline{e}_4$$

.

and so $\underline{x}$ is a linear combination of $\underline{e}_1,\underline{e}_2,\underline{e}_3,\underline{e}_4$ and hence B spans $\mathbb{R}^4$.

So, B is basis for $\mathbb{R}^4$.

Example 9.

Suppose $\underline{e_i}$ is the vector in $\mathbb{R}^n$ with zero in every coordinate except the i^{th} coordinate where there is a one. Then $B=\{\underline{e_1},\underline{e_2},\dots,\underline{e_n}\}$ is a basis for $\mathbb{R}^n$. The proof of this is similar to example 8. Also, this basis is called the standard basis for $\mathbb{R}^n$.

Theorem 1.

Suppose $\underline{y_1},\underline{y_2},\dots,\underline{y_w}$ are vectors in $S(\underline{u_1},\underline{u_2},\dots,\underline{u_k})$. If $w>k$, then $\underline{y_1},\underline{y_2},\dots,\underline{y_w}$ are linearly dependent.

Remark 2.

If $\underline{u_1},\underline{u_2},\dots,\underline{u_k}$ is a basis for a subspace T of $\mathbb{R}^n$, then $T=S(\underline{u_1},\underline{u_2},\dots,\underline{u_k})$.

Theorem 2.

Suppose $\{\underline{u_1},\underline{u_2},\dots,\underline{u_k}\}$ and $\{\underline{y_1},\underline{y_2},\dots,\underline{y_r}\}$ are two bases for a subspace T of $\mathbb{R}^n$. Then $k=r$. The unique number of vectors in a basis for T is called the dimension of T and is denoted by $\dim T$.

Example 10.

$\mathbb{R}^2$ has dimension 2 because $\{(1,0),(0,1)\}$ is a basis for $\mathbb{R}^2$. Also, $\mathbb{R}^n$ has dimension n because $\{\underline{e_1},\underline{e_2},\dots,\underline{e_n}\}$ is a basis for $\mathbb{R}^n$.

Theorem 3.

Suppose S is a subspace of $\mathbb{R}^n$ and suppose that C is a subset of S such that the number of elements in C is $\dim S$. Then C is linearly independent $\iff$ C spans S.

Remark 3.

Theorem 3 shows that if the number of elements in C is $\dim S$, then we only have to check one of the two conditions (i), (ii) in definition 7 in order to check if C is a basis for S.

Example 11

Prove that $C = \{(-1,0,3),(0,2,-2),(0,0,5)\}$ is a basis for $\mathbb{R}^3$.

Proof.

First, note that $\mathbb{R}^3$ is a subspace of $\mathbb{R}^3$ and that C is a subset of $\mathbb{R}^3$ such that the number of elements in C is $\dim \mathbb{R}^3 = 3$. Then, remark 3 shows that if we can show C is linearly independent, then C will be a basis for $\mathbb{R}^3$.

So, we will prove that C is linearly independent and then we will be done. So, suppose

$$\alpha_1(-1,0,3) + \alpha_2(0,2,-2) + \alpha_3(0,0,5) = (0,0,0) \qquad (*)$$

for some $\alpha_i \in \mathbb{R}$ for $1 \le i \le k$

Then,

$$(*) \iff (-\alpha_1, 2\alpha_2, 3\alpha_1 - 2\alpha_2 + 5\alpha_3) = (0,0,0)$$

$$\iff \alpha_1 = 0,\ 2\alpha_2 = 0,\ 3\alpha_1 - 2\alpha_2 + 5\alpha_3 = 0$$

$$\iff \alpha_1 = 0,\ \alpha_2 = 0,\ \alpha_3 = 0$$

Hence C is linearly independent and we are done.

Section 6.2 – Row Space, Column Space and Rank.

Remark 4.

(i) Suppose A is a $k \times n$ matrix. We can consider each row of A as a vector in $\mathbb{R}^n$ (called a row vector). Similarly, we can consider each column of A as a vector in $\mathbb{R}^k$ (called a column vector).

(ii) If W is a spanning set for L, then we say that L is spanned by W.