MT234P – MULTIVARIABLE CALCULUS –2022

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Lecture 14

Remark 13.

The method of Lagrange Multipliers only produces candidates for absolute maxima and absolute minima of f subject to the constraint $g(x, y) = 0$. There may no such absolute maxima and there may be no such absolute minima and the next example will illustrate this.

Example 8.

(i) Use Lagrange multipliers to find the candidates for absolute maxima and absolute minima of $f(x, y) = x + y$ subject to the constraint $xy = 16$.

(ii) Show that there are no absolute maxima and no absolute minima of $f(x, y) = x + y$ subject to the constraint $xy = 16$.

Solution.

(i) Let $g(x, y) = xy - 16$ so that the constraint $xy = 16$ can be written in the form $g(x, y) = 0$.

So, by remark 12, in order to find candidates (a, b) for the absolute maxima and absolute minima of $f(x, y) = x + y$ subject to the constraint $g(x, y) = 0$, we look for points (a, b) and real numbers λ that satisfy

$$\nabla f_{|(a,b)} = \lambda \nabla g_{|(a,b)} \quad \text{and} \quad g(a, b) = 0 \qquad\qquad (*)$$

Now, $(*)$ gives us

$$(\vec{i} + \vec{j})_{|(a,b)} = \lambda(y\vec{i} + x\vec{j})_{|(a,b)} \quad \text{and} \quad ab - 16 = 0$$

$$\Rightarrow 1 = \lambda b, \quad 1 = \lambda a, \quad ab = 16$$

$$\Rightarrow \lambda a = \lambda(\frac{16}{a})$$

$$\Rightarrow \lambda(a - \frac{16}{a}) = 0$$

$$\Rightarrow \lambda = 0 \quad \text{or} \quad a = \pm 4$$

We now look at the two cases separately:

CASE 1: Suppose $\lambda = 0$. Then $1 = \lambda a = 0$ which is impossible.

CASE 2: Suppose $a = \pm 4$. This implies $b = \pm 4 \quad (**)$.

Now, using $(**)$ we get that the candidates for the required absolute maxima and absolute minima are $(4, 4)$, $(-4, -4)$. So, we have finished (i).

(ii) We will now show that there are no absolute maxima and no absolute minima of $f(x, y) = x + y$ subject to the constraint $xy = 16$.

Notice that the farther you go from the origin on the hyperbola $xy = 16$ in the first quadrant, the larger the sum $x + y$ is. This means that there are no absolute maxima for $x + y$ subject to the constraint $xy = 16$.

Similarly, notice that the farther you go from the origin on the hyperbola $xy = 16$ in the fourth quadrant, the smaller the sum $x + y$ is. This means that there are no absolute minima for $x + y$ subject to the constraint $xy = 16$. We are now finished with (ii).

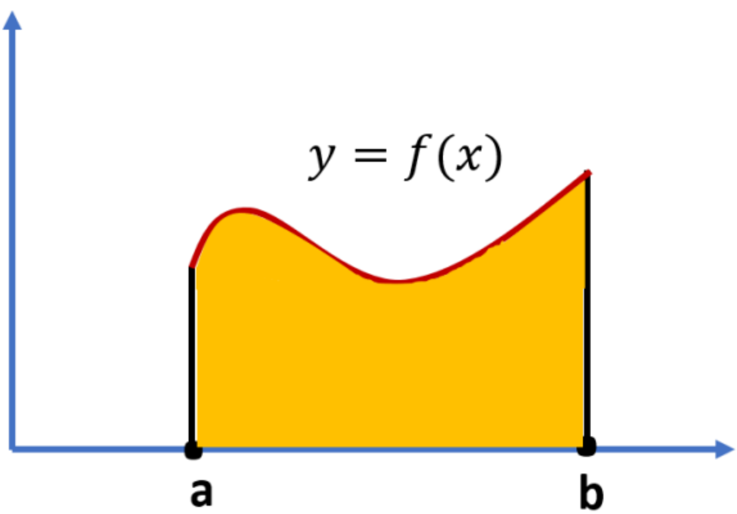
Chapter 4 – Multiple Integration.

Section 4.1 – Multiple Integrals.

Remark 1.

It will be helpful to recall the situation for functions of one variable before we motivate the definition of integrals of functions of two variables. There are many important and powerful applications of integrals to science, engineering and other areas.

Recall that definite integrals of the form $\int_a^b f(x)\,dx$ were used to find the area of the shaded region below.



The shaded region above is bounded by the curve $y = f(x)$, the x–axis and the vertical lines $x = a$, $x = b$. The definition of $\int_a^b f(x)\,dx$ can be motivated as follows:

We will first motivate the definition of the area of the shaded region above (when f is continuous and non–negative on $[a, b]$). We partition the closed interval $[a, b]$ into n subintervals $[x_{i-1}, x_i]$ (for $1 \leq i \leq n$), each of length $\dfrac{b-a}{n}$, by selecting points $x_0 < x_1 < x_2 < \ldots < x_{n-1} < x_n$ where $x_0 = a$ and $x_n = b$. The vertical lines $x = x_i$ divide the shaded region above into vertical strips. We approximate each strip by a rectangle with base $[x_{i-1}, x_i]$ and height $f(w_i)$ where $f(w_i)$ is the absolute minimum value of f on $[x_{i-1}, x_i]$. Let $\Delta x = \dfrac{b-a}{n}$, so that the sum of the areas of these rectangles is

$$\sum_{i=1}^{n} f(w_i)\Delta x \qquad (*)$$

We say that $(*)$ approximates the area of the shaded region above. Note that this approximation may not necessarily be a good approximation if n is small. However, we expect the approximation to improve as n increases (i.e. as Δx approaches zero). With this as motivation, we define the area of the shaded region to be

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} f(w_i)\Delta x$$

We now consider any function f on $[a, b]$. Motivated by the discussion above regarding area, we partition the closed interval $[a, b]$ into n subintervals $[x_{i-1}, x_i]$ (for $1 \leq i \leq n$), by selecting points $x_0 < x_1 < x_2 < \ldots < x_{n-1} < x_n$ where $x_0 = a$ and $x_n = b$. $P = \{x_0, x_1, \ldots, x_n\}$ is called the corresponding partition of $[a, b]$. Let $\Delta x_i = x_i - x_{i-1}$ denote the length of the i^{th} subinterval. Select a random point c_i in $[x_i, x_{i-1}]$ and consider the so called Riemann sum:

$$\sum_{i=1}^{n} f(c_i)\Delta x_i$$

The norm of the partition P above is the length of the longest subinterval and is denoted by $||P||$. We are now ready to define the definite integral $\int_a^b f(x)\, dx$:

We say

$$\lim_{||P||\to 0} \sum_{i=1}^{n} f(c_i)\Delta x_i = L \qquad (**)$$

if given any $\epsilon > 0$, there exists $\delta > 0$ such that

$$||P|| < \delta \;\Rightarrow\; \left| \sum_{i=1}^{n} f(c_i)\Delta x_i - L \right| < \epsilon$$

for any choice of the c_i in $[x_{i-1}, x_i]$.

If L exists in (**) above, then we define the definite integral of f over $[a, b]$ by

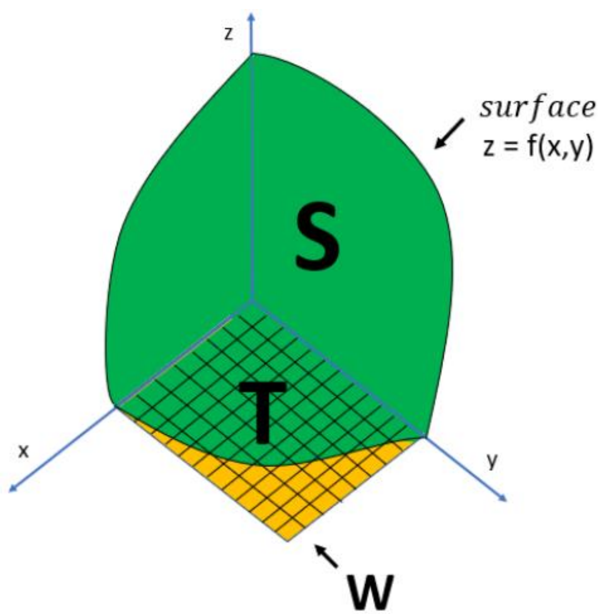
$$\int_a^b f(x)\, dx = L$$

Remark 2.

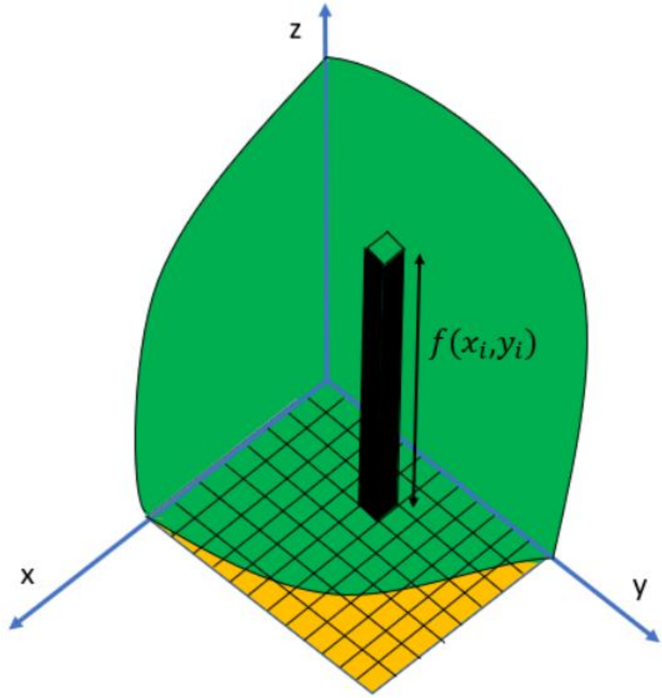
Suppose f is a continuous function on $[a, b]$ and suppose $f(x) \geq 0$, for all $x \in [a, b]$. Then, based on the above discussion it is no surprise that $\int_a^b f(x)\, dx$ is the area of the region bounded by the curve $y = f(x)$, the x–axis and the vertical lines $x = a,\ x = b$.

Remark 3.

The approach for defining the integral of a function of two variables will be similar in some ways to the one variable case in remark 1. Suppose f is a continuous function and non–negative on a region T in the xy–plane. We will first try to find the volume of a three–dimensional solid region S bounded above by the surface $z = f(x, y)$ and bounded below by the region T in the xy–plane in the picture below.



We start by superimposing a rectangular grid W over T. The n rectangles lying entirely in T form an inner partition Δ, whose norm $||\Delta||$ is defined as the length of the longest diagonal of the n rectangles. We then select a point (x_i, y_i) in each rectangle and form the rectangular prism with height $f(x_i, y_i)$ as in the picture below.


z
$f(x_i, y_i)$
x
y