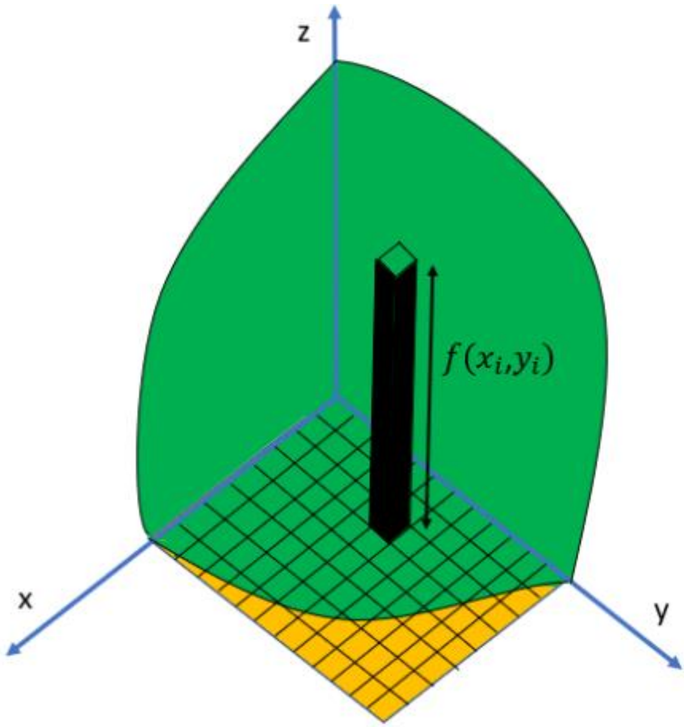
Fiacre Ó Cairbre

Lecture 15

.



We denote the area of the i^{th} rectangle by ΔA_i so that the volume of the i^{th} prism is $f(x_i, y_i)\Delta A_i$. We say that the so called Riemann sum

$$\sum_{i=1}^{n} f(x_i, y_i)\Delta A_i$$

approximates the volume of S. Note that this approximation may not necessarily be a good approximation if $||\Delta||$ is big. However, we expect the approximation to improve as $||\Delta||$ approaches zero. With this as motivation, we define the volume of S to be

$$\lim_{||\Delta||\to 0} \sum_{i=1}^{n} f(x_i, y_i)\Delta A_i$$

where this limit is defined in a similar way as $(**)$ in remark 1.

With the above as motivation (and using similar notation as above), we are now ready to define the so called double integral of f over T as follows:

Definition 1 – Double integral.

Suppose $f(x, y)$ is defined on a closed, bounded region T in the xy–plane. The double integral of f over T is defined as

$$\int_T\int f(x,y)\;dA = \lim_{||\Delta||\to 0}\sum_{i=1}^{n} f(x_i, y_i)\Delta A_i$$

If this limit exists, we say that f is integrable over T.

On account of the discussion above, it is not surprising that a double integral is used to find the volume of a solid region. Here is the definition of the volume of a solid region:

Definition 2 – Volume of a solid region.

Suppose f is integrable over a region T in the xy–plane and suppose $f(x, y) \geq 0$ for all $(x, y) \in T$. Then, the volume of the solid region lying above T and below the surface $z = f(x, y)$ is given by

$$V = \int_T\int f(x,y)\;dA$$

Theorem 1 – Properties of Double Integrals.

Suppose f and g are continuous functions on a closed bounded region T in the xy–plane and suppose k is a constant. Then,

(i) $\displaystyle\int_T\int kf(x,y)\;dA = k\int_T\int f(x,y)\;dA$

(ii) $\displaystyle\int_T\int (f(x,y)+g(x,y))\;dA = \int_T\int f(x,y)\;dA + \int_T\int g(x,y)\;dA$

(iii) $\displaystyle\int_T\int f(x,y)\;dA \geq 0$, if $f(x,y) \geq 0$

(iv) $\displaystyle\int_T\int f(x,y)\;dA = \int_{T_1}\int f(x,y)\;dA + \int_{T_2}\int f(x,y)\;dA$

where T is the union of two disjoint sets T_1 and T_2.

Remark 4.

A definite integral of a function of one variable, $\int_a^b g(x)\;dx$, will now be called a single integral (to distinguish it from a double integral).

Theorem 2 – Fubini's Theorem.

Suppose f is a continuous function on a region T in the xy–plane.

(i) If $T = \{(x, y) \in \mathbb{R}^2 : a \leq x \leq b,\; g_1(x) \leq y \leq g_2(x)\}$, where g_1 and g_2 are continuous on $[a, b]$, then

$$\int_T \int f(x,y)\, dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\, dydx \qquad (*)$$

We will discuss the expression on the right hand side of $(*)$ in remark 5 below.

(ii) If $T = \{(x,y) \in \mathbb{R}^2 : c \leq y \leq d,\ h_1(y) \leq x \leq h_2(y)\}$, where h_1 and h_2 are continuous on $[c,d]$, then

$$\int_T \int f(x,y)\, dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\, dxdy \qquad (**)$$

We will discuss the expression on the right hand side of $(**)$ in remark 5 below.

Remark 5.

The two integrals on the right in $(*)$ and $(**)$ above are called iterated integrals. Fubini's theorem says that if T is horizontally simple (i.e. $a \leq x \leq b$), then the double integral in (i) can be calculated by performing two single integrals, one after the other – you integrate with respect to (w.r.t.) y first (treating x as constant, like we did with partial derivatives) and then you integrate w.r.t. x last (treating y as constant). This will be made clear in example 1 below.

Similarly, if T is vertically simple (i.e. $c \leq y \leq d$), then the double integral in (ii) can be calculated by performing two single integrals, one after the other – you integrate w.r.t. x first (treating y as constant) and then you integrate w.r.t. y last (treating x as constant). This will be made clear in remark 6 below.

In the examples below the notation

$$[w(x)]_a^b$$

is the usual notation for $w(b) - w(a)$.

Example 1. Find the volume of the solid lying below $z = 4 - x - y$ and above the square T given $0 \leq x \leq 1$ and $1 \leq y \leq 2$.

Solution.

By definition 2, the required volume is

$$V = \int_T \int (4 - x - y)\, dA$$

$$= \int_{x=0}^1 \left(\int_{y=1}^2 (4 - x - y)\, dy \right) dx \qquad \text{by Fubini's Theorem (i)}$$

$$= \int_{x=0}^1 \left[4y - xy - \frac{y^2}{2} \right]_1^2 dx$$

$$= \int_0^1 \left[(8 - 2x - 2) - (4 - x - \frac{1}{2}) \right] \, dx$$

$$= \int_0^1 \left(\frac{5}{2} - x \right) \, dx$$

$$= \left[\frac{5x}{2} - \frac{x^2}{2} \right]_0^1 = 2$$

Remark 6.

In example 1 we could use Fubini's Theorem (ii) and integrate w.r.t. y last, as follows

$$V = \int_T \int (4 - x - y) \, dA$$

$$= \int_{y=1}^{2} \left(\int_{x=0}^{1} (4 - x - y) \, dx \right) dy \qquad \text{by Fubini's Theorem (ii)}$$

$$= \int_{y=1}^{2} \left[4x - \frac{x^2}{2} - yx \right]_0^1 \, dy$$

$$= \int_1^2 \left[(4 - \frac{1}{2} - y) - (0 - 0 - 0) \right] \, dy$$

$$= \int_1^2 \left(\frac{7}{2} - y \right) \, dy$$

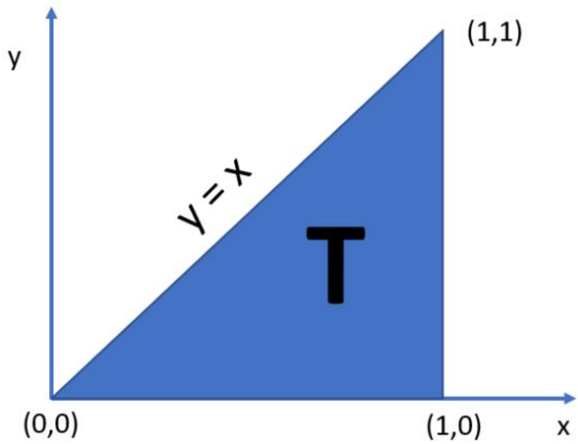
$$= \left[\frac{7y}{2} - \frac{y^2}{2} \right]_1^2 = 2$$

Example 2.

Find $\int_T \int xy \, dA$, where T is the triangle with vertices $(0,0), (1,0), (1,1)$.

Solution.

It's important to draw a good picture here.



Consequently, we see that $T = \{(x, y) : 0 \leq x \leq 1,\ 0 \leq y \leq x\}$. Using Fubini's theorem (i) we get that:

$$\int_T\int xy\ dA = \int_{x=0}^{1}\left(\int_{y=0}^{x} xy\ dy\right)\ dx$$
$$= \int_0^1 \left[\frac{xy^2}{2}\right]_0^x\ dx$$
$$= \int_0^1 \frac{x^3}{2}\ dx$$
$$= \left[\frac{x^4}{8}\right]_0^1$$

Remark 7.

In example 2, we also see that from the picture, $T = \{(x, y) : 0 \leq y \leq 1,\ y \leq x \leq 1\}$. and so using Fubini's theorem (ii) we get that:

$$\int_T\int xy\ dA = \int_{y=0}^{1}\left(\int_{x=y}^{1} xy\ dx\right)\ dy$$
$$= \int_0^1 \left[\frac{x^2y}{2}\right]_y^1\ dy$$
$$= \int_0^1 \left(\frac{y}{2} - \frac{y^3}{2}\right)\ dy$$
$$= \left[\frac{y^2}{4} - \frac{y^4}{8}\right]_0^1$$
$$= \frac{1}{8}$$