# MT234P – MULTIVARIABLE CALCULUS – 2022

Fiacre Ó Cairbre

## Lecture 17

### Section 4.2 – Surface Area.

**Definition 4.**

Suppose $f(x, y)$ and its partial derivatives are continuous on the closed region $T$ in the $xy$–plane. Then, the area of the surface given by $z = f(x, y)$ over $T$ is given by

$$\int_T\int \sqrt{1 + f_x^2 + f_y^2}\, dA$$

where

$$f_x = \frac{\partial f}{\partial x}, \quad f_x^2 = (\frac{\partial f}{\partial x})^2, \quad f_y = \frac{\partial f}{\partial y}, \quad f_y^2 = (\frac{\partial f}{\partial y})^2$$

**Example 7.**

Find the surface area of the portion of the surface $z = \frac{1}{2}(x^2 - 2y)$ lying above the triangular region $T$ in the $xy$-plane bounded by the lines $y = 0, x = 2$ and $y = 3x$.

**Solution.**

We see that $T = \{(x, y) : 0 \leq x \leq 2,\ 0 \leq y \leq 3x\}$.

Let $f(x, y) = \frac{1}{2}(x^2 - 2y)$. Then

$$f_x = x \quad \text{and} \quad f_y = -1$$

So, the required surface area is

$$S = \int_T\int \sqrt{1 + f_x^2 + f_y^2}\, dA$$

$$= \int_0^2 \left( \int_0^{3x} \sqrt{2 + x^2}\, dy \right) dx$$

$$= \int_0^2 \left[ y\sqrt{2 + x^2} \right]_0^{3x} dx$$

$$= \int_0^2 \left(3x\sqrt{2+x^2}\right)\, dx \qquad (*)$$

Now use substitution on $(*)$ with $u = 2 + x^2$ to get

$$(*) = \frac{3}{2}\int_2^6 \sqrt{u}\, du$$

$$= \left[u^{\frac{3}{2}}\right]_2^6$$

$$= 6\sqrt{6} - 2\sqrt{2}$$

**Example 8.**

Find the surface area of the portion of the surface $z = 2x + 3y$ lying above the rectangular region $T$ given by $-1 \le x \le 2,\ 0 \le y \le 2$.

**Solution.**

Let $f(x, y) = 2x + 3y$. Then

$$f_x = 2 \quad \text{and} \quad f_y = 3$$

So, the required surface area is

$$S = \int_T\int \sqrt{1 + f_x^2 + f_y^2}\, dA$$

$$= \int_{-1}^2 \left(\int_0^2 \sqrt{1+4+9}\, dy\right) dx$$

$$= \int_{-1}^2 \left[y\sqrt{14}\right]_0^2\, dx$$

$$= \int_{-1}^2 \left(2\sqrt{14}\right)\, dx$$

$$= \left[2x\sqrt{14}\right]_{-1}^2\, dx$$

$$= 6\sqrt{14}$$

**Section 4.3 – Line Integrals.**

**Remark 9.**

We will first discuss what a piecewise smooth curve is. Suppose $C$ is a curve in the $xy$-plane given by the parametric equations $x = x(t), y = y(t)$ for $a \leq t \leq b$. An example of such a curve is the circle with centre $(0, 0)$ and radius 1 which has parametric equations $x = \cos t,\ y = \sin t$ for $0 \leq t \leq 2\pi$.

We now say that $C$ is smooth if

$$\frac{dx}{dt} \quad \text{and} \quad \frac{dy}{dt}$$

are continuous on $[a, b]$. Similarly, suppose $B$ is a curve in $\mathbb{R}^3$ given by the parametric equations $x = x(t), y = y(t), z = z(t)$ for $a \leq t \leq b$. We say that $B$ is smooth if

$$\frac{dx}{dt}, \frac{dy}{dt} \quad \text{and} \quad \frac{dz}{dt}$$

are continuous on $[a, b]$.

A curve $W$ is piecewise smooth if the interval $[a, b]$ can be subdivided into a finite number of subintervals, on each of which $W$ is smooth.

**Example 9.**

Consider the curve $C$ given by $x = \cos t,\ y = \sin t$, for $0 \leq t \leq 2\pi$. Then, $C$ is smooth. $C$ is the circle of radius 1 with centre at the origin (as mentioned above).

Consider the curve $W$ given by $x = \cos t,\ y = \sin t,\ z = t$ for $0 \leq t \leq 4\pi$. Then, $W$ is smooth. $W$ is a helix.