MT251P – Lecture 10

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Example 7.

Suppose $\underline{x} = (3, -2, 4, 6, 0)$ and $\underline{y} = (0, 1, 0, -2, 5)$ in $\mathbb{R}^5$.

(a) $\underline{x} + \underline{y} = (3, -1, 4, 4, 5)$

(b) $||\underline{x}|| = \sqrt{9 + 4 + 16 + 36} = \sqrt{65}$

(c) $\underline{x}.\underline{y} = -14$

Theorem 4.

Suppose $\underline{x} = (x_1, x_2, \ldots, x_n)$, $\underline{y} = (y_1, y_2, \ldots, y_n)$, $\underline{u} = (u_1, u_2, \ldots, u_n)$ and $\underline{w} = (w_1, w_2, \ldots, w_n) \in \mathbb{R}^n$. Then

(a) $\underline{y} - \underline{x}$ is the vector from $(x_1, x_2, \ldots, x_n)$ to $(y_1, y_2, \ldots, y_n)$. Also, the vector from $(x_1, x_2, \ldots, x_n)$ to $(y_1, y_2, \ldots, y_n)$ is parallel to the vector from $(u_1, u_2, \ldots, u_n)$ to $(w_1, w_2, \ldots, w_n)$ $\Longleftrightarrow$ $\underline{y} - \underline{x} = t(\underline{w} - \underline{u})$, for some $t \in \mathbb{R}$.

(b) The line L passing through $(x_1, x_2, \ldots, x_n)$ and $(y_1, y_2, \ldots, y_n)$ is given by

$$L = \{(x_1, x_2, \ldots, x_n) + t(y_1 - x_1, y_2 - x_2, \ldots, y_n - x_n) : t \in \mathbb{R}\}$$

L is also the line passing through $(x_1, x_2, \ldots, x_n)$ parallel to the vector $\underline{y} - \underline{x}$ and is called the parametric equation of L.

Furthermore the line M passing through $(x_1, x_2, \ldots, x_n)$ and parallel to the vector $(u_1, u_2, \ldots, u_n)$ is given by

$$M = \{(x_1, x_2, \ldots, x_n) + t(u_1, u_2, \ldots, u_n) : t \in \mathbb{R}\}$$

Example 8.

(a) Find the parametric equation of the line L passing through $(1, 2, -1)$ and $(2, 0, 2)$.

(b) Is $(3, -2, 5)$ on L? Is $(3, 0, 1)$ on L?

Solution.

(a) $L = \{(1, 2, -1) + t(1, -2, 3) : t \in \mathbb{R}\}$

(b) $(3, -2, 5)$ is on L because it corresponds to $t = 2$ in (a). $(3, 0, 1)$ is not on L because there is no $t \in \mathbb{R}$ satisfying

$$1 + t = 3, \quad 2 - 2t = 0, \quad -1 + 3t = 1 \qquad (*)$$

Definition 5.

The non–zero vectors $\underline{x}, \underline{y} \in \mathbb{R}^n$ are said to be perpendicular if the angle between them is $\frac{\pi}{2}$.

We also define the zero vector to be perpendicular to any vector.

Theorem 5.

Suppose $\underline{x}, \underline{y} \in \mathbb{R}^n$. Then

(a) $\underline{x}.\underline{y} = 0 \iff \underline{x}$ and $\underline{y}$ are perpendicular

Theorem 6 – Cauchy–Schwartz Inequality.

$|\underline{x}.\underline{y}| \leq ||\underline{x}|| ||\underline{y}||$, for all $\underline{x}, \underline{y} \in \mathbb{R}^n$ $\qquad (i)$

Also, equality holds in (i) above $\iff \underline{x} = t\underline{y}$, for some $t \in \mathbb{R}$ $\qquad (ii)$

Proof.

First we note that

$$(\underline{x} + s\underline{y}).(\underline{x} + s\underline{y}) \geq 0, \text{ for all } s \in \mathbb{R}$$

and so

$$\underline{x}.\underline{x} + s\underline{y}.\underline{x} + s\underline{x}.\underline{y} + s^2\underline{y}.\underline{y} \geq 0, \text{ for all } s \in \mathbb{R}$$

$$\Rightarrow ||\underline{x}||^2 + 2s\underline{x}.\underline{y} + s^2||\underline{y}||^2 \geq 0, \text{ for all } s \in \mathbb{R} \qquad (*)$$

This means that the left hand side of $(*)$ (as a quadratic equation in s) has at most one real root. So, by the quadratic formula we get

$$4(\underline{x}.\underline{y})^2 - 4||\underline{y}||^2||\underline{x}||^2 \leq 0$$

$$\Rightarrow (\underline{x}.\underline{y})^2 \leq ||\underline{x}||^2||\underline{y}||^2$$

$$\Rightarrow |\underline{x}.\underline{y}| \leq ||\underline{x}|| ||\underline{y}|| \qquad (**)$$

and so we have proved (i).

Now equality holds in $(**)$

$$\iff \text{ the quadratic in } s \text{ in } (*) \text{ has exactly one real root}$$

$$\iff (\underline{x} + q\underline{y}).(\underline{x} + q\underline{y}) = 0, \text{ for exactly one } q \in \mathbb{R}$$

$$\iff \underline{x} = -q\underline{y}$$

and so we have proved (ii) and we are done.

Remark 10.

The Cauchy–Schwartz inequality implies that for all non–zero $\underline{x}, \underline{y} \in \mathbb{R}^n$, we have

$$-1 \leq \frac{\underline{x}.\underline{y}}{||\underline{x}||||\underline{y}||} \leq 1$$

Hence, there is a unique θ such that

$$\cos\theta = \frac{\underline{x}.\underline{y}}{||\underline{x}||||\underline{y}||} \quad \text{and} \quad 0 \leq \theta \leq \pi \qquad (a)$$