Fiacre Ó Cairbre

**Lecture 2**

**Example 7.**

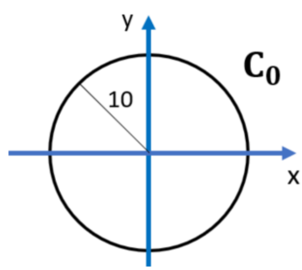
Suppose $f(x,y) = 100 - x^2 - y^2$. Plot the level curves

$$f(x,y) = 0, \quad f(x,y) = 91$$

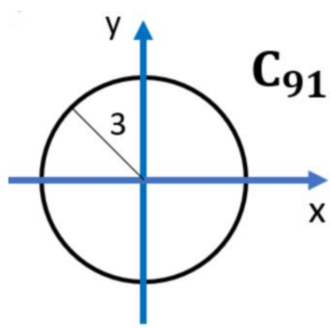
in the domain of $f$ in $\mathbb{R}^2$. Also, draw the graph of the function $f$.

**Solution.**

The level curve $f(x,y) = 0$ is the set $C_0 = \{(x,y) \in \mathbb{R}^2 : f(x,y) = 0\} = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = 100\}$ which is the circle with centre $(0,0)$ and radius 10.

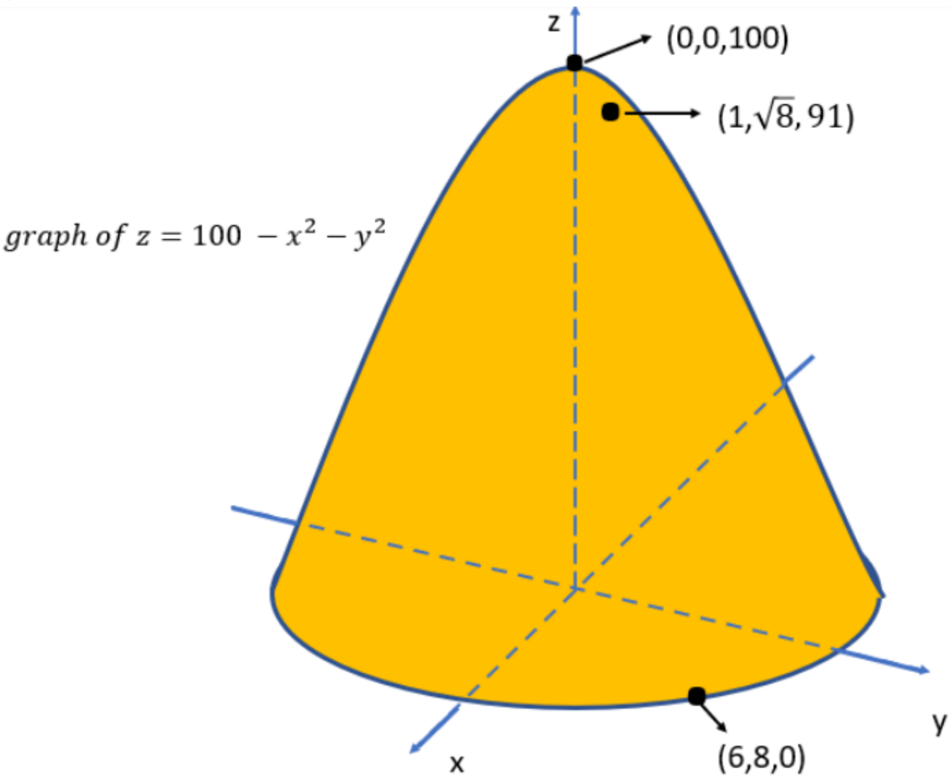



Similarly, the level curve $f(x,y) = 91$ is the set $C_{91} = \{(x,y) \in \mathbb{R}^2 : f(x,y) = 91\}$ which is the circle with centre $(0,0)$ and radius 3.




We will now draw the graph of $f$ in $\mathbb{R}^3$. Note that $f(0,0) = 100$ and so $(0,0,100)$ is on the graph. Note also that $(0,0,100)$ is the top point on the surface $z = f(x,y)$. Also, another example of a point on the graph of $f$ is $(6,8,0)$ because $0 = 100 - 6^2 - 8^2$.

Note that $(6,8)$ is on the level curve $C_0$ in $\mathbb{R}^2$ and $(6,8,0)$ is on the graph of $f$. Similarly, note that $(1,\sqrt{8})$ is on the level curve $C_{91}$ and $(1,\sqrt{8},91)$ is on the graph of $f$.




The domain of $f(x,y)$ is $\mathbb{R}^2$. Note that the level curve $f(x,y)=100$ is the set consisting of the single point $(0,0)$. The picture above shows the surface $z=100-x^2-y^2$ for $z\geq 0$. The surface continues on down for $z<0$. For example, when $z=-21$, we have $x^2+y^2=121$ and so $(1,\sqrt{120},-21)$ is a point on the surface $z=f(x,y)$. Also, the level curve $f(x,y)=-21$ is the set

$$C_{-21}=\{(x,y)\in\mathbb{R}^2 : f(x,y)=-21\}$$

$$=\{(x,y)\in\mathbb{R}^2 : x^2+y^2=121\}$$

which is the circle with centre $(0,0)$ and radius 11.

For each fixed $w\leq 100$, the level curve $C_w=\{(x,y)\in\mathbb{R}^2 : f(x,y)=w\}$ is the circle with centre at the origin $(0,0)$ and radius $\sqrt{100-w}$.

This explains why the surface looks the way it does – because as $w$ increases up through the $z$ values on the $z$–axis, the level curves $f(x,y)=w$ are circles with smaller and smaller radius. Note that the points on the graph of $f$ are on the surface in the picture and not inside. For example, $(0,0,0)$ is inside but not on the surface because $x=0,\ y=0,\ z=0$ doesn't satisfy the equation $z=100-x^2-y^2$. However, $(1,\sqrt{8},91)$ is on the surface (as in the picture) because $x=1,\ y=\sqrt{8},\ z=91$ satisfies the equation $z=100-x^2-y^2$.

**Example 8.**

Suppose $H = \{(x, y) \in \mathbb{R}^2 : y \geq 6\}$. Then $\text{Int}(H) = \{(x, y) \in \mathbb{R}^2 : y > 6\}$ and $\text{Bdy}(H) = \{(x, y) \in \mathbb{R}^2 : y = 6\}$

**Definition 6.**

(i) An open ball in $\mathbb{R}^3$ is a set of the form

$$S = \{(x, y, z) \in \mathbb{R}^3 : (x - a)^2 + (y - b)^2 + (z - c)^2 < t^2\}$$

which has centre $(a, b, c)$ in $\mathbb{R}^2$ and radius $t > 0$.

(ii) A closed ball in $\mathbb{R}^3$ is a set of the form

$$W = \{(x, y, z) \in \mathbb{R}^3 : (x - a)^2 + (y - b) + (z - c)^2 \leq t^2\}$$

which has centre $(a, b, c)$ in $\mathbb{R}^3$ and radius $t > 0$.

**Definition 7.**

(i) A point $(x, y, z)$ in a subset $T$ of $\mathbb{R}^3$ is called an interior point of $T$ if $(x, y, z)$ is the centre of an open ball that is a subset of $T$.

(ii) A point $(x, y, z)$ is a boundary point of a subset $W$ of $\mathbb{R}^3$ if every open ball with centre $(x, y, z)$ contains points that are not in $W$ and also contains points that are in $W$. Note that the boundary point $(x, y, z)$ itself need not be an element of $W$.

(iii) The interior of a subset $X$ of $\mathbb{R}^3$ is the set of all interior points of $X$. Denote the set of interior points of $X$ by $\text{Int}(X)$.

(iv) The boundary of a subset $L$ of $\mathbb{R}^3$ is the set of all boundary points of $L$. Denote the set of boundary points of $L$ by $\text{Bdy}(L)$.

(v) A subset $G$ of $\mathbb{R}^3$ is called open if and only if $\text{Int}(G) = G$.

(vi) A subset $Z$ of $\mathbb{R}^3$ is called closed if and only if $\text{Bdy}(Z)$ is a subset of $Z$.

**Example 9.**

(a) $A = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 < 1\}$ is open because $\text{Int}(A) = A$.

(b) $C = \{(x, y, z) \in \mathbb{R}^3 : z \leq 0\}$ is closed because $\text{Bdy}(C)$ is a subset of $C$.

**Definition 8.**

Suppose $f : \mathbb{R}^3 \to \mathbb{R}$ is a function and $w \in \mathbb{R}$. Then, the set

$$S_w = \{(x, y, z) \in \mathbb{R}^3 : f(x, y, z) = w\}$$

is called a level surface of $f$. The level surface $S_w$ above is also called the level surface $f(x, y, z) = w$. Notice that a level surface of $f$ is a subset of the domain of $f$.

**Definition 9.**

Suppose $f : \mathbb{R}^3 \to \mathbb{R}$ is a function. Then, the set

$$G = \{(x, y, z, q) \in \mathbb{R}^4 : (x, y, z) \text{ is in the domain of } f \text{ and } q = f(x, y, z)\}$$

is called the graph of $f$. Note that the graph of $f$ is a subset of $\mathbb{R}^4$.

**Example 10.**

Describe the level surfaces of $f(x, y, z) = \sqrt{x^2 + y^2 + z^2}$.

**Solution.**

Suppose $w \in \mathbb{R}$ and $w \geq 0$. Then, the level surface $f(x, y, z) = w$ is the set

$$S_w = \{(x, y, z) \in \mathbb{R}^3 : f(x, y, z) = w\}$$

$$= \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = w^2\}$$

which is the sphere with centre $(0, 0, 0)$ and radius $w$.