MT234P – MULTIVARIABLE CALCULUS – 2022

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Lecture 13

Remark 11.

Many important and powerful applications of mathematics to science, engineering and other areas involve finding absolute maxima and absolute minima of functions of more than one variable. The theory in our last few lectures can help to find such absolute maxima and absolute minima in certain situations. The next section will discuss another concept that can help find absolute maxima and absolute minima in certain situations.

Section 3.2 – Lagrange Multipliers.

Remark 12 – The method of Lagrange Multipliers.

Suppose $f(x,y)$ and $g(x,y)$ are differentiable functions. The candidates for the absolute maxima and absolute minima of $f(x,y)$ subject to the constraint $g(x,y) = 0$ are the points (a,b) which satisfy

$$\nabla f_{|(a,b)} = \lambda \nabla g_{|(a,b)} \quad \text{and} \quad g(a,b) = 0 \quad \text{for some} \ \ \lambda \in \mathbb{R}$$

Example 7.

Use Lagrange multipliers to find the greatest value and smallest value of $f(x,y) = xy$ on the ellipse $x^2 + 2y^2 = 1$. Also, find the points where f has this greatest value and this smallest value.

Solution.

We need to find the absolute maxima and absolute minima of $f(x,y) = xy$ subject to the constraint $g(x,y) = 0$, where $g(x,y) = x^2 + 2y^2 - 1$.

So, by remark 12, in order to find candidates (a,b) for the absolute maxima and absolute minima of xy subject to the constraint $x^2 + 2y^2 - 1 = 0$, we look for points (a,b) and real numbers λ that satisfy

$$\nabla f_{|(a,b)} = \lambda \nabla g_{|(a,b)} \quad \text{and} \quad g(a,b) = 0 \qquad\qquad (*)$$

Now, $(*)$ gives us

$$(y\vec{i} + x\vec{j})_{|(a,b)} = \lambda(2x\vec{i} + 4y\vec{j})_{|(a,b)} \quad \text{and} \quad a^2 + by^2 - 1 = 0$$

$$\Rightarrow b = 2\lambda a, \qquad a = 4\lambda b, \qquad a^2 + 2b^2 - 1 = 0$$

$$\Rightarrow b = 8\lambda^2 b$$

$$\Rightarrow b(1-8\lambda^2) = 0$$

$$\Rightarrow b = 0 \quad \text{or} \quad \lambda = \pm\frac{1}{\sqrt{8}}$$

We now look at the two cases separately:

CASE 1: Suppose $b = 0$. Then $a = 4\lambda b = 0$ which is impossible because $a^2 + 2b^2 = 1$.

CASE 2: Suppose $\lambda = \pm\frac{1}{\sqrt{8}}$. Then $b = 2\lambda a$ implies that

$$b = \pm\frac{a}{\sqrt{2}} \qquad (**)$$

Now, $a^2 + 2b^2 = 1$ and so $(**)$ implies that $a^2 + a^2 = 1$ which gives $a = \pm\frac{1}{\sqrt{2}}$.

Now, using $(**)$ we get that the candidates for the required absolute maxima and absolute minima are

$$(\frac{1}{\sqrt{2}}, \frac{1}{2}), \quad (\frac{1}{\sqrt{2}}, -\frac{1}{2}), \quad (-\frac{1}{\sqrt{2}}, \frac{1}{2}), \quad (-\frac{1}{\sqrt{2}}, -\frac{1}{2})$$

We evaluate f at these candidates to get

$$f(\frac{1}{\sqrt{2}}, \frac{1}{2}) = \frac{1}{\sqrt{8}}, \quad f(\frac{1}{\sqrt{2}}, -\frac{1}{2}) = -\frac{1}{\sqrt{8}}, \quad f(-\frac{1}{\sqrt{2}}, \frac{1}{2}) = -\frac{1}{\sqrt{8}}, \quad f(-\frac{1}{\sqrt{2}}, -\frac{1}{2}) = \frac{1}{\sqrt{8}}$$

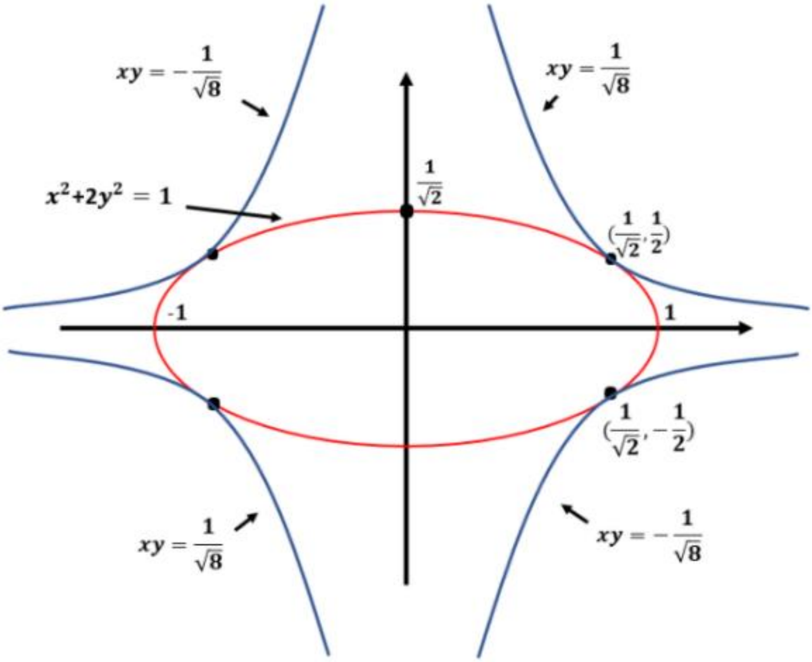
So, the greatest value of xy on the ellipse $x^2 + 2y^2 = 1$ is $\frac{1}{\sqrt{8}}$ and the points where f has this greatest value are

$$(\frac{1}{\sqrt{2}}, \frac{1}{2}), \quad (-\frac{1}{\sqrt{2}}, -\frac{1}{2})$$

Similarly, the smallest value of xy on the ellipse $x^2 + 2y^2 = 1$ is $-\frac{1}{\sqrt{8}}$ and the points where f has this smallest value are

$$(\frac{1}{\sqrt{2}}, -\frac{1}{2}), \quad (-\frac{1}{\sqrt{2}}, \frac{1}{2})$$

We have answered the question fully. We will now do something extra and discuss some geometry related to our work above. In the picture below, we see the ellipse $x^2 + 2y^2 = 1$ and some level curves of $f(x, y) = xy$. Note that the level curves of xy are the hyperbolas $xy = c$, for some $c \in \mathbb{R}$.



Notice that we were looking for the greatest value of xy and the smallest value of xy subject to the constraint that (x, y) also lies on the ellipse $x^2 + 2y^2 = 1$. Now, note that the farther the level curves $xy = c$ are away from the origin $(0,0)$, then the larger the absolute value of xy is. So, geometrically, we can say that we were looking for the level curves (i.e. hyperbolas $xy = c$) that intersect the ellipse above and also lie the farthest away from the origin.

So, which hyperbolas $xy = c$ (that intersect the ellipse above) will lie the farthest away from the origin? We see that the hyperbolas that just graze the ellipse are the ones that will achieve this condition of intersecting the ellipse above and being the farthest away from the origin. These hyperbolas will be $xy = \frac{1}{\sqrt{8}}$ and $xy = -\frac{1}{\sqrt{8}}$ and furthermore the points of intersection between these hyperbolas and the ellipse will be

$$(\frac{1}{\sqrt{2}}, \frac{1}{2}), \quad (-\frac{1}{\sqrt{2}}, -\frac{1}{2}), \quad (\frac{1}{\sqrt{2}}, -\frac{1}{2}), \quad (-\frac{1}{\sqrt{2}}, \frac{1}{2})$$

This all corresponds to the two facts (proved earlier)

(i) that the greatest value of xy on the ellipse $x^2 + 2y^2 = 1$ is $\frac{1}{\sqrt{8}}$ and the points where f has this greatest value are

$$(\frac{1}{\sqrt{2}}, \frac{1}{2}), \quad (-\frac{1}{\sqrt{2}}, -\frac{1}{2})$$

(ii) that the smallest value of xy on the ellipse $x^2 + 2y^2 = 1$ is $-\frac{1}{\sqrt{8}}$ and the points where f has this smallest value are

$$(\frac{1}{\sqrt{2}}, -\frac{1}{2}), \quad (-\frac{1}{\sqrt{2}}, \frac{1}{2})$$