MT251P – Lecture 19

Chapter 6 – Subspaces, Dimension and Rank.

Section 6.1 – Linear Independence and Basis.

Definition 1.

The vectors $\underline{u_1}, \underline{u_2}, \ldots, \underline{u_k} \in \mathbb{R}^n$ are called linearly independent if the only $\alpha_i \in \mathbb{R}$ for $1 \leq i \leq k$ that satisfy

$$\alpha_1 \underline{u_1} + \alpha_2 \underline{u_2} + \cdots + \alpha_k \underline{u_k} = \underline{0}$$

are $\alpha_1 = \alpha_2 = \cdots = \alpha_k = 0$

Definition 2.

If the vectors $\underline{u_1}, \underline{u_2}, \ldots, \underline{u_k} \in \mathbb{R}^n$ are not linearly independent, then we call them linearly dependent.

Example 1.

(a) $(1,0,0), (0,2,0), (0,0,-1)$ are linearly independent in $\mathbb{R}^3$ since

$$\alpha_1(1,0,0) + \alpha_2(0,2,0) + \alpha_3(0,0,-1) = (0,0,0)$$

$$\iff (\alpha_1, 2\alpha_2, -\alpha_3) = (0,0,0)$$

$$\iff \alpha_1 = \alpha_2 = \alpha_3 = 0$$

(b) $(2,0,0), (0,2,0), (2,4,0)$ are linearly dependent in $\mathbb{R}^3$ since

$$(2,0,0) + 2(0,2,0) + (-1)(2,4,0) = (0,0,0)$$

Example 2.

Are $(0,2,1), (-3,1,1), (-6,6,4)$ linearly independent in $\mathbb{R}^3$?

Solution.

$$\alpha_1(0,2,1) + \alpha_2(-3,1,1) + \alpha_3(-6,6,4) = (0,0,0)$$

$$\iff -3\alpha_2 - 6\alpha_3 = 0, \quad 2\alpha_1 + \alpha_2 + 6\alpha_3 = 0, \quad \alpha_1 + \alpha_2 + 4\alpha_3 = 0$$

This is a system of three linear equations in $\alpha_1, \alpha_2, \alpha_3$. One can show there are infinitely many solutions for $\alpha_1, \alpha_2, \alpha_3$ and so $(0,2,1), (-3,1,1), (-6,6,4)$ are linearly dependent.

Definition 3.

A non–empty subset S of $\mathbb{R}^n$ is called a subspace of $\mathbb{R}^n$ if the following two conditions are satisfied:

(i) $\underline{x} + \underline{y} \in S, \quad \text{for all} \quad \underline{x}, \underline{y} \in S$

(ii) $\alpha \underline{x} \in S, \quad \text{for all} \quad \alpha \in \mathbb{R}, \quad \underline{x} \in S$

Example 3.

Consider $S = \{(0, t) : t \text{ is a prime number}\}$ as a subset of $\mathbb{R}^2$. Is S a subspace of $\mathbb{R}^2$?

Solution.

No, because $(0, 3), (0, 5) \in S$ but $(0, 3) + (0, 5) = (0, 8) \notin S$.

Example 4.

Consider $S = \{(t, 0, 0) : t \text{ is an even integer}\}$ as a subset of $\mathbb{R}^3$. Is S a subspace of $\mathbb{R}^3$?

Solution.

No, because $\frac{1}{2}(2, 0, 0) \notin S$.

Example 5.

Consider $S = \{(0, 0, 0, 0)\}$ as a subset of $\mathbb{R}^4$. Is S a subspace of $\mathbb{R}^4$?

Solution.

Yes, because the two conditions in definition 3 are satisfied.

Definition 4.

The vectors $\underline{u_1}, \underline{u_2}, \ldots, \underline{u_k}$ span (or generate) a subspace S of $\mathbb{R}^n$ if every $\underline{x} \in S$ can be written in the form:

$$\underline{x} = \alpha_1 \underline{u_1} + \alpha_2 \underline{u_2} + \cdots + \alpha_k \underline{u_k}, \quad \text{for some} \quad \alpha_i \in \mathbb{R}, \quad 1 \leq i \leq k$$

In this case we call $\{\underline{u_1}, \underline{u_2}, \ldots, \underline{u_k}\}$ a spanning set of S.

Definition 5.

Suppose $\underline{u_1}, \underline{u_2}, \ldots, \underline{u_k}$ are vectors in $\mathbb{R}^n$. A linear combination of $\underline{u_1}, \underline{u_2}, \ldots, \underline{u_k}$ is an expression of the form:

$$\alpha_1 \underline{u_1} + \alpha_2 \underline{u_2} + \cdots + \alpha_k \underline{u_k}, \quad \text{where} \quad \alpha_i \in \mathbb{R}, \quad 1 \leq i \leq k$$