MT251P – Lecture 16

Definition 13.

The identity $n \times n$ matrix is denoted by I_n and is the $n \times n$ matrix with all the entries along the main diagonal equal to one and all other entries equal to zero.

Example 15.

$$I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Remark 7.

If A is any $k \times n$ matrix, then $I_k A = A$ and $AI_n = A$.

Definition 14.

(i) A symmetric matrix B is a square matrix such that $B^T = B$.

(ii) A diagonal matrix B is a square matrix such that all the entries not on the main diagonal are zero.

(iii) A lower triangular matrix is a square matrix such that all the entries above the main diagonal are zero. An upper triangular matrix is a square matrix such that all the entries below the main diagonal are zero.

Example 16.

$$\begin{pmatrix} 1 & 4 & 7 \\ 4 & 2 & -1 \\ 7 & -1 & 5 \end{pmatrix} \quad \text{is symmetric}$$

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 3 \end{pmatrix} \quad \text{is a diagonal matrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 3 & 2 & 0 \\ -1 & 3 & 4 \end{pmatrix} \quad \text{is lower triangular}$$

Chapter 5 – The Inverse and Determinant of a Matrix.

Remark 1.

Let $M_{k\times n}(\mathbb{R})$ denote the set of $k \times n$ matrices with real entries. So for example

$$\begin{pmatrix} 0 & -1 & 4 \\ 2 & 3 & 1 \end{pmatrix} \in M_{2\times 3}(\mathbb{R})$$

Definition 1.

Suppose $A \in M_{n\times n}(\mathbb{R})$. Then, we say that A is invertible if there is a matrix $B \in M_{n\times n}(\mathbb{R})$ such that

$$AB = I_n = BA$$

In this case B is called an inverse of A.

Theorem 1.

Suppose $A \in M_{n\times n}(\mathbb{R})$ is invertible. Then,

(i) If B and C are both inverses of A, then $B = C$, i.e. the inverse of A is unique. Hence, we call B the inverse of A and we denote it by A^{-1}.

(ii) A^{-1} is invertible and $(A^{-1})^{-1} = A$

(iii) Suppose $W \in M_{n\times n}(\mathbb{R})$ is invertible. Then, AW is invertible and $(AW)^{-1} = W^{-1}A^{-1}$.

Remark 2.

Here is a proof of Theorem 1(i):

First note that $AB = AC = I_n$. So, we have

$$AB = AC \Rightarrow B(AB) = B(AC)$$

$$\Rightarrow (BA)B = (BA)C$$

$$\Rightarrow I_nB = I_nC$$

$$\Rightarrow B = C$$

Theorem 2.

Suppose $A \in M_{2\times 2}(\mathbb{R})$ with

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Then, A is invertible $\iff ad - bc \neq 0$

Also, if $ad - bc \neq 0$, then

$$A^{-1} = \frac{1}{ad - bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

Example 1.

Suppose

$$A = \begin{pmatrix} 2 & -1 \\ 0 & 4 \end{pmatrix}$$

Then, A is invertible and

$$A^{-1} = \frac{1}{8}\begin{pmatrix} 4 & 1 \\ 0 & 2 \end{pmatrix}$$

Definition 2.

We know from Theorem 1 in chapter 4 that any matrix A is row equivalent to a unique matrix B which is in RREF. We call this matrix B the RREF of A.

Theorem 3.

An $n \times n$ matrix A is invertible $\iff$ the RREF of $A = I_n$.

Also, in this case where A is invertible, the collection of elementary row operations that take you from A to I_n will also take you from I_n to A^{-1}.

Example 2.

Suppose $A = \begin{pmatrix} 3 & 0 & 2 \\ 2 & 0 & -2 \\ 0 & 1 & 1 \end{pmatrix}$

Does A^{-1} exist? If it does, then find A^{-1}.

Solution.

$$A = \begin{pmatrix} 3 & 0 & 2 \\ 2 & 0 & -2 \\ 0 & 1 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I_3$$

Interchange R_2 with R_1 (in both matrices A and I_3) to get

$$\begin{pmatrix} 2 & 0 & -2 \\ 3 & 0 & 2 \\ 0 & 1 & 1 \end{pmatrix} \quad \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Replace R_1 with $\frac{1}{2}R_1$ (in both matrices) to get

$$\begin{pmatrix} 1 & 0 & -1 \\ 3 & 0 & 2 \\ 0 & 1 & 1 \end{pmatrix} \quad \begin{pmatrix} 0 & \frac{1}{2} & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$