8. Set Theory

Definition 8.1. *(1) A* set *is a collection of distinct objects. The objects are called* elements. *If A is a set and x is an element in A, then we write $x \in A$. If x does not belong to A, then we write $x \notin A$.*

(2) Let A and B be sets. We say B is a subset *of A, or B is* contained *in A, and write $B \subseteq A$, if for all $x \in B$ we have $x \in A$. If $B \subseteq A$ and there is some $x \in A$ such that $x \notin B$, then we say B is a* proper subset *of A and write $B \subsetneq A$.*

(3) Two sets A and B are called equal *and we write $A = B$, if they have the same elements, or in other words, $A \subseteq B$ and $B \subseteq A$.*

(4) The set that contains no element is called empty set. *We denoted it by $\emptyset$.*

Remark 8.2. *We can describe a set using set notation. For instance we can list the elements in the set as in $A = \{1, 2, 3\}$ or $\mathbb{N} = \{1, 2, 3, \ldots\}$. Or we may describe a set of elements that satisfy a certain property P, e.g. $\{x \in \mathbb{N} : x \text{ satisfies } P\}$.*

Example 8.3. *(1) $\mathbb{N} = \{1, 2, 3, \ldots\}$ denotes the set of natural numbers. Then $10 \in \mathbb{N}$ and $1/2 \notin \mathbb{N}$.*
(2) $\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$ denotes the set of integers.
(3) $\mathbb{Q} = \{\frac{a}{b} : a, b \in \mathbb{Z}, b \neq 0\}$ denotes the set of rational numbers.
(4) $\mathbb{N} \subsetneq \mathbb{Z} \subsetneq \mathbb{Q}$
(5) $\{x \in \mathbb{N} : x < 5\} = \{1, 2, 3, 4\}$
(6) $\{x \in \mathbb{Z} : x^2 + x = 2\} = \{1, -2\}$
(7) $\{3n : \ n \in \mathbb{N}\}$ describes the set of natural numbers that are multiples of 3.
(8) Let A be a set. Then $\emptyset \subseteq A$ and $A \subseteq A$.

Definition 8.4. *Let A and B be sets.*
(1) The union *of A and B is the set*

$$A \cup B := \{x : \ x \in A \text{ or } x \in B\}$$

(2) The intersection *of A and B is the set*

$$A \cap B := \{x : \ x \in A \text{ and } x \in B\}$$

(3) The complement *of B relative to A is the set*

$$A \setminus B := \{x : \ x \in A \text{ and } x \notin B\}$$

Example 8.5. *(1)* $\{1,2,3\} \cup \{3,4,5\} = \{1,2,3,4,5\}$
(2) $\{1,2,3\} \cap \{3,4,5\} = \{3\}$
(3) $\{1,2,3\} \setminus \{3,4,5\} = \{1,2\}$
(4) $\mathbb{N} \cup \{-n : n \in \mathbb{N}\} = \mathbb{Z} \setminus \{0\}$
(5) $\{2n : \ n \in \mathbb{N}\} \cap \{3n : \ n \in \mathbb{N}\} = \{6n : \ n \in \mathbb{N}\}$

Definition 8.6. *If the intersection of two sets A and B is empty, then A and B are called* disjoint*.*

Example 8.7. *(1) The alphabet and $\mathbb{Z}$ are disjoint sets.*
(2) The even and the odd integers are disjoint, that is, $\{2n : \ n \in \mathbb{Z}\} \cap \{2n+1 : n \in \mathbb{Z}\} = \emptyset$.

Definition 8.8. *Let I be a set, and for each $i \in I$, let A_i be a set. Then*
(1) $\displaystyle\bigcup_{i \in I} A_i := \{x : \ x \in A_i, \text{ for some } i \in I\}$.
(2) $\displaystyle\bigcap_{i \in I} A_i := \{x : \ x \in A_i, \text{ for all } i \in I\}$.

The set I is called index set*.*

Example 8.9. *(1) Let $A_n := \{0, \ldots, n\}$, for all $n \in \mathbb{N}$. We claim that $\displaystyle\bigcup_{n \in \mathbb{N}} A_n = \mathbb{N}$. Clearly $A_n \subseteq \mathbb{N}$, for all $n \in \mathbb{N}$ and so $\displaystyle\bigcup_{n \in \mathbb{N}} A_n \subseteq \mathbb{N}$. On the other hand, if $x \in \mathbb{N}$, then $x \in A_x$, and so $\mathbb{N} \subseteq \displaystyle\bigcup_{n \in \mathbb{N}} A_n$. This proves the claim.*
(2) Let $A_n := \{k \in \mathbb{N} : k \geq n\}$, for all $n \in \mathbb{N}$. We claim that $\displaystyle\bigcap_{n \in \mathbb{N}} A_n = \emptyset$. Assume $\displaystyle\bigcap_{n \in \mathbb{N}} A_n \neq \emptyset$ and let $x \in \displaystyle\bigcap_{n \in \mathbb{N}} A_n$. Then $x \in A_n$, for all $n \in \mathbb{N}$. Since $A_0 = \mathbb{N}$, it follows that $x \in \mathbb{N}$. But then $x \notin A_{x+1}$, which is a contradiction, and therefore $\displaystyle\bigcap_{n \in \mathbb{N}} A_n = \emptyset$.

Definition 8.10. *Let A be a set. Then*
$$\mathcal{P}(A) := \{X : \ X \subseteq A\},$$
the set of all subsets of A, is called power set *of A.*

Example 8.11. *Let $A = \{0,1,2\}$. Then*
$$\mathcal{P}(A) = \{\emptyset, \{0\}, \{1\}, \{2\}, \{0,1\}, \{0,2\}, \{1,2\}, A\}.$$