3 Limits and Continuity

3.1 Introduction to Limits of Functions

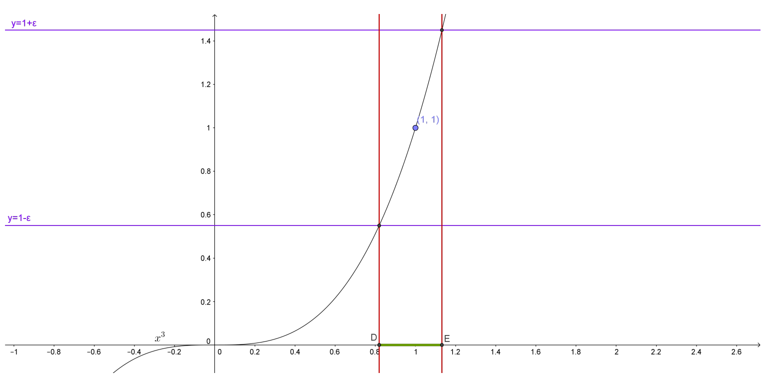
In this section we will consider the definition of the limit of a function. That is what does it mean to say that $\lim_{x \to c} f(x) = L$?

Intuitively we would say that $\lim_{x \to c} f(x) = L$ if we can make $f(x)$ as close as we like to L for all values of x sufficiently close to c. From our experience with limits of sequences we know that we need to be able to make the distance between $f(x)$ and L arbitrarily small for all x sufficiently close to c. We know that the distance between $f(x)$ and L is $|f(x) - L|$ and saying that we can make this arbitrarily small means that we can make it smalled than any small positive number ie $\lim_{x \to c} f(x) = L$ if given any $\epsilon > 0$ we have $|f(x) - L| < \epsilon$ for all x sufficiently close to c. Now the distance between x and c is just $|x - c|$, so 'x sufficiently close to c' means $0 < |x - c| < \delta$ for some small δ. Thus we get the following definition:

Definition 3.1. We say that $\lim_{x \to c} f(x) = L$ if for all $\epsilon > 0$ there exists $\delta > 0$ such that $|f(x) - L| < \epsilon$ whenever $0 < |x - c| < \delta$.

Note: δ will usually depend on ϵ in a similar way to the dependence of N on ϵ in the limit of sequences definition. Note also that in the definition we have $0 < |x - c| < \delta$ so what happens to $f(x)$ at $x = c$ is not relevant. In the limit definition, we are interested in what happens to f close to c but not at c itself.

Remark In the picture below, we consider the function $f(x) = x^3$ and we would like to prove that $\lim_{x \to 1} f(x) = 1$. We need to show that given $\epsilon > 0$ we can find $\delta > 0$ such that $|x^3 - 1| < \epsilon$ if $0 < |x - 1| < \delta$. From the graph we can see that $1 - \epsilon < f(x) < 1 + \epsilon$ if $x \in [D, E]$. So we need to find δ so that $(1 - \delta, 1 + \delta) \subset [D, E]$. That looks possible from our picture. Can you prove it? Watch the video on the definition of limits of functions and try the Geogebra applet for some visualisations of this definition in action.



Example 3.2. Show that $\lim_{x \to 1} 2x + 1 = 3$.

Solution: Here $f(x) = 2x + 1$, $L = 3$, and $c = 1$. So we need to show that for all $\epsilon > 0$ there exists a δ such that $|2x + 1 - 3| < \epsilon$ whenever $0 < |x - 1| < \delta$, that is $|2x - 2| < \epsilon$ whenever $0 < |x - 1| < \delta$.

Let $\epsilon > 0$ be given. Now $|2x-2| < \epsilon$ iff $2|x-1| < \epsilon$ or $|x-1| < \frac{\epsilon}{2}$. So we need to find δ so that $|x-1| < \frac{\epsilon}{2}$ whenever $0 < |x-1| < \delta$. Let's let $\delta = \frac{\epsilon}{2}$.
So if $0 < |x-1| < \delta = \frac{\epsilon}{2}$, then $2|x-1| < \epsilon$ so $|2x+1-3| < \epsilon$, as required. Thus we have shown that $\lim_{x\to 1} 2x+1 = 3$.
Remark: Notice that we are using the same kinds of arguments here as we did in the chapter on limits of sequences. We start the proof with 'Let $\epsilon > 0$ be given', then we demonstrate a choice of δ (this might involve some roughwork), and finally show that our choice of δ works.

Exercise 3.3. *Your turn:* Use the $\epsilon - \delta$ definition to show that

1. $\lim_{x\to 2} 2x+1 = 5$.
2. $\lim_{x\to 0} x^2 = 0$.
3. $\lim_{x\to 1} x^3 = 1$. [Note this is a bit trickier than the examples we have seen so far]

Example 3.4. Let $f(x) = \dfrac{x^2-9}{x-3}$. Show that $\lim_{x\to 3} f(x) = 6$.

Solution: Note that

$$f(x) = \begin{cases} x+3 & \text{if } x \neq 3 \\ \text{undefined} & \text{if } x = 3 \end{cases}$$

We need to show that for all $\epsilon > 0$ there exists a $\delta > 0$ such that

$$|f(x) - 6| < \epsilon \text{ if } 0 < |x-3| < \delta.$$

Let $\epsilon > 0$ be given. Notice that $0 < |x-3|$ means that $x \neq 3$. So if $0 < |x-3| < \delta$ we need to guarantee that $|f(x) - 6| = |x+3-6| = |x-3| < \epsilon$. Let's choose $\delta = \epsilon$ then if $0 < |x-3| < \delta$ we have $0 < |x-3| < \epsilon$ and so $|f(x) - 6| = |x-3| < \epsilon$. Thus $\lim_{x\to 3} f(x) = 6$.
Note that it was crucial here that we did not need to consider what happened at $x = 3$ itself.
Let's look at a related example:

Example 3.5. Let $g(x) = \begin{cases} x+3 & \text{if } x \neq 3 \\ 1 & \text{if } x = 3 \end{cases}$. Show that $\lim_{x\to 3} g(x) = 6$.

Solution: Here we need to show that for all $\epsilon > 0$ there exists $\delta > 0$ such that $|g(x)-6| < \epsilon$ if $0 < |x-3| < \delta$.
Let $\epsilon > 0$ be given.
As in the example above we have $|g(x) - 6| = |x+3-6| = |x-3|$ if $x \neq 3$. If we let $\delta = \epsilon$ we get that if

$$0 < |x-3| < \delta \text{ then } |g(x) - 6| = |x-3| < \delta = \epsilon$$

as required. So $\lim_{x\to 3} g(x) = 6$.
Note that it was important here again that we did not need to consider what happened at $x = 3$ itself. Also $\lim_{x\to 3} g(x) \neq g(3)$. Draw a picture to illustrate what is happening in both of these examples or look at the related videos.

Example 3.6. Prove that $\lim_{x \to c} x = c$.

Solution: This might seem obvious but we still need to check that we can prove it from the definition. Here $f(x) = x$ and $L = c$.

Let $\epsilon > 0$ be given. We need to show that we can find a $\delta > 0$ such that $|f(x) - L| = |x - c| < \epsilon$ whenever $0 < |x - c| < \delta$.

Let's choose $\delta = \epsilon$. Then if $0 < |x - c| < \delta$, we have $|x - c| < \delta = \epsilon$ as required. Thus $\lim_{x \to c} x = c$.

Example 3.7. Prove that $\lim_{x \to 2} x^2 = 4$.

Solution: Here $f(x) = x^2$, $L = 4$, $c = 2$. So for all $\epsilon > 0$ we must find a $\delta > 0$ such that $|x^2 - 4| < \epsilon$ if $0 < |x - 2| < \delta$.

Let's look at $|x^2 - 4| < \epsilon$. We can write $x^2 - 4 = (x - 2)(x + 2)$ so $|x^2 - 4| < \epsilon$ iff $|(x - 2)(x + 2)| < \epsilon$.

Let's estimate $|x + 2|$: Suppose we chose $\delta < 1$, so if $|x - 2| < \delta$ we have $|x - 2| < 1$ or $-1 < x - 2 < 1$ so $3 < x + 2 < 5$. Thus if $\delta < 1$ then $|x + 2| < 5$. Now

$$\begin{aligned} |x^2 - 4| &= |x - 2| \times |x + 2| \\ &< 5|x - 2| \qquad \text{if } \ |x - 2| < 1 \\ &< \epsilon \qquad\qquad\ \ \text{if } \ |x - 2| < \frac{\epsilon}{5} \end{aligned}$$

So let's choose $\delta = Min\{1, \frac{\epsilon}{5}\}$. Then if $|x - 2| < \delta$ we have

$$\begin{aligned} |x^2 - 4| &= |x - 2| \times |x + 2| \\ &< 5|x - 2| \qquad \text{since } \ |x - 2| < \delta \le 1 \\ &< \epsilon \qquad\qquad\ \ \text{since } \ |x - 2| < \delta \le \frac{\epsilon}{5} \end{aligned}$$

Thus $\lim_{x \to 2} x^2 = 4$.

Exercise 3.8. *Your turn:*

1. Prove that $\lim_{x \to 0} x^2 = 0$.
2. Prove that $\lim_{x \to c} x^2 = c^2$.

Remark: Exercise 2 above would be easy if we could say

$$\lim_{x \to c} x^2 = \lim_{x \to c} (x \times x) = \lim_{x \to c} x \times \lim_{x \to c} x = c^2$$

however we have not yet proved that $\lim_{x \to c} (f(x) \times g(x)) = \lim_{x \to c} f(x) \times \lim_{x \to c} g(x)$.

Recall that Theorem 2.15 gave us a similar result for sequences. The results in the next section will help us to 'convert' results about limits of sequences to results about limits of functions.

3.2 Limit Theorems

Theorem 3.9. *Given $f : \mathbb{R} \to \mathbb{R}$ and $c \in \mathbb{R}$ the following statements are equivalent:*

1. $\lim_{x\to c} f(x) = L$
2. *For all sequences $\{x_n\}$ in $\mathbb{R}$ satisfying $x_n \neq c$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} x_n = c$ we have*

$$\lim_{n\to\infty} f(x_n) = L$$

Proof: Let's begin by showing that *1.* $\Rightarrow$ *2.*
Assume that $\lim_{x\to c} f(x) = L$ and let $\{x_n\}$ be a sequence in $\mathbb{R}$ satisfying $x_n \neq c$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} x_n = c$. We must show that $\lim_{n\to\infty} f(x_n) = L$.
Let $\epsilon > 0$ be given. Then since $\lim_{x\to c} f(x) = L$ there exists $\delta > 0$ such that

$$|f(x) - L| < \epsilon \qquad \text{if } 0 < |x - c| < \delta. \quad (*)$$

We also know that $\lim_{n\to\infty} x_n = c$ so for this δ there exists $N \in \mathbb{N}$ such that

$$|x_n - c| < \delta \qquad \text{if } n \geq N.$$

Therefore given $\epsilon > 0$ we have found $N \in \mathbb{N}$ such that if $n \geq N$ we have $|x_n - c| < \delta$ and thus $|f(x_n) - L| < \epsilon$ (by $(*)$ above). This means that $\lim_{n\to\infty} f(x_n) = L$. Thus *1.* $\Rightarrow$ *2.*

Now let's show that *2.* $\Rightarrow$ *1.*
Assume that for all sequences $\{x_n\}$ in $\mathbb{R}$ satisfying $x_n \neq c$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} x_n = c$ we have $\lim_{n\to\infty} f(x_n) = L$.
Let's suppose that *1.* is not true and (hopefully!) find a contradiction.
If *1.* is not true then $\lim_{x\to c} f(x) \neq L$ so there exists some $\epsilon_1 > 0$ such that for all $\delta > 0$ $|f(x) - L| > \epsilon_1$ for some x such that $0 < |x - c| < \delta$.
Let $\delta_n = \frac{1}{n}$ and for each $n \in \mathbb{N}$ choose a real number x_n such that $0 < |x_n - c| < \delta_n$ and $|f(x_n) - L| > \epsilon_1$. (Note by assumption, there is at least one such x_n for each n.)
Clearly $\lim_{n\to\infty} x_n = c$ (since $|x_n - c| < \frac{1}{n}$) and since $0 < |x_n - c|$ we have that $x_c \neq c$ for all $n \in \mathbb{N}$.
However $\lim_{n\to\infty} f(x_n) \neq L$ since we have $|f(x_n) - L| > \epsilon_1$ for all $n \in \mathbb{N}$. This is a contradiction of *2.*, therefore our assumption that $\lim_{x\to c} f(x) \neq L$ is wrong. Thus *2.* $\Rightarrow$ *1.*

Remark: Stop here and think how you might use this theorem!

If $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$, let's define the functions $(f+g)(x) = f(x) + g(x)$, $f.g(x) = f(x) \times g(x)$ and $\frac{f}{g}(x) = \frac{f(x)}{g(x)}$.

Theorem 3.10. *Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ and $c \in \mathbb{R}$. Suppose that $\lim_{x\to c} f(x) = L$ and $\lim_{x\to c} g(x) = M$, then:*

1. $\lim_{x\to c} kf(x) = kL$ *for all* $k \in \mathbb{R}$.
2. $\lim_{x\to c} (f+g)(x) = L + M$.
3. $\lim_{x\to c} f.g(x) = LM$.
4. $\lim_{x\to c} \dfrac{f}{g}(x) = \dfrac{L}{M}$ *provided that* $M \neq 0$.

Proof: The proof follows from Theorem 2.15 and Theorem 3.9. We will prove *3.* here. You should do the other parts yourselves.
Proof of *3.*: We need to show that if $\lim_{x\to c} f(x) = L$ and $\lim_{x\to c} g(x) = M$, then $\lim_{x\to c} f.g(x) = LM$. Let $\{x_n\}$ be a sequence in $\mathbb{R}$ such that $\lim_{n\to\infty} x_n = c$ and $x_n \neq c$ for all $n \in \mathbb{N}$. Then

$$\begin{aligned}
\lim_{x\to c} f.g(x) &= \lim_{n\to\infty} f.g(x_n) \qquad \text{(by Thm 3.9)} \\
&= \lim_{n\to\infty} [f(x_n) \times g(x_n)] \\
&= \lim_{n\to\infty} f(x_n) \times \lim_{n\to\infty} g(x_n) \qquad \text{(by Thm 2.15)} \\
&= L \times M \qquad \text{(by Thm 3.9)}
\end{aligned}$$

Example 3.11. We can use Theorem 3.9 to prove the following:

1. If $p(x)$ is a polynomial then $\lim_{x\to c} p(x) = p(c)$.

   Since we know $\lim_{x\to c} x = c$ we can use part 3 of the theorem to get $\lim_{x\to c} x^n = c^n$ for all $n \in \mathbb{N}$. Part one of the theorem tell us that $\lim_{x\to c} kx^n = kc^n$ for all $k \in \mathbb{R}$. If $p(x)$ is a polynomial then $p(x) = a_0 + a_1x + a_2x^2 + ...a_nx^n$ so we can use part 2 of the theorem to get

   $$\begin{aligned}
   \lim_{x\to c} p(x) &= \lim_{x\to c} a_0 + \lim_{x\to c} a_1x + \lim_{x\to c} a_2x^2 + ... \lim_{x\to c} a_nx^n \\
   &= a_0 + a_1c + a_2c^2 + ...a_nc^n \\
   &= p(c)
   \end{aligned}$$

2. If $c \neq 0$ then $\lim_{x\to c} \dfrac{1}{x} = \dfrac{1}{c}$ by part 4.

3. If $p(x)$ and $q(x)$ are polynomials then $\lim_{x\to c} \dfrac{p(x)}{q(x)} = \dfrac{p(c)}{q(c)}$ as long as $q(c) \neq 0$.

Another way that Theorem 3.9 can be used is to prove that limits do not exist:

Theorem 3.12 (Divergence Criterion)**.** *Let $f : \mathbb{R} \to \mathbb{R}$ be a function and let $c \in \mathbb{R}$. If there exist two sequences $\{x_n\}$ and $\{y_n\}$ in $\mathbb{R}$ with $x_n \neq c$, $y_n \neq c$ for all $n \in \mathbb{N}$ and*

$$\lim_{n\to\infty} x_n = c \textit{ and } \lim_{n\to\infty} y_n = c \textit{ but } \lim_{n\to\infty} f(x_n) \neq \lim_{n\to\infty} f(y_n)$$

then $\lim_{x\to c} f(x)$ does not exist.

Proof: This is an easy corollary of Theorem 3.9. Suppose that $\lim_{x\to c} f(x)$ does exist, say $\lim_{x\to c} f(x) = L$.
Then by Theorem 3.9 we have that $\lim_{n\to\infty} f(x_n) = L = \lim_{n\to\infty} f(y_n)$ but this is a contradiction since $\lim_{n\to\infty} f(x_n) \neq \lim_{n\to\infty} f(y_n)$.
Thus $\lim_{x\to c} f(x)$ does not exist.

Example 3.13. Show that $\lim_{x\to 0} \sin(\frac{1}{x})$ does not exist.

Solution: Let $f(x) = \sin(\frac{1}{x})$. Let $\{x_n\} = \{\frac{1}{2n\pi}\}$ and $\{y_n\} = \{\frac{1}{2n\pi + \frac{\pi}{2}}\}$.
Then $\lim_{n\to\infty} x_n = 0 = \lim_{n\to\infty} y_n$ and $x_n \neq 0$, $y_n \neq 0$ for all $n \in \mathbb{N}$.
But $\sin(\frac{1}{x_n}) = \sin(2n\pi) = 0$ for all $n \in \mathbb{N}$
and $\sin(\frac{1}{y_n}) = \sin(2n\pi + \frac{\pi}{2}) = 1$ for all $n \in \mathbb{N}$.
So $\lim_{n\to\infty} f(x_n) = 0$ and $\lim_{n\to\infty} f(y_n) = 1$. Thus by the Divergence Criterion $\lim_{x\to 0} \sin(\frac{1}{x})$ does not exist. (Note: Think about the graph of $f(x)$, does it give you an insight as to why this limit does not exist?)

Example 3.14. Let $f(x) = \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x \leq 0 \end{cases}$. Show that $\lim_{x\to 0} f(x)$ does not exist.

Solution: This function is sometimes called the diving-board function - draw its graph to see why.
Consider the sequences $\{x_n\} = \{\frac{1}{n}\}$ and $\{y_n\} = \{-\frac{1}{n}\}$. Then $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = 0$ and for all $n \in N$ we have $x_n \neq 0$ and $y_n \neq 0$.
However $\lim_{n\to\infty} f(x_n) = 1$ and $\lim_{n\to\infty} f(y_n) = -1$ so by the Divergence criterion, $\lim_{x\to 0} f(x)$ does not exist.
Does this solution remind you of anything from your Calculus courses?

Exercise 3.15. *Your turn:*

1. Give a formal definition of the left and right-hand limits $\lim_{x\to c-} f(x)$ and $\lim_{x\to c^+} f(x)$.

2. Prove that $\lim\limits_{x\to c} f(x) = L$ if and only if $\lim\limits_{x\to c^-} f(x) = L = \lim\limits_{x\to c^+} f(x)$

We can use our techniques to prove theorems about limits and order:

Theorem 3.16. *Let $f, g : \mathbb{R} \to \mathbb{R}$ be functions and let $c \in \mathbb{R}$. If $f(x) \leq g(x)$ for all $x \in \mathbb{R} - \{c\}$ and if $\lim\limits_{x\to c} f(x)$ and $\lim\limits_{x\to c} g(x)$ exist then*

$$\lim_{x\to c} f(x) \leq \lim_{x\to c} g(x).$$

Proof: By Theorem 2.16 if $\{a_n\}$ and $\{b_n\}$ are sequences with $a_n \leq b_n$ for all $n \in \mathbb{N}$, and if $\lim\limits_{n\to\infty} a_n$ and $\lim\limits_{n\to\infty} b_n$ exist then $\lim\limits_{n\to\infty} a_n \leq \lim\limits_{n\to\infty} b_n$.
Let $\{x_n\}$ be a sequence in $\mathbb{R}$ such that $\lim\limits_{n\to\infty} x_n = c$ but $x_n \neq c$ for all $n \in \mathbb{N}$. By Theorem 3.9 we have:

$$\lim_{x\to c} f(x) = \lim_{n\to\infty} f(x_n)$$
$$\lim_{x\to c} g(x) = \lim_{n\to\infty} g(x_n)$$

and since $f(x_n) \leq g(x_n)$ for all $n \in N$ Theorem 2.16 tells us that

$$\lim_{n\to\infty} f(x_n) \leq \lim_{n\to\infty} g(x_n).$$

Thus

$$\begin{aligned} \lim_{x\to c} f(x) &= \lim_{n\to\infty} f(x_n) \\ &\leq \lim_{n\to\infty} g(x_n) \\ &= \lim_{x\to c} g(x). \end{aligned}$$

This theorem has the following two corollaries. Try to prove them yourself!

Corollary 3.17. *Let $f : \mathbb{R} \to \mathbb{R}$ be a function and let $c \in \mathbb{R}$. Suppose that $\lim\limits_{x\to c} f(x)$ exists. If there exists $a, b \in \mathbb{R}$ such that $a \leq f(x) \leq b$ for all $x \in \mathbb{R} - \{c\}$ then*

$$a \leq \lim_{x\to c} f(x) \leq b.$$

In particular, if $f(x) \geq 0$ for all $x \in \mathbb{R} - \{c\}$ then $\lim\limits_{x\to c} f(x) \geq 0$.

Corollary 3.18. *Let $f, g, h : \mathbb{R} \to \mathbb{R}$ be functions and let $c \in \mathbb{R}$. Suppose that for all $x \in \mathbb{R} - \{c\}$ we have $f(x) \leq g(x) \leq h(x)$. If $\lim\limits_{x\to c} f(x) = L = \lim\limits_{x\to c} h(x)$ then*

$$\lim_{x\to c} g(x) = L.$$

[Note: This is sometimes called the Squeeze Theorem or the Flyswatter Principle.]

3.3 Continuity

You have probably seen the definition of a continuous function previously - most likely in the form 'f is continuous at $x = c$ if $\lim_{x \to c} f(x) = f(c)$'. Thus the formal definition of continuity at a point c is:

Definition 3.19. The function $f : \mathbb{R} \to \mathbb{R}$ is *continuous* at $c \in \mathbb{R}$ if for all $\epsilon > 0$ there exists $\delta > 0$ such that

$$|f(x) - f(c)| < \epsilon \text{ if } |x - c| < \delta.$$

If $A \subset \mathbb{R}$, we say that f is *continuous on* A if f is continuous at all points $c \in A$. If f is continuous on $\mathbb{R}$, we say f is continuous everywhere or simply that it is continuous.

Example 3.20. Some examples of continuous and discontinuous functions:

1. All polynomials are continuous on $\mathbb{R}$ by Example 3.11 (1).

2. All rational functions are continuous except where the denominator is 0 by Example 3.11 (3).

3. If $\lim_{x \to c} f(x)$ exists but is not equal to $f(c)$ then the discontinuity at c is called *removable*. For example

$$f(x) = \begin{cases} x + 3 & \text{if } x \neq 3 \\ 7 & \text{if } x = 3 \end{cases}$$

   has a removable discontinuity at $x = 3$.

4. If $\lim_{x \to c^-} f(x) \neq \lim_{x \to c^+} f(x)$ then we say that f has a *jump discontinuity* at $x = c$. For example

$$f(x) = \begin{cases} -1 & \text{if } x \leq 0 \\ 1 & \text{if } x > 0 \end{cases}$$

   has a jump discontinuity at $x = 0$.

5. If $\lim_{x \to c} f(x)$ does not exist for a different reason to 4. above then we say that f has an *essential discontinuity* at $x = c$. For example

$$f(x) = \begin{cases} \sin(\frac{1}{x}) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

   has an essential discontinuity at $x = 0$.

Exercise 3.21. *Your Turn:* Prove that $f(x) = |x|$ is continuous on $\mathbb{R}$.

Example 3.22. In Example 3.13 we saw that $f(x) = \sin(\frac{1}{x})$ does not have a limit at $c = 0$ and therefore is not continuous at $c = 0$. What about

$$g(x) = \begin{cases} x\sin(\frac{1}{x}) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}?$$

Solution: We claim that $g(x)$ is continuous at $c = 0$.
Consider $|g(x) - g(0)| = |x\sin(\frac{1}{x}) - 0| = |x||\sin(\frac{1}{x})| \leq |x|$, since $|\sin(\frac{1}{x})| \leq 1$ for all $x \in \mathbb{R} - \{0\}$. And when $x = 0$, $|g(x) - g(0)| = 0 = |x|$. So $|g(x) - g(0)| \leq |x|$ for all $x \in \mathbb{R}$.
Let $\epsilon > 0$ be given. Let $\delta = \epsilon$. Then

$$|g(x) - g(0)| \leq |x| < \epsilon \text{ if } |x| < \delta.$$

Therefore g is continuous at $c = 0$.

We can use Theorem 3.9 to characterise continuity in terms of sequences:

Theorem 3.23. *The function $f : \mathbb{R} \to \mathbb{R}$ is continuous at $c \in \mathbb{R}$ iff for all sequences $\{x_n\}$ in $\mathbb{R}$ with $\lim\limits_{n\to\infty} x_n = c$ we have $\lim\limits_{n\to\infty} f(x_n) = f(c)$.*

Proof: Suppose that $f : \mathbb{R} \to \mathbb{R}$ is continuous at c, then by definition $\lim\limits_{x\to c} f(x) = f(c)$. Then by Theorem 3.9 we know that if $\{x_n\}$ is a sequence in $\mathbb{R}$ with $\lim\limits_{n\to\infty} x_n = c$ we have

$$\lim_{n\to\infty} f(x_n) = \lim_{x\to c} f(x) = f(c).$$

Now suppose that for all sequences $\{x_n\}$ in $\mathbb{R}$ with $\lim\limits_{n\to\infty} x_n = c$ we have $\lim\limits_{n\to\infty} f(x_n) = f(c)$. Then we can use Theorem 3.9 again to see that

$$\lim_{x\to c} f(x) = \lim_{n\to\infty} f(x_n) = f(c).$$

That is, f is continuous at $x = c$.
We can use this theorem to prove that functions are not continuous at a point:

Corollary 3.24 (Discontinuity Criterion)**.** *The function f is not continuous at $c \in \mathbb{R}$ iff there exists a sequence $\{x_n\}$ in $\mathbb{R}$ such that $\lim\limits_{n\to\infty} x_n = c$ but $\lim\limits_{n\to\infty} f(x_n) \neq f(c)$.*

Example 3.25 (Dirichlet's Function)**.** Define Dirichlet's Function to be

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \in \mathbb{R} - \mathbb{Q} \end{cases}.$$

We claim that f is not continuous at any $c \in \mathbb{R}$.
Solution: Let $c \in \mathbb{R} - \mathbb{Q}$, then for all $n \in \mathbb{N}$ there exists at least one rational number $x_n \in (c, c+\frac{1}{n})$ by Theorem 1.15. Consider the sequence $\{x_n\}$. Clearly this sequence converges to c. But $f(x_n) = 1$ for all $n \in \mathbb{N}$ so

$$\lim_{n\to\infty} f(x_n) = 1 \neq 0 = f(c).$$

Thus by the Discontinuity Criterion, we have that f is not continuous at any irrational number c.

What if $c \in \mathbb{Q}$? Then we know that for all $n \in \mathbb{N}$ there exists an irrational number $y_n \in (c, c + \frac{1}{n})$ [proved on homework]. Now consider the sequence $\{y_n\}$. This sequence converges to c but $f(y_n) = 0$ for all $n \in \mathbb{N}$ so

$$\lim_{n\to\infty} f(y_n) = 0 \neq 1 = f(c).$$

Thus by the Discontinuity Criterion, we have that f is not continuous at any rational number c. So f is not continuous at any real number. [Try to picture what the graph of f looks like.]

We can use the definition of continuity and Theorem 3.10 to get:

Theorem 3.26. *Assume that $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are continuous at $c \in \mathbb{R}$. Then:*

1. *$kf(x)$ is continuous at c for all $k \in \mathbb{R}$.*
2. *$(f + g)(x)$ is continuous at c.*
3. *$f.g(x)$ is continuous at c.*
4. *$\dfrac{f}{g}(x)$ is continuous at c provided that $g(c) \neq 0$.*

Example 3.27. Let $f(x) = \sqrt{x}$. Then f is continuous at all $c > 0$.

Solution: Let $c > 0$. We need to show that for any $\epsilon > 0$ we can find $\delta > 0$ such that

$$|\sqrt{x} - \sqrt{c}| < \epsilon \text{ if } |x - c| < \delta.$$

Now for $x > 0$ we have

$$|\sqrt{x} - \sqrt{c}| = |\sqrt{x} - \sqrt{c}||\frac{\sqrt{x} + \sqrt{c}}{\sqrt{x} + \sqrt{c}}| = \frac{|x - c|}{\sqrt{x} + \sqrt{c}} \leq \frac{|x - c|}{\sqrt{c}}$$

So $|\sqrt{x} - \sqrt{c}| < \epsilon$ if $\dfrac{|x - c|}{\sqrt{c}} < \epsilon$.

Let $\epsilon > 0$ be given. Let's choose $\delta = \epsilon\sqrt{c}$. then

$$|\sqrt{x} - \sqrt{c}| \leq \frac{|x - c|}{\sqrt{c}} < \epsilon \text{ if } |x - c| < \delta$$

Therefore $f(x) = \sqrt{x}$ is continuous at each $c > 0$.

[Note: Here the choice of δ depended on c, so at $c = 1$ we would choose $\delta = \epsilon$ and at $c = \frac{1}{4}$ we would need $\delta = \frac{\epsilon}{2}$ etc.]

Question: Now that we know that $f(x) = \sqrt{x}$ is continuous on $(0, \infty)$ and we know that $g(x) = x^2 + 1$ is continuous for all x, can we conclude that $h(x) = \sqrt{x^2 + 1}$ is continuous everywhere? We need the following:

Theorem 3.28. *Let A and B be subsets of $\mathbb{R}$. Suppose that $f : A \to \mathbb{R}$ is continuous at $a \in A$ and $g : B \to \mathbb{R}$ is continuous at $b = f(a)$. Then $g \circ f$ is continuous at a.*

Proof: We need to show that given $\epsilon > 0$ there exists $\delta > 0$ such that

$$|g(f(x)) - g(f(a))| < \epsilon \text{ if } |x - a| < \delta$$

or

$$|g(f(x)) - g(b)| < \epsilon \text{ if } |x - a| < \delta.$$

Let $\epsilon > 0$ be given. Since g is continuous at b we can find $\delta_1 > 0$ such that

$$|g(y) - g(b)| < \epsilon \text{ if } |y - b| < \delta_1$$

so

$$|g(y) - g(f(a))| < \epsilon \text{ if } |y - f(a)| < \delta_1.$$

Since f is continuous at a, for this choice of δ_1 we can find $\delta_2 > 0$ such that

$$|f(x) - f(a)| < \delta_1 \text{ if } |x - a| < \delta_2.$$

Let $\delta = \delta_2$. So if $|x - a| < \delta = \delta_2$ we have $|f(x) - f(a)| < \delta_1$ thus

$$|g(f(x)) - g(f(a))| < \epsilon.$$

Therefore $g \circ f$ is continuous at $x = a$, as required.

Example 3.29. We can use this theorem to show:

1. If f is continuous on a set A then so is $|f|$.
2. Suppose f is continuous on a set A and $f(A) \subset (0, \infty)$, then $h(x) = \sqrt{f(x)}$ is continuous on A.

3.4 Continuity on a Closed Interval

We would like to be able to talk about functions being continuous on a closed interval. For example it seems natural to say that $f(x) = \sqrt{x}$ is continuous on $[0, 1]$. But what does it mean to say that this function is continuous at $x = 0$ since it is not defined for any negative numbers? We will need to return to the idea of right and left hand limits.

Definition 3.30. Let $f : \mathbb{R} \to \mathbb{R}$ be a function and let $c \in \mathbb{R}$. We say that $\lim\limits_{x \to c^+} f(x) = L$ if for all $\epsilon > 0$ there exists $\delta > 0$ such that $|f(x) - L| < \epsilon$ if $0 < x - c < \delta$. In this case we call L the *right-hand limit* of f at c.

Note that here we are concerned with values of x such that $0 < x - c$ which means $x > c$. Similarly, for left-hand limits we consider values of x st $0 < c - x$, that is $x < c$.

Definition 3.31. Let $f : \mathbb{R} \to \mathbb{R}$ be a function and let $c \in \mathbb{R}$. We say that $\lim_{x \to c^-} f(x) = L$ if for all $\epsilon > 0$ there exists $\delta > 0$ such that $|f(x) - L| < \epsilon$ if $0 < c - x < \delta$. In this case we call L the *left-hand limit* of f at c.

We can think of these limits in terms of sequences also:

Theorem 3.32. *Let $f : \mathbb{R} \to \mathbb{R}$. Let $c \in \mathbb{R}$. Then the following are equivalent:*

1. $\lim_{x \to c^+} f(x) = L$
2. *For all sequences $\{x_n\}$ in $\mathbb{R}$ satisfying $x_n > c$ for all $n \in \mathbb{N}$ and $\lim_{n \to \infty} x_n = c$ we have*

$$\lim_{n \to \infty} f(x_n) = L$$

Exercise 3.33. *Your turn:* Prove Theorem 3.32 and formulate the equivalent result for left-hand limits.

Theorem 3.34. *Let $f : \mathbb{R} \to \mathbb{R}$. Let $c \in \mathbb{R}$. Then $\lim_{x \to c} f(x) = L$ if and only if*

$$\lim_{x \to c^-} f(x) = L = \lim_{x \to c^+} f(x)$$

Proof: If $\lim_{x \to c} f(x) = L$ then given any $\epsilon > 0$ there exists $\delta > 0$ such that

$$|f(x) - L| < \epsilon \text{ if } 0 < |x - c| < \delta$$

but this means that $|f(x) - L| < \epsilon$ if $0 < x - c < \delta$ i.e. $\lim_{x \to c^+} f(x) = L$ and $|f(x) - L| < \epsilon$ if $0 < c - x < \delta$ i.e. $\lim_{x \to c^-} f(x) = L$.

Let's assume that $\lim_{x \to c^-} f(x) = L = \lim_{x \to c^+} f(x)$. Let $\epsilon > 0$ be given. Then there exists $\delta_1 > 0$ such that

$$|f(x) - L| < \epsilon \text{ if } 0 < x - c < \delta_1$$

and $\delta_2 > 0$ such that

$$|f(x) - L| < \epsilon \text{ if } 0 < c - x < \delta_2.$$

Let $\delta = min\{\delta_1, \delta_2\}$. Then if $0 < |x - c| < \delta$ we have $|f(x) - L| < \epsilon$. That is $\lim_{x \to c} f(x) = L$.

Definition 3.35. Let $[a, b]$ be an interval in $\mathbb{R}$ and $f : [a, b] \to \mathbb{R}$ be a function. We say that f is continuous on $[a, b]$ if it is continuous at all $c \in (a, b)$ and $\lim_{x \to a^+} f(x) = f(a)$ and $\lim_{x \to b^-} f(x) = f(b)$.

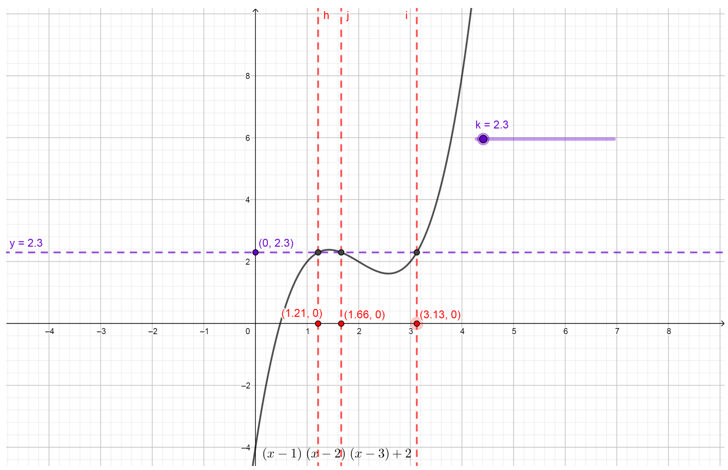
Note that we say that f is *right-continuous* at $x = c$ if $\lim_{x \to c^+} f(x) = f(c)$ and is *left-continuous* at $x = c$ if $\lim_{x \to c^-} f(x) = f(c)$. So f is continuous on $[a, b]$ if it is continuous on (a, b), right-continuous at a, and left-continuous at b.

Example 3.36. The function $f(x) = \sqrt{x}$ is continuous on $[0, 1]$.

3.5 Properties of Continuous Functions

We will investigate the properties of functions which are continuous on closed intervals. Our first result is:

Theorem 3.37 (The Intermediate Value Theorem)**.** *Let f be a continuous function on $[a, b]$ and let k be any number lying between $f(a)$ and $f(b)$. Then there exists at least one $c \in (a, b)$ such that $f(c) = k$.*



Proof: Without loss of generality let $f(a) < f(b)$. Denote $[a, b]$ by $I_1 = [A_1, B_1]$. Let p_1 be the midpoint of I_1 (i.e. $p_1 = \frac{A_1+B_1}{2}$). If $f(p_1) = k$ we are done.
If $f(p_1) > k$ let $I_2 = [A_2, B_2]$ be $[A_1, p_1]$.
If $f(p_1) < k$ let $I_2 = [A_2, B_2]$ be $[p_1, B_1]$.
In either case, we have:
1. $[A_2, B_2] \subset [A_1, B_1]$,
2. $B_2 - A_2 = \frac{1}{2}(B_1 - A_1)$,
3. $f(A_2) < k < f(B_2)$.

Now repeat this process to get intervals $[A_3, B_3]$, $[A_4, B_4]$,...
If at any stage the midpoint p_n has the property that $f(p_n) = k$ we stop.
Otherwise we get a sequence of closed intervals $\{[A_n, B_n]\}$ such that for all $n \in \mathbb{N}$
1. $[A_{n+1}, B_{n+1}] \subset [A_n, B_n]$,
2. $B_n - A_n = (\frac{1}{2})^{n-1}(B_1 - A_1)$,
3. $f(A_n) < k < f(B_n)$.

By 1. above we see that $\{A_n\}$ is increasing and bounded above by b so by the Monotone Convergence Theorem it is convergent. Let's say $\lim_{n\to\infty} A_n = A$. Since $A_n \leq b$ for all $n \in \mathbb{N}$ we know that $A \leq b$.
Similarly, we see that $\{B_n\}$ is decreasing and bounded below by a so by the Monotone Convergence Theorem it is convergent. Let's say $\lim_{n\to\infty} B_n = B$. Since $B_n \geq a$ for all $n \in \mathbb{N}$ we know that $B \geq a$.

Now consider

$$
\begin{aligned}
B - A &= \lim_{n\to\infty} B_n - \lim_{n\to\infty} A_n \\
&= \lim_{n\to\infty} (B_n - A_n) \\
&= (b-a) \lim_{n\to\infty} (\frac{1}{2})^{n-1} \\
&= 0
\end{aligned}
$$

So $A = B$. Now let $c = A = B$. We have $a \leq B = c = A \leq b$ so $c \in [a, b]$. Since f is continuous at c we have

$$\lim_{n\to\infty} f(A_n) = f(c) \text{ and } \lim_{n\to\infty} f(B_n) = f(c).$$

Now we know that $f(A_n) < k$ for all $n \in \mathbb{N}$ so $\lim_{n\to\infty} f(A_n) \leq k$ and $f(B_n) > k$ for all $n \in \mathbb{N}$ so $\lim_{n\to\infty} f(B_n) \geq k$. This gives $f(c) \leq k$ and $f(c) \geq k$ which means that $f(c) = k$.
Also since $f(a) < k < f(b)$ we have that $c \neq a$ and $c \neq b$ so $c \in (a, b)$ as required.

Example 3.38 (Finding Roots and Solutions)**.** Suppose that $f : [a, b] \to \mathbb{R}$ with $f(a) < 0$ and $f(b) > 0$ (or $f(a) > 0$ and $f(b) < 0$). Then using the Intermediate Value Theorem (with $k = 0$) we can see that there exists at least one $c \in (a, b)$ with $f(c) = 0$.

For example we can show that the equation $x^5 + x - 1 = 0$ has a solution in $[0, 1]$. Let $f(x) = x^5 + x - 1$. Clearly f is continuous on $[0, 1]$ since it is a polynomial. We can see that $f(0) = -1 < 0 < 1 = f(1)$ so by the Intermediate Value Theorem there is a $c \in (0, 1)$ such that $f(c) = 0$, that is, c is a solution to the equation $x^5 + x - 1 = 0$.

Example 3.39 (Fixed Points of Functions)**.** We say that a function f has a *fixed point* at $x = c$ if $f(c) = c$. Suppose that $f : [0, 1] \to [0, 1]$ is a continuous function, then f has a fixed point.

To prove this note that if $f(0) = 0$ or $f(1) = 1$ then we are done. So let's assume that $0 < f(0)$ and $f(1) < 1$. Consider $g(x) = f(x) - x$. Since f is continuous on $[0, 1]$ then so is g. Also $g(0) = f(0) - 0 > 0$ and $g(1) = f(1) - 1 < 0$, so by the Intermediate Value Theorem there is at least one c in (0.1) st $g(c) = 0$ that is $f(c) - c = 0$ or $f(c) = c$. Such a c is a fixed point of f.

Exercise 3.40. *Your turn:*

1. Try to visualise the situation in Example 3.31 above. What does the existence of a fixed point mean for the graph of f?

2. Suppose f is continuous on an interval $[a, b]$ with $f(a) > 0$ and $f(b) < 0$. Can you say anything about the number of times the graph of f crosses the x-axis? What about the number of times it touches the x-axis?

Definition 3.41. We say that a function $f : A \to \mathbb{R}$ is *bounded* on A if there exists $M > 0$ such that $|f(x)| \leq M$ for all $x \in A$.

Example 3.42. We can see that:

1. The function $f(x) = 3x$ is bounded on $[0, 1]$ but not bounded on $[0, \infty)$.

2. The function $g(x) = \frac{1}{x}$ is not bounded on $(0, 1]$.

Theorem 3.43. *Let $a, b \in \mathbb{R}$ and let $f : [a, b] \to \mathbb{R}$ be a continuous function. Then f is bounded on $[a, b]$.*

Proof: Suppose that f is not bounded on $[a, b]$. Then for all $n \in \mathbb{N}$ there exists $x_n \in [a, b]$ such that $|f(x_n)| > n$. So $\{x_n\}$ is a sequence in $[a, b]$ and is therefore bounded. By the Bolzano-Weierstrass Theorem it has a convergent subsequence $\{x_{n_k}\}$. Let $x = \lim\limits_{k\to\infty} x_{n_k}$. Since $a \leq x_{n_k} \leq b$ for all $k \in \mathbb{N}$ we have $a \leq x \leq b$ i.e. $x \in [a, b]$. Therefore f is continuous at x and so $\lim\limits_{k\to\infty} f(x_{n_k}) = f(x)$ by Theorem 3.22. So $\{f(x_{n_k})\}$ is a convergent sequence and is thus bounded. But this is a contradiction since $|f(x_{n_k})| > n_k \geq k$ for all $k \in \mathbb{N}$. Therefore our assumption that f is not bounded on $[a, b]$ is false.

Definition 3.44. Let $A \subset \mathbb{R}$ and $f : A \to \mathbb{R}$ be a function. We say that f has an *absolute maximum* on A if there exists $z \in A$ such that $f(z) \geq f(x)$ for all $x \in A$. We call $f(z)$ the absolute maximum value of f on A. We say that f has an *absolute minimum* on A if there exists $w \in A$ such that $f(w) \leq f(x)$ for all $x \in A$. We call $f(w)$ the absolute minimum value of f on A. Absolute maximum and absolute minimum values are called *extreme values* of f on A.

Example 3.45. We can see that:

1. The function $f(x) = 3x$ has an absolute maximum at $x = 1$ and an absolute minimum at $x = 0$ on $[0, 1]$.

2. The function $f(x) = 3x$ has neither absolute maximum nor absolute minimum on $(0, 1)$.

3. The function $g(x) = x^2$ has an absolute minimum at $x = 0$ but no absolute maximum on $\mathbb{R}$.

4. The function $h(x) = \frac{1}{x}$ has no absolute maximum and no absolute minimum on $(0, \infty)$.

5. It is possible for a function to attain its absolute maximum or minimum value at more than one point, for example on $\mathbb{R}$ $f(x) = \cos(x)$ has an absolute maximum on $\mathbb{R}$ at $x = 0, 2\pi, ...$ and an absolute minimum at $x = \pi, 3\pi, ...$

Our example shows that it is possible for a function to be continuous on a set A but not to have absolute maximum or minimum values on that set. Our last theorem shows that if A is a closed interval then f will always attain absolute maximum or minimum values there.

Theorem 3.46 (Extreme Values Theorem)**.** *Let $a, b \in \mathbb{R}$ and let $f : [a, b] \to \mathbb{R}$ be a continuous function. Then f has an absolute maximum and an absolute minimum on $[a, b]$.*

Proof: Let $I = [a, b]$ and $f(I) = \{f(x)|x \in [a, b]\}$. Theorem 3.43 tells us that $f(I)$ is a bounded set. Let $s = lub\{f(I)\}$ and $t = glb\{f(I)\}$ (by the Axiom of Completeness we know that s and t exist). We claim that there exist $z, w \in I$ such that $f(z) = s$ and $f(w) = t$.

We will prove that z exists here (the proof that w exists will be similar and is an exercise for you!).

Since $s = lub\{f(I)\}$ then $s - \frac{1}{n}$ is not an upper bound for $f(I)$. So there exists $x_n \in I$ such that

$$s - \frac{1}{n} < f(x_n) \leq s.$$

We can find such an x_n for each $n \in \mathbb{N}$ so we get a sequence $\{x_n\}$ in I. Since each $x_n \in I = [a, b]$, the sequence $\{x_n\}$ is bounded and by the Bolzano-Weierstrass Theorem it has a convergent subsequence $\{x_{n_k}\}$. Let $z = \lim\limits_{k\to\infty} x_{n_k}$. Now $a \leq x_{n_k} \leq b$ for all $k \in \mathbb{N}$ so $a \leq \lim\limits_{k\to\infty} x_{n_k} \leq b$, that is $z \in I$.

Therefore f is continuous at z, and so $\lim\limits_{k\to\infty} f(x_{n_k}) = f(z)$ by Theorem 3.22.

But $s - \frac{1}{n_k} < f(x_{n_k}) \leq s$ and so by Theorem 2.16 $s \leq \lim\limits_{k\to\infty} f(x_{n_k}) \leq s$ that is

$$s = \lim_{k\to\infty} f(x_{n_k}) = f(z).$$

This we have found $z \in I$ such that f has an absolute maximum on I at z, as required.