MT251P – Lecture 2

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Section 1.2 – Proof.

The starting point for mathematics is a collection of accepted statements (also called axioms). One is not required to prove these axioms because they are accepted. How do we move on to obtain new mathematical statements? This is where deductive reasoning comes in. Deductive reasoning works in the following way:

We start with premises (which are accepted statements) and we then derive conclusions with certainty.

The last word certainty in the above makes deductive reasoning a very special type of reasoning.

Here is an example of deductive reasoning: Suppose my premises are that

All students in the class have brown hair. (*)

and

Brian is a student in the class. (**)

Then, I can conclude with certainty that

Brian has brown hair.

Mathematical proof.

The classical Greeks decided (around 600 BC) that in order to prove something in mathematics one needs to use deductive reasoning to make the conclusion. This is still the requirement for a mathematical proof today and is what distinguishes a mathematical proof from other forms of 'proof'.

Example 5.

Prove that the square of any odd integer is an odd integer.

Solution.

Here, the set of integers is the usual set $\{\ldots, -2, -1, 0, 1, 2, 3, \ldots\}$. Suppose n is an odd integer. Then we have that

$$n = 2k + 1, \quad \text{for some integer} \quad k$$

Thus

$$n^2 = (2k + 1)^2$$

$$= 4k^2 + 4k + 1$$

$$= 2(2k^2 + 2k) + 1$$

and so n^2 is odd and we are done.

Example 6.

Using a similar approach as in example 5, try to prove that the square of an even integer is an even integer.

Remark 3. The converse of a statement.

Suppose P is the statement that n is an odd integer and Q is the statement that n^2 is an odd integer. Then, in example 5 above we proved a statement of the form

$$P \Rightarrow Q$$

which means P implies Q (or in other words – if P is true, then Q is true). Suppose we want to prove the statement

$$Q \Rightarrow P$$

i.e. if n^2 is an odd integer, then n is an odd integer. This statement $Q \Rightarrow P$ is called the converse of $P \Rightarrow Q$. We will now prove $Q \Rightarrow P$ in the following example:

Example 7.

If n^2 is an odd integer, then prove that n is an odd integer.

Solution.

If n^2 is an odd integer, then n has to be odd because otherwise n is even and so (by example 6) n^2 is even which is impossible.

Remark 4 – Equivalent statements.

Using the P and Q from remark 3 we have now proved that

$$P \Rightarrow Q \qquad \text{and} \qquad Q \Rightarrow P \qquad (*)$$

The combination of the two statements in (*) above can be more concisely written as

$$P \iff Q$$

and so

$$n \text{ is odd} \iff n^2 \text{ is odd}$$

and we say

$$n \text{ is odd if and only if } n^2 \text{ is odd}$$

We say that the statements P and Q are equivalent when $P \iff Q$

Remark 5 – Counterexamples.

Counterexamples can be used to show that a certain statment is not valid. For example, -2 is a counterexample to the statement that $n^2 \geq 0$ implies $n \geq 0$.

Proof by contradiction.

Proof by contradiction works as follows:

Suppose we want to prove that the statement P is true. Well, we try to prove this by actually trying to prove that P is not false. How do we try to prove P is not false? Well, we suppose that P is false and try to show it leads to a contradiction and this will mean that the statement – P is false – cannot hold. Hence P must be true and we are done.

Example 8 – Prove that $\sqrt{2}$ is irrational.

Proof.

Now, we want to prove that $\sqrt{2}$ is irrational and so (using proof by contradiction) we will start by supposing $\sqrt{2}$ is not irrational (i.e. $\sqrt{2}$ is rational).

So, there are integers a, b with $b \neq 0$ such that

$$\sqrt{2} = \frac{a}{b}$$

Now, by dividing out any relevant common divisors of a and b, we can assume that the greatest common divisor of a and b is 1.

$$\sqrt{2} = \frac{a}{b}$$

$$\Rightarrow 2 = \frac{a^2}{b^2}$$

$$\Rightarrow a^2 = 2b^2$$

$$\Rightarrow a^2 \text{ is even}$$

$$\Rightarrow a \text{ is even}$$

$$\Rightarrow a = 2c, \quad \text{for some integer } c$$

$$\Rightarrow 4c^2 = 2b^2$$

$$\Rightarrow 2c^2 = b^2$$

$$\Rightarrow b^2 \ \text{ is even}$$

$$\Rightarrow b \ \text{ is even}$$

$$\Rightarrow 2 \ \text{ is a common divisor of } \ a \ \text{ and } \ b$$

which is a contradiction because the greatest common divisor of a and b is 1.

So, our initial assumption that $\sqrt{2}$ is rational is false. Hence $\sqrt{2}$ is irrational and we are done.