MT251P – Lecture 21

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Definition 8.

(i) The row space of a $k \times n$ matrix A is the subspace $R(A)$ of $\mathbb{R}^n$ that is spanned by the k row vectors of A. The row rank of A is defined to be $\dim R(A)$.

(ii) The column space of a $k \times n$ matrix A is the subspace $C(A)$ of $\mathbb{R}^k$ that is spanned by the n column vectors of A. The column rank of A is defined to be $\dim C(A)$.

Remark 5.

A $k \times n$ matrix A can be considered as a function from $\mathbb{R}^n$ to $\mathbb{R}^k$ as follows:

Suppose

$$\underline{x} = \begin{pmatrix} x_1 \\ x_2 \\ . \\ . \\ x_n \end{pmatrix}$$

is a column vector in $\mathbb{R}^n$. Then,

$$\underline{y} = A\underline{x} \qquad (*)$$

will give a column vector in $\mathbb{R}^k$.

In $(*)$ above $A\underline{x}$ means the product of the $k \times n$ matrix A by the $n \times 1$ matrix $\underline{x}$.

Example 12.

Consider the 2×3 matrix

$$A = \begin{pmatrix} 2 & -1 & 3 \\ 4 & 1 & 2 \end{pmatrix}$$

and suppose

$$\underline{x} = \begin{pmatrix} 2 \\ -2 \\ 5 \end{pmatrix} \in \mathbb{R}^3$$

Then,

$$A\underline{x} = \begin{pmatrix} 2 & -1 & 3 \\ 4 & 1 & 2 \end{pmatrix} \begin{pmatrix} 2 \\ -2 \\ 5 \end{pmatrix} = \begin{pmatrix} 21 \\ 16 \end{pmatrix} \in \mathbb{R}^2$$

Definition 9.

Suppose A is a $k \times n$ matrix. Then, the image space of A is denoted by $Im(A)$ and is defined as:

$$Im(A) = \{A\underline{x} : \underline{x} \in \mathbb{R}^n\}$$

Remark 6.

In definition 9, suppose G_i is the i^{th} column of A considered as a column vector in $\mathbb{R}^k$ for $1 \leq i \leq n$ and suppose

$$\underline{x} = \begin{pmatrix} x_1 \\ x_2 \\ . \\ . \\ x_n \end{pmatrix} \in \mathbb{R}^n$$

Then, $A\underline{x} = x_1 G_1 + x_2 G_2 + \cdots + x_n G_n$ and so $Im(A)$ is the set of all linear combinations of the n column vectors of A. Hence, $Im(A) = C(A)$.

Definition 10.

Suppose A is $k \times n$ matrix. Then, the rank of A is denoted by rank (A) and is defined to be $\dim Im(A)$ which is also $\dim C(A)$.

Definition 11.

Suppose A is $k \times n$ matrix. Then, the kernel of A is denoted by $\ker(A)$ and is defined as:

$$\ker(A) = \{\underline{x} \in \mathbb{R}^n : A\underline{x} = 0\}$$

Definition 12.

One can show that $\ker(A)$ is a subspace of $\mathbb{R}^n$. The nullity of A is then defined to be $\dim \ker(A)$.

Theorem 4 – Rank–Nullity Theorem.

Suppose A is $k \times n$ matrix. Then

$$\dim \ker A + \dim ImA = n$$

i.e. rank $A+$ nullity of $A = n$.

Theorem 5.

Suppose A is a $k \times n$ matrix. Then, the row rank of A is equal to the column rank of A. Also, the row rank of A is equal to the rank of A

Remark 7 – How do we find the rank of a matrix?

Suppose A is a $k \times n$ matrix. Here is the strategy for finding rank A: You first perform elementary row operations on A and stop when you have an REF matrix C. Then, rank A is the number of non–zero rows in C.

Example 13.

Find the rank of the matrix

$$A = \begin{pmatrix} 2 & -1 & 3 \\ 1 & -2 & -3 \\ 5 & -4 & 3 \end{pmatrix}$$

Solution.

Interchange R_1 with R_2 to get

$$\begin{pmatrix} 1 & -2 & -3 \\ 2 & -1 & 3 \\ 5 & -4 & 3 \end{pmatrix}$$

Replace R_2 with $R_2 - 2R_1$ and replace R_3 with $R_3 - 5R_1$ to get

$$\begin{pmatrix} 1 & -2 & -3 \\ 0 & 3 & 9 \\ 0 & 6 & 18 \end{pmatrix}$$

Replace R_3 with $R_3 - 2R_2$ to get

$$\begin{pmatrix} 1 & -2 & -3 \\ 0 & 3 & 9 \\ 0 & 0 & 0 \end{pmatrix}$$

Replace R_2 with $\frac{1}{3}R_2$ to get

$$C = \begin{pmatrix} 1 & -2 & -3 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{pmatrix}$$

C is in REF and so the rank of A is the number of non–zero rows in C. Hence, rank $A = 2$.

Theorem 6.

Suppose A is an $n \times n$ matrix. Then, A is invertible $\iff$ rank $A = n$.