7. The Real Numbers and the Complex Numbers

Observe that the rational numbers can now be described as

$$\mathbb{Q} = \left\{ \frac{a}{b} : a, b \in \mathbb{Z},\ b \neq 0,\ \gcd(a,b) = 1 \right\}.$$

Lemma 7.1. *There is no $x \in \mathbb{Q}$ such that $x^2 = 2$, that is, the number $\sqrt{2}$ is not a rational number.*

Proof: Let $x \in \mathbb{Q}$ such that $x^2 = 2$. We write $x = a/b$, where $a, b \in \mathbb{Z}$, $b \neq 0$ and a, b are coprime. Then $a^2 = 2b^2$. Hence 2 divides a and thus 4 divides $a^2 = 2b^2$. Now 2 divides b contradicting that a and b are coprime.

□

Remark 7.2. (Real Numbers)
The above Lemma implies that equation $x^2 = 2$ has not solution in $\mathbb{Q}$. Thus one introduces the **real numbers** *as the limit of converging sequences of rational numbers. For instance the sequences $(x_n)_{n \geq 1}$, with $x_1 = 1$ and $x_{n+1} = 4/(x_n + (2/x_n))$, for $n \geq 2$, converges to $\sqrt{2}$. The real numbers are 'ordered' in the sense that $x \leq y$ or $y \leq x$, for any pair x, y of real numbers. The real numbers can be represented on the number line.*

Remark 7.3. (Complex Numbers)
The equation $x^2 = -1$, have no solutions $x \in \mathbb{R}$. This leads to the introduction of the **complex number**, *denoted by $\mathbb{C}$, where*

$$\mathbb{C} := \{a + bi : \ a, b \in \mathbb{R}\},$$

where we define $i^2 = -1$. We say $a + bi \neq c + di$, for $a, b, c, d \in \mathbb{R}$, unless $a = c$ and $b = d$. So, $4 - i \neq 2 + 3i$. For complex numbers $z = a + bi$ and $w = c + di$ we define an addition and multiplication:

$$z + w := (a + c) + (b + d)i, \quad zw := (ac - bd) + (ad + bc)i.$$

For instance

$$(2 + i) + (-1 + 2i) = 1 + 3i \text{ and } (2 + i)(-1 + 2i) = -4 + 3i.$$

These operations are commutative and associative, they have inverses and multiplication distributes over addition.

Complex numbers can be represented in the **complex plane**, *where the x-axis represents the complex numbers with zero imaginary part and the y-axis represents the 'pure' complex numbers with zero real part. Then we can identify z with the point (a, b) in the plane.*

Definition 7.4. *Let $z = a + bi$ be a complex number. Then*

(1) $\mathrm{Re}(z) := a$ *is called* **real part** *of z*
(2) $\mathrm{Im}(z) := b$ *is called* **imaginary part** *of z*
(3) $\overline{z} := a - bi$ *is called* **complex conjugate** *of z*
(4) $|z| := \sqrt{a^2 + b^2}$ *is called* **modulus** *or length of z*

Remark 7.5. *(1) There is a copy of $\mathbb{R}$ embedded in $\mathbb{C}$, as we can identify $x \in \mathbb{R}$ with $z := x + 0i \in \mathbb{C}$. The zero element of the complex numbers is $0 + 0i$ and the one element is $1 + 0i$. Unlike $\mathbb{R}$, the complex numbers are not ordered.*

(2) Addition of complex numbers is like vector addition. Furthermore, for $z = a + bi$, the additive inverse is

$$-z = (-a) + (-b)i = -a - bi.$$

(3) Note that $z = a + bi$ is the complex conjugate of $\overline{z}$. Next one checks that $z\overline{z} = (a^2 + b^2) + 0i$. In particular, $z\overline{z} \in \mathbb{R}$ and $z\overline{z} = |z|^2$. Note that $z\overline{z} = 0$ if and only if $a = b = 0$, that is, $z = 0$. Thus whenever $z \neq 0$ we have $z \cdot (\overline{z}/z\overline{z}) = 1$, that is,

$$z^{-1} = \overline{z}/z\overline{z} = \frac{a}{a^2 + b^2} - \frac{b}{a^2 + b^2}i.$$

For instance

$$\frac{1}{1-i} = \frac{1}{1-i} \cdot \frac{1+i}{1+i} = \frac{1+i}{2} = \frac{1}{2} + \frac{1}{2}i$$

and

$$\frac{2+3i}{1-i} = (2+3i) \cdot \left(\frac{1}{2} + \frac{1}{2}i\right) = \left(1 - \frac{3}{2}\right) + \left(\frac{3}{2} + 1\right) i = \frac{-1}{2} + \frac{5}{2}i$$

(4) Let $z, w \in \mathbb{C}$. Then $\overline{zw} = \overline{z} \cdot \overline{w}$. From this it follows that $|zw|^2 = (zw)\overline{zw} = (z\overline{z})(w\overline{w}) = |z|^2|w|^2$, and so $|zw| = |z||w|$.

(5) Every complex number z is uniquely identified by $|z|$ and its angle θ, in radians, between the positive real axis and z, traced out counter-clockwise. Recall that for all $x \in \mathbb{R}$ we have

$$\exp(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} \quad \sin(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \quad \cos(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$

If we extend these definitions to complex numbers, then we get that

$$\exp(xi) = \cos(x) + \sin(x)i.$$

We write e^{xi} *for* $\exp(xi)$. *Note that* $|e^{xi}| = \cos(x)^2 + \sin(x)^2 = 1$, *for any* $x \in \mathbb{R}$. *This gives the* **exponential form** *of* z *as*

$$z = re^{\theta i},$$

where $r = |z|$ *and* θ *is unique modulo* 2π.

Take for instance $z = 2 + 3i$. *Then* $|z| = \sqrt{4+9} = \sqrt{13}$. *Hence*

$$z = \sqrt{13} \cdot \left(\frac{2}{\sqrt{13}} + \frac{3}{\sqrt{13}} i \right) = \sqrt{13} \cdot (\cos(\theta) + \sin(\theta)i),$$

for some $\theta \in \mathbb{R}$. *We calculate* $\theta = \arccos\left(\frac{2}{\sqrt{13}}\right) = 0.983$ rad *or* $\theta = 56.3°$. *Since* $\sin(\theta) = \frac{3}{\sqrt{13}}$, *we have* $z = \sqrt{13} \cdot e^{0.983 \cdot i}$. *What about* $w = 2 - 3i$. *Then* $|w| = \sqrt{4+9} = \sqrt{13}$ *and*

$$w = \sqrt{13} \cdot \left(\frac{2}{\sqrt{13}} - \frac{3}{\sqrt{13}} i \right) = \sqrt{13} \cdot (\cos(\eta) + \sin(\eta)i),$$

for some $\eta \in \mathbb{R}$. *Again we calculate* $\eta = \arccos\left(\frac{2}{\sqrt{13}}\right) = 0.983$ rad *or* $\eta = 56.3°$ *and* $\sin(\eta) = \frac{3}{\sqrt{13}} \neq -\frac{3}{\sqrt{13}}$. *Let's switch the sign of* η, *that is, we set* $\eta := -0.983$ rad. *Then* $\cos(\eta) = \frac{2}{\sqrt{13}}$ *and* $\sin(\eta) = -\frac{3}{\sqrt{13}}$. *Hence* $w = \sqrt{13} \cdot e^{-0.983 \cdot i} = \sqrt{13} \cdot e^{(2\pi - 0.983) \cdot i} = \sqrt{13} \cdot e^{5.3 \cdot i}$. *Here* $\eta = 5.3$ rad *or* $\eta = 303.7°$.

(6) Now for complex numbers $z = |z|e^{\theta i}$ *and* $w = |w|e^{\eta i}$ *it follows that*

$$zw = |z|e^{\theta i} \cdot |w|e^{\eta i} = |z||w|e^{\theta i} \cdot e^{\eta i} = |zw|e^{(\theta+\eta)i}.$$

Theorem 7.6 (DeMoivre's Theorem)**.** *For all natural numbers* n *and real numbers* θ *we have*

$$(\cos(\theta) + \sin(\theta)i)^n = \cos(n\theta) + \sin(n\theta)i.$$

Proof: The formula can be verified using induction.

$\square$

Definition 7.7. *Let* $n \geq 1$ *be an integer. We say* $z \in \mathbb{C}$ *is a* n*-th root of unity if* $z^n = 1$.

Theorem 7.8. *Let* $n \geq 1$ *be a integer. Then there are* n *distinct* n*-th roots of unity, namely the elements of*

$$\left\{ e^{\frac{2\pi k}{n} i} = \cos\left(\frac{2\pi k}{n}\right) + \sin\left(\frac{2\pi k}{n}\right) i : k = 0, 1, \ldots, n-1 \right\}.$$

Example 7.9. *The three third roots of unity are*

$$
\begin{aligned}
e^{0i} &= \cos(0) + \sin(0)i = 1 \\
e^{\frac{2\pi}{3}i} &= \cos\left(\frac{2\pi}{3}\right) + \sin\left(\frac{2\pi}{3}\right)i = -\frac{1}{2} + \frac{\sqrt{3}}{2}i \\
e^{\frac{4\pi}{3}i} &= \cos\left(\frac{4\pi}{3}\right) + \sin\left(\frac{4\pi}{3}\right)i = -\frac{1}{2} - \frac{\sqrt{3}}{2}i
\end{aligned}
$$