MT234P – MULTIVARIABLE CALCULUS – 2022

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Lecture 5

Definition 11.

A function $f(x, y)$ is said to be continuous at (a, b) if the following conditions are satisfied:

(i) $f(a, b)$ is defined.

(ii) $\lim_{(x,y)\to(a,b)} f(x, y)$ exists.

(iii) $\lim_{(x,y)\to(a,b)} f(x, y) = f(a, b)$.

A function is called continuous if it's continuous at every point in its domain.

Example 18.

Consider f defined as follows:

$$f(x, y) = \frac{2xy}{x^2 + y^2}, \quad \text{for} \quad (x, y) \neq (0, 0) \qquad \text{and} \quad f(0, 0) = 0$$

Prove that f is continuous at every point in $\mathbb{R}^2$ except $(0, 0)$.

Solution.

We know from example 15 that $\lim_{(x,y)\to(0,0)} f(x, y)$ doesn't exist and so f is not continuous at $(0, 0)$.

If $(a, b) \neq (0, 0)$, then by Theorem 1

$$\lim_{(x,y)\to(a,b)} \frac{2xy}{x^2 + y^2} = \frac{2ab}{a^2 + b^2} = f(a, b)$$

So, f is continuous at (a, b).

Theorem 3.

Suppose $f(x, y)$ is continuous at (a, b) and suppose $g : \mathbb{R} \to \mathbb{R}$ is continuous at $f(a, b)$. Then, $h = g \circ f$ is continuous at (a, b).

Here, $g \circ f$ denotes the composition function g after f, which means that $(g \circ f)(x, y) = g(f(x, y))$.

Example 19.

$h(x, y) = e^{3x-y}$ is continuous at every point $(a, b) \in \mathbb{R}^2$.

Proof.

Let $f(x,y) = 3x - y$ and let $g(x) = e^x$. Then, $h = g \circ f$. Theorem 1 gives

$$\lim_{(x,y)\to(a,b)} (3x - y) = 3a - b$$

So, $\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$ and hence f is continuous at (a,b). Also, g is continuous and so by Theorem 3 we get that h is continuous at (a,b).

Section 1.4 – Partial Derivatives.

Definition 12.

Suppose $g : \mathbb{R} \to \mathbb{R}$ is a function. We say that L is the limit of $g(x)$ as x approaches a and we write

$$\lim_{x\to a} g(x) = L$$

if for every $\epsilon > 0$, there exists a corresponding $\delta > 0$ such that

$$0 < |x - a| < \delta \;\Rightarrow\; |g(x) - L| < \epsilon$$

Remark 6.

Note how definition 12 is similar to definition 10. The intuition behind definition 12 is similar to the intuition behind definition 10, i.e. L is the limit of $g(x)$ as x approaches a if we can make $g(x)$ as close as we like to L by taking x sufficiently close to a but not equal to a.

Example 20.

As an example of limits of functions of one variable, we will discuss the derivative of a function because we will need it for partial derivatives.

Suppose $g : \mathbb{R} \to \mathbb{R}$. Then, the derivative of $g(x)$ is denoted by $g'(x)$ and the value of $g'(x)$ at $a \in \mathbb{R}$ is defined as

$$g'(a) = \lim_{h\to 0} \frac{g(a+h) - g(a)}{h}$$

We also denote $g'(a)$ by $\dfrac{d}{dx} g(x)_{|x=a}$

Now, we will find $f'(x)$ and $f'(3)$, where $f(x) = x^2$.

Well, for $x \in \mathbb{R}$, we have

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h}$$

$$= \lim_{h \to 0} \frac{x^2 + 2xh + h^2 - x^2}{h}$$

$$= \lim_{h \to 0} \frac{2xh + h^2}{h}$$

$$= \lim_{h \to 0} (2x + h), \quad \text{because we can assume } \ h \neq 0 \ \text{ in the limit as } \ h \ \text{ approaches } \ 0$$

$$= 2x$$

So, we get $f'(x) = 2x$ and hence $f'(3) = 6$ and we are done.

Definition 13.

Suppose $f : \mathbb{R}^2 \to \mathbb{R}$. The partial derivative of $f(x, y)$ with respect to x is denoted by $\dfrac{\partial f}{\partial x}$ and the value of $\dfrac{\partial f}{\partial x}$ at the point (a, b) is defined as

$$\frac{\partial f}{\partial x}_{|(a,b)} = \frac{d}{dx} f(x, b)_{|x=a} = \lim_{h \to 0} \frac{f(a+h, b) - f(a, b)}{h} \qquad (*)$$

Remark 7.

Note that in definition 13, we can consider b as a constant and so $q(x) = f(x, b)$ can be considered as a function from $\mathbb{R}$ to $\mathbb{R}$ and so $(*)$ in definition 13 is saying that

$$\frac{\partial f}{\partial x}_{|(a,b)} = \frac{d}{dx} q(x)_{|x=a}$$

So, the way to find $\dfrac{\partial f}{\partial x}$ is to treat the y variable as a constant and differentiate with respect to x.

Example 21.

If $f(x, y) = x^3 - 3x^2y + 2$, then

$$\frac{\partial f}{\partial x} = 3x^2 - 6xy + 0 = 3x^2 - 6xy \quad \text{and} \quad \frac{\partial f}{\partial x}_{|(2,-3)} = 12 + 36 = 48$$

Definition 14.

Suppose $f : \mathbb{R}^2 \to \mathbb{R}$. The partial derivative of $f(x, y)$ with respect to y is denoted by $\dfrac{\partial f}{\partial y}$ and the value of $\dfrac{\partial f}{\partial y}$ at the point (a, b) is defined as

$$\frac{\partial f}{\partial y}_{|(a,b)} = \frac{d}{dy} f(a, y)_{|y=b} = \lim_{h\to 0} \frac{f(a, b+h) - f(a, b)}{h} \qquad (*)$$

Remark 8.

Note that in definition 14, we can consider a as a constant and so $p(y) = f(a, y)$ can be considered as a function from $\mathbb{R}$ to $\mathbb{R}$ and so $(*)$ in definition 14 is saying that

$$\frac{\partial f}{\partial y}_{|(a,b)} = \frac{d}{dy} p(y)_{|y=b}$$

So, the way to find $\dfrac{\partial f}{\partial y}$ is to treat the x variable as a constant and differentiate with respect to y.

Example 22.

If $f(x, y) = x^3 - 3x^2y + 2$, then

$$\frac{\partial f}{\partial y} = -3x^2 \quad \text{and} \quad \frac{\partial f}{\partial y}_{|(2,-3)} = -12$$

Example 23.

If $g(x, y) = \arctan(x^2y) + \arctan(xy^2)$, then

$$\frac{\partial g}{\partial x} = \frac{2xy}{1 + x^4y^2} + \frac{y^2}{1 + x^2y^4}$$

and

$$\frac{\partial g}{\partial y} = \frac{x^2}{1 + x^4y^2} + \frac{2xy}{1 + x^2y^4}$$

Remark 9.

For functions of more than two variables, we define partial derivatives in a similar way and we evaluate partial derivatives in a similar way as the next example will show.

Example 24.

If $f(x, y, z) = \ln(x + 2y + 3z)$, then treat y and z as constant and differentiate with respect to x to get

$$\frac{\partial f}{\partial x} = \frac{1}{x + 2y + 3z}$$