Fiacre Ó Cairbre

Lecture 11

Remark 7 – Strategy for finding absolute maxima and absolute minima of a continuous function f on a closed bounded set T in $\mathbb{R}^2$.

Step 1.

List the interior points of T that are candidates for local maxima and local minima of f. These points are the critical points of f. Then, evaluate f at these points.

Step 2.

Check the boundary points of T and evaluate f at the candidates for absolute maxima and absolute minima of f.

Step 3.

Select the greatest value of f from steps 1 and 2 above. This will give the absolute maximum value of f on T. The points where f takes on this absolute maximum value, will be the absolute maxima of f on T.

Select the smallest value of f from steps 1 and 2 above. This will give the absolute minimum value of f on T. The points where f takes on this absolute minimum value, will be the absolute minima of f on T.

Remark 8.

Before we do an example using remark 7, we will discuss the case for functions of one variable on a closed interval $[p, q]$ in $\mathbb{R}$ because we will need that case.

Suppose $g(x)$ is a function of one variable on a closed interval $[p, q]$ in $\mathbb{R}$. Then, $g(w)$ is a local maximum value of g if $g(w) \geq g(x)$, for all domain points x in an open interval with centre w. We call w a local maximum. $g(v)$ is a local minimum value of g if $g(v) \leq g(x)$, for all domain points x in an open interval with centre v. We call v a local minimum.

Suppose $g(x)$ has a local maximum or local minimum at $t \in (p, q)$. Also, suppose $g'(t)$ exists. Then, $g'(t) = 0$.

Suppose z is in (p, q) where $g'(z) = 0$ or where $g'(z)$ doesn't exist. Then, z is called a critical point of g.

$g(k)$ is an absolute maximum value of g if $g(k) \geq g(x)$, for all domain points x. We call k an absolute maximum. $g(m)$ is an absolute minimum value of g if $g(m) \leq g(x)$, for all domain points x. We call m an absolute minimum.

Remark 9.

Using the definitions in remark 8, we now give the three steps for finding absolute maxima and absolute minima of a continuous function g on a closed interval $[p, q]$ in $\mathbb{R}$.

Step 1.

Evaluate g at the critical points.

Step 2.

Evaluate g at the endpoints p, q of $[p, q]$.

Step 3.

Select the greatest value of g from steps 1 and 2 above. This will give the absolute maximum value of g on $[p, q]$. The points where g takes on this absolute maximum value, will be the absolute maxima of g on $[p, q]$.

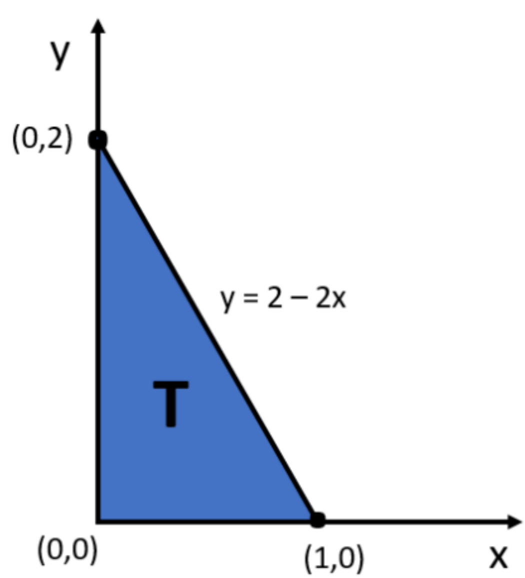
Select the smallest value of g from steps 1 and 2 above. This will give the absolute minimum value of g on $[p, q]$. The points where g takes on this absolute minimum value, will be the absolute minima of g on $[p, q]$.

Example 4.

Find the absolute maxima and absolute minima of $f(x, y) = x^2 + y^2$ on the closed triangular region T bounded by the lines $x = 0$, $y = 0$ and $y + 2x = 2$ in the first quadrant. Also, find the absolute maximum value and the absolute minimum value of f on T.

Solution.

Denote the closed triangular region by T. It is helpful to draw a picture of T below.



We see that $T = \{(x, y) \in \mathbb{R}^2 : y \leq 2 - 2x,\ x \geq 0,\ y \geq 0\}$. Note that T is a closed bounded subset of $\mathbb{R}^2$ and that f is continuous on T and so we can use the strategy in remark 7.

Step 1 – Interior points of T.

Note that $\frac{\partial f}{\partial x} = 2x$ and $\frac{\partial f}{\partial y} = 2y$. So,

$$\frac{\partial f}{\partial x}_{|(a,b)} = 0 = \frac{\partial f}{\partial y}_{|(a,b)} \iff (a,b) = (0,0)$$

However, $(0,0)$ is not an interior point of T and so step 1 produces no critical points.

Step 2 – Boundary points of T.

The boundary of T consists of the three sides of the triangle and we will take one side at a time.

(i) On the line segment joining $(0,0)$ to $(1,0)$ we have $y = 0$ and so $f(x,y) = f(x,0) = x^2$, which may be considered as a function of one variable on $[0,1]$. From remark 9 we have that the candidates for the absolute maxima and absolute minima for this function of one variable are the endpoints $0,1$ and the x–values in $(0,1)$, where

$$0 = \frac{d}{dx}(x^2) \Rightarrow x = 0$$

and so there are no such x–values in $(0,1)$. So, the only candidates for the absolute maxima and absolute minima for this function of one variable are the endpoints $0,1$. So, the candidates for the absolute maxima and absolute minima for $f(x,y)$ are $(0,0)$ and $(1,0)$ and we evaluate f at these points to get

$$f(0,0) = 0 \quad \text{and} \quad f(1,0) = 1 \qquad (*)$$

(ii) On the line segment joining $(0,0)$ to $(0,2)$ we have $x = 0$ and so $f(x,y) = f(0,y) = y^2$, which may be considered as a function of one variable on $[0,2]$. From remark 9 we have that the candidates for the absolute maxima and absolute minima for this function of one variable are the endpoints $0,2$ and the y–values in $(0,2)$, where

$$0 = \frac{d}{dy}(y^2) \Rightarrow y = 0$$

and so there are no such y–values in $(0,2)$. So, the only candidates for the absolute maxima and absolute minima for this function of one variable are the endpoints $0,2$. So, the candidates for absolute maxima and absolute minima for $f(x,y)$ are $(0,0)$ and $(0,2)$ and we evaluate f at these points to get

$$f(0,0) = 0 \quad \text{and} \quad f(0,2) = 4 \qquad (**)$$

(iii) On the line segment joining $(1,0)$ to $(0,2)$ we have $y = 2 - 2x$ and so $f(x,y) = x^2 + (2-2x)^2 = x^2 + 4 - 8x + 4x^2 = 5x^2 - 8x + 4$, which may be considered as a function of one variable on $[0,1]$. From remark 9 we have that the candidates for the absolute maxima and absolute minima for this function of one variable are the endpoints $0,1$ and the x–values in $(0,1)$, where

$$0 = \frac{d}{dx}(5x^2 - 8x + 4) = 10x - 8 \Rightarrow x = \frac{4}{5}$$

So, the only candidates for the absolute maxima and absolute minima for this function of one variable are the endpoints $0, 1$ and $\frac{4}{5}$. So, the candidates for absolute maxima and absolute minima for $f(x, y)$ are $(0, 2)$, $(1, 0)$ and $(\frac{4}{5}, \frac{2}{5})$ and we evaluate f at these points to get

$$f(0,2) = 4, \quad f(1,0) = 1 \quad \text{and} \quad f(\frac{4}{5}, \frac{2}{5}) = \frac{4}{5} \qquad (***)$$

Step 3.

The relevant values from steps 1 and 2 are in $(*)$, $(**)$, $(***)$ and so the relevant values are

$$f(0,0) = 0, \quad f(1,0) = 1, \quad f(0,2) = 4, \quad f(\frac{4}{5}, \frac{2}{5}) = \frac{4}{5} \qquad (****)$$

Select the greatest value of f from $(****)$ to get that the absolute maximum value of f on T is 4. The points where f takes on this absolute maximum value, will be the absolute maxima of f on T and so $(0, 2)$ is the only absolute maximum of f on T.

Select the smallest value of f from $(****)$ to get that the absolute minimum value of f on T is 0. The points where f takes on this absolute minimum value, will be the absolute minima of f on T and so $(0, 0)$ is the only absolute minimum of f on T.