MT251P – Lecture 1

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Chapter 1 – Problem solving, Proof and Propositional Logic.

Section 1.1 – Problem Solving.

Mathematics has many features. The first feature we will discuss is problem solving.

1. Try to understand the problem first.

2. It may be helpful to draw a diagram.

3. You should never say that you cannot solve a problem after just trying for a few minutes.

4. Following on from 3 above, it's good to have more than one approach to a problem (if that one doesn't work).

5. The more problems you work on, the better you will (hopefully) become at problem solving.

6. If one approach doesn't work in a particular problem, you might still learn something from it.

Example 1.

Problem: – Consider a sports tournament (like tennis) where each game between two players produces one winner and one loser. The winner progresses to the next round and the loser drops out of the tournament. Eventually, there is a final between two players and the winner is called the champion. Suppose there are 91 players at the start of the tournament. How many games have been played in the tournament when the champion lifts the trophy?

Solution: –

What are we looking for? We are looking for a number (i.e. the number of games played). Now, we will show that there is something else (different from games) that has the same total number as number of games. Well, when the champion lifts the trophy at the end of the tournament, we know that the number of losers is exactly the same as the number of games played. The number of losers is 90 and so the number of games played is 90 and the problem is solved.

Remark 1.

Notice how the answer in example 1 above is independent of the structure of the tournament which might seem a little surprising. Also, example 1 can be generalised to n players (instead of 91). Try and convince yourself that the answer then is $n - 1$.

Example 2.

Problem: – Suppose there are twenty statements on a page and for $1 \leq k \leq 20$, the k^{th} statement reads

— There are exactly k false statements on this page. —

Which of the k statements are true and which are false?

Solution. – The first thing to notice is that at most one of the k statements can be true because we cannot have exactly t false statements and exactly w false statements where $t \neq w$. So, we have either exactly one true statement or no true statements.

Now, it's impossible to have no true statements because then all twenty statements are false which means that the 20^{th} statement is true which is a contradiction.

So, there is exactly one true statement which means exactly nineteen statements are false which means that the 19^{th} statement is true.

So, the answer is that the 19^{th} statement is true and all the other statements are false.

Example 3.

Problem: – A perfect number is defined as a positive integer that is the sum of all its positive divisors excluding itself. So, for example 6 is a perfect number because 1, 2, 3, 6 are the positive divisors of 6 and $6 = 1 + 2 + 3$. Find the next perfect number after 6. The classical Greeks (around 600 BC) were the first to consider perfect numbers and they called them perfect because they really liked them.

Solution: – We start checking numbers after 6. Notice that we can skip over prime numbers because the only positive divisors of a prime number p are 1 and p. After a little while we see that 28 is the next perfect number because its positive divisors are 1, 2, 4, 7, 14, 28 and $28 = 1 + 2 + 4 + 7 + 14$. I won't ask you to find the next perfect number after 28 and you will see why later.

Example 4 – First to 100 wins.

Problem: – Consider the following game between two players, A and B. Let $S = \{1, 2, 3, \ldots, 10\}$. A chooses any number from S and announces it. B chooses any number from S and adds it to A's number and announces the result. For, example, if A announced 6 and B chose 3, then B announces 9. A then chooses any number from S and adds it to the previous announcement and announces the result. For example, continuing on from above, if A chose 8, then A announces 17 (=$9 + 8$). Continue like this and the first player to announce 100 wins. Notice that during the game A and B can choose any number from S and it doesn't matter if the number was chosen before by A or B. Find a strategy that guarantees you win if you go first.

Solution: – Many people approach this problem by starting at the start of the game. However, it's more beneficial to start at the end of the game as follows. If you can announce the number 89, then you are guaranteed to win because you can announce 100 after your opponent's announcement. In a similar way, working back, we see that if you can announce the number 78, then you are guaranteed that you can announce 89 and so you are guaranteed to win. Continue working back like this and we see that the following strategy will guarantee a win:

You start by announcing 1 and then you announce 12 (no matter what your opponent announces), and then you announce 23, 34, 45, 56, 67, 78, 89, 100 to guarantee a win.

Remark 2.

There are still many unsolved problems in mathematics. Some of these problems have been around for thousands of years. Here is one example of an unsolved problem:

– Is there an odd perfect number? –