MT251P – Lecture 9

Fiacre Ó Cairbre

Three–dimensional vectors.

Remark 6.

Suppose we are in three–dimensional space given by

$$\mathbb{R}^3 = \{(x, y, z) : x, y, z \in \mathbb{R}\}$$

and suppose

$$A = (x_1, y_1, z_1),\ B = (x_2, y_2, z_2) \in \mathbb{R}^3$$

The vector $\underline{u} = \vec{AB}$ can be written as

$$\underline{u} = u_1 i + u_2 j + u_3 k, \quad \text{where} \quad u_1 = x_2 - x_1,\ u_2 = y_2 - y_1,\ u_3 = z_2 - z_1$$

$\underline{u}$ is called the displacement vector.

$0i + 0j + 0k$ is called the zero vector.

Theorem 2.

Suppose $\underline{v} = v_1 i + v_2 j + v_3 k$, $\underline{w} = w_1 i + w_2 j + w_3 k$ and $t \in \mathbb{R}$.

(a) $\underline{v} + \underline{w} = (v_1 + w_1)i + (v_2 + w_2)j + (v_3 + w_3)k$.

(b) $t\underline{v} = tv_1 i + tv_2 j + tv_3 k$.

(c) The magnitude (or length) of $\underline{v}$ is denoted by $||\underline{v}||$ and satisfies

$$||\underline{v}|| = \sqrt{{v_1}^2 + {v_2}^2 + {v_3}^2}$$

(d) The dot (or scalar) product of $\underline{v}$ and $\underline{w}$ is denoted by $\underline{v}.\underline{w}$ and is defined by

$$\underline{v}.\underline{w} = v_1 w_1 + v_2 w_2 + v_3 w_3$$

(e) Suppose neither $\underline{v}$ nor $\underline{w}$ is the zero vector. Then, $\underline{v}.\underline{v} = ||\underline{v}||||\underline{v}|| \cos\theta$, where θ is the angle between $\underline{v}$ and $\underline{w}$ and $0 \le \theta \le \pi$.

Definition 2.

The non–zero vectors $\underline{v}, \underline{w}$ are said to be perpendicular if the angle between $\underline{v}$ and $\underline{w}$ is $\frac{\pi}{2}$.

We also define the zero vector to be perpendicular to any vector.

Remark 7.

Two vectors $\underline{v}$, $\underline{w}$ in $\mathbb{R}^3$ are perpendicular $\iff$ $\underline{v}.\underline{w} = 0$

Definition 3.

Suppose $\underline{u} = u_1 i + u_2 j + u_3 k$ and $\underline{w} = w_1 i + w_2 j + w_3 k$. The cross product of $\underline{u}$ and $\underline{w}$ is denoted by $\underline{u} \times \underline{w}$ and is defined by

$$\underline{u} \times \underline{w} = (u_2 w_3 - u_3 w_2)i + (u_3 w_1 - u_1 w_3)j + (u_1 w_2 - u_2 w_1)k$$

Example 5.

Consider the vectors $\underline{u} = 3i - 2j + k$ and $\underline{w} = i + 3j + 3k$. Find

(a) $\underline{u} + \underline{w}$

(b) $||\underline{w}||$

(c) $\underline{u}.\underline{w}$

(d) $\underline{u} \times \underline{w}$

Solution.

(a) $\underline{u} + \underline{w} = 4i + j + 4k$

(b) $||\underline{w}|| = \sqrt{1+9+9} = \sqrt{19}$

(c) $\underline{u}.\underline{w} = 0$

(d) $\underline{u} \times \underline{w} = ((-2)(3) - (1)(3))i + ((1)(1) - (3)(3))j + ((3)(3) - (-2)(1))k = -9i - 8j + 11k$

Remark 8.

If $\underline{u}$ and $\underline{w}$ are vectors in $\mathbb{R}^3$, then $\underline{u} \times \underline{w}$ gives a vector which is perpendicular to both $\underline{u}$ and $\underline{w}$. In example 5, $\underline{m} = \underline{u} \times \underline{w}$ is perpendicular to both $\underline{u}$ and $\underline{w}$ because $\underline{m}.\underline{u} = 0$ and $\underline{m}.\underline{w} = 0$.

Example 6.

Consider the vectors $\underline{u} = i + 2j + k$ and $\underline{w} = 3i + j + 2k$. Find the angle θ between $\underline{u}$ and $\underline{w}$.

Solution.

$\underline{u}.\underline{w} = 7,\ ||\underline{u}|| = \sqrt{6}, ||\underline{w}|| = \sqrt{14}$ and so

$$7 = \sqrt{6}\sqrt{14}\cos\theta \quad \Rightarrow \quad \cos\theta = \frac{7}{\sqrt{84}} \quad \Rightarrow \quad \theta = \arccos(\frac{7}{\sqrt{84}})$$

Remark 9.

Suppose $A = (0,0,0),\ B = (x_1, x_2, x_3)$. The vector $\underline{v} = \vec{AB}$ is called a position vector.

Section 3.2 – Vectors in $\mathbb{R}^n$.

Definition 4.

For $n \geq 1$ we define

$$\mathbb{R}^n = \{(x_1, x_2, \ldots, x_n) : x_i \in \mathbb{R} \text{ for } 1 \leq i \leq n\}$$

The vector with initial point at the origin, $(0, 0, \ldots, 0)$, and terminal point $(x_1, x_2, \ldots, x_n)$ is called a position vector and is denoted by

$$\underline{x} = (x_1, x_2, \ldots, x_n)$$

Theorem 3.

Suppose $(x_1, x_2, \ldots, x_n)$ and $(y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$ and $k \in \mathbb{R}$.

(a) $\underline{x} + \underline{y} = (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n)$

(b) $k\underline{x} = (kx_1, kx_2, \ldots, kx_n)$

(c) $||\underline{x}|| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$

(d) $\underline{x}.\underline{y} = x_1y_1 + x_2y_2 + \cdots + x_ny_n$

(e) $\underline{x}.\underline{y} = ||\underline{x}||||\underline{y}|| \cos\theta$, where θ is the angle between $\underline{x}$ and $\underline{y}$ and $0 \leq \theta \leq \pi$ and neither $\underline{x}$ nor $\underline{y}$ is the zero vector $(0, 0, \ldots, 0)$