MT251P – Lecture 13

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This theorem should have appeared after the Cauchy Schwartz inequality in Lecture 10.

Theorem – The Triangle Inequality.

$||\underline{x}+\underline{y}|| \leq ||\underline{x}|| + ||\underline{y}||$, for all $\underline{x},\ \underline{y} \in \mathbb{R}^n$.

Now, we return to chapter 4 and continue from the end of lecture 12.

Definition 2.

A system of k linear equations in the n variables $x_1, x_2, \dots\ x_n$ is given by:

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2$$

$$\vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots$$

$$a_{k1}x_1 + a_{k2}x_2 + \cdots + a_{kn}x_n = b_k$$

where a_{ij}, b_i are constants for $1 \leq i \leq k,\ 1 \leq j \leq n$.

A solution of the above system of equations is a list of n numbers $s_1, s_2, \dots, s_n$ such that $x_1 = s_1, x_2 = s_2, \dots, x_n = s_n$ gives a solution to every equation in the above system. The set of all solutions is called the solution set.

Two systems of linear equations are called equivalent if they have the same solution set.

Definition 3.

A $k \times n$ matrix is a rectangular array of numbers with k rows (horizontal) and n columns (vertical).

Example 5.

$$\begin{pmatrix} 2 & 4 \\ -1 & 0 \\ 2 & 1 \end{pmatrix} \quad \text{is a} \quad 3 \times 2 \quad \text{matrix}$$

Remark 1.

A $k \times n$ matrix A can be written in the form:

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ . & . & . & . \\ . & . & . & . \\ a_{k1} & a_{k2} & \dots & a_{kn} \end{pmatrix}$$

where a_{ij} is the entry in the i^{th} row and j^{th} column. We can also write $A = [a_{ij}]$.

Definition 4.

Two matrices $A = [a_{ij}]$ and $B = [bij]$ are defined to be equal if A and B have the same number of rows (say k) and the same number of columns (say n) and $a_{ij} = b_{ij}$, for $1 \leq i \leq k,\ 1 \leq j \leq n$.

Definition 5.

An elementary row operation on a matrix A consists of one of the following operations, where R_i denotes the i^{th} row of A:

(a) Interchange R_i and R_j.

(b) Replace R_i with cR_i, where $c \neq 0$.

(c) Replace R_i with $R_i + dR_j$, where $i \neq j$ and d is some constant.

Definition 6.

We say that two matrices are row equivalent if one is obtained from the other by a collection of elementary row operations.

A row in a matrix is called a zero row if all the entries in the row are 0. A row that is not a zero row is called a non–zero row.

Definition 7.

A $k \times n$ matrix B is said to be in reduced row echelon form (RREF) if satisfies the following conditions:

1. In a non–zero row, the first non–zero entry is a 1. We call this 1 a leading 1.

2. If there any zero rows, then they are grouped together at the bottom of the matrix.

3. In any two successive non–zero rows, the leading 1 in the lower row occurs further to the right than the leading 1 in the higher row.

4. Each column that contains a leading 1 (from some row) has zeros everywhere else.

Definition 8.

A $k \times n$ matrix B is said to be in row echelon form (REF) if it satisfies conditions 1,2 and 3 in definition 7 (but doesn't necessarily satisfy condition 4).

Example 6.

$$\begin{pmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & -1 \end{pmatrix} \qquad \text{is in RREF}$$

Theorem 1.

Any matrix A is row equivalent to a unique matrix B which is in RREF.

Section 4.2 – How to find a solution set of a system of linear equations.

Remark 2.

We now look at finding the solution set of the following system linear equations S in definition 2:

First we define the augmented matrix A' of S given by

$$A' = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ . & . & . & . & . \\ . & . & . & . & . \\ a_{k1} & a_{k2} & \dots & a_{kn} & b_k \end{pmatrix}$$

Now, any elementary row operation on A' gives the augmented matrix of a system of linear equations that is equivalent to S.

We perform on A' a collection of elementary row operations until we obtain a matrix C which is in RREF. It will be convenient to read off the solution set of the system of linear equations V corresponding to C and this will be the same as the solution set of S. This strategy is called Gauss–Jordan elimination.

Example 7.

Use Gauss–Jordan elimination to find the solution set of the following system T of linear equations:

$$x_1 + 2x_2 + x_3 = 3$$

$$2x_1 + 5x_2 - x_3 = -4$$

$$3x_1 - 2x_2 - x_3 = 5$$