MT232P - Analysis

Assignment #4

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Question 1

Let $x_1 = 1$, and set $x_{n+1} = \sqrt{2 + x_n}$ for all $n \in \mathbb{N}$. Use the Monotone Convergence Theorem to show that $\{x_n\}_1^\infty$ converges, and find $\lim_{n\to\infty} x_n$.

Solution

To use the Monotone Convergence Theorem, a sequence must be monotone. For a sequence to be monotone, we need to show that it is increasing and bounded. To show that the sequence is bounded we can use induction.
Base case:

$$x_2 = \sqrt{2+1} > x_1 = 1$$

$$\sqrt{3} > 1$$

Assume: $x_{n+2} > x_{n+1}$
Then:

$$x_{n+3} > x_{n+2}$$

$$\sqrt{2 + x_{n+2}} > \sqrt{2 + x_{n+1}}$$

$$2 + x_{n+2} > 2 + x_{n+1}$$

$$x_{n+2} > x_{n+1} \forall n \in \mathbb{N}$$

Because x_n is strictly increasing, we know that if we prove it is also bounded, then we will know that it is monotone.
For a sequence to be bounded, $\exists M \in \mathbb{N}$ such that $|x_n| < M \forall n \in \mathbb{N}$
We can show that $x_n < 2 \forall n \in \mathbb{N}$ by induction
Base case:

$$n = 1, x_n = 1 < 2$$

Assume:

$$x_n < 2$$

, then we must show that

$$x_{n+1} < 2$$

$$\sqrt{2 + x_n} < 2$$

$$2 + x_n < 4$$

$$x_n < 2$$

, which is true by our assumption. Therefore since the sequence is increasing and bounded it is also monotone.Then by using the Monotone Convergence Theorem we know that x_n converges.
Since x_n converges, we know that $\lim_{n\to\infty} x_n$ exists. Let L = $\lim_{n\to\infty} x_n$, then since $x_{n+1} = \sqrt{2 + x_n}$:

$$x_{n+1}^2 = 2 + x_n$$

$$\lim_{n\to\infty} (x_{n+1})^2 = L^2$$

$$\lim_{n\to\infty} 2 + x_n = 2 + L$$

, because ($x_{n+1}^2 = 2 + x_n$), we know that $L^2 = 2 + L$. Then to find L we use the -b formula to get $L_1 = 2$ and $L_2 = -1$, however since x_n is increasing and $x_1 = 1$, then the sequence can't converge to -1, therefore x_n converges to 2

Question 2

Show that if every subsequence of $\{a_n\}_1^\infty$ has itself a subsequence which converges to 0, then $\{a_n\}_1^\infty$ converges to 0

Solution

Let a_{n_k} be a subsequence of a_n. We know that a_{n_k} has a subsequence of it's own $a_{n_{k_j}}$. This new subsequence also has to be a subsequence of a_n

Assume a_n doesn't converge to 0. We know that some subsequences of a_n converge to 0, meaning that if a_n doesn't converge to 0, then it must also have a subsequence which also doesn't converge to 0. Let's call this subsequence a_{n_s}. If we show that this subsequence also converges to 0, then a_n must also converge to 0. Because a_{n_s} is a subsequence of a_n, then it has a subsequence convergent to 0. By assumption, a_{n_s} doesn't converge to 0. Therefore it must have a subsequence which doesn't converge to 0. Call this $a_{n_{s_k}}$. This is also a subsequence of a_n. Then it must have a subsequence that converges to 0 even tough that does not.

If we repeat the steps above, eventually we will get that the subsequence must converge to 0.

Question 3

Assume $\{a_n\}_1^\infty$ and $\{b_n\}_1^\infty$ are Cauchy sequences. Use a triangle inequality argument to prove directly from the definition of a Cauchy sequence that $\{c_n\}_1^\infty$, where $c_n = |a_n - b_n|$,is also a Cauchy sequence.

Solution

Using the triangle inequality to prove that $c_{n n\in\mathbb{N}}$ is also a Cauchy sequence, where $c_n = |a_n - b_n|$. To show that c_n is a Cauchy sequence, we need to show that $\forall\varepsilon > 0, \exists N \in \mathbb{N}$ such that $|c_n - c_m| < \varepsilon$, where $n, m > N$. We know that $\forall\varepsilon > 0, \exists N$ such that $m, n > N$. Then:

$$|a_n - a_m| < \frac{\varepsilon}{2}, |b_n - b_m| < \frac{\varepsilon}{2}$$

We can show this as:

$$|a_{n_1} - a_{m_1}| < \frac{\varepsilon}{2}, |b_{n_1} - b_{m_1}| < \frac{\varepsilon}{2}$$

if $m_1, n_1 > N_1$ and

$$|a_{n_2} - a_{m_2}| < \frac{\varepsilon}{2}, |b_{n_2} - b_{m_2}| < \frac{\varepsilon}{2}$$

if $m_2, n_2 > N_2$. Let $N = \max\{N_1, N_2\}$, then

$$|a_n - a_m| < \frac{\varepsilon}{2}, |b_n - b_m| < \frac{\varepsilon}{2}$$

if $m, n > N$ Therefore we can show the following: If n_1:

$$|a - b| \geq ||a| - |b||$$

(The Triangle Inequality)

$$\begin{aligned}
|c_n - c_m| &= ||a_n - b_n| - |a_m - b_m|| \\
&\leq |a_n - b_n - a_m + b_m| \\
&\leq |a_n - a_m - (b_n - b_m)| \\
&\leq |a_n - a_m| + |b_n - b_m| \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\
&< \varepsilon
\end{aligned}$$

if n, m > N

Question 4

What is the value of $\lim_{x\to 1} \frac{1}{1+\sqrt{x}}$? Prove your assertion using an $\epsilon - \delta$ argument.

Solution

Let $f(x) = \frac{1}{1+\sqrt{x}}$. Let $h(x) = 1 + \sqrt{x}$, and $g(x) = 1$, then $\frac{g}{h}(x) = f(x)$. Therefore $\frac{\lim_{x\to 1} g(x)}{\lim_{x\to 1} h(x)} = \lim_{x\to 1} f(x)$. The limit of $g(x) = 1 \forall x$, The limit of $h(x)$ we can guess that is $h(1) = 2$ from the formula. To prove it, we must show that $\forall \varepsilon > 0, \exists \delta$ such that if $0 < |x - 1| < \delta$. Then:

$$|h(x) - 2| < \varepsilon$$

$$|\sqrt{x} - 1| < \varepsilon$$

$$-\varepsilon < \sqrt{x} - 1 < \varepsilon$$

$$(1 - \varepsilon)^2 < x < (1 + \varepsilon)^2$$

$$(1 - \varepsilon)^2 - 1 < x - 1 < (1 + \varepsilon)^2 - 1$$

$$|x - 1| < \delta$$

and

$$|x - 1| < \max\left\{|(1 - \varepsilon)^2 - 1|, |(1 + \varepsilon)^2 - 1\right\}$$

Then $\delta = \max\left\{|(1 - \varepsilon)^2 - 1|, |(1 + \varepsilon)^2 - 1\right\}$
Therefore we have proven $\lim_{x\to 1} h(x) = 2$.
Therefore since $\lim_{x\to 1} h(x) = 2$ and $\lim_{x\to 1} g(x) = 1$ and $\lim_{x\to 1} f(x) = \frac{\lim_{x\to 1} g(x)}{\lim_{x\to 1} h(x)}$. Then by Theorem 3.10.4, $\lim_{x\to 1} f(x) = \frac{\lim_{x\to 1} g(x)}{\lim_{x\to 1} h(x)} = \frac{1}{2}$