11. Conditional Probability

Example 11.1. *Consider an urn with five red and five blue balls. We pick all ten balls, one at a time, without replacement. Prior to the experiment the chance to pick blue as the i-th ball is $\frac{1}{2}$. However this chance changes as the experiment progresses. Assume we pick a blue first ball. Then the chance to pick blue as the second ball drops to $\frac{4}{9}$. In fact, if A is the event to pick a blue second ball and B is the event to pick a blue first ball, then the chance that A happens, given that B has happened, is identical to the $\frac{P(A\cap B)}{P(B)}$.*

Definition 11.2. *In a finite probability space $(\Omega, \mathcal{P}(\Omega), P)$ let $A, B \subseteq \Omega$ such that $P(B) \neq 0$. Then the* **conditional probability** *of A given B is defined as*

$$P(A \mid B) := \frac{P(A\cap B)}{P(B)}.$$

The idea is that $P(A \mid B)$ gives the probability that event A occurs, given that we know that B has occurred.

Example 11.3. *We roll a fair die twice, with respective results X_1 and X_2. Assuming that $X_1+X_2 = 8$, what is the probability of there having been thrown (i) a one, (ii) a two and (iii) a four? Let $\Omega = \{(X_1, X_2) : X_1, X_2 \in \{1, \ldots, 6\}\}$. Also let $B = \{(X_1, X_2) \in \Omega : X_1 + X_2 = 8\}$. Then*

$$B = \{(2, 6), (3, 5), (4, 4), (5, 3), (6, 2)\},$$

and $P(B) = \frac{5}{36}$. Next, for $i = 1, \ldots, 6$, et A_i be the event that the number i is thrown at least once. Then

$$A_1 \cap B = \emptyset, \quad A_2 \cap B = \{(2, 6), (6, 2)\} \quad A_4 \cap B = \{(4, 4)\}.$$

Hence

$$P(A_1 \mid B) = \frac{P(A_1 \cap B)}{P(B)} = \frac{0}{\frac{5}{36}} = 0$$
$$P(A_2 \mid B) = \frac{P(A_2 \cap B)}{P(B)} = \frac{\frac{2}{36}}{\frac{5}{36}} = \frac{2}{5}$$
$$P(A_4 \mid B) = \frac{P(A_4 \cap B)}{P(B)} = \frac{\frac{1}{36}}{\frac{5}{36}} = \frac{1}{5}$$

Theorem 11.4 (Law of Total Probability)**.** *In a finite probability space $(\Omega, \mathcal{P}(\Omega), P)$ let $B_1, \ldots, B_n$ be disjoint events with $P(B_i) > 0$, for all $i = 1, \ldots, n$ and*

$\Omega = B_1 \cup \cdots \cup B_n$. *Then for any event A we have*

$$P(A) = \sum_{i=1}^{n} P(A \mid B_i) P(B_i).$$

Proof. Note that $A = (A \cap B_1) \cup \cdots \cup (A \cap B_n)$ is a union or pairwise disjoint sets, and so

$$P(A) = \sum_{i=1}^{n} P(A \cap B_i) = \sum_{i=1}^{n} P(A \mid B_i) P(B_i).$$

$\square$

Example 11.5. *(1) In a game a player X throws a fair die and receives a bag with 100 marbles where r are red and the rest blue. If X throws 1, 2 or 3, then $r = 45$, if X throws 4 or 5, then $r = 60$. If X throws 6, then $r = 75$. Now X picks a marble at random from the bag. The player wins if the marble is red. What is the chance of winning?*

Let R be the event that the chosen marble is red and let B_r be the event that the bag contains r red marbles. Then $P(R \mid B_r) = \frac{r}{100}$. Overall

$$P(R) = \sum_{r \in \{45,60,75\}} P(R \mid B_r) P(B_r) = \frac{45}{100} \cdot \frac{1}{2} + \frac{60}{100} \cdot \frac{1}{3} + \frac{75}{100} \cdot \frac{1}{6} = 0.55$$

(2) We consider families with two children (girl and/or boy) and assume all outcomes $\{GG, GB, BG, BB\}$ are equally likely. What is the probability that both children are girls given that (a) the first born is a girl, (b) the family has a girl? Let $A = \{GG\}$, $B = \{GG, GB\}$ and $C = \{GG, GB, BG\}$. Then

$$P(A) = \frac{1}{4}, \qquad P(B) = \frac{1}{2}, \qquad P(C) = \frac{3}{4}$$

(a) We want $P(A \mid B) = \frac{P(A \cap B)}{P(B)} = \frac{P(A)}{P(B)} = \frac{\frac{1}{4}}{\frac{1}{2}} = \frac{1}{2}$.

(b) We want $P(A \mid C) = \frac{P(A \cap C)}{P(C)} = \frac{P(A)}{P(C)} = \frac{\frac{1}{4}}{\frac{3}{4}} = \frac{1}{3}$.

Theorem 11.6 (Bayes' Rule)**.** *In a finite probability space $(\Omega, \mathcal{P}(\Omega), P)$ let A and B be events such that $P(A) \neq 0$. Then*

$$P(B \mid A) = \frac{P(A \mid B) \cdot P(B)}{P(A)}$$

Example 11.7. *(1) We continue with Example 11.5(1). What is the probability that bag B_r was chosen, given that the player won, that is, picked a red marble? We ask for $P(B_r \mid R)$. We have*

$$P(B_r \mid R) = \frac{P(R \mid B_r) \cdot P(B_r)}{P(R)} = \frac{\frac{r}{100}}{0.55} \cdot P(B_r) = \frac{r}{55} \cdot P(B_r)$$

Hence

$$P(B_{45} \mid R) = 0.4091, \qquad P(B_{60} \mid R) = 0.3636, \qquad P(B_{75} \mid R) = 0.2273$$

(2) We continue with Example 11.5(2). Let us assume that girls are called Lara with a very small probability $\alpha << 0.1$. Assume that a family has a girl named Lara. What is the probability of the family having two girls. Let L be the event of there being a child named Lara. We want $P(GG \mid L)$. Note that

$$\begin{aligned} &P(L \mid BB) = 0 \\ &P(L \mid GB) = P(L \mid BG) = \alpha \\ &P(L \mid GG) = \alpha + (1-\alpha)\alpha = 2\alpha - \alpha^2 \end{aligned}$$

By the Law of Total Probability we have

$$\begin{aligned} P(L) &= (P(L \mid BB) + P(L \mid GB) + P(L \mid BG) + P(L \mid GG)) \cdot \frac{1}{4} \\ &= (4\alpha - \alpha^2) \cdot \frac{1}{4}. \end{aligned}$$

Now by Bayes' Rule

$$P(GG \mid L) = \frac{P(L \mid GG) \cdot P(GG)}{P(L)} = \frac{(2\alpha - \alpha^2) \cdot \frac{1}{4}}{(4\alpha - \alpha^2) \cdot \frac{1}{4}} = \frac{2-\alpha}{4-\alpha} \approx \frac{1}{2}$$

12. Independence

Definition 12.1. *Let $(\Omega, \mathscr{E}, P)$ be a probability space. We call two events A, B* **independent** *if*

$$P(A \cap B) = P(A) \cdot P(B).$$

Remark 12.2. *Note that A, B are independent if and only if $P(A) = P(A \mid B)$ and $P(B) = P(B \mid A)$.*

Example 12.3. *(1) Suppose we pick at random a number n from the set $\{1, \ldots, 10\}$, that is, each outcome is equally likely. Let A be the event that n is less than 7 and let B be the event that n is even. Then*

$$A = \{1, 2, 3, 4, 5, 6\}, \quad B = \{2, 4, 6, 8, 10\} \text{ and } A \cap B = \{2, 4, 6\},$$

and so

$$P(A) = \frac{6}{10} = 0.6, \quad P(B) = \frac{1}{2} = 0.5 \text{ and } P(A \cap B) = \frac{3}{10} = 0.3.$$

Hence $P(A \cap B) = P(A) \cdot P(B)$ and so A and B are independent.

However if we let A be the event that n is less than 8, then $P(A) = 0.7$ and A and B are dependent, because $P(A \mid B) = \frac{P(A \cap B)}{P(B)} = 0.6$.

(2) A fair coin is tossed twice. The outcome of the two tosses are independent. However, next let two equally good snooker players X and Y face each other in a match. We assume the first frame is a $50/50$ affair. But since losing a frame increases the pressure, the loser's chances of winning the next frame drop by one percent. Next let A be the event that player X wins the first game and B the event that player X wins the second game. Then

$$P(A) = \frac{1}{2}, \text{ and } P(B) = \frac{1}{2} \cdot \frac{49}{100} + \frac{1}{2} \cdot \frac{51}{100} = \frac{1}{2},$$

and so $P(A) \cdot P(B) = \frac{1}{4}$. However

$$P(A \cap B) = \frac{1}{2} \cdot \frac{51}{100} = \frac{51}{200} = \frac{1}{4} + \frac{1}{200}.$$

Hence the outcome of successive games is dependent. Also note that

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)} = \frac{1}{2} + \frac{1}{100} = \frac{P(A \cap B)}{P(A)} = P(B \mid A).$$