MT234P – MULTIVARIABLE CALCULUS – 2022

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Lecture 19

Example 12.

Find $\int_C (x^2+y^2-2z)\,ds$, where C is the curve with parametric equations

$$x=t,\ y=-3t,\ z=2t, \quad \text{for} \quad t\in[0,1]$$

Solution.

Let $f(x,y,z)=x^2+y^2-2z$. Now

$$x(t)=t \Rightarrow x'(t)=1$$

$$y(t)=-3t \Rightarrow y'(t)=-3$$

$$z(t)=2t \Rightarrow z'(t)=2$$

So,

$$\sqrt{(x'(t))^2+(y'(t))^2+(z'(t))^2}$$

$$=\sqrt{1+9+4}$$

$$=\sqrt{14}$$

Hence,

$$\int_C (x^2+y^2-2z)\,ds$$

$$=\int_0^1 (t^2+9t^2-4t)\sqrt{14}\,dt$$

$$=\sqrt{14}\int_0^1 (10t^2-4t)\,dt$$

$$= \sqrt{14}\left[\frac{10t^3}{3} - 2t^2\right]_0^1$$

$$= \sqrt{14}\left(\frac{4}{3}\right)$$

$$= \frac{4\sqrt{14}}{3}$$

Example 13.

Find $\int_C (x + \sqrt{y} - z^2)\, ds$, where C is the curve with parametric equations

$$x = t,\ y = t^2,\ z = 0, \quad \text{for} \quad t \in [0, 1]$$

Solution.

Let $f(x, y, z) = x + \sqrt{y} - z^2$. Now,

$$x(t) = t \Rightarrow x'(t) = 1$$

$$y(t) = t^2 \Rightarrow y'(t) = 2t$$

$$z(t) = 0 \Rightarrow z'(t) = 0$$

So,

$$\sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}$$

$$= \sqrt{1 + 4t^2}$$

Hence,

$$\int_C (x + \sqrt{y} - z^2)\, ds$$

$$= \int_0^1 (t + \sqrt{t^2} - 0)\sqrt{1 + 4t^2}\, dt$$

$$= 2\int_0^1 t\sqrt{1 + 4t^2}\, dt \qquad (*)$$

Use the substitution rule on $(*)$ with $u = 1 + 4t^2$ to get

$$
\begin{aligned}
(*) &= \frac{1}{4}\int_1^5 \sqrt{u}\,du \\
&= \frac{1}{4}\left[\frac{2}{3}u^{\frac{3}{2}}\right]_1^5 \\
&= \frac{1}{4}\left[\frac{2}{3}5^{\frac{3}{2}} - \frac{2}{3}\right] \\
&= \frac{1}{6}(5\sqrt{5}-1)
\end{aligned}
$$

Section 4.4 – Surface Integrals.

Remark 13.

We have already met a variety of different types of integrals in previous lectures. Now we will meet what is called a surface integral which will relate to integrating a function over a surface. All these types of integrals have many important and powerful applications in Science, Engineering and other areas.

We will now discuss how to integrate a function over surface. To motivate the idea we can consider an electric charge distributed over a surface S given by $z = f(x,y)$. Now suppose that the function $h(x,y,z)$ gives the electric charge per unit area (i.e. charge density) at each point (x,y,z) on S. Then, the total charge on S can be calculated in the following way:

Suppose R is the vertical projection of S onto the xy-plane (i.e. $R = \{(x,y,0) : (x,y,z) \in S\}$). We partition R into small rectangles A_k, $1 \leq k \leq n$, like we did before in the definition of a double integral. Denote the area of A_k by ΔA_k. Directly above A_k lies a patch B_k of S with area ΔB_k and this patch can be approximated with a parallelogram shaped piece E_k of the tangent plane that has area ΔE_k. One can show that ΔE_k can be approximated by

$$\sqrt{1 + f_x^2(x_k, y_k) + f_y^2(x_k, y_k)}\Delta A_k$$

where $(x_k, y_k, 0)$ is a point in A_k. The total charge over B_k can then be approximated by

$$h(x_k, y_k, z_k)\sqrt{1 + f_x^2(x_k, y_k) + f_y^2(x_k, y_k)}\Delta A_k$$

The total charge over S can then be approximated by

$$\sum_{k=1}^{n} h(x_k, y_k, z_k)\sqrt{1 + f_x^2(x_k, y_k) + f_y^2(x_k, y_k)}\Delta A_k \qquad (*)$$

The approximation gets better as the rectangles A_k get smaller. So, we now define the surface integral of h over S to be the limit of $(*)$ as the length of the longest diagonal of the n rectangles A_k goes to zero. We denote the surface integral of h over S by

$$\int_S \int h(x, y, z)\, dS$$

The limit above is also the double integral

$$\int_R \int h(x, y, f(x, y))\sqrt{1 + f_x^2 + f_y^2}\, dA$$

This motivates the following theorem which is useful for calculating surface integrals.

Theorem 6.

Suppose S is a surface given by $z = f(x, y)$ and suppose R is the vertical projection of S onto the xy-plane as above. If f, f_x, f_y are continuous on R and $g(x, y, z)$ is continuous on S, then the surface integral

$$\int_S \int g(x, y, z)\, dS$$

is given by

$$\int_S \int g(x, y, z)\, dS = \int_R \int g(x, y, f(x, y))\sqrt{1 + f_x^2 + f_y^2}\, dA$$

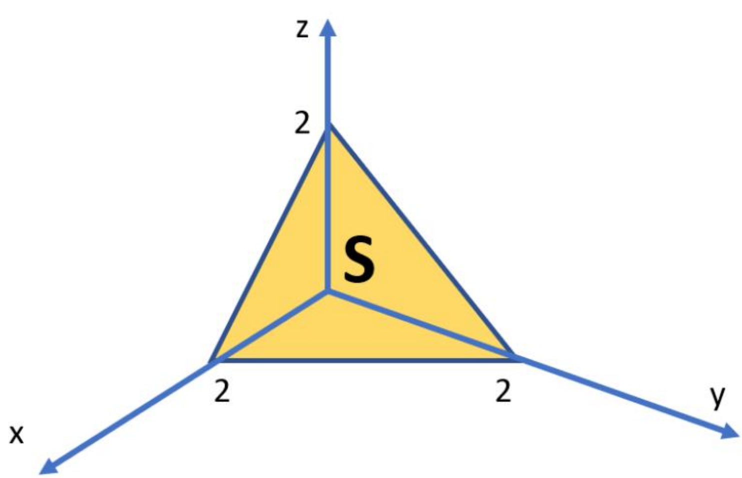
where the integral over R is a double integral as before.

Example 14.

Find the surface integral, $\int_S \int (xy + z)\, dS$, where S is that part of the plane $x + y + z = 2$ in the first octant (i.e. $x \geq 0, y \geq 0, z \geq 0$).

Solution.

See the picture for S below.



We have that $z = 2 - x - y$ and so we let $f(x, y) = 2 - x - y$ and $g(x, y, z) = xy + z$.