MT251P – Lecture 4

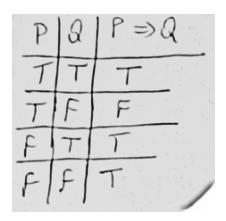
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Implication and Conditional Statements.

Conditional statements of the form 'If P, then Q' are constructed using the implication operation \Rightarrow

Note that P implies Q (i.e. $P \Rightarrow Q$) is false if and only if P is true and Q is false.

The truth table for $P \Rightarrow Q$ is



Check that $P \Rightarrow Q$ is equivalent to $(\neg P) \lor Q$.

Remark 9.

The 'if and only if' statement is denoted by $P \iff Q$ and can now be defined as

$$P \iff Q = (P \Rightarrow Q) \land (Q \Rightarrow P)$$

Remark 10.

The operation \neg has precedence over \land which has precedence over \lor . For example

$$\neg P \wedge Q \ \text{means} \ (\neg P) \wedge Q \ \text{and doesn't mean} \ \neg (P \wedge Q)$$

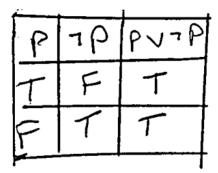
Check that $(\neg P) \land Q$ is not equivalent to $\neg (P \land Q)$ by showing they don't have identical truth tables.

Tautology.

A formula is a tautology if it takes the value T for all possible values of its variables.

Example 14.

 $P \vee \neg P$ is a tautology because its truth table is:



Example 15.

Suppose P,Q,R are three propositions. Then

$$(P \wedge Q) \wedge R \sim P \wedge (Q \wedge R) \tag{*}$$

Proof:

P	Q	R	PAQ	(PAQ)AR
1	T	1	+	T
T	T	F	ŀ	F
F	F	T	L)	۲
<u> </u>	۴	F	F	F
F	T	T,	F	F
NF.	T	F	F	F
ع	F	\vdash	F	F
16	F	F	F	F

P	Q	R	@ NR	PA(QAR)
T	1	1	۲	T
T	7	L		F
T	F	T	F	K
5	F	1	Д	L
F	T	T	T	Ц
۶	T	T	Ţ	u
F	F	T	F	F
۶	P	F	P	٦

Remark 11.

One can show that the following equivalences hold:

Idempotency: $P \lor P \sim P$, $P \land P \sim P$

Associativity: $P \lor (Q \lor R) \sim (P \lor Q) \lor R$, $P \land (Q \land R) \sim (P \land Q) \land R$,

Commutativity: $P \lor Q \sim Q \lor P$, $P \land Q \sim Q \land P$

Distributivity: $P \wedge (Q \vee R) \sim (P \wedge Q) \vee (P \wedge R), \quad P \vee (Q \wedge R) \sim (P \vee Q) \wedge (P \vee R)$

Identity: Here we consider T to denote a formula which always has the value T and we consider F to denote a formula which always has the value F. Then, we have $P \vee F \sim P$, $P \vee T \sim T$, $P \wedge F \sim F$, $P \wedge T \sim P$

Complements: $P \vee \neg P \sim T$, $P \wedge \neg P \sim F$

Involution: $\neg(\neg P) \sim P$

De Morgan's Laws: $\neg(P \lor Q) \sim \neg P \land \neg Q$, $\neg(P \land Q) \sim \neg P \lor \neg Q$

Remark 12.

The equivalences in remark 11 can be used to prove other equivalences and to simplify propositional expressions by using the following two rules:

- (i) If we replace a variable in two equivalent formulas by the same arbitrary formula in both, then we will end up with two equivalent formulas again.
- (ii) A subformula within a formula can be replaced by an equivalent formula.