Fiacre Ó Cairbre

Lecture 7

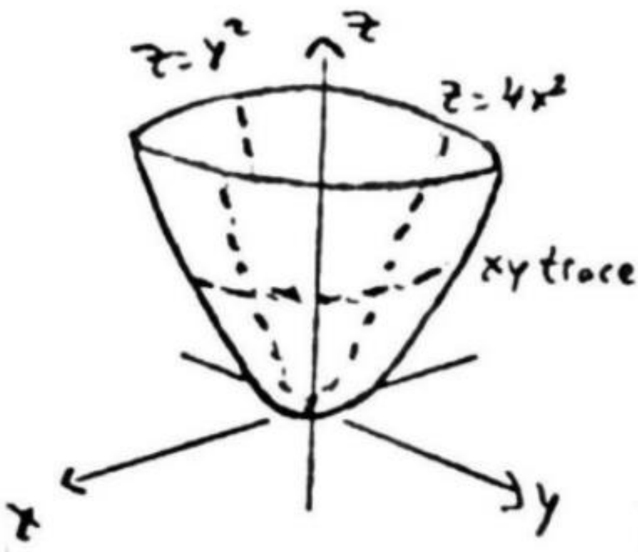
Example 27.

Sketch the surface given by $z - y^2 - 4x^2 = 0$.

We first write the equation in the form

$$z = 4x^2 + y^2$$

and so we have an elliptic paraboloid with the picture below. Note that the trace in the $z = 4$ plane (parallel to the xy–plane) is the ellipse $x^2 + \frac{y^2}{4} = 1$. The trace in the xz–plane is the parabola $z = 4x^2$. The trace in the yz–plane is the parabola $z = y^2$. See the picture below.



Example 28 – Ellipsoid.

The equation is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

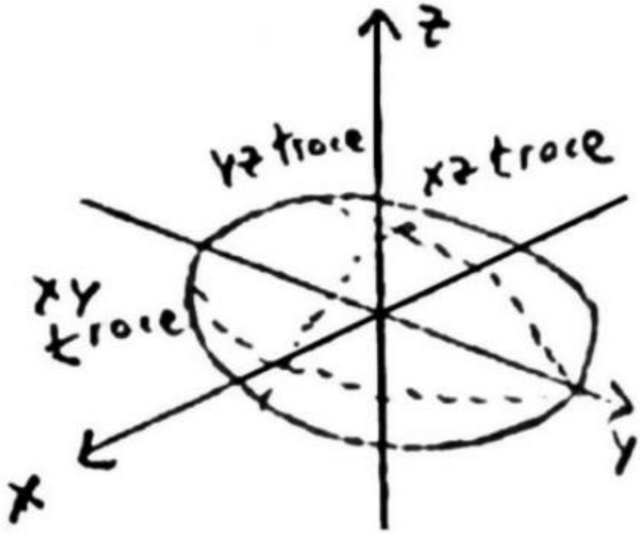
where $a, b, c \in \mathbb{R}$. The relevant traces in a plane

parallel to the xy–plane are ellipses

parallel to the xz–plane are ellipses

parallel to the yz–plane are ellipses

See the picture below.



Example 29 – Elliptic Cone.

The equation is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0$$

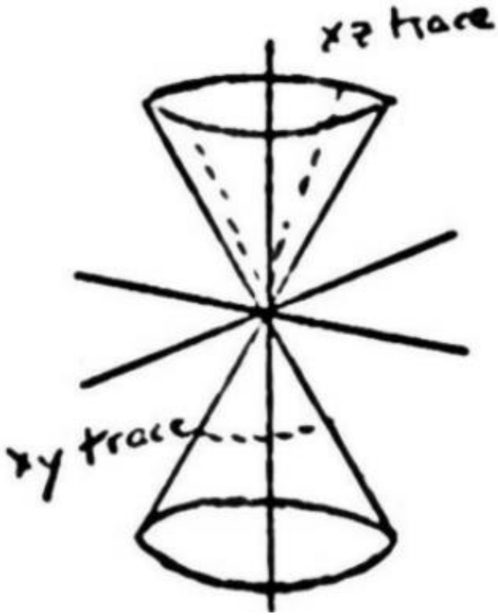
where $a, b, c \in \mathbb{R}$. The relevant traces in a plane

parallel to the xy–plane are ellipses

parallel to the xz–plane are hyperbolas

parallel to the yz–plane are hyperbolas

See the picture below.



Example 30 – Hyperbolic Paraboloid.

The equation is

$$z = \frac{y^2}{b^2} - \frac{x^2}{a^2}$$

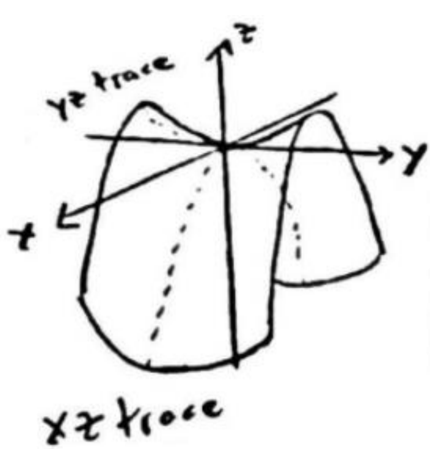
where $a, b \in \mathbb{R}$. The relevant traces in a plane

parallel to the xy–plane are hyperbolas

parallel to the xz–plane are parabolas

parallel to the yz–plane are parabolas

See the picture below.



Example 31 – Hyperboloid of one sheet.

The equation is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$$

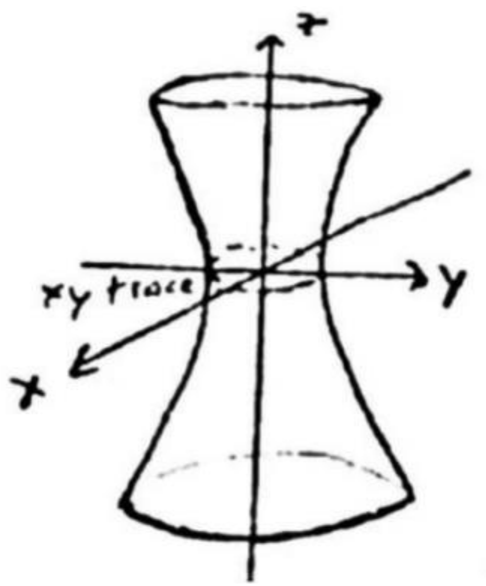
where $a, b, c \in \mathbb{R}$. The relevant traces in a plane

parallel to the xy–plane are ellipses

parallel to the xz–plane are hyperbolas

parallel to the yz–plane are hyperbolas

See the picture below.



Example 32 – Hyperboloid of two sheets.

The equation is

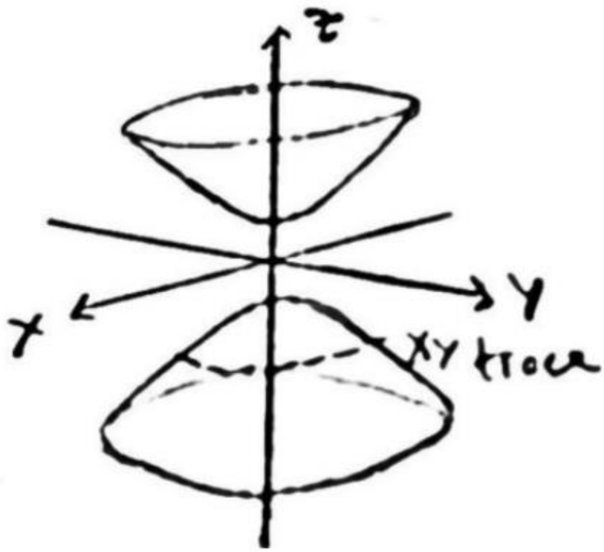
$$\frac{z^2}{c^2} - \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

where $a, b, c \in \mathbb{R}$. The relevant traces in a plane

parallel to the xy–plane are ellipses

parallel to the xz–plane are hyperbolas parallel to the yz–plane are hyperbolas

See the picture below.



Section 1.6 – Chain Rule.

Remark 15.

We will first discuss some notation. Suppose $w = f(x)$ is a differentiable function of one variable. Then we can denote the derivative $w'(x)$ by $\dfrac{dw}{dx}$.

Recall the chain rule for functions of one variable which states that if $w = f(x)$ is a differentiable function and $x = g(t)$ is a differentiable function of one variable, then $w(t) = (f \circ g)(t)$ is the composition of f after g and $w(t)$ is differentiable and

$$\frac{dw}{dt} = \frac{dw}{dx}\frac{dx}{dt}$$

We will now discuss various chain rules for functions of more than one variable.

Theorem 8 – Chain Rule.

Suppose $w = f(x, y)$ is a differentiable function of two variables and also suppose x and y are both differentiable functions of the one variable t. Then w is a differentiable function of the one variable t and

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt}$$

Example 33.

If $w = \ln(x^2 + y^2)$, $x = e^{-t}$, $y = e^t$, then

$$(i) \qquad \frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt}$$

$$= \frac{2x}{x^2+y^2}(-e^{-t}) + \frac{2y}{x^2+y^2}(e^t)$$

$$= \frac{2(e^{2t}-e^{-2t})}{e^{2t}+e^{-2t}}$$

(ii) When $t = 1$ we get $\dfrac{dw}{dt} = \dfrac{2(e^2-e^{-2})}{e^2+e^{-2}}$

Theorem 9 – Chain Rule for three variables.

Suppose $w = f(x, y, z)$ is a differentiable function of three variables and also suppose x, y and z are all differentiable functions of the one variable t. Then w is a differentiable function of the one variable t and

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}$$

Example 34.

If $w = 2xy + z$ and $x = \cos t$, $y = \sin t$, $z = t$, then

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}$$

$$= -2y(\sin t) + 2x(\cos t) + 1$$

$$= -2\sin^2 t + 2\cos^2 t + 1$$

Theorem 10 – Another Chain Rule.

Suppose $w = f(x, y)$ is a differentiable function of two variables and also suppose $x = g(r, s)$, $y = h(r, s)$ are both differentiable functions of the two variables r, s. Then w is a differentiable function of the two variables r, s and

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r}$$

and

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s}$$

Example 35.

If $w = x^2 + y^2$ and $x = 2r - 3s$, $y = r + 5s$, then

$$(i) \quad \frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r} = 2x(2) + 2y(1) = 10r - 2s$$

$$(ii) \quad \frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} = 2x(-3) + 2y(5) = -2r + 68s$$