MT251P – Lecture 17

Continuation of example 2.

$$\begin{pmatrix} 1 & 0 & -1 \\ 3 & 0 & 2 \\ 0 & 1 & 1 \end{pmatrix} \qquad \begin{pmatrix} 0 & \frac{1}{2} & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Replace R_2 with $R_2 - 3R_1$ (in both matrices) to get

$$\begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & 5 \\ 0 & 1 & 1 \end{pmatrix} \qquad \begin{pmatrix} 0 & \frac{1}{2} & 0 \\ 1 & -\frac{3}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Interchange R_3 with R_2 (in both matrices) to get

$$\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 5 \end{pmatrix} \qquad \begin{pmatrix} 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \\ 1 & -\frac{3}{2} & 0 \end{pmatrix}$$

Replace R_1 with $R_1 + \frac{1}{5}R_3$ and replace R_2 with $R_2 - \frac{1}{5}R_3$ (in both matrices) to get

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{pmatrix} \qquad \begin{pmatrix} 0.2 & 0.2 & 0 \\ -0.2 & 0.3 & 1 \\ 1 & -\frac{3}{2} & 0 \end{pmatrix}$$

Replace R_3 with $\frac{1}{5}R_3$ (in both matrices) to get

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} 0.2 & 0.2 & 0 \\ -0.2 & 0.3 & 1 \\ 0.2 & -0.3 & 0 \end{pmatrix}$$

So, the RREF of $A = I_3$ and hence A is invertible. So, A^{-1} exists and

$$A^{-1} = \begin{pmatrix} 0.2 & 0.2 & 0 \\ -0.2 & 0.3 & 1 \\ 0.2 & -0.3 & 0 \end{pmatrix}$$

Theorem 4.

Suppose $A,\ B \in M_{n\times n}(\mathbb{R})$. Then

$$AB = I_n \iff BA = I_n$$

Remark 3.

A consequence of theorem 4 is that if you want to check that $C = A^{-1}$, then you only need to check that $AC = I_n$ and you don't need to check that $CA = I_n$.

Section 5.2 – The Determinant of a Matrix.

Definition 3.

Suppose A is an $n \times n$ matrix. The determinant of A is denoted by $\det(A)$ and is defined as follows:

If $A = (a_{11})$ is a 1×1 matrix, then $\det(A) = a_{11}$.

If $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ is a 2×2 matrix then $\det(A) = a_{11}a_{22} - a_{12}a_{21}$

Now, assume that the determinant has been defined for all $(n-1) \times (n-1)$ matrices and let $A = [a_{ij}]$ be an $n \times n$ matrix with a_{ij} in the i^{th} row and j^{th} column of A.

How do we find $\det(A)$?

Let $D(i,j)$ be the $(n-1) \times (n-1)$ matrix obtained by deleting row i and column j of A.

Let $M_{i,j} = \det D(i,j)$

Let $C_{i,j} = (-1)^{i+j} M_{i,j}$.

$C_{i,j}$ is called the ij cofactor of A.

Then we define

$$\det(A) = \sum_{j=1}^{n} a_{1j} C_{1j}$$

Example 3.

Suppose

$$A = \begin{pmatrix} 4 & 1 & 9 \\ 3 & 1 & -2 \\ -1 & 1 & 3 \end{pmatrix}$$

Find $\det(A)$.

Solution.

$$C_{11} = M_{11} = \det D(1,1) = \det \begin{pmatrix} 1 & -2 \\ 1 & 3 \end{pmatrix} = 5$$

$$C_{12} = -M_{12} = -\det D(1,2) = -\det \begin{pmatrix} 3 & -2 \\ -1 & 3 \end{pmatrix} = -7$$

$$C_{13} = M_{13} = \det D(1,3) = \det \begin{pmatrix} 3 & 1 \\ -1 & 1 \end{pmatrix} = 4$$

So,

$$\det(A) = \sum_{j=1}^{3} a_{1j}C_{1j} = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$$

$$= 4C_{11} + C_{12} + 9C_{13}$$

$$= 20 - 7 + 36 = 49$$

Theorem 5.

Suppose $A = [a_{ij}]$ is an $n \times n$ matrix with a_{ij} in the i^{th} row and j^{th} column of A. Then

(i) $\det A = \sum_{j=1}^{n} a_{ij}C_{ij}$, for any i satisfying $1 \le i \le n$ (this is called expansion along the i^{th} row).

Also, $\det A = \sum_{i=1}^{n} a_{ij}C_{ij}$, for any j satisfying $1 \le j \le n$ (this is called expansion along the j^{th} column).

(ii) $\det A = \det A^T$

(iii) If A is upper triangular (i.e. $a_{ij} = 0$ whenever $i > j$), then $\det A = a_{11}a_{22}a_{33}\ldots a_{nn}$.

If A is lower triangular (i.e. $a_{ij} = 0$ whenever $i < j$), then $\det A = a_{11}a_{22}a_{33}\ldots a_{nn}$.

(iv) If B is the matrix obtained from A by multiplying one row of A by a scalar s, then $\det B = s \det A$.

(v) If B is the matrix obtained from A by swapping two rows of A, then $\det B = -\det A$.

(vi) If two rows of A are identical, then $\det A = 0$.

(vii) If A has a zero row, then $\det A = 0$

(viii) If B is the matrix obtained from A by adding s times one row of A to another row of A, then $\det B = \det A$.

(ix) Properties (iv), (v), (vi), (vii) and (viii) remain valid if the word, row, is replaced everywhere by the word, column,

(x) $\det(AB) = \det A \det B$, where B is any $n \times n$ matrix. This is a very useful property of determinants.