**MT234P – MULTIVARIABLE CALCULUS – 2022**

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**Lecture 8**

## Chapter 2 – Directional Derivatives, Gradient and Tangent Planes.

### Section 2.1 – Directional Derivatives and Gradient.

**Remark 1.**

Suppose $f(x, y, z)$ is a differentiable function on some set $D$ in $\mathbb{R}^3$. Suppose $(x_0, y_0, z_0)$ is a point in $D$ and suppose $\underline{u} = u_1\vec{i} + u_2\vec{j} + u_3\vec{k}$ is a unit vector, i.e. a vector of length one.

Then, recall that the line through $(x_0, y_0, z_0)$ parallel to $\underline{u}$ is given by

$$M = \{(x_0, y_0, z_0) + t(u_1, u_2, u_3) \ : \ t \in \mathbb{R}\}$$

So, $(x, y, z)$ is on the line through $(x_0, y_0, z_0)$ parallel to $\underline{u} \iff$

$$x = x_0 + tu_1, \quad y = y_0 + tu_2, \quad z = z_0 + tu_3, \quad t \in \mathbb{R} \qquad (*)$$

The equations in $(*)$ are called the parametric equations for the line through $(x_0, y_0, z_0)$ parallel to $\underline{u}$.

Now, the parameter $t$ in $(*)$ measures the 'directed distance' along the line $M$ from $(x_0, y_0, z_0)$ to $(x_0 + tu_1, y_0 + tu_2, z_0 + tu_3)$

We wish to find $\frac{df}{dt}$ which is the rate at which $f$ changes with respect to distance in the direction of $\underline{u}$ at $(x_0, y_0, z_0)$.

Now, by the chain rule (theorem 9 in chapter 1) we have

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}$$

$$= \frac{\partial f}{\partial x}u_1 + \frac{\partial f}{\partial y}u_2 + \frac{\partial f}{\partial z}u_3$$

$$= (\frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}j + \frac{\partial f}{\partial z}\vec{k}).\underline{u} \qquad (**)$$

**Definition 1.**

The gradient of $f(x, y, z)$ is denoted by $\nabla f$ and is defined as

$$\nabla f = \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j} + \frac{\partial f}{\partial z}\vec{k}$$

We also denote the value of $\nabla f$ at the point $(x_0, y_0, z_0)$ by $\nabla f_{|(x_0,y_0,z_0)}$.

**Remark 2.**

Remark 1(**) now says that $\frac{df}{dt} = \nabla f.\underline{u}$ and so at $(x_0, y_0, z_0)$ we get that

$$\frac{df}{dt}_{|(x_0,y_0,z_0)} = \nabla f_{|(x_0,y_0,z_0)}.\underline{u} \qquad (***)$$

where $\frac{df}{dt}_{|(x_0,y_0,z_0)}$ denotes the value of $\frac{df}{dt}$ at the point $(x_0, y_0, z_0)$.

**Definition 2.**

The derivative of $f(x, y, z)$ at $(x_0, y_0, z_0)$ in the direction of the unit vector $\underline{u}$ is denoted by $(D_{\underline{u}}f)_{|(x_0,y_0,z_0)}$ and is defined as $(***)$ in remark 2, and so

$$(D_{\underline{u}}f)_{|(x_0,y_0,z_0)} = \nabla f_{|(x_0,y_0,z_0)}.\underline{u} \qquad (i)$$

This derivative of $f$ at $(x_0, y_0, z_0)$ in the direction of $\underline{u}$ is also called the directional derivative of $f$ at $(x_0, y_0, z_0)$ in the direction of $\underline{u}$.

If $\underline{w}$ is not a unit vector, then the directional derivative of $f$ at $(x_0, y_0, z_0)$ in the direction of $\underline{w}$ is $\nabla f_{|(x_0,y_0,z_0)}.\underline{u}$, where $\underline{u}$ is the unit vector in the direction of $\underline{w}$, i.e. $\underline{u} = \frac{\underline{w}}{||\underline{w}||}$, where $||\underline{w}||$ denotes the length of $\underline{w}$.

If we wish to consider the directional derivative as a function, then we can denote it by $D_{\underline{u}}f$ and its value at $(x_0, y_0, z_0)$ is $(D_{\underline{u}}f)_{|(x_0,y_0,z_0)}$

**Example 1.**

Find the directional derivative of $f(x, y, z) = 3e^x \cos yz$ at $(0, 0, 0)$ in the direction of $\underline{w} = -2\vec{i} + \vec{j} + 2\vec{k}$.

**Solution.**

The unit vector $\underline{u}$ in the direction of $\underline{w}$ is

$$\underline{u} = \frac{\underline{w}}{||\underline{w}||} = \frac{-2\vec{i} + \vec{j} + 2\vec{k}}{\sqrt{9}} = -\frac{2}{3}\vec{i} + \frac{1}{3}\vec{j} + \frac{2}{3}\vec{k}$$

Note that

$$\frac{\partial f}{\partial x}_{|(0,0,0)} = 3e^x \cos yz_{|(0,0,0)} = 3$$

$$\frac{\partial f}{\partial y}_{|(0,0,0)} = -3e^x z \sin yz_{|(0,0,0)} = 0$$

$$\frac{\partial f}{\partial z}_{|(0,0,0)} = -3e^x y \sin yz_{|(0,0,0)} = 0$$

So, the directional derivative of $f$ at $(0,0,0)$ in the direction of $\underline{w}$ is $\nabla f_{|(0,0,0)}.\underline{u}$

$$= \frac{\partial f}{\partial x}_{|(0,0,0)}(-\frac{2}{3}) + \frac{\partial f}{\partial y}_{|(0,0,0)}(\frac{1}{3}) + \frac{\partial f}{\partial y}_{|(0,0,0)}(\frac{2}{3})$$

$$= -2$$

**Remark 3.**

Definition 2(i) implies

$$(D_{\underline{u}}f)_{|(x_0,y_0,z_0)} = ||\nabla f||_{|(x_0,y_0,z_0)} \cos\theta$$

where $\theta$ is the angle between $\nabla f_{|(x_0,y_0,z_0)}$ and $\underline{u}$.

**Theorem 1.**

The following statements hold for the directional derivative:

(i) Using the notation in remark 3, we have that the directional derivative has its largest positive value when $\cos\theta = 1$, i.e. when $\underline{u}$ is in the direction of $\nabla f$. So, $f$ increases most rapidly, at any point in its domain, in the direction of $\nabla f$. In this case the directional derivative is $||\nabla f||$.

(ii) Using remark 3, we have that the directional derivative has its largest negative value when $\cos\theta = -1$, i.e. when $\underline{u}$ is in the direction of $-\nabla f$. So, $f$ decreases most rapidly, at any point in its domain, in the direction of $-\nabla f$. In this case the directional derivative is $-||\nabla f||$.

(iii) Any direction $\underline{u}$ that is perpendicular to $\nabla f$ is a direction of zero change in $f$ because in that case $\cos\theta = 0$ in remark 3.

**Example 2.**

Find the directions in which $f(x,y,z) = \ln(xy) + \ln(xz) + \ln(yz)$ increases and decreases most rapidly at $(1,1,1)$. Also, at what rate does $f$ change in these directions?

**Solution.**

First note that

$$\nabla f_{|(1,1,1)} = (\frac{2}{x}\vec{i} + \frac{2}{y}\vec{j} + \frac{2}{z}\vec{k})_{|(1,1,1)} = 2\vec{i} + 2\vec{j} + 2\vec{k}$$

Theorem 1(i) says that $f$ increases most rapidly in the direction of $\underline{u}$ where $\underline{u}$ is in the direction of $\nabla f$. So, $\underline{u}$ is the unit vector in the direction of $\nabla f$. Hence,

$$\underline{u} = \frac{1}{\sqrt{3}}\vec{i} + \frac{1}{\sqrt{3}}\vec{j} + \frac{1}{\sqrt{3}}\vec{k}$$

and so $f$ increases most rapidly in the direction of $\frac{1}{\sqrt{3}}\vec{i} + \frac{1}{\sqrt{3}}\vec{j} + \frac{1}{\sqrt{3}}\vec{k}$

Similarly, using theorem 1(ii) we get that $f$ decreases most rapidly in the direction of $-\frac{1}{\sqrt{3}}\vec{i} - \frac{1}{\sqrt{3}}\vec{j} - \frac{1}{\sqrt{3}}\vec{k}$

Now to find the rates at which $f$ changes in these two directions. Well, by theorem 1(i) and (ii) we get that the rates of change in the respective directions are $||\nabla f||$ and $-||\nabla f||$ and so we get $2\sqrt{3}$ and $-2\sqrt{3}$.

**Remark 4.**

So far in section 2.1 we have considered functions of three variables (starting with remark 1). We can do the same with functions of two variables. So, suppose $f(x, y)$ is a differentiable function on some set $D$ in $\mathbb{R}^2$ (like in the first line of remark 1).

Then, proceed as in remark 1 with the only difference being that we will have no $u_3\vec{k}$, no $z_0$ and no $z$ in $(*)$ in remark 1. Definition 1 will then be replaced by

**Definition 3.**

The gradient of $f(x, y)$ is denoted by $\nabla f$ and is defined as

$$\nabla f = \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j}$$

We also denote the value of $\nabla f$ at the point $(x_0, y_0)$ by $\nabla f_{|(x_0,y_0)}$.

Definition 2 will then be replaced by

**Definition 4.**

The derivative of $f(x, y)$ at $(x_0, y_0)$ in the direction of the unit vector $\underline{u}$ is denoted by $(D_{\underline{u}}f)_{|(x_0,y_0)}$ and is defined as

$$(D_{\underline{u}}f)_{|(x_0,y_0)} = \nabla f_{|(x_0,y_0)}.\underline{u}$$

This derivative of $f$ at $(x_0, y_0)$ in the direction of $\underline{u}$ is also called the directional derivative of $f$ at $(x_0, y_0)$ in the direction of $\underline{u}$.

If $\underline{w}$ is not a unit vector, then the directional derivative of $f$ at $(x_0, y_0)$ in the direction of $\underline{w}$ is $\nabla f_{|(x_0,y_0)}.\underline{u}$, where $\underline{u}$ is the unit vector in the direction of $\underline{w}$, i.e. $\underline{u} = \dfrac{\underline{w}}{||\underline{w}||}$, where $||\underline{w}||$ denotes the length of $\underline{w}$.

If we wish to consider the directional derivative as a function, then we can denote it by $D_{\underline{u}}f$ and its value at $(x_0, y_0)$ is $(D_{\underline{u}}f)_{|(x_0,y_0)}$

Theorem 1 will be replaced by

**Theorem 2.**

For $f(x, y)$, the following statements hold for the directional derivative:

(i) We have that the directional derivative has its largest positive value when $\cos\theta = 1$, i.e. when $\underline{u}$ is in the direction of $\nabla f$. So, $f$ increases most rapidly, at any point in its domain, in the direction of $\nabla f$. In this case the directional derivative is $||\nabla f||$.

(ii) Similarly, we have that the directional derivative has its largest negative value when $\cos\theta = -1$, i.e. when $\underline{u}$ is in the direction of $-\nabla f$. So, $f$ decreases most rapidly, at any point in its domain, in the direction of $-\nabla f$. In this case the directional derivative is $-||\nabla f||$.

(iii) Any direction $\underline{u}$ that is perpendicular to $\nabla f$ is a direction of zero change in $f$ because in that case $\cos\theta = 0$.