Example 14

Is $A = \begin{pmatrix} 2 & -1 & 3 \\ 1 & -2 & -3 \\ 5 & -4 & 3 \end{pmatrix}$ invertible?

Solution.

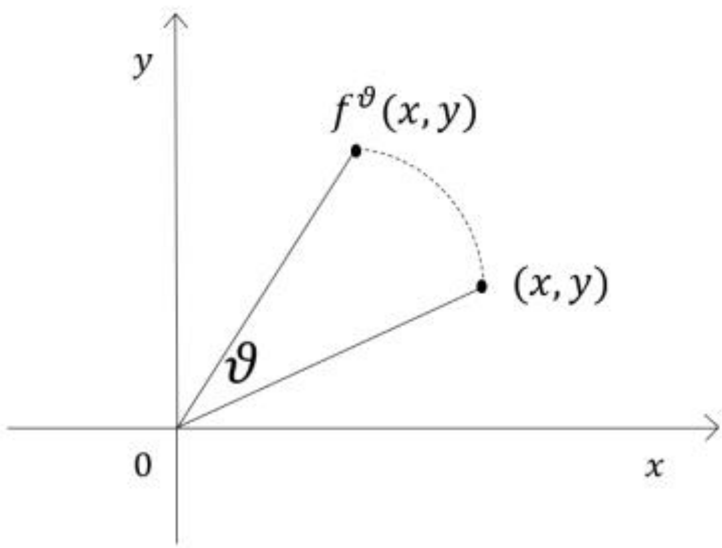
Note that from example 13 we have that rank $A = 2$ and so theorem 6 implies that A is not invertible.

Chapter 7 – Applications of Matrices to Geometry.

Section 7.1 – Rotations.

Remark 1.

Rotations appear in many important applications of mathematics. Consider the usual xy plane denoted by $\mathbb{R}^2 = \{(x, y) : x, y \in \mathbb{R}\}$ and consider the anti–clockwise rotation about the origin $(0, 0)$ through an angle $\theta \in [0, 2\pi)$. Denote this rotation by f^θ.



Theorem 1.

Using the notation in remark 1, we have that

$$f^\theta(x, y) = (x\cos\theta - y\sin\theta, x\sin\theta + y\cos\theta), \quad \text{for all} \quad (x, y) \in \mathbb{R}^2$$

Proof.

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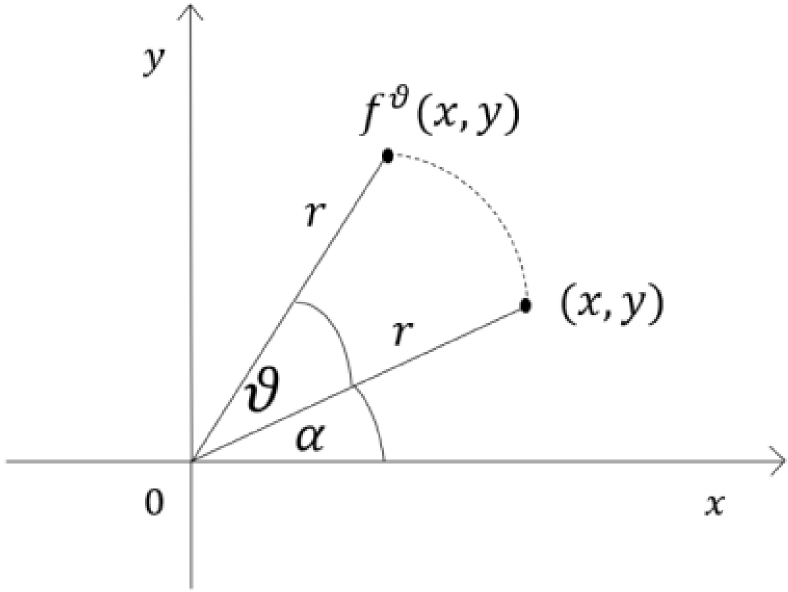


Let α be the angle that the line joining (x, y) to the origin, makes with the positive x-axis. Let r be the distance from (x, y) to the origin. Then,

$$x = r\cos\alpha \qquad \text{and} \qquad y = r\sin\alpha$$

Now,

$$f^{\theta}(x, y) = (a, b), \qquad \text{where} \quad a = r\cos(\alpha+\theta),\ b = r\sin(\alpha+\theta)$$

So,

$$a = r(\cos\alpha\cos\theta - \sin\alpha\sin\theta)$$

$$= x\cos\theta - y\sin\theta$$

and

$$b = r(\sin\alpha\cos\theta + \sin\theta\cos\alpha)$$

$$= y\cos\theta + x\sin\theta$$

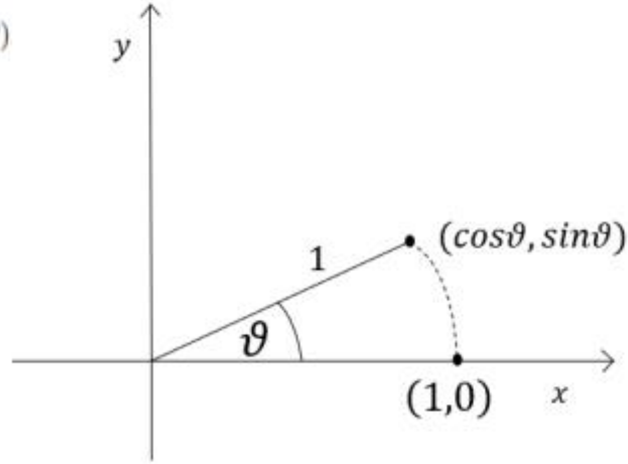
and so

$$f^{\theta}(x, y) = (x\cos\theta - y\sin\theta, x\sin\theta + y\cos\theta)$$

and we are done.

Example 1.

$f^{\theta}(1,0) = (\cos\theta, \sin\theta)$



Remark 2.

Consider the following 2×2 matrix

$$A_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

Notice that

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x\cos\theta - y\sin\theta \\ x\sin\theta + y\cos\theta \end{pmatrix}$$

and so

$$A_\theta \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x\cos\theta - y\sin\theta \\ x\sin\theta + y\cos\theta \end{pmatrix}$$

So, the rotation f^{θ} corresponds to multiplication on the left by the matrix A_θ. This shows how matrices arise naturally in rotations and it's an example of an application of matrices to geometry.

Remark 3.

Note that in theorem 1 we write the elements of $\mathbb{R}^2$ as row (or horizontal) vectors and in remark 2 we write the elements of $\mathbb{R}^2$ as column (or vertical) vectors.

Remark 4.

Many important problems in science, engineering, computer animation, special effects in movies, space navigation etc. involve rotations.

Example 2.

Suppose $\theta = \frac{\pi}{2}$. Then

$$A_\theta = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

So, the anti–clockwise rotation about the origin $(0, 0)$ through an angle $\frac{\pi}{2}$ corresponds to multiplication on the left by the matrix

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

Example 3.

Suppose $\theta = \frac{\pi}{4}$. Then

$$A_\theta = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

So, the anti–clockwise rotation about the origin $(0, 0)$ through an angle $\frac{\pi}{4}$ corresponds to multiplication on the left by the matrix

$$\begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$