MT234P – MULTIVARIABLE CALCULUS – 2022

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Lecture 3

Section 1.3 – Limits and Continuity.

Definition 10.

We say that a function $f(x, y)$ approaches the limit L as (x, y) approaches (a, b) and we write

$$\lim_{(x,y)\to(a,b)} f(x, y) = L$$

if for every $\epsilon > 0$, there exists a corresponding value $\delta > 0$ such that

$$0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta \ \Rightarrow\ |f(x, y) - L| < \epsilon$$

We say that L is the limit of $f(x, y)$ as (x, y) approaches (a, b).

Remark 2.

The intuition behind definition 10 is that L is the limit of $f(x, y)$ as (x, y) approaches (a, b) if we can make $f(x, y)$ as close as we like to L by taking (x, y) sufficiently close to (a, b) but not equal to (a, b).

Remark 3. Note that (a, b) in definition 10 can be any interior point of the domain of f or any boundary point of the domain of f. Also, note that a boundary point of the domain of f need not be in the domain of f. The points (x, y) that approach (a, b) in definition 10 have to be in the domain of f.

Theorem 1.

Suppose $\lim\limits_{(x,y)\to(a,b)} f(x, y) = L_1$ and $\lim\limits_{(x,y)\to(a,b)} g(x, y) = L_2$

Then, the following hold:

(i) $\lim\limits_{(x,y)\to(a,b)} x = a$

(ii) $\lim\limits_{(x,y)\to(a,b)} y = b$

(iii) $\lim\limits_{(x,y)\to(a,b)} k = k, \quad \text{for any} \quad k \in \mathbb{R}$

(iv) $\lim\limits_{(x,y)\to(a,b)} (f(x, y) + g(x, y)) = L_1 + L_2$

(v) $\lim\limits_{(x,y)\to(a,b)}(f(x,y)-g(x,y)) = L_1 - L_2$

(vi) $\lim\limits_{(x,y)\to(a,b)} f(x,y)g(x,y) = L_1L_2$

(vii) $\lim\limits_{(x,y)\to(a,b)} \dfrac{f(x,y)}{g(x,y)} = \dfrac{L_1}{L_2}$ if $L_2 \neq 0$

(viii) $\lim\limits_{(x,y)\to(a,b)} \sqrt[n]{f(x,y)} = \sqrt[n]{L_1}$, if $\sqrt[n]{L_1} \in \mathbb{R}$

Example 11.

Find $\lim\limits_{(x,y)\to(3,-4)} \sqrt{x^2+y^2}$

Solution.

First note that $\lim\limits_{(x,y)\to(3,-4)} x^2 = 9$ and $\lim\limits_{(x,y)\to(3,-4)} y^2 = 16$, by parts (i), (ii) and (vi) in Theorem 1.

So, $\lim\limits_{(x,y)\to(3,-4)} (x^2+y^2) = 25$, by part (iv) in Theorem 1.

Finally, $\lim\limits_{(x,y)\to(3,-4)} \sqrt{x^2+y^2} = 5$, by part (viii) in Theorem 1.

Example 12.

Find $\lim\limits_{(x,y)\to(0,0)} f(x,y)$, where $f(x,y) = \dfrac{2x^2-2xy}{\sqrt{x}-\sqrt{y}}$

Solution.

Note that we cannot use Theorem 1(vii) because $\lim\limits_{(x,y)\to(0,0)} (\sqrt{x}-\sqrt{y}) = 0$ from Theorem 1(i), (ii), (v) and (viii).

We have

$$\lim_{(x,y)\to(0,0)} \frac{2x^2-2xy}{\sqrt{x}-\sqrt{y}} = \lim_{(x,y)\to(0,0)} \frac{(2x^2-2xy)(\sqrt{x}+\sqrt{y})}{(\sqrt{x}-\sqrt{y})(\sqrt{x}+\sqrt{y})} \qquad (*)$$

$$= \lim_{(x,y)\to(0,0)} \frac{2x(x-y)(\sqrt{x}+\sqrt{y})}{x-y} \qquad (**)$$

$$= \lim_{(x,y)\to(0,0)} 2x(\sqrt{x}+\sqrt{y}) \qquad (***)$$

$$= 0 \qquad (****)$$

Note that we get $(*)$ by multiplying the numerator and denominator of $f(x,y)$ by $\sqrt{x}+\sqrt{y}$ which is allowed because $\sqrt{x}+\sqrt{y}$ is never 0 because we can assume $(x,y) \neq (0,0)$ from definition 10 and remark 2.

Recall from remark 3 that the points (x, y) that approach $(0, 0)$ have to be in the domain of f. So, we get $(***)$ by dividing the numerator and denominator of the function in $(**)$ by $x - y$ which is allowed because $x - y$ is never 0 because the line $x - y = 0$ is not in the domain of f.

Finally, we get $(****)$ by using Theorem 1(i), (ii), (iii), (iv), (vi) and (viii).

Example 13.

Find $\displaystyle\lim_{(x,y)\to(0,0)} \frac{8xy^2}{x^2 + y^2}$ if it exists.

Solution.

Note that we cannot use Theorem 1(vii) because $\displaystyle\lim_{(x,y)\to(0,0)} (x^2 + y^2) = 0$ from Theorem 1(i), (ii), (iv) and (vi).