MT234P – MULTIVARIABLE CALCULUS – 2022

Fiacre Ó Cairbre

Lecture 4

Example 13 continued.

Suppose $f(x, y) = \dfrac{8xy^2}{x^2 + y^2}$. We will prove that $\lim\limits_{(x,y)\to(0,0)} f(x, y) = 0$ by showing that definition 10 is satisfied with $L = 0$ and $(a, b) = (0, 0)$.

So, suppose $\epsilon > 0$. Note that

$$\left|\frac{8xy^2}{x^2 + y^2}\right| = \frac{8|x|y^2}{x^2 + y^2}$$

$$\leq 8|x|$$

$$= 8\sqrt{x^2}$$

$$\leq 8\sqrt{x^2 + y^2} \qquad (*)$$

So, if we choose $\delta = \dfrac{\epsilon}{8}$, then we have that

$$0 < \sqrt{x^2 + y^2} < \delta \;\Rightarrow\; |f(x, y) - 0| = \left|\frac{8xy^2}{x^2 + y^2}\right|$$

$$\leq 8\sqrt{x^2 + y^2} \;\text{ by }\; (*)$$

$$< 8\delta$$

$$= \epsilon$$

So, we have shown that for every $\epsilon > 0$ there exists a corresponding $\delta > 0$ such that

$$0 < \sqrt{x^2 + y^2} < \delta \;\Rightarrow\; |f(x, y) - 0| < \epsilon$$

and we are done because definition 10 is satisfied with $L = 0$ and $(a, b) = (0, 0)$ and so $\lim\limits_{(x,y)\to(0,0)} f(x, y) = 0$

Example 14.

Suppose $f(x,y) = \dfrac{y}{x}$. Does $\displaystyle\lim_{(x,y)\to(0,0)} f(x,y)$ exist?

Solution.

We will show that $\displaystyle\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist.

First note that $f(x,y) = 0$ for any points (x,y) on the line $y = 0$ excluding $(0,0)$. Also, note that $f(x,y) = 1$ for any points (x,y) on the line $y = x$ excluding $(0,0)$.

So, any open ball with centre $(0,0)$ and positive radius will contain points where $f(x,y) = 0$ and will contain points where $f(x,y) = 1$. $\qquad (*)$

Now, suppose $L \in \mathbb{R}$. Also, suppose $\epsilon = \dfrac{1}{4}$. Then, there is no corresponding $\delta > 0$ such that

$$0 < \sqrt{x^2 + y^2} < \delta \ \Rightarrow \ |f(x,y) - L| < \epsilon \qquad (**)$$

because $(*)$ and $(**)$ mean that $|0 - L| < \dfrac{1}{4}$ and $|1 - L| < \dfrac{1}{4}$ which implies $|1 - 0| \leq |1 - L| + |L - 0| < \dfrac{1}{2}$ which is impossible.

This means that definition 10 cannot be satisfied for $L \in \mathbb{R}$ and so $\displaystyle\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist and we are done.

Remark 4.

The idea behind the solution in example 14 is that the limit doesn't exist because $f(x,y)$ approaches two different values (0 and 1) as (x,y) approaches $(0,0)$ along two different paths ($y = 0$ and $y = x$) in the domain of f. We will now state this idea as a theorem.

Theorem 2 – Two path test for non–existence of a limit.

Suppose a function $f(x,y)$ approaches two different values as (x,y) approaches (a,b) along two different paths in the domain of f. Then, $\displaystyle\lim_{(x,y)\to(a,b)} f(x,y)$ does not exist.

Example 15.

Prove that $\displaystyle\lim_{(x,y)\to(0,0)} \frac{2xy}{x^2 + y^2}$ does not exist.

Solution.

Suppose $f(x,y) = \dfrac{2xy}{x^2 + y^2}$

Now, for every $m \in \mathbb{R}$, we have that $f(x,y) = \dfrac{2m}{1 + m^2}$ on the line $y = mx$, excluding $(0,0)$, because if $y = mx$ with $x \neq 0$, then

$$f(x,y) = \frac{2xy}{x^2+y^2} = \frac{2x(mx)}{x^2+(mx)^2} = \frac{2mx^2}{(1+m^2)x^2} = \frac{2m}{1+m^2}$$

So, f approaches two different values (0 and $\frac{4}{5}$) as (x,y) approaches $(0,0)$ along two different paths ($y=0$ and $y=2x$) in the domain of f.

Hence, by Theorem 2, we have that $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ does not exist and we are done.

Remark 5.

In this chapter, you may assume all functions are real valued unless otherwise stated.

Example 16.

Prove that $\lim\limits_{(x,y)\to(0,0)} \frac{y^2-x^4}{x^4+y^2}$ does not exist.

Solution.

Suppose $f(x,y) = \frac{y^2-x^4}{x^4+y^2}$

Now, for every $m \in \mathbb{R}$, we have that $f(x,y) = \frac{m^2-1}{m^2+1}$ on the path $y = mx^2$, excluding $(0,0)$, because if $y = mx^2$ with $x \neq 0$, then

$$f(x,y) = \frac{m^2x^4-x^4}{x^4+m^2x^4} = \frac{(m^2-1)x^4}{(1+m^2)x^4} = \frac{m^2-1}{m^2+1}$$

So, f approaches two different values (0 and $\frac{3}{5}$) as (x,y) approaches $(0,0)$ along two different paths ($y=x^2$ and $y=2x^2$) in the domain of f.

Hence, by Theorem 2, we have that $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ does not exist and we are done.

Example 17.

Find $\lim\limits_{(x,y)\to(0,2)} \frac{x-xy+4}{x^3y-5xy-y^2}$ if it exists.

Solution.

Using all parts of Theorem 1, except part (viii), we get that

$$\lim_{(x,y)\to(0,2)} \frac{x-xy+4}{x^3y-5xy-y^2} = \frac{0-0+4}{0-0-4} = -1$$