13. Discrete Random Variables

Definition 13.1. *Let* $(\Omega, \mathscr{E}, P)$ *be a probability space. A function* $X : \Omega \to \mathbb{R}$ *is called* **random variable**. *We say* X *is* **discrete** *if its range* R_X *is a countable set, that is, it is either finite or bijective to the natural numbers.*

Example 13.2. *(1) We toss a fair coin* 3 *times. Then*

$$\Omega = \{HHH, HHT, HTH, THH, THT, HTT, TTH, TTT\}.$$

Let X *be the number of heads per outcome. Then for instance* $X(HHH) = 3$ *and* $X(THT) = 1$. *Furthermore* $R_X = \{0, 1, 2, 3\}$. *Other possible random variables are for instance*

(1) Y *is the number of tails per outcome; here* $R_Y = \{0, 1, 2, 3\}$
(2) $Z = X - Y$ *is the number of heads minus the number of tails; here* $R_Z = [-3, 3] \cap \mathbb{Z}$.

(2) We toss a fair coin until the first heads appears. If X *is the number of throws, then* $R_X = \{1, 2, 3, \ldots\}$.

(3) Let X *be the random variable that gives the exact moment from now in minutes until the next rainfall occurs in a certain place. Then* $R_X = \mathbb{N}$.

Definition 13.3. *Let* $(\Omega, \mathscr{E}, P)$ *be a probability space and* $X : \Omega \to \mathbb{R}$ *a discrete random variable with range* R_X. *Then the function* $P_X : R_X \to [0, 1]$, *where*

$$P_X(x) = P(X = x) = P(\{\omega \in \Omega : X(\omega) = x\}),$$

for every $x \in R_X$, *is called the* **probability mass function** *(PMF) of* X.

The **cumulative distribution function** *(CDF) of* X *is defined as*

$$F_X(x) = P(X \le x), \text{ for all } x \in \mathbb{R}.$$

Example 13.4. *(1) We toss a fair coin* 3 *times. Let* X *be the number of heads per outcome. Then* $R_X = \{0, 1, 2, 3\}$ *and the PMF of* X *is*

$$P_X(0) = P(X = 0) = P(\{TTT\}) = \frac{1}{8}$$
$$P_X(1) = P(X = 1) = P(\{TTH, THT, HTT\}) = \frac{3}{8}$$
$$P_X(2) = P(X = 2) = P(\{HHT, HTH, THH\}) = \frac{3}{8}$$
$$P_X(3) = P(X = 3) = P(\{HHH\}) = \frac{1}{8}$$

Furthermore

$$F_X(x) = \begin{cases} 0, & x \in (-\infty, 0) \\ \frac{1}{8}, & x \in [0, 1) \\ \frac{1}{2}, & x \in [1, 2) \\ \frac{7}{8}, & x \in [2, 3) \\ 1, & x \in [3, \infty) \end{cases}$$

(2) We toss a coin, where $P(H) = p$, for $0 < p < 1$, until the first heads appears. If X is the number of throws, then $R_X = \{1, 2, 3, \ldots\}$. Now

$$P_X(1) = p, \qquad P_X(2) = (1-p)p, \qquad P_X(3) = (1-p)^2 p, \qquad \ldots$$

In fact, we have $P_X(k) = p(1-p)^{k-1}$, for all $k \in R_X$. Hence, if the coin is fair, then $P_X(k) = \frac{1}{2^k}$, for all $k \in R_X$.

Next observe that, for all integers $n \geq 1$,

$$\begin{aligned} F_X(n) &= \sum_{k=1}^{n} P_X(k) = p \cdot \sum_{k=1}^{n} (1-p)^{k-1} = (1-(1-p)) \cdot \sum_{k=1}^{n} (1-p)^{k-1} \\ &= \sum_{k=1}^{n} (1-p)^{k-1} - \sum_{k=1}^{n} (1-p)^{k} = 1 - (1-p)^n \end{aligned}$$

Lemma 13.5. *Let $(\Omega, \mathscr{E}, P)$ be a probability space with discrete random variable $X : \Omega \to \mathbb{R}$. Then*

(1) $\displaystyle\sum_{x \in R_X} P_X(x) = 1$,

(2) $P(a < X \leq b) = F_X(b) - F_X(a)$, *for real $a < b$.*

Proof. Part (1) follows immediately from Lemma 9.6. For part (2) observe that $P(X \leq b) = P(X \leq a) + P(a < X \leq b)$. $\square$

Example 13.6. *Consider the PMF $P_X(k) = p(1-p)^{k-1}$, for all integers $k \geq 1$, where $0 < p < 1$, from Example 13.4 (2). Then*

$$\sum_{k=1}^{\infty} P_X(k) = \lim_{n\to\infty} \sum_{k=1}^{n} P_X(k) = \lim_{n\to\infty} F_X(n) = \lim_{n\to\infty} (1 - (1-p)^n) = 1$$

Furthermore

$$\begin{aligned} P(10 < X \leq 100) &= F_X(100) - F_X(10) = (1-(1-p)^{100}) - (1-(1-p)^{10}) \\ &= (1-p)^{10} - (1-p)^{100} = (1-p)^{10} \cdot \big(1-(1-p)^{90}\big) \end{aligned}$$

Definition 13.7. *We call discrete random variables $X_1, \dots, X_n$* **independent** *if, for all sets $A_1, \dots, A_n$ in $\mathbb{R}$,*

$$P(X_1 \in A_1, \dots, X_n \in A_n) = P(X_1 \in A_1) \cdot \ldots \cdot P(X_n \in A_n).$$

Example 13.8. *We throw a blue die and red die ten times each independent of each other. Let X count the number of red 6's and let Y count the number of blue odd numbers. We have*

$$\begin{aligned} P(X < 2, Y > 1) &= P(X < 2) \cdot P(Y > 0) = (P_X(0) + P_X(1)) \cdot (1 - P_Y(0)) \\ &= \left(\left(\frac{5}{6}\right)^{10} + 10 \cdot \frac{1}{6} \cdot \left(\frac{5}{6}\right)^9 \right) \cdot \left(1 - \left(\frac{1}{2}\right)^{10} \right) \\ &= \left(\frac{5}{6}\right)^9 \cdot \left(\frac{5}{2}\right) \cdot \left(1 - \left(\frac{1}{2}\right)^{10} \right) = 0.484 \end{aligned}$$

14. Special Random Variables and Distributions

Bernoulli Random Variables. A Bernoulli trial, is an experiment with two possible outcomes, called success and failure, that is, $\Omega = \{success, failure\}$. Assume that success happens with probability p, for some $0 < p < 1$. Next let $X : \Omega \to \mathbb{R}$ such that $X(failure) = 0$ and $X(success) = 1$. Then $R_X = \{0, 1\}$ and

$$P_X(x) = \begin{cases} p, & x = 1 \\ 1 - p, & x = 0 \\ 0, & \text{otherwise} \end{cases}$$

We say X is a **Bernoulli Random Variable** / has a **Bernoulli distribution**, with parameter p, and write $X \sim \text{Bernoulli}(p)$.

Example 14.1. *(1) Toss a coin and let X record 1 for heads and 0 for tails. If the coin is fair, then X has a Bernoulli distribution with parameter $\frac{1}{2}$, that is, $X \sim \text{Bernoulli}(\frac{1}{2})$.*

(2) We roll a die and let X record 1 if the number is 5 or 6, and 0 otherwise. Then X is a Bernoulli Random Variable with parameter $p = 1/3$.

Binomial Distribution. Consider an experiment of n independent Bernoulli trials, each with the same probability of success p. Let X be the random variable that records the number of successes. Then $R_X = \{0, 1, \ldots, n\}$. Let $k \in R_X$. Note that there are $\binom{n}{k}$ different ways to record precisely k successes. Furthermore each such outcome occurs with probability $p^k \cdot (1-p)^{n-k}$. Hence

$$P_X(k) = \begin{cases} \binom{n}{k} p^k (1-p)^{n-k}, & k = 0, 1, \ldots, n \\ 0, & \text{otherwise} \end{cases}$$

We say X has a **Binomial distribution**, with parameters n and p, and write $X \sim \text{Binomial}(n, p)$.

Example 14.2. *(1) Toss a coin 100 times, where head represents a success and let X record the number of successes. If the coin is fair, then X has a Binomial distribution with parameters 100 and $\frac{1}{2}$, that is, $X \sim \text{Binomial}(100, \frac{1}{2})$*

(2) Rolling a die we call the outcomes 5 or 6 a success. We roll four times and let X record the number of successes. Then $X \sim \text{Binomial}(4, \frac{1}{3})$ and

k	0	1	2	3	4
$P_X(k)$	0.1975	0.3951	0.2963	0.0988	0.0123

Geometric Distribution. Consider an experiment where we run independent Bernoulli trials, each with probability of success p, until the first success is observed. Let X count the number of trials, that is, the experiment terminates after X trails. As seen in Example 13.4 (2) we get

$$P_X(k) = \begin{cases} p(1-p)^{k-1}, & k = 1, 2, 3 \dots \\ 0, & \text{otherwise} \end{cases}$$

We say X has a **Geometric distribution**, with parameter p, and write $X \sim \text{Geometric}(p)$. For instance, let $p = \frac{1}{3}$,

k	1	2	3	4
$P_X(k)$	$\frac{1}{3} = 0.333$	$\frac{2}{9} = 0.222$	$\frac{4}{27} = 0.148$	$\frac{8}{81} = 0.0987$

Pascal Distribution. In a variant, Bernoulli trials are repeated until there are r successes. Again let X record the number of trials. Then

$$P_X(k) = \begin{cases} \binom{k-1}{r-1} p^r (1-p)^{k-r}, & k = r, r+1, r+2, r+3 \dots \\ 0, & \text{otherwise} \end{cases}$$

In this case X has **Pascal distribution**, with parameters r and p, and write $X \sim \text{Pascal}(r, p)$.

Example 14.3. *We throw a fair coin until we have thrown three Heads. If X counts the number of throws needed, then $X \sim$ Pascal$(3, \frac{1}{2})$. We have*

k	0	1	2	3	4	5	6	7	8
$P_X(k)$	0	0	0	0.125	0.1875	0.1875	0.1563	0.1172	0.082

Poisson Distribution. Let $\lambda > 0$ be real and $X_n \sim \text{Binomial}(n, p_n)$, where $p_n = \frac{\lambda}{n}$, for integers $n \geq 1$. Set $q_n = 1 - p_n$. Then for fixed $k \in \mathbb{N}$ we have

$$P(X_n = k) = \binom{n}{k} p_n^k q_n^{n-k} = \frac{\lambda^k}{k!} \cdot \frac{n(n-1)\dots(n-k+1)}{n^k} \cdot \frac{(1-\lambda/n)^n}{(1-\lambda/n)^k}$$

Now let $n \to \infty$. Then, one can show that,

$$\frac{n(n-1)\dots(n-k+1)}{n^k} \to 1, \quad (1-\lambda/n)^n \to e^{-\lambda}, \quad (1-\lambda/n)^k \to 1.$$

So

$$P(X_n = k) \to \frac{\lambda^k e^{-\lambda}}{k!}.$$

We say a random variable X has **Poisson distribution** with parameter $\lambda > 0$, and write $X \sim \text{Poisson}(\lambda)$, if

$$P_X(k) = \begin{cases} \frac{\lambda^k}{k!}e^{-\lambda}, & k = 0, 1, 2, 3, \ldots \\ 0, & \text{otherwise} \end{cases}$$

Lemma 14.4. *The above function is a probability mass function on the set* $\mathbb{N}$.

Proof. We have $\displaystyle\sum_{k=0}^{\infty} P_X(k) = \sum_{k=0}^{\infty} \frac{\lambda^k}{k!}e^{-\lambda} = e^{-\lambda} \cdot \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} = e^{-\lambda} \cdot e^{\lambda} = 1$ □

Remark 14.5. *The Poisson distribution can be used to approximate a Binomial distribution for large n and small p. Furthermore the Poisson distribution is used when we count the occurrences of certain events in time or space, knowing that on average they occur λ times.*

Example 14.6. *(1) Two dice are rolled* 100 *times, and the number of double sixes is counted by X. Each dice roll is a Bernoulli trial with $p = 1/36$. So $X \sim \text{Binomial}(n = 100, p = 1/36)$. Since n is large relative to p, we can approximate X as a Poisson random variable, with $\lambda = p \cdot n = \frac{25}{9}$. Thus*

$$P_X(k) \approx \frac{(\frac{25}{9})^k}{k!}e^{-\frac{25}{9}}$$

We have

k	0	1	2	3	4	5	6	7
Binomial$(X \leq k)$	0.06	0.231	0.472	0.698	0.854	0.94	0.979	0.993
Poisson$(X \leq k)$	0.062	0.235	0.475	0.697	0.851	0.937	0.977	0.992

(2) A small store has ten customers per hour on average. Let X be the number of customers arriving between 10am to 11.30am? What is the probability of there being more than 12, but no more than 16 customers during that time? We assume that X is Poisson distributed with parameter $\lambda = 1.5 \cdot 10 = 15$ and want $P(12 < X \leq 16)$. We have

$$\begin{aligned} P(12 < X \leq 16) &= \sum_{k=13}^{16} P_X(k) = \sum_{k=13}^{16} \frac{15^k}{k!}e^{-15} \\ &= e^{-15} \cdot \left(\frac{15^{13}}{13!} + \frac{15^{14}}{14!} + \frac{15^{15}}{15!} + \frac{15^{16}}{16!}\right) = 0.3965 \end{aligned}$$