MT234P – MULTIVARIABLE CALCULUS – 2022

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Lecture 9

Section 2.2 – Tangent Planes.

Remark 5. Recall theorem 5, definition 15 and theorem 6 from chapter 1, where we discussed the definition of a differentiable function of two variables. The situation for a function of three variables is similar and we now give the relevant two theorems and definition.

Theorem 3.

Suppose $f : \mathbb{R}^3 \to \mathbb{R}$. Suppose that the partial derivatives $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}$ are defined on an open set W containing the point (a, b, c). Suppose that $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}$ are continuous at (a, b, c). Define Δm as $\Delta m = f(a + \Delta x, b + \Delta y, c + \Delta z) - f(a, b, c)$ so that Δm is the change in f as (x, y, z) goes from (a, b, c) to $(a + \Delta x, b + \Delta y, c + \Delta z)$ in W.

Then, we have the following:

$$\Delta m = \frac{\partial f}{\partial x}_{|(a,b,c)} \Delta x + \frac{\partial f}{\partial y}_{|(a,b,c)} \Delta y + \frac{\partial f}{\partial z}_{|(a,b,c)} \Delta z$$

$$+\epsilon_1 \Delta x + \epsilon_2 \Delta y + \epsilon_3 \Delta z, \text{where } \epsilon_1,\ \epsilon_2,\ \epsilon_3 \to 0 \text{ as } \ \Delta x,\ \Delta y \text{ and } \Delta z \to 0 \ (**)$$

Definition 5.

Suppose $f : \mathbb{R}^3 \to \mathbb{R}$. We say f is differentiable at (a, b, c) if $\frac{\partial f}{\partial x}_{|(a,b,c)}, \frac{\partial f}{\partial y}_{|(a,b,c)}$ and $\frac{\partial f}{\partial z}_{|(a,b,c)}$ exist and $(**)$ in theorem 3 holds for f at (a, b, c). We say f is differentiable if it's differentiable at every point in its domain.

Theorem 4.

Suppose $f : \mathbb{R}^3 \to \mathbb{R}$. If $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$ and $\frac{\partial f}{\partial y}$ are continuous on an open set W, then f is differentiable at every point in W.

Remark 6.

Recall that for a differentiable function of one variable, $g(x)$, the derivative of g gives the slope of a tangent line. Here we will see how the gradient of a differentiable function of three variables, $f(x, y, z)$, will give a vector perpendicular to a so called tangent plane (which will be defined later).

Remark 7.

Consider a curve in $\mathbb{R}^3$ given by $g(t)\vec{i} + h(t)\vec{j} + q(t)\vec{k},\ \ t \in \mathbb{R}$. Then, we say that this curve is differentiable at $t = t_0$ if g, h, q are all differentiable functions at $t = t_0$. We say that this curve is differentiable if it's differentiable at all t in its domain.

Example 3.

Consider the curve in $\mathbb{R}^3$ given by $\cos t\ \vec{i} + \sin t\ \vec{j} + t\ \vec{k},\ \ t \in \mathbb{R}$. Then, this curve is differentiable and will give a helix in $\mathbb{R}^3$.

Remark 8.

Suppose $f(x, y, z)$ is a differentiable function of three variables. Suppose (x_0, y_0, z_0) is a point on the level surface $f(x, y, z) = c$, for some $c \in \mathbb{R}$. Suppose $g(t)\vec{i} + h(t)\vec{j} + q(t)\vec{k}$, $t \in \mathbb{R}$ is any differentiable curve on the level surface $f(x, y, z) = c$.

Then, $f(g(t), h(t), q(t)) = c$. Let $x = g(t),\ y = h(t),\ z = q(t)$. Now, using the chain rule we get

$$0 = \frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt} = \nabla f.\underline{v}$$

where $\underline{v} = \dfrac{dx}{dt}\vec{i} + \dfrac{dy}{dt}\vec{j} + \dfrac{dz}{dt}\vec{k}$ is the curve's so called 'velocity vector' or 'tangent vector'.

So, ∇f is perpendicular to the velocity vector at every point on the curve. All the velocity vectors at (x_0, y_0, z_0) are perpendicular to ∇f at (x_0, y_0, z_0) and so all the tangent lines to these curves lie in the plane containing (x_0, y_0, z_0) that is perpendicular to ∇f. We call this plane the tangent plane of the level surface at (x_0, y_0, z_0). So, the tangent plane at the point (x_0, y_0, z_0) of the level surface $f(x, y, z) = c$ is the plane containing (x_0, y_0, z_0) that is perpendicular to ∇f.

The line through (x_0, y_0, z_0) that is perpendicular to the tangent plane is called the surface's normal line at (x_0, y_0, z_0). So, the normal line of the level surface $f(x, y, z) = c$ at the point (x_0, y_0, z_0) is parallel to ∇f. We now gather this information to form the following definition:

Definition 6.

Suppose $f(x, y, z)$ is a differentiable function. The tangent plane at the point (x_0, y_0, z_0) of the level surface $f(x, y, z) = c$ is the plane with equation

$$\frac{\partial f}{\partial x}_{|(x_0,y_0,z_0)}(x - x_0) + \frac{\partial f}{\partial y}_{|(x_0,y_0,z_0)}(y - y_0) + \frac{\partial f}{\partial z}_{|(x_0,y_0,z_0)}(z - z_0) = 0$$

The normal line of the level surface $f(x, y, z) = c$ at the point (x_0, y_0, z_0) on the level surface $f(x, y, z) = c$ is the line with parametric equations

$$x = x_0 + \frac{\partial f}{\partial x}_{|(x_0,y_0,z_0)} t, \qquad y = y_0 + \frac{\partial f}{\partial y}_{|(x_0,y_0,z_0)} t, \qquad z = z_0 + \frac{\partial f}{\partial z}_{|(x_0,y_0,z_0)} t, \qquad t \in \mathbb{R}$$

Example 4.

Find the equations for the tangent plane and normal line at $(1, -1, 3)$ of the level surface $x^2 + 2xy - y^2 + z^2 = 7$.

Solution.

Let $f(x, y, z) = x^2 + 2xy - y^2 + z^2$ and let $(x_0, y_0, z_0) = (1, -1, 3)$. Now,

$$\frac{\partial f}{\partial x}_{|(1,-1,3)} = (2x + 2y)_{|(1,-1,3)} = 0$$

$$\frac{\partial f}{\partial y}_{|(1,-1,3)} = (2x - 2y)_{|(1,-1,3)} = 4$$

$$\frac{\partial f}{\partial z}_{|(1,-1,3)} = (2z)_{|(1,-1,3)} = 6$$

So, the equation of the required tangent plane is $4(y + 1) + 6(z - 3) = 0$ which gives the plane $2y + 3z = 7$. The equation of the required normal line is $x = 1,\ y = -1 + 4t,\ z = 3 + 6t, \quad t \in \mathbb{R}$.

Chapter 3 – Maxima and Minima.

Section 3.1 – Derivative tests.

Definition 1.

Suppose $f(x, y)$ is a function of two variables and suppose f is defined on a subset T of $\mathbb{R}^2$ with $(a, b) \in T$. Then

(i) $f(a, b)$ is a local maximum value of f if $f(a, b) \geq f(x, y)$, for all domain points (x, y) in an open ball with centre (a, b). We call (a, b) a local maximum.

(ii) $f(a, b)$ is a local minimum value of f if $f(a, b) \leq f(x, y)$, for all domain points (x, y) in an open ball with centre (a, b). We call (a, b) a local minimum.

Remark 1.

Note that if (a, b) is a local maximum of f, then it may not necessarily be the case that $f(a, b) \geq f(x, y)$, for all points (x, y) in the domain of f.