2. Principle of Induction

Note that by (A1) and (A2) we get for every $m \in \mathbb{N}$ that

$$S(m) = S(m+0) = m + S(0) = m + 1.$$

Hence the Axiom of Induction can be rephrased as follows:

(PA5) If K is a subset of $\mathbb{N}$ such that
(a) $0 \in K$ and
(b) if $n \in K$ then $n+1 \in K$,
then $K = \mathbb{N}$.

Theorem 2.1. (Principle of Induction)
Let $n_0 \in \mathbb{Z}$ and let $P(n)$ be a property which can be true or false, for all $n \in \mathbb{Z}$ with $n_0 \leq n$. Furthermore assume that $P(n_0)$ is true and if $P(n)$ is true for some $n \in \mathbb{Z}$ with $n_0 \leq n$, then $P(n+1)$ is true. Then $P(n)$ is true, for all integers n with $n_0 \leq n$.

Proof. Set $K := \{n \in \mathbb{N} : P(n_0 + n) \text{ is true.}\}$. As $P(n_0)$ is true we have $0 \in K$. Furthermore if $n \in K$, then $P(n_0 + n)$ is true and so by assumption $P(n_0 + n + 1)$ is true. Consequently, by the axiom of induction, $n + 1 \in K$ and thus $K = \mathbb{N}$. $\square$

Example 2.2. *(1) Show that $\sum_{i=1}^{n} i = 1+2+\ldots+n = \frac{n(n+1)}{2}$, for all integers $n \geq 1$. The statement is true for $n = 1$, as the sum on the left and the fraction on the right both equal one. Next suppose the statement is true for some $n \geq 1$. We need to show it holds for $n + 1$. We have*

$$\sum_{i=1}^{n+1} i = (n+1) + \sum_{i=1}^{n} i = (n+1) + \frac{n(n+1)}{2} = (n+1) \cdot \left(1 + \frac{n}{2}\right)$$
$$= (n+1) \cdot \left(\frac{n+2}{2}\right) = \frac{(n+1)(n+2)}{2}.$$

Hence the statement is true for $n+1$. In particular, the statement is true for all $n \geq 1$.

(2) For $n, k \in \mathbb{N}$ such that $0 \leq k \leq n$ we define the **binomial coefficient**

$$\binom{n}{k} := \frac{n!}{k!(n-k)!}$$

We claim that all binomial coefficient are integers. This statement is quickly verified to be true if $n = 0$ or $k = 0$. Henceforth we assume that neither are zero. If $n = 1$, then $k = 1$ and as $\binom{1}{1} = 1$. Hence the statement holds for

$n = 1$. Next assume the statement is true for some integer $n \geq 1$ and all integers k with $0 \leq k \leq n$. Then for all $1 \leq k \leq n+1$ we have

$$\begin{aligned}\binom{n+1}{k} &= \frac{(n+1)!}{k!(n+1-k)!} = \frac{n! \cdot ((n+1)-k+k)}{k!(n+1-k)!} \\ &= \frac{n!}{k!(n-k)!} + \frac{n!}{(k-1)!(n+1-k)!} = \binom{n}{k} + \binom{n}{k-1}\end{aligned}$$

Note that both $\binom{n}{k}$ and $\binom{n}{k-1}$ are integers and consequently so is $\binom{(n+1)}{k}$.

Corollary 2.3. (Strong Induction)
Let $n_0 \in \mathbb{Z}$ and let $P(n)$ be a property which can be true or false, for all $n \in \mathbb{Z}$ with $n_0 \leq n$. Furthermore assume that $P(n_0)$ is true and that $P(n+1)$ is true, for some $n \in \mathbb{Z}$, whenever $P(k)$ is true for all integers k with $n_0 \leq k \leq n$. Then $P(n)$ is true, for all integers n with $n_0 \leq n$.

Example 2.4. *Set $T_1 = 1, T_2 = 3$ and recursively $T_n = T_{n-1} + T_{n-2}$, for $n \geq 3$ (Lucas Sequence). Prove that*

$$T_n < \left(\frac{7}{4}\right)^n, \quad \textit{for all } n \geq 1.$$

As

$$T_1 = 1 < \frac{7}{4} \textit{ and } T_2 = 3 < \frac{49}{16} = \left(\frac{7}{4}\right)^2,$$

the statement holds for 1 and 2. Next suppose it holds for all integers k where $1 \leq k \leq n$, for some $n \geq 2$. Then

$$\begin{aligned}T_{n+1} = T_n + T_{n-1} &< \left(\frac{7}{4}\right)^n + \left(\frac{7}{4}\right)^{n-1} = \left(\frac{7}{4}\right)^{n-1} \cdot \left(\frac{7}{4} + 1\right) \\ &= \left(\frac{7}{4}\right)^{n-1} \cdot \left(\frac{11}{4}\right) = \left(\frac{7}{4}\right)^{n-1} \cdot \left(\frac{44}{16}\right) < \left(\frac{7}{4}\right)^{n-1} \cdot \left(\frac{49}{16}\right) \\ &= \left(\frac{7}{4}\right)^{n-1} \cdot \left(\frac{7}{4}\right)^2 = \left(\frac{7}{4}\right)^{n+1}\end{aligned}$$

Hence the statement holds for $n+1$ and thus for all $n \geq 1$.

Theorem 2.5. (*Well-Ordering Principle*) *Let S be a non-empty subset of $\mathbb{N}$. Then S has a least element, that is, there is some $l \in S$ so that $l \leq s$, for all $s \in S$.*

Proof: Let S be a subset of $\mathbb{N}$ without a least element. Then surely $0 \notin S$. Now set

$$K := \{n \in \mathbb{N} : n \notin S\}$$

We say $P(n)$ is true for some $n \in \mathbb{N}$, if $n \in K$. Hence $P(0)$ is true. Next let $n \geq 0$ and assume that $P(k)$ is true for all integers k with $0 \leq k \leq n$. If $n+1 \in S$, then $n+1$ will be a least element of S, as otherwise there must be an $s \in S$ with $s < n+1$. But then $P(s)$ is true by assumption, that is, $s \in K$ in contradiction to $s \in S$. Therefore $n+1 \notin S$, that is, $n+1 \in K$. Now by Corollary 2.3 we get that $K = \mathbb{N}$. In particular, $S = \emptyset$.

□

Corollary 2.6. *There is no $a \in \mathbb{Z}$ such that $0 < a < 1$.*

Proof: Assume there is such an $a \in \mathbb{Z}$ and let S be the set of all such a. Then S is a non-empty subset of $\mathbb{N}$ and as such contains a least element l. Note that $l \in \mathbb{N} \setminus \{0\}$. As $l < 1$, there is $x \in \mathbb{N} \setminus \{0\}$ such that $l + x = 1$. Multiplying l onto this equation gives $l^2 + xl = l$. Clearly $xl \in \mathbb{N} \setminus \{0\}$ and so $l^2 < l$. We also have $0 < l^2 < 1$, that is $l^2 \in S$. But this contradicts the minimality of l. In particular there is no a as described.

□

Remark 2.7. *The integers do not satisfy the Well-Ordering Principle, as there are subsets S of $\mathbb{Z}$ that do not contain a smallest element, take for instances $S = \mathbb{Z}$ or the subset of negative integers. In particular any number system containing the integers, such as $\mathbb{Q}$ for instance, cannot satisfy the Well-Ordering Principle either.*