MT234P – MULTIVARIABLE CALCULUS – 2022

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Lecture 1

Chapter 1 – Limits and Partial Derivatives.

Section 1.1 – A little bit of the history of calculus.

Calculus was created independently by Newton and Leibniz in the late 1600s. The ideas in calculus have turned out to be incredibly powerful in solving an abundance of important problems in science, engineering, finance, meteorology, navigation and many other areas.

Section 1.2 – Functions of two or more variables.

Remark 1. Many functions depend on more than one variable. For example, the volume of a cylinder is $\pi r^2 h$ and depends on the two variables, r, h, where r is the radius of the cylinder and h is the height of the cylinder.

Example 1. One can think of the volume above as the function.

$$f : \mathbb{R}^2 \to \mathbb{R}$$

$$(r, h) \to \pi r^2 h$$

where the domain of f is $\{(r, h) \in \mathbb{R}^2 : r > 0, h > 0\}$

One sees that $f(2, 3) = 12\pi$

Example 2.

If $f(x, y) = \dfrac{y}{(y - x)^2}$, then the domain of f is

$$\{(x, y) \in \mathbb{R}^2 : x \neq y\}$$

and the range of f is $\mathbb{R}$ because if $t \in \mathbb{R}$, then $t = f(t - 1, t)$

Example 3.

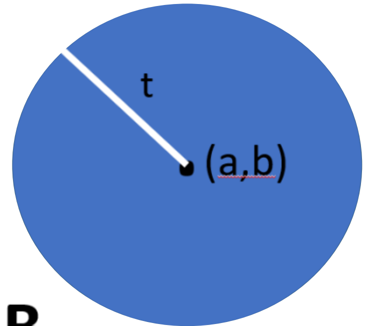
If $g(x, y, z) = e^{\sin(xyz)}$, then the domain of g is $\mathbb{R}^3$ and the range of g is $[e^{-1}, e]$ in $\mathbb{R}$.

Definition 1.

(i) An open ball in $\mathbb{R}^2$ is a set of the form

$$B = \{(x, y) \in \mathbb{R}^2 : (x-a)^2 + (y-b)^2 < t^2\}$$

which has centre (a, b) in $\mathbb{R}^2$ and radius $t > 0$.

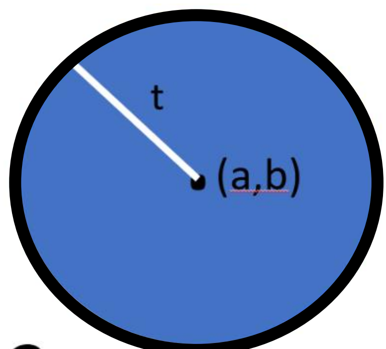


B

(ii) A closed ball in $\mathbb{R}^2$ is a set of the form

$$C = \{(x, y) \in \mathbb{R}^2 : (x-a)^2 + (y-b) \leq t^2\}$$

which has centre (a, b) in $\mathbb{R}^2$ and radius $t > 0$.



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Definition 2.

(i) A point (x, y) in a subset T of $\mathbb{R}^2$ is called an interior point of T if (x, y) is the centre of an open ball that is a subset of T.

(ii) A point (x, y) is a boundary point of a subset W of $\mathbb{R}^2$ if every open ball with centre (x, y) contains points that are not in W and also contains points that are in W. Note that the boundary point (x, y) itself need not be an element of W.

(iii) The interior of a subset X of $\mathbb{R}^2$ is the set of all interior points of X. Denote the set of interior points of X by $\text{Int}(X)$.

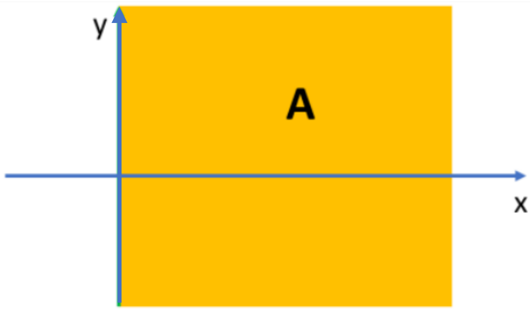
(iv) The boundary of a subset L of $\mathbb{R}^2$ is the set of all boundary points of L. Denote the set of boundary points of L by $\text{Bdy}(L)$.

(v) A subset G of $\mathbb{R}^2$ is called open if and only if $\text{Int}(G) = G$.

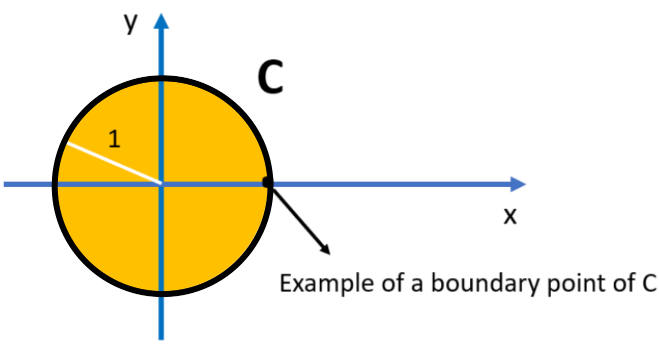
(vi) A subset Z of $\mathbb{R}^2$ is called closed if and only if $\text{Bdy}(Z)$ is a subset of Z.

Example 4.

(a) $A = \{(x, y) \in \mathbb{R}^2 : x > 0\}$ is open because $\mathrm{Int}(A) = A$.



(b) $C = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$ is closed because $\mathrm{Bdy}(C)$ is a subset of C.



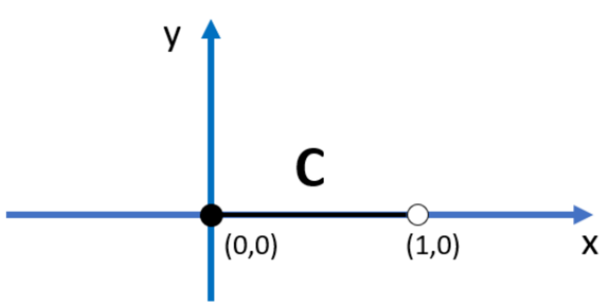
(c) $\mathbb{R}^2$ is open and $\mathbb{R}^2$ is closed.

Example 5.

Give an example of a set in $\mathbb{R}^2$ that is neither open nor closed.

Solution.

$C = \{(x, y) \in \mathbb{R}^2 : y = 0 \text{ and } 0 \leq x < 1\}$ is not open because $(0, 0)$ is an element of C that is not an interior point of C.



C is not closed because $(1, 0)$ is a boundary point of C that is not an element of C. So, C is neither open nor closed.

Definition 3.

A subset T of $\mathbb{R}^2$ is called bounded if it is a subset of an open ball. A subset L of $\mathbb{R}^2$ is called unbounded if it is not bounded.

Example 6.

(i) $C = \{(x, y) \in \mathbb{R}^2 : 0 \leq x \leq 1,\ 3 \leq y < 8\}$ is bounded because it is a subset of the open ball with centre (0,0) and radius 10.

(ii) $G = \{(x, y) \in \mathbb{R}^2 : y < 4\}$ is unbounded because it is not a subset of an open ball.

Definition 4.

Suppose $f : \mathbb{R}^2 \to \mathbb{R}$ is a function. Then, the set

$$C_w = \{(x, y) \in \mathbb{R}^2 : f(x, y) = w, \quad \text{for some} \quad w \in \mathbb{R}\}$$

is called a level curve of f. The level curve C_w above is also called the level curve $f(x, y) = w$. Notice that a level curve of f is a subset of the domain of f.

Definition 5.

Suppose $f : \mathbb{R}^2 \to \mathbb{R}$ is a function. Then, the set

$$G = \{(x, y, z) \in \mathbb{R}^3 : (x, y) \ \text{ is in the domain of } \ f \ \text{ and } \ z = f(x, y)\}$$

is called the graph of f. The graph of f is also called the surface $z = f(x, y)$.

Note that the graph of f is a subset of $\mathbb{R}^3$. Also, note that a level curve of f is a subset of $\mathbb{R}^2$. So, a level curve of f is not a subset of the graph of f.