# MT251P – Lecture 12

Fiacre Ó Cairbre

**Solution to example 14.**

Let

$$A = (1, 0, 0), \quad B = (1, 2, 1), \quad C = (2, 1, 0)$$

Then, let

$$\underline{n} = \vec{AB} \times \vec{AC}$$

Then, $\underline{n}$ will be perpendicular to both $\vec{AB}$ and $\vec{AC}$. Also, $\underline{n}$ can be taken to be a normal vector to $\alpha$. Now,

$$\underline{n} = (2j + k) \times (i + j)$$

$$= -i + j - 2k$$

and so the equation of $\alpha$ is

$$-(x - 1) + (y - 0) - 2(z - 0) = 0$$

$$\Rightarrow -x + y - 2z + 1 = 0$$

$$\Rightarrow x - y + 2z = 1$$

**Example 15.**

Find the parametric equation of the line $L$ which is the intersection of the two planes

$$x + y + z = 1 \quad \text{and} \quad x + 2y + 3z = 2$$

**Solution.**

$$\underline{n} = i + j + k \quad \text{is a normal vector to the plane} \quad x + y + z = 1$$

$$\underline{s} = i + 2j + 3k \quad \text{is a normal vector to the plane} \quad x + 2y + 3z = 2$$

Now, a vector parallel to $L$ is perpendicular to both $\underline{n}$ and $\underline{s}$. So, we can take $\underline{n} \times \underline{s}$ as a vector parallel to $L$.

Now,

$$\underline{n} \times \underline{s} = i - 2j + k$$

and so $i-2j+k$ is a vector parallel to $L$. We will now show how to find a point on $L$. Well, one can show that any line in $\mathbb{R}^3$ must have at least one point with either $x=0$, $y=0$ or $z=0$.

Try $z=0$ to get

$$x+y=1 \quad \text{and} \quad x+2y=2 \qquad (*)$$

$x=0$, $y=1$ will satisfy $(*)$. So, $(0,1,0)$ is a point on $L$.

So, overall, we have that $L$ is a line containing the point $(0,1,0)$ and parallel to the vector $i-2j+k$. Hence, the parametric equation of $L$ is

$$L = \{(0,1,0) + t(1,-2,1) : t \in \mathbb{R}\}$$

**Theorem 7 – Shortest distance from a point to a plane.**

Suppose $(x_0, y_0, z_0)$ is a point not on the plane $ax+by+cz+d=0$. Then, the shortest distance from $(x_0, y_0, z_0)$ to this plane is

$$\left| \frac{ax_0 + by_0 + cz_0 + d}{\sqrt{a^2+b^2+c^2}} \right|$$

**Example 16.**

Find the shortest distance from $(2,2,3)$ to the plane $2x+2y-3z+3=0$.

**Solution.**

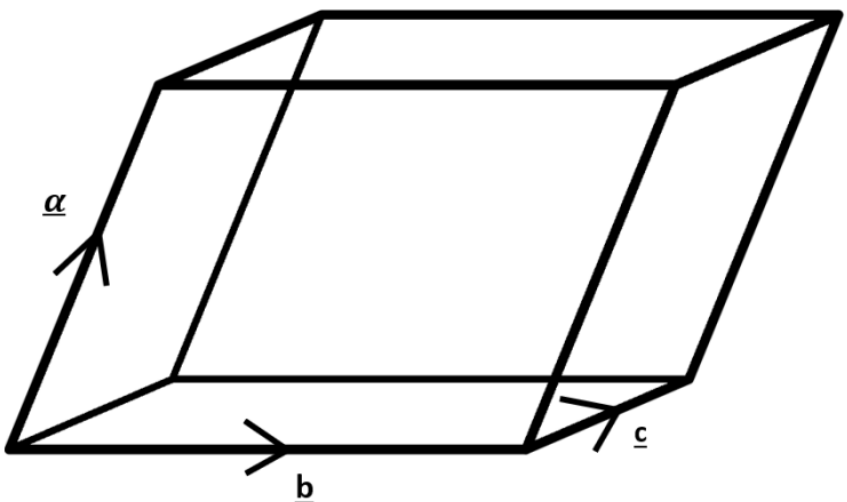
The required shortest distance is

$$\left| \frac{2(2)+2(2)+3(-3)+3}{\sqrt{2^2+2^2+(-3)^2}} \right|$$

$$= \frac{2}{\sqrt{17}}$$

**Volume of a Parallelepiped in $\mathbb{R}^3$.**

A parallelepiped is a three–dimensional generalisation of a parallelogram as in the picture. A parallelepiped is determined by the three vectors $\underline{a}$, $\underline{b}$, $\underline{c}$.




**Theorem 8.**

The volume of the above parallelepiped is

$$|\underline{a}.(\underline{b} \times \underline{c})|$$

$\underline{a}.(\underline{b} \times \underline{c})$ is called the triple product of the three vectors $\underline{a}, \underline{b}, \underline{c}$.

**Example 17.**

Find the volume of the parallelepiped determined by $\underline{a} = (1, 0, -1)$, $\underline{b} = (2, 1, 2)$, $\underline{c} = (3, 2, -1)$.

**Solution.**

The required volume $V$ is given by

$$V = |\underline{a}.(\underline{b} \times \underline{c})|$$

Now, $\underline{b} \times \underline{c} = -5i + 8j + k$ and so $\underline{a}.(\underline{b} \times \underline{c}) = -6$. Hence, $V = 6$.

## Chapter 4 – Matrices.

### Section 4.1 – Systems of linear equations.

**Definition 1.**

A linear equation in the $n$ variables $x_1, x_2, \ldots, x_n$ is given by

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$$

where $b,\ a_i$ are constants for $1 \leq i \leq n$.

**Example 1.**

$2x_1 + 3x_2 = 5$ is a linear equation. This is also the equation of a line in $\mathbb{R}^2$.

**Example 2.**

$2x_1 + 3x_2 - x_3 = 5$ is a linear equation. This is also the equation of a plane in $\mathbb{R}^3$.

**Example 3.**

(i) $3x - \frac{1}{2}y + 2 = 0$ is a linear equation.

(ii) $x^2 + 3y = 5$ is not a linear equation.

(iii) $\cos x + 4y - z = 2$ is not a linear equation.

**Example 4.**

Note that two lines in $\mathbb{R}_2$:

$$ax + by = c \quad \text{and} \quad rx + sy = t \quad \text{where} \quad a, b, c, r, s, t \quad \text{are constants}$$

intersect in either no points, exactly one point or or infinitely many points.