15. Expected Value, Variance, Standard Deviation

Definition 15.1. *Let X be a discrete random variable with range R_X. The* **expected value, mean** *or* **average** *of X, denoted by $\mathbb{E}(X)$, is defined as*

$$\mathbb{E}(X) = \sum_{k \in R_X} k \cdot P_X(k)$$

Example 15.2. *(1) We throw a die, that is, $\Omega = \{1,2,3,4,5,6\}$ and let X denote the outcome, that is, $X(\omega) = \omega$. Then*

$$\mathbb{E}(X) = \sum_{k=1}^{6} k \cdot \underbrace{P_X(k)}_{=\frac{1}{6}} = \frac{1}{6} \cdot (1+2+3+4+5+6) = \frac{21}{6} = 3.5$$

(2) Let $X \sim \text{Bernoulli}(p)$. Then $R_X = \{0,1\}$ and

$$\mathbb{E}(X) = 0 \cdot P_X(0) + 1 \cdot P_X(1) = 0 \cdot (1-p) + 1 \cdot p = p$$

(3) Let $X \sim \text{Geometric}(p)$. Then $R_X = \{1,2,3,\dots\}$ and $P_X(k) = p(1-p)^{k-1}$, for $k \in R_X$. Then

$$\mathbb{E}(X) = \sum_{k=1}^{\infty} k \cdot P_X(k) = p \cdot \underbrace{\sum_{k=1}^{\infty} k \cdot (1-p)^{k-1}}_{=\frac{1}{(1-(1-p))^2}} = \frac{1}{p}$$

(4) Let $X \sim \text{Poisson}(\lambda)$. Then $R_X = \{0,1,2,\dots\}$ and $P_X(k) = \frac{e^{-\lambda}\lambda^k}{k!}$, for $k \in R_X$. Then

$$\begin{aligned}\mathbb{E}(X) &= \sum_{k=1}^{\infty} k \cdot P_X(k) = e^{-\lambda} \cdot \sum_{k=1}^{\infty} \frac{\lambda^k}{(k-1)!} = \lambda \cdot e^{-\lambda} \cdot \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} \\ &= \lambda \cdot e^{-\lambda} \cdot e^{\lambda} = \lambda\end{aligned}$$

Lemma 15.3. *Let X and $X_1, \dots, X_n$ be random variables and $a, b \in \mathbb{R}$. Then*

(1) $\mathbb{E}(a \cdot X + b) = a \cdot \mathbb{E}(X) + b$
(2) $\mathbb{E}(X_1 + \ldots + X_n) = \mathbb{E}(X_1) + \ldots + \mathbb{E}(X_n)$

Proof. (1) Let $Y = a \cdot X + b$. Then $R_Y = \{a \cdot x + b : \ x \in R_X\}$. Also $P(Y = a \cdot x + b) = P(X = x)$, for all $x \in R_X$. Now

$$\mathbb{E}(a \cdot X + b) = \sum_{y \in \mathbb{R}_Y} y \cdot P_X(y) = \sum_{x \in \mathbb{R}_X} (a \cdot x + b) \cdot P_X(x) = a \cdot \mathbb{E}(X) + b$$

(2) We show the case $n = 2$. First note that

$$\mathbb{E}(X_1) = \sum_{k_1 \in R_{X_1}} k_1 \cdot P(X_1 = k_1) = \sum_{k_1 \in R_{X_1}} k_1 \cdot \sum_{k_2 \in R_{X_2}} P(X_1 = k_1, X_2 = k_2)$$
$$= \sum_{\substack{k_1 \in R_{X_1} \\ k_2 \in R_{X_2}}} k_1 \cdot P(X_1 = k_1, X_2 = k_2)$$

and likewise

$$\mathbb{E}(X_2) = \sum_{\substack{k_1 \in R_{X_1} \\ k_2 \in R_{X_2}}} k_2 \cdot P(X_1 = k_1, X_2 = k_2)$$

Then

$$\mathbb{E}(X_1 + X_2) = \sum_{\substack{k_1 \in R_{X_1} \\ k_2 \in R_{X_2}}} (k_1 + k_2) \cdot P(X_1 = k_1, X_2 = k_2)$$
$$= \sum_{\substack{k_1 \in R_{X_1} \\ k_2 \in R_{X_2}}} k_1 \cdot P(X_1 = k_1, X_2 = k_2) + \sum_{\substack{k_1 \in R_{X_1} \\ k_2 \in R_{X_2}}} k_2 \cdot P(X_1 = k_1, X_2 = k_2)$$
$$= \mathbb{E}(X_1) + \mathbb{E}(X_2)$$

□

Example 15.4. *(1) Let* $X \sim \text{Binomial}(n, p)$. *Then* $X = \sum_{i=1}^{n} Y_i$, *where* $Y_i \sim \text{Bernoulli}(p)$, *for all* $i = 1, \ldots, n$. *Now*

$$\mathbb{E}(X) = \mathbb{E}(Y_1 + \ldots + Y_n) = n \cdot \mathbb{E}(Y_1) = n \cdot p$$

(2) Let $X \sim \text{Pascal}(r, p)$. *Then* $X = \sum_{i=1}^{r} Y_i$, *where* $Y_i \sim \text{Geometric}(p)$, *for all* $i = 1, \ldots, r$. *Now*

$$\mathbb{E}(X) = \mathbb{E}(Y_1 + \ldots + Y_r) = r \cdot \mathbb{E}(Y_1) = \frac{r}{p}$$

Definition 15.5. *Let* X *be a discrete random variable. The* **variance** *of* X, *denoted by* $\text{Var}(X)$, *is defined as*

$$\text{Var}(X) = \mathbb{E}([X - \mathbb{E}(X)]^2)$$

Furthermore, the **standard deviation** *of* X, *denoted by* σ_X, *is defined as*

$$\sigma_X = \sqrt{\text{Var}(X)}$$

Remark 15.6. *Variance and standard deviation indicate how far or how close* X *is distributed around its average.*

Lemma 15.7. *Let X be a discrete random variable. Then*

$$\mathrm{Var}(X) = \mathbb{E}(X^2) - [\mathbb{E}(X)]^2$$

Proof. We have

$$\begin{aligned}\mathrm{Var}(X) &= \mathbb{E}([X - \mathbb{E}(X)]^2) = \mathbb{E}([X^2 - 2X\mathbb{E}(X) + [\mathbb{E}(X)]^2) \\ &= \mathbb{E}(X^2) - 2\mathbb{E}(X) \cdot \mathbb{E}(X) + \mathbb{E}([\mathbb{E}(X)]^2) = \mathbb{E}(X^2) - 2\mathbb{E}(X)^2 + [\mathbb{E}(X)]^2 \\ &= \mathbb{E}(X^2) - [\mathbb{E}(X)]^2\end{aligned}$$

□

Example 15.8. *(1) We throw a die, that is, $\Omega = \{1, 2, 3, 4, 5, 6\}$ and let X denote the outcome. Then $\mathbb{E}(X) = 3.5$. Furthermore*

$$\mathbb{E}(X^2) = \sum_{k=1}^{6} k^2 \cdot P_X(k) = \frac{1}{6} \cdot (1 + 4 + 9 + 16 + 25 + 36) = \frac{91}{6} = 15.166...$$

Hence $\mathrm{Var}(X) = \mathbb{E}(X^2) - [\mathbb{E}(X)]^2 = 15.166 - (3.5)^2 = 2.91$. *Furthermore* $\sigma_X = \sqrt{2.91} = 1.708$.

(2) Let $X \sim \mathrm{Bernoulli}(p)$. Then $R_X = \{0, 1\}$ and $\mathbb{E}(X) = p$. Furthermore

$$\mathbb{E}(X^2) = 0^2 \cdot P_X(0) + 1^2 \cdot P_X(1) = 0 \cdot (1 - p) + 1 \cdot p = p$$

and so $\mathrm{Var}(X) = p - p^2 = p(1 - p)$.

Lemma 15.9. (1) *Let X be a random variable and $a, b \in \mathbb{R}$. Then*

$$\mathrm{Var}(a \cdot X + b) = a^2 \cdot \mathrm{Var}(X).$$

(2) *Let $X_1, \ldots, X_n$ be independent random variables. Then*

$$\mathrm{Var}(X_1 + \ldots + X_n) = \mathrm{Var}(X_1) + \ldots + \mathrm{Var}(X_n).$$

Proof. (1) Set $Y = a \cdot X + b$. Then $\mathbb{E}(Y) = a \cdot \mathbb{E}(X) + b$. Now

$$\begin{aligned}\mathrm{Var}(Y) &= \mathbb{E}[(Y - \mathbb{E}(Y))^2] = \mathbb{E}[(a \cdot X + b - a \cdot \mathbb{E}(X) - b)^2] \\ &= \mathbb{E}[a^2 \cdot (X - \mathbb{E}(X))^2] = a^2 \cdot \mathbb{E}[(X - \mathbb{E}(X))^2] = a^2 \cdot \mathrm{Var}(X)\end{aligned}$$

□

Example 15.10. *Let $X \sim \mathrm{Binomial}(n, p)$. Then $X = \sum_{i=1}^{n} Y_i$, where $Y_i \sim \mathrm{Bernoulli}(p)$, for all $i = 1, \ldots, n$, and all Y_i are independent. Now*

$$\mathrm{Var}(X) = \mathrm{Var}(Y_1 + \ldots + Y_n) = n \cdot \mathrm{Var}(Y_1) = n \cdot p \cdot (1 - p)$$