DEPARTMENT OF MATHEMATICS: COURSE MT232P

PROBLEM SHEET 2

DEADLINE: 4pm Monday 14 November

1. Let X be the set of all possible words in the English language. Prove that X is countable.

2. Prove that the set of all lines through the origin in $\mathbb{R}^2$ is uncountable.

3. Let X be the open interval $(0,1) \subset \mathbb{R}$ and let $S = \{(x,y) \in \mathbb{R}^2 \,|\, 0 < x, y < 1\}$ be the open unit square.

   (a) Find an injective function from X to S.

   (b) Use the fact that every real number has a decimal expansion to find an injective function from S to X.

   (c) Taken together, what can be deduced from (a) and (b) about the cardinality of the sets X and S? Justify your answer.

4. Let $A \subset \mathbb{R}$. Prove that if a is an upper bound for A and if $a \in A$ then $a = lub(A)$.

5. Let $A \subset \mathbb{R}$ and suppose that A is bounded below. Define

$$B = \{b \in \mathbb{R} \,|\, b \text{ is a lower bound for } A\}.$$

   (a) Prove that $lub(B) \in B$.

   (b) Prove that $lub(B) = glb(A)$.

   (c) Use part (b) to show that all non-empty sets of real numbers that are bounded below have a greatest lower bound.