Fiacre Ó Cairbre

Lecture 6

Example 24 continued

Treat x and z as constant and differentiate with respect to y to get

$$\frac{\partial f}{\partial y} = \frac{2}{x+2y+3z}$$

Treat x and y as constant and differentiate with respect to z to get

$$\frac{\partial f}{\partial z} = \frac{3}{x+2y+3z}$$

Remark 10.

We can differentiate twice to get second order partial derivatives. For example,

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$$

Example 25.

If $f(x,y) = x\cos y + ye^x$, then

$$\frac{\partial f}{\partial x} = \cos y + ye^x$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = ye^x$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = -\sin y + e^x$$

$$\frac{\partial f}{\partial y} = -x\sin y + e^x$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = -x\cos y$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = -\sin y + e^x$$

Theorem 4 – Mixed Partial Derivatives Theorem.

Suppose $f : \mathbb{R}^2 \to \mathbb{R}$. Suppose $f(x, y)$ and its partial derivatives

$$\frac{\partial f}{\partial x}, \quad \frac{\partial f}{\partial y}, \quad \frac{\partial^2 f}{\partial x \partial y}, \quad \frac{\partial^2 f}{\partial y \partial x}$$

are defined in an open set containing (a, b) and are also continuous at (a, b). Then

$$\frac{\partial^2 f}{\partial x \partial y}_{|(a,b)} = \frac{\partial^2 f}{\partial y \partial x}_{|(a,b)}$$

Section 1.5 – Differentiability.

Remark 11.

Suppose $g : \mathbb{R} \to \mathbb{R}$. Suppose g is differentiable at a, i.e. $g'(a)$ exists.

Then $g'(a) = \lim\limits_{h \to 0} \dfrac{g(a+h) - g(a)}{h}$.

Let $\Delta x = h$ and so Δx denotes the change in x as x goes from a to $a + \Delta x$. Similarly, let Δy denote the corresponding change in $g(x)$ as x goes from a to $a + \Delta x$, i.e. $\Delta y = g(a + \Delta x) - g(a)$.

Then,

$$g'(a) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} \quad \text{and so} \quad 0 = \lim_{\Delta x \to 0} \left(\frac{\Delta y}{\Delta x} - g'(a)\right)$$

Let $\epsilon = \dfrac{\Delta y}{\Delta x} - g'(a)$ and so $\lim\limits_{\Delta x \to 0} \epsilon = 0$

Then

$$\Delta y = g'(a)\Delta x + \epsilon \Delta x, \;\text{ where }\; \epsilon \to 0 \;\text{ as }\; \Delta x \to 0 \qquad (*)$$

It is this property $(*)$ that we will generalise to functions of two variables.

Theorem 5.

Suppose $f:\mathbb{R}^2 \to \mathbb{R}$. Suppose that the partial derivatives $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$ are defined on an open set W containing the point (a,b). Suppose that $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$ are continuous at (a,b). Define Δz as $\Delta z = f(a+\Delta x, b+\Delta y) - f(a,b)$ so that Δz is the change in f as (x,y) goes from (a,b) to $(a+\Delta x, b+\Delta y)$ in W.

Then, we have the following generalisation of $(*)$:

$$\Delta z = \frac{\partial f}{\partial x}_{|(a,b)}\Delta x + \frac{\partial f}{\partial y}_{|(a,b)}\Delta y + \epsilon_1 \Delta x + \epsilon_2 \Delta y, \text{where } \epsilon_1 \text{ and } \epsilon_2 \to 0 \text{ as } \Delta x \text{ and } \Delta y \to 0 \; (**)$$

Definition 15.

Suppose $f:\mathbb{R}^2 \to \mathbb{R}$. We say f is differentiable at (a,b) if $\dfrac{\partial f}{\partial x}_{|(a,b)}$ and $\dfrac{\partial f}{\partial y}_{|(a,b)}$ exist and $(**)$ holds for f at (a,b). We say f is differentiable if it's differentiable at every point in its domain.

Theorem 6.

Suppose $f:\mathbb{R}^2 \to \mathbb{R}$. If $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$ are continuous on an open set W, then f is differentiable at every point in W.

Theorem 7.

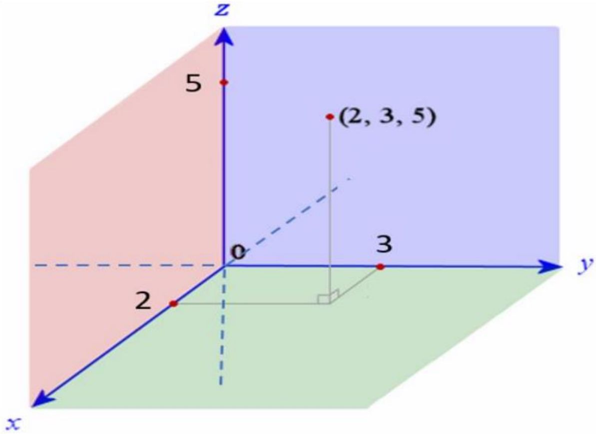
Suppose $f:\mathbb{R}^2 \to \mathbb{R}$. If f is differentiable at (a,b), then f is continuous at (a,b).

Section 1.6 – Quadrics.

In this section we will group some surfaces under the heading of quadric surfaces.

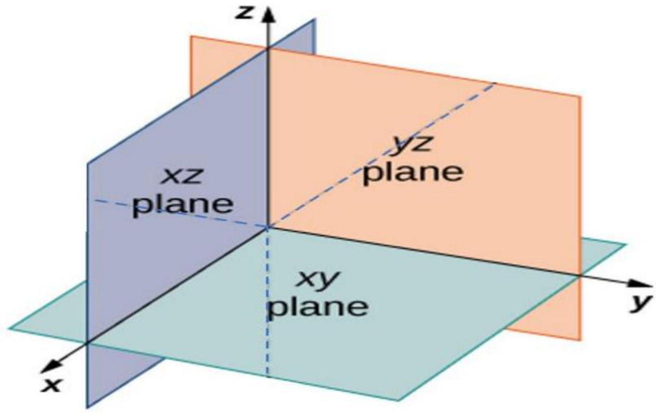
Remark 12.

Recall how to plot points in $\mathbb{R}^3$.



Remark 13.

Recall planes in $\mathbb{R}^3$.



Remark 14.

The equation of a quadric surface in $\mathbb{R}^3$ is a second degree equation of the form

$$kx^2 + ly^2 + mz^2 + nxy + pxz + qyz + rx + sy + tz + u = 0$$

where $k, l, m, n, p, q, r, s, t, u \in \mathbb{R}$

So, the quadric surface is the set of points:

$$\{(x, y, z) \in \mathbb{R}^3 : kx^2 + ly^2 + mz^2 + nxy + pxz + qyz + rx + sy + tz + u = 0\}$$

The intersection of the surface with a plane is called the trace of the surface in the plane. In order to visualise the surface, it can be useful to find the traces in planes parallel to the xy-plane, planes parallel to the xz–plane and planes parallel to the yz-plane. Note that a plane parallel to the xy–plane has the equation $z = t$, for some constant $t \in \mathbb{R}$. Similarly, a plane parallel to the xz–plane has the equation $y = s$, for some constant $s \in \mathbb{R}$. Finally, a plane parallel to the yz–plane has the equation $x = w$, for some constant $w \in \mathbb{R}$.

Example 26 – Elliptic paraboloid.

The equation is

$$z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$$

where $a, b \in \mathbb{R}$. The relevant traces in a plane

parallel to the xy–plane are ellipses $\quad (*)$

parallel to the xz–plane are parabolas $\quad (**)$

parallel to the yz–plane are parabolas

The reason for (*) above is because when $z = t$, for some constant $t \in \mathbb{R}$, then $t = \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2}$ is an ellipse. The reason for (**) above is because when $y = s$, for some constant $s \in \mathbb{R}$, then $z = \dfrac{x^2}{a^2} + \dfrac{s^2}{b^2}$ is a parabola. See the picture below.