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MT231P: Integration

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Contents

1. Sets and numbers

Informally a set is a collection of distinct things, which are called the elements of the set. In particular if the elements are listed, the order in which they occur is unimportant. Elements of a set may themselves be sets. We say that a is an element of A, if A is a set, one of whose elements is a. We also say that a belongs to A, or that A contains a. This is written symbolically as $a \in A$.

Two sets A and B are equal if they contain the same elements. This means that the statement $a \in A$ is equivalent to the statement $a \in B$.

The notion of a set is familiar but beware! There are logical paradoxes which must be avoided e.g. there is no set of all sets.

The empty set is the unique set ϕ or $\{\}$ which satisfies $a \notin \{\}$, for all a.

The set A is a subset of the set B, written $A \subseteq B$, if $a \in A$ implies that $a \in B$.

Trivial but useful: $A = B$ iff $A \subseteq B$ and $B \subseteq A$.

Let $P(x)$ be a mathematical statement which depends on a variable x. Assume that for any specific x, $P(x)$ is either true or false. Given a set X, the notation $S = \{x \in X \mid P(x)\}$ defines S as a subset of X.

Intersection, union, complement, symmetric difference.

Venn diagrams are useful to represent a small number of sets.

Cartesian product of sets A and B is the set $A \times B = \{(a, b) \mid a \in A, b \in B\}$ of pairs.

Example: $\mathbb{R} \times \mathbb{R}$ is the Cartesian plane: represents all points in a 2-dimensional plane.

Numbers

The set of natural numbers/ counting numbers is $\mathbb{N} = \{1, 2, 3, \ldots\}$.

Arithmetic: addition and multiplication.

We will use the principal of induction: suppose that $S \subseteq \mathbb{N}$ and $1 \in S$, and whenever $s \in S$ then $s + 1 \in S$. Then $S = \mathbb{N}$.

The set of integers or whole numbers is $\mathbb{Z} = \{0, \pm 1, \pm 2, \pm 3, \ldots\}$.

Meaning of $\leq$ and $<$.

The integers are totally ordered by $<$ i.e. given $n \neq m \in \mathbb{Z}$ either $n < m$ or $m < n$.

We speak of the *ring of integers*. This is $\mathbb{Z}$, together with the familiar arithmetic operations of addition and multiplication.

Rational numbers: $\mathbb{Q} = \{\frac{a}{b} \mid a, b \in \mathbb{Z}, b \neq 0\}$.

Equality: $\frac{a}{b} = \frac{c}{d}$ iff $ad = bc$ (an identity in $\mathbb{Z}$).

We can think of $\frac{a}{b}$ as the set of pairs $\{(c, d) \in \mathbb{Z} \times \mathbb{Z} \mid ad = bc\}$.

Example: $\frac{1}{2}$ can be identified with $\{(1, 2), (-1, -2), (2, 4), (-2, -4), (3, 6), (-3, -6), \ldots\}$.

The rationals are totally ordered: given $\frac{a}{b}$ and $\frac{c}{d}$ we write $\frac{a}{b} < \frac{c}{d}$ if $b, d > 0$ and $ad < bc$. Addition and multiplication of rationals:

$$\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}, \quad \frac{a}{b}\frac{c}{d} = \frac{ac}{bd}.$$

The relation between $\leq$ and arithmetic: given $a, b, c \in \mathbb{Q}$ with $a \leq b$. Then $a + c \leq b + c$. Also $ac \leq bc$, if $0 \leq c$. On the other hand $bc \leq ac$ if $c \leq 0$.

We speak of the *field of rational numbers*. This is $\mathbb{Q}$, together with the arithmetic operations of addition and multiplication. We call it a field, because each non-zero rational has a multiplicative inverse.

We will use an informal definition of the real numbers: we can place all rational numbers on a straight line. These numbers do not fill up the line. The reals are all numbers on the line (what does that mean?).

See MT232P Analysis 1, in Semester 2, for a more rigourous definition of real number.

Dedekind cut: a pair of sets of rational numbers A, B such that every rational is either in A or in B, $a < b$ for all $a \in A$ and $b \in B$ and A has no largest member.

Idea: (A, B) represents the point on the number line dividing A from B.

$\mathbb{Q}$ is the set of all (A, B) where B has a smallest member.

Example: 0 is represented by $A = \{x \in \mathbb{Q} \mid x < 0\}$ and $B = \{x \in \mathbb{Q} \mid 0 \leq x\}$. Notice that 0 is the smallest member of B.

Example: $\sqrt{2}$ is represented by $A = \{\frac{a}{b} \mid b > 0, a^2 < 2b\}$ and $B = \{\frac{a}{b} \mid b > 0, a^2 \geq 2b\}$.

An irrational number is a real number which is not rational. So $\sqrt{2}$ is irrational, as there is no $\frac{a}{b} \in \mathbb{Q}$ such that $\left(\frac{a}{b}\right)^2 = 2$.

Real numbers, decimal and geometric representation. A decimal number represents a rational number iff it has only finitely many non-zero digits or if its digits are periodic, after some position.

Certain rational numbers have exactly two decimal representations: e.g. $1 = 0.9999'$.

Consider $[0, 1]$ and $x \in [0, 1]$. Write $x = 0.d_1d_2d_3d_4\ldots$, with integers $0 \leq d_i \leq 9$ if $\frac{d_1}{10} \leq x \leq \frac{d_1+1}{10}$ and $\frac{d_1}{10} + \frac{d_2}{10^2} \leq x \leq \frac{d_1}{10} + \frac{d_2+1}{10^2}$, e.t.c.

Note: given $b > 1 \in \mathbb{N}$, every natural number (and hence every integer) has a base b representation. Likewise every real number has a base b representation.

Axioms of arithmetic: $+$ and $\times$ for $\mathbb{R}$. They satisfy the same aritmetic properties as in $\mathbb{Z}$ and $\mathbb{Q}$.

Note that, like $\mathbb{Q}$, the real numbers $\mathbb{R}$ form a field, the field of real numbers.

The axioms of order. $a \leq b$ then $a + c \leq b + c$ and if $0 \leq c$ then $ac \leq bc$.

A set S of real numbers is bounded above if there exists $r \in \mathbb{R}$ such that $s \leq r$, for all $s \in S$. We say that r is an upper bound for S. We call r a supremum of S if r is an upper bound for S, and $r < r'$, whenever r' is an upper bound for S. So supremum is the same as *least upper bound*.

Bounded below. Infimum. Bounded. The following assumption about $\mathbb{R}$ is called the *completeness axiom*:

Axiom 1. *Every non-empty set of real numbers which is bounded above has a supremum. Every non-empty set of real numbers which is bounded below has an infimum.*

The supremum $\sup(S)$ and the infimum $\inf(S)$.

Example: $\sup\{\frac{1}{n} \mid n \in \mathbb{N}\} = 1$ and $\inf\{\frac{1}{n} \mid n \in \mathbb{N}\} = 0$.

Example: $\sup\{s \in \mathbb{R} \mid 0 < s, s^2 < 2\} = \sqrt{2}$ and $\inf\{s \in \mathbb{R} \mid 0 < s, s^2 < 2\} = 0$.

A real number is said to be irrational if it is not rational. The set of irrational numbers does not form a ring.

Density: between any two distinct rationals there is an irrational. Between any two distinct irrationals there is a rational.

Positive and negative real numbers. Absolute value.

Intervals of real numbers: $[a, b], (a, b], [a, b), (a, b)$. Use of $\pm\infty$.

Inequalities: $|x - 2| < 1$ has solution set $(1, 3)$ and $|x - 3| \geq 2$ has solution set $(-\infty, 1] \cup [5, \infty)$.

3. Function

A function f consists of 3 things: a set X called the domain of f, a set Y called the codomain of f and an assignment to each $x \in X$ of an element $f(x) \in Y$. Write $y = f(x)$ to express the fact that $y \in Y$ is the image of $x \in X$ under f. Write $f : X \to Y$ to express that f is a function with domain X and codomain Y.

Note: a function g is equal to f if g also has domain X and codomain Y and $g(x) = f(x)$, for all $x \in X$.

Note that given $y \in Y$, there can be several $x \in X$ such that $f(x) = y$, or no such x. The set of $y \in Y$ for which $y = f(x)$ for some $x \in X$ is called the image of f, denoted $\mathrm{Im}(f)$.

Example: $y = x^2$ defines a function $\mathbb{R} \to \mathbb{R}$, the squaring map. The domain and codomain are both $\mathbb{R}$ but the image of this function is the set $[0, \infty)$ of non-negative real numbers.

Example: $f : [0, \infty) \to [0, \infty)$, with $f(x) = x^2$ is a function with domain and codomain $[0, \infty)$. It is different from the squaring function on $\mathbb{R}$, as it has a different domain and a different codomain.

The set of ordered pairs $\{(x, y) \in X \times Y \mid y = f(x)\}$ is called the graph of f, denoted $\mathrm{Gr}(f)$. So f is determined by $\mathrm{Gr}(f)$, together with knowledge of the codomain Y.

Given $y \in Y$, define $f^{-1}(y) := \{x \in X \mid y = f(x)\}$. So $f^{-1}(y)$ is a subset of X (possibly the empty set), for all $y \in Y$. We call the set of all such preimages $\{f^{-1}(y) \mid y \in Y\}$ the kernel of f, denoted $\mathrm{Ker}(f)$.

Note that $\mathrm{Ker}(f)$ forms a partition of X.

Example: the squaring map on $\mathbb{R}$ has kernel $\{\{r, -r\} \mid r \in \mathbb{R}\} \cup \{\{\}\}$. So the sets in the kernel are: the empty set $\{\}$, the singleton set $\{0\}$ and the 2-elements subsets $\{r, -r\}$, for non-zero $r \in \mathbb{R}$.

Example: say $f(x) = \sqrt{2x + 4}$, where $\sqrt{\ }$ means positive real square root. This formula makes sense for $x \geq -2$. So the domain of f is $[-2, \infty)$. We could take f to have codomain $\mathbb{R}$. The smallest possible codomain for a function f defined by this formula is $[0, \infty)$.

Example: say $f(z) = \sqrt{2z+4}$, for all complex numbers z. The formula does not define a function, as $w = f(z)$, for a complex number w if and only if $w^2 = 2z + 4$, and there are usually 2 possible w's ($w = 0$ if $z = -2$ is exceptional). So in order for the formula to define a function, we would need to specify a choice of w. This can be complicated!

Convention for domains. The functions $f(x) = \frac{x^2}{x}$ and $g(x) = x$ are not the same. They have the same values for $x \neq 0$ but $f(0)$ is not defined. So $f(x)$ and $g(x)$ have different domains. We call $x = 0$ a removable singularity of $f(x)$. We say that $g(x)$ is an extension of $f(x)$ from $\mathbb{R}\backslash\{0\}$ to $\mathbb{R}$.

Given a function $f : X \to Y$ and a subset S of X, the restriction of f to S is the function $f_S : S \to Y$ defined by $y = f(x)$, for all $x \in S$.

A function f is said to be injective if whenever $x_1, x_2 \in X$ and $f(x_1) = f(x_2)$ then $x_1 = x_2$. Equivalently f is injective iff $|f^{-1}(y)| \leq 1$ for all $y \in Y$.

A function f is said to be surjective if whenever $\mathrm{Im}(f) = Y$ i.e. for all $y \in Y$ there is $x \in X$ such that $y = f(x)$. Equivalently f is surjective iff $|f^{-1}(y)| \geq 1$ for all $y \in Y$.

A function is said to be bijective iff it is both injective and surjective.

Example: there is a bijection $(0, \infty) \to \mathbb{R}$.

Exercise: find a bijection $[0, \infty) \to \mathbb{R}$.

Fact: strictly increasing functions $\mathbb{R} \to \mathbb{R}$ are bijective.

Every injective function has an inverse: if E is the image of D under f, then $f^{-1} : E \to \mathbb{R}$ is the function defined by $f^{-1}(x) = y$, where $x = f(y)$, for all $x \in E$.

Example: $f(x) = x^2$ maps $\mathbb{R}$ onto $[0, \infty)$. It restricts to a bijection on $[0, \infty)$. The inverse function of this bijection is $g(x) = \sqrt{x}$.

Example: $f(x) = \frac{x}{\sqrt{x^2+1}}$ has domain $\mathbb{R}$. It is a bijection from $\mathbb{R}$ onto $(-1, 1)$. The inverse function is $f^{-1}(x) = \frac{x}{\sqrt{1-x^2}}$.

Example: $f(x) = x + \frac{1}{x} : \mathbb{R}\backslash\{0\} \to \mathbb{R}\backslash(-2, 2)$. It is injective on each interval $(-\infty, -1], [-1, 0), (0, 1]$ and $[1, \infty)$. There are two algebraic expressions defining inverse functions: $y = \frac{x+\sqrt{x^2-4}}{2}$ maps $|x| \geq 2$ onto $[-1, 0) \cup [1, \infty)$ and $y = \frac{x-\sqrt{x^2-4}}{2}$ maps $|x| \geq 2$ onto $(-\infty, -1] \cup (0, 1]$.

Elementary functions

The functions $y = x$ and $y = \frac{1}{x}$. The former has domain $\mathbb{R}$ and the latter has domain $\mathbb{R}\backslash\{0\}$.

The function $y = x^2$ is two-to-one on $\mathbb{R}\backslash\{0\}$. Restricting gives a bijection $[0, \infty) \to [0, \infty)$. We denote the inverse function by $y = \sqrt{x}$; for $x \geq 0$, $y = \sqrt{x}$ iff $y \geq 0$ and $y^2 = x$. So generally $\sqrt{x}$ is taken to be the positive square-root of x.

Power functions: x^n, x^{-n} and $x^{1/n}$ for $n \in \mathbb{N}$. Also x^q for $q \in \mathbb{Q}$.

What about x^r for $r \in \mathbb{R}$?

Composition of functions: $g \circ f$ and $g(f(x))$.

Example: $f(x) = 1 - x^2$ and $g(x) = \sqrt{x}$. Then f has domain $\mathbb{R}$ and g has domain $\mathbb{R}_{\geq 0}$. Also $g \circ f(x) = \sqrt{1 - x^2}$ for $x \in [-1, 1]$ and $f \circ g(x) = 1 - x$, for $x \geq 0$.

Sum, difference, product and quotient of functions.

Polynomial functions: linear, quadratic, cubic.

Rational functions $\frac{P(x)}{Q(x)}$. Horizontal and vertical asymptotes. If $P(x)$ and $Q(x)$ have no common factor then each zero of $Q(x)$ is a vertical asymptote. Linear asymptotes: the rational function $g(x) = \frac{x^3+1}{x^2+2x}$ is asymptotic to $x - 2$ in the sense that $\lim_{x\to\infty}(g(x) - (x - 2)) = 0$.

$\lim_{x\to\infty}(\sqrt{x^2 + 1} - x) = 0$ and $\lim_{x\to\infty}(\sqrt{x^2 + x} - x - \frac{1}{2}) = 0$.

4. Limits

Limits of sequences

A (real) sequence is an infinite list of (real) numbers

$$t_1, t_2, t_3, t_4, \ldots$$

Write this compactly as (t_n) or $(t_n)_{n=1}^{\infty}$. The first term is t_1, the second t_2 and in general t_n is the n-th term, for $n = 1, 2, 3, \ldots$.

Alternatively, a sequence is a function $t : \mathbb{N} \to \mathbb{R}$. We write t_n for the function value at n. So $t_n = t(n)$.

Frequently we describe a sequence by just giving a formula for the n-th e.g. $(1/n)$ or (n^2). It is customary to use letters like n, m for integer or natural number variables.

We say that (t_n) converges to $L \in \mathbb{R}$ if the terms t_n get closer and closer to L, as n gets larger and larger. We write this as $\lim\limits_{n\to\infty} t_n = L$ and call L the limit of (t_n).

Example: $\lim(1/n) = 0$.

Example: $\lim(n^2)$ does not exist. In fact, we can say $\lim(n^2) = +\infty$.

Precise definition: given a sequence (t_n) and a real number L, then $\lim\limits_{n\to\infty} t_n = L$ if for every $\epsilon > 0$ there exists (an integer) $N > 0$ such that $|t_n - L| < \epsilon$ for all $n \geq N$.

Triangle inequality: given 3 real numbers a, b, c we have $|a - b| \leq |a - c| + |c - b|$.

Uniqueness of limits: say $\lim(t_n) = L$ and $\lim(t_n) = L'$, where L, L' are real numbers. We claim that $L = L'$. For otherwise $|L - L'| > 0$. Choose $\epsilon > 0$ with $\epsilon < |L - L'|/2$. Then from the definition, there exists $N > 0$ such that $|t_n - L| < \epsilon$ for all $n \geq N$. Likewise there exists $N' > 0$ such that $|t_n - L'| < \epsilon$ for all $n \geq N'$. Without loss of generality (WLOG) $N > N'$. So for all $n > N$, we have $|t_n - L| < \epsilon$ and $|t_n - L'| < \epsilon$. Then we get the contradiction

$$|L - L'| \leq |L - t_n| + |t_n - L'| < \epsilon + \epsilon < |L - L'|.$$

Example: $\lim(n^{1/n}) = 1$. Proof: show first that $n^{1/n} > 1$ for all $n > 1$. So write $n^{1/n} = 1 + k_n$ for some $k_n > 0$. Then $n = (1 + k_n)^n$. Show that $n - 1 > \frac{1}{2}n(n-1)k_n^2$. So

$k_n^2 \leq 2/n$. Given $\epsilon > 0$, choose N so that $2/N < \epsilon^2$. Then $2/n < \epsilon^2$ for all $n \geq N$. So $k_n \leq \epsilon$ for all $n \geq N$.

Example: $\lim(1 + 1/n)^n = e$.

Example: $\lim(\sin(n)/n) = 0$. See the Squeeze theorem below, and the fact that $|\sin n| \leq 1$.

Theorem 2 (Bolzano-Weierstrass)**.** *Every bdd sequence has a convergent subsequence.*

Call (t_n) a Cauchy sequence if for each $\epsilon > 0$ there exists N such that $|t_n - t_m| < \epsilon$ for all $n, m \geq N$.

A sequence is convergent if and only if it is Cauchy.

We are interested in the behaviour of a function $f(x)$ at a real number c, provided that there are domain points of f arbitrarily close to, but distinct from c.

We call c a *cluster point* of the domain of f if for each $\delta > 0$ there exists x with $0 < |x - c| < \delta$ such that $f(x)$ is defined.

If c is a cluster point of the domain of f, we say that $\lim\limits_{x \to c} f(x)$ exists and equals a real number L if $f(x)$ gets closer and closer to L for all points in the domain of f which are sufficiently close to c.

More precisely: we write $\lim\limits_{x \to c} f(x) = L$ if given $\epsilon > 0$ there exists $\delta > 0$ such that if $|x - c| < \delta$, $x \neq c$ and $f(x)$ is defined, then $|f(x) - L| < \epsilon$.

Let $f : D \to \mathbb{R}$ be a real valued function, where $D \subseteq \mathbb{R}$.

Recall that we say that f converges to $L \in \mathbb{R}$ at the cluster point c of D, and write $\lim_{x\to c} f(x) = L$, if for all real $\epsilon > 0$, there exists $\delta > 0$ such that for all $x \in \mathrm{Dom}(f)$ with $|x - c| < \delta$, we have $|f(x) - L| < \epsilon$.

Example: $\lim_{x\to c} x = c$.

Example: $\lim_{x\to 0} \frac{1}{x}$ does not exist.

Example: $\lim_{x\to 0} \frac{x^2}{x} = 0$, even though 0 is not in the domain of $\frac{x^2}{x}$.

Example: $\lim_{x\to 0} \sin(1/x)$ does not exist but $\lim_{x\to 0} x \sin(1/x) = 0$.

Exercise: what does $\lim_{x\to\infty} f(x) = L$ mean?

Exercise: $\lim_{x\to\infty} x \sin(1/x) = 1$.

Sum, difference, product and quotient rule for limits.

Theorem 3 (Squeeze theorem)**.** *Suppose $f, g, h : A \to \mathbb{R}$, c is a cluster point of A and*

$$f(x) \le g(x) \le h(x), \quad \text{for all } x \in A, x \neq c.$$

If $\lim_{x\to c} f(x) = \lim_{x\to c} h(x) = L$, then $\lim_{x\to c} g(x) = L$.

Continuity

Let $c \in D$. We say that f is continuous (cts) at c if $\lim_{x\to c} f(x) = f(c)$.

We say that f is **continuous** if it is cts at each point of its domain. We say that f is cts on $I \subseteq D$ if f is cts at each $c \in I$.

We will be interested in the case that I is an interval of $\mathbb{R}$.

Example: polynomial functions and rational functions are cts.

Example: the inverse of a cts bijection is cts.

Thomae's function: Define $h : [0, 1] \to \mathbb{R}$ by

$$h(x) = \begin{cases} \frac{1}{b}, & \text{if } x = \frac{a}{b} \text{ is rational.} \\ 0, & \text{if } x \text{ is irrational.} \end{cases}$$

Then h is cts at each irrational but not cts at any rational.

Question: suppose we define

$$g(x) = \begin{cases} 1, & \text{if } x \text{ is rational.} \\ 0, & \text{if } x \text{ is irrational.} \end{cases}$$

At what points is $g(x)$ cts?

Question: suppose we define

$$f(x) = \begin{cases} x, & \text{if } x \text{ is rational.} \\ 0, & \text{if } x \text{ is irrational.} \end{cases}$$

At what points is $f(x)$ cts?

Example: $\frac{\sin(x)}{x}$ is not defined at $x = 0$. However $\lim_{x \to 0} \frac{\sin(x)}{x} = 1$. Thus the following function is cts on $\mathbb{R}$:

$$s(x) = \begin{cases} \frac{\sin(x)}{x}, & \text{if } x \in \mathbb{R} \text{ and } x \neq 0. \\ 1, & \text{if } x = 0. \end{cases}$$

Similarly $\lim_{x \to 0} \frac{1-\cos(x)}{x} = 0$. We can see this numerically.

Example: $\frac{1}{x}$ is cts at each point of its domain, but is not defined at $x = 0$. In fact, there is no $L \in \mathbb{R}$ such that the following function is cts:

$$f(x) = \begin{cases} \frac{1}{x}, & \text{if } x \in \mathbb{R} \text{ and } x \neq 0. \\ L, & \text{if } x = 0. \end{cases}$$

Sums, difference, products, quotients of continuous functions.

Pinching Lemma. Example: $\lim_{x \to 0} x \sin(1/x) = 0$.

Theorem 4 (Intermediate Value Theorem)**.** *Suppose that f is continuous on $[a, b]$ and v is a number between $f(a)$ and $f(b)$. Then there exists $c \in [a, b]$ such that $f(c) = v$.*

Theorem 5 (Min-max theorem)**.** *Suppose that $f : [a, b] \to \mathbb{R}$ is continuous. Then f attains its maximum and minimum values on $[a, b]$.*

Proof. First show that $f([a, b])$ is bounded.

Next let $s = \sup f([a, b])$, a real number.

For each $n > 0$, $s - \frac{1}{n}$ is not an upper bound for $f(I)$. So choose $x_n \in [a, b]$ such that $s - \frac{1}{n} < f(x_n) \leq s$. Now (x_n) is a bdd sequence. So by Bolzano-Weierstrass it has a convergent subsequence (x_{n_r}). Let $x = \lim(x_{n_r})$. Then $x \in [a, b]$. As f is cts, $f(x) = \lim f(x_{n_r})$. Show that $f(x) = s$.

Repeat with $\operatorname{Inf} f([a, b])$. $\square$

Theorem 6 (Chain rule for continuity)**.** *Suppose that $g : I \to \mathbb{R}$ and $f : J \to \mathbb{R}$ and $g(I) \subseteq J$. If $c \in I$ and g is cts at c, and f is cts at $g(c)$, then $f \circ g$ is cts at c.*

Note: if $f : I \to J$ is a cts bijection, then the inverse function $f^{-1} : J \to I$ is cts.

Uniform continuity: we say that f is uniformly continuous on I if for each $\epsilon > 0$ there exists $\delta > 0$ such that $|f(x_1) - f(x_2)| < \epsilon$ whenever $x_1, x_2 \in I$ and $|x_1 - x_2| < \delta$.

Note: if $I = [a, b]$ is a closed bdd interval, then f is cts on $[a, b]$ iff f is uniformly cts on $[a, b]$.

Example: $\frac{1}{x}$ is cts on $(0, 1]$, is uniformly cts on each interval $[\epsilon, 1]$, for $0 < \epsilon < 1$, but it is not uniformly cts on $(0, 1]$.

6. Differentiation

What is a curve in the plane? A curve is the image of a function from an interval into the plane, $\phi : I \to \mathbb{R}^2$. The curve is said to be continuous if ϕ is continuous, and simple if ϕ is injective.

Example: Say I is an interval and $f : I \to \mathbb{R}$. Then the graph of f defines a curve; the associated ϕ is $\phi(x) = (x, f(x))$, for all $x \in I$.

Given a curve C in the plane, and a point P on the curve, we say that a line L is tangent to C at P if L 'just touches' C at P. Alternatively 'L is the line through two points on the curve which are infinitesmally close to P'. What do these definitions really mean?

A function f is differentiable at x_0, with derivative m if the line $y = f(x_0) + m(x - x_0)$ is tangent to the graph $y = f(x)$ at x_0.

Given $f : I \to \mathbb{R}$, where I is an interval, we say that $f(x)$ is differentiable at $c \in I$ if $\lim_{x\to c} \frac{f(x)-f(c)}{x-c}$ exists. If the limit exists, we denote it by $f'(c)$. If $f(x)$ is differentiable at every point of I, we let $f'(x)$ denote the corresponding function $c \mapsto f'(c)$.

Theorem 7. *If $f'(c)$ exists then f is continuous at c.*

For suppose that $f(x)$ is defined in the interval I containing c, then $f(x) - f(c) = \frac{f(x)-f(c)}{x-c}(x - c)$. So taking limits we get

$$\lim_{x\to c}(f(x) - f(c)) = \lim_{x\to c}\left(\frac{f(x) - f(c)}{x - c}\right)\lim_{x\to c}(x - c) = f'(c).0 = 0.$$

Converse is false: $f(x) = x \sin(1/x)$ is continuous at 0 but not differentiable at 0.

Examples: $x^{1/3}$ has a vertical tangent line at 0. Next $x^{2/3}$ has a double tangent line or cusp at 0. Finally $|x|$ has a corner at 0; the difference quotient has different left and right limits -1 and 1. So none of these functions is diff at 0.

The Cantor function $c : [0, 1] \to [0, 1]$ is defined as follows: Express $x \in [0, 1]$ in base 3. If x contains a 1, replace every digit strictly after the first 1 by 0. Replace any remaining 2's with 1's. Interpret the result as a binary number. The result is $c(x)$. Then $c(x)$ is cts, non-constant, and its derivative exists and equals 0 almost everywhere.

7. Rules for differentiation, Caratheodory theorem, sine and cosine

Weierstrass showed that $f(x) = \sum_{n=0}^{\infty} \frac{1}{2^n} \cos(3^n x)$ is cts but not diff at any real number.

Given functions f and g, there are sum, difference, product and quotient functions:

$$f + g, \quad f - g, \quad fg, \quad \frac{f}{g}.$$

Exercise: $(f+g)' = f' + g'$ and $(\lambda f)' = \lambda(f')$, for each $\lambda \in \mathbb{R}$.

Lemma 8 (Leibnitz)**.** *Let f and g be diff on D. Then fg is diff on D and $(fg)' = f'g + fg'$.*

Example: $\frac{dx^n}{dx} = nx^{n-1}$, for $n = 1, 2, \ldots$. This can be proved by induction on n.

Exercise: Say f is diff on D and $f \neq 0$ on D. Then $(1/f)$ is diff on D and $(1/f)' = -f'/f^2$.

Exercise: Let f and g be diff on D and say g does not vanish on D. Then f/g is diff on D and $(f/g)' = \frac{f'g - fg'}{g^2}$.

Theorem 9 (Carathéodory)**.** *Suppose that $f : I \to \mathbb{R}$ and $c \in I$. Then $f'(c)$ exists iff there exists $\phi : I \to \mathbb{R}$ such that ϕ is cts at c and $f(x) - f(c) = \phi(x)(x - c)$, for all $x \in I$. If ϕ exists then $f'(c) = \phi(c)$.*

Example: $f(x) = x^3$. For $c \in \mathbb{R}$ we have $x^3 - c^3 = (x^2 + cx + c^2)(x - c)$. So take $\phi(x) = x^2 + cx + c^2$, whence $f'(c) = 3c^2$.

Trignometric functions: we say that a point (x, y) on the unit circle $x^2 + y^2 = 1$ makes an angle of θ radians with the positive X-axis if the arc from $(1, 0)$ to (x, y) has length θ. So as (x, y) rotates ccw around the circle, starting at $(1, 0)$, the angle increases from 0 radians to 2π radians. The cosine and sine functions are the coordinate functions $x = \cos(\theta)$ and $y = \sin(\theta)$, for $0 \leq \theta < 2\pi$. So $\cos^2(\theta) + \sin^2(\theta) = 1$, for all θ.

Power series expressions for cosine and sine:

$\cos(x) = 1 - \frac{x}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots + (-1)^n \frac{x^{2n}}{(2n)!} \pm \ldots,$

$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \ldots \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} \pm \ldots.$

As a consequence $\cos'(x) = -\sin(x)$ and $\sin'(x) = \cos(x)$. Also $\tan'(x) = \sec^2(x) = 1 + \tan^2(x)$.

8. Mean Value Theorem, Chain Rule

An open interval is any interval of the real numbers of the form (a, b), where $a < b$.

We say that a function f is increasing on an interval I if

$$x < x' \quad \text{implies that} \quad f(x) < f(x'), \quad \text{for all } x, x' \in I.$$

Example: x^2 is increasing on $[0, 1]$ but not on any interval $[-r, 1]$, for $r > 0$.

The following important result is a consequence of Carathéodory's theorem:

Corollary 10. *Say f is differentiable at c and $f'(c) > 0$. Then f is increasing on some open interval containing c.*

More precisely, the conclusion is that there is an $\epsilon > 0$ such that $f(x) < f(x')$, for all $x, x' \in (c - \epsilon, c + \epsilon)$, so long as $x < x'$.

Exercise: if $f'(c) < 0$ then f is decreasing on some open interval containing c.

We say that f has a relative max value at a point c in its domain if there are a, b with $a < c < b$ and $f(x) \leq f(c)$, for all $x \in (a, b)$. Relative min value. Relative extremum. Global extremum.

Theorem 11. *Say $f : [a, b] \to \mathbb{R}$, $a < c < b$ and f has a relative extremum at c. Then $f'(c) = 0$ or $f'(c)$ does not exist.*

Corollary 12 (Mean Value Theorem)**.** *Say $f : [a, b] \to \mathbb{R}$ is cts on $[a, b]$ and diff on (a, b). Then there exists $c \in (a, b)$ such that $f'(c) = \frac{f(b)-f(a)}{b-a}$.*

Proof. Replace f by $g(x) = f(x) - \frac{b-x}{b-a} f(a) - \frac{x-a}{b-a} f(b)$. So g is cts on $[a, b]$ and diff on (a, b). In addition $g(a) = g(b) = 0$. If $g \equiv 0$, $f'(x) = \frac{f(b)-f(a)}{b-a}$, for all x. So we are done.

Say $g \not\equiv 0$. Then g has an extremum in (a, b). So there is $c \in (a, b)$ such that $g'(c) = 0$. $\square$

Theorem 13 (Rolle's Theorem)**.** *Say $f : [a, b] \to \mathbb{R}$ is cts on $[a, b]$ and diff on (a, b) and $f(a) = f(b)$. Then there exists $c \in (a, b)$ such that $f'(c) = 0$.*

Theorem 14 (Chain rule for differentiation)**.** *Suppose that $f : I \to \mathbb{R}$, $g : J \to \mathbb{R}$ and $g(J) \subseteq I$. Let $c \in J$. If g is diff at c and f is diff at $g(c)$, then $f \circ g$ is diff at c and $(f \circ g)'(c) = f'(g(c))g'(c)$.*

Proof: this follows easily from Carathéodory's Theorem.

Inverse functions: Say f is a real valued function which is injective on an interval I. Set $J = f(I)$. Then the inverse function of f is the function $g : J \to \mathbb{R}$, such that $g(y) = x$, where $y = f(x)$, for all $y \in J$.

Notice that for g, we have reversed the roles of x and y. But we can still write $g(x) = y$ where $f(y) = x$.

Example: $f(x) = x^2$, for all $x \in [0, \infty)$. Then $g(x) = \sqrt{x}$, for all $x \in [0, \infty)$.

Example: $f(x) = x + \frac{1}{x}$ is defined on $\mathbb{R}\backslash\{0\}$ but not injective. However it is injective on $[1, \infty)$ (to see this, show that $f(x_1) = f(x_2)$ iff $x_2 = x_2$ or $x_2 = \frac{1}{x_1}$). The inverse of f on $[1, \infty)$ is a function $g : [2, \infty) \to \mathbb{R}$ such that $g(x) = \frac{x+\sqrt{x^2-1}}{2}$, for all $x \in [2, \infty)$.

Derivative of inverse: Say $f : I \to \mathbb{R}$ is invertible, with inverse function $g : J \to \mathbb{R}$. For $c \in J$, say f is diff at $g(c)$ and $f'(g(c)) \neq 0$. Then g is diff at c and

$$g'(c) = \frac{1}{f'(g(c))}.$$

Example: $g(x) = \frac{x+\sqrt{x^2-1}}{2}$ is diff on $(2, \infty)$.

Example: $\frac{d\arccos(x)}{dx} = \frac{-1}{\sqrt{1-x^2}}$, $\frac{d\arcsin(x)}{dx} = \frac{1}{\sqrt{1-x^2}}$. Notice that $\arccos(x)+\arcsin(x) = \pi/2$.

Example: $\frac{d\arctan(x)}{dx} = \frac{1}{1+x^2}$.

Derivative of a power: f^n has derivative $nf^{n-1}f'$ for all $n \in \mathbb{Z}$.

Say $n > 0$ and set $g = f^{-n}$. Then $1 = f^n g$. So the product rule gives

$$0 = nf^{n-1}f'g + f^n g'.$$

Solving for g' we get

$$g' = \frac{-nf^{n-1}g}{f^n} = \frac{-ng}{f} = -nf^{-n-1}.$$

Implicit differentiation: Say $x = y + \frac{1}{y}$. Then $y^2 - xy + 1 = 0$. So $2y\frac{dy}{dx} - y - x\frac{dy}{dx} = 0$ or $\frac{dy}{dx} = \frac{y}{2y-x}$.

Example: $(x-1)(x^2+y^2) = -2x^2$ implicitly defines y as a function of x, for $x \in [-1, 1)$.

Let I be a closed bounded interval. A partition of I is a finite set $P : x_0 < x_1 < \cdots < x_n$ of real numbers such that $I = [x_0, x_n]$. Then P divides I into subintervals $I_i = [x_i, x_{i-1}]$, for $i = 1, \ldots, n$. The norm of P is $\max |x_i - x_{i-1}|$, written $||P||$. The smaller $||P||$ is the finer the partition is.

A tagged partition is a partition P together with a set of tags $t_i \in I_i$. We write $\dot{P}$ to denote a tagged partition on P.

Now suppose that $f : I \to \mathbb{R}$ is a function. The Riemann sum $S(f, \dot{P})$ associated with $\dot{P}$ is $\sum_{i=1}^{n} f(t_i)(x_i - x_{i-1})$. This approximates the 'area under the graph' of f.

We say that f is Riemann integrable on I, and write $f \in R(I)$, if there is a real number L such that for each $\epsilon > 0$ there exists $\delta > 0$ so that $|S(f, \dot{P}) - L| < \epsilon$ for all tagged partitions $\dot{P}$ with $||P|| < \delta$. If $f \in R([a, b])$, we write $L = \int_a^b f$ or $\int_a^b f(x)dx$.

A **step function** is $f : I \to \mathbb{R}$ such that there is a partition P of I such that f is constant on each subinterval of P.

Example: constant functions and step functions are integrable.

Cauchy Criterion: $f \in R(I)$ iff for every $\epsilon > 0$ there exists $\delta > 0$ such that $|S(f, \dot{P}) - S(f, \dot{Q})| < \epsilon$ for all tagged partitions $\dot{P}, \dot{Q}$ with $||P||, ||Q|| < \delta$.

Step functions are integrable.

The characteristic function $\phi(r) = \begin{cases} 1, & r \in \mathbb{Q} \\ 0, & r \notin \mathbb{Q} \end{cases}$ of $\mathbb{Q}$ is not integrable on $[0, 1]$.

Thomae's function $h(a/b) = 1/b$ is integrable on $[0, 1]$ and indeed $\int_0^1 h = 0$.

Squeeze Theorem: $f \in R(I)$ iff for every $\epsilon > 0$ there exist $g, h \in R(I)$ with $g \le f \le h$ and $\int_I (g - h) < \epsilon$.

We say that f is bounded (bdd) on a closed interval I if there exists $L > 0$ such that $|f(x)| < L$ for all $x \in I$.

Theorem 15. *If f is integrable on $[a, b]$ then f is bounded on $[a, b]$.*

Note f cts on $[a, b]$ then f is bdd on $[a, b]$. Also f is uniformly cts on $[a, b]$.

Theorem 16. *If f is cts on $[a, b]$ then f is Riemann integrable on $[a, b]$.*

First show: given $\epsilon > 0$ there exists $\delta > 0$ such that if $u, v \in [a, b]$ and $|u - v| < \delta$ then $|f(u) - f(v)| < \epsilon$ i.e. a function which is cts on a closed bdd interval is uniformly cts.

Now given $\epsilon > 0$, choose $\delta > 0$ such that as above if $u, v \in I$ and $|u - v| < \delta$ then $|f(u) - f(v)| < \epsilon/(b - a)$. Now suppose that P is a partition with $||P|| < \delta$. Let g be the step function given by the min of f on P and let h be the step function given by the max of f on P. Then $g \leq f \leq h$. Also $g, h \in R([a, b])$ and $\int_a^b (g - h) < \epsilon$. So $f \in R([a, b])$, by the squeeze lemma.

Example: Cantor's function $c(x)$ is integrable on $[0, 1]$. What is $\int_0^1 c(x)dx$?

Exercise: suppose that f is cts on $[a, b]$, that $f \geq 0$ and that $\int_a^b f = 0$.
Prove that $f(x) = 0$ for all $x \in [a, b]$.

We say that f is monotone increasing on $[a, b]$ if $f(x_1) \leq f(x_2)$ whenever $a \leq x_1 \leq x_2 \leq b$. Similarly f is monotone decreasing. We say that f is *monotone* if it is monotone increasing or monotone decreasing.

Fact: if f is monotone on $[a, b]$ then f is integrable on $[a, b]$.

Theorem 17. *Say $f : [a, b] \to \mathbb{R}$ and $c \in (a, b)$. Then f is integrable on $[a, b]$ iff its restrictions are integrable on $[a, c]$ and $[c, b]$, respectively, in which case $\int_a^b f = \int_a^c f + \int_c^b f$.*

Define $\int_b^a f = -\int_a^b f$, for $a < b$ and $f \in R([a, b])$.

10. The Fundamental Theorem of Calculus

Recall the Mean Value Theorem is

Theorem 18. *Say $f : [a,b] \to \mathbb{R}$ is cts on $[a,b]$ and diff on (a,b). Then there exists $c \in (a,b)$ such that $f'(c) = \frac{f(b)-f(a)}{b-a}$.*

Theorem 19 (Fundamental Theorem of Calculus, 1st Form)**.** *Let f and $F : [a,b] \to \mathbb{R}$ such that $F'(x) = f(x)$ for all $x \in [a,b]$ and $f \in R([a,b])$. Then $\int_a^b f(x)dx = F(b) - F(a)$.*

(In fact we only require $F'(x) = f(x)$ on all but finitely many elements of $[a,b]$.)

Proof uses: given $P : x_0, \ldots, x_n$, and i, there exists $u_i \in [x_{i-1}, x_i]$ such that $F(x_i) - F(x_{i-1}) = F'(u_i)(x_i - x_{i-1})$ by the Mean Value Theorem. Thus

$$F(b) - F(a) = \sum_{i=1}^{n} f(u_i)(x_i - x_{i-1}).$$

Theorem 20 (Fundamental Theorem of Calculus, 2nd Form)**.** *Let f be integrable on $[a,b]$ and set $F(x) = \int_a^x f(t)dt$ for all $x \in [a,b]$. Suppose that f is cts at $c \in [a,b]$. Then F is diff at c and $F'(c) = f(c)$.*

(We call F the indefinite integral of f with basepoint a)

Corollary: f cts on $[a,b]$. Then its indefinite integral is diff on $[a,b]$, with derivative f.

Example: $\int \cos(x)dx = \sin(x) + C$, $\int \sin(x)dx = -\cos(x) + C$ and $\int \tan(x)dx = -\ln(\cos(x)) + C$.

Example: $\int \tan^2(x)dx = \tan(x) - x + C$, using the fact that $\frac{d\tan(x)}{dx} = \tan^2(x) + 1$.

Let h be Thomae's function on $[0,1]$. Then $H(x) = \int_0^x h$ is identically zero on $[0,1]$. So $H'(x) = 0$. But $H'(x) \neq h(x)$, for all rational $x \in [0,1]$. Note that h is not cts at any rational and is cts at every irrational.

Example: Say ϕ is diff and define $F(x) = \displaystyle\int_a^x f(t)dt$. So $F(\phi(x)) = \displaystyle\int_a^{\phi(x)} f(t)dt$. Then

$$\frac{dF(\phi(x))}{dx} = \frac{dF}{dx}(\phi(x))\phi'(x) = f(\phi(x))\phi'(x).$$

So $\displaystyle\int_a^{\phi(x)} f(t)dt$ has derivative $f(\phi(x))\phi'(x)$.

Integration by substitution

Suppose that $\phi : [a,b] \to \mathbb{R}$ is diff, and ϕ' is cts on $[a,b]$. Let I be an interval containing $\phi[a,b]$ and let $f : I \to \mathbb{R}$ be cts. Then

$$\int_a^b f(\phi(t))\phi'(t)dt = \int_{\phi(a)}^{\phi(b)} f(x)dx$$

Example: the substitution $t = u^2$ gives $\displaystyle\int_{a^2}^{x^2} f(t)dt = \int_a^x 2uf(u^2)du$. So $\displaystyle\int_{a^2}^{x^2} f(t)dt$ has derivative $2xf(x^2)$.

Example: $\displaystyle\int_1^4 \frac{\sin(\sqrt{t})}{\sqrt{t}}dt = 2(\cos(1) - \cos(2))$.

Example: $\displaystyle\int_0^8 \frac{\cos(\sqrt{x+1})}{\sqrt{x+1}}dx = 2\int_1^3 \cos(u)du = 2\sin(3) - 2\sin(1)$.

Example: $\displaystyle\int_a^b f(x+c)dx = \int_{a+c}^{b+c} f(x)dx$.

Example: $\displaystyle\int_a^b xf(x^2)dx = \frac{1}{2}\int_{a^2}^{b^2} f(x)dx$.

Recall that the harmonic sum $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges (meaning that the sequence with n-th term $t_n := \sum_{k=1}^{n} \frac{1}{k}$ does not have a limit). In fact $t_n > \frac{k+1}{2}$, for $n > \frac{1}{2^k}$.

Define $L(x) = \int_1^x \frac{1}{t} dt$ for $x > 0$.

Note: let $\phi(x) = \frac{1}{x}$, for $x > 0$. Then $\phi : (0, 1] \to [1, \infty)$ is a bijection, and $\phi'(x) = -\frac{1}{x^2}$. Substitution gives

$$L(x) = \int_1^x \frac{1}{t} dt = \int_{\phi(1)}^{\phi(x)} \frac{1}{\phi(t)} \phi'(t) dt = -\int_1^{1/x} \frac{1}{t} dt = -L(1/x).$$

Interpreting $L(x)$ as 'the area under the graph $y = 1/x$ from 1 to x', we see that that for $x \in [n, n+1]$ we have

$$\frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n+1} < L(x) < 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}.$$

Note that $L'(x) = \frac{1}{x}$. So L is monotone increasing and cts on $(0, \infty)$. Also $L(1) = 0$. Now $L(x) \to \infty$, as $x \to \infty$. So L maps $[1, \infty)$ bijectively onto $[0, \infty)$, and thus also $(0, 1]$ bijectively onto $(-\infty, 0]$. So $L : (0, \infty) \to \mathbb{R}$ is a bijection.

Let $E(x) : \mathbb{R} \to (0, \infty)$ be the inverse of $L(x)$. The formula for the derivative of an inverse function gives

$$E'(x) = \frac{1}{L'(E(x))} = E(x).$$

We call $L(x)$ the (natural) logarithm function, we call $E(x)$ the exponential function. These functions are denoted by $\ln(x)$ and $\exp(x)$ or e^x, respectively. Thus

$$\frac{d \ln(x)}{dx} = \frac{1}{x}, \qquad \frac{de^x}{dx} = e^x, \qquad \ln(e^x) = x, \qquad e^{\ln(x)} = x.$$

N.B.: $\exp(x)$ is the unique function on $\mathbb{R}$ satisfying $\exp'(x) = \exp(x)$ and $\exp(0) = 1$.

Definition: $e = E(1)$, Euler's number. Then $e \approx 2.71828182846\ldots$. It is known that e is irrational.

Note: $\exp(-x) = \exp(-\ln(\exp(x))) = \exp(\ln(1/\exp(x))) = 1/\exp(x)$.

Theorem 21. $\exp(x + y) = \exp(x)\exp(y)$ *for all* $x, y \in \mathbb{R}$ *and* $\ln(xy) = \ln(x) + \ln(y)$, *for all* $x, y \in (0, \infty)$.

Proof. Fix $y \in \mathbb{R}$ and define $G(x) = \frac{\exp(x+y)}{\exp(y)}$. Then $G'(x) = \frac{\exp(x+y)}{\exp(y)} = G(x)$ and $G(0) = 1$. So $G(x) = \exp(x)$, by uniqueness. Next we have

$$\ln(xy) = \ln\left(\exp(\ln(x))\exp(\ln(y))\right) = \ln(\exp(\ln(x) + \ln(y))) = \ln(x) + \ln(y).$$

$\square$

Define $x^0 = 1$, $x^1 = x$ and recursively $x^{n+1} = xx^n$, for $n = 1, 2, 3, \ldots$. Now $x \mapsto x^n$ is a bijection from $[0, \infty)$ to itself. Define $x^{1/n}$ as the inverse function. So $x^{1/n} > 0$ satisfies $(x^{1/n})^n = x$ and $(x^n)^{1/n} = x$.

Now suppose that $a/b \in \mathbb{Q}$ with $a, b > 0$. Define $x^{a/b} := (x^{1/b})^a$. This happens to equal $(x^a)^{1/b}$.

What about x^r, for $x > 0$, and $r \in \mathbb{R}$? We can define

$$x^r := \exp(r \ln(x)).$$

In particular $e^r = \exp(r)$.

Note: $(xy)^r = \exp(r\ln(xy)) = \exp(r\ln(x) + r\ln(y)) = \exp(r\ln(x))\exp(r\ln(y)) = x^r y^r$.

Note: $(x/y)^r = \exp(r\ln(x/y)) = \exp(r\ln(x) - r\ln(y)) = \exp(r\ln(x))/\exp(r\ln(y)) = x^r/y^r$.

Theorem 22. *$x^{a+b} = x^a x^b$, $x^{ab} = (x^a)^b$ and $x^{-a} = 1/x^a$.*

Theorem 23. *Let $f(x) = x^r$, for $x \in (0, \infty)$. Then f is diff and $f'(x) = rx^{r-1}$.*

For $f(x) = \exp(r\ln(x))$ is a composition of diff functions. The chain rule gives $f'(x) = \exp(r\ln(x))r/x = rx^r/x = rx^{r-1}$.

Theorem 24. *Let $f(x) = r^x$, for $r \in (0, \infty)$. Then f is diff on $\mathbb{R}$ and $f'(x) = \ln(r)f(x)$ i.e.*

$$\frac{dr^x}{dx} = \ln(r)r^x.$$

For $f(x) = \exp(x\ln(r))$. So $f'(x) = \ln(r)\exp(x\ln(r)) = \ln(r)r^x$.

Cosine and Sine

The unit circle is the set $\{(x,y) \in \mathbb{R} \mid x^2+y^2=1\}$. The length of its perimeter is 2π and its area is π. For the moment we define π to be its area. Its easy to estimate that $3 < \pi < 4$ by considering squares inside and outside the circle.

Consider a point (x,y) on the unit circle; $-1 \leq x \leq 1$ and $x^2+y^2=1$. Secretly $x = \cos\theta$ and $y = \sin\theta$, where θ is the angle subtended. But imagine we don't know what an angle is!

Consider the area of the circular arc $\Lambda(x)$ defined by (x,y). We give a geometric argument to show that $\Lambda'(x) = -\frac{1}{2\sqrt{1-x^2}}$ for $-1 < x < 1$.

Define $A(x) = \pi - \displaystyle\int_{-1}^{x} \frac{dt}{\sqrt{1-t^2}}$, for $x \in [-1,1]$. Then $A(-1) = \pi = 2\Lambda(-1)$ and $A'(x) = -\frac{1}{\sqrt{1-x^2}} = 2\Lambda'(x)$. So $A(x) = 2\Lambda(x)$ for all $x \in [-1,1]$. In particular $A : [-1,1] \to [0,\pi]$ is a bijection.

Let $C : [0,\pi] \to [-1,1]$ be the inverse function. Thus $C(\theta) = x$ iff $\theta = A(x)$. Also

$$C'(\theta) = \frac{1}{A'(C(\theta))} = -\sqrt{1 - C(\theta)^2}.$$

Set $S(\theta) = -C'(\theta)$. Then $S^2(\theta) + C^2(\theta) = 1$. So if (x,y) subtends area $A(x)$ then $y = S(x)$. Now $S(\theta) = 0$ iff $|C(\theta)| = 1$ iff $\theta = 0, \pi$.

Differentiate $S^2(\theta) + C^2(\theta) = 1$, to get $2S(x)S'(x) - 2C(x)S(x) = 0$, and $S(x) = 0$ iff $x = 0$. So for $x \neq 0, \pi$, we have $S'(x) = C(x)$.

Of course, $C(\theta)$ is really $\cos(\theta)$ and $S(\theta)$ is really $\sin(\theta)$.

Then $A(x)$ is $\arccos(x) : [-1,1] \to [0,\pi]$. Likewise $\sin(\theta)$ has an inverse $\arcsin(x) : [-1,1] \to [-\pi/2, \pi/2]$.

Note: $A(-1) = \pi$, $A(0) = \pi/2$ and $A(1) = 0$. So $\cos(\pi) = -1$, $\cos(\pi/2) = 0$ and $\cos(0) = 1$. Thus also $\sin(\pi) = \sin(0) = 0$ and $\sin(\pi/2) = 1$.

Note: $\cos(-x) = \cos(x)$, $\cos(x + 2\pi) = \cos(x)$ and $\cos(x+\pi) = -\cos(x)$.

Note: $\sin(x) = \cos(x - \pi/2)$ and $\cos(x) = \sin(x + \pi/2)$.

Note: $\sin(-x) = -\sin(x)$, $\sin(x+2\pi) = \sin(x)$ and $\sin(x+\pi) = -\sin(x)$.

Note: $\sin(a+b) = \sin(a)\cos(b) + \cos(a)\sin(b)$.

Tan and Cotan

Define $\tan(x) = \frac{\sin(x)}{\cos(x)}$. So

$$\tan'(x) = \frac{\cos^2(x) + \sin^2(x)}{\cos^2(x)} = \frac{1}{\cos^2(x)}, \quad \text{for } x \in (-\pi/2, \pi/2).$$

Now $\tan : (-\pi/2, \pi/2) \to \mathbb{R}$ is a monotone increasing function. Its inverse function is the monotone increasing function $\arctan : \mathbb{R} \to (-\pi/2, \pi/2)$. Interestingly the inverse function has two horizontal asymptotes. Also

$$\arctan'(x) = \frac{1}{1/\cos^2(\arctan(x))} = \cos^2(\arctan(x)) = \frac{1}{1+x^2}.$$

A useful identity

Set $t = \tan(x/2)$. Then $x = 2\arctan(t)$. So $dx = \frac{2}{1+t^2}dt$. Also

$$t = \frac{\sin(x/2)}{\cos(x/2)} = \frac{\sin(x/2)\cos(x/2)}{\cos^2(x/2)} = \frac{\sin(x)}{1+\cos(x)}$$

Next

$$\tan(a+b) = \frac{\sin(a)\cos(b) + \cos(a)\sin(b)}{\cos(a)\cos(b) - \sin(a)\sin(b)} = \frac{\frac{\sin(a)}{\cos(a)} + \frac{\sin(b)}{\cos(b)}}{1 - \frac{\sin(a)}{\cos(a)}\frac{\sin(b)}{\cos(b)}} = \frac{\tan(a) + \tan(b)}{1 - \tan(a)\tan(b)}.$$

Taking $a = b = x/2$, we get $\tan(x) = \frac{2t}{1-t^2}$. Write $\sin(x) = 2tA(t)$ and $\cos(x) = (1 - t^2)A(t)$. As $\sin^2(x) + \cos^2(x) = 1$, we get $1/A^2(t) = 4t^2 + (1-t^2)^2 = t^4 + 2t^2 + 1 = (1+t^2)^2$. So $A(t) = 1 + t^2$ and hence

$$\sin(x) = \frac{2t}{1+t^2}, \quad \cos(x) = \frac{1-t^2}{1+t^2}, \quad \text{where } t = \tan(x/2).$$

Example:

$$\begin{aligned}
\int_a^b \frac{1}{\sin(x)} dx &= \int_{\tan(a/2)}^{\tan(b/2)} \frac{1+t^2}{2t} \frac{2}{1+t^2} dt \\
&= \int_{\tan(a/2)}^{\tan(b/2)} \frac{1}{t} dt \\
&= \ln(t)\Big|_{\tan(a/2)}^{\tan(b/2)} \\
&= \ln(\tan(b/2)) - \ln(\tan(a/2)).
\end{aligned}$$

Note: $\cot(x) = \frac{\cos(x)}{\sin(x)}$ and $\csc(x) = \frac{1}{\cos(x)}$. Check that $\cot(x) + \csc(x) = \cot(x/2)$.

Example: $\int \frac{1}{\sin(x)+\cos(x)} dx$.

Theorem 25. *Suppose that $f : [a, b] \to \mathbb{R}$ is Riemann integrable and that $f \geq 0$. Then the area enclosed by $x = a, x = b, y = 0$ and $y = f(x)$ equals $\int_a^b f(x)dx$.*

Example: area under $\sin(x)$ between $x = 0$ and $x = \pi$ is $\int_0^\pi \sin(x)dx = -\cos(x)|_0^\pi = 2$.

Theorem 26. *Suppose that $\phi : [a, b] \to \mathbb{R}$ and $f : [\phi(a), \phi(b)] \to \mathbb{R}$ such that ϕ is diff, ϕ' is cts on $[a, b]$ and f is cts on $[\phi(a), \phi(b)]$. Then*

$$\int_a^b f(\phi(t))\phi'(t)dt = \int_{\phi(a)}^{\phi(b)} f(x)dx$$

Proof. We know that f is RI on $[\phi(a), \phi(b)]$. Set $F(x) = \int_{\phi(a)}^x f(s)ds$. Then Fun Thm Calc (2nd form) implies that $F'(x) = f(x)$, for all $x \in [\phi(a), \phi(b)]$. Next consider the composition $F \circ \phi : [a, b] \to \mathbb{R}$. The chain rule gives $(F \circ \phi)'(t) = f(\phi(t))\phi'(t)$, for all $t \in [a, b]$. Then Fun Thm Calc (1st form) implies that

$$\int_a^b f(\phi(t))\phi'(t)dt = F(\phi(b)) - F(\phi(a)) = \int_{\phi(a)}^{\phi(b)} f(x)dx.$$

$\square$

Example: the substitution $t = u^2$ gives $\int_{a^2}^{x^2} f(t)dt = \int_a^x 2uf(u^2)du$. So $\int_{a^2}^{x^2} f(t)dt$ has derivative $2xf(x^2)$.

Example: $\int_1^4 \frac{\sin(\sqrt{t})}{\sqrt{t}}dt = 2(\cos(1) - \cos(2))$. Use $u = \sqrt{x+1}$.

Example: $\int_0^8 \frac{\cos(\sqrt{x+1})}{\sqrt{x+1}}dx = 2\int_1^3 \cos(u)du = 2\sin(3) - 2\sin(1)$. Use $u = \sqrt{x+1}$.

Exercise: $\int_a^b f(x+c)dx = \int_{a+c}^{b+c} f(x)dx$ and $\int_a^b xf(x^2)dx = \frac{1}{2}\int_{a^2}^{b^2} f(x)dx$.

Example: $\int_0^a \sqrt{a^2 - x^2}dx$. For this set $x = a\sin(t)$. Then $\sqrt{a^2 - x^2} = a\cos(t)$ and $dx = a\cos(t)dt$. So $\int_0^a \sqrt{a^2 - x^2}dx = \int_0^{\pi/2} a^2\cos^2(t)dt = \frac{a^2}{2}(t + \sin(t)\cos(t))\Big|_0^{\pi/2} = \frac{a^2\pi}{4}$.

Exercise: determine $\int \sqrt{a^2 - x^2}dx$ using $x = a\tan(t)$.

A useful identity

Set $t = \tan(x/2)$. Then $x = 2\arctan(t)$. So $dx = \frac{2}{1+t^2}dt$. Also

$$t = \frac{\sin(x/2)}{\cos(x/2)} = \frac{\sin(x/2)\cos(x/2)}{\cos^2(x/2)} = \frac{\sin(x)}{1+\cos(x)}$$

Next

$$\tan(a+b) = \frac{\sin(a)\cos(b)+\cos(a)\sin(b)}{\cos(a)\cos(b)-\sin(a)\sin(b)} = \frac{\frac{\sin(a)}{\cos(a)}+\frac{\sin(b)}{\cos(b)}}{1-\frac{\sin(a)}{\cos(a)}\frac{\sin(b)}{\cos(b)}} = \frac{\tan(a)+\tan(b)}{1-\tan(a)\tan(b)}.$$

Taking $a = b = x/2$, we get $\tan(x) = \frac{2t}{1-t^2}$. Thus

$$\sin(x) = \frac{2t}{1+t^2}, \quad \cos(x) = \frac{1-t^2}{1+t^2}, \quad \text{where } t = \tan(x/2).$$

Note: $\cot(x) = \frac{\cos(x)}{\sin(x)}$, $\sec(x) = 1/\cos(x)$ and $\csc(x) = 1/\sin(x)$. Check that $\cot(x) + \csc(x) = \cot(x/2)$.

Example: $\displaystyle\int \sec(x)dx = \int \frac{1+t^2}{1-t^2}\frac{2}{1+t^2}dt = \int \frac{2}{1-t^2}dt$. Now $\frac{2}{1-t^2} = \frac{1}{1-t} + \frac{1}{1+t}$. So the integral is $-\ln(1-t) + \ln(1+t) = \ln\frac{1+t}{1-t}$. Finally,

$$\frac{1+t}{1-t} = \frac{(1+t)^2}{1-t^2} = \frac{1+t^2}{1-t^2} + \frac{2t}{1-t^2}.$$

Thus we get the famous identity $\displaystyle\int \sec(x)dx = \ln(\sec(x)+\tan(x)) + C$.

Exercise: $\displaystyle\int_a^b \csc(x)dx = \ln(\tan(b/2)) - \ln(\tan(a/2))$.

Exercise: $\displaystyle\int \frac{1}{\sin(x)+\cos(x)}dx$.

Antiderivatives

Given functions $f, F : D \to \mathbb{R}$, we say that F is the antiderivative of f if $F'(x) = f(x)$, for all $x \in D$. Any two antiderivatives of f differ by a constant function. So we use the notation $F(x) + C$ to denote any antiderivative of f.

If f is RI, we write

$$\int f(x)dx = F(x) + C.$$

to indicate that $F(x) = \displaystyle\int_a^x f(t)dt$, for some $a \in D$, and all $x \in D$. Then we call C the constant of integration.

Suppose that $f : A \to B$ is an invertible function, with inverse $g : B \to A$. Suppose also that F is an anti-derivative of f. Then Laisant proved that

$$G(x) = xg(x) - F(g(x)) \quad \text{is an antiderivative of } g.$$

Example: we know that e^x has inverse $\ln(x)$. Also e^x is an anti-derivative of e^x. So $x\ln(x) - x$ is an anti-derivative of $\ln(x)$.

Example: $\sin : [-\pi/2, \pi/2] \to [-1, 1]$ has inverse $\arcsin(x)$, and $\cos(x)$ is an antiderivative of $\sin(x)$. So $\displaystyle\int \arcsin(x)dx = x\arcsin(x) - \cos(\arcsin(x)) + C$.

Note that $e^{-x^2}, \sin(x^2), \frac{sin(x)}{x}, \frac{1}{\ln(x)}$ and x^x are each R.I. on their domains. So each has antiderivatives, yet the antiderivatives cannot be expressed in terms of standard functions.

Example: $F(x) = x^2\sin(1/x)$ is diff on $\mathbb{R}$, with derivative

$$f(x) = \begin{cases} 2x\sin(1/x) - \cos(1/x), & x \neq 0, \\ 0, & x = 0. \end{cases}$$

Notice that f is cts on $\mathbb{R}\backslash\{0\}$, but $\lim_{x\to 0} f(x) \neq f(0)$ i.e. f is not cts at $x = 0$.

Note: Volterra's function is a function $F : \mathbb{R} \to \mathbb{R}$ such that F is diff on $\mathbb{R}$, F' is bdd but F' is not Riemann-integrable on any interval of $\mathbb{R}$. Uses Cantor's set and copies of the function $x^2\sin(1/x)$

Integration by parts

Theorem 27. *Suppose that $u, v : [a,b] \to \mathbb{R}$ are diff on $[a,b]$. Then*

$$\int_a^b u dv = uv \Big|_a^b - \int_a^b v du.$$

Proof. Note that $(uv)'(x) = u'(x)v(x) + u(x)v'(x)$. Then we have $uv \mid_a^b = \int_a^b (uv)'(x)dx = \int_a^b (u(x)v'(x) + u'(x)v(x))dx$. $\square$

Example: For $\int xe^x dx$, take $u = x, dv = e^x dx$. Then $du = dx$ and $v = e^x$. So

$$\int xe^x dx = [xe^x] - \int e^x dx = (x-1)e^x + C.$$

Example: show that $\int x\tan^2(x)dx = \frac{-x^2}{2} + x\tan(x) + \ln(\cos(x)) + C$, using $u = x$ and $dv = \tan^2(x)dx$.

Example: $\int \ln(x)dx$. Let $u = \ln(x), dv = dx$. Then $du = \frac{1}{x}dx$ and $v = x$. So

$$\int \ln(x)dx = [x\ln(x)] - \int dx = x(\ln(x) - 1).$$

Example: $\int \arctan(x)dx$. Let $u = \arctan(x)$ and $dv = dx$. Then $du = \frac{1}{1+x^2}dx$ and $v = x$ and

$$\int \arctan(x)dx = [x\arctan(x)] - \int \frac{x}{1+x^2}dx = x\arctan(x) - \frac{1}{2}\ln(1+x^2) + C.$$

Partial fractions

Example: $\int \frac{x^3+3x^2}{x^2+1}dx$. Write $x^3 + 3x^2 = (x+3)(x^2+1) - (x+3)$. So

$$\begin{aligned}\int \frac{x^3+3x^2}{x^2+1}dx &= \int (x+3)dx - \int \frac{x}{x^2+1}dx - \int \frac{3}{x^2+1}dx \\ &= (\frac{1}{2}x^2 + 3x) - \frac{1}{2}\ln(x^2+1) - 3\arctan(x) + C.\end{aligned}$$

Example:

$$\begin{aligned}\int \frac{x}{2x-1}dx &= \frac{1}{2}\int \frac{2x-1}{2x-1}dx + \frac{1}{2}\int \frac{1}{2x-1}dx \\ &= \frac{x}{2} + \frac{1}{4}\ln|2x-1| + C.\end{aligned}$$

Example: $x^2-5x+6=(x-3)(x-2)$. Write $\frac{x+4}{x^2-5x+6}=\frac{A}{x-3}+\frac{B}{x-2}$. Then $A(x-2)+B(x-3)=x+4$. So $A+B=1$ and $-2A-3B=4$. Solve to get $A=7, B=-6$. Thus

$$\int \frac{x+4}{x^2-5x+6}dx=\int \frac{7}{x-3}dx-\int \frac{6}{x-2}dx=7\ln|x-3|-6\ln|x-2|+C.$$

Example: $\frac{x^2+3x+2}{x(x^2+1)}=\frac{A}{x}+\frac{Bx+C}{x^2+1}$. Then $A(x^2+1)+x(Bx+C)=x^2+3x+2$. So $A+B=1$, $C=3$ and $A=2$. Thus $B=-1$. Thus

$$\int \frac{x^2+3x+2}{x(x^2+1)}dx=\int \frac{2}{x}dx+\int \frac{-x+3}{x^2+1}dx=2\ln|x|-\frac{1}{2}\ln(x^2+1)+3\arctan(x^2+1)+C.$$

Recall we defined the limit of a sequence $\lim(a_n)$. We also defined the limit of a function $f(x)$, denoted $\lim_{x\to c} f(x)$ at a cluster point c of its domain. We can extend this notion to limits at infinity in various ways.

Given a function $f : [a, \infty) \to \mathbb{R}$, and a real number L, we write $\lim_{x\to+\infty} f(x) = L$ if for all $\epsilon > 0$, there is a real number N such that $|f(x) - L| < \epsilon$, for all $x > N$. Note that $y = L$ is then said to be a horizontal asymptote to the graph $y = f(x)$.

Example: $\lim_{x\to+\infty} \dfrac{\ln(x)}{x} = 0$ and $\lim_{x\to+\infty} \dfrac{x^2}{x^2+3x+2} = 1$.

We write $\lim_{x\to+\infty} f(x) = \infty$ if for all $M > 0$, there is a real number N such that $|f(x)| > M$, for all $x > N$. Similarly define $\lim_{x\to+\infty} f(x) = \pm\infty$

Example: $\lim_{x\to+\infty} \ln(x) = +\infty$ and $\lim_{x\to+\infty} \ln(\ln(x)) = +\infty$.

Next suppose that c is a cluster point of the domain of f. We write $\lim_{x\to c} f(x) = \infty$ if for all $M > 0$, there is $\delta > 0$ such that $|f(x)| > M$, for all x with $0 < |x - c| < \delta$. Note that $x = c$ is then said to be a vertical asymptote to $y = f(x)$.

Exercise: say what is meant by $\lim_{x\to-\infty} f(x) = L$, and $\lim_{x\to-\infty} f(x) = \pm\infty$ if $(-\infty, a]$ is contained in the domain of f.

Improper integrals

Suppose that $f : [a, \infty) \to \mathbb{R}$ is a function which is integrable on each interval $[a, b]$, for $a < b$. Then we define

$$\int_a^\infty f(x)dx = \lim_{b\to\infty} \int_a^b f(x)dx.$$

We say that $\int_a^\infty f(x)dx$ converges, diverges to $\pm\infty$ or does not exist as $\lim_{b\to\infty} \int_a^b f(x)dx$ converges, diverges to $\pm\infty$ or does not exist, respectively.

Similarly if $f : (-\infty, b) \to \mathbb{R}$ is a function which is integrable on each interval $[a, b]$, for $a < b$, we define

$$\int_{-\infty}^b f(x)dx = \lim_{a\to-\infty} \int_a^b f(x)dx.$$

We say that $\int_{-\infty}^b f(x)dx$ converges, diverges to $\pm\infty$ or does not exist as $\lim_{b\to\infty} \int_a^b f(x)dx$ converges, diverges to $\pm\infty$ or does not exist, respectively.

Example: Consider $\int_1^\infty \frac{1}{x^s}dx$. Suppose that $s > 1$. Then

$$\int_1^b \frac{1}{x^s}dx = \left.\frac{x^{1-s}}{1-s}\right|_1^b = \frac{1}{1-s}\left(\frac{1}{b^{s-1}} - 1\right).$$

Thus

$$\int_1^\infty \frac{1}{x^s}dx = \frac{1}{s-1}.$$

Next suppose that $s = 1$. Then

$$\int_1^b \frac{1}{x}dx = \ln(x)_1^b = \ln(b).$$

So

$$\int_1^\infty \frac{1}{x}dx = \infty.$$

Exercise: if $s < 1$ then $\int_1^\infty \frac{1}{x^s}dx = \infty$.

Example: $\int_0^\infty \cos(x)dx$ does not exist.

Example: $\int_{-\infty}^\infty \frac{\cos(x)}{x^2+1}dx = \frac{\pi}{e}$.

The normal distribution with mean μ and variance σ has probability density function $N(x) = \frac{1}{\sqrt{2\pi\sigma^2}}e^{-\frac{(x-\mu)^2}{2\sigma^2}}$. When $\mu = 0$ and $\sigma = 1$, we must have $\int_{-\infty}^\infty N(x)dx = 1$. Thus

$$\sqrt{\pi} = \int_{-\infty}^\infty e^{-x^2}dx.$$

Next let $f : (a,b] \to \mathbb{R}$ be a function which is integrable on $(c,b]$, for all $c \in (a,b]$. Then

$$\int_a^b f(x)dx := \lim_{c\to a+}\int_c^b f(x)dx.$$

Example: consider $\int_0^1 \frac{1}{x^s}dx$. For $s < 1$ we have

$$\int_\epsilon^1 \frac{1}{x^s}dx = \frac{1}{1-s}\left(1 - \frac{1}{\epsilon^{s-1}}\right) = \frac{1-\epsilon^{1-s}}{1-s}.$$

Thus $\int_0^1 \frac{1}{x^s}dx = \frac{1}{1-s}$.

Exercise: $\int_\epsilon^1 \frac{1}{x}dx = -\ln(\epsilon)$. So $\int_0^1 \frac{1}{x^s}dx = -\infty$.

16. Hyperbolic functions, Area between curves

Sinh and cosh

Recall DeMoivre's Theorem:

$$(\cos(x) + i\sin(x))^n = \cos(nx) + i\sin(nx), \quad \text{for all } n = 1, 2, 3, \ldots.$$

In fact, this is a consequences of Euler's formula:

$$e^{ix} = \cos(x) + i\sin(x), \quad \text{for all } x \in \mathbb{R}.$$

Thus $(\cos(x) + i\sin(x))^r = \cos(rx) + i\sin(rx)$, for all $r \in \mathbb{R}$. Moreover $\{e^{ix} \mid x \in [0, 2\pi)\}$ is the set of points on the unit circle in the complex plane. As x ranges over $\mathbb{R}$, e^{ix} winds itself around the unit circle an infinite number of times (one circuit every interval of length 2π).

All these identities are consequences of the Taylor series representations of e^x, $\cos(x)$ and $\sin(x)$, extended from the real variable x to a complex variable z.

Now $e^{-ix} = \cos(x) - i\sin(x)$, as cos is an even function and sin is an odd function. Putting this together, we gets

$$\cos(x) = \frac{e^{ix} + e^{-ix}}{2}, \quad \sin(x) = \frac{e^{ix} - e^{-ix}}{2i}$$

Thus $\tan(x) = \sin(x)/\cos(x)$ can be written as

$$\tan(x) = \frac{e^{ix} - e^{-ix}}{i(e^{ix} + e^{-ix})}$$

In analogy with the above formulas, we define the hyperbolic functions:

$$\cosh(x) = \frac{e^x + e^{-x}}{2}, \quad \sinh(x) = \frac{e^x - e^{-x}}{2}, \quad \tanh(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}}.$$

Then it is clear that $\dfrac{d\cosh(x)}{dx} = \sinh(x)$ and $\dfrac{d\sinh(x)}{dx} = \cosh(x)$. Moreover,

$$\begin{aligned}
\cosh^2(x) &= \tfrac{1}{2}(\cosh(2x) + 1) \\
\sinh^2(x) &= \tfrac{1}{2}(\cosh(2x) - 1) \\
\cosh(x)\sinh(x) &= \tfrac{1}{2}\sinh(2x) \\
\cosh^2(x) - \sinh^2(x) &= 1.
\end{aligned}$$

Application

The substitution $x = a\sinh(t)$ is useful in certain types of integrals. First notice that $dx = a\cosh(t)dt$.

Now $a^2\cosh^2(t) - a^2\sinh^2(t) = a^2$. So $a\cosh(t) = \sqrt{a^2+x^2}$.

To get t in terms of x, use $e^t = \cosh(t) + \sinh(t)$. So $ae^t = \sqrt{a^2+x^2} + x$, from which it follows that $t = \ln(\sqrt{a^2+x^2} + x) - \ln(a)$.

Next $\frac{a^2}{2}\sinh(2t) = (a\sinh(t))(a\cosh(t)) = x\sqrt{a^2+x^2}$.

Example: compute $\int \sqrt{a^2+x^2}dx$.

Set $x = a\sinh(t)$. Then $dx = a\cosh(t)dt$. So

$$\begin{aligned}\int \sqrt{a^2+x^2}dx = a^2\int \cosh^2(t)dt &= \frac{a^2}{2}\int(\cosh(2t)+1)dt \\ &= \frac{a^2}{2}\left(\frac{1}{2}\sinh(2t)+t\right)+C \\ &= \frac{1}{2}\left(x\sqrt{a^2+x^2}+a^2\ln(x+\sqrt{a^2+x^2})\right)+C,\end{aligned}$$

where we have folded the constant term $-\dfrac{a^2}{2}\ln(a)$ into C.

Area bounded by curves

Suppose that f and g are continuous functions. Define $A := \{(x,y) \mid g(x) \le y \le f(x)\}$. If A is a bounded region of the plane, we say that A is the region bounded by the curves $y = f(x)$ and $y = g(x)$. Then if A is contained between $x = a$ and $x = b$, the area of A is given by

$$\int_a^b (f(x) - g(x))dx.$$

Example: (Jan 2020, Q(3)(b)) Compute the area of the region which is bounded by the curves $y = x^3 - 5x^2 + 1$ and $y = x^2 - 9x + 1$.

Solution: Let $f(x) = x^3 - 5x^2 + 1$ and $g(x) = x^2 - 9x + 1$, respectively. Then $f - g = x^3 - 6x^2 + 9x = x(x-3)^2$. So $f(x) = g(x)$ for $x = 0$ or $x = 3$, and $f - g \ge 0$ on $[0,3]$. It follows that the area bounded is

$$\int_0^3 x(x-3)^2dx = \int_0^3(x^3-6x^2+9x)dx = \left[\frac{x^4}{4} - 2x^3 + \frac{9}{2}x^2\right]_0^3 = \frac{81}{4} - 54 + \frac{81}{2} = \frac{27}{4} = 6.75.$$

Example: (Jan 2015, Q(3)(b)) Determine the area of the region that is bounded by the curves $y = x^3 - 3x^2 + 1$ and $y = x^2 - 4x + 1$.

Solution: Let $f(x) = x^3 - 3x^2 + 1$ and $g(x) = x^2 - 4x + 1$. Then $f(x) - g(x) = x^3 - 4x^2 + 4x = x(x^2 - 4x + 4) = x(x-2)^2$. So the area bounded is

$$\int_0^2 (f(x) - g(x))dx = \int_0^2 (x^3 - 4x^2 + 4x)dx = \left[\frac{x^4}{4} - \frac{4}{3}x^3 + 2x^2\right]_0^2 = 4 - \frac{32}{3} + 8 = \frac{4}{3}.$$

Many functions which are of interest are integrable but do not have antiderivatives which are expressible in any straightforward way. For this reason, there are many methods to numerically estimate the value of definite integrals. This is part of an important branch of applied mathematics called numerical analysis.

It is really important to note that each numerical integration rule comes with an error estimate, which depends on having an upper bound on some derivative. An estimate is useless, nay dangerous, if you cannot bound the error in the estimate!

The following Taylor remainder theorem generalizes Rolle's theorem, and is very useful:

Lemma 28. *Suppose that $f : [a, b] \to \mathbb{R}$ is such that $f', f'', \ldots, f^{n+1}$ all exist and that $x, c \in [a, b]$. Then there exists $z \in [a, b]$ such that*

$$f(x) = f(c) + f'(c)(x - c) + \frac{f''(c)}{2}(x - c)^2 + \cdots + \frac{f^n(c)}{n!}(x - c)^n + \frac{f^{n+1}(z)}{(n+1)!}(x - c)^{n+1}.$$

Corollary 29. *Suppose that $f : [a, b] \to \mathbb{R}$ is such that f'' exists and $|f''(x)| \leq K$. Then for all $x, c \in [a, b]$ we have*

$$|f(x) - f(c) - f'(c)(x - c)| \leq \frac{K}{2}(x - c)^2.$$

We begin with the simplest method, which is called the mid-point rule. This is based on the idea that for small intervals of length h, the area under the graph $y = f(x)$ is approximately $hf(c)$, where c is the mid-point of the interval.

Theorem 30. *Let $f : [a, b] \to \mathbb{R}$ be cts and let $n > 1$ be an integer. Set $h := \frac{b-a}{n}$ and consider the midpoints $m_i = a + (i - \frac{1}{2})h$, for $i = 1, 2, \ldots, n$. Then*

$$\int_a^b f(x)dx \quad \approx \quad \underbrace{h\left(f(m_1) + f(m_2) + \cdots + f(m_n)\right)}_{M_n}$$

If f has a cts second order derivative on $[a, b]$, and $|f''(x)| \leq K$, then

$$\left|\int_a^b f(x)dx - M_n\right| \leq \frac{K(b-a)^3}{24n^2}.$$

Proof. We divide $[a, b]$ into n subintervals $[x_i, x_{i+1}]$ where $x_i = a+ih$, $i = 0, \ldots, n$. So m_i is the mid-point of $[x_{i-1}, x_i]$. If n is large, h is small enough so that $f(x) \approx f(m_i)$ on $[x_{i-1}, x_i]$, by continuity of f. Thus $\int_{x_{i-1}}^{x_i} f(x)dx \approx hf(m_i)$. Adding all these approximations gives us $M_n \approx \int_a^b f(x)dx$.

Now $\int_{x_{i-1}}^{x_i} f'(m_i)(x-m_i)dx = 0$ and $\int_{x_{i-1}}^{x_i} f(m_i)dx = hf(m_i)$. Also the corollary gives

$$|f(x) - f(m_i) - f'(m_i)(x-m_i)| \leq \frac{K}{2}(x-m_i)^2.$$

Thus

$$\begin{aligned}
\left|\int_{x_{i-1}}^{x_i} f(x)dx - hf(m_i)\right| &= \left|\int_{x_{i-1}}^{x_i} \left(f(x) - f(m_i) - f(m_i)'(x-m_i)\right) dx\right| \\
&\leq \frac{K}{2}\int_{x_{i-1}}^{x_i} (x-m_i)^2 dx \\
&= \frac{K}{2}\frac{(x-m_i)^3}{3}\bigg|_{x_{i-1}}^{x_i} = \frac{Kh^3}{24}
\end{aligned}$$

Thus finally, we can estimate

$$\left|\int_a^b f(x)dx - M_n\right| = \left|\sum_{i=1}^n \left(\int_{x_{i-1}}^{x_i} f(x)dx - hf(m_i)\right)\right| \leq \sum_{i=1}^n \frac{Kh^3}{24} = \frac{K(b-a)^3}{24n^2}.$$

□

There is another rule which gives a similar error, but is easier to use. The idea is to imagine that $y = f(x)$ is close to being a straight line between the points $(x_1, f(x_1))$ and $(x_2, f(x_2))$, so long as $|x_2 - x_1|$ is small. Now the area bounded by $x = x_1, x = x_2, y = 0$ and the straight line from $(x_1, f(x_1))$ and $(x_2, f(x_2))$ is $\frac{(x_2-x_1)(f(x_1)+f(x_2))}{2}$. The following is called the Trapezoidal rule:

Theorem 31. *Let $f : [a,b] \to \mathbb{R}$ be cts and let $n > 1$ be an integer. Set $h := \frac{b-a}{n}$ and consider the points $x_i = a + ih$, for $i = 0, 1, 2, \ldots, n$. Then*

$$\int_a^b f(x)dx \quad \approx \quad \underbrace{\frac{h}{2}\left(f(a) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(b)\right)}_{T_n}$$

If f has a cts second order derivative on $[a,b]$, and $|f''(x)| \leq K$, then

$$\left|\int_a^b f(x)dx - T_n\right| \leq \frac{K(b-a)^3}{12n^2}.$$

The next method for numerical integration is based on the following idea:

Lemma 32. *Let $h > 0$ and let $y_1, y_2, y_3 \in \mathbb{R}$. Then there is a unique quadratic $p(x) = ax^2 + bx + c$ which contains the 3 points $(-h, y_1), (0, y_2)$ and (h, y_3). Moreover*

$$\int_{-h}^{h} p(x)dx = \frac{h}{3}\left(y_1 + 4y_2 + y_3\right).$$

Proof. This is a special case of a result called Lagrange interpolation. Check that $p(x)$ works with

$$a = \frac{y_1 - 2y_2 + y_3}{2h^2}, \quad b = \frac{y_3 - y_1}{2h}, \quad c = y_2$$

Now

$$\int_{-h}^{h} p(x)dx = \left[\frac{ax^3}{3} + \frac{bx^2}{2} + cx\right]_{-h}^{h} = \left[\frac{2ah^3}{3} + 2ch\right] = \frac{h}{3}\left[y_1 + 4y_2 + y_3\right].$$

$\square$

Now the next rule is based on approximating the graph $y = f(x)$ on an interval $[a, b]$ by the unique quadratic function $cx^2 + dx + e$ which passes through the 3 points $(a, f(a)), (\frac{a+b}{2}, f(\frac{a+b}{2}))$ and $(d, f(d))$. Using the Lemma, we get

$$\int_a^b f(x)dx \quad \approx \quad \frac{b-a}{6}\left(f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right)$$

The following is called Simpson's rule:

Theorem 33. *Let $f : [a, b] \to \mathbb{R}$ be cts and let $n > 1$ be an even integer. Set $h := \frac{b-a}{n}$ and consider the points $x_i = a + ih$, for $i = 0, 1, 2, \ldots, n$. Also set $y_i := f(x_i)$, for all i. Then*

$$\int_a^b f(x)dx \quad \approx \quad \underbrace{\frac{h}{3}\left(y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + \cdots + 2y_{n-2} + 4y_{n-1} + y_n\right)}_{S_n}$$

If f has a cts fourth order derivative on $[a, b]$, and $|f^{(4)}(x)| \le K$, then

$$\left|\int_a^b f(x)dx - S_n\right| \le \frac{K(b-a)^5}{180n^4}.$$

Example: consider the integral $I = \int_0^1 \sqrt{x}e^x dx$. Numerically integrating this directly gives a poor estimate, because the derivative of $\sqrt{x}e^x$ is unbounded close to 0. We can get around this problem by using the substitution $x = t^2$. Then $I = 2\int_0^1 t^2 e^{t^2} dt$. As $t^2e^{t^2}$ has bounded derivatives on $[0, 1]$, we get good estimates using our methods.

Compare the graphs $y = \sqrt{x}e^x$ and $y = 2x^2e^{x^2}$ on Desmos.

Example: consider the improper integral $I = \int_0^\infty \frac{dx}{\sqrt{x^4 + x^2 + 2}}$. We cannot estimate this directly by numerical integration. But we can use the following trick, based on the fact that $x \to 1/x$ is a bijection of $[1, \infty)$ onto $(0, 1]$. First

$$\int_0^\infty \frac{dx}{\sqrt{x^4 + x^2 + 2}} = \int_0^1 \frac{dx}{\sqrt{x^4 + x^2 + 2}} + \int_1^\infty \frac{dx}{\sqrt{x^4 + x^2 + 2}}$$

Now substitute $x = 1/t$ in the second integral. Then $dx = -dt/t^2$. So

$$\int_1^\infty \frac{dx}{\sqrt{x^4 + x^2 + 2}} = \int_1^0 \frac{-dt}{t^2\sqrt{t^{-4} + t^{-2} + 2}} = \int_0^1 \frac{dt}{\sqrt{2t^4 + t^2 + 1}}$$

Thus

$$I = \int_0^1 \left(\frac{1}{\sqrt{x^4 + x^2 + 2}} + \frac{1}{\sqrt{2x^4 + x^2 + 1}} \right) dx.$$

This is a proper integral. So our numerical methods can be applied.

Recall that the area of a circle of radius r is

$$\begin{aligned}\int_{-r}^{r}\sqrt{r^2-x^2}dx &= 4\int_0^r \sqrt{r^2-x^2}dx, \quad \text{which upon setting } x=\sin(t)\\ &= 4r^2\int_0^{\pi/2}\cos^2(t)dt = 2r^2\int_0^{\pi/2}(\cos(2t)+1)dt\\ &= \pi r^2.\end{aligned}$$

Theorem 34. *Suppose that S is a solid in $\mathbb{R}^3$ such that $a \le x \le b$ whenever $(x,y,z)\in S$. For each x_0, let $A(x_0)$ be the area of the intersection of S with the plane $x = x_0$. Then*

$$\mathrm{Vol}(S) = \int_a^b A(x)dx.$$

Example: A four sided pyramid P has height h and a rectangular base of area A. Take the vertical as the x-axis. Then $A(x_0) = (1-\frac{x_0}{h})^2 A$, for $x_0 \in [0,h]$. So

$$\mathrm{Vol}(P) = \int_0^h \left(1-\frac{x}{h}\right)^2 A dx = \frac{-h}{3}\left[\left(1-\frac{x}{h}\right)^3\right]_0^h = \frac{Ah}{3}.$$

Example: Let S be a sphere of radius r. So $-r \le x \le r$ and $A(x) = \pi(r^2-x^2)$. Thus

$$\mathrm{Vol}(S) = \int_{-r}^{r}\pi(r^2-x^2)dx = \pi\left[r^2x - \frac{x^3}{3}\right]_{-r}^{r} = 2\left(r^3-\frac{r^3}{3}\right) = \frac{4\pi r^3}{3}.$$

We now consider the volume of a solid formed by rotating a region bounded by a curve about an axis. The first technique is called the *disk method*:

Theorem 35. *Let $f : [a,b] \to \mathbb{R}_{\ge 0}$ be a function and let R be the region bounded by the x-axis, the graph of f and the lines $x = a$ and $x = b$. Let S be the solid generated by revolving R about the x-axis. Then*

$$\mathrm{Vol}(S) = \pi\int_a^b f(x)^2 dx.$$

Example: consider the region R bounded by $f(x) = 1/x$, along the interval $[1,\infty)$. The solid S generated by revolving R about the x-axis is an infinitely long horn-shape H. Then

$$\mathrm{Vol}(H) = \pi\int_0^\infty \frac{1}{x^2}dx = \pi\left.\frac{-1}{x}\right|_1^\infty = \pi\left(1-\lim_{x\to\infty}\frac{-1}{x}\right) = \pi.$$

Remark: the surface area of H is $\text{Area}(H) = 2\pi \int_0^\infty \frac{1}{x} dx$ which is $+\infty$. So it would require an ∞ of paint to cover H. But if H were hollow, one could fill it with π paint.

A washer is like a disk but has a central hole cut out:

Example: Let S be the ring-shaped solid obtained by rotating the region bounded by $y = x^2$ and $y = 1$ about the line $y = 2$. First note that we obtain the same solid if we rotate the region bounded by $y = x^2 - 2$ and $y = -1$ between $x = -1$ and $x = 1$ about the x-axis. This is the difference of the volumes of two solids of revolution:

$$\begin{aligned}
\text{Vol}(S) &= \pi \int_{-1}^{1} ((x^2 - 2)^2 - 1) dx = \pi \int_{-1}^{1} (x^4 - 4x^2 + 3) dx \\
&= \pi \left[\frac{x^5}{5} - \frac{4x^3}{3} + 3x \right]_{-1}^{1} = 2\pi \left(\frac{1}{5} - \frac{4}{3} + 3 \right) \\
&= \frac{56\pi}{15}.
\end{aligned}$$

For solids got by revolving around the y-axis, the following is useful:

Theorem 36. *Let $f : [a,b] \to \mathbb{R}_{\geq 0}$ be a function and let R be the region bounded by the x-axis, the graph of f and the lines $x = a$ and $x = b$. Let S be the solid generated by revolving R about the y-axis. Then*

$$\mathrm{Vol}(S) = 2\pi \int_a^b x f(x) dx.$$

Example: Let S be the ring-shaped solid obtained by rotating the region bounded by $y = x^2$ and $y = 1$ about the line $y = 2$ (as in previous example). Rotating the figure by a quarter turn, we get the same solid S by rotating the region R bounded by $y^2 = (2 - x)$, the lines $x = 1$ and $x = 2$, about the y axis. So

$$\mathrm{Vol}(S) = 2\pi \int_1^2 x\left(2\sqrt{2-x}\right) dx = 4\pi \left[\frac{-2}{15}(2-x)^{3/2}(3x+4)\right]_1^2 = \frac{56}{15}\pi.$$

This uses $\displaystyle\int x\sqrt{2-x}dx = \frac{-2}{15}(2-x)^{3/2}(3x+4) + C.$

Example: consider the disk of radius a centered at $(b, 0)$ where $b > a > 0$. Rotating this disk about the y-axis produces a torus T. The boundary of the disk is described by $(x-b)^2 + y^2 = a^2$. So we are rotating the region bounded between $y = -\sqrt{a^2 - (x-b)^2}$ and $y = \sqrt{a^2 - (x-b)^2}$, for $b - a \leq x \leq b + a$. Thus

$$\begin{aligned}
\mathrm{Vol}(T) &= 2\pi \int_{b-a}^{b+a} x 2\sqrt{a^2 - (x-b)^2} dx, \quad u = x - b, \\
&= 4\pi \int_{-a}^{a} (u+b)\sqrt{a^2 - u^2} du \\
&= 8b\pi \int_0^a \sqrt{a^2 - u^2} du, \quad \text{as } \sqrt{a^2 - u^2} \text{ is even and } u \text{ is odd} \\
&= 8b\pi \left[\frac{a^2}{2}\arcsin(u/a) + \frac{u}{2}\sqrt{a^2 - u^2}\right]_0^a = 8b\pi \frac{a^2}{2}\arcsin(1) \\
&= 2a^2 b\pi^2.
\end{aligned}$$

Length of a curve (will not be examined)

Now we consider curves in the plane. First we need to define what we mean by a curve.

Definition 37. *A curve is the image C of a continuous function $\gamma : I \to \mathbb{R}^2$ from a closed interval $I \subseteq \mathbb{R}$ to the plane. We call γ a parametrization of the curve.*

Remarks:

- C is the set of points $\gamma(t) \mid t \in I\}$, not the function γ.
- If $I = [a, b]$ we say that C is a curve going from $\gamma(a)$ to $\gamma(b)$.
- The curve C may be parametrized by many different functions γ.

Example: let $f : [a, b] \to \mathbb{R}$ be a continuous function. Then the graph $\mathrm{Gr}(f)$ of f is a curve. Here $\gamma : [a, b] \to \mathrm{Gr}(f)$ is defined by $\gamma(t) = (t, f(t))$, for all $t \in [a, b]$.

Example: the unit circle $x^2 + y^2 = 1$ is a curve. A standard way of parametrizing the circle is as $\gamma(\theta) = (\cos(\theta), \sin(\theta))$, for all $\theta \in [0, \pi]$.

We could also parametrize $x^2 + y^2 = 1$ using $\zeta : [0, 4] \to \mathbb{R}^2$:

$$\zeta(t) = \begin{cases} (1-t, \sqrt{1-(1-t)^2}), & t \in [0,2] \\ (t-3, -\sqrt{1-(t-3)^2}), & t \in [2,4] \end{cases}$$

We now discuss a Riemann sum which allows us to define what we mean by the length of a curve.

Suppose that C is a curve going from A to B. A polygonal skeleton $\mathcal{P}$ of C is a finite set $A = P_0$, $P_1, \ldots$, $P_n = B$ of points, in order, belonging to C. Denote by $|P_{i-1}P_i|$ the distance of the straight line from P_{i-1} to P_i. Then the the length of $\mathcal{P}$ is

$$L(\mathcal{P}) = \sum_{i=1}^{n} |P_{i-1}P_i|.$$

Definition 38. *We call C a rectifiable curve if $\{L(\mathcal{P}) \mid \mathcal{P}$ a skeleton of $C\}$ is a bounded subset of $\mathbb{R}$. When C is rectifiable, we define its length as*

$$\mathrm{L}(C) = \sup\{L(\mathcal{P}) \mid \mathcal{P} \text{ a skeleton of } C\}.$$

Example: the graph of $x \sin(1/x)$, for $x \in [0, 1]$ is not rectifiable.

Example: the Koch snowflake (Niels Fabian Helge von Koch) is not rectifiable.

Example: recall Cantor's function $c : [0,1] \to [0,1]$. Ludwig Scheeffer showed that the length of $\mathrm{Gr}(c)$ is 2.

It is not a surprise, but in many cases $L(C)$ can be represented by an integral.

Theorem 39. *Let $f : [a,b] \to \mathbb{R}$ be a function such that f' is cts. Let $C = \mathrm{Gr}(f)$. The C is rectifiable and*

$$L(C) = \int_a^b \sqrt{1 + f'(x)^2}dx.$$

Proof. Omitted, but it requires the mean value theorem. □

Example: The length of the curve $y = x^{2/3}$ from $x = 0$ to $x = 8$ is given by

$$L(C) = \int_0^8 \sqrt{1 + \frac{4}{9}x^{-2/3}}dx = \int_0^8 \frac{\sqrt{9x^{2/3}+4}}{3x^{1/3}}dx$$

Now set $u = 9x^{2/3} + 4$. Then $\dfrac{dx}{3x^{1/3}} = \dfrac{du}{18}$. Also if $x = 8$ then $u = 40$, and if $x = 0$ then $u = 0$. So

$$L(C) = \int_0^{40} \frac{\sqrt{u}}{18}du = \left[\frac{1}{27}u^{3/2}\right]_0^{40} = \frac{80}{27}\sqrt{10} \approx 9.369712.$$

We note that the straightline distance from $(0,0)$ to $(8,4)$ is $\sqrt{80} \approx 8.94427$.

Example: consider the circle $C : x^2 + y^2 = r^2$ of radius r centered at $(0,0)$. By symmetry, the length of C (circumference of the circle) is 4 times the length of the arc in the first quadrant. Now for $x \in [0,r]$, this arc is parametrized by $y = \sqrt{r^2 - x^2}$. As $\dfrac{dy}{dx} = \dfrac{-x}{\sqrt{r^2-x^2}}$, we get

$$L(C) = 4\int_0^r \sqrt{1 + \frac{x^2}{r^2-x^2}}dx = 4\int_0^r \frac{r}{\sqrt{r^2-x^2}}dx = 4r\left[\arcsin\left(\frac{x}{r}\right)\right]_0^r = 2\pi r.$$