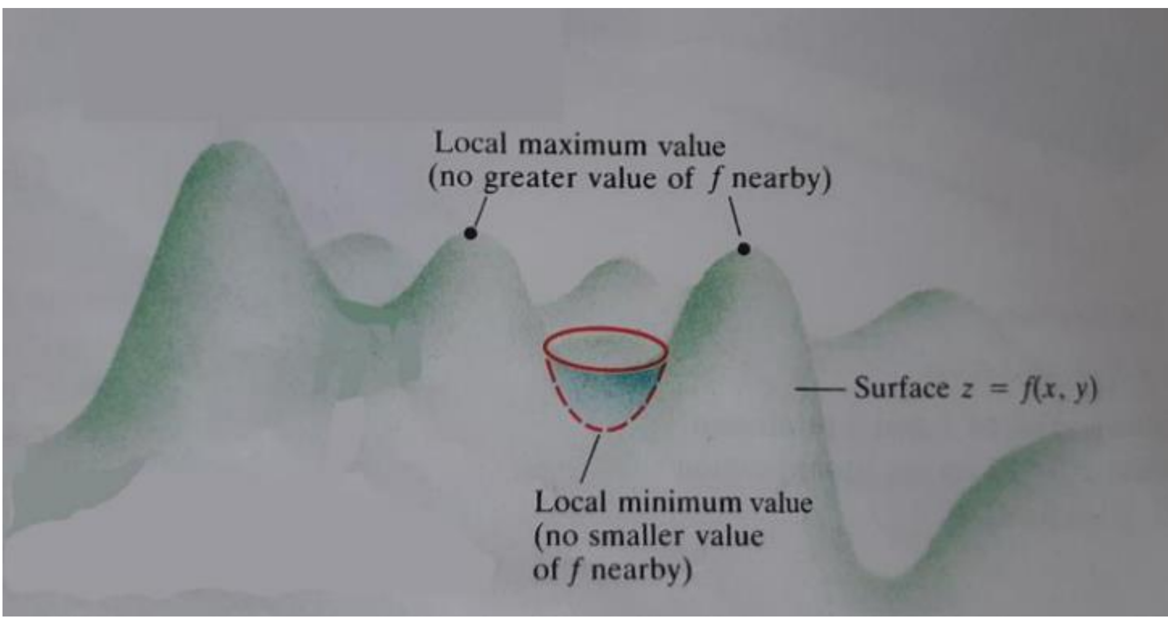
Fiacre Ó Cairbre

Lecture 10

Remark 1 continued.

Intuitively, one can think of an example of a surface $z = f(x, y)$ like a mountain range in the picture below. Note that, in the picture, a local maximum value of f is like a single mountain peak but is not necessarily the highest mountain peak in the whole mountain range. Similarly, if (a, b) is a local minimum of f, then it may not necessarily be the case that $f(a, b) \leq f(x, y)$, for all points (x, y) in the domain of f.



Theorem 1 – First Derivative test for local maxima and local minima.

Suppose $f(x, y)$ has a local maximum or local minimum at an interior point (a, b) of its domain. Also, suppose that $\frac{\partial f}{\partial x}_{|(a,b)}$ and $\frac{\partial f}{\partial y}_{|(a,b)}$ both exist. Then,

$$\frac{\partial f}{\partial x}_{|(a,b)} = 0 = \frac{\partial f}{\partial y}_{|(a,b)}$$

Definition 2.

Suppose (a,b) is an interior point of the domain of $f(x,y)$ with $\frac{\partial f}{\partial x}_{|(a,b)} = 0 = \frac{\partial f}{\partial y}_{|(a,b)}$ or where one or both of $\frac{\partial f}{\partial x}_{|(a,b)}$, $\frac{\partial f}{\partial y}_{|(a,b)}$ don't exist. Then, (a,b) is called a critical point of f.

Remark 2.

Theorem 1 says that the only candidates for local maxima and local minima of $f(x,y)$ are critical points of f and boundary points of the domain of f.

Definition 3.

Suppose (a,b) is a critical point of a differentiable function $f(x,y)$. Then (a,b) is called a saddle point of f if in every open ball, with centre (a,b), there are domain points (x,y) where $f(x,y) > f(a,b)$ and there are also domain points (z,w) where $f(a,b) > f(z,w)$.

Example 1.

Find the local maxima and local minima (if any) of $f(x,y) = x^2 + y^2 - 4y + 9$.

Solution.

The domain of f is $\mathbb{R}^2$ and so there are no boundary points of the domain of f. So, by remark 2, the only candidates for local maxima and local minima are critical points.

Note that $\frac{\partial f}{\partial x} = 2x$ and $\frac{\partial f}{\partial y} = 2y - 4 \qquad (*)$.

So, $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ exist everywhere and so the only critical points are where $\frac{\partial f}{\partial x} = 0 = \frac{\partial f}{\partial y}$. So, by $(*)$ we have that $(0,2)$ is the only critical point.

Note that $f(x,y) = x^2 + (y-2)^2 + 5$ and so $f(x,y)$ is never less than $5 = f(0,2)$. Hence, $(0,2)$ is indeed a local minimum and it's the only local minimum of f. Also, there are no local maxima of f and we are done.

Example 2.

Find the local maxima and local minima (if any) of $f(x,y) = 3x^2 - y^2$.

Solution.

The domain of f is $\mathbb{R}^2$ and so there are no boundary points of the domain of f. So, the only candidates for local maxima and local minima are critical points.

Note that $\frac{\partial f}{\partial x} = 6x$ and $\frac{\partial f}{\partial y} = -2y \qquad (*)$.

So, $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ exist everywhere and so the only critical points are where $\frac{\partial f}{\partial x} = 0 = \frac{\partial f}{\partial y}$. So, by $(*)$ we have that $(0,0)$ is the only critical point.

However, every open ball, with centre $(0,0)$, contains points $(x,0)$, with $x \neq 0$, where $f(x,0) = 3x^2 > f(0,0)$ and also contains points $(0,y)$, with $y \neq 0$, where $f(0,y) = -y^2 <$

$f(0,0)$, So, $(0,0)$ is neither a local maximum nor local minimum. Consequently, f has no local maxima and no local minima and we are done.

Note that $(0,0)$ is actually a saddle point of f.

Theorem 2 – Second derivative test for local maxima and local minima.

Suppose $f(x,y)$ and $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$ are continuous on an open ball with centre (a,b). Also, suppose that $\frac{\partial f}{\partial x}_{|(a,b)} = 0 = \frac{\partial f}{\partial y}_{|(a,b)}$

Then,

(i) f has a local maximum at (a,b) if

$$\frac{\partial^2 f}{\partial x^2}_{|(a,b)} < 0 \quad \text{and} \quad \frac{\partial^2 f}{\partial x^2}_{|(a,b)} \frac{\partial^2 f}{\partial y^2}_{|(a,b)} - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2_{|(a,b)} > 0$$

(ii) f has a local minimum at (a,b) if

$$\frac{\partial^2 f}{\partial x^2}_{|(a,b)} > 0 \quad \text{and} \quad \frac{\partial^2 f}{\partial x^2}_{|(a,b)} \frac{\partial^2 f}{\partial y^2}_{|(a,b)} - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2_{|(a,b)} > 0$$

(iii) f has a saddle point at (a,b) if

$$\frac{\partial^2 f}{\partial x^2}_{|(a,b)} \frac{\partial^2 f}{\partial y^2}_{|(a,b)} - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2_{|(a,b)} < 0$$

(iv) The test is inconclusive if

$$\frac{\partial^2 f}{\partial x^2}_{|(a,b)} \frac{\partial^2 f}{\partial y^2}_{|(a,b)} - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2_{|(a,b)} = 0$$

Remark 3.

In Theorem 2, the expression

$$\frac{\partial^2 f}{\partial x^2}_{|(a,b)} \frac{\partial^2 f}{\partial y^2}_{|(a,b)} - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2_{|(a,b)}$$

is called the Hessian of f.

Remark 4.

Look back at example 30 in chapter 1 where $z = \frac{y^2}{b^2} - \frac{x^2}{a^2} = f(x,y)$ and you will see that $(0,0)$ is a saddle point because of theorem 2(iii) above. You will also see how it looks like $(0,0)$ is on a saddle in the picture in example 30 in chapter 1.

Example 3.

Find the local maxima, local minima and saddle points (if any) of $f(x, y) = 3y^2 - 2y^3 - 3x^2 + 6xy$.

Solution.

The domain of f is $\mathbb{R}^2$ and so there are no boundary points of the domain of f. So, the only candidates for local maxima, local minima and saddle points are critical points.

Note that $\dfrac{\partial f}{\partial x} = -6x + 6y$ and $\dfrac{\partial f}{\partial y} = 6y - 6y^2 + 6x \qquad (*)$.

So, $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$ exist everywhere and so the only critical points are where $\dfrac{\partial f}{\partial x} = 0 = \dfrac{\partial f}{\partial y}$. So, by $(*)$ above we have that (x, y) is a critical point $\iff$

$$-6x + 6y = 0 = 6y - 6y^2 + 6x$$

$$\iff x = y \;\text{ and }\; 2y - y^2 = 0$$

$$\iff x = y \;\text{ and }\; y(y-2) = 0$$

$$\iff x = y \;\text{ and }\; y = 0, 2$$

So, the only critical points are $(0, 0)$ and $(2, 2)$.

Note that $\dfrac{\partial^2 f}{\partial x^2} = -6$, $\dfrac{\partial^2 f}{\partial y^2} = 6 - 12y$ and $\dfrac{\partial^2 f}{\partial x \partial y} = 6$

and so

$$\frac{\partial^2 f}{\partial x^2}_{|(0,0)} \frac{\partial^2 f}{\partial y^2}_{|(0,0)} - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2_{|(0,0)} = -72$$

and

$$\frac{\partial^2 f}{\partial x^2}_{|(2,2)} \frac{\partial^2 f}{\partial y^2}_{|(2,2)} - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2_{|(2,2)} = 72$$

Hence, by theorem 2, $(0, 0)$ is a saddle point of f and $(2, 2)$ is a local maximum of f. Also, there are no other local maxima, local minima or saddle points of f.

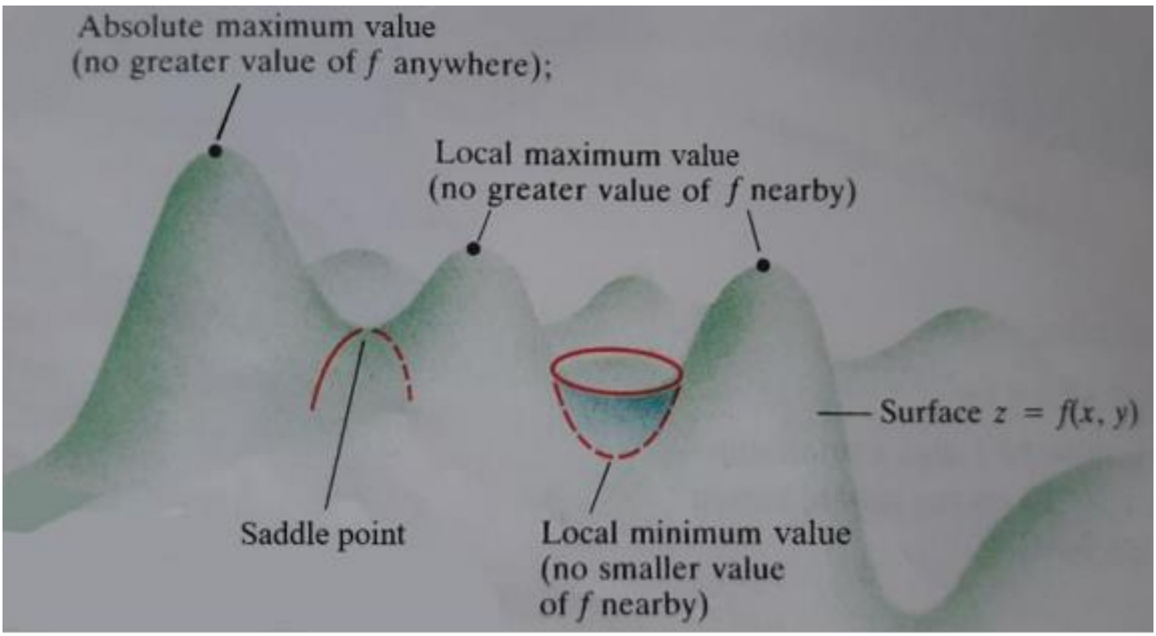
Definition 4.

(i) $f(a, b)$ is an absolute maximum value of f if $f(a, b) \geq f(x, y)$, for all (x, y) in the dcmain of f. We say that (a, b) is an absolute maximum of f.

(ii) $f(a,b)$ is an absolute minimum value of f if $f(a,b) \leq f(x,y)$, for all (x,y) in the domain of f. We say that (a,b) is an absolute minimum of f.

Remark 5.

We will now add to remark 1. Intuitively, one can think of an example of a surface $z = f(x,y)$ like a mountain range in the picture below (like after remark 1). Note that, in the picture below, an absolute maximum value of f is the highest mountain peak in the whole mountain range.



Remark 6.

If $g(x)$ is a continuous function of one variable on a closed bounded interval $[a,b]$ in $\mathbb{R}$, then g has at least one absolute maximum in $[a,b]$ and g has at last one absolute minimum in $[a,b]$.

The following states something similar for functions of two variables.

Theorem 3.

Suppose $f(x,y)$ is a continuous function on a closed bounded subset T of $\mathbb{R}^2$. Then f has at least one absolute maximum in T and f has at least one absolute minimum in T.