

# **MT251P - Foundations of Euclidean Geometry**

## **Assignment #4**

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## Question 1

In each case below, state whether the statement is true or false. Justify your answer in each case.

### Part A

There are infinitely many  $4 \times 4$  matrices that are not invertible.

### Solution

Since a square matrix is invertible if and only if its determinant is non-zero. Since there are infinitely many possible values for the elements of a  $4 \times 4$  matrix, there are also infinitely many matrices that have a determinant of zero ie there are infinite amount of matrices with a zero row and any matrix with a zero row has  $\det = 0$  and therefore not invertible. Therefore the statement is True

### Part B

There is a  $4 \times 4$  invertible matrix  $A$  such that  $A^3 = 2A^2$  and  $\det A = 2$ .

### Solution

$$A^3 = 2A^2$$

$$A(AA) = 2(AA)$$

Since  $A$  is invertible, then  $A^{-1}$  exists. So:

$$A(AA)(A^{-1}A^{-1}) = 2(AA)(A^{-1}A^{-1})$$

Since  $AA^{-1} = I_4 = A^{-1}A$ , then:

$$A(AA^{-1})(AA^{-1}) = 2(AA^{-1})(AA^{-1})$$

We also know that  $AA^{-1} = I_4$  since  $A$  is a  $4 \times 4$  matrix. So:

$$A(I_4)(I_4) = 2(I_4)(I_4)$$

$$A = 2(I_4)$$

$$\det(A) = \det(2(I_4))$$

$$\det(A) = 16$$

Since  $16 \neq 2$ , then the statement is False.

### Part C

There is a  $4 \times 4$  matrix  $A$  such that  $A^2 = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

### Solution

$$\det(A^2) = \det(A) \det(A) = \det(A)^2$$

Since  $\det(A) \in \mathbb{R}$ , then  $\det(A^2) \geq 0$

However,

$$\begin{aligned} \det(A^2) &= -1 \det \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ &= -1 \det \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ &= -1 \end{aligned}$$

This gives a contradiction. Therefore the statement is False.

## Question 2

### Part A

Prove that  $\det(A^{-1}BA) = \det(B)$ , for all  $n \times n$  matrices  $A, B$ , where  $A$  is invertible and  $n > 1$ .

### Solution

Let  $A$  be an invertible  $n \times n$  matrix, and let  $B$  be any  $n \times n$  matrix. We have  $\det(A^{-1}BA) = \det(A^{-1}) \det(B) \det(A) = \frac{1}{\det(A)} \det(B) \det(A) = \det(B)$ , so  $\det(A^{-1}BA) = \det(B)$

### Part B

Suppose  $\underline{a} = i + 2j - k$ ,  $\underline{b} = i + 3j + k$  and  $\underline{c} = 3i + 8j + 4k$ . Find  $||\underline{w}||^2$  if  $\underline{w} \in \mathbb{R}^3$  such that  $\underline{w} \cdot \underline{a} = 3$ ,  $\underline{w} \cdot \underline{b} = 5$  and  $\underline{w} \cdot \underline{c} = 17$ .

### Solution

Since  $\underline{w}$  is in  $\mathbb{R}^3$ , it must be represented in the form of  $\underline{w} = xi + yj + zk$  for some  $x, y, z \in \mathbb{R}$ .

We have the following:

$$\underline{w} \cdot \underline{a} = (xi + yj + zk) \cdot (i + 2j - k) = x + 2y - z = 3$$

$$\underline{w} \cdot \underline{b} = (xi + yj + zk) \cdot (i + 3j + k) = x + 3y + z = 5$$

$$\underline{w} \cdot \underline{c} = (xi + yj + zk) \cdot (3i + 8j + 4k) = 3x + 8y + 4z = 17$$

Now we can solve for  $x, y, z$  using a matrix:

$$\begin{pmatrix} 1 & 2 & -1 & 3 \\ 1 & 3 & 1 & 5 \\ 3 & 8 & 4 & 17 \end{pmatrix}$$

Replace  $R_2$  with  $R_2 - R_1$

$$\begin{pmatrix} 1 & 2 & -1 & 3 \\ 0 & 1 & 2 & 2 \\ 3 & 8 & 4 & 17 \end{pmatrix}$$

Replace  $R_3$  with  $R_3 - 3R_1$

$$\begin{pmatrix} 1 & 2 & -1 & 3 \\ 0 & 1 & 2 & 2 \\ 0 & 2 & 7 & 8 \end{pmatrix}$$

Replace  $R_3$  with  $R_3 - 2R_2$

$$\begin{pmatrix} 1 & 2 & -1 & 3 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 3 & 4 \end{pmatrix}$$

Replace  $R_1$  with  $R_1 - 2R_2$

$$\begin{pmatrix} 1 & 0 & -5 & -1 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 3 & 4 \end{pmatrix}$$

Replace  $R_3$  with  $\frac{1}{3}R_3$

$$\begin{pmatrix} 1 & 0 & -5 & -1 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 1 & \frac{4}{3} \end{pmatrix}$$

Replace  $R_2$  with  $R_2 - 2R_3$

$$\begin{pmatrix} 1 & 0 & -5 & -1 \\ 0 & 1 & 0 & -\frac{2}{3} \\ 0 & 0 & 1 & \frac{4}{3} \end{pmatrix}$$

Replace  $R_1$  with  $R_1 - (-5)R_3$

$$\begin{pmatrix} 1 & 0 & 0 & \frac{17}{3} \\ 0 & 1 & 0 & -\frac{2}{3} \\ 0 & 0 & 1 & \frac{4}{3} \end{pmatrix}$$

Then:  $x = \frac{17}{3}, y = -\frac{2}{3}, z = \frac{4}{3}$

Therefore  $||\underline{w}|| = \sqrt{x^2 + y^2 + z^2} = \sqrt{\frac{17^2}{3} - \frac{2^2}{3} + \frac{4^2}{3}} = \frac{\sqrt{309}}{3}$

Then:  $||\underline{w}||^2 = (\frac{\sqrt{309}}{3})^2 = \frac{103}{3}$

### Question 3

Find the solution set of the following system of linear equations:

$$x_1 - 4x_2 + 3x_3 = 0$$

$$2x_1 - 6x_2 + 10x_3 = 6$$

$$x_1 - 2x_2 + 7x_3 = 5$$

**Solution**

We can solve for  $x_1, x_2, x_3$  using a matrix:

$$\begin{pmatrix} 1 & -4 & 3 & 0 \\ 2 & -6 & 10 & 6 \\ 1 & -2 & 7 & 5 \end{pmatrix}$$

Replace  $R_2$  with  $R_2 - 2R_1$

$$\begin{pmatrix} 1 & -4 & 3 & 0 \\ 0 & 2 & 4 & 6 \\ 1 & -2 & 7 & 5 \end{pmatrix}$$

Replace  $R_3$  with  $R_3 - R_1$

$$\begin{pmatrix} 1 & -4 & 3 & 0 \\ 0 & 2 & 4 & 6 \\ 0 & 2 & 4 & 5 \end{pmatrix}$$

Replace  $R_3$  with  $R_3 - R_2$

$$\begin{pmatrix} 1 & -4 & 3 & 0 \\ 0 & 2 & 4 & 6 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Notice how the last row states:  $0 = -1$ , therefore the system has no solutions.

**Question 4**

Find the solution set of the following system of linear equations:

$$4x - 6y + 8z = 8$$

$$x + 2y - 5z = 2$$

$$y + 4x - 6z = 8$$

**Solution**

We can solve for  $x, y, z$  using a matrix:

$$\begin{pmatrix} 4 & -6 & 8 & 8 \\ 1 & 2 & -5 & 2 \\ 4 & 1 & -6 & 8 \end{pmatrix}$$

Replace  $R_1$  with  $R_2$

$$\begin{pmatrix} 1 & 2 & -5 & 2 \\ 4 & -6 & 8 & 8 \\ 4 & 1 & -6 & 8 \end{pmatrix}$$

Replace  $R_2$  with  $R_2 - 4R_1$

$$\begin{pmatrix} 1 & 2 & -5 & 2 \\ 0 & -14 & 28 & 0 \\ 4 & 1 & -6 & 8 \end{pmatrix}$$

Replace  $R_3$  with  $R_3 - 4R_1$

$$\begin{pmatrix} 1 & 2 & -5 & 2 \\ 0 & -14 & 28 & 0 \\ 0 & -7 & 14 & 0 \end{pmatrix}$$

Replace  $R_2$  with  $\frac{1}{2}R_2$

$$\begin{pmatrix} 1 & 2 & -5 & 2 \\ 0 & -7 & 14 & 0 \\ 0 & -7 & 14 & 0 \end{pmatrix}$$

Replace  $R_3$  with  $R_3 - R_2$

$$\begin{pmatrix} 1 & 2 & -5 & 2 \\ 0 & -7 & 14 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Replace  $R_2$  with  $-\frac{1}{7}R_2$

$$\begin{pmatrix} 1 & 2 & -5 & 2 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Replace  $R_1$  with  $R_1 - 2R_2$

$$\begin{pmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Notice how the matrix is in RREF and the last row states  $0 = 0$ , therefore the system has infinitely many solutions. Then the system of linear equations is:

$$x - z = 2$$

$$y - 2z = 0$$

By Remark 4,  $x, y$  are leading variables, and  $z$  is a free variable. We say that  $z = s$ , where  $s$  can be any real number. Then we can represent  $x, y$  in terms of  $z$ :

$$x = 2 + z = 2 + s$$

$$y = 2z = 2s$$

Therefore the solution set of the system of equations is:

$$\{(2 + s, 2s, s) : s \in \mathbb{R}\}$$

## Question 5

Find  $A^{-1}$  if  $A = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 4 & -3 & 8 \end{pmatrix}$

### Solution

$$A = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 4 & -3 & 8 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I_3$$

Replace  $R_1$  with  $R_2$  in both matrices

$$\begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 4 & -3 & 8 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Replace  $R_3$  with  $R_3 - 4R_1$  in both matrices

$$\begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & -3 & -4 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & -4 & 1 \end{pmatrix}$$

Replace  $R_3$  with  $R_3 - (-3)R_2$  in both matrices

$$\begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 3 & -4 & 1 \end{pmatrix}$$

Replace  $R_3$  with  $\frac{1}{2}R_3$  in both matrices

$$\begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ \frac{3}{2} & -2 & \frac{1}{2} \end{pmatrix}$$

Replace  $R_2$  with  $R_2 - 2R_3$  in both matrices

$$\begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ -2 & 4 & -1 \\ \frac{3}{2} & -2 & \frac{1}{2} \end{pmatrix}$$

Replace  $R_1$  with  $R_1 - 3R_3$  in both matrices

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -\frac{9}{2} & 7 & -\frac{3}{2} \\ -2 & 4 & -1 \\ \frac{3}{2} & -2 & \frac{1}{2} \end{pmatrix}$$

Since A is in RREF, then:

$$A^{-1} = \begin{pmatrix} -\frac{9}{2} & 7 & -\frac{3}{2} \\ -2 & 4 & -1 \\ \frac{3}{2} & -2 & \frac{1}{2} \end{pmatrix}$$