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Lecture 16

Example 3.

Find $\int_T \int 5y \, dA$, where T is the region on the xy–plane bounded by the curves $y = x^2 - 3$ and $y = -2x$.

Solution.

It's important to draw a good picture. We first find the points of intersection of the two curves $x^2 - 3$ and $-2x$.

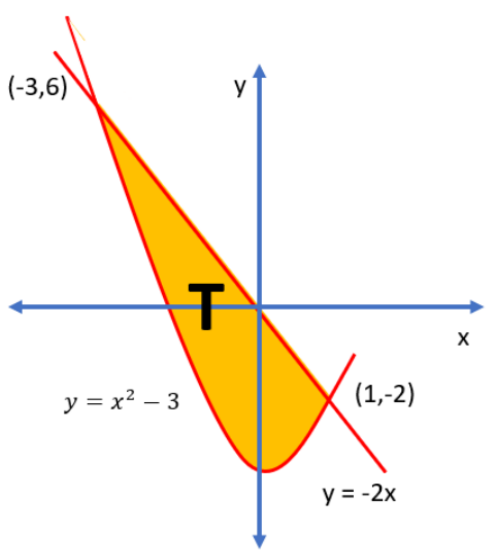
$$x^2 - 3 = -2x$$

$$\Rightarrow x^2 + 2x - 3 = 0$$

$$\Rightarrow (x + 3)(x - 1) = 0$$

$$\Rightarrow x = -3, 1$$

.
So, $(-3, 6)$ and $(1, -2)$ are the two points of intersection.



So, $T = \{(x, y) : -3 \leq x \leq 1,\ x^2 - 3 \leq y \leq -2x\}$.

Hence, by Fubini's theorem (i) we have

$$\int_T\int 5y\,dA = \int_{-3}^{1}\left(\int_{x^2-3}^{-2x} 5y\,dy\right)dx$$

$$= \int_{-3}^{1}\left[\frac{5y^2}{2}\right]_{x^2-3}^{-2x} dx$$

$$= \int_{-3}^{1}\left(10x^2 - \frac{5}{2}(x^4 - 6x^2 + 9)\right) dx$$

$$= \left[\frac{10x^3}{3} - \frac{x^5}{2} + \frac{15x^3}{3} - \frac{45x}{2}\right]_{-3}^{1}$$

$$= \left[\frac{25x^3}{3} - \frac{x^5}{2} - \frac{45x}{2}\right]_{-3}^{1}$$

$$= \left(\left(\frac{25}{3} - \frac{1}{2} - \frac{45}{2}\right) - \left(-225 + \frac{243}{2} + \frac{135}{2}\right)\right)$$

$$= \frac{25}{3} - \frac{424}{2} + 225$$

$$= \frac{25}{3} + 13$$

$$= \frac{64}{3}$$

Remark 8.

In some cases you need to reverse the order of integration (when using Fubini's Theorem) because it may not be possible to integrate without reversing the order. The next example illustrates this.

Example 4. Find $\int_0^1 \int_x^1 e^{y^2}\,dydx$

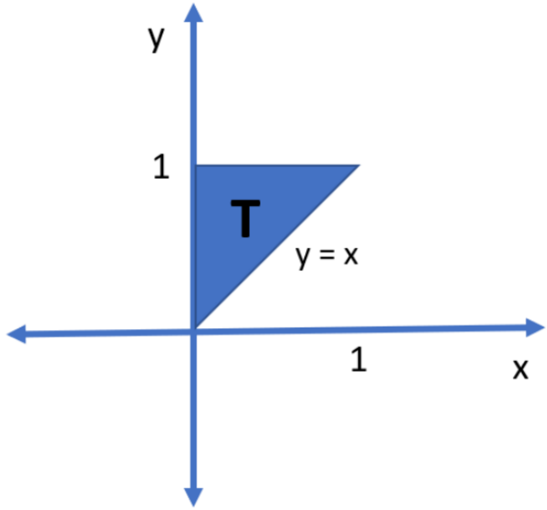
Solution.

There is no elementary function whose derivative is e^{y^2} and so one cannot integrate w.r.t. y first in the above iterated integral. Elementary functions are essentially the type of functions you have studied so far. In order to try to reverse the order of integration we need to look at the relevant region T in Fubini's theorem.

By Fubini's theorem (i) we have:

$$\int_0^1 \int_x^1 e^{y^2}\,dydx = \int_T\int e^{y^2}\,dA$$

where $T = \{(x, y) : 0 \le x \le 1,\ x \le y \le 1\}$. Here is a picture of T.



Note that we can also write T as $T = \{(x, y) : 0 \le y \le 1,\ 0 \le x \le y\}$. So, by Fubini's theorem (ii) we can integrate w.r.t. x first to get

$$\int_T\int e^{y^2}\, dA = \int_0^1 \int_0^y e^{y^2}\, dx dy$$

$$= \int_0^1 \left[xe^{y^2}\right]_0^y dy$$

$$= \int_0^1 ye^{y^2}\, dy$$

For this single integral, use the substitution rule with $u = y^2$ so

$$\int_0^1 ye^{y^2}\, dy = \frac{1}{2}\int_0^1 e^u\, du$$

$$= \frac{1}{2}\left[e^u\right]_0^1$$

$$= \frac{1}{2}(e - 1)$$

Definition 3.

The average value of a function f over the region T is

$$\frac{1}{S}\int_T\int f(x, y)\, dA$$

where S is the area of T.

Example 5.

Find the average value of $f(x, y) = \sin(x + y)$ over the region T given by $0 \leq x \leq \pi,\ 0 \leq y \leq \pi$.

Solution.

The average value is

$$\frac{1}{\pi^2} \int_T \int \sin(x+y)\, dA$$

$$= \frac{1}{\pi^2} \int_0^\pi \left(\int_0^\pi \sin(x+y)\, dy \right) dx$$

$$= \frac{1}{\pi^2} \int_0^\pi [-\cos(x+y)]_0^\pi \ dx$$

$$= \frac{1}{\pi^2} \int_0^\pi (-\cos(x+\pi) + \cos x)\ dx$$

$$= \frac{1}{\pi^2} [-\sin(x+\pi) + \sin x]_0^\pi$$

$$= 0$$

Example 6.

Find the average value of $g(x, y) = x\cos(xy)$ over the region T given by $0 \leq x \leq \pi,\ 0 \leq y \leq 1$.

Solution.

The average value is

$$\frac{1}{\pi} \int_T \int x\cos(xy)\, dA$$

$$= \frac{1}{\pi} \int_0^\pi \left(\int_0^1 x\cos(xy)\, dy \right) dx$$

$$= \frac{1}{\pi} \int_0^\pi [\sin(xy)]_0^1 \ dx$$

$$= \frac{1}{\pi} \int_0^\pi (\sin x - 0)\ dx$$

$$= \frac{1}{\pi} [-\cos x]_0^\pi$$

$$= \frac{1}{\pi}(1 + 1)$$

$$= \frac{2}{\pi}$$