Fiacre Ó Cairbre

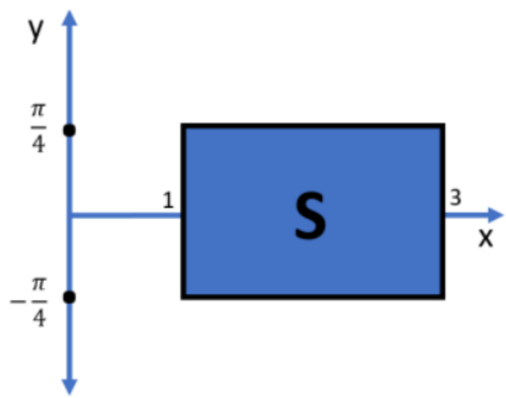
## Lecture 12

**Example 5.**

Find the absolute maxima and absolute minima of $f(x,y) = (4x - x^2)\cos y$ on the closed rectangular region $S$ given by $1 \le x \le 3$, $-\frac{\pi}{4} \le y \le \frac{\pi}{4}$. Also, find the absolute maximum value and the absolute minimum value of $f$ on $S$.

**Solution.**

Denote the rectangular region by $S$. It is helpful to draw a picture of $S$ below.




We see that $S = \{(x,y) \in \mathbb{R}^2 : 1 \le x \le 3,\ -\frac{\pi}{4} \le y \le \frac{\pi}{4}\}$ Note that $S$ is a closed bounded subset of $\mathbb{R}^2$ and that $f$ is continuous on $S$ and so we can use the strategy in remark 7.

**Step 1 – Interior points of $S$.**

Note that $\dfrac{\partial f}{\partial x} = (4-2x)\cos y$ and $\dfrac{\partial f}{\partial y} = -(4x - x^2)\sin y$. So,

$$\frac{\partial f}{\partial x}_{|(a,b)} = 0 = \frac{\partial f}{\partial y}_{|(a,b)} \quad \text{and} \quad (a,b) \in \text{Int}(S) \iff (a,b) = (2,0)$$

So, $(2,0)$ is the only critical point and $f(2,0) = 4$

**Step 2 – Boundary points of $S$.**

The boundary of $S$ consists of the four sides of the rectangle and we will take one side at a time.

(i) On the line segment joining $(1, -\frac{\pi}{4})$ to $(3, -\frac{\pi}{4})$ we have $f(x,y) = f(x, -\frac{\pi}{4}) = \frac{1}{\sqrt{2}}(4x - x^2)$, which may be considered as a function of one variable on $[1,3]$. From remark 9 we have that the candidates for the absolute maxima and absolute minima for this function of one variable are the endpoints $1, 3$ and the $x$–values in $(1, 3)$, where

$$0 = \frac{d}{dx}\frac{1}{\sqrt{2}}(4x - x^2) \Rightarrow 4 - 2x = 0 \Rightarrow x = 2$$

So, the only candidates for the absolute maxima and absolute minima for this function of one variable are the endpoints $1, 3$ and $2$. So, the candidates for the absolute maxima and absolute minima for $f(x, y)$ are $(1, -\frac{\pi}{4})$, $(3, -\frac{\pi}{4})$ and $(2, -\frac{\pi}{4})$ and we evaluate $f$ at these points to get

$$f(1, -\frac{\pi}{4}) = \frac{3}{\sqrt{2}}, \quad f(3, -\frac{\pi}{4}) = \frac{3}{\sqrt{2}}, \quad f(2, -\frac{\pi}{4}) = \frac{4}{\sqrt{2}} \qquad (*)$$

(ii) On the line segment joining $(1, \frac{\pi}{4})$ to $(3, \frac{\pi}{4})$ we have $f(x, y) = f(x, \frac{\pi}{4}) = \frac{1}{\sqrt{2}}(4x - x^2)$, which may be considered as a function of one variable on $[1, 3]$. From remark 9 we have that the candidates for the absolute maxima and absolute minima for this function of one variable are the endpoints $1, 3$ and the $x$–values in $(1, 3)$, where

$$0 = \frac{d}{dx}\frac{1}{\sqrt{2}}(4x - x^2) \Rightarrow 4 - 2x = 0 \Rightarrow x = 2$$

So, the only candidates for the absolute maxima and absolute minima for this function of one variable are the endpoints $1, 3$ and $2$. So, the candidates for the absolute maxima and absolute minima for $f(x, y)$ are $(1, \frac{\pi}{4})$, $(3, \frac{\pi}{4})$ and $(2, \frac{\pi}{4})$ and we evaluate $f$ at these points to get

$$f(1, \frac{\pi}{4}) = \frac{3}{\sqrt{2}}, \quad f(3, \frac{\pi}{4}) = \frac{3}{\sqrt{2}}, \quad f(2, \frac{\pi}{4}) = \frac{4}{\sqrt{2}} \qquad (**)$$

(iii) On the line segment joining $(3, -\frac{\pi}{4})$ to $(3, \frac{\pi}{4})$ we have $f(x, y) = f(3, y) = 3\cos y$, which may be considered as a function of one variable on $[-\frac{\pi}{4}, \frac{\pi}{4}]$. From remark 9 we have that the candidates for the absolute maxima and absolute minima for this function of one variable are the endpoints $-\frac{\pi}{4}, \frac{\pi}{4}$ and the $y$–values in $(-\frac{\pi}{4}, \frac{\pi}{4})$, where

$$0 = \frac{d}{dy}3\cos y \Rightarrow -3\sin y = 0 \Rightarrow y = 0$$

So, the only candidates for the absolute maxima and absolute minima for this function of one variable are the endpoints $-\frac{\pi}{4}, \frac{\pi}{4}$ and $0$. So, the candidates for the absolute maxima and absolute minima for $f(x, y)$ are $(3, -\frac{\pi}{4})$, $(3, \frac{\pi}{4})$ and $(3, 0)$, and we evaluate $f$ at these points to get

$$f(3, -\frac{\pi}{4}) = \frac{3}{\sqrt{2}}, \quad f(3, \frac{\pi}{4}) = \frac{3}{\sqrt{2}}, \quad f(3, 0) = 3 \qquad (***)$$

(iv) On the line segment joining $(1, -\frac{\pi}{4})$ to $(1, \frac{\pi}{4})$ we have $f(x, y) = f(1, y) = 3\cos y$, which may be considered as a function of one variable on $[-\frac{\pi}{4}, \frac{\pi}{4}]$. From remark 9 we have that the candidates for the absolute maxima and absolute minima for this function of one variable are the endpoints $-\frac{\pi}{4}, \frac{\pi}{4}$ and the $y$–values in $(-\frac{\pi}{4}, \frac{\pi}{4})$, where

$$0 = \frac{d}{dy} 3\cos y \Rightarrow -3\sin y = 0 \Rightarrow y = 0$$

So, the only candidates for the absolute maxima and absolute minima for this function of one variable are the endpoints $-\frac{\pi}{4}, \frac{\pi}{4}$ and 0. So, the candidates for the absolute maxima and absolute minima for $f(x, y)$ are $(1, -\frac{\pi}{4})$, $(1, \frac{\pi}{4})$ and $(1,0)$, and we evaluate $f$ at these points to get

$$f(1, -\frac{\pi}{4}) = \frac{3}{\sqrt{2}}, \quad f(1, \frac{\pi}{4}) = \frac{3}{\sqrt{2}}, \quad f(1, 0) = 3 \qquad (****)$$

**Step 3.**

The relevant values from steps 1 and 2 are in step 1 and $(*)$, $(**)$, $(***)$, $(****)$ and so the relevant values are

$$f(2,0) = 4, \quad f(1, -\frac{\pi}{4}) = \frac{3}{\sqrt{2}}, \quad f(3, -\frac{\pi}{4}) = \frac{3}{\sqrt{2}}, \quad f(2, -\frac{\pi}{4}) = \frac{4}{\sqrt{2}}, \quad f(1, \frac{\pi}{4}) = \frac{3}{\sqrt{2}}$$

$$f(3, \frac{\pi}{4}) = \frac{3}{\sqrt{2}}, \quad f(2, \frac{\pi}{4}) = \frac{4}{\sqrt{2}}, \quad f(3,0) = 3, \quad f(1,0) = 3 \qquad (*****)$$

Select the greatest value of $f$ from $(*****)$ to get that the absolute maximum value of $f$ on $T$ is 4. The points where $f$ takes on this absolute maximum value, will be the absolute maxima of $f$ on $T$ and so $(2, 0)$ is the only absolute maximum of $f$ on $T$.

Select the smallest value of $f$ from $(*****)$ to get that the absolute minimum value of $f$ on $T$ is $\frac{3}{\sqrt{2}}$. The points where $f$ takes on this absolute minimum value, will be the absolute minima of $f$ on $T$ and so the absolute minima of $f$ of $T$ are

$$(1, -\frac{\pi}{4}), \quad (1, \frac{\pi}{4}), \quad (3, -\frac{\pi}{4}), \quad (3, \frac{\pi}{4})$$

**Remark 10.**

Our next example will give an application of our theory.

**Example 6.**

A flat rectangular plate is given by $1 \le x \le 3$, $-\frac{\pi}{4} \le y \le \frac{\pi}{4}$. The plate is heated so that the temperature at the point $(x, y)$ is $(4x - x^2)\cos y$. Find the temperatures at the hottest and coldest points on the plate. Also, find the points on the plate where the hottest and coldest temperatures occur.

**Solution.**

Suppose $f(x, y) = (4x - x^2)\cos y$. Then, we can use the results of step 3 in example 5 to get that the temperature at the hottest point on the plate is 4 this occurs at the point $(2, 0)$ on the plate.

Also, we can use the results of step 3 in example 5 to get that the temperature at the coldest point on the plate is $\frac{3}{\sqrt{2}}$ and this occurs at the points

$$(1, -\frac{\pi}{4}), \quad (1, \frac{\pi}{4}), \quad (3, -\frac{\pi}{4}), \quad (3, \frac{\pi}{4})$$

on the plate.