MT251P – Lecture 14

Fiacre Ó Cairbre

Example 7.

Use Gauss–Jordan elimination to find the solution set of the following system T of linear equations:

$$x_1 + 2x_2 + x_3 = 3$$

$$2x_1 + 5x_2 - x_3 = -4$$

$$3x_1 - 2x_2 - x_3 = 5$$

Solution.

The augmented matrix for T is

$$\begin{pmatrix} 1 & 2 & 1 & 3 \\ 2 & 5 & -1 & -4 \\ 3 & -2 & -1 & 5 \end{pmatrix}$$

We perform elementary row operations on this augmented matrix. Replace R_2 with $R_2 - 2R_1$ and replace R_3 with $R_3 - 3R_1$ to get

$$\begin{pmatrix} 1 & 2 & 1 & 3 \\ 0 & 1 & -3 & -10 \\ 0 & -8 & -4 & -4 \end{pmatrix}$$

Replace R_3 with $R_3 + 8R_2$ to get

$$\begin{pmatrix} 1 & 2 & 1 & 3 \\ 0 & 1 & -3 & -10 \\ 0 & 0 & -28 & -84 \end{pmatrix}$$

Replace R_3 with $-\frac{1}{28}R_3$ to get

$$\begin{pmatrix} 1 & 2 & 1 & 3 \\ 0 & 1 & -3 & -10 \\ 0 & 0 & 1 & 3 \end{pmatrix}$$

Replace R_1 with $R_1 - R_3$ and replace R_2 with $R_2 + 3R_3$ to get

$$\begin{pmatrix} 1 & 2 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3 \end{pmatrix}$$

Replace R_1 with $R_1 - 2R_2$ to get

$$C = \begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3 \end{pmatrix}$$

which is in RREF.

The system of linear equations V corresponding to C is:

$$x_1 = 2$$

$$x_2 = -1$$

$$x_3 = 3$$

The solution set of V is $x_1 = 2,\ x_2 = -1,\ x_3 = 3$. Hence, by remark 3, the solution set of the original system of linear equations T is also

$$x_1 = 2,\ x_2 = -1,\ x_3 = 3$$

Example 8.

Find the solution set of the following system T of linear equations:

$$x_1 - 3x_2 + 5x_3 = 3$$

$$x_1 - 2x_2 + 7x_3 = 5$$

$$2x_1 - 6x_2 + 10x_3 = 5$$

Solution.

The augmented matrix for T is

$$\begin{pmatrix} 1 & -3 & 5 & 3 \\ 1 & -2 & 7 & 5 \\ 2 & -6 & 10 & 5 \end{pmatrix}$$

Replace R_2 with $R_2 - R_1$ and replace R_3 with $R_3 - 2R_1$ to get

$$E = \begin{pmatrix} 1 & -3 & 5 & 3 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Note that E is not in RREF but we can actually stop here without having to go to RREF because the solution set of the system of linear equations W corresponding to E can be read off conveniently as follows:

The system of linear equations W is:

$$x_1 - 3x_2 + 5x_3 = 3$$

$$x_2 + 2x_3 = 2$$

$$0 = -1$$

and so W has no solutions. So, T has no solutions.

Remark 3.

If in remark 2 we stop at REF instead of RREF, then the strategy is called Gaussian Elimination.

Example 9.

Use Gaussian elimination to find the solution set of the following system U of linear equations:

$$x_1 + 2x_2 + x_3 = 3$$

$$3x_1 - x_2 - 3x_3 = -1$$

$$2x_1 + 3x_2 + x_3 = 4$$

The augmented matrix for U is

$$\begin{pmatrix} 1 & 2 & 1 & 3 \\ 3 & -1 & -3 & -1 \\ 2 & 3 & 1 & 4 \end{pmatrix}$$

Replace R_2 with $R_2 - 3R_1$ and replace R_3 with $R_3 - 2R_1$ to get

$$\begin{pmatrix} 1 & 2 & 1 & 3 \\ 0 & -7 & -6 & -10 \\ 0 & -1 & -1 & -2 \end{pmatrix}$$

Interchange R_3 and R_2 to get

$$\begin{pmatrix} 1 & 2 & 1 & 3 \\ 0 & -1 & -1 & -2 \\ 0 & -7 & -6 & -10 \end{pmatrix}$$

Replace R_2 with $-R_2$ to get

$$\begin{pmatrix} 1 & 2 & 1 & 3 \\ 0 & 1 & 1 & 2 \\ 0 & -7 & -6 & -10 \end{pmatrix}$$

Replace R_3 with $R_3 + 7R_2$ to get

$$C = \begin{pmatrix} 1 & 2 & 1 & 3 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 4 \end{pmatrix}$$

which is in REF (but not RREF).

The system of linear equations V corresponding to C is:

$$x_1 + 2x_2 + x_3 = 3$$

$$x_2 + x_3 = 2$$

$$x_3 = 4$$

The solution set of V is $x_3 = 4, \quad x_2 = 2 - x_3 = -2, \quad x_1 = 3 - 2x_2 - x_3 = 3$. So, the solution set of U is also

$$x_1 = 3,\ x_2 = -2,\ x_3 = 4$$

Section 4.3 – Free variable case.

Remark 4.

Suppose, as in remark 2, the augmented matrix C is in RREF. If there are less non–zero rows in C than variables, then we are in what is called the free variable case. In this case. there will be infinitely many solutions and here is how we will read off the required solution set:

If a leading 1 in some row in C occurs in column k, then we say the variable x_k is a leading variable. Otherwise, we say x_k is a free variable. We can assign any real number value to the free variables and we can write the leading variables in terms of the free variables. So, the values of the leading variables will be determined by the values of the free variables.

Theorem 3.

Suppose S is a system of linear equations as in definition 2. Then, S has either no solution, a unique solution or infinitely many solutions.

Example 10.

Find the solution set of the following system W of linear equations:

$$x_1 + 6x_2 + 3x_4 = 2$$

$$12x_2 + 2x_1 + x_3 + 6x_4 = 0$$

$$3x_1 + x_3 + 18x_2 + 9x_4 = 2$$

Solution.

The augmented matrix for W is

$$\begin{pmatrix} 1 & 6 & 0 & 3 & 2 \\ 2 & 12 & 1 & 6 & 0 \\ 3 & 18 & 1 & 9 & 2 \end{pmatrix}$$