Report of Programming Task 2 of the course "Introduction to Optimization" - Fall 2024

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1 Team Information

- Team leader: Nikita Zagainov — 5

  Managed team work, Contributed to the algorithm implementation, Wrote report

- Team member 1: Ilyas Galiev — 5 Contributed to the algorithm implementation

- Team member 2: Arthur Babkin — 5

  Contributed to the algorithm implementation, Wrote QA tests

- Team member 3: Nikita Menshikov — 5

  Contributed to the algorithm implementation, Adapted problem from previous assignment

- Team member 4: Sergey Aitov — 5

  Contributed to the algorithm implementation, Contributed to QA testing

2 Link to the product

[Project source code](#)

3 Programming language

Python

Ingredient	Tomato	Cucumber	Bell Pepper	Lettuce Leaf	Onion
Cost, rub/kg	130	100	155	85	50
Nutritious value, ckal/kg	200	160	260	150	400
Max weight in salad, kg	0.6	0.6	0.6	0.2	0.05
Fats, proportion	0.004	0.005	0.006	0.003	0.004

Таблица 1: Ingredients and their properties

4   Linear programming problem

We aim to maximize nutritious value of salad given constraints on cost of its ingredients, maximum fats concentration, and weight of each individual component

- Our problem is maximization problem

- Objective function & constraints:

$$\text{maximize } c^T x$$

subject to

$$Ax \leq b$$

where:

$$A = \begin{bmatrix} 130 & 100 & 155 & 85 & 50 \\ 0.004 & 0.005 & 0.006 & 0.003 & 0.004 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$b = \begin{bmatrix} 200 \\ 1 \\ 0.6 \\ 0.6 \\ 0.6 \\ 0.2 \\ 0.05 \end{bmatrix}$$

2

$$c = \begin{bmatrix} 200 \\ 160 \\ 260 \\ 150 \\ 400 \end{bmatrix}$$

However, since the algorithm we implemented solves maximization problem

$$\text{maximize } c^T x$$

subject to

$$Ax = b$$

We manually introduce slack variables to convert inequality constraints to equality constraints.

5 Output & Results

We tested our implementation of interior point method by comparing its outputs with scipy implementation, and all tests show that outputsof both methods are the same on multiple tests, including original problem.
The method is applicable to our problem:

$$\text{Problem is bounded: True}$$
$$x : [0.2115, \ 0.6, \ 0.6, \ 0.2, \ 0.05]$$
$$f : 344.3$$

The results match with our previous simplex method implementation.

6 Code

```python
import numpy as np
from typing import Tuple


def pivot_col(tableau: np.ndarray, tol: float) -> int:
    last_row = tableau[-1, :-1]
    if np.all(last_row >= -tol):
        return -1
    return np.argmin(last_row)


def pivot_row(tableau: np.ndarray, tol: float, col: int) -> int:
    rhs = tableau[:-1, -1]
    lhs = tableau[:-1, col]
```

3

```python
15        ratios = np.full_like(rhs, np.inf)
16        valid = lhs > tol
17        ratios[valid] = rhs[valid] / lhs[valid]
18        if np.all(ratios == np.inf):
19            return -1
20        return np.argmin(ratios)
21
22
23    def find_basic_solution(
24        A: np.ndarray, b: np.ndarray, tol: float = 1e-6
25    ) -> Tuple[bool, np.ndarray]:
26        """
27        Performs Phase I of the simplex method to find a basic feasible
             solution.
28
29        Args:
30            A: Coefficient matrix of the constraints (m x n).
31            b: Right-hand side vector of the constraints (m,).
32            tol: Tolerance for determining feasibility.
33
34        Returns:
35            A tuple containing:
36            - A boolean indicating whether a feasible solution was found.
37            - A numpy array representing the basic feasible solution (if
                 found).
38        """
39        m, n = A.shape
40        A_phase1 = np.hstack([A, np.eye(m)])
41        c_phase1 = np.concatenate([np.zeros(n), np.ones(m)])
42        B = list(range(n, n + m))
43
44        tableau = np.hstack([A_phase1, b.reshape(-1, 1)])
45        tableau = np.vstack([tableau, np.concatenate([c_phase1, [0]])])
46
47        for i in range(m):
48            tableau[-1, :] -= tableau[i, :]
49
50        while True:
51            col = np.argmin(tableau[-1, :-1])
52            if tableau[-1, col] >= -tol:
53                break
54
55            ratios = []
56            for i in range(m):
57                if tableau[i, col] > tol:
58                    ratio = tableau[i, -1] / tableau[i, col]
59                    ratios.append((ratio, i))
60            if not ratios:
61                return False, None
62            _, row = min(ratios)
```

```python
            pivot = tableau[row, col]
            tableau[row, :] /= pivot
            for i in range(m + 1):
                if i != row:
                    tableau[i, :] -= tableau[i, col] * tableau[row, :]

            B[row] = col

    basic_solution = np.zeros(n + m)
    basic_solution[B] = tableau[:m, -1]
    if np.any(basic_solution[n:] > tol):
        return False, None

    x_basic = basic_solution[:n]
    x_basic = x_basic + 2e-5 # this is to avoid zero gradients on first
        step
    return True, x_basic


def interior_point(
    A: np.ndarray,
    b: np.ndarray,
    c: np.ndarray,
    alpha: float = 0.5,
    tol: float = 1e-6,
    max_iters: int = 100000,
) -> Tuple[bool, np.ndarray]:
    """
    Performs the interior point method to solve a linear programming
        problem.

    Args:
        A: Coefficient matrix of the constraints (m x n).
        b: Right-hand side vector of the constraints (m,).
        c: Coefficient vector of the objective function to be maximized
            (n,).
        alpha: Step size for the Newton step.
        tol: Tolerance for determining convergence.
        max_iters: Maximum number of iterations to perform.

    Returns:
        A tuple containing:
        - A boolean indicating whether a feasible solution was found.
        - A numpy array representing the optimal solution (if found).
    """
    bounded, x = find_basic_solution(A, b)
    if not bounded:
        return False, None
    prev_x = None
```

```python
        n_iters = 0

        while prev_x is None or np.linalg.norm(x - prev_x) > tol:
            n_iters += 1
            if n_iters > max_iters:
                return False, None, None

            D = x.copy()
            x_hat = x * (1 / D)
            A_hat = A * D
            c_hat = c * D
            P = np.eye(A_hat.shape[1]) - A_hat.T @ np.linalg.inv(A_hat @
                A_hat.T) @ A_hat

            c_proj = P @ c_hat
            nu = np.abs(np.min(c_proj)) if np.any(c_proj < 0) else 1
            x_hat = x_hat + (alpha / nu) * c_proj
            prev_x = x.copy()
            x = D * x_hat

        return True, x, c @ x


def main():
    np.set_printoptions(precision=4, suppress=True)

    A = np.array(
        [
            [130, 100, 155, 85, 50],
            [0.004, 0.005, 0.006, 0.003, 0.004],
            [1, 0, 0, 0, 0],
            [0, 1, 0, 0, 0],
            [0, 0, 1, 0, 0],
            [0, 0, 0, 1, 0],
            [0, 0, 0, 0, 1],
        ],
        dtype=np.float32,
    )
    b = np.array([200, 0.01, 0.6, 0.6, 0.6, 0.2, 0.05], dtype=np.float32)
    c = np.array([200, 160, 260, 150, 400], dtype=np.float32)

    # since we are maximizing c.T @ x s. t. A @ x <= b, we need to
        introduce slack variables:
    A_slack = np.hstack([A, np.eye(A.shape[0])])
    c_slack = np.concatenate([c, np.zeros(A.shape[0])])

    feasible, x, f = interior_point(A_slack, b, c_slack)
    if feasible:
        print(f"Optimal x: {x[: c.shape[0]]}, optimal value: {f}")
    else:
```

```
158            print("No feasible solution found.")
159
160
161  if __name__ == "__main__":
162      main()
```