STAT 430-1, Fall 2024

HW1

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Contents

Exercise 1 Intersections and Unions of σ-algebras

1.(a)

Say we have a family of σ-algebras indexed by $\mathcal{I}$: $\{(E,\mathcal{E}_i)\}_{i\in\mathcal{I}}$. Then we verify that $\bigcap_{i\in\mathcal{I}}\mathcal{E}_i$ is a σ-algebra.

- **Empty set**: Since $\forall i\in\mathcal{I}$ we have $\emptyset, E\in\mathcal{E}_i$, we have $\emptyset, E\in\bigcap_{i\in\mathcal{I}}\mathcal{E}_i$.
- **Closed under complement**: For any $A\in\bigcap_{i\in\mathcal{I}}\mathcal{E}_i$, we have $A\in\mathcal{E}_i,\forall i\in\mathcal{I}$. Thus $A^c\in\mathcal{E}_i,\forall i\in\mathcal{I}$, which implies $A^c\in\bigcap_{i\in\mathcal{I}}\mathcal{E}_i$.
- **Closed under σ-union**: For any $\{A_n\}_{n\in\mathbb{N}}\subseteq\bigcap_{i\in\mathcal{I}}\mathcal{E}_i$, we have $A_n\in\mathcal{E}_i,\forall i\in\mathcal{I}$. Thus $\bigcup_{n\in\mathbb{N}}A_n\in\mathcal{E}_i,\forall i\in\mathcal{I}$, which implies $\bigcup_{n\in\mathbb{N}}A_n\in\bigcap_{i\in\mathcal{I}}\mathcal{E}_i$.

By verifying the three properties, we conclude that $\bigcap_{i\in\mathcal{I}}\mathcal{E}_i$ is a σ-algebra.

1.(b)

We construct the following example for union of σ-algebras:

$$\begin{aligned} E =&\{1,2,3\},\\ \mathcal{E}_1 =&\{\emptyset,\{1\},\{2,3\},E\},\\ \mathcal{E}_2 =&\{\emptyset,\{1,2\},\{3\},E\}. \end{aligned}$$

Then we have $\mathcal{E}_1\cup\mathcal{E}_2=\{\emptyset,\{1\},\{2,3\},\{1,2\},\{3\},E\}$, which is not a σ-algebra. For example, $\{1\}\cup\{3\}=\{1,3\}\notin\mathcal{E}_1\cup\mathcal{E}_2$ (not closed under union).

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Actually the σ-algebra generated by $\mathcal{E}_1 \cup \mathcal{E}_2$ is

$$\mathcal{E}_1 \wedge \mathcal{E}_2 = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, E\}.$$

Exercise 2 Increasing algebras and σ-algebras

2.(a)

If $\mathcal{E}_n$ is algebra, we verify the three properties of algebra for $\mathcal{E} = \bigcup_{n=1}^{\infty} \mathcal{E}_n$:

- **Empty set**: Since $\emptyset \in \mathcal{E}_n, \forall n \in \mathbb{N}$, we have $\emptyset \in \mathcal{E}$.
- **Closed under complement**: For any $A \in \mathcal{E}$, meaning that $\exists n_0 \in \mathbb{N}^+$ s.t. $A \in \mathcal{E}_n$, $\forall n \geq n_0$. Then since $\mathcal{E}_n$ is algebra, we have $A^c \in \mathcal{E}_n, \forall n \geq n_0$, which implies $A^c \in \mathcal{E} = \bigcup_{n=1}^{\infty} \mathcal{E}_n$.
- **Cloed under finite union**: It suffices to verify closure under union of 2. For $A, B \in \mathcal{E}$, we have $\exists n_1, n_2 \in \mathbb{N}^+$ s.t. $A \in \mathcal{E}_{n_1}, B \in \mathcal{E}_{n_2}$. Without loss of generality, we assume $n_1 \leq n_2$. Then we have $A, B \in \mathcal{E}_{n_2}$, then using closure under finite union of $\mathcal{E}_{n_2}$, we have $A \cup B \in \mathcal{E}_n, \forall n \geq n_2$. This implies $A \cup B \in \mathcal{E}$.

Thus we conclude that $\mathcal{E} = \bigcup_{n=1}^{\infty} \mathcal{E}_n$ is an algebra.

2.(b)

Consider the following example:

$$\mathcal{E}_n = \sigma(\emptyset, \{1\}, \{2\}, \cdots, \{n\}, \mathbb{N}^+)$$

for which all with $\Omega = \mathbb{N}^+$. However we find that $\bigcup_{n=1}^{\infty} \mathcal{E}_n$ is not a σ-algebra by noticing the following:

$$\{2i\} \in \mathcal{E} = \bigcup_{n=1}^{\infty} \mathcal{E}_n, \forall i \in \mathbb{R}, \text{ but } \bigcup_{i \in \mathbb{N}^+} \{2i\} = \{2, 4, 6, \cdots\} \notin \mathcal{E}.$$

which implies that $\mathcal{E}$ is not closed under countable union. Thus $\bigcup_{n=1}^{\infty} \mathcal{E}_n$ is not a σ-algebra.

Exercise 3 Borel σ-algebra on $\mathbb{R}$

3.(a)

- Any σ-algebra containing all open sets must contain all open intervals, which is trivial. (And definitely we have $\mathcal{B}_{\mathbb{R}}$ being a σ-algebra, and it contains all open intervals.)
- Then consider any σ-algebra $\mathcal{A}$ containing all open intervals. In this case, for any open sets $O \subseteq \mathbb{R}$, by definition, $\exists$ intervals $\{I_i\}_{i \in \mathbb{N}^+}$ s.t.

  $$O = \bigcup_{i \in \mathbb{N}^+} I_i.$$

  On the other hand, since $\mathcal{A}$ contains all open intervals, i.e. $I_i \in \mathcal{A}, \forall i \in \mathbb{N}^+$. Thus $O = \bigcup_{i \in \mathbb{N}^+} I_i \in \mathcal{A}$, which implies that $\mathcal{A}$ contains all open sets.

By summarizing the two points, we conclude that: any σ-algebra containing all open sets must also contain all open intervals, and vice versa. Thus the Borel σ-algebra $\mathcal{B}_{\mathbb{R}}$, as the intersection of all σ-algebras containing all open sets, is also the smallest σ-algebra containing all open intervals.

3.(b)

We verify the following cases:

- $(-\infty, \mathbf{x})$: Since $(-\infty, x)$ is open, we have $(-\infty, x) \in \mathcal{B}_{\mathbb{R}}$.
- $(-\infty, \mathbf{x}]$: We can write $(-\infty, x] = \bigcap_{n=1}^{\infty}(-\infty, x + \frac{1}{n})$, thus $(-\infty, x] \in \mathcal{B}_{\mathbb{R}}$.
- $(\mathbf{x}, \mathbf{y}]$: We can write $(x, y] = (-\infty, y] \cap (x, y+1)$, thus $(x, y] \in \mathcal{B}_{\mathbb{R}}$.
- $[\mathbf{x}, \mathbf{y}]$: Similarly, we can write $[x, y] = (-\infty, y] \cap [x, +\infty)$, thus $[x, y] \in \mathcal{B}_{\mathbb{R}}$.
- $\{\mathbf{x}\}$: We can write $\{x\} = (-\infty, x] \cap [x, +\infty)$, thus $\{x\} \in \mathcal{B}_{\mathbb{R}}$.

3.(c)

We first prove the following lemma:

▶ For a collection of intervals $G = \{g_i\}_{i\in\mathcal{I}}$ (could be uncountable). If any open set $O \subseteq \mathbb{R}$ can be written as **countable** operations of g_i (operations include union, intersection, and complement), then $\mathcal{B}_{\mathbb{R}}$ is also the generated σ-algebra by G.

Proof. The proof is similar to that in 3.1.

- Since G is a collection of intervals, then any σ-algebra containing all open sets must also contain all elements in G.
- On the other hand, for any σ-algebra $\mathcal{A}$ containing all elements in G, then for any open set $O \subseteq \mathbb{R}$, by assumption, O can be written as a countable operation of g_i, i.e. $O \in \mathcal{A}$. While $g_i \in \mathcal{A}$ gives that this countable operation is also in $\mathcal{A}$. Thus $O \in \mathcal{A}$, which implies that $\mathcal{A}$ contains all open sets.

By summarizing the two points, we conclude that $\mathcal{B}_{\mathbb{R}}$ is also the generated σ-algebra by G. □

Then it suffices to show that in the four cases listed in the questions, any open sets can be written as countable operations of the intervals. Further since open sets are countable unions of open intervals, **it suffices to show that open intervals can be written as countable operations of the intervals given in eeach of the four cases.**

(i) Any $(a, b) = \big((-\infty, a]^{\complement}\big) \cap \big(\bigcup_{i=1}^{\infty}(-\infty, b - \frac{1}{i}]\big)$

(ii) Any $(a, b) = \bigcup_{i=1}^{\infty}(a, b - \frac{1}{i}]$

(iii) Any $(a, b) = \bigcup_{i=1}^{\infty}[a + \frac{1}{i}, b - \frac{1}{i}]$

(iv) Any $(a, b) = \big((a, \infty)\big) \cap \big(\bigcup_{i=1}^{\infty}(b - \frac{1}{i}, \infty)^{\complement}\big)$

Further, to upgrade to rational number cases, we may just notice that any real number x could be written as a monotone (say, increasing) limit of rational number sequence $\{x_n\}_{n\in\mathbb{N}^+}$ $x_n \in \mathbb{Q}$ (for example, using decimal numeral system). In this way, we can obtain any interval with real ends by countable operations of intervals with rational ends. Thus the Borel σ-algebra $\mathcal{B}_\mathbb{R}$ is generated by the four cases listed in the question.

Exercise 4 Borel σ-algebra and continuous functions

We have the following two points:

- Consider any σ-algebra $\tilde{\mathcal{E}}$ that makes all continuous functions measurable. $\tilde{\mathcal{E}} \subseteq \mathcal{B}_{\mathbb{R}^n}$.

  Proof. For any continuous function $f : (\mathbb{R}^n, \mathcal{B}_{\mathbb{R}^n}) \mapsto (\mathbb{R}, \mathcal{B}_\mathbb{R})$, we can check for

  $$f^{-1}((-\infty, x)) = \{\omega \in \mathbb{R}^n : f(\omega) < x\} \in \mathcal{B}_{\mathbb{R}^n}$$

  which is obvious form the definition of continuous functions. Thus we have $\tilde{\mathcal{E}} \subseteq \mathcal{B}_{\mathbb{R}^n}$. □

  So further, denote $\mathcal{E}$ the minimal σ-algebra that makes all continuous functions measurable, we have also $\mathcal{E} \subseteq \mathcal{B}_{\mathbb{R}^n}$.

- If a σ-algebra $\mathcal{E}$ is the smallest one that makes all continuous functions measurable, then $\mathcal{B}_{\mathbb{R}^n} \subseteq \mathcal{E}$.

  Proof. It suffices to show that for such $\mathcal{E}$, it has to contain all closed sets. So consider any given closed set $C \subseteq \mathbb{R}^n$ and a corresponding continuous function $f_C(\omega)$ defined as the distance of ω to C, i.e. $f_C(\omega) = \inf_{x\in C} \|\omega - x\|$. Then f_C is continuous, and thus $f_C^{-1}(\{0\}) = C \in \mathcal{E}$. Such construction is possible for any closed set $C \in \mathbb{R}^n$, so $\mathcal{E}$ contains all closed sets, while these closed sets generate $\mathcal{B}_{\mathbb{R}^n}$. So we have $\mathcal{B}_{\mathbb{R}^n} \subseteq \mathcal{E}$. □

Combining the above two points, we conclude that $\mathcal{B}_{\mathbb{R}^n} \subseteq \mathcal{E} \subseteq \mathcal{B}_{\mathbb{R}^n}$, which implies that $\mathcal{E} = \mathcal{B}_{\mathbb{R}^n}$. i.e. the Borel σ-algebra on $\mathbb{R}^n$ is the smallest σ-algebra that makes all continuous functions measurable.

Exercise 5 Restrictions and traces of measures

5.(a)

We verify the conditions of a measure:

- **Empty set**: Since μ is a measure, we have $\nu(\emptyset) = \mu(\emptyset \cap D) = \mu(\emptyset) = 0$
- **Countable additivity**: For any disjoint sets $\{A_n\}_{n\in\mathbb{N}} \subseteq \mathcal{E}$, we have also $\{A_n \cap D\}_{n\in\mathbb{N}} \subseteq \mathcal{E}$ being disjoint. Then

  $$\nu(\biguplus_{n\in\mathbb{N}} A_n) = \mu\left(\left(\biguplus_{n\in\mathbb{N}} A_n\right) \cap D\right) = \mu\left(\biguplus_{n\in\mathbb{N}} (A_n \cap D)\right) = \sum_{n\in\mathbb{N}} \mu(A_n \cap D) = \sum_{n\in\mathbb{N}} \nu(A_n).$$

Thus ν is a measure on $(E, \mathcal{E})$.

5.(b)

We verify the conditions of a measure:

- **Empty set**: Since $\mathcal{D} = \mathcal{E} \cap D$, thus $\emptyset \in \mathcal{D}$. We have $\nu(\emptyset) = \mu(\emptyset) = 0$.
- **Countable additivity**: For any disjoint sets $\{A_n\}_{n\in\mathbb{N}} \subseteq \mathcal{D} \subseteq \mathcal{E}$, we thus have $\nu(\biguplus_{n\in\mathbb{N}} A_n) = \mu\left(\biguplus_{n\in\mathbb{N}} A_n\right) = \sum_{n\in\mathbb{N}} \mu(A_n) = \sum_{n\in\mathbb{N}} \nu(A_n)$.

Thus ν is a measure on $(D, \mathcal{D})$.

Appendix References

- I discussed Question 2.(b) with Yikun Li (yikunli2028@u.northwestern.edu).
- For Question 4, I referred to the following StackExchange post: StackExchange Post.
- Wikipedia pages on measurable functions and Borel sets.
- Textbooks, including Durrett and Billingsley.