# STAT 430-1, Fall 2024
## HW2

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### Exercise 1  Atoms

Denote by $K_n$ the set of atoms with mass greater than $1/n$, then we have

$$\mu(E) \geq \mu(K_n) \geq \frac{|K_n|}{n} \Rightarrow |K_n| \leq n\mu(E)$$

which means that $K_n$ is finite for any $n \in \mathbb{N}^+$. And we also note that $\{K_n\}_{n \in \mathbb{N}^+}$ is an increasing sequence of sets. By definition of stones, we have the set of all stones being $\mathcal{S} = \bigcup_{n=1}^{\infty} K_n = \biguplus_{n=1}^{\infty} K_n \backslash K_{n-1}$. i.e. $\{K_n \backslash K_{n-1}\}_{n \in \mathbb{N}^+}$ is a partition of $\mathcal{S}$.

**If** $S$ is uncountable, then there exists $n_0 \in \mathbb{N}^+$ such that $K_{n_0} \backslash K_{n_0-1}$ is uncountable, an thus $K_{n_0} = \bigcup_{k=1}^{n_0} K_{n_0} \backslash K_{n_0-1}$ is also uncountable, which contradicts the fact that $|K_{n_0}| \leq n\mu(E) < \infty$.

Thus $S$ is countable.

### Exercise 2  Fatou's Lemma

Note that $\{\inf_{m \geq n} f_m\}_{n \in \mathbb{N}}$ is a monotone increasing sequence of functions. Using monotone convergence theorem, we have

$$
\begin{aligned}
\mu\left(\liminf_{n \to \infty} f_n\right) &= \mu\left(\lim_{n \to \infty} \inf_{m \geq n} f_m\right) \\
&= \lim_{n \to \infty} \mu\left(\inf_{m \geq n} f_m\right) \\
&\leq \lim_{n \to \infty} \inf_{m \geq n} \mu(f_m) \\
&= \liminf_{n \to \infty} \mu(f_n)
\end{aligned}
$$

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## Exercise 3    Dominated convergence theorem

From above, we already have $\mu(f) \leq \liminf_{n\to\infty} \mu(f_n)$. Now it suffice to show the other direction.

Note that we have dominance condition $-g \leq f_n \leq g$, $\forall n \in \mathbb{N}^+$, thus we can consider the following sequence of functions: $\{g - f_n\}_{n\in\mathbb{N}^+}$, which has

$$0 \leq g - f_n \leq 2g$$

thus is still integrable. By Fatou's lemma, we have

$$\mu(g - f) \leq \liminf_{n\to\infty} \mu(g - f_n) \Rightarrow \mu(f) \geq \limsup_{n\to\infty} \mu(f_n)$$

which concludes the proof.

## Exercise 4    On interchanging limits and integration

- **Left hand side**: we have

$$\lim_{\lambda\to 0} \int_0^\infty f_\lambda(x) \, \mathrm{d}x = \lim_{\lambda\to 0} \int_0^\infty \lambda e^{-\lambda x} \, \mathrm{d}x = \lim_{\lambda\to 0} 1 = 1$$

- **Right hand side**: we have

$$\int_0^\infty \lim_{\lambda\to 0} f_\lambda(x) \, \mathrm{d}x = \int_0^\infty \lim_{\lambda\to 0} \lambda e^{-\lambda x} \, \mathrm{d}x = \int_0^\infty 0 \, \mathrm{d}x = 0$$

Thus the two sides are not equal, and the limit cannot be interchanged with the integral.

To explain why this happens, we illustrate the following that there is no dominating function for the sequence of functions $\{f_\lambda(x)\}_{\lambda>0}$: for any $x \in \mathbb{R}^+$, and any $\varepsilon \in (0,1)$, we can find $\lambda_\varepsilon$ s.t. $\lambda_\varepsilon e^{-\lambda_\varepsilon x} = \varepsilon$, thus to dominate the sequence, we need to have $g(x) \geq \varepsilon$, $\forall x \in \mathbb{R}^+$, which makes $g(x)$ un-integrable. Thus there is no dominating function for the sequence of functions, and the condition of Lebesgue's dominated convergence theorem is not satisfied.

## Exercise 5    On interchanging integrals

- We have

$$I = \int_0^\infty \mathrm{d}x \int_0^\infty \mathrm{d}y f(x,y) = \int_0^\infty \mathrm{d}x \begin{cases} x - 1, & x \in [0,1] \\ 0, & x \in (1,\infty) \end{cases}$$
$$= \int_0^1 \mathrm{d}x (x-1) = -\frac{1}{2}$$

- We have

$$J = \int_0^\infty \mathrm{d}y \int_0^\infty \mathrm{d}x f(x,y) = \int_0^\infty \mathrm{d}y \begin{cases} 1 - y, & y \in [0,1] \\ 0, & y \in (1,\infty) \end{cases}$$
$$= \int_0^1 \mathrm{d}y (1-y) = \frac{1}{2}$$

i.e. we got $I \neq J$, and the integrals cannot be interchanged. We can explain this by noticing that the condition of Fubini's theorem is not satisfied: the function $f(x, y)$ is not integrable on $\mathbb{R}^2$:

$$\int_0^\infty \mathrm{d}x \int_0^\infty \mathrm{d}y \, |f(x, y)| = \infty.$$