

STAT 430-1, Fall 2024

HW5

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Notation: I use $\hat{=}$ for Fourier transform and $\hat{=}$ for inverse Fourier transform. i.e. $f(x) \hat{=} \phi(t)$ and $\phi(t) \hat{=} f(x)$. $\text{Res}_f(x)$ for the residue of $f(x)$ at x . Dirac delta function (at zero) is denoted as $\delta(x)$ s.t. $\int_{-\infty}^{\infty} \delta(x) dx = 1$.

Exercise 1 Characteristic functions

(a) Note that function $t \mapsto 1/(1+t^2)$ has inverse Fourier transform as follows:

$$1/(1+t^2) \hat{=} \frac{1}{2\pi} \int_{\mathbb{R}} \frac{1}{1+t^2} e^{-ixt} dt = \begin{cases} \frac{2\pi i}{2\pi} \text{Res}_{e^{-ixt}/(1+t^2)}(i), & x > 0 \\ \frac{2\pi i}{2\pi} \text{Res}_{e^{-ixt}/(1+t^2)}(-i), & x \leq 0 \end{cases} = e^{-|x|}/2$$

which is the density function of a two-sided exponential distribution.

(b) Not a characteristic function by noticing that $t \mapsto e^{-t^4}$ is not bounded.

(c) Not a characteristic function by noticing that $\sin(t)|_{t=0} = 0 \neq 1$, thus does not satisfy the condition of Bochner's theorem.

(d) Note that $t \mapsto \cos(t)$ has inverse Fourier transform as follows:

$$\cos(t) \hat{=} \frac{1}{2\pi} \int_{\mathbb{R}} \cos(t) e^{-ixt} dt = \frac{1}{2\pi} \int_{\mathbb{R}} \frac{e^{it} + e^{-it}}{2} e^{-ixt} dt = \frac{1}{2} (\delta(x+1) + \delta(x-1)) \sim \text{Unif}\{\pm 1\}$$

a uniform distribution on $\{\pm 1\}$.

(e) Note that $t \mapsto \frac{1 + \cos(t)}{2}$ has inverse Fourier transform as follows:

$$\frac{1 + \cos(t)}{2} \hat{=} \frac{1}{2\pi} \int_{\mathbb{R}} \frac{1 + \cos(t)}{2} e^{-ixt} dt = \frac{1}{2\pi} \int_{\mathbb{R}} \frac{1 + \frac{e^{it} + e^{-it}}{2}}{2} e^{-ixt} dt = \frac{1}{4} (\delta(x-1) + \delta(x+1) + 2\delta(x))$$

which is a discrete r.v. with $P(X=1) = P(X=-1) = 1/4$ and $P(X=0) = 1/2$.

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Exercise 2 Die roll plus uniform

(a) For a fair die X , we have

$$X \sim \text{Unif}\{1, 2, 3, 4, 5, 6\} \doteq \frac{1}{6} \sum_{k=1}^6 e^{ikt}$$

(b) For Y uniform on $[0, 1]$, we have

$$Y \sim \text{Unif}[0, 1] \doteq \frac{e^{it} - 1}{it}$$

(c) For $Z = X + Y$, we have

$$X + Y \doteq \frac{1}{6} \sum_{k=1}^6 e^{ikt} \cdot \frac{e^{it} - 1}{it} = \frac{1}{6} \sum_{k=1}^6 \frac{e^{i(k+1)t} - e^{ikt}}{it} = \frac{e^{i7t} - e^{it}}{6it} \doteq \text{Unif}[1, 7]$$

(d) For $W = X - Y$, we have

$$X - Y \doteq \frac{1}{6} \sum_{k=1}^6 e^{ikt} \cdot \frac{\overline{e^{it} - 1}}{it} = \frac{1}{6} \sum_{k=1}^6 e^{ikt} \cdot \frac{e^{-it} - 1}{-it} = \frac{1}{6} \sum_{k=1}^6 \frac{e^{i(k-1)t} - e^{ikt}}{-it} = \frac{e^{i6t} - 1}{6it} \doteq \text{Unif}[0, 6]$$

Exercise 3 Difference of two i.i.d. random variables

Note that for two i.i.d. r.v. X, Y with characteristic function $\phi(t)$, we have

$$X - Y \doteq \phi(t) \cdot \overline{\phi(t)} = |\phi(t)|^2 \geq 0$$

however for uniform distribution $Z \sim \text{Unif}[-1, 1]$ we know that

$$Z \sim \text{Unif}[-1, 1] \doteq \frac{\sin t}{t} \text{ is not non-negative}$$

which yields contradiction. Thus Z cannot be represented as the difference of two i.i.d. r.v.s.

Exercise 4 Trigonometric identities

We define the following:

$$X_n \sim \text{Unif}\{\pm \frac{1}{2^n}\} \doteq \frac{1}{2} (e^{i/2^n t} + e^{-i/2^n t}) = \cos(\frac{t}{2^n})$$

$$Y_n := \sum_{i=1}^n X_i \sim \text{Unif}\{\frac{k}{2^n}\}_{k=-2^n+1}^{2^n-1}$$

$$Z \sim \text{Unif}[-1, 1] \doteq \frac{\sin t}{t}$$

- We note that $Z \stackrel{d}{=} \text{Unif}(-\frac{1}{2}, \frac{1}{2}) + X_1$ (which is easy to verify following the same proof as in Exercise. 2).

We thus have

$$\begin{aligned} \frac{\sin t}{t} &\doteq Z \stackrel{d}{=} \text{Unif}(-\frac{1}{2}, \frac{1}{2}) + X_1 \doteq \frac{\sin(t/2)}{t/2} \cdot \cos(t/2) \\ \Rightarrow \frac{\sin t}{t} &= \frac{\sin(t/2)}{t/2} \cdot \cos(t/2) \end{aligned}$$

- We further have $Y_n \xrightarrow{d} Z$ by noticing the following:

$$\forall \xi \in [-1, 1] : \mathbb{P}(Y_n \geq \xi) \xrightarrow{n \rightarrow \infty} \mathbb{P}(Z \geq \xi).$$

Thus we can conclude convergence in characteristic functions:

$$\begin{aligned} \prod_{i=1}^n \cos\left(\frac{t}{2^i}\right) &= \sum_{i=1}^n X_i = Y_n \xrightarrow{d} Z = \frac{\sin t}{t} \\ \Rightarrow \frac{\sin t}{t} &= \prod_{i=1}^{\infty} \cos\left(\frac{t}{2^i}\right), \quad \forall t \in \mathbb{R}. \end{aligned}$$

Exercise 5 Poisson approximation to the binomial; exponential approximation to the geometric

(This time we prove them "by hand")

5.(a) Poisson approximation to the Binomial distribution

We have $\forall k \in \mathbb{N}^+$:

$$\begin{aligned} \mathbb{P}(\text{Bin}(n, p_n) = k) &= \binom{n}{k} p_n^k (1 - p_n)^{n-k} \\ &= \frac{1}{k!} \frac{n!}{(n-k)!} \cdot \frac{1}{n^n} (np_n)^k (n - np_n)^{n-k} \\ &= \frac{1}{k!} (n^k + o(n^k)) \cdot \frac{1}{n^n} (\lambda + o(1))^k (n - \lambda + o(1))^{n-k} \\ &= \frac{(\lambda + o(1))^k}{k!} \frac{(n^k + o(n^k)) \cdot (n - \lambda + o(1))^{n-k}}{n^n} \\ &= \frac{(\lambda + o(1))^k}{k!} \left(1 - \frac{\lambda}{n} + o(n^{-1})\right)^n \\ &\xrightarrow{n \rightarrow \infty} \frac{\lambda^k}{k!} e^{-\lambda} \sim \text{Poi}(\lambda) \end{aligned}$$

Thus we have proved the Poisson approximation to the Binomial distribution.

5.(b) Exponential approximation to the Geometric distribution

We have $\forall t \geq 0$ that

$$\mathbb{P}(pX_p > t) = \mathbb{P}(\text{Geom}(p) > t/p) = \sum_{k=\lceil t/p \rceil}^{\infty} (1-p)^{k-1} p = (1-p)^{\lfloor t/p \rfloor}$$

Note that

$$(1-p)^{t/p} \xrightarrow{p \rightarrow 0} e^{-t}, \quad (1-p)^{t/p-1} \xrightarrow{p \rightarrow 0} e^{-t}$$

we have

$$\mathbb{P}(pX_p > t) = (1-p)^{\lfloor t/p \rfloor} \xrightarrow{p \rightarrow 0} e^{-t} = \mathbb{P}(\text{Exp}(1) > t)$$

Thus we have proved the Exponential approximation to the Geometric distribution.