STAT 430-1, Fall 2024
HW5

Tuorui Peng[1]

Contents

Notation: I use $\fallingdotseq$ for Fourier transform and $\risingdotseq$ for inverse Fourier transform. i.e. $f(x) \fallingdotseq \phi(t)$ and $\phi(t) \risingdotseq f(x)$. $\mathrm{Res}_f(x)$ for the residue of $f(x)$ at x. Dirac delta function (at zero) is denoted as $\delta(x)$ s.t. $\int_{-\varepsilon}^{\varepsilon} \delta(x)\,\mathrm{d}x = 1$, $\forall \varepsilon > 0$.

Exercise 1 Characteristic functions

(a) Note that function $t \mapsto 1/(1+t^2)$ has inverse Fourier transform as follows:

$$1/(1+t^2) \risingdotseq \frac{1}{2\pi}\int_{\mathbb{R}} \frac{1}{1+t^2} e^{-ixt}\,\mathrm{d}t = \begin{cases} \dfrac{2\pi i}{2\pi}\mathrm{Res}_{e^{-ixt}/(1+t^2)}(i), & x > 0 \\ \dfrac{2\pi i}{2\pi}\mathrm{Res}_{e^{-ixt}/(1+t^2)}(-i), & x \leq 0 \end{cases} = e^{-|x|}/2$$

which is the density function of a two-sided exponential distribution.

(b) Not a characteristic function by noticing that the characteristic function $t \mapsto e^{-t^4} := \phi_X(t)$ yields a 0 second moment:

$$\mathbb{E}\left[X^2\right] = \phi_X''(0) = \frac{\mathrm{d}^2}{\mathrm{d}t^2} e^{-t^4}\Big|_{t=0} = 0$$

which means that X is actually a degenerate r.v. with $X = 0$ a.s., which should actually correspond to a characteristic function of $t \mapsto 1$ instead. Thus the contradiction arises, so $t \mapsto e^{-t^4}$ is not a characteristic function.

Reference: I checked the proof in this Mathexchange post.

(c) Not a characteristic function by noticing that $\sin(t)\big|_{t=0} = 0 \neq 1$, thus does not satisfy the condition of Bochner's theorem.

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[1] TuoruiPeng2028@u.northwestern.edu

(d) Note that $t \mapsto \cos(t)$ has inverse Fourier transform as follows:

$$\cos(t) \eqqcolon \frac{1}{2\pi}\int_{\mathbb{R}} \cos(t)e^{-ixt}\,\mathrm{d}t = \frac{1}{2\pi}\int_{\mathbb{R}} \frac{e^{it}+e^{-it}}{2}e^{-ixt}\,\mathrm{d}t = \frac{1}{2}(\delta(x+1)+\delta(x-1)) \sim \mathrm{Unif}\{\pm 1\}$$

a uniform distribution on $\{\pm 1\}$.

(e) Note that $t \mapsto \dfrac{1+\cos(t)}{2}$ has inverse Fourier transform as follows:

$$\frac{1+\cos(t)}{2} \eqqcolon \frac{1}{2\pi}\int_{\mathbb{R}} \frac{1+\cos(t)}{2}e^{-ixt}\,\mathrm{d}t = \frac{1}{2\pi}\int_{\mathbb{R}} \frac{1+\dfrac{e^{it}+e^{-it}}{2}}{2}e^{-ixt}\,\mathrm{d}t = \frac{1}{4}\big(\delta(x-1)+\delta(x+1)+2\delta(x)\big)$$

which is a discrete r.v. with $P(X=1)=P(X=-1)=1/4$ and $P(X=0)=1/2$.

Exercise 2 Die roll plus uniform

(a) For a fair die X, we have

$$X \sim \mathrm{Unif}\{1,2,3,4,5,6\} \risingdotseq \frac{1}{6}\sum_{k=1}^{6} e^{ikt}$$

(b) For Y uniform on $[0,1]$, we have

$$Y \sim \mathrm{Unif}[0,1] \risingdotseq \frac{e^{it}-1}{it}$$

(c) For $Z = X+Y$, we have

$$X+Y \risingdotseq \frac{1}{6}\sum_{k=1}^{6} e^{ikt}\cdot\frac{e^{it}-1}{it} = \frac{1}{6}\sum_{k=1}^{6}\frac{e^{i(k+1)t}-e^{ikt}}{it} = \frac{e^{i7t}-e^{it}}{6it} \fallingdotseq \mathrm{Unif}[1,7]$$

(d) For $W = X-Y$, we have

$$X-Y \risingdotseq \frac{1}{6}\sum_{k=1}^{6} e^{ikt}\cdot\overline{\frac{e^{it}-1}{it}} = \frac{1}{6}\sum_{k=1}^{6} e^{ikt}\cdot\frac{e^{-it}-1}{-it} = \frac{1}{6}\sum_{k=1}^{6}\frac{e^{i(k-1)t}-e^{ikt}}{-it} = \frac{e^{i6t}-1}{6it} \fallingdotseq \mathrm{Unif}[0,6]$$

Exercise 3 Difference of two i.i.d. random variables

Note that for two i.i.d. r.v. X, Y with characteristic function $\phi(t)$, we have

$$X-Y \risingdotseq \phi(t)\cdot\overline{\phi(t)} = |\phi(t)|^2 \geq 0$$

however for uniform distribution $Z \sim \mathrm{Unif}[-1,1]$ we know that

$$Z \sim \mathrm{Unif}[-1,1] \risingdotseq \frac{\sin t}{t} \text{ is not non-negative}$$

which yields contradiction. Thus Z cannot be represented as the difference of two i.i.d. r.v.s.

Exercise 4 Trigonometric identities

We define the following:

$$\begin{aligned} X_n \sim & \mathrm{Unif}\{\pm\frac{1}{2^n}\} \risingdotseq \frac{1}{2}\big(e^{i/2^n t} + e^{-i/2^n t}\big) = \cos(\frac{t}{2^n}) \\ Y_n := & \sum_{i=1}^n X_i \sim \mathrm{Unif}\{\frac{k}{2^n}\}_{k=-2^n+1}^{2^n-1} \\ Z \sim & \mathrm{Unif}[-1,1] \risingdotseq \frac{\sin t}{t} \end{aligned}$$

- We note that $Z \overset{\mathrm{d}}{=} \mathrm{Unif}(-\frac{1}{2},\frac{1}{2}) + X_1$ (which is easy to verify following the same proof as in Exercise. 2). We thus have

$$\begin{aligned} \frac{\sin t}{t} & \fallingdotseq Z \overset{\mathrm{d}}{=} \mathrm{Unif}(-\frac{1}{2},\frac{1}{2}) + X_1 \risingdotseq \frac{\sin(t/2)}{t/2}\cdot\cos(t/2) \\ \Rightarrow \frac{\sin t}{t} & = \frac{\sin(t/2)}{t/2}\cdot\cos(t/2) \end{aligned}$$

- We further have $Y_n \xrightarrow{\mathrm{d}} Z$ by noticing the following:

$$\forall \xi \in [-1,1] : \mathbb{P}\left(Y_n \geq \xi\right) \underset{n\to\infty}{\to} \mathbb{P}\left(Z \geq \xi\right).$$

Thus we can conclude convergence in characteristic functions:

$$\begin{aligned} \prod_{i=1}^n \cos(\frac{t}{2^i}) & \fallingdotseq \sum_{i=1}^n X_i = Y_n \xrightarrow{\mathrm{d}} Z \risingdotseq \frac{\sin t}{t} \\ \Rightarrow \frac{\sin t}{t} & = \prod_{i=1}^\infty \cos(\frac{t}{2^i}), \quad \forall t \in \mathbb{R}. \end{aligned}$$

Exercise 5 Poisson approximation to the binomial; exponential approximation to the geometric

(This time we prove them "by hand")

5.(a) Poisson approximation to the Binomial distribution

We have $\forall k \in \mathbb{N}^+$:

$$\begin{aligned} \mathbb{P}\left(\mathrm{Bin}(n,p_n) = k\right) = & \binom{n}{k} p_n^k (1-p_n)^{n-k} \\ = & \frac{1}{k!}\frac{n!}{(n-k)!}\cdot\frac{1}{n^n}(np_n)^k(n-np_n)^{n-k} \\ = & \frac{1}{k!}(n^k + o(n^k))\cdot\frac{1}{n^n}(\lambda+o(1))^k(n-\lambda+o(1))^{n-k} \\ = & \frac{(\lambda+o(1))^k}{k!}\frac{(n^k+o(n^k))\cdot(n-\lambda+o(1))^{n-k}}{n^n} \\ = & \frac{(\lambda+o(1))^k}{k!}\big(1-\frac{\lambda}{n}+o(n^{-1})\big)^n \\ \xrightarrow{n\to\infty} & \frac{\lambda^k}{k!}e^{-\lambda} \sim \mathrm{Poi}(\lambda) \end{aligned}$$

Thus we have proved the Poisson approximation to the Binomial distribution.

5.(b) Exponential approximation to the Geometric distribution

We have $\forall t \geq 0$ that

$$\mathbb{P}\left(pX_p > t\right) = \mathbb{P}\left(\mathrm{Geom}(p) > t/p\right) = \sum_{k=\lceil t/p \rceil}^{\infty} (1-p)^{k-1} p = (1-p)^{\lfloor t/p \rfloor}$$

Note that

$$(1-p)^{t/p} \xrightarrow{p\to 0} e^{-t}, \quad (1-p)^{t/p-1} \xrightarrow{p\to 0} e^{-t}, \quad (1-p)^{t/p-1} \leq (1-p)^{\lfloor t/p \rfloor} \leq (1-p)^{t/p}$$

we have

$$\mathbb{P}\left(pX_p > t\right) = (1-p)^{\lfloor t/p \rfloor} \xrightarrow{p\to 0} e^{-t} = \mathbb{P}\left(\mathrm{Exp}(1) > t\right)$$

Thus we have proved the Exponential approximation to the Geometric distribution.