STAT 430-1, Fall 2024
HW3

Tuorui Peng[1]

Contents

Exercise 1 Convolution

Notation: I use $\fallingdotseq$ for Fourier transform and $\risingdotseq$ for inverse Fourier transform. i.e. $f(x) \fallingdotseq \phi(t)$ and $\phi(t) \risingdotseq f(x)$.

We utilize the relation between convolution and Fourier transform to solve the convolution problem. For the given distributions $f(x)$ we have their characteristic functions $\phi(t)$ as follows:

- **Binomial:**
$$f(x) = \binom{n}{x} p^x (1-p)^{n-x} \fallingdotseq (p + (1-p)e^{it})^n$$

- **Poisson:**
$$f(x) = \frac{\lambda^x}{x!} e^{-\lambda} \fallingdotseq e^{\lambda(e^{it}-1)}$$

- **Geometric:**
$$f(x) = p(1-p)^x \fallingdotseq \frac{p}{1-(1-p)e^{it}}$$

- **Normal:**
$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \fallingdotseq e^{it\mu - \frac{1}{2}\sigma^2 t^2}$$

- **Exponential:**
$$f(x) = \lambda e^{-\lambda x} \fallingdotseq \frac{\lambda}{\lambda - it}$$

[1] TuoruiPeng2028@u.northwestern.edu

Using the transformation that

$$f_{X+Y}(\xi) \risingdotseq \phi_X(t) \cdot \phi_Y(t)$$

we have the following for summation of two distributions:

- **Binomial:**

$$\begin{aligned} X \sim \mathrm{Binom}(n,p) &\risingdotseq (p+(1-p)e^{it})^n := \phi_X(t) \\ Y \sim \mathrm{Binom}(m,p) &\risingdotseq (p+(1-p)e^{it})^m := \phi_Y(t) \\ \phi_X(t)\cdot\phi_Y(t) = (p+(1-p)e^{it})^{n+m} &\fallingdotseq \mathrm{Binom}(n+m,p) \end{aligned}$$

- **Poisson:**

$$\begin{aligned} X \sim \mathrm{Poisson}(\lambda) &\risingdotseq e^{\lambda(e^{it}-1)} := \phi_X(t) \\ Y \sim \mathrm{Poisson}(\mu) &\risingdotseq e^{\mu(e^{it}-1)} := \phi_Y(t) \\ \phi_X(t)\cdot\phi_Y(t) = e^{(\lambda+\mu)(e^{it}-1)} &\fallingdotseq \mathrm{Poisson}(\lambda+\mu) \end{aligned}$$

- **Geometric:**

$$\begin{aligned} X \sim \mathrm{Geometric}(p) &\risingdotseq \frac{p}{1-(1-p)e^{it}} := \phi_X(t) \\ Y \sim \mathrm{Geometric}(p) &\risingdotseq \frac{p}{1-(1-p)e^{it}} := \phi_Y(t) \\ \phi_X(t)\cdot\phi_Y(t) = \left(\frac{p}{1-(1-p)e^{it}}\right)^2 &\fallingdotseq \mathrm{Negative\,Binom}(2,p) \end{aligned}$$

- **Normal:**

$$\begin{aligned} X \sim \mathrm{Normal}(m_1,\sigma_1^2) &\risingdotseq e^{im_1t-\frac{1}{2}\sigma_1^2t^2} := \phi_X(t) \\ Y \sim \mathrm{Normal}(m_2,\sigma_2^2) &\risingdotseq e^{im_2t-\frac{1}{2}\sigma_2^2t^2} := \phi_Y(t) \\ \phi_X(t)\cdot\phi_Y(t) = e^{i(m_1+m_2)t-\frac{1}{2}(\sigma_1^2+\sigma_2^2)t^2} &\fallingdotseq \mathrm{Normal}(m_1+m_2,\sigma_1^2+\sigma_2^2) \end{aligned}$$

- **Exponential:**

$$\begin{aligned} X \sim \mathrm{Exponential}(\lambda) &\risingdotseq \frac{\lambda}{\lambda-it} := \phi_X(t) \\ Y \sim \mathrm{Exponential}(\mu) &\risingdotseq \frac{\mu}{\mu-it} := \phi_Y(t) \end{aligned}$$

  the summation is not any known distribution, we just give the CDF here:

$$\begin{aligned} \mathbb{P}\left(X+Y\leq t\right) =& \int_0^\infty \mathrm{d}x \int_0^{t-x} \mathrm{d}y\,\lambda e^{-\lambda x}\mu e^{-\mu y} \\ =& 1-\frac{\mu e^{-\lambda t}-\lambda e^{-\mu t}}{\mu-\lambda}. \end{aligned}$$

  At $\mu\to\lambda$ limit we have

$$\mathbb{P}\left(X+Y\leq t\right) = 1-\lambda t e^{-\lambda t}.$$

  which is the CDF of Gamma distribution $\Gamma(2,\lambda)$.

Exercise 2 Cauchy-Schwarz Inequality

2.(a)

Consider the following:

$$\begin{aligned}0 \leq &\mathbb{E}\left[\left(|X|\,\mathbb{E}\left[Y^2\right]-|Y|\,\mathbb{E}\left[|XY|\right]\right)^2\right]\\ =&\mathbb{E}\left[|X|^2\,\mathbb{E}\left[Y^2\right]^2-2\,|XY|\,\mathbb{E}\left[Y^2\right]\mathbb{E}\left[|XY|\right]+|Y|^2\,\mathbb{E}\left[|XY|\right]^2\right]\\ =&\mathbb{E}\left[Y^2\right]\left\{\mathbb{E}\left[X^2\right]\mathbb{E}\left[Y^2\right]-\mathbb{E}\left[|XY|\right]^2\right\}\end{aligned}$$

thus we have

$$\mathbb{E}\left[|XY|\right] \leq \sqrt{\mathbb{E}\left[X^2\right]\mathbb{E}\left[Y^2\right]}.$$

2.(b)

WLOG we assume X and Y are both mean-zero. Using Jensen's inequality we have

$$\mathrm{Cov}(X,Y) =\mathbb{E}\left[XY\right] \leq \left|\mathbb{E}\left[XY\right]\right| \leq \mathbb{E}\left[|XY|\right] \leq \sqrt{\mathbb{E}\left[X^2\right]\mathbb{E}\left[Y^2\right]} = \sqrt{\mathrm{Var}(X)\mathrm{Var}(Y)}.$$

Exercise 3 Chernoff Bound

3.(a)

Using the Markov inequality we have

$$\mathbb{P}\left(X \geq m\right) =\mathbb{P}\left(e^{tX} \geq e^{tm}\right) \leq \frac{\mathbb{E}\left[e^{tX}\right]}{e^{tm}}.$$

3.(b)

Using Jensen's inequality we have

$$\begin{aligned}\mathbb{E}\left[e^{tX}\right] =&\prod_{i=1}^{n}\mathbb{E}\left[e^{tX_i}\right]\\ =&\prod_{i=1}^{n}\left(1-p_i+p_ie^t\right)\\ =&\exp\left[n\cdot\frac{1}{n}\sum_{i=1}^{n}\log\left(1-p_i+p_ie^t\right)\right]\\ \leq&\exp\left[n\cdot\log\frac{1}{n}\sum_{i=1}^{n}\left(1-p_i+p_ie^t\right)\right]\\ =&\exp\left[n\cdot\log\left(e^tp+1-p\right)\right]\\ =&\left(e^tp+1-p\right)^n.\end{aligned}$$

3.(c)

By Chernoff bound we have

$$\mathbb{P}\left(X \geq \mu+\lambda\right) \leq e^{-t(\mu+\lambda)}\left(e^t p+1-p\right)^n, \quad \forall t>0.$$

Denote $\xi := p+\frac{\lambda}{n}$, we optimize the bound by setting the minimizer $t_0=\log\frac{1-p}{p}\frac{\xi}{1-\xi}$, which gives

$$\begin{aligned}\mathbb{P}\left(X \geq \mu+\lambda\right) \leq & e^{-t_0(\mu+\lambda)}\left(e^{t_0} p+1-p\right)^n \\ = & \exp\left[-n(\xi\log\frac{\xi}{p}+(1-\xi)\log\frac{1-\xi}{1-p})\right] \\ = & \exp\left[-nH_p(\xi)\right]=\exp\left[-nH_p(p+\frac{\lambda}{n})\right]\end{aligned}$$

3.(d)

It suffices to prove that $H_p(p+\frac{\lambda}{n})\geq 2\frac{\lambda^2}{n}$. To illustrate this, we consider the function $f(x)=H_p(x)-2(x-p)^2=x\log\frac{x}{p}+(1-x)\log\frac{1-x}{1-p}-2(x-p)^2$, we have

$$\begin{aligned}\frac{\mathrm{d}f(x)}{\mathrm{d}x} = & \log\frac{x}{p}-\log\frac{1-x}{1-p}+4p-4x \\ \frac{\mathrm{d}^2 f(x)}{\mathrm{d}x^2} = & \frac{1}{x}+\frac{1}{1-x}-4\geq 0.\end{aligned}$$

Thus by further noticing that $f'(x)<0$ at $[0,p]$ and $f'(x)>0$ at $[p,1]$ and $f(p)=0$, we have $f(x)\geq 0$ for $x\in[0,1]$, which implies $H_p(x)\geq 2(x-p)^2$ for $x\in[0,1]$, thus we have

$$\mathbb{P}\left(X \geq \mu+\lambda\right) \leq \exp\left[-nH_p(p+\frac{\lambda}{n})\right]\leq\exp\left[-\frac{2\lambda^2}{n}\right]$$

Exercise 4 On the LLN

4.(a)

We have

$$\mathbb{E}\left[X_n\right]=n^2\cdot\frac{1}{n^2}=1.$$

4.(b)

Note that we have the following:

$$\forall n\geq N,\ X_n=0 \Rightarrow \lim_{n\to\infty}\frac{1}{n}\sum_{n=1}^{\infty}X_n=0.$$

which implies

$$\bigcap_{n=N}^{\infty}\{X_n=0\}\subset\left\{\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}X_i=0\right\}.$$

On the other hand, by the Borel-Cantelli lemma we have

$$\begin{aligned}\mathbb{P}\left(X_n = n^2\right) = \frac{1}{n^2} \Rightarrow \sum_{n=1}^{\infty} \mathbb{P}\left(X_n = n^2\right) = \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty \\ \Rightarrow & \mathbb{P}\left(X_n = n^2 \text{ i.o.}\right) = 0 \\ \Rightarrow & \mathbb{P}\left(\bigcap_{N=1}^{\infty} \bigcup_{n=N}^{\infty} \{X_n = n^2\}\right) = 0 \\ \Rightarrow & \mathbb{P}\left(\bigcup_{N=1}^{\infty} \bigcap_{n=N}^{\infty} \{X_n = 0\}\right) = 1\end{aligned}$$

which implies

$$\mathbb{P}\left(\lim_{n\to\infty} \frac{1}{n} \sum_{n=1}^{\infty} X_n = 0\right) \geq \mathbb{P}\left(\bigcap_{n=N}^{\infty} \{X_n = 0\}\right), \quad \forall N \geq 1.$$

then we have

$$\mathbb{P}\left(\lim_{n\to\infty} \frac{1}{n} \sum_{n=1}^{\infty} X_n = 0\right) \geq 1 \Rightarrow \frac{1}{n} \sum_{n=1}^{\infty} X_n \xrightarrow{\text{a.s.}} 0.$$

Exercise 5 SLLN assuming finite fourth moment

5.(a)

By Chebyshev's inequality we have

$$\mathbb{P}\left(\left|\frac{1}{n}\sum_{i=1}^{n}(X_i - m)\right| > \frac{1}{r}\right) \leq \frac{\mathbb{E}\left[\left(\sum_{i=1}^{n}(X_i - m)\right)^4\right]}{n^4/r^4}$$

So we need to consider the fourth moment of $\sum_{i=1}^{n}(X_i - m)$, as follows: (denote $Y_i = X_i - m$ the centered random variable) since $\mathbb{E}Y_i = 0$ we would only have the terms of $\mathbb{E}Y^4$ and $(\mathbb{E}Y^2)^2$ in the expansion of $\mathbb{E}(\sum_{i=1}^{n} Y_i)^4$, thus we have

$$\mathbb{E}(\sum_{i=1}^{n}(X_i - m))^4 = \mathbb{E}(\sum_{i=1}^{n} Y_i)^4 = n\mathbb{E}Y^4 + \binom{n}{2}(\mathbb{E}Y^2)^2 := Dn + Cn^2, \quad C, D \in \mathbb{R}^+.$$

thus we have

$$\mathbb{P}\left(\left|\frac{1}{n}\sum_{i=1}^{n}(X_i - m)\right| > \frac{1}{r}\right) \leq \frac{\mathbb{E}\left[\left(\sum_{i=1}^{n}(X_i - m)\right)^4\right]}{n^4/r^4} = \frac{Dn + Cn^2}{n^4/r^4} \leq \frac{\tilde{C}r^4}{n^2}$$

where $\tilde{C}$ is some universal constant.

5.(b)

Note that for any given $r \in \mathbb{N}^+$ we have

$$\sum_{n=1}^{\infty} \mathbb{P}\left(\left| \frac{1}{n} \sum_{i=1}^{n} (X_i - m) \right| > \frac{1}{r} \right) \leq \sum_{n=1}^{\infty} \frac{\tilde{C} r^4}{n^2} < \infty.$$

thus by Borel-Cantelli lemma we have

$$\mathbb{P}\left(\limsup_{n\to\infty} \left| \frac{1}{n} \sum_{i=1}^{n} (X_i - m) \right| > \frac{1}{r} \right) = 0 \Rightarrow \mathbb{P}\left(\limsup_{n\to\infty} \left| \frac{1}{n} \sum_{i=1}^{n} (X_i - m) \right| = 0 \right) = 1 \Rightarrow \frac{1}{n} \sum_{i=1}^{n} (X_i - m) \xrightarrow{\text{a.s.}} 0.$$