STAT 430-1, Fall 2024
HW2

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Exercise 1 Atoms

Denote by K_n the set of atoms with mass greater than $1/n$, then we have

$$\mu(E) \geq \mu(K_n) \geq \frac{|K_n|}{n} \Rightarrow |K_n| \leq n\mu(E)$$

which means that K_n is finite for any $n \in \mathbb{N}^+$. And we also note that $\{K_n\}_{n\in\mathbb{N}^+}$ is an increasing sequence of sets. By definition of stones, we have the set of all stones being $\mathcal{S} = \bigcup_{n=1}^{\infty} K_n = \biguplus_{n=1}^{\infty} K_n \backslash K_{n-1}$. i.e. $\{K_n\backslash K_{n-1}\}_{n\in\mathbb{N}^+}$ is a partition of $\mathcal{S}$.

If S is uncountable, then there exists $n_0 \in \mathbb{N}^+$ such that $K_{n_0}\backslash K_{n_0-1}$ is infinite (otherwise $\mathcal{S} = \biguplus_{n=1}^{\infty} K_n\backslash K_{n-1}$ would be countable), and thus $K_{n_0} = \cup_{k=1}^{n_0} K_{n_0}\backslash K_{n_0-1}$ is also infinite, which contradicts the fact that $|K_{n_0}| \leq n\mu(E) < \infty$ is actually finite.

Thus S is countable.

Exercise 2 Fatou's Lemma

Note that $\{\inf_{m\geq n} f_m\}_{n\in\mathbb{N}}$ is a monotone increasing sequence of functions. Using monotone convergence theorem, we have

$$\begin{aligned}
\mu\left(\liminf_{n\to\infty} f_n\right) =& \mu\left(\lim_{n\to\infty} \inf_{m\geq n} f_m\right) \\
=& \lim_{n\to\infty} \mu\left(\inf_{m\geq n} f_m\right) \\
\leq& \lim_{n\to\infty} \inf_{m\geq n} \mu(f_m) \\
=& \liminf_{n\to\infty} \mu(f_n)
\end{aligned}$$

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Exercise 3 Dominated convergence theorem

From above, we already have $\mu(f) \le \liminf_{n\to\infty} \mu(f_n)$. Now it suffice to show the other direction.

Note that we have dominance condition $-g \le f_n \le g$, $\forall n \in \mathbb{N}^+$, thus we can consider the following sequence of functions: $\{g - f_n\}_{n\in\mathbb{N}^+}$, which has

$$0 \le g - f_n \le 2g$$

thus is still integrable. By Fatou's lemma, we have

$$\mu(g - f) \le \liminf_{n\to\infty} \mu(g - f_n) \Rightarrow \mu(f) \ge \limsup_{n\to\infty} \mu(f_n)$$

which concludes the proof.

Exercise 4 On interchanging limits and integration

- **Left hand side**: we have

$$\lim_{\lambda\to 0} \int_0^\infty f_\lambda(x)\,\mathrm{d}x = \lim_{\lambda\to 0} \int_0^\infty \lambda e^{-\lambda x}\,\mathrm{d}x = \lim_{\lambda\to 0} 1 = 1$$

- **Right hand side**: we have

$$\int_0^\infty \lim_{\lambda\to 0} f_\lambda(x)\,\mathrm{d}x = \int_0^\infty \lim_{\lambda\to 0} \lambda e^{-\lambda x}\,\mathrm{d}x = \int_0^\infty 0\,\mathrm{d}x = 0$$

Thus the two sides are not equal, and the limit cannot be interchanged with the integral.

To explain why this happens, we illustrate the following that there is no dominating function for the sequence of functions $\{f_\lambda(x)\}_{\lambda>0}$: for any $x \in \mathbb{R}^+$, and any given $\varepsilon \in (0,1)$, we note that function $\lambda \mapsto \lambda e^{-\lambda x}$ reaches maximum of $\dfrac{1}{ex}$ at $\lambda = \dfrac{1}{x}$, thus consider the function sequence:

$$\mathcal{F} := \{f_{\tilde{\lambda}},\, \tilde{\lambda} = \frac{1}{n}\}_{n\in\mathbb{N}^+}$$

which shares the same limit as the orginal sequence. For this sequence we notice that:

$$\text{for } x \in [n-1, n),\ \text{ we have } f_{1/n}(x) > f_{1/n}(n) = \frac{1}{ne},$$

thus to bound all $f_{\tilde{\lambda}} \in \mathcal{F}$, we must have some g satisfying

$$g(x) > \frac{1}{ne}, \quad x \in [n-1, n), \quad n \in \mathbb{N}^+$$

thus

$$\int_0^\infty g(x)\,\mathrm{d}x = \sum_{n=1}^\infty \int_{n-1}^n g(x)\,\mathrm{d}x \ge \sum_{n=1}^\infty \frac{1}{ne} = \infty$$

which is not integrable on $\mathbb{R}^+$ by noticing that $\sum_{n=1}^\infty \dfrac{1}{n} = \infty$. Thus we cannot find a dominating function for the sequence $\{f_\lambda(x)\}_{\lambda>0}$, thus the limit cannot be interchanged with the integral.

Exercise 5 On interchanging integrals

- We have

$$I = \int_0^\infty \mathrm{d}x \int_0^\infty \mathrm{d}y f(x,y) = \int_0^\infty \mathrm{d}x \begin{cases} x-1, & x \in [0,1] \\ 0, & x \in (1,\infty) \end{cases}$$
$$= \int_0^1 \mathrm{d}x (x-1) = -\frac{1}{2}$$

- We have

$$J = \int_0^\infty \mathrm{d}y \int_0^\infty \mathrm{d}x f(x,y) = \int_0^\infty \mathrm{d}y \begin{cases} 1-y, & y \in [0,1] \\ 0, & y \in (1,\infty) \end{cases}$$
$$= \int_0^1 \mathrm{d}y (1-y) = \frac{1}{2}$$

i.e. we got $I \neq J$, and the integrals cannot be interchanged. We can explain this by noticing that the condition of Fubini's theorem is not satisfied: the function $f(x,y)$ is not integrable on $\mathbb{R}^2$:

$$\int_0^\infty \mathrm{d}x \int_0^\infty \mathrm{d}y \, |f(x,y)| = \infty.$$