STAT 430-1, Fall 2024

HW6

Tuorui Peng[1]

Contents

Notation: I use $\fallingdotseq$ for Fourier transform and $\risingdotseq$ for inverse Fourier transform. i.e. $f(x) \fallingdotseq \phi(t)$ and $\phi(t) \risingdotseq f(x)$. $\mathrm{Res}_f(x)$ for the residue of $f(x)$ at x. Dirac delta function (at zero) is denoted as $\delta(x)$ s.t. $\int_{-\varepsilon}^{\varepsilon} \delta(x)\,\mathrm{d}x = 1$, $\forall \varepsilon > 0$.

Exercise 1 Self-normalized sum

- **Denominator:** By SLLN and continuous mapping theorem, we have

$$\sqrt{\frac{\sum_{i=1}^n X_i^2}{n}} \xrightarrow{\text{a.s.}} \sqrt{\mathbb{E}X_1^2} = \sqrt{1/3}$$

- **Numerator:** By CLT, we have

$$\frac{\sum_{i=1}^n X_i}{\sqrt{n} \cdot var(X_1)} = \frac{\sum_{i=1}^n X_i}{\sqrt{n}\sqrt{1/3}} \xrightarrow{\mathcal{D}} N(0,1)$$

Combining the two, and by Slutsky's theorem, we have

$$\frac{\sum_{i=1}^n X_i}{\sqrt{\sum_i = 1^n X_i^2}} = \sqrt{\frac{1}{3}} \frac{\sum_{i=1}^n X_i/\sqrt{n}\sqrt{1/3}}{\sqrt{\sum_{i=1}^n X_i^2/n}} \xrightarrow{\text{d}} N(0,1)$$

Exercise 2 Geometric mean

2.(a)

It suffices to show that $\log G_n = \dfrac{1}{n}\sum_{i=1}^n \log X_i \xrightarrow{\text{a.s.}} -1$ by continuous mapping theorem.

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[1] TuoruiPeng2028@u.northwestern.edu

To show that we notice that with $X_i \overset{i.i.d.}{\sim} \text{Unif}(0,1)$, we have $-\log X_i \overset{i.i.d.}{\sim} \text{Exp}(1)$, thus we have by SLLN

$$\frac{1}{n}\sum_{i=1}^{n} \log X_i \xrightarrow{\text{a.s.}} -\mathbb{E}\left[\text{Exp}(1)\right] = -1$$

thus finishes the proof that $G_n \xrightarrow{\text{a.s.}} e^{-1}$.

2.(b)

Still use the log transformation, we have

$$\log(eG_n)^{\sqrt{n}} = \sqrt{n}\left(1 + \frac{1}{n}\sum_{i=1}^{n}\log X_i\right) \xrightarrow[(1)]{\text{d}} N(0,1)$$

in which (1) is by CLT by noticing that $\mathbb{E}\left[\log X_1\right] = -1$ and $\text{Var}\left[\log X_1\right] = 1$. Thus by continuous mapping theorem applied to $\xi \mapsto \exp \xi$, we have

$$(eG_n)^{\sqrt{n}} \xrightarrow{\text{d}} \exp N(0,1) \sim \text{LogNormal}(0,1)$$

in which LogNormal$(0,1)$ is the log-normal distribution, with density function

$$f_{\text{LogNormal}(0,1)}(x) = \frac{1}{x\sqrt{2\pi}} \exp\left(-\frac{(\log x)^2}{2}\right)$$

Exercise 3 Weak convergence for finitely supported distributions

"$\Rightarrow$" Given that $\mathbb{P}\left(X_n = k\right) \xrightarrow{n\to\infty} \mathbb{P}\left(X = k\right), \forall k \in S$, we have also that $\forall k \in S$:

$$\mathbb{P}\left(X_n \le k\right) = \sum_{i=1}^{k} \mathbb{P}\left(X_n = i\right) \xrightarrow{n\to\infty} \sum_{i=1}^{k} \mathbb{P}\left(X = i\right) = \mathbb{P}\left(X \le k\right)$$

thus we have $X_n \xrightarrow{\text{d}} X$.

"$\Leftarrow$" Given that $X_n \xrightarrow{\text{d}} X$, we have that $\forall k \in S$:

$$\mathbb{P}\left(X_n = k\right) = \mathbb{P}\left(X_n \le k\right) - \mathbb{P}\left(X_n \le k-1\right) \xrightarrow{n\to\infty} \mathbb{P}\left(X \le k\right) - \mathbb{P}\left(X \le k-1\right) = \mathbb{P}\left(X = k\right)$$

(with a trivial fix for $k = 0$ that $\mathbb{P}\left(X_n = 0\right) \to \mathbb{P}\left(X = 0\right)$ automatically holds). Thus we have $\mathbb{P}\left(X_n = k\right) \xrightarrow{n\to\infty} \mathbb{P}\left(X = k\right), \forall k \in S$.

Exercise 4 Coupling Poisson distributions

Note that for independent Poisson random variables W, Z with parameters ν_1, ν_2, we have $W + Z \sim \text{Poisson}(\nu_1 + \nu_2)$. The proof is as follows:

Proof. Using characteristic function, we have

$$W + Z \risingdotseq \exp\left(\nu_1(e^{it} - 1)\right)\exp\left(\nu_2(e^{it} - 1)\right) = \exp\left((\nu_1 + \nu_2)(e^{it} - 1)\right) \fallingdotseq \text{Poisson}(\nu_1 + \nu_2).$$

$\square$

Thus for the Poisson distributed random variables X, Y with parameter λ, μ, respectively, we can construct a $\delta \sim \text{Poisson}(\mu - \lambda) \geq 0$ and by the above property we have a coupling of X, Y that

$$Y' = X' + \delta \sim \text{Poisson}(\lambda + \mu), \qquad X' \sim \text{Poisson}(\lambda), \qquad \delta \sim \text{Poisson}(\mu - \lambda) \geq 0.$$

and thus proves that Y first-order stochastically dominates X.

Exercise 5 Coupling Exponential distributions

Using log transform, we have that

$$\text{Exp}(\lambda) \sim -\frac{1}{\lambda} \log \text{Unif}(0, 1)$$

thus we can construct a coupling of exponential random variables U, V with parameters λ, μ, respectively, that

$$V' = -\frac{\mu}{1} \log e^{-\lambda U'} = \frac{\lambda}{\mu} U' \sim \text{Exp}(\mu), \quad U' \sim \text{Exp}(\lambda)$$

in which we notice that $0 < \lambda < \mu \Leftrightarrow V' < U'$, thus we have V first-order stochastically dominates U.