

STAT 430-2 2025 Winter

HW4

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Exercise 1 Exercise 6.1.14

1.(a)

We have

$$\begin{aligned}
 \mathbb{P}(X_{n+1} = 1 | X_1^n) &= \mathbb{E} [\mathbf{1}_{X_{n+1}=1} | X_1^n] \\
 &= \mathbb{E}_\theta [\mathbb{E} [\mathbf{1}_{X_{n+1}=1} | X_1^n, \theta]] \\
 &= \mathbb{E}_\theta [1 - \theta] \\
 &= 1/2
 \end{aligned}$$

1.(b)

We have

$$\begin{aligned}
 \mathbb{P}(S_{n+1} = s | S_1^n) &= \mathbb{P}(S_{n+1} = s | X_1^n) \\
 &= \mathbb{P}\left(X_{n+1} = s - \sum_{i=1}^n X_i | X_1^n\right) \\
 &= \mathbb{P}(X_{n+1} = s - S_n | S_n)
 \end{aligned}$$

from the previous part. Thus we have S_n being Markov.

Exercise 2 Exercise 6.1.18

2.(a)

By property of sets we have $\Gamma_n \rightarrow \Gamma$ thus $\mathbf{1}_{\Gamma_n} \xrightarrow{\text{a.s.}} \mathbf{1}_\Gamma$, then by Lévy's upward theorem we have:

$$\begin{aligned}
 \mathbb{P}(\Gamma_n | X_n) &= \mathbb{E} [\mathbf{1}_{\Gamma_n} | X_n] \\
 &= \mathbb{E} [\mathbf{1}_{\Gamma_n} | \mathcal{F}_n] \\
 &\xrightarrow[L_1]{\text{a.s.}} \mathbb{E} [\mathbf{1}_\Gamma | \mathcal{F}_\infty] \\
 &= \mathbf{1}_\Gamma
 \end{aligned}$$

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2.(b)

Denote $K := \{\omega : X_n(\omega) \in A_n \text{ i.o.}\}$. Then we have that $\forall N > 0, \exists n > N$ s.t. $\mathbb{P}(\Gamma_n \cap K | X_n) \geq \eta > 0$. On the other hand we have

$$\eta < \mathbb{P}(\Gamma_n \cap K | X_n) \xrightarrow{\text{a.s.}} \mathbb{P}(\Gamma \cap K | X_\infty) = \mathbf{1}_{\Gamma \cap K} = 1 = \mathbb{P}(\Gamma \cap K)$$

which gives $\mathbb{P}(K \setminus \Gamma) = 0$. In the above we applied Lévy's upward theorem to $\Gamma_n \cap K | X_n$.

2.(c)

Use $A_n \equiv A$ and $B_n \equiv B$ and we have using the precedence:

$$\begin{aligned} 1 &\leq \mathbb{P}(\{X_n \in A \text{ finitely often}\} \cup (\{X_n \in A \text{ i.o.}\} \setminus \Gamma) \cup \Gamma) \\ &\leq \mathbb{P}(\{X_n \in A \text{ finitely often}\} \cup \Gamma) + \mathbb{P}(\{X_n \in A \text{ i.o.}\} \setminus \Gamma) \\ &= \mathbb{P}(\{X_n \in A \text{ finitely often}\} \cup \Gamma) + 0 \\ &= \mathbb{P}(\{X_n \in A \text{ finitely often}\}) + \mathbb{P}(\Gamma) \end{aligned}$$

where $\Gamma = \{X_n \in B \text{ i.o.}\}$ so thus we have proved the claim.

Exercise 3 Exercise 6.1.19

We prove the result for symmetric SRW directly. Denote $\tau = \inf\{k : \omega_k \geq b\}$ and

$$h_k(\omega) = \sum_{i=-\infty}^{+\infty} \mathbf{1}_{\omega_{n-k}=b+i}, \quad k \in [n]$$

and by Strong Markov Property (SMP) we have

$$\begin{aligned} \mathbb{P}\left(\max_{k \leq n} \omega_k \geq b\right) &= \mathbb{E}[\mathbf{1}_{\tau \leq n}] \\ &= \mathbb{E}\left[\mathbf{1}_{\tau \leq n} \sum_{i=-\infty}^{+\infty} \mathbf{1}_{\omega_n=i}\right] \\ &= \mathbb{E}\left[\mathbf{1}_{\tau \leq n} \mathbb{E}\left[\sum_{i=-\infty}^{+\infty} \mathbf{1}_{\theta^{n \wedge \tau} \omega_{n-n \wedge \tau}=b+i} | \mathcal{F}_{n \wedge \tau}\right]\right] \\ &= \mathbb{E}\left[\mathbf{1}_{\tau \leq n} \mathbb{E}\left[h_{n \wedge \tau}(\theta^{n \wedge \tau} \omega) | \mathcal{F}_{n \wedge \tau}\right]\right] \\ &\stackrel{\text{SMP}}{=} \mathbb{E}\left[\mathbf{1}_{\tau \leq n} \mathbb{E}_{X_{n \wedge \tau}}[h_{n \wedge \tau}(\omega)]\right] \\ &= \mathbb{E}\left[\mathbf{1}_{\tau \leq n} \mathbb{E}_{X_{n \wedge \tau}}\left[\mathbf{1}_{\omega_{n-n \wedge \tau}=0} + 2 \sum_{i=1}^{\infty} \mathbf{1}_{\omega_{n-n \wedge \tau}=i}\right]\right] \\ &= \mathbb{E}[\mathbf{1}_{\omega_n=b}] + 2 \mathbb{E}\left[\sum_{i=1}^{\infty} \mathbf{1}_{\omega_n=b+i}\right] \\ &= \mathbb{P}(\omega_n = b) + 2 \mathbb{P}(\omega_n > b) \end{aligned}$$

Exercise 4 Exercise 6.2.2

4.(a)

We use $h_r = \mathbf{1}_{\omega_{n-r} \in B}$ and apply for stopping time $T_{y,r}$ the SMP to obtain that

$$\begin{aligned}
 \mathbb{P}_x(X_n \in B, T_{y,r} \leq n) &= \mathbb{E}[\mathbf{1}_{X_n \in B} \mathbf{1}_{T_{y,r} \leq n}] \\
 &= \mathbb{E}[\mathbf{1}_{T_{y,r} \leq n} \mathbb{E}[h_{T_{y,r}}(\theta^{T_{y,r}} \omega) | \mathcal{F}_{T_{y,r}}]] \\
 &\stackrel{\text{SMP}}{=} \mathbb{E}[\mathbf{1}_{T_{y,r} \leq n} \mathbb{E}_y[h_{T_{y,r}}(\omega)]] \\
 &= \sum_{k=0}^{n-r} \mathbb{P}_x(T_{y,r} = n - k) \mathbb{P}_y(X_{n-k} \in B)
 \end{aligned}$$

4.(b)

Making relabeling $k \mapsto n - k$ and $B = \{y\}$ and we have

$$\mathbb{P}_x(X_n = y) = \sum_{k=r}^n \mathbb{P}_x(T_{y,r} = k) \mathbb{P}_y(X_{n-k} = y)$$

4.(c)

We have

$$\begin{aligned}
 \text{R.H.S.} &= \sum_{n=r}^{l+r} \mathbb{P}_y(X_n = y) \\
 &= \sum_{n=r}^{l+r} \sum_{k=r}^n \mathbb{P}_y(T_{y,r} = k) \mathbb{P}_y(X_{n-k} = y) \\
 &= \sum_{k=r}^{l+r} \mathbb{P}_y(T_{y,r} = k) \sum_{n=k}^{l+r} \mathbb{P}_y(X_{n-k} = y) \\
 &\leq \sum_{k=r}^{l+r} \mathbb{P}_y(T_{y,r} = k) \sum_{n=k}^{l+k} \mathbb{P}_y(X_{n-k} = y) \\
 &= \sum_{k=r}^{l+r} \mathbb{P}_y(T_{y,r} = k) \sum_{j=0}^l \mathbb{P}_y(X_j = y) \\
 &\leq \sum_{j=0}^l \mathbb{P}_y(X_j = y)
 \end{aligned}$$

Exercise 5 Exercise 6.2.5

5.(a)

Since $\mathbb{P}_{x \notin C}(\tau_C < \infty) > 0$, we know that $\exists N > 0$ and some ε s.t.

$$\mathbb{P}_{x \notin C}(\tau_C < N) > \varepsilon$$

and thus we have

$$\mathbb{P}_{x \notin C}(\tau_C \geq N) \leq 1 - \varepsilon$$

Consider applying the SMP to $h := \mathbf{1}_{\tau > N}$ we have

$$\begin{aligned} \mathbb{E}[\mathbf{1}_{\tau > (k+1)N}] &= \mathbb{E}[\mathbf{1}_{\tau > kN} \mathbb{E}[h(\theta^{kN}\omega) | \mathcal{F}_{kN}]] \\ &= \mathbb{E}[\mathbf{1}_{\tau > kN} \mathbb{E}_{x \notin C}[h(\omega)]] \\ &= \mathbb{E}[\mathbf{1}_{\tau > kN} \mathbb{P}_{x \notin C}(\tau_C \geq N)] \\ &\leq \mathbb{E}[\mathbf{1}_{\tau > kN}(1 - \varepsilon)] \\ &\dots \leq (1 - \varepsilon)^k \mathbb{E}_y[\mathbf{1}_{\tau > N}] \\ &\leq (1 - \varepsilon)^k \end{aligned}$$

5.(b)

By Borel-Cantelli lemma we have $\mathbb{P}(\tau_C < \infty) = 1$ since

$$\sum_{k=1}^{\infty} \mathbb{P}_{x \notin C}(\tau_C \geq kN) < \infty$$

Then we have for any $x = X_0 \notin C$ that:

$$\begin{aligned} g(x) &= \mathbb{P}_{X_0=x}(\tau_A < \tau_B) \\ &= \mathbb{P}_{X_0}(X_{\tau_C} \in A) \\ &= \sum_{y \in \mathbb{S}} \mathbb{P}_{X_0}(X_1 = y) \mathbb{P}_{X_1=y}(X_{\tau_C} \in A) \\ &= \sum_{y \in \mathbb{S}} p(x, y) g(y) \end{aligned}$$

thus $g(\cdot)$ is harmonic on $\mathbb{S} \setminus C$.

5.(c)

Note that we have $X_{n \wedge \tau_C - 1} \notin C$

By the harmonic property we have that

$$\begin{aligned} \mathbb{E}[g(X_{n \wedge \tau_C + 1}) | \mathcal{F}_n] &= \mathbb{E}[g(X_{n \wedge \tau_C + 1})(\mathbf{1}_{\tau_C \leq n} + \mathbf{1}_{\tau_C > n}) | \mathcal{F}_n] \\ &= g(X_{n \wedge \tau_C}) \mathbf{1}_{\tau_C \leq n} + \mathbb{E}[g(X_{n \wedge \tau_C + 1}) \mathbf{1}_{\tau_C > n} | \mathcal{F}_n] \\ &= g(X_{n \wedge \tau_C}) \mathbf{1}_{\tau_C \leq n} + \mathbb{E}[g(\theta^1 X_{n \wedge \tau_C}) \mathbf{1}_{\tau_C > n} | \mathcal{F}_n] \\ &\stackrel{\text{SMP}}{=} g(X_{n \wedge \tau_C}) \mathbf{1}_{\tau_C \leq n} + \mathbb{E}_{X_{n \wedge \tau_C} = x \notin C}[g(X_{n \wedge \tau_C + 1})] \mathbf{1}_{\tau_C > n} \\ &= g(X_{n \wedge \tau_C}) \mathbf{1}_{\tau_C \leq n} + g(X_{n \wedge \tau_C}) \mathbf{1}_{\tau_C > n} \\ &= g(X_{n \wedge \tau_C}) \end{aligned}$$

thus we have $g(X_{n \wedge \tau_C})$ is a M.G.

5.(d)

We have for such M.G. $g(X_{n \wedge \tau_C})$ with $X_0 = x$ that:

$$\begin{aligned} g(x) &= \mathbb{E} [g(X_{0 \wedge \tau_C})] \\ &= \mathbb{E} [g(X_{n \wedge \tau_C})] \\ &= \mathbb{E} [g(X_{\tau_C})] \\ &= \mathbb{E} [\mathbf{1}_{X_{\tau_C} \in A}] \\ &= \mathbb{P}_x (\tau_A < \tau_B) \end{aligned}$$

combine with previous results that $\mathbb{P}_x (\tau_A < \tau_B)$ satisfies the condition of bounded harmonic on $x \notin C$ we have that it's the only one that satisfies the condition.

5.(e)

Similar to (c) we prove the following:

$$\begin{aligned} \mathbb{E} [M_{n+1} | \mathcal{F}_n] &= \mathbb{E} [M_{n+1} \mathbf{1}_{\tau_C \leq n} | \mathcal{F}_n] + \mathbb{E} [M_{n+1} \mathbf{1}_{\tau_C > n} | \mathcal{F}_n] \\ &= M_n \mathbf{1}_{\tau_C \leq n} + \mathbb{E} [M_{n+1} \mathbf{1}_{\tau_C > n} | \mathcal{F}_n] \\ &= M_n \mathbf{1}_{\tau_C \leq n} + (n + 1 + \mathbb{E} [f(X_{n+1}) | \mathcal{F}_n]) \mathbf{1}_{\tau_C > n} \\ &= M_n \mathbf{1}_{\tau_C \leq n} + (n + 1 + \mathbb{E}_{X_n = x \notin C} [f(X_{n+1})]) \mathbf{1}_{\tau_C > n} \\ &= M_n \mathbf{1}_{\tau_C \leq n} + (n + 1 + f(X_n) - 1) \mathbf{1}_{\tau_C > n} \\ &= M_n \end{aligned}$$

thus M_n is a M.G. On the other hand we notice that

$$\begin{aligned} \mathbb{E} [M_n] &= \mathbb{E} [n \wedge \tau_C + f(X_{n \wedge \tau_C})] \\ &\leq \mathbb{E} [\tau_C] + \sum_{x \in \mathbb{S}} f(x) \mathbb{P} (X_{n \wedge \tau_C} = x) \\ &\leq \mathbb{E} [\tau_C] + \sum_{x \in \mathbb{S} \setminus C} f(x) + \sum_{x \in C} 0 < \infty \end{aligned}$$

then by DCT we have that

$$\begin{aligned} \mathbb{E}_x [\tau_C] &= \mathbb{E} \left[\lim_{n \rightarrow \infty} n \wedge \tau_C + f(X_{n \wedge \tau_C}) \right] \\ &= \lim_{n \rightarrow \infty} \mathbb{E} [n \wedge \tau_C + f(X_{n \wedge \tau_C})] \\ &= \mathbb{E}_x [0 + f(X_{0 \wedge \tau_C})] \\ &= f(x) \end{aligned}$$

Exercise 6 Exercise 6.2.8

6.(a)

We have that

$$\begin{aligned}
 \rho_{xz} &= \mathbb{E}_x [\mathbf{1}_{T_z < \infty}] \\
 &\geq \mathbb{E}_x [\mathbf{1}_{T_z < \infty} \mathbf{1}_{T_y < T_z}] \\
 &= \mathbb{E}_x [\mathbf{1}_{T_y < T_z} \mathbb{E} [\theta^{T_y} \mathbf{1}_{T_z < \infty} | \mathcal{F}_{T_y}]] \\
 &\stackrel{\text{SMP}}{=} \mathbb{E}_x [\mathbf{1}_{T_y < T_z} \mathbb{E}_y [\mathbf{1}_{T_z < \infty}]] \\
 &= \rho_{xy} \rho_{yz} > 0
 \end{aligned}$$

6.(b)

We have

- $x \leftrightarrow x$ by convention;
- $x \leftrightarrow y$ implies $y \leftrightarrow x$ by symmetry of the condition $\rho_{xy} > 0, \rho_{yx} > 0$;
- $x \leftrightarrow y$ and $y \leftrightarrow z$ implies $x \leftrightarrow z$ by the previous part.

6.(c)

- If C_1 leads to C_2 and C_2 leads to C_3 , then there exists $x, y \in C_2$ s.t. $x \leftrightarrow y$ and thus for any $z \in C_1$ and $w \in C_3$ we have $z \rightarrow x \rightarrow y \rightarrow w$, thus $z \leftrightarrow w$, i.e. C_1 leads to C_3 .
- If C_1 leads to C_2 and C_2 leads to C_1 , then for any $x \in C_1$ and $y \in C_2$ we have $x \rightarrow y$ and $y \rightarrow x$, which turns out to be $x \leftrightarrow y$, which is a contradiction, so if C_1 leads to C_2 , then C_2 cannot lead to C_1 .

Exercise 7 Exercise 6.2.11

7.(a)

We have $\forall k$ and any bounded function g that

$$\begin{aligned}
 \mathbb{E}_y [g(r_k, X_{R_{k-1}}, \dots, X_{R_k-1}) | \mathcal{F}_{R_{k-1}}] &= \mathbb{E}_y [\theta^{R_{k-1}} g(\omega) | \mathcal{F}_{R_{k-1}}] \\
 &\stackrel{\text{SMP}}{=} \mathbb{E}_y [g(\omega)] \\
 &= \mathbb{E}_y [g(r_0, X_0, \dots, X_{R_0-1})]
 \end{aligned}$$

and notice that $\mathcal{F}_{k-1} \supseteq \sigma(r_1^k, X_1^{R_k-1})$ we have that $(r_k, X_{R_{k-1}}, \dots, X_{R_k-1})$ are independent and identically distributed as $(r_0, X_0, \dots, X_{R_0-1})$.

7.(b)

Similarly we have

$$\begin{aligned}\mathbb{E}_\nu [g(r_k, X_{R_{k-1}}, \dots, X_{R_k-1}) | \mathcal{F}_{R_{k-1}}] &= \mathbb{E}_\nu [\mathbb{E} [\theta^{R_{k-1}} g(\omega) | \mathcal{F}_{R_{k-1}}] | \mathcal{F}_{R_1}] \\ &= \mathbb{E}_\nu [\mathbb{E}_y [\theta^{R_1} g(\omega) | \mathcal{F}_{R_1}]] \\ &= \mathbb{E}_y [g(r_2, X_{R_1}, \dots, X_{R_2-1})]\end{aligned}$$

thus proved that $(r_k, X_{R_{k-1}}, \dots, X_{R_k-1})$ are independent and identically distributed as $(r_1, X_{R_1}, \dots, X_{R_2-1})$, $k \geq 2$.

Exercise 8 Exercise 6.2.18

Denote the initial distribution as $X_0 \sim \nu$.

8.(a)

Take $\nu = \delta_0$, then we have

$$\begin{aligned}0 &= \mathbb{E}_\nu [X_0] = 0 \\ &= \mathbb{E}_\nu [X_1] \\ &= \sum_{i=0}^N \mathbb{P}(X_0 = 0) p(0, i) \\ &\Rightarrow p(0, i) = 0, \quad i \neq 0\end{aligned}$$

the above argument is similar for $\nu = \delta_N$. Together we have $p(0, 0) = p(N, N) = 1$.

8.(b)

We have for any $x \notin \{0, N\}$:

$$\begin{aligned}1 - \rho_{xx} &= \mathbb{P}_x (\tau_x = \infty) \\ &\geq \mathbb{E}_x [\mathbf{1}_{\tau_x = \infty} \mathbf{1}_{T_{\{0, N\}} < \infty}] \\ &= \mathbb{E}_x [\mathbf{1}_{\tau_{\{0, N\}} < \infty} \mathbb{E} [\theta^{\tau_{\{0, N\}}} \mathbf{1}_{T_x = \infty} | \mathcal{F}_{\tau_{\{0, N\}}}]] \\ &\stackrel{\text{SMP}}{=} \mathbb{E}_x [\mathbf{1}_{\tau_{\{0, N\}} < \infty} \mathbb{E}_{X_{\tau_{\{0, N\}}}} [\mathbf{1}_{\tau_x = \infty}]] \\ &= \mathbb{E}_x [\mathbf{1}_{\tau_{\{0, N\}} < \infty}] > 0\end{aligned}$$

thus proved that any $x \notin \{0, N\}$ is transient.

Then consider for any x and choose $\nu = \delta_x$. We have

$$\mathbb{E}_x [X_{\tau_{\{0, N\}}}] = \mathbb{E}_x [X_0] = x$$

on the other hand we know that

$$\mathbb{E}_x \left[X_{\tau_{\{0,N\}}} \right] = \mathbb{P}_x (\tau_0 < \tau_N) \times 0 + \mathbb{P}_x (\tau_0 > \tau_N) \times N = N \mathbb{P}_x (\tau_0 > \tau_N)$$

together we have $\mathbb{P}_x (\tau_0 > \tau_N) = x/N$.

8.(c)

We have for S_n denoting the symmetric SRW that $\{S_n^2 - n\}$ is a M.G. by checking the following:

$$\mathbb{E} [S_{n+1}^2 - (n+1) | \mathcal{F}_n] = \mathbb{E} [S_n^2 + \xi_{n+1}^2 + 2S_n \xi_{n+1} - (n+1) | S_n] = S_n^2 - n$$

and thus we have

$$\mathbb{E}_x [S_{\tau_{\{0,N\}}}^2 - \tau_{\{0,N\}}] = \mathbb{E}_x [S_{0 \wedge \tau_{\{0,N\}}}^2 - 0 \wedge \tau_{\{0,N\}}] = x^2$$

on the other hand we have

$$\mathbb{E}_x [S_{\tau_{\{0,N\}}}^2 - \tau_{\{0,N\}}] = \frac{x}{N} (N^2 - \mathbb{E}_x [\tau_{\{0,N\}}]) + \frac{N-x}{N} (0 - \mathbb{E}_x [\tau_{\{0,N\}}])$$

together we have $\mathbb{E}_x [\tau_{\{0,N\}}] = x(N-x)$.

Exercise 9 6.2.22

We construct $h(x) = \mathbf{1}_{x \notin [-r,r]}$ and we have that $G_{1/2} = [-r,r] \cap \mathbb{Z}$. We verify the super-harmonic as follows: for any $x \in \mathbb{Z} \setminus G_r$ we have

$$\begin{aligned} 1 &= h(x) \\ &\geq \sum_{m=-r}^r \mathbb{P} (\xi_i = m) \\ &= \sum_{y=x-r}^{x+r} p(x, y) \\ &\geq \sum_{y=x-r}^{x+r} p(x, y) h(y) \end{aligned}$$

since we have bounded difference of the chain. Thus by prop 6.2.21 we know that S_n is a recurrent chain.

Exercise 10 6.2.23

- If constant function, WLOG $f(x) \equiv 1$, is super-harmonic, then we use the previous argument that choose $h(x) = \mathbf{1}_{x \in \mathbb{S} \setminus F}$ where F is some finite set in \mathbb{S} , in this way we have h being super-harmonic on $\mathbb{S} \setminus F$ and thus by prop 6.2.21 we have that S_n is recurrent.

- If S_n is recurrent, we consider the stopping time $\tau_u := \inf_n \{S_n = u\}$ where u is some state in \mathbb{S} and by recurrence we have $\mathbb{P}_x(\tau_u < \infty) = 1$ for any $x \in \mathbb{S}$, then we have that a super-harmonic function f should be s.t. $f(S_n)$ is a super-martingale, using the condition we see that $\forall x \in \mathbb{S}$:

$$\begin{aligned} f(x) &= \mathbb{E}_x[f(S_0)] \\ &\geq \mathbb{E}_u[f(S_{\tau_u})] \\ &= f(u) \end{aligned}$$

which means that f can only be a constant function.