

STAT 430-2 2025 Winter

HW3

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Exercise 1 5.3.3

We prove the following: $(a) \Rightarrow (b)$, $(b) \Rightarrow (e)$, $(d) \Rightarrow (e)$, $(c) \Rightarrow$ Doob's convergence. Combined with the original condition in Doob's convergence that: $(e) \Rightarrow$ Doob's convergence, we have the equivalence of all conditions.

$(a) \Rightarrow (b)$ If $\lim_n \mathbb{E}[|X_n|] = M$, then fix any $\varepsilon > 0$, there $\exists N_0$, such that $\forall n > N_0$, $\mathbb{E}[|X_n|] < M + \varepsilon$. Then

$$\sup_n \mathbb{E}[|X_n|] \leq \sup_{k=1}^{N_0} \mathbb{E}[|X_k|] \vee (M + \varepsilon) < \infty$$

Thus (b) holds.

$(b) \Rightarrow (e)$ Note that $|x| = x_+ + x_-$, we thus have $\forall n$:

$$\begin{aligned} \mathbb{E}[|X_n|] &= \mathbb{E}[(X_n)_+] + \mathbb{E}[(X_n)_-] \\ &\geq \mathbb{E}[(X_n)_+] \end{aligned}$$

thus $\sup_n \mathbb{E}[|X_n|] < \infty$ would yield $\sup_n \mathbb{E}[(X_n)_+] < \infty$, which is (e) .

$(d) \Rightarrow (e)$ Idea is the same as $(a) \Rightarrow (b)$ by just replacing $|X_n|$ with $(X_n)_+$.

$(c) \Rightarrow$ Doob In the prove of Doob's convergence, we have in Doob's up-crossing:

$$\mathbb{E}[U_n[a, b]] \leq \frac{1}{b-a} \mathbb{E}[(X_n - a)_+] \leq \frac{1}{(b-a)} (|a| + \mathbb{E}[(X_n)_+])$$

then applying Fatou's lemma, we have:

$$\mathbb{E}\left[\liminf_n U_n[a, b]\right] \leq \frac{1}{(b-a)} (|a| + \liminf_n \mathbb{E}[(X_n)_+]) < \infty$$

then the rest of the proof is the same as Doob's convergence, thus (c) implies Doob's convergence.

Exercise 2 5.3.6

We verify that $X_n := W_n + an - (K + a)N_{n-1}$ is a sup-MG as follows:

$$\begin{aligned} \mathbb{E}[X_{n+1} - X_n | \mathcal{F}_n] &= \mathbb{E}[W_{n+1} + a(n+1) - (K+a)N_n - W_n - an + (K+a)N_{n-1} | \mathcal{F}_n] \\ &= \begin{cases} \text{if } W_n \geq b, & \mathbb{E}[W_{n+1} - W_n + a | \mathcal{F}_n] \mathbb{1}_{W_n \geq b} \leq 0 \\ \text{if } W_n < b, & \mathbb{E}[W_{n+1} - W_n - K | \mathcal{F}_n] \leq 0 \end{cases} \end{aligned}$$

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and finite expectation is easily verified. Thus X_n is a supermartingale. Further we notice that X_n have uniformly bounded difference $2(K+a)$, thus by (prop 5.3.5) we have that

$$\mathbb{P} \left(\lim_n X_n \text{ exists } \bigcup_n \liminf X_n = -\infty \right) = 1$$

Now consider the event that $\{N_\infty = \text{some } \nu < \infty\}$ We note that in this case, we have for large n :

$$\liminf_n X_n = \liminf_n W_n + an - (K+a)\nu \rightarrow \infty$$

since W_n is non-negative. Thus

$$\mathbb{P}(N_\infty < \infty) \leq \mathbb{P} \left(\liminf_n X_n = \infty \right) = \mathbb{P} \left(\left(\lim_n X_n \text{ exists } \bigcup_n \liminf X_n = -\infty \right)^c \right) = 0$$

Exercise 3 5.3.10

We verify that $W_n := \frac{X_n + 1}{\prod_{k=1}^{n-1} (Y_k + 1)}$ is a supermartingale as follows:

$$\begin{aligned} \mathbb{E}[W_{n+1} - W_n | \mathcal{F}_n] &= \mathbb{E} \left[\frac{X_{n+1} + 1}{\prod_{k=1}^n (Y_k + 1)} - \frac{X_n + 1}{\prod_{k=1}^{n-1} (Y_k + 1)} \middle| \mathcal{F}_n \right] \\ &= \frac{1}{\prod_{k=1}^n (Y_k + 1)} \mathbb{E}[X_{n+1} + 1 - (X_n + 1)(Y_n + 1) | \mathcal{F}_n] \leq 0 \end{aligned}$$

in which some technical conditions are: $\log \prod_{k=1}^\infty (Y_k + 1) = \sum_{k=1}^\infty \log(Y_k + 1) \leq \sum_{k=1}^\infty Y_k < \infty$, and $\mathbb{E}[|W_n|] \leq \mathbb{E}[|X_n|] < \infty$. Thus we have that W_n is a (non-negative) supermartingale. Thus directly we have that $\lim_n W_n$ exists a.s., denote it as W_∞ .

Then note that for each $\omega \in \Omega$:

$$\prod_{k=1}^{n-1} (Y_k(\omega) + 1) < \infty, \quad \text{a.s.}$$

and is an increasing sequence, thus it has a limit $Z(\omega)$. Combine the fact that $W_n \xrightarrow{\text{a.s.}} W_\infty$, we have

$$X_n \xrightarrow{\text{a.s.}} W_\infty Z - 1$$

Exercise 4 5.3.21

Note that $\mathcal{F}_k^A := \sigma(\mathcal{A}, Z_0, \dots, Z_{k-1})$ and $\mathcal{F}_k^B := \sigma(\mathcal{B}, Y_0, \dots, Y_{k-1})$ are both filtrations in \mathcal{F} , and also we have that

$$\begin{aligned} Y_k &= \mathbb{E}[X | \mathcal{F}_k^A] \text{ is U.I. M.G.} \\ Z_k &= \mathbb{E}[X | \mathcal{F}_k^B] \text{ is U.I. M.G.} \end{aligned}$$

which are easy to verify. Thus we have by Theorem 5.3.12 that there exists some Y_∞, Z_∞ s.t.

$$\begin{aligned} Y_k &= \mathbb{E}[X | \mathcal{F}_k^A] \xrightarrow[L_1]{\text{a.s.}} Y_\infty \\ Z_k &= \mathbb{E}[X | \mathcal{F}_k^B] \xrightarrow[L_1]{\text{a.s.}} Z_\infty \end{aligned}$$

Now follow similar idea as Theorem 5.3.15, we would obtain that $Y_\infty \in m\mathcal{F}_\infty^A$ and then $Y_\infty = \mathbb{E}[X|\mathcal{F}_\infty^A]$ where $\mathcal{F}_\infty^A = \sigma(\mathcal{A}, Z_0, \dots)$; similarly we have $Z_\infty = \mathbb{E}[X|\mathcal{F}_\infty^B]$ where $\mathcal{F}_\infty^B = \sigma(\mathcal{B}, Y_0, \dots)$.

Then we have:

$$\begin{aligned}\mathbb{E}[Y_\infty|\mathcal{F}_\infty^B] &= \mathbb{E}[\mathbb{E}[X|\mathcal{F}_\infty^A]|\mathcal{F}_\infty^B] \\ &= \mathbb{E}[\mathbb{E}[X|\mathcal{F}_\infty^B]|\mathcal{F}_\infty^A] \\ &= \mathbb{E}[Z_\infty|\mathcal{F}_\infty^A] = Z_\infty\end{aligned}$$

and vice versa $\mathbb{E}[Z_\infty|\mathcal{F}_\infty^A] = Y_\infty$. Together we have

$$Y_\infty = \mathbb{E}[Z_\infty|\mathcal{F}_\infty^A] = \mathbb{E}[\mathbb{E}[Y_\infty|\mathcal{F}_\infty^B]|\mathcal{F}_\infty^A]$$

thus we must have $Y_\infty = Z_\infty$ a.s., then apply the U.I. property again we have $Y_n - Z_n \xrightarrow[L_1]{\text{a.s.}} 0$.

Exercise 5 5.3.24

5.(a)

For any fixed W , there exists some n_W such that $W \in L^2\mathcal{F}_{n_W}$, then we have for $n > n_W$ that

$$\begin{aligned}\mathbb{E}[WZ_n] &= \mathbb{E}[\mathbb{E}[WZ_n|\mathcal{F}_{n_W}]] \\ &= \mathbb{E}\left[\mathbb{E}\left[\frac{W}{\sqrt{n}}\sum_{k=1}^{n_W}\xi_k|\mathcal{F}_{n_W}\right] + \frac{\mathbb{E}[W|\mathcal{F}_{n_W}]}{\sqrt{n}}\sum_{k=n_W+1}^{\infty}\xi_k\right] \rightarrow 0\end{aligned}$$

since the first term decay to 0 as $n \rightarrow \infty$, the second term simply has mean 0.

5.(b)

Define $W_n := \mathbb{E}[W|\mathcal{F}_n]$, in this way we can write that $WZ_n = W_mZ_n + (W - W_m)Z_n$. Then we consider the two parts:

Part 1: For any fixed m we have by (a) that

$$\mathbb{E}[W_mZ_n] \rightarrow 0, \quad (n \rightarrow \infty)$$

Part 2: We have

$$\begin{aligned}\limsup_{n \rightarrow \infty} \mathbb{E}[|(W - W_m)Z - n|] &\leq (\lim_{n \rightarrow \infty} \mathbb{E}[|Z_n|]) \cdot \mathbb{E}[|W - W_m|] \\ &= \text{const} \cdot \mathbb{E}[|W - W_m|]\end{aligned}$$

since by CLT, we would have $Z_n \xrightarrow{d} N(0, 1)$.

Together we have for any $m \in \mathbb{N}^+$,

$$\limsup_{n \rightarrow \infty} \mathbb{E}[|WZ_n|] \leq \text{const} \cdot \mathbb{E}[|W - W_m|]$$

further we notice that by the definition of W_m we have $W_m \xrightarrow[L_1]{\text{a.s.}} W$ thus $\mathbb{E} [|W - W_m|] \rightarrow 0$, thus we have $\limsup_{n \rightarrow \infty} \mathbb{E} [|W Z_n|] = 0$.

Now if there exists some $\tilde{Z} \in \mathcal{F}$ s.t. $Z \xrightarrow{L_2} \tilde{Z}$, which means that

$$\mathbb{E} \left[\left| Z_n - \tilde{Z} \right|^2 \right] \rightarrow 0$$

we would see that

$$\text{L.H.S.} = \mathbb{E} [Z_n^2] + \mathbb{E} [\tilde{Z}^2] - 2\mathbb{E} [Z_n \tilde{Z}] \rightarrow 0$$

however we have $\liminf_n \mathbb{E} [Z_n^2] \geq \mathbb{E} [\liminf_n Z_n^2] > 0$ by CLT, and that $\mathbb{E} [Z_n \tilde{Z}] \rightarrow 0$ by the above argument, thus we have a contradiction. Thus Z does not converge in L_2 .

Exercise 6 5.4.7

We first prove (5.4.6): Since \max is a convex function, we have that $Y_n := \max\{X_n, -1\}$ is still a sub-M.G. Now assume that $\mathbb{E} [\sup |Y_n|] < \infty$ i.e. Y_n is integrable. We consider the stopping time defined as:

$$\tau := \inf\{n : Y_n < 0\}$$

and since $\inf_n X_n < 0$ a.s., we have $\tau < \infty$ a.s., for which $Y_\tau < 0$, and thus we further have by Doob's optional stopping theorem that

$$0 > \mathbb{E} [Y_\tau] \geq \mathbb{E} [Y_0] \geq 0$$

which is a contradiction. Thus we have that $\mathbb{E} [\sup |Y_n|] = \infty$. Now since $\sup |Y_n| = \max\{\sup Y_+, \sup Y_-\}$ while $\sup Y_- \leq 1$, we have

$$\begin{aligned} \infty &= \mathbb{E} [\sup |Y_n|] \leq \mathbb{E} [\max\{\sup \max\{X_n, -1\}_+, 1\}] \\ &\leq \mathbb{E} \left[\max\{1, \sup_{X_n}\} \right] \end{aligned}$$

for this to hold, we must have $\mathbb{E} [\sup X_n] = \infty$. Thus we have proved (5.4.6).

Now using the lemma, we solve (5.4.7).

6.(a)

For random walk, we know that we have $S_n/\sqrt{n} \xrightarrow{d} N(0, 1)$ which is a non-degenerate distribution. Thus we have that $\mathbb{P} (\lim_n S_n \text{ exists}) = 0$, because for the event $\{\lim_n S_n \text{ exists}\}$, we must have that $S_n(\omega)/\sqrt{n} \rightarrow 0$.

For such M.G. with bounded difference, by (prop 5.3.5) we have that

$$\lim_n S_n \text{ exists, or } \liminf_n S_n = -\infty, \limsup_n S_n = \infty, \quad \text{a.s.}$$

and from the above argument we have that the first case is w.p. 0, thus we have that $\limsup_n S_n = \infty$ & $\liminf_n S_n = -\infty$ a.s., which means that $\tau_b < \infty$ a.s.

6.(b)

Note that we have $S_{\tau_b} \geq b > 0$, i.e. $\sup S_{n \wedge \tau_b} > 0$. And since τ_b is a stopping time, we also have that $S_{n \wedge b}$ is a M.G. (with $S_{0 \wedge \tau_b} = 0$), thus by the lemma we have that $\mathbb{E} [\min\{S_n : n \leq \tau_b\}] = \mathbb{E} [\inf_n S_{n \wedge \tau_b}] = \infty$.

Exercise 7 5.4.10

7.(a)

First we have that

$$\begin{aligned}
 \mathbb{E} [S_\tau] &= \mathbb{E} \left[\sum_{k=1}^{\infty} \xi_k I_{k \leq \tau} \right] \\
 &\leq \mathbb{E} \left[\sum_{k=1}^{\infty} |\xi_k| I_{k \leq \tau} \right] \\
 &\stackrel{\text{monotone}}{=} \sum_{k=1}^{\infty} \mathbb{E} [|\xi_k|] \mathbb{P} (k \leq \tau) \\
 &\stackrel{\text{converge}}{=} \mathbb{E} [|\xi_1|] \sum_{k=1}^{\infty} \mathbb{P} (k \leq \tau) = \mathbb{E} [|\xi_1|] \mathbb{E} [\tau] < \infty
 \end{aligned}$$

Then by DCT, we can apply the above argument to S_τ directly and obtain that $\mathbb{E} [S_\tau] = \mathbb{E} [\xi_1] \mathbb{E} [\tau]$.

7.(b)

Assume $\mathbb{E} [\xi_i] = 0$ (otherwise we can replace ξ_i with $\xi_i - \mathbb{E} [\xi_i]$), which would not influence the result. Now it suffice to prove $\mathbb{E} [S_\tau^2] = \text{var}(\xi_1) \mathbb{E} [\tau]$. To do so we consider $S_{n \wedge \tau}$ for which we have

$$S_{n \wedge \tau} = \sum_{k=1}^n \xi_k I_{k \leq \tau}$$

and we have

$$\mathbb{E} [S_{n \wedge \tau}^2] = \mathbb{E} \left[\sum_{k=1}^n \xi_k^2 I_{k \leq \tau} \right] = \text{var}(\xi_1) \sum_{k=1}^n \mathbb{P} (k \leq \tau) = \text{var}(\xi_1) \mathbb{E} [\tau] < \infty$$

for which right hand side does not depend on n . Thus here we can apply Doob's L_p convergence theorem ($p = 2$) to obtain

$$\mathbb{E} [S_\tau^2] = \mathbb{E} \left[\lim_n S_{n \wedge \tau}^2 \right] = \text{var}(\xi_1) \mathbb{E} [\tau]$$

7.(c)

If $\xi \geq 0$, then we have

$$\begin{aligned}\mathbb{E}[S_\tau] &= \mathbb{E}\left[\sum_{k=1}^{\infty} \xi_k I_{k \leq \tau}\right] \\ &\stackrel{\text{monotone}}{=} \sum_{k=1}^{\infty} \mathbb{E}[\xi_k] \mathbb{P}(k \leq \tau) \\ &\stackrel{\text{converge}}{=} \mathbb{E}[\xi_1] \sum_{k=1}^{\infty} \mathbb{P}(k \leq \tau) = \mathbb{E}[\xi_1] \mathbb{E}[\tau]\end{aligned}$$

The proof of Wald's second is still the same.

Exercise 8 5.4.11

8.(a)

In 5.4.7 we already proved it (by noting that SRW is also a M.G. with bounded difference).

8.(b)

By the similar argument as in (coroll 5.4.8) in which we use the M.G. $M_n := \exp(\lambda S_n)/M(\lambda)^n$ we have

$$S_{\tau_{a,b}} = \begin{cases} -a, & w.p. \frac{e^{\lambda b} - 1}{e^{\lambda b} - e^{-\lambda a}} \\ b, & w.p. \frac{1 - e^{-\lambda a}}{e^{\lambda b} - e^{-\lambda a}} \end{cases}$$

where $\lambda = \log \frac{1-p}{p}$. Thus we have

$$\begin{aligned}\mathbb{E}[S_{\tau_{a,b}}] &= -a \cdot \frac{e^{\lambda b} - 1}{e^{\lambda b} - e^{-\lambda a}} + b \cdot \frac{1 - e^{-\lambda a}}{e^{\lambda b} - e^{-\lambda a}} \\ &= \mathbb{E}[\xi_1] \mathbb{E}[\tau_{a,b}] \\ &= (2p - 1) \mathbb{E}[\tau_{a,b}]\end{aligned}$$

and we have

$$\mathbb{E}[\tau_{a,b}] = \frac{1}{2p - 1} \left(-a \cdot \frac{e^{\lambda b} - 1}{e^{\lambda b} - e^{-\lambda a}} + b \cdot \frac{1 - e^{-\lambda a}}{e^{\lambda b} - e^{-\lambda a}} \right)$$

where $\lambda = \log \frac{1-p}{p}$.

And for the case that $p = 1/2$, i.e. $\lambda = 0$, we use Wald's second to obtain that

$$\begin{aligned}\mathbb{E}[\tau_{a,b}] &= \mathbb{E}[S_\tau^2] / \text{var}(\xi_1) \\ &= \mathbb{E}[S_\tau^2] \\ &= ab\end{aligned}$$

8.(c)

We have

$$\tau_{a,b} = \inf\{n : S_n = -a\} \wedge \inf\{n : S_n = b\} \xrightarrow{a \rightarrow \infty} \inf\{n : S_n = b\} = \tau_b$$

(and the convergence is monotone, which is trivial). We thus have by Wald's first that

$$\mathbb{E} [\tau_b] = \mathbb{E} [S_{\tau_b}] / \mathbb{E} [\xi_1] = b / (2p - 1)$$

8.(d)

For $p \geq 1/2$, it's a submartingale, thus by 5.4.7 we have $\tau_b < \infty$ a.s..

8.(e)

$$\begin{aligned} \mathbb{E} [Y_{n+1} | \mathcal{F}_n] &= (S_n^4 + 6S_n^2 + 1) - 6(n+1)(S_n^2 + 1) + c_1(n^2 + 2n + 1) + c_2(n+1) \\ &= S_n^4 - 6nS_n^2 + c_1n^2 + c_2n + (1 - 6(n+1) + c_1(2n+1) + c_2) \end{aligned}$$

to make it a M.G., we have $c_1 = 3, c_2 = 2$, thus we have

$$Y_n := S_n^4 - 6nS_n^2 + 3n^2 + 2n$$

is a M.G.

For which we have

$$\begin{aligned} \mathbb{E} [Y_{\tau_{b,b}}] &= \mathbb{E} [\mathbb{E} [Y_k | \tau_{b,b} = k]] \\ &= b^4 - 6b^2 \mathbb{E} [\tau_{b,b}] + 3\mathbb{E} [\tau_{b,b}^2] + 2\mathbb{E} [\tau_{b,b}] \\ &= b^4 - 6b^2 \cdot b^2 + 3\mathbb{E} [\tau_{b,b}^2] + 2b^2 \\ &= \mathbb{E} [Y_0] = 0 \end{aligned}$$

where the last step is by Doob's optional stopping theorem. Now we have

$$\mathbb{E} [\tau_{b,b}^2] = \frac{5b^4 - 2b^2}{3}$$

Exercise 9 5.4.12

9.(a)

Since $M_n = \exp [\lambda S_n] / \mathbb{E} [\exp [\lambda n \xi_1]]$ is a M.G., then by Doob's optional stopping theorem we have

$$\begin{aligned} 1 &= \mathbb{E} [M_0] \\ &= \mathbb{E} [M_{\tau_b}] \\ &= \mathbb{E} [\exp [\lambda S_{\tau_b}] / \mathbb{E} [\exp [\lambda \xi_1]]^{\tau_b}] \\ &= e^{\lambda b} \mathbb{E} [M(\lambda)^{-\tau_b}] \end{aligned}$$

9.(b)

Denote $s = M(\lambda)^{-1} \in (0, 1)$ for some λ , and use $b = 1$ in the above result then we have

$$\mathbb{E}[s^{\tau_1}] = e^{-\lambda}$$

and that

$$\mathbb{E}[\tau_b] = e^{-\lambda b} = \mathbb{E}[s^{\tau_1}]^b$$

Now we revisit $s = M(\lambda)^{-1} = \frac{1}{pe^\lambda + qe^{-\lambda}}$, and we have

$$\begin{aligned} e^{-\lambda} &= \frac{1}{2qs} \frac{2qe^{-\lambda}}{pe^\lambda + qe^{-\lambda}} \\ &= \frac{1}{2qs} \left(1 - \frac{pe^\lambda - qe^{-\lambda}}{pe^\lambda + qe^{-\lambda}} \right) \\ &= \frac{1}{2qs} \left(1 - \sqrt{\frac{(pe^\lambda - qe^{-\lambda})^2}{(pe^\lambda + qe^{-\lambda})^2}} \right) \\ &= \frac{1}{2qs} \left(1 - \sqrt{1 - 4 \frac{pe^\lambda \cdot qe^{-\lambda}}{pe^\lambda + qe^{-\lambda}}} \right) \\ &= \frac{1}{2qs} \left(1 - \sqrt{1 - 4pqs^2} \right) \end{aligned}$$

substitute this back to the previous result we have

$$\mathbb{E}[s^{\tau_1}] = e^{-\lambda} = \frac{1}{2qs} \left(1 - \sqrt{1 - 4pqs^2} \right), \quad s = M(\lambda)^{-1} \in (0, 1), \quad p \geq 1/2$$

9.(c)

In 5.4.11 (c) we had that $\tau_{a,b} \xrightarrow{a \rightarrow \infty} \tau_b$ a.s.. Thus we have

$$\mathbb{P}(S_{\tau_{a,b}} = b) \rightarrow \mathbb{P}(S_{\tau_b} = b) = \mathbb{P}(\tau_b < \infty) = \frac{1}{e^{\lambda_\star b}} = \exp[-\lambda_\star b]$$

where $\lambda_\star = \log \frac{1-p}{p}$.

9.(d)

Continuing from the previous result, we have

$$\mathbb{P}(S_{\tau_b} = b) = \exp[-\lambda_\star b], \quad b \in \mathbb{N}^+$$

which gives us for $Z := 1 + \max_{k \geq 0} S_k$ that:

$$\begin{aligned}
 \mathbb{P}(Z = b + 1) &= \mathbb{P}\left(\max_{k \geq 0} S_k = b\right) \\
 &= \mathbb{P}(\tau_b < \infty, \tau_{b+1} = \infty) \\
 &= \mathbb{P}(\tau_b < \infty, \text{after reaching } b, \text{ never reach } b + 1) \\
 &= \mathbb{P}(\tau_b < \infty) (1 - \mathbb{P}(\tau_1 = \infty)) \\
 &= \exp[-\lambda_* b] (1 - \exp[-\lambda_*]) \\
 &\sim \text{Geo}(1 - \exp[-\lambda_*])
 \end{aligned}$$

Exercise 10 5.4.14

Denote $26 = \kappa$

10.(a)

We have

$$\mathbb{E}[M_{n+1} | \mathcal{F}_n] = \frac{1}{\kappa} M_n \times \kappa + \left(1 - \frac{1}{\kappa}\right) \times 0 = M_n$$

thus M_n is a martingale.

The construction of total gain would look like:

$$M_n = \sum_n (\kappa^{10}(\kappa \mathbf{1}_{\text{PROBABILITY}} - 1) + \kappa^9(\kappa \mathbf{1}_{\text{PROBABILIT}} - 1) + \cdots + \kappa^1(\kappa \mathbf{1}_{\text{PR}} - 1) + \kappa^0(\kappa \mathbf{1}_{\text{P}} - 1))$$

10.(b)

We see that $\hat{\tau}$ is equivalent to that $\hat{\tau} := \inf\{n \geq 11 : \text{gain of a gamblers} = \kappa^{11}\}$ (i.e. when the gambler win 11 consecutive games, he gets the word ‘PROBABILITY’). WHEN The gambler wins 11 consecutive games, there are other $\hat{\tau} - 1$ gamblers losing. And as we mentioned earlier, we have M_n being a martingale, thus by Doob’s optional stopping theorem we have

$$0 = \mathbb{E}[M_0] = \mathbb{E}[M_{\hat{\tau}}] = \mathbb{E}[\mathbb{E}[M_k | \tau = k]] = \sum_{k=0}^{10} (\kappa^{k+1} - \kappa^k) - (\mathbb{E}[\hat{\tau}] - 1)$$

And it gives that

$$\mathbb{E}[\hat{\tau}] = \kappa^{11} = 26^{11}$$

For ‘ABRACADABRA’ = ‘ABRA’ + ‘CAD’ + ‘ABRA’, we have that

$$\begin{aligned}
 0 = \mathbb{E}[M_0] &= \sum_{k=0}^{10} (\kappa^{k+1} - \kappa^k) + \sum_{k=0}^3 (\kappa^{k+1} - \kappa^k) - (\mathbb{E}[\tilde{\tau}] - 2) = \kappa^{11} + \kappa^4 - \mathbb{E}[\tilde{\tau}] \\
 \Rightarrow \mathbb{E}[\tilde{\tau}] &= \kappa^{11} + \kappa^4 = 26^{11} + 26^4
 \end{aligned}$$

where $+\kappa^4$ for that, when reaching ‘ABRACADABRA’, there is also another gambler who was winning at the first 4 letters ‘ABRA’.