STAT 430-2 2025 Winter
HW3

Tuorui Peng[1]

Exercise 1 5.3.3

We prove the following: $(a) \Rightarrow (b)$, $(b) \Rightarrow (e)$, $(d) \Rightarrow (e)$, $(c) \Rightarrow$ Doob's convergence. Combined with the original condition in Doob's convergence that: $(e) \Rightarrow$ Doob's convergence, we have the equivalence of all conditions.

$(a) \Rightarrow (b)$ If $\lim_n \mathbb{E}[|X_n|] = M$, then fix any $\varepsilon > 0$, there $\exists N_0$, such that $\forall n > N_0$, $\mathbb{E}[|X_n|] < M + \varepsilon$. Then

$$\sup_n \mathbb{E}[|X_n|] \leq \sup_{k=1}^{N_0} \mathbb{E}[|X_k|] \vee (M+\varepsilon) < \infty$$

Thus (b) holds.

$(b) \Rightarrow (e)$ Note that $|x| = x_+ + x_-$, we thus have $\forall n$:

$$\begin{aligned}\mathbb{E}[|X_n|] =& \mathbb{E}[(X_n)_+] + \mathbb{E}[(X_n)_-] \\ \geq& \mathbb{E}[(X_n)_+]\end{aligned}$$

thus $\sup_n \mathbb{E}[|X_n|] < \infty$ would yield $\sup_n \mathbb{E}[(X_n)_+] < \infty$, which is (e).

$(d) \Rightarrow (e)$ Idea is the same as $(a) \Rightarrow (b)$ by just replacing $|X_n|$ with $(X_n)_+$.

$(c) \Rightarrow$ Doob In the prove of Doob's convergence, we have in Doob's up-crossing:

$$\mathbb{E}[U_n[a,b]] \leq \frac{1}{b-a}\mathbb{E}[(X_n - a)_+] \leq \frac{1}{(b-a)}\big(|a| + \mathbb{E}[(X_n)_+]\big)$$

then applying Fatou's lemma, we have:

$$\mathbb{E}\left[\liminf_n U_n[a,b]\right] \leq \frac{1}{(b-a)}\big(|a| + \liminf_n \mathbb{E}[(X_n)_+]\big) < \infty$$

then the rest of the proof is the same as Doob's convergence, thus (c) implies Doob's convergence.

Exercise 2 5.3.6

We verify that $X_n := W_n + an - (K+a)N_{n-1}$ is a sup-MG as follows:

$$\begin{aligned}\mathbb{E}[X_{n+1} - X_n | \mathcal{F}_n] =& \mathbb{E}[W_{n+1} + a(n+1) - (K+a)N_n - W_n - an + (K+a)N_{n-1} | \mathcal{F}_n] \\ =& \begin{cases} \text{if } W_n \geq b, & \mathbb{E}[W_{n+1} - W_n + a | \mathcal{F}_n] \mathbb{1}_{W_n \geq b} \leq 0 \\ \text{if } W_n < b, & \mathbb{E}[W_{n+1} - W_n - K | \mathcal{F}_n] \leq 0 \end{cases}\end{aligned}$$

[1] TuoruiPeng2028@u.northwestern.edu

and finite expectation is easily verified. Thus X_n is a supermartingale. Further we notice that X_n have uniformly bounded difference $2(K+a)$, thus by (prop 5.3.5) we have that

$$\mathbb{P}\left(\lim_n X_n \text{ exists} \bigcup \liminf_n X_n = -\infty\right) = 1$$

Now consider the event that $\{N_\infty = \text{some } \nu < \infty\}$ We note that in this case, we have for large n:

$$\liminf_n X_n = \liminf_n W_n + an - (K+a)\nu \to \infty$$

since W_n is non-negative. Thus

$$\mathbb{P}\left(N_\infty < \infty\right) \leq \mathbb{P}\left(\liminf_n X_n = \infty\right) = \mathbb{P}\left(\left(\lim_n X_n \text{ exists} \bigcup \liminf_n X_n = -\infty\right)^{\complement}\right) = 0$$

Exercise 3 5.3.10

We verify that $W_n := \dfrac{X_n + 1}{\prod_{k=1}^{n-1}(Y_k+1)}$ is a supermartingale as follows:

$$\begin{aligned}\mathbb{E}\left[W_{n+1} - W_n|\mathcal{F}_n\right] =& \mathbb{E}\left[\frac{X_{n+1}+1}{\prod_{k=1}^{n}(Y_k+1)} - \frac{X_n+1}{\prod_{k=1}^{n-1}(Y_k+1)}|\mathcal{F}_n\right] \\ =& \frac{1}{\prod_{k=1}^{n}(Y_k+1)}\mathbb{E}\left[X_{n+1}+1-(X_n+1)(Y_n+1)|\mathcal{F}_n\right] \leq 0\end{aligned}$$

in which some technical conditions are: $\log\prod_{k=1}^{\infty}(Y_k+1) = \sum_{k=1}^{\infty}\log(Y_k+1) \leq \sum_{k=1}^{\infty} Y_k < \infty$, and $\mathbb{E}\left[|W_n|\right] \leq \mathbb{E}\left[|X_n|\right] < \infty$. Thus we have that W_n is a (non-negative) supermartingalem. Thus directly we have that $\lim_n W_n$ exists a.s., denote it as W_∞.

Then note that for each $\omega \in \Omega$:

$$\prod_{k=1}^{n-1}(Y_k(\omega)+1) < \infty, \quad \text{a.s.}$$

and is an increasing sequence, thus it has a limit $Z(\omega)$. Combine the fact that $W_n \xrightarrow{\text{a.s.}} W_\infty$, we have

$$X_n \xrightarrow{\text{a.s.}} W_\infty Z - 1$$

Exercise 4 5.3.21

Note that $\mathcal{F}_k^A := \sigma(\mathcal{A}, Z_0, \ldots, Z_{k-1})$ and $\mathcal{F}_k^B := \sigma(\mathcal{B}, Y_0, \ldots, Y_{k-1})$ are both filtrations in $\mathcal{F}$, and also we have that

$$\begin{aligned}Y_k = \mathbb{E}\left[X|\mathcal{F}_k^A\right] &\text{ is U.I. M.G.}\\ Z_k = \mathbb{E}\left[X|\mathcal{F}_k^B\right] &\text{ is U.I. M.G.}\end{aligned}$$

which are easy to verify. Thus we have by Theorem 5.3.12 that there exists some Y_∞, Z_∞ s.t.

$$\begin{aligned}Y_k = \mathbb{E}\left[X|\mathcal{F}_k^A\right] \xrightarrow[L_1]{\text{a.s.}} Y_\infty\\ Z_k = \mathbb{E}\left[X|\mathcal{F}_k^B\right] \xrightarrow[L_1]{\text{a.s.}} Z_\infty\end{aligned}$$

Now follow similar idea as Theorem 5.3.15, we would obtain that $Y_\infty \in m\mathcal{F}_\infty^A$ and then $Y_\infty = \mathbb{E}\left[X|\mathcal{F}_\infty^A\right]$ where $\mathcal{F}_\infty^A = \sigma(\mathcal{A}, Z_0, \ldots)$; similarly we have $Z_\infty = \mathbb{E}\left[X|\mathcal{F}_\infty^B\right]$ where $\mathcal{F}_\infty^B = \sigma(\mathcal{B}, Y_0, \ldots)$.

Then we have:

$$\begin{aligned} Y_\infty =& \mathbb{E}\left[Y_\infty|\mathcal{F}_\infty^B\right] \\ =& \mathbb{E}\left[Y_\infty|\mathcal{F}_\infty^A\right] = \mathbb{E}\left[X|\mathcal{F}_\infty^A\right] \end{aligned}$$

where the first line since $Y_k \xrightarrow{\text{a.s.}} Y_\infty$ and the second line since $Y_\infty \in \mathcal{F}_\infty^A$. Together we have

$$Y_\infty = \mathbb{E}\left[Y_\infty|\mathcal{F}_\infty^B\right] = \mathbb{E}\left[\mathbb{E}\left[Y_\infty|\mathcal{F}_\infty^A\right]|\mathcal{F}_\infty^B\right] = \mathbb{E}\left[X|\sigma(\mathcal{F}_\infty^A, \mathcal{F}_\infty^B)\right] = Z_\infty$$

thus we must have $Y_\infty = Z_\infty$ a.s., then apply the U.I. property again we have $Y_n - Z_n \xrightarrow[L_1]{\text{a.s.}} 0$.

Exercise 5 5.3.24

5.(a)

We first prove for $W \in L^2\mathcal{F}_{n_W}$ fixed then we have for $n > n_W$ that

$$\begin{aligned} \mathbb{E}\left[WZ_n\right] =& \mathbb{E}\left[\mathbb{E}\left[WZ_n|\mathcal{F}_{n_W}\right]\right] \\ =& \mathbb{E}\left[\mathbb{E}\left[\frac{W}{\sqrt{n}}\sum_{k=1}^{n_W}\xi_k|\mathcal{F}_{n_W}\right] + \frac{\mathbb{E}\left[W|\mathcal{F}_{n_W}\right]}{\sqrt{n}}\sum_{k=n_W+1}^{\infty}\xi_k\right] \to 0 \end{aligned}$$

since the first term decay to 0 as $n \to \infty$, and the second term simply has mean 0.

Then we upgrade to $W \in L^2\mathcal{F}_\infty$. Define $W_n := \mathbb{E}\left[W|\mathcal{F}_n\right]$, in this way we can write that $WZ_n = W_mZ_n + (W - W_m)Z_n$. Then we consider the two parts:

Part 1: For any fixed m we have by preceding that

$$\mathbb{E}\left[W_mZ_n\right] \to 0, \quad (n \to \infty)$$

Part 2: We have

$$\begin{aligned} \limsup_{n\to\infty} \mathbb{E}\left[|(W - W_m)Z_n|\right] \leq& (\lim_{n\to\infty}\sqrt{\mathbb{E}\left[|Z_n|^2\right] \cdot \mathbb{E}\left[|W - W_m|^2\right]}) \\ =& \text{const} \cdot \mathbb{E}\left[|W - W_m|^2\right] \end{aligned}$$

since by CLT, we would have $Z_n \xrightarrow{\text{d}} N(0,1)$.

Together we have for any $m \in \mathbb{N}^+$,

$$\limsup_{n\to\infty} \mathbb{E}\left[|WZ_n|\right] \leq \text{const} \cdot \mathbb{E}\left[|W - W_m|^2\right]$$

further we notice that by the definition of W_m and using Doob's L_2 convergence we have $W_m \xrightarrow{L_2} W$ thus $\mathbb{E}\left[|W - W_m|^2\right] \to 0$, thus we have $\limsup_{n\to\infty} \mathbb{E}\left[|WZ_n|\right] = 0$.

5.(b)

Now for $Z \in \mathcal{F}$, we know that the above convergence works for $\mathbb{E}\left[Z|\mathcal{F}_\infty\right] \in \mathcal{F}_\infty$. Then we have

$$\mathbb{E}\left[WZ_n\right] = \mathbb{E}\left[\mathbb{E}\left[WZ_n|\mathcal{F}_\infty\right]\right] = \mathbb{E}\left[Z_n\mathbb{E}\left[W|\mathcal{F}_\infty\right]\right] \to 0$$

Now if there exists some $\tilde{Z} \in \mathcal{F}$ s.t. $Z \xrightarrow{L_2} \tilde{Z}$, which means that

$$\mathbb{E}\left[\left|Z_n - \tilde{Z}\right|^2\right] \to 0$$

we would see that

$$\text{L.H.S.} = \mathbb{E}\left[Z_n^2\right] + \mathbb{E}\left[\tilde{Z}^2\right] - 2\mathbb{E}\left[Z_n\tilde{Z}\right] \to 0$$

however we have $\liminf_n \mathbb{E}\left[Z_n^2\right] \geq \mathbb{E}\left[\liminf_n Z_n^2\right] > 0$ by CLT, and that $\mathbb{E}\left[Z_n\tilde{Z}\right] \to 0$ by the above argument, thus we have a contradiction. Thus Z does not converge in L_2.

Exercise 6 5.4.7

We first prove (5.4.6): Since max is a convex function, we have that $Y_n := \max\{X_n, -1\}$ is still a sub-M.G. Now assume that $\mathbb{E}\left[\sup|Y_n|\right] < \infty$ i.e. Y_n is integrable. We consider the stopping time defined as:

$$\tau := \inf\{n : Y_n < 0\}$$

and since $\inf_n X_n < 0$ a.s., we have $\tau < \infty$ a.s., for which $Y_\tau < 0$, and thus we further have by Doob's optional stopping theorem that

$$0 > \mathbb{E}\left[Y_\tau\right] \geq \mathbb{E}\left[Y_0\right] \geq 0$$

which is a contradiction. Thus we have that $\mathbb{E}\left[\sup|Y_n|\right] = \infty$. Now since $\sup|Y_n| = \max\{\sup Y_+, \sup Y_-\}$ while $\sup Y_- \leq 1$, we have

$$\begin{aligned}\infty = \mathbb{E}\left[\sup|Y_n|\right] \leq& \mathbb{E}\left[\max\{\sup\max\{X_n, -1\}_+, 1\}\right]\\ \leq& \mathbb{E}\left[\max\{1, \sup X_n\}\right]\end{aligned}$$

for this to hold, we must have $\mathbb{E}\left[\sup X_n\right] = \infty$. Thus we have proved (5.4.6).

Now using the lemma, we solve (5.4.7).

6.(a)

For random walk, we know that we have $S_n/\sqrt{n} \xrightarrow{\text{d}} N(0,1)$ which is a non-degenerate distribution. Thus we have that $\mathbb{P}\left(\lim_n S_n \text{ exists}\right) = 0$, because for the event $\{\lim_n S_n \text{ exists}\}$, we must have that $S_n(\omega)/\sqrt{n} \to 0$.

For such M.G. with bounded difference, by (prop 5.3.5) we have that

$$\lim_n S_n \text{ exists }, or \liminf_n S_n = -\infty,\ \limsup_n S_n = \infty, \quad \text{a.s.}$$

and from the above argument we have that the first case is w.p. 0, thus we have that $\limsup_n S_n = \infty \& \liminf_n S_n = -\infty$ a.s., which means that $\tau_b < \infty$ a.s.

6.(b)

Note that we have $S_{\tau_b} \geq b > 0$, i.e. $\sup S_{n\wedge\tau_b} > 0$. And since τ_b is a stopping time, we also have that $S_{n\wedge b}$ is a M.G. (with $S_{0\wedge\tau_b} = 0$), thus by the lemma we have that $\mathbb{E}\left[\min\{S_n : n \leq \tau_b\}\right] = \mathbb{E}\left[\inf_n S_{n\wedge\tau_b}\right] = \infty$.

Exercise 7 5.4.10

7.(a)

First we have that

$$\begin{aligned}
\mathbb{E}\left[S_\tau\right] =& \mathbb{E}\left[\sum_{k=1}^{\infty} \xi_k I_{k\leq\tau}\right] \\
\leq& \mathbb{E}\left[\sum_{k=1}^{\infty} \left|\xi_k\right|_k I_{k\leq\tau}\right] \\
\overset{\text{monotone}}{\underset{\text{converge}}{=}}& \sum_{k=1}^{\infty} \mathbb{E}\left[|\xi_k|\right] \mathbb{P}\left(k \leq \tau\right) \\
=& \mathbb{E}\left[|\xi_1|\right] \sum_{k=1}^{\infty} \mathbb{P}\left(k \leq \tau\right) = \mathbb{E}\left[|\xi_1|\right] \mathbb{E}\left[\tau\right] < \infty
\end{aligned}$$

Then by DCT, we can apply the above argument to S_τ directly and obtain that $\mathbb{E}\left[S_\tau\right] = \mathbb{E}\left[\xi_1\right] \mathbb{E}\left[\tau\right]$.

7.(b)

Assume $\mathbb{E}\left[\xi_i\right] = 0$(otherwise we can replace ξ_i with $\xi_i - \mathbb{E}\left[\xi_i\right]$), which would not influence the result. Now it suffice to prove $\mathbb{E}\left[S_\tau^2\right] = var(\xi_1)\mathbb{E}\left[\tau\right]$. To do so we consider $S_{n\wedge\tau}$ for which we have

$$S_{n\wedge\tau} = \sum_{k=1}^{n} \xi_k I_{k\leq\tau}$$

and we have

$$\mathbb{E}\left[S_{n\wedge\tau}^2\right] = \mathbb{E}\left[\sum_{k=1}^{n} \xi_k^2 I_{k\leq\tau}\right] = var(\xi_1) \sum_{k=1}^{n} \mathbb{P}\left(k \leq \tau\right) = var(\xi_1)\mathbb{E}\left[\tau\right] < \infty$$

for which right hand side does not depend on n. Thus here we can apply Doob's L_p convergence theorem ($p = 2$) to obtain

$$\mathbb{E}\left[S_\tau^2\right] = \mathbb{E}\left[\lim_n S_{n\wedge\tau}^2\right] = var(\xi_1)\mathbb{E}\left[\tau\right]$$

7.(c)

If $\xi \geq 0$, then we have

$$\begin{aligned}\mathbb{E}\left[S_\tau\right] =&\mathbb{E}\left[\sum_{k=1}^{\infty}\xi_k I_{k\leq\tau}\right]\\ \overset{\text{monotone}}{\underset{\text{converge}}{=}}&\sum_{k=1}^{\infty}\mathbb{E}\left[\xi_k\right]\mathbb{P}\left(k\leq\tau\right)\\ =&\mathbb{E}\left[\xi_1\right]\sum_{k=1}^{\infty}\mathbb{P}\left(k\leq\tau\right)=\mathbb{E}\left[\xi_1\right]\mathbb{E}\left[\tau\right]\end{aligned}$$

The proof of Wald's second is still the same.

Exercise 8 5.4.11

8.(a)

In 5.4.7 we already proved it (by noting that SRW is also a M.G. with bounded difference).

8.(b)

By the similar argument as in (coroll 5.4.8) in which we use the M.G. $M_n := \exp(\lambda S_n)/M(\lambda)^n$ we have

$$S_{\tau_{a,b}} = \begin{cases} -a, & w.p.\dfrac{e^{\lambda b}-1}{e^{\lambda b - e^{-\lambda a}}} \\ b, & w.p.\dfrac{1-e^{-\lambda a}}{e^{\lambda b - e^{-\lambda a}}} \end{cases}$$

where $\lambda = \log\dfrac{1-p}{p}$. Thus we have

$$\begin{aligned}\mathbb{E}\left[S_{\tau_{a,b}}\right] =& -a\cdot\frac{e^{\lambda b}-1}{e^{\lambda b}-e^{-\lambda a}} + b\cdot\frac{1-e^{-\lambda a}}{e^{\lambda b}-e^{-\lambda a}}\\ =&\mathbb{E}\left[\xi_1\right]\mathbb{E}\left[\tau_{a,b}\right]\\ =&(2p-1)\mathbb{E}\left[\tau_{a,b}\right]\end{aligned}$$

and we have

$$\mathbb{E}\left[\tau_{a,b}\right] = \frac{1}{2p-1}\Big(-a\cdot\frac{e^{\lambda b}-1}{e^{\lambda b}-e^{-\lambda a}} + b\cdot\frac{1-e^{-\lambda a}}{e^{\lambda b}-e^{-\lambda a}}\Big)$$

where $\lambda = \log\dfrac{1-p}{p}$.

And for the case that $p = 1/2$, i.e. $\lambda = 0$, we use Wald's second to obtain that

$$\begin{aligned}\mathbb{E}\left[\tau_{a,b}\right] =&\mathbb{E}\left[S_\tau^2\right]/var(\xi_1)\\ =&\mathbb{E}\left[S_\tau^2\right]\\ =&ab\end{aligned}$$

8.(c)

We have

$$\tau_{a,b} = \inf\{n : S_n = -a\} \wedge \inf\{n : S_n = b\} \underset{a\to\infty}{\to} \inf\{n : S_n = b\} = \tau_b$$

(and the convergence is monotone, which is trivial). We thus have by Wald's first that

$$\mathbb{E}\left[\tau_b\right] = \mathbb{E}\left[S_{\tau_b}\right]/\mathbb{E}\left[\xi_1\right] = b/(2p-1)$$

8.(d)

For $p \geq 1/2$, it's a submartingale, thus by 5.4.7 we have $\tau_b < \infty$ a.s..

8.(e)

$$\begin{aligned}\mathbb{E}\left[Y_{n+1}|\mathcal{F}_n\right] =& \big(S_n^4 + 6S_n^2 + 1\big) - 6(n+1)\big(S_n^2 + 1\big) + c_1(n^2 + 2n + 1) + c_2(n+1)\\ =& S_n^4 - 6nS_n^2 + c_1 n^2 + c_2 n + \big(1 - 6(n+1) + c_1(2n+1) + c_2\big)\end{aligned}$$

to make it a M.G., we have $c_1 = 3, c_2 = 2$, thus we have

$$Y_n := S_n^4 - 6nS_n^2 + 3n^2 + 2n$$

is a M.G.

For which we have

$$\begin{aligned}\mathbb{E}\left[Y_{\tau_{b,b}}\right] =& \mathbb{E}\left[\mathbb{E}\left[Y_k|\tau_{b,b} = k\right]\right]\\ =& b^4 - 6b^2\mathbb{E}\left[\tau_{b,b}\right] + 3\mathbb{E}\left[\tau_{b,b}^2\right] + 2\mathbb{E}\left[\tau_{b,b}\right]\\ =& b^4 - 6b^2 \cdot b^2 + 3\mathbb{E}\left[\tau_{b,b}^2\right] + 2b^2\\ =& \mathbb{E}\left[Y_0\right] = 0\end{aligned}$$

where the last step is by Doob's optional stopping theorem. Now we have

$$\mathbb{E}\left[\tau_{b,b}^2\right] = \frac{5b^4 - 2b^2}{3}$$

Exercise 9 5.4.12

9.(a)

Since $M_n = \exp\left[\lambda S_n\right]/\mathbb{E}\left[\exp\left[\lambda n\xi_1\right]\right]$ is a M.G., then by Doob's optional stopping theorem we have

$$\begin{aligned}1 =& \mathbb{E}\left[M_0\right]\\ =& \mathbb{E}\left[M_{\tau_b}\right]\\ =& \mathbb{E}\left[\exp\left[\lambda S_{\tau_b}\right]/\mathbb{E}\left[\exp\left[\lambda \xi_1\right]\right]^{\tau_b}\right]\\ =& e^{\lambda b}\mathbb{E}\left[M(\lambda)^{-\tau_b}\right]\end{aligned}$$

9.(b)

Denote $s = M(\lambda)^{-1} \in (0,1)$ for some λ, and use $b = 1$ in the above result then we have

$$\mathbb{E}\left[s^{\tau_1}\right] = e^{-\lambda}$$

and that

$$\mathbb{E}\left[\tau_b\right] = e^{-\lambda b} = \mathbb{E}\left[s^{\tau_1}\right]^b$$

Now we revisit $s = M(\lambda)^{-1} = \frac{1}{pe^{\lambda}+qe^{-\lambda}}$, and we have

$$\begin{aligned}
e^{-\lambda} =& \frac{1}{2qs}\frac{2qe^{-\lambda}}{pe^{\lambda}+qe^{-\lambda}} \\
=& \frac{1}{2qs}\left(1 - \frac{pe^{\lambda}-qe^{-\lambda}}{pe^{\lambda}+qe^{-\lambda}}\right) \\
=& \frac{1}{2qs}\left(1 - \sqrt{\frac{(pe^{\lambda}-qe^{-\lambda})^2}{(pe^{\lambda}+qe^{-\lambda})^2}}\right) \\
=& \frac{1}{2qs}\left(1 - \sqrt{1 - 4\frac{pe^{\lambda}\cdot qe^{-\lambda}}{pe^{\lambda}+qe^{-\lambda}}}\right) \\
=& \frac{1}{2qs}\left(1 - \sqrt{1-4pqs^2}\right)
\end{aligned}$$

substitute this back to the previous result we have

$$\mathbb{E}\left[s^{\tau_1}\right] = e^{-\lambda} = \frac{1}{2qs}\left(1 - \sqrt{1-4pqs^2}\right), \quad s = M(\lambda)^{-1} \in (0,1), \quad p \geq 1/2$$

9.(c)

In 5.4.11 (c) we had that $\tau_{a,b} \overset{a\to\infty}{\to} \tau_b$ a.s.. Thus we have

$$\mathbb{P}\left(S_{\tau_{a,b}} = b\right) \to \mathbb{P}\left(S_{\tau_b} = b\right) = \mathbb{P}\left(\tau_b < \infty\right) = \frac{1}{e^{\lambda_\star b}} = \exp\left[-\lambda_\star b\right]$$

where $\lambda_\star = \log\dfrac{1-p}{p}$.

9.(d)

Continuing from the previous result, we have

$$\mathbb{P}\left(S_{\tau_b} = b\right) = \exp\left[-\lambda_\star b\right], \quad b \in \mathbb{N}^+$$

which gives us for $Z := 1 + \max_{k\geq 0} S_k$ that:

$$\begin{aligned}\mathbb{P}(Z = b+1) =& \mathbb{P}\left(\max_{k\geq 0} S_k = b\right)\\ =&\mathbb{P}(\tau_b < \infty, \tau_{b+1} = \infty)\\ =&\mathbb{P}(\tau_b < \infty, \text{after reaching } b, \text{never reach } b+1)\\ =&\mathbb{P}(\tau_b < \infty)\big(1 - \mathbb{P}(\tau_1 = \infty)\big)\\ =&\exp[-\lambda_\star b]\big(1 - \exp[-\lambda_\star]\\ \sim&\mathrm{Geo}(1 - \exp[-\lambda_\star])\end{aligned}$$

Exercise 10 5.4.14

Denote $26 = \kappa$

10.(a)

We have

$$\mathbb{E}[M_{n+1}|\mathcal{F}_n] = \frac{1}{\kappa} M_n \times \kappa + (1 - \frac{1}{\kappa}) \times 0 = M_n$$

thus M_n is a martingale.

The construction of total gain would look like:

$$M_n = \sum_n \big(\kappa^{10}(\kappa \mathbf{1}_{\text{PROBABILITY}} - 1) + \kappa^9(\kappa \mathbf{1}_{\text{PROBABILIT}} - 1) + \cdots + \kappa^1(\kappa \mathbf{1}_{\text{PR}} - 1) + \kappa^0(\kappa \mathbf{1}_{\text{P}} - 1)\big)$$

10.(b)

We see that $\hat{\tau}$ is equivalent to that $\hat{\tau} := \inf\{n \geq 11 : \text{gain of a gamblers } = \kappa^{11}\}$ (i.e. when the gambler win 11 conserсutive games, he gets the word 'PROBABILITY'). WHEN The gambler wins 11 consercutive games, there are other $\hat{\tau} - 1$ gamblers losing. And as we mentioned earlier, we have M_n being a martingale, thus by Doob's optional stopping theorem we have

$$0 = \mathbb{E}[M_0] = \mathbb{E}[M_{\hat{\tau}}] = \mathbb{E}[\mathbb{E}[M_k|\tau = k]] = \sum_{k=0}^{10}(\kappa^{k+1} - \kappa^k) - (\mathbb{E}[\hat{\tau}] - 1)$$

And it gives that

$$\mathbb{E}[\hat{\tau}] = \kappa^{11} = 26^{11}$$

For 'ABRACADABRA' = 'ABRA' + 'CAD' + 'ABRA', we have that

$$\begin{aligned}0 =& \mathbb{E}[M_0] = \sum_{k=0}^{10}(\kappa^{k+1} - \kappa^k) + \sum_{k=0}^{3}(\kappa^{k+1} - \kappa^k) - (\mathbb{E}[\tilde{\tau}] - 2) = \kappa^{11} + \kappa^4 - \mathbb{E}[\tilde{\tau}]\\ \Rightarrow \mathbb{E}[\tilde{\tau}] =& \kappa^{11} + \kappa^4 = 26^{11} + 26^4\end{aligned}$$

where $+\kappa^4$ for that, when reaching 'ABRACADABRA', there is also another gambler who was winning at the first 4 letters 'ABRA'.