

STAT 430-2 2025 Winter

HW1

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Exercise 1 4.2.13

Note that we can write:

$$kX \sim \text{Unif}(0, k) \stackrel{d}{=} \text{Unif}\{0, 1, \dots, k-1\} + \text{Unif}(0, 1) := N + U \quad N \perp\!\!\!\perp U$$

And we have

$$\begin{aligned} \mathbb{E}[X|Y] &= \frac{1}{k} \mathbb{E}[kX | kX - [kX]] \\ &= \frac{1}{k} \mathbb{E}[N + U | U] \\ &= \frac{1}{k} (U + \mathbb{E}[N]) \\ &= \frac{1}{k} (Y + \frac{k-1}{2}) \end{aligned}$$

Exercise 2 4.2.14

We can write

$$\begin{aligned} \mathbb{E}[X|Y] &= \mathbb{E}[X \mathbf{1}_{X \leq t} + X \mathbf{1}_{X > t} | \max(X, t)] \\ &= \mathbb{E}[X \mathbf{1}_{\max(X, t)=t} + X \mathbf{1}_{\max(X, t)=X} | \max(X, t)] \\ &= \mathbf{1}_{Y=t} \mathbb{E}[X | X \leq t] + Y \mathbf{1}_{Y > t} \end{aligned}$$

The other side of $Z = \min(X, t)$ is symmetric to the above, so we have

$$\mathbb{E}[X|Z] = \mathbf{1}_{Z=t} \mathbb{E}[X | X \geq t] + Z \mathbf{1}_{Z < t}$$

Exercise 3 4.2.15

3.(a)

We can directly write

$$\begin{aligned} Z|\theta = 1 &= [X, Y] \stackrel{d}{=} P \times P \\ Z|\theta = 0 &= [Y, X] \stackrel{d}{=} P \times P \end{aligned}$$

where $X, Y \stackrel{i.i.d.}{\sim} P$. i.e. $Z \perp\!\!\!\perp \theta$.

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3.(b)

We have

$$\begin{aligned}
 \mathbb{E}[g(X, Y)|Z] &= \mathbb{E}[\mathbb{E}[g(X, Y)|\theta, Z]|\theta] \\
 &= \mathbb{P}(\theta = 1) \mathbb{E}[g(X, Y)|\theta = 1, Z] + \mathbb{P}(\theta = 0) \mathbb{E}[g(X, Y)|\theta = 0, Z] \\
 &= pg(Z_1, Z_2) + (1 - p)g(Z_2, Z_1)
 \end{aligned}$$

Exercise 4 4.2.16

4.(a)

Note that from the property of conditional variance:

$$\mathbb{E}[X|\mathcal{G}] = \arg \min_{g \in \mathcal{G}} \mathbb{E}[(X - g)^2]$$

Then we have for $\mathcal{G}_1 \subseteq \mathcal{G}_2$:

$$\begin{aligned}
 \mathbb{E}[var(X|\mathcal{G}_2)] &= \mathbb{E}\left[\arg \min_{g \in \mathcal{G}_2} \mathbb{E}[(X - g)^2]\right] \\
 &\leq \mathbb{E}\left[\arg \min_{g \in \mathcal{G}_1} \mathbb{E}[(X - g)^2]\right] \\
 &= \mathbb{E}[var(X|\mathcal{G}_1)]
 \end{aligned}$$

4.(b)

We have

$$\begin{aligned}
 var(X) &= \mathbb{E}[\mathbb{E}[(X - \mathbb{E}[X])^2|\mathcal{G}]] \\
 &= \mathbb{E}[\mathbb{E}[(X - \mathbb{E}[X|\mathcal{G}] + \mathbb{E}[X|\mathcal{G}] - \mathbb{E}[X])^2|\mathcal{G}]] \\
 &= \mathbb{E}[\mathbb{E}[(X - \mathbb{E}[X|\mathcal{G}])^2|\mathcal{G}]] + \mathbb{E}[(\mathbb{E}[X|\mathcal{G}] - \mathbb{E}[X])^2|\mathcal{G}] \\
 &= \mathbb{E}[var(X|\mathcal{G})] + var(\mathbb{E}[X|\mathcal{G}])
 \end{aligned}$$

Exercise 5 4.2.17

5.(a)

We have

$$\begin{aligned}
\mathbb{E}[X] &= \mathbb{E} \left[\mathbb{E} \left[\sum_{i=1}^N \xi_i \mid N \right] \right] \\
&= \sum_{j=1}^{\infty} \mathbb{P}(N = j) \sum_{i=1}^j \mathbb{E}[X_i] \\
&= \sum_{i=1}^{\infty} \sum_{j=i}^{\infty} \mathbb{P}(N = j) \mathbb{E}[\xi_i] \\
&= \sum_{i=1}^{\infty} \mathbb{E}[\xi_i] \sum_{j=i}^{\infty} \mathbb{P}(N = j) < \infty
\end{aligned}$$

5.(b)

We have

$$\begin{aligned}
\mathbb{E}[X^2] &= \mathbb{E} \left[\mathbb{E} \left[\left(\sum_{i=1}^N \xi_i \right)^2 \mid N \right] \right] \\
&= \sum_{j=1}^{\infty} \mathbb{P}(N = j) (j \cdot \text{var}(\xi_i) + 2j^2 \mathbb{E}[\xi_i]^2) \\
&= \sum_{i=1}^{\infty} \sum_{j=i}^{\infty} \mathbb{P}(N = j) (\text{var}(\xi_i) + 2j \mathbb{E}[\xi_i]^2) \\
&= \text{var}(\xi_i) \mathbb{E}[N] + \mathbb{E}[\xi_i]^2 \mathbb{E}[N^2] < \infty \\
\Rightarrow \text{var}(X) &= \mathbb{E}[X^2] - \mathbb{E}[X]^2 \\
&= \text{var}(\xi_i) \mathbb{E}[N] + \mathbb{E}[\xi_i]^2 \mathbb{E}[N^2] - \mathbb{E}[\xi_i]^2 \mathbb{E}[N]^2 \\
&= \text{var}(\xi_i) \mathbb{E}[N] + \mathbb{E}[\xi_i]^2 \text{var}(N)
\end{aligned}$$

Exercise 6 4.2.22

6.(a)

We have for $\mathcal{G} \subseteq \mathcal{F}$:

$$\begin{aligned}
\mathbb{E}[|X|^p \mid \mathcal{G}] &= \int |x|^p \, d\mathbb{P}(|X| \mid \mathcal{G}) \\
&\stackrel{\text{integration by parts}}{=} \int p |x|^{p-1} \mathbb{P}(|X| > x \mid \mathcal{G}) \, dx
\end{aligned}$$

6.(b)

Further we have

$$\begin{aligned}
 \mathbb{E}[|X|^p | \mathcal{G}] &= \int p |x|^{p-1} \mathbb{P}(|X| > x | \mathcal{G}) \, dx \\
 &= \int_0^a + \int_a^\infty p |x|^{p-1} \mathbb{P}(|X| > x | \mathcal{G}) \, dx \\
 &\geq \int_0^a p |x|^{p-1} \mathbb{P}(|X| > a | \mathcal{G}) \, dx \\
 &= \mathbb{P}(|X| > a | \mathcal{G}) a^p \\
 \Rightarrow \mathbb{P}(|X| > a | \mathcal{G}) &\leq a^{-p} \mathbb{E}[|X|^p | \mathcal{G}]
 \end{aligned}$$

Exercise 7 4.2.23

Using similar argument as Proposition 1.3.17 in textbook, we have by Cauchy-Schwarz inequality:

$$\begin{aligned}
 1 &= \frac{1}{p} + \frac{1}{q} = \frac{\|X\|_p^p}{p \|X\|_p^p} + \frac{\|Y\|_q^q}{q \|X\|_q^q} \\
 &\geq \frac{\mathbb{E}[|XY|]}{\|X\|_p \|Y\|_q} \\
 \Rightarrow \mathbb{E}[|XY|] &\leq \|X\|_p \|Y\|_q
 \end{aligned}$$

Exercise 8 4.3.13

8.(a)

As we stated previously in 4.2.16, for any given N ,

$$\mathbb{E}[(\mathbb{E}[X | \mathcal{G}_m] - \mathbb{E}[X | \mathcal{G}_N])^2], \quad m \geq N$$

is a increasing function in m (because $\mathbb{E}[(X - \mathbb{E}[X | \mathcal{G}_m])^2]$ is decreasing). On the other hand since $X \in L^2(\mathcal{F})$, this quantity is bounded. The above means that $\mathbb{E}[(\mathbb{E}[X | \mathcal{G}_m] - \mathbb{E}[X | \mathcal{G}_N])^2]$ has some limit, i.e.

$$\mathbb{E}[(\mathbb{E}[X | \mathcal{G}_m] - \mathbb{E}[X | \mathcal{G}_n])^2] \rightarrow 0, \quad m, n \rightarrow \infty$$

i.e. it's a cauchy sequence.

8.(b)

For Hilbert space $L^2(\mathcal{G})$, we see that $\exists h$ s.t. the cauchy sequence $\mathbb{E}[X | \mathcal{G}_n]$ converges to h in $L^2(\mathcal{F})$. And by the Orthogonal Projection Theorem, we have this h being the unique minimizer that reaches $\min_{g \in \mathcal{G}} \mathbb{E}[(X - g)^2 | \mathcal{G}]$. i.e.

$$\mathbb{E}[X | \mathcal{G}_m] \rightarrow h = \mathbb{E}[X | \mathcal{G}]$$

Exercise 9 4.4.8

9.(a)

We have:

$$\mathbb{E}[\xi_1|S_n] = \dots = \mathbb{E}[\xi_n|S_n]$$

thus

$$\mathbb{E}[\xi_1|S_n] = n^{-1} \mathbb{E}\left[\sum_{i=1}^n \xi_i|S_n\right] = n^{-1} S_n$$

9.(b)

Consider transformation $(\xi_1, \xi_2) \mapsto (\xi_1, u) = (\xi_1, \xi_1 + \xi_2)$, which gives

$$\begin{aligned} f_{\xi_1, u}(\xi_1, u) &= \lambda^2 e^{-\lambda u} \mathbf{1}_{\xi_1 > 0, u > \xi_1} \\ f_u(u) &= \lambda^2 u e^{-\lambda u} \mathbf{1}_{u > 0} \end{aligned}$$

And we have

$$f(\xi_1|u) = \frac{1}{u} \mathbf{1}_{\xi_1 > 0, u > \xi_1} \sim \text{Unif}(0, u)$$

i.e.

$$\mathbb{P}(\xi_1 \leq b|S_2) = \frac{b}{S_2} \mathbf{1}_{0 < b < S_2}$$

Exercise 10 4.4.10

Note that we have

$$\text{cov}\left(X - \frac{\mathbb{E}[XY]}{\mathbb{E}[Y^2]}Y, Y\right) = 0$$

and both $X - \frac{\mathbb{E}[XY]}{\mathbb{E}[Y^2]}Y$ and Y are gaussian, thus are independent. Now we have

$$\begin{aligned} \mathbb{E}[X|Y] &= \mathbb{E}\left[X - \frac{\mathbb{E}[XY]}{\mathbb{E}[Y^2]}Y + \frac{\mathbb{E}[XY]}{\mathbb{E}[Y^2]}Y|Y\right] \\ &= 0 + \frac{\mathbb{E}[XY]}{\mathbb{E}[Y^2]}Y = \rho Y \end{aligned}$$