STAT 430-2 2025 Winter
HW1

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Exercise 1 4.2.13

Note that we can write:

$$kX \sim \mathrm{Unif}(0,k) \overset{\mathrm{d}}{=} \mathrm{Unif}\{0,1,\dots,k-1\} + \mathrm{Unif}(0,1) := N + U \quad N \perp\!\!\!\perp U$$

And we have

$$\begin{aligned}\mathbb{E}\left[X|Y\right] =& \frac{1}{k}\mathbb{E}\left[kX|kX-[kX]\right]\\ =& \frac{1}{k}\mathbb{E}\left[N+U|U\right]\\ =& \frac{1}{k}(U+\mathbb{E}\left[N\right])\\ =& \frac{1}{k}(Y+\frac{k-1}{2})\end{aligned}$$

Exercise 2 4.2.14

We can write

$$\begin{aligned}\mathbb{E}\left[X|Y\right] =& \mathbb{E}\left[X\mathbf{1}_{X\leq t} + X\mathbf{1}_{X<t}|\max(X,t)\right]\\ =& \mathbb{E}\left[X\mathbf{1}_{\max(X,t)=t} + X\mathbf{1}_{\max(X,t)=X}|\max(X,t)\right]\\ =& \mathbf{1}_{Y=t}\mathbb{E}\left[X|X\leq t\right] + Y\mathbf{1}_{Y>t}\end{aligned}$$

The other side of $Z=\min(X,t)$ is symmetric to the above, so we have

$$\mathbb{E}\left[X|Z\right] = \mathbf{1}_{Z=t}\mathbb{E}\left[X|X\geq t\right] + Z\mathbf{1}_{Z<t}$$

Exercise 3 4.2.15

3.(a)

We can directly write

$$\begin{aligned}Z|\theta=1 =& [X,Y] \overset{\mathrm{d}}{=} P\times P\\ Z|\theta=0 =& [Y,X] \overset{\mathrm{d}}{=} P\times P\end{aligned}$$

where $X,Y \overset{i.i.d.}{\sim} P$. i.e. $Z \perp\!\!\!\perp \theta$.

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3.(b)

We have

$$\begin{aligned}\mathbb{E}\left[g(X,Y)|Z\right] =&\mathbb{E}\left[\mathbb{E}\left[g(X,Y)|\theta,Z\right]|\theta\right]\\ =&\mathbb{P}\left(\theta=1\right)\mathbb{E}\left[g(X,Y)|\theta=1,Z\right]+\mathbb{P}\left(\theta=0\right)\mathbb{E}\left[g(X,Y)|\theta=0,Z\right]\\ =&pg(Z_1,Z_2)+(1-p)g(Z_2,Z_1)\end{aligned}$$

Exercise 4 4.2.16

4.(a)

Note that from the property of conditional variance:

$$\mathbb{E}\left[X|\mathcal{G}\right]=\arg\min_{g\in\mathcal{G}}\mathbb{E}\left[(X-g)^2\right]$$

Then we have for $\mathcal{G}_1\subseteq\mathcal{G}_2$:

$$\begin{aligned}\mathbb{E}\left[var(X|\mathcal{G}_2)\right]=&\mathbb{E}\left[\arg\min_{g\in\mathcal{G}_2}\mathbb{E}\left[(X-g)^2\right]\right]\\ \leq&\mathbb{E}\left[\arg\min_{g\in\mathcal{G}_1}\mathbb{E}\left[(X-g)^2\right]\right]\\ =&\mathbb{E}\left[var(X|\mathcal{G}_1)\right]\end{aligned}$$

4.(b)

We have

$$\begin{aligned}var(X)=&\mathbb{E}\left[\mathbb{E}\left[(X-\mathbb{E}\left[X\right])^2|\mathcal{G}\right]\right]\\ =&\mathbb{E}\left[\mathbb{E}\left[(X-\mathbb{E}\left[X|\mathcal{G}\right]+\mathbb{E}\left[X|\mathcal{G}\right]-\mathbb{E}\left[X\right])^2|\mathcal{G}\right]\right]\\ =&\mathbb{E}\left[\mathbb{E}\left[(X-\mathbb{E}\left[X|\mathcal{G}\right])^2|\mathcal{G}\right]\right]+\mathbb{E}\left[(\mathbb{E}\left[X|\mathcal{G}\right]-\mathbb{E}\left[X\right])^2|\mathcal{G}\right]\\ =&\mathbb{E}\left[var(X|\mathcal{G})\right]+var(\mathbb{E}\left[X|\mathcal{G}\right])\end{aligned}$$

Exercise 5 4.2.17

5.(a)

We have

$$\begin{aligned}
\mathbb{E}\left[X\right] =&\mathbb{E}\left[\mathbb{E}\left[\sum_{i=1}^{N}\xi_i|N\right]\right]\\
=&\sum_{j=1}^{\infty}\mathbb{P}\left(N=j\right)\sum_{i=1}^{j}\mathbb{E}\left[X_i\right]\\
=&\sum_{i=1}^{\infty}\sum_{j=i}^{\infty}\mathbb{P}\left(N=j\right)\mathbb{E}\left[\xi_i\right]\\
=&\sum_{i=1}^{\infty}\mathbb{E}\left[\xi_i\right]\sum_{j=i}^{\infty}\mathbb{P}\left(N=j\right)<\infty
\end{aligned}$$

5.(b)

We have

$$\begin{aligned}
\mathbb{E}\left[X^2\right] =&\mathbb{E}\left[\mathbb{E}\left[\left(\sum_{i=1}^{N}\xi_i\right)^2|N\right]\right]\\
=&\sum_{j=1}^{\infty}\mathbb{P}\left(N=j\right)\left(j\cdot var(\xi_i)+2j^2\mathbb{E}\left[\xi_i\right]^2\right)\\
=&\sum_{i=1}^{\infty}\sum_{j=i}^{\infty}\mathbb{P}\left(N=j\right)\left(var(\xi_i)+2j\mathbb{E}\left[\xi_i\right]^2\right)\\
=&var(\xi_i)\mathbb{E}\left[N\right]+\mathbb{E}\left[\xi_i\right]^2\mathbb{E}\left[N^2\right]<\infty\\
\Rightarrow var(X) =&\mathbb{E}\left[X^2\right]-\mathbb{E}\left[X\right]^2\\
=&var(\xi_i)\mathbb{E}\left[N\right]+\mathbb{E}\left[\xi_i\right]^2\mathbb{E}\left[N^2\right]-\mathbb{E}\left[\xi_i\right]^2\mathbb{E}\left[N\right]^2\\
=&var(\xi_i)\mathbb{E}\left[N\right]+\mathbb{E}\left[\xi_i\right]^2 var(N)
\end{aligned}$$

Exercise 6 4.2.22

6.(a)

We have for $\mathcal{G}\subseteq\mathcal{F}$:

$$\begin{aligned}
\mathbb{E}\left[\left|X\right|^p|\mathcal{G}\right] =&\int\left|x\right|^p\,\mathrm{d}\mathbb{P}\left(\left|X\right||\mathcal{G}\right)\\
\overset{\text{integration by parts}}{=}&\int p\left|x\right|^{p-1}\mathbb{P}\left(\left|X\right|>x|\mathcal{G}\right)\,\mathrm{d}x
\end{aligned}$$

6.(b)

Further we have

$$
\begin{aligned}
\mathbb{E}\left[|X|^p\,|\mathcal{G}\right] =& \int p\,|x|^{p-1}\,\mathbb{P}\left(|X|>x|\mathcal{G}\right)\,\mathrm{d}x\\
=& \int_0^a + \int_a^\infty p\,|x|^{p-1}\,\mathbb{P}\left(|X|>x|\mathcal{G}\right)\,\mathrm{d}x\\
\geq& \int_0^a p\,|x|^{p-1}\,\mathbb{P}\left(|X|>a|\mathcal{G}\right)\,\mathrm{d}x\\
=& \mathbb{P}\left(|X|>a|\mathcal{G}\right)a^p\\
\Rightarrow \mathbb{P}\left(|X|>a|\mathcal{G}\right) \leq& a^{-p}\mathbb{E}\left[|X|^p\,|\mathcal{G}\right]
\end{aligned}
$$

Exercise 7 4.2.23

Using similar argument as Proposition 1.3.17 in textbook, we have by Cauchy-Schwarz inequality:

$$
\begin{aligned}
1 = \frac{1}{p} + \frac{1}{q} =& \frac{\|X\|_p^p}{p\,\|X\|_p^p} + \frac{\|Y\|_q^q}{q\,\|X\|_q^q}\\
\geq& \frac{\mathbb{E}\left[|XY|\right]}{\|X\|_p\,\|Y\|_q}\\
\Rightarrow \mathbb{E}\left[|XY|\right] \leq& \|X\|_p\,\|Y\|_q
\end{aligned}
$$

Exercise 8 4.3.13

8.(a)

As we stated previously in 4.2.16, for any given N,

$$
\mathbb{E}\left[\left(\mathbb{E}\left[X|\mathcal{G}_m\right]-\mathbb{E}\left[X|\mathcal{G}_N\right]\right)^2\right],\quad m\geq N
$$

is a increasing function in m (because $\mathbb{E}\left[(X-\mathbb{E}\left[X|\mathcal{G}_m\right])^2\right]$ is decreasing). On the other hand since $X\in L^2(\mathcal{F})$, this quantity is bounded. The above means that $\mathbb{E}\left[\left(\mathbb{E}\left[X|\mathcal{G}_m\right]-\mathbb{E}\left[X|\mathcal{G}_N\right]\right)^2\right]$ has some limit, i.e.

$$
\mathbb{E}\left[\left(\mathbb{E}\left[X|\mathcal{G}_m\right]-\mathbb{E}\left[X|\mathcal{G}_n\right]\right)^2\right]\to 0,\quad m,n\to\infty
$$

i.e. it's a cauchy sequence.

8.(b)

For Hilbert space $L^2(\mathcal{G})$, we see that $\exists h$ s.t. the cauchy sequence $\mathbb{E}\left[X|\mathcal{G}_n\right]$ converges to h in $L^2(\mathcal{F})$. And by the Orthogonal Projection Theorem, we have this h being the unique minimizer that reaches $\min_{g\in\mathcal{G}}\mathbb{E}\left[(X-g)^2|\mathcal{G}\right]$. i.e.

$$
\mathbb{E}\left[X|\mathcal{G}_m\right]\to h=\mathbb{E}\left[X|\mathcal{G}\right]
$$

Exercise 9 4.4.8

9.(a)

We have:

$$\mathbb{E}\left[\xi_1|S_n\right] = \ldots = \mathbb{E}\left[\xi_n|S_n\right]$$

thus

$$\mathbb{E}\left[\xi_1|S_n\right] = n^{-1}\mathbb{E}\left[\sum_{i=1}^{n}\xi_i|S_n\right] = n^{-1}S_n$$

9.(b)

Consider transformation $(\xi_1,\xi_2) \mapsto (\xi_1,u) = (\xi_1,\xi_1+\xi_2)$, which gives

$$\begin{aligned} f_{\xi_1,u}(\xi_1,u) =& \lambda^2 e^{-\lambda u}\nVdash_{\xi_1>0,u>\xi_1} \\ f_u(u) =& \lambda^2 u e^{-\lambda u}\nVdash_{u>0} \end{aligned}$$

And we have

$$f(\xi_1|u) = \frac{1}{u}\mathbf{1}_{\xi_1>0,u>\xi_1} \sim \mathrm{Unif}(0,u)$$

i.e.

$$\mathbb{P}\left(\xi_1 \leq b|S_2\right) = \frac{b}{S_2}\nVdash_{0<b<S_2}$$

Exercise 10 4.4.10

Note that we have

$$cov(X - \frac{\mathbb{E}\left[XY\right]}{\mathbb{E}\left[Y^2\right]}Y, Y) = 0$$

and both $X - \frac{\mathbb{E}\left[XY\right]}{\mathbb{E}\left[Y^2\right]}Y$ and Y are gaussian, thus are independent. Now we have

$$\begin{aligned} \mathbb{E}\left[X|Y\right] =& \mathbb{E}\left[X - \frac{\mathbb{E}\left[XY\right]}{\mathbb{E}\left[Y^2\right]}Y + \frac{\mathbb{E}\left[XY\right]}{\mathbb{E}\left[Y^2\right]}Y|Y\right] \\ =& 0 + \frac{\mathbb{E}\left[XY\right]}{\mathbb{E}\left[Y^2\right]}Y = \rho Y \end{aligned}$$