STAT 430-2 2025 Winter
HW4

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Exercise 1 Exercise 6.1.14

1.(a)

We have

$$\begin{aligned}\mathbb{P}\left(X_{n+1}=1|X_1^n\right) =&\mathbb{E}\left[\mathbf{1}_{X_{n+1}=1}|X_1^n\right]\\ =&\mathbb{E}_\theta\left[\mathbb{E}\left[\mathbf{1}_{X_{n+1}=1}|X_1^n,\theta\right]\right]\\ =&\mathbb{E}_\theta\left[1-\theta\right]\\ =&1/2\end{aligned}$$

1.(b)

We have

$$\begin{aligned}\mathbb{P}\left(S_{n+1}=s|S_1^n\right) =&\mathbb{P}\left(S_{n+1}=s|X_1^n\right)\\ =&\mathbb{P}\left(X_{n+1}=s-\sum_{i=1}^n X_i|X_1^n\right)\\ =&\mathbb{P}\left(X_{n+1}=s-S_n|S_n\right)\end{aligned}$$

from the previous part. Thus we have S_n being Markov.

Exercise 2 Exercise 6.1.18

2.(a)

By property of sets we have $\Gamma_n\to\Gamma$ thus $\mathbf{1}_{\Gamma_n}\xrightarrow{\text{a.s.}}\mathbf{1}_\Gamma$, then by Lévy's upward theorem we have:

$$\begin{aligned}\mathbb{P}\left(\Gamma_n|X_n\right) =&\mathbb{E}\left[\mathbf{1}_{\Gamma_n}|X_n\right]\\ =&\mathbb{E}\left[\mathbf{1}_{\Gamma_n}|\mathcal{F}_n\right]\\ \xrightarrow[L_1]{\text{a.s.}}&\mathbb{E}\left[\mathbf{1}_\Gamma|\mathcal{F}_\infty\right]\\ =&\mathbf{1}_\Gamma\end{aligned}$$

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2.(b)

Denote $K := \{\omega : X_n(\omega) \in A_n \, i.o.\}$. Then we have that $\forall N > 0$, $\exists n > N$ s.t. $\mathbb{P}\left(\Gamma_n \cap K | X_n\right) \geq \eta > 0$. On the other hand we have

$$\eta < \mathbb{P}\left(\Gamma_n \cap K | X_n\right) \xrightarrow{\text{a.s.}} \mathbb{P}\left(\Gamma \cap K | X_\infty\right) = \mathbf{1}_{\Gamma \cap K} = 1 = \mathbb{P}\left(\Gamma \cap K\right)$$

which gives $\mathbb{P}\left(K \backslash \Gamma\right) = 0$. In the above we applied Lévy's upward theorem to $\Gamma_n \cap K | X_n$.

2.(c)

Use $A_n \equiv A$ and $B_n \equiv B$ and we have using the precedence:

$$\begin{aligned}
1 \leq & \mathbb{P}\left(\{X_n \in A \text{ finitely often}\} \cup \left(\{X_n \in A \, i.o.\} \backslash \Gamma\right) \cup \Gamma\right) \\
\leq & \mathbb{P}\left(\{X_n \in A \text{ finitely often}\} \cup \Gamma\right) + \mathbb{P}\left(\{X_n \in A \, i.o.\} \backslash \Gamma\right) \\
= & \mathbb{P}\left(\{X_n \in A \text{ finitely often}\} \cup \Gamma\right) + 0 \\
= & \mathbb{P}\left(\{X_n \in A \text{ finitely often}\}\right) + \mathbb{P}\left(\Gamma\right)
\end{aligned}$$

where $\Gamma = \{X_n \in B \, i.o.\}$ so thus we have proved the claim.

Exercise 3 Exercise 6.1.19

We prove the result for symmetric SRW directly. Denote $\tau = \inf\{k : \omega_k \geq b\}$ and

$$h_k(\omega) = \sum_{i=-\infty}^{+\infty} \mathbf{1}_{\omega_{n-k} = b+i}, \quad k \in [n]$$

and by Strong Markov Property (SMP) we have

$$\begin{aligned}
\mathbb{P}\left(\max_{k \leq n} \omega_k \geq b\right) = & \mathbb{E}\left[\mathbf{1}_{\tau \leq n}\right] \\
= & \mathbb{E}\left[\mathbf{1}_{\tau \leq n} \sum_{\imath=-\infty}^{+\infty} \mathbf{1}_{\omega_n = \imath}\right] \\
= & \mathbb{E}\left[\mathbf{1}_{\tau \leq n} \mathbb{E}\left[\sum_{i=-\infty}^{+\infty} \mathbf{1}_{\theta^{n \wedge \tau} \omega_{n - n\wedge\tau} = b+i} | \mathcal{F}_{n \wedge \tau}\right]\right] \\
= & \mathbb{E}\left[\mathbf{1}_{\tau \leq n} \mathbb{E}\left[h_{n\wedge\tau}(\theta^{n\wedge\tau}\omega) | \mathcal{F}_{n\wedge\tau}\right]\right] \\
\overset{\text{SMP}}{=} & \mathbb{E}\left[\mathbf{1}_{\tau \leq n} \mathbb{E}_{X_{n\wedge\tau}}\left[h_{n\wedge\tau}(\omega)\right]\right] \\
= & \mathbb{E}\left[\mathbf{1}_{\tau \leq n} \mathbb{E}_{X_{n\wedge\tau}}\left[\mathbf{1}_{\omega_{n-n\wedge\tau}=0} + 2\sum_{i=1}^{\infty} \mathbf{1}_{\omega_{n-n\wedge\tau}=i}\right]\right] \\
= & \mathbb{E}\left[\mathbf{1}_{\omega_n = b}\right] + 2\mathbb{E}\left[\sum_{i=1}^{\infty} \mathbf{1}_{\omega_n = b+i}\right] \\
= & \mathbb{P}\left(\omega_n = b\right) + 2\mathbb{P}\left(\omega_n > b\right)
\end{aligned}$$

Exercise 4 Exercise 6.2.2

4.(a)

We use $h_r = \mathbf{1}_{\omega_{n-r}\in B}$ and apply for stopping time $T_{y,r}$ the SMP to obtain that

$$\begin{aligned}\mathbb{P}_x\left(X_n \in B, T_{y,r} \le n\right) =& \mathbb{E}\left[\mathbf{1}_{X_n\in B}\mathbf{1}_{T_{y,r}\le n}\right]\\ =& \mathbb{E}\left[\mathbf{1}_{T_{y,r}\le n}\mathbb{E}\left[h_{T_{y,r}}(\theta^{T_{y,r}}\omega)|\mathcal{F}_{T_{y,r}}\right]\right]\\ \overset{\text{SMP}}{=}& \mathbb{E}\left[\mathbf{1}_{T_{y,r}\le n}\mathbb{E}_y\left[h_{T_{y,r}}(\omega)\right]\right]\\ =& \sum_{k=0}^{n-r}\mathbb{P}_x\left(T_{y,r}=n-k\right)\mathbb{P}_y\left(X_{n-k}\in B\right)\end{aligned}$$

4.(b)

Making relabeling $k \mapsto n-k$ and $B=\{y\}$ and we have

$$\mathbb{P}_x\left(X_n=y\right) = \sum_{k=r}^{n}\mathbb{P}_x\left(T_{y,r}=k\right)\mathbb{P}_y\left(X_{n-k}=y\right)$$

4.(c)

We have

$$\begin{aligned}\text{R.H.S.} &= \sum_{n=r}^{l+r}\mathbb{P}_y\left(X_n=y\right)\\ &= \sum_{n=r}^{l+r}\sum_{k=r}^{n}\mathbb{P}_y\left(T_{y,r}=k\right)\mathbb{P}_y\left(X_{n-k}=y\right)\\ &= \sum_{k=r}^{l+r}\mathbb{P}_y\left(T_{y,r}=k\right)\sum_{n=k}^{l+r}\mathbb{P}_y\left(X_{n-k}=y\right)\\ &\le \sum_{k=r}^{l+r}\mathbb{P}_y\left(T_{y,r}=k\right)\sum_{n=k}^{l+k}\mathbb{P}_y\left(X_{n-k}=y\right)\\ &= \sum_{k=r}^{l+r}\mathbb{P}_y\left(T_{y,r}=k\right)\sum_{j=0}^{l}\mathbb{P}_y\left(X_j=y\right)\\ &\le \sum_{j=0}^{l}\mathbb{P}_y\left(X_j=y\right)\end{aligned}$$

Exercise 5 Exercise 6.2.5

5.(a)

Since $\mathbb{P}_{x\notin C}\left(\tau_C<\infty\right)>0$, we know that $\exists N>0$ and some ε s.t.

$$\mathbb{P}_{x\notin C}\left(\tau_C<N\right)>\varepsilon$$

and thus we have

$$\mathbb{P}_{x\notin C}\left(\tau_C \geq N\right) \leq 1-\varepsilon$$

Consider applying the SMP to $h := \mathbf{1}_{\tau>N}$ we have

$$\begin{aligned}
\mathbb{E}\left[\mathbf{1}_{\tau>(k+1)N}\right] =&\mathbb{E}\left[\mathbf{1}_{\tau>kN}\mathbb{E}\left[h(\theta^{kN}\omega)|\mathcal{F}_{kN}\right]\right]\\
=&\mathbb{E}\left[\mathbf{1}_{\tau>kN}\mathbb{E}_{x\notin C}\left[h(\omega)\right]\right]\\
=&\mathbb{E}\left[\mathbf{1}_{\tau>kN}\mathbb{P}_{x\notin C}\left(\tau_C\geq N\right)\right]\\
\leq&\mathbb{E}\left[\mathbf{1}_{\tau>kN}(1-\varepsilon)\right]\\
\ldots\leq& (1-\varepsilon)^k\mathbb{E}_y\left[\mathbf{1}_{\tau>N}\right]\\
\leq&(1-\varepsilon)^k
\end{aligned}$$

5.(b)

By Borel-Cantelli lemma we have $\mathbb{P}\left(\tau_C<\infty\right)=1$ since

$$\sum_{k=1}^{\infty}\mathbb{P}_{x\notin C}\left(\tau_C\geq kN\right)<\infty$$

Then we have for any $x=X_0\notin C$ that:

$$\begin{aligned}
g(x) =&\mathbb{P}_{X_0=x}\left(\tau_A<\tau_B\right)\\
=&\mathbb{P}_{X_0}\left(X_{\tau_C}\in A\right)\\
=&\sum_{y\in\mathbb{S}}\mathbb{P}_{X_0}\left(X_1=y\right)\mathbb{P}_{X_1=y}\left(X_{\tau_C}\in A\right)\\
=&\sum_{y\in\mathbb{S}}p(x,y)g(y)
\end{aligned}$$

thus $g(\,\cdot\,)$ is harmonic on $\mathbb{S}\backslash C$.

5.(c)

Note that we have $X_{n\wedge\tau_C-1}\notin C$

By the harmonic proporty we have that

$$\begin{aligned}
\mathbb{E}\left[g(X_{n\wedge\tau_C+1})|\mathcal{F}_n\right] =&\mathbb{E}\left[g(X_{n\wedge\tau_C+1})(\mathbf{1}_{\tau_C\leq n}+\mathbf{1}_{\tau_C>n})|\mathcal{F}_n\right]\\
=&g(X_{n\wedge\tau_C})\mathbf{1}_{\tau_C\leq n}+\mathbb{E}\left[g(X_{n\wedge\tau_C+1})\mathbf{1}_{\tau_C>n}|\mathcal{F}_n\right]\\
=&g(X_{n\wedge\tau_C})\mathbf{1}_{\tau_C\leq n}+\mathbb{E}\left[g(\theta^1X_{n\wedge\tau_C})\mathbf{1}_{\tau_C>n}|\mathcal{F}_n\right]\\
\overset{\text{SMP}}{=}&g(X_{n\wedge\tau_C})\mathbf{1}_{\tau_C\leq n}+\mathbb{E}_{X_{n\wedge\tau_C}=x\notin C}\left[g(X_{n\wedge\tau_C+1})\right]\mathbf{1}_{\tau_C>n}\\
=&g(X_{n\wedge\tau_C})\mathbf{1}_{\tau_C\leq n}+g(X_{n\wedge\tau_C})\mathbf{1}_{\tau_C>n}\\
=&g(X_{n\wedge\tau_C})
\end{aligned}$$

thus we have $g(X_{n\wedge\tau_C})$ is a M.G.

5.(d)

We have for such M.G. $g(X_{n\wedge\tau_C})$ with $X_0 = x$ that:

$$\begin{aligned} g(x) =& \mathbb{E}\left[g(X_{0\wedge\tau_C})\right] \\ =& \mathbb{E}\left[g(X_{n\wedge\tau_C})\right] \\ =& \mathbb{E}\left[g(X_{\tau_C})\right] \\ =& \mathbb{E}\left[\mathbf{1}_{X_{\tau_C}\in A}\right] \\ =& \mathbb{P}_x\left(\tau_A < \tau_B\right) \end{aligned}$$

combine with previous results that $\mathbb{P}_x\left(\tau_A < \tau_B\right)$ satisfies the condition of bounded harmonic on $x \notin C$ we have that it's the only one that satisfies the condition.

5.(e)

Similar to (c) we prove the following:

$$\begin{aligned} \mathbb{E}\left[M_{n+1}|\mathcal{F}_n\right] =& \mathbb{E}\left[M_{n+1}\mathbf{1}_{\tau_C\leq n}|\mathcal{F}_n\right] + \mathbb{E}\left[M_{n+1}\mathbf{1}_{\tau_C>n}|\mathcal{F}_n\right] \\ =& M_n\mathbf{1}_{\tau_C\leq n} + \mathbb{E}\left[M_{n+1}\mathbf{1}_{\tau_C>n}|\mathcal{F}_n\right] \\ =& M_n\mathbf{1}_{\tau_C\leq n} + (n+1+\mathbb{E}\left[f(X_{n+1})|\mathcal{F}_n\right])\mathbf{1}_{\tau_C>n} \\ =& M_n\mathbf{1}_{\tau_C\leq n} + (n+1+\mathbb{E}_{X_n=x\notin C}\left[f(X_{n+1})\right])\mathbf{1}_{\tau_C>n} \\ =& M_n\mathbf{1}_{\tau_C\leq n} + (n+1+f(X_n)-1)\mathbf{1}_{\tau_C>n} \\ =& M_n \end{aligned}$$

thus M_n is a M.G. On the other hand we notice that

$$\begin{aligned} \mathbb{E}\left[M_n\right] =& \mathbb{E}\left[n\wedge\tau_C + f(X_{n\wedge\tau_C})\right] \\ \leq& \mathbb{E}\left[\tau_C\right] + \sum_{x\in\mathbb{S}} f(x)\mathbb{P}\left(X_{n\wedge\tau_C} = x\right) \\ \leq& \mathbb{E}\left[\tau_C\right] + \sum_{x\in\mathbb{S}\setminus C} f(x) + \sum_{x\in C} 0 < \infty \end{aligned}$$

then by DCT we have that

$$\begin{aligned} \mathbb{E}_x\left[\tau_C\right] =& \mathbb{E}\left[\lim_{n\to\infty} n\wedge\tau_C + f(X_{n\wedge\tau_C})\right] \\ =& \lim_{n\to\infty}\mathbb{E}\left[n\wedge\tau_C + f(X_{n\wedge\tau_C})\right] \\ =& \mathbb{E}_x\left[0 + f(X_{0\wedge\tau_C})\right] \\ =& f(x) \end{aligned}$$

Exercise 6 Exercise 6.2.8

6.(a)

We have that

$$\begin{aligned}
\rho_{xz} =& \mathbb{E}_x \left[\mathbf{1}_{T_z<\infty}\right] \\
\geq& \mathbb{E}_x \left[\mathbf{1}_{T_z<\infty}\mathbf{1}_{T_y<T_z}\right] \\
=& \mathbb{E}_x \left[\mathbf{1}_{T_y<T_z}\mathbb{E}\left[\theta^{T_y}\mathbf{1}_{T_z<\infty}|\mathcal{F}_{T_y}\right]\right] \\
\overset{\text{SMP}}{=}& \mathbb{E}_x \left[\mathbf{1}_{T_y<T_z}\mathbb{E}_y\left[\mathbf{1}_{T_z<\infty}\right]\right] \\
=& \rho_{xy}\rho_{yz} > 0
\end{aligned}$$

6.(b)

We have

- $x \leftrightarrow x$ by convention;
- $x \leftrightarrow y$ implies $y \leftrightarrow x$ by symmetry of the condition $\rho_{xy} > 0$, $\rho_{yx} > 0$;
- $x \leftrightarrow y$ and $y \leftrightarrow z$ implies $x \leftrightarrow z$ by the previous part.

6.(c)

- If C_1 leads to C_2 and C_2 leads to C_3, then there exists $x, y \in C_2$ s.t. $x \leftrightarrow y$ and thus for any $z \in C_1$ and $w \in C_3$ we have $z \to x \to y \to w$, thus $z \leftrightarrow w$, i.e. C_1 leads to C_3.
- If C_1 leads to C_2 and C_2 leads to C_1, then for any $x \in C_1$ and $y \in C_2$ we have $x \to y$ and $y \to x$, which turns out to be $x \leftrightarrow y$, which is a contradiction, so if C_1 leads to C_2, then C_2 cannot lead to C_1.

Exercise 7 Exercise 6.2.11

7.(a)

We have $\forall k$ and any bounded function g that

$$\begin{aligned}
\mathbb{E}_y \left[g(r_k, X_{R_{k-1}}, \dots X_{R_k-1})|\mathcal{F}_{R_{k-1}}\right] =& \mathbb{E}_y \left[\theta^{R_{k-1}}g(\omega)|\mathcal{F}_{R_{k-1}}\right] \\
\overset{\text{SMP}}{=}& \mathbb{E}_y \left[g(\omega)\right] \\
=& \mathbb{E}_y \left[g(r_0, X_0, \dots X_{R_0-1})\right]
\end{aligned}$$

and notice that $\mathcal{F}_{k-1} \supseteqq \sigma(r_1^k, X_1^{R_k-1})$ we have that $(r_k, X_{R_{k-1}}, \dots X_{R_k-1})$ are independent and identically distributed as $(r_0, X_0, \dots X_{R_0-1})$.

7.(b)

Similarly we have

$$\begin{aligned}\mathbb{E}_\nu\left[g(r_k, X_{R_{k-1}}, \dots X_{R_k-1})|\mathcal{F}_{R_{k-1}}\right] =&\mathbb{E}_\nu\left[\mathbb{E}\left[\theta^{R_{k-1}}g(\omega)|\mathcal{F}_{R_{k-1}}\right]|F_{R_1}\right]\\ =&\mathbb{E}_\nu\left[\mathbb{E}_y\left[\theta^{R_1}g(\omega)\right]|\mathcal{F}_{R_1}\right]\\ =&\mathbb{E}_y\left[g(r_2, X_{R_1}, \dots X_{R_2-1})\right]\end{aligned}$$

thus proved that $(r_k, X_{R_{k-1}}, \dots X_{R_k-1})$ are independent and identically distributed as $(r_1, X_{R_1}, \dots X_{R_2-1})$, $k \geq 2$.

Exercise 8 Exercise 6.2.18

Denote the initial distribution as $X_0 \sim \nu$.

8.(a)

Take $\nu = \delta_0$, then we have

$$\begin{aligned}0 =&\mathbb{E}_\nu\left[X_0\right] = 0\\ =&\mathbb{E}_\nu\left[X_1\right]\\ =&\sum_{i=0}^{N}\mathbb{P}\left(X_0 = 0\right)p(0, i)\\ \Rightarrow& p(0, i) = 0, \quad i \neq 0\end{aligned}$$

the above argument is similar for $\nu = \delta_N$. Together we have $p(0,0) = p(N,N) = 1$.

8.(b)

We have for any $x \notin \{0, N\}$:

$$\begin{aligned}1 - \rho_{xx} =&\mathbb{P}_x\left(\tau_x = \infty\right)\\ \geq&\mathbb{E}_x\left[\mathbf{1}_{\tau_x=\infty}\mathbf{1}_{T_{\{0,N\}}<\infty}\right]\\ =&\mathbb{E}_x\left[\mathbf{1}_{\tau_{\{0,N\}}<\infty}\mathbb{E}\left[\theta^{\tau_{\{0,N\}}}\mathbf{1}_{T_x=\infty}|\mathcal{F}_{\tau_{\{0,N\}}}\right]\right]\\ \overset{\text{SMP}}{=}&\mathbb{E}_x\left[\mathbf{1}_{\tau_{\{0,N\}}<\infty}\mathbb{E}_{X_{\tau_{\{0,N\}}}}\left[\mathbf{1}_{\tau_x=\infty}\right]\right]\\ =&\mathbb{E}_x\left[\mathbf{1}_{\tau_{\{0,N\}}<\infty}\right] > 0\end{aligned}$$

thus proved that any $x \notin \{0, N\}$ is trancient.

Then consider for any x and choose $\nu = \delta_x$. We have

$$\mathbb{E}_x\left[X_{\tau_{\{0,N\}}}\right] =\mathbb{E}_x\left[X_0\right] = x$$

on the other hand we know that

$$\mathbb{E}_x\left[X_{\tau_{\{0,N\}}}\right] = \mathbb{P}_x\left(\tau_0 < \tau_N\right) \times 0 + \mathbb{P}_x\left(\tau_0 > \tau_N\right) \times N = N\mathbb{P}_x\left(\tau_0 > \tau_N\right)$$

together we have $\mathbb{P}_x\left(\tau_0 > \tau_N\right) = x/N$.

8.(c)

We have for S_n denoting the symmeric SRW that $\{S_n^2 - n\}$ is a M.G. by checking the following:

$$\mathbb{E}\left[S_{n+1}^2 - (n+1)|\mathcal{F}_n\right] = \mathbb{E}\left[S_n^2 + \xi_{n+1}^2 + 2S_n\xi_{n+1} - (n+1)|S_n\right] = S_n^2 - n$$

and thus we have

$$\mathbb{E}_x\left[S^2_{\tau_{\{0,N\}}} - \tau_{\{0,N\}}\right] = \mathbb{E}_x\left[S^2_{0\wedge\tau_{\{0,N\}}} - 0 \wedge \tau_{\{0,N\}}\right] = x^2$$

on the other hand we have

$$\mathbb{E}_x\left[S^2_{\tau_{\{0,N\}}} - \tau_{\{0,N\}}\right] = \frac{x}{N}(N^2 - \mathbb{E}_x\left[\tau_{\{0,N\}}\right]) + \frac{N-x}{N}(0 - \mathbb{E}_x\left[\tau_{\{0,N\}}\right])$$

together we have $\mathbb{E}_x\left[\tau_{\{0,N\}}\right] = x(N-x)$.

Exercise 9 6.2.22

We construct $h(x) = \mathbf{1}_{x\notin[-r,r]}$ and we have that $G_{1/2} = [-r,r] \cap \mathbb{Z}$. We verify the super-harmonic as follows: for any $x \in \mathbb{Z}\backslash G_r$ we have

$$\begin{aligned} 1 =& h(x) \\ \geq& \sum_{m=-r}^{r} \mathbb{P}\left(\xi_i = m\right) \\ =& \sum_{y=x-r}^{x+r} p(x,y) \\ \geq& \sum_{y=x-r}^{x+r} p(x,y)h(y) \end{aligned}$$

since we have bounded difference of the chain. Thus by prop 6.2.21 we know that S_n is a recurrent chain.

Exercise 10 6.2.23

- If constant function, WLOG $f(x) \equiv 1$, is super-harmonic, then we use the previous argument that choose $h(x) = \mathbf{1}_{x\in\mathbb{S}\backslash F}$ where F is some finite set in $\mathbb{S}$, in this way we have h being super-harmonic on $\mathbb{S}\backslash F$ and thus by prop 6.2.21 we have that S_n is recurrent.
- A super-harmonic function f should be s.t. $f(S_n)$ is a super-martingale, using the condition we see that $\forall x \in \mathbb{S}$:

$$f(S_n) \xrightarrow{\text{a.s.}} Y$$

and in $f(S_n)$ we have that for any $x \in \mathbb{S}$ that

$$\mathbb{P}_x\left(T_x < \infty\right) = 1$$

then $\mathbb{P}\left(S_n(\omega) = x\, i.o.\right) = 1$, thus all $f(x)$ should be equal to each other, which gives that f is a constant function.