STAT 430-2 2025 Winter
HW4

Tuorui Peng[1]

Exercise 1 Exercise 6.1.14

1.(a)

We have

$$\begin{aligned}
\mathbb{P}\left(X_{n+1}=1|X_1^n\right) =& \mathbb{E}\left[\mathbf{1}_{X_{n+1}=1}|X_1^n\right]\\
=&\mathbb{E}_\theta\left[\mathbb{E}\left[\mathbf{1}_{X_{n+1}=1}|X_1^n,\theta\right]\right]\\
=&\mathbb{E}_\theta\left[1-\theta\right]\\
=&1/2
\end{aligned}$$

1.(b)

We have

$$\begin{aligned}
\mathbb{P}\left(S_{n+1}=s|S_1^n\right) =& \mathbb{P}\left(S_{n+1}=s|X_1^n\right)\\
=&\mathbb{P}\left(X_{n+1}=s-\sum_{i=1}^n X_i|X_1^n\right)\\
=&\mathbb{P}\left(X_{n+1}=s-S_n|S_n\right)
\end{aligned}$$

from the previous part. Thus we have S_n being Markov.

Exercise 2 Exercise 6.1.18

2.(a)

By property of sets we have $\Gamma_n\to\Gamma$ thus $\mathbf{1}_{\Gamma_n}\xrightarrow{\text{a.s.}}\mathbf{1}_\Gamma$, then by Lévy's upward theorem we have:

$$\begin{aligned}
\mathbb{P}\left(\Gamma_n|X_n\right) =& \mathbb{E}\left[\mathbf{1}_{\Gamma_n}|X_n\right]\\
=&\mathbb{E}\left[\mathbf{1}_{\Gamma_n}|\mathcal{F}_n\right]\\
\xrightarrow[L_1]{\text{a.s.}}&\mathbb{E}\left[\mathbf{1}_{\Gamma}|\mathcal{F}_\infty\right]\\
=&\mathbf{1}_\Gamma
\end{aligned}$$

[1] TuoruiPeng2028@u.northwestern.edu

2.(b)

Denote $K := \{\omega : X_n(\omega) \in A_n\, i.o.\}$. Then we have that $\forall N > 0$, $\exists n > N$ s.t. $\mathbb{P}\left(\Gamma_n \cap K | X_n\right) \geq \eta > 0$. On the other hand we have

$$\eta < \mathbb{P}\left(\Gamma_n \cap K | X_n\right) \xrightarrow{\text{a.s.}} \mathbb{P}\left(\Gamma \cap K | X_\infty\right) = \mathbf{1}_{\Gamma \cap K} = 1 = \mathbb{P}\left(\Gamma \cap K\right)$$

which gives $\mathbb{P}\left(K \backslash \Gamma\right) = 0$. In the above we applied Lévy's upward theorem to $\Gamma_n \cap K | X_n$.

2.(c)

Use $A_n \equiv A$ and $B_n \equiv B$ and we have using the precedence:

$$\begin{aligned}
1 \leq & \mathbb{P}\left(\{X_n \in A \text{ finitely often}\} \cup (\{X_n \in A\, i.o.\} \backslash \Gamma) \cup \Gamma\right) \\
\leq & \mathbb{P}\left(\{X_n \in A \text{ finitely often}\} \cup \Gamma\right) + \mathbb{P}\left(\{X_n \in A\, i.o.\} \backslash \Gamma\right) \\
= & \mathbb{P}\left(\{X_n \in A \text{ finitely often}\} \cup \Gamma\right) + 0 \\
= & \mathbb{P}\left(\{X_n \in A \text{ finitely often}\}\right) + \mathbb{P}\left(\Gamma\right)
\end{aligned}$$

where $\Gamma = \{X_n \in B\, i.o.\}$ so thus we have proved the claim.

Exercise 3 Exercise 6.1.19

We prove the result for symmetric SRW directly. Denote $\tau = \inf\{k : \omega_k \geq b\}$ and

$$h_k(\omega) = \sum_{i=-\infty}^{+\infty} \mathbf{1}_{\omega_{n-k}=b+i}, \quad k \in [n]$$

and by Strong Markov Property (SMP) we have

$$\begin{aligned}
\mathbb{P}\left(\max_{k \leq n} \omega_k \geq b\right) = & \mathbb{E}\left[\mathbf{1}_{\tau \leq n}\right] \\
= & \mathbb{E}\left[\mathbf{1}_{\tau \leq n} \sum_{\imath=-\infty}^{+\infty} \mathbf{1}_{\omega_n=\imath}\right] \\
= & \mathbb{E}\left[\mathbf{1}_{\tau \leq n} \mathbb{E}\left[\sum_{i=-\infty}^{+\infty} \mathbf{1}_{\theta^{n\wedge\tau}\omega_{n-n\wedge\tau}=b+i} | \mathcal{F}_{n\wedge\tau}\right]\right] \\
= & \mathbb{E}\left[\mathbf{1}_{\tau \leq n} \mathbb{E}\left[h_{n\wedge\tau}(\theta^{n\wedge\tau}\omega) | \mathcal{F}_{n\wedge\tau}\right]\right] \\
\overset{\text{SMP}}{=} & \mathbb{E}\left[\mathbf{1}_{\tau \leq n} \mathbb{E}_{X_{n\wedge\tau}}\left[h_{n\wedge\tau}(\omega)\right]\right] \\
= & \mathbb{E}\left[\mathbf{1}_{\tau \leq n} \mathbb{E}_{X_{n\wedge\tau}}\left[\mathbf{1}_{\omega_{n-n\wedge\tau}=0} + 2\sum_{i=1}^{\infty} \mathbf{1}_{\omega_{n-n\wedge\tau}=i}\right]\right] \\
= & \mathbb{E}\left[\mathbf{1}_{\omega_n=b}\right] + 2\mathbb{E}\left[\sum_{i=1}^{\infty} \mathbf{1}_{\omega_n=b+i}\right] \\
= & \mathbb{P}\left(\omega_n = b\right) + 2\mathbb{P}\left(\omega_n > b\right)
\end{aligned}$$

Exercise 4 Exercise 6.2.2

4.(a)

We use $h_r = \mathbf{1}_{\omega_{n-r}\in B}$ and apply for stopping time $T_{y,r}$ the SMP to obtain that

$$\begin{aligned}
\mathbb{P}_x\left(X_n \in B, T_{y,r} \le n\right) =& \mathbb{E}\left[\mathbf{1}_{X_n\in B}\mathbf{1}_{T_{y,r}\le n}\right] \\
=& \mathbb{E}\left[\mathbf{1}_{T_{y,r}\le n}\mathbb{E}\left[h_{T_{y,r}}(\theta^{T_{y,r}}\omega)|\mathcal{F}_{T_{y,r}}\right]\right] \\
\overset{\text{SMP}}{=}& \mathbb{E}\left[\mathbf{1}_{T_{y,r}\le n}\mathbb{E}_y\left[h_{T_{y,r}}(\omega)\right]\right] \\
=& \sum_{k=0}^{n-r}\mathbb{P}_x\left(T_{y,r} = n-k\right)\mathbb{P}_y\left(X_{n-k}\in B\right)
\end{aligned}$$

4.(b)

Making relabeling $k \mapsto n-k$ and $B = \{y\}$ and we have

$$\mathbb{P}_x\left(X_n = y\right) = \sum_{k=r}^{n}\mathbb{P}_x\left(T_{y,r} = k\right)\mathbb{P}_y\left(X_{n-k} = y\right)$$

4.(c)

We have

$$\begin{aligned}
\text{R.H.S.} &= \sum_{n=r}^{l+r}\mathbb{P}_y\left(X_n = y\right) \\
&= \sum_{n=r}^{l+r}\sum_{k=r}^{n}\mathbb{P}_y\left(T_{y,r} = k\right)\mathbb{P}_y\left(X_{n-k} = y\right) \\
&= \sum_{k=r}^{l+r}\mathbb{P}_y\left(T_{y,r} = k\right)\sum_{n=k}^{l+r}\mathbb{P}_y\left(X_{n-k} = y\right) \\
&\le \sum_{k=r}^{l+r}\mathbb{P}_y\left(T_{y,r} = k\right)\sum_{n=k}^{l+k}\mathbb{P}_y\left(X_{n-k} = y\right) \\
&= \sum_{k=r}^{l+r}\mathbb{P}_y\left(T_{y,r} = k\right)\sum_{j=0}^{l}\mathbb{P}_y\left(X_j = y\right) \\
&\le \sum_{j=0}^{l}\mathbb{P}_y\left(X_j = y\right)
\end{aligned}$$

Exercise 5 Exercise 6.2.5

5.(a)

Since $\mathbb{P}_{x\notin C}\left(\tau_C < \infty\right) > 0$, we know that $\exists N > 0$ and some ε s.t.

$$\mathbb{P}_{x\notin C}\left(\tau_C < N\right) > \varepsilon$$

and thus we have

$$\mathbb{P}_{x\notin C}\left(\tau_C \geq N\right) \leq 1-\varepsilon$$

Consider applying the SMP to $h := \mathbf{1}_{\tau>N}$ we have

$$\begin{aligned}
\mathbb{E}\left[\mathbf{1}_{\tau>(k+1)N}\right] =& \mathbb{E}\left[\mathbf{1}_{\tau>kN}\mathbb{E}\left[h(\theta^{kN}\omega)|\mathcal{F}_{kN}\right]\right]\\
=&\mathbb{E}\left[\mathbf{1}_{\tau>kN}\mathbb{E}_{x\notin C}\left[h(\omega)\right]\right]\\
=&\mathbb{E}\left[\mathbf{1}_{\tau>kN}\mathbb{P}_{x\notin C}\left(\tau_C\geq N\right)\right]\\
\leq&\mathbb{E}\left[\mathbf{1}_{\tau>kN}(1-\varepsilon)\right]\\
\ldots\leq&\left(1-\varepsilon\right)^k
\end{aligned}$$

5.(b)

By Borel-Cantelli lemma we have $\mathbb{P}\left(\tau_C<\infty\right)=1$ since

$$\sum_{k=1}^{\infty}\mathbb{P}_{x\notin C}\left(\tau_C\geq kN\right)<\infty$$

Then we have for any $x=X_0\notin C$ that:

$$\begin{aligned}
g(x) =&\mathbb{P}_{X_0=x}\left(\tau_A<\tau_B\right)\\
=&\mathbb{P}_{X_0}\left(X_{\tau_C}\in A\right)\\
=&\sum_{y\in\mathbb{S}}\mathbb{P}_{X_0}\left(X_1=y\right)\mathbb{P}_{X_1=y}\left(X_{\tau_C}\in A\right)\\
=&\sum_{y\in\mathbb{S}}p(x,y)g(y)
\end{aligned}$$

thus $g(\,\cdot\,)$ is harmonic on $\mathbb{S}\backslash C$.

5.(c)

Note that we have $X_{n\wedge\tau_C-1}\notin C$

By the harmonic proporty we have that

$$\left(X_{n\wedge\tau_C+1}\right) =\mathbf{1_{\tau_C\leq n}}$$