STAT 430-2 2025 Winter
HW2

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Exercise 1 5.1.8

1.(a)

We have

$$\begin{aligned}\mathbb{E}\left[D_n\right] =&\mathbb{E}\left[\mathbb{E}\left[X_n - X_{n-1}|\mathcal{F}_{n-1}\right]\right]\\ =&\mathbb{E}\left[\mathbb{E}\left[X_n|\mathcal{F}_{n-1}\right] - X_{n-1}\right]\\ =&0, \qquad n\geq 1\end{aligned}$$

thus D_n are mean-zero and we have (assume WLOG $n > m$)

$$\begin{aligned}cov(D_n, D_m) =&\mathbb{E}\left[D_n D_m\right]\\ =&\mathbb{E}\left[\mathbb{E}\left[D_n D_m|\mathcal{F}_m\right]\right]\\ =&\mathbb{E}\left[D_m\mathbb{E}\left[D_n|\mathcal{F}_m\right]\right]\\ =&0\end{aligned}$$

thus D_n are uncorrelated.

1.(b)

We have

$$\begin{aligned}\mathbb{E}\left[X_l Y_l|\mathcal{F}_n\right] - X_n Y_n =&\mathbb{E}\left[X_l Y_l|\mathcal{F}_n\right] - X_n\mathbb{E}\left[Y_l|\mathcal{F}_n\right] - Y_n\mathbb{E}\left[X_l|\mathcal{F}_n\right] + X_n Y_n\\ =&\mathbb{E}\left[(X_l - X_n)(Y_l - Y_n)|\mathcal{F}_n\right]\end{aligned}$$

so we proved the first part.

Take $l = n+1$ we have for each n:

$$\mathbb{E}\left[X_{n+1}Y_{n+1}|\mathcal{F}_n\right] - X_n Y_n = \mathbb{E}\left[(X_{n+1} - X_n)(Y_{n+1} - Y_n)|\mathcal{F}_n\right]$$

then repeat for $n+1$ to l we have

$$\begin{aligned}\mathbb{E}\left[X_l Y_l|\mathcal{F}_n\right] - X_n Y_n =&\mathbb{E}\left[\ldots\mathbb{E}\left[X_l Y_l|\mathcal{F}_{l-1}\right] - X_{l-1}Y_{l-1} + \ldots X_{n+1}Y_{n+1} - X_n Y_n|\ldots|\mathcal{F}_n\right]\\ =&\sum_{k=n+1}^{l}\mathbb{E}\left[(X_k - X_{k-1})(Y_k - Y_{k-1})|\mathcal{F}_n\right]\end{aligned}$$

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1.(c)

By the above, in which we let $Y_i = X_i$, we have

$$\begin{aligned}
C^2 \geq \mathbb{E}\left[X_l^2|\mathcal{F}_0\right] - X_0^2 =& \sum_{k=1}^{l}\mathbb{E}\left[(X_k - X_{k-1})^2|\mathcal{F}_0\right]\\
=& \sum_{k=1}^{l}\mathbb{E}\left[D_k^2\right]
\end{aligned}$$

then we have the following:

$$\begin{aligned}
\mathbb{E}\left[(\sum_{k=1}^{l} D_k^2)^2\right] =& \mathbb{E}\left[\sum_{k=1}^{l} D_k^4\right] + 2\mathbb{E}\left[\sum_{i=1}^{l}\sum_{j=1+1}^{l} D_i^2 D_j^2\right]\\
\leq& \mathbb{E}\left[\sum_{k=1}^{l}(2C)^2 D_k^2\right] + 2\mathbb{E}\left[\sum_{i=1}^{l} D_i^2 \mathbb{E}\left[\sum_{j=i+1}^{l} D_j^2|\mathcal{F}_i\right]\right]\\
\leq& 4C^2C^2 + 2C^4 = 6C^4
\end{aligned}$$

Exercise 2 5.1.12

- For $\theta \wedge \tau$:

$$\begin{aligned}
\{\omega : \theta \wedge \tau(\omega) \leq n\} =& \{\omega : \theta(\omega) \leq n \text{ or } \tau(\omega) \leq n\}\\
=& \{\omega : \theta(\omega) \leq n\} \cup \{\omega : \tau(\omega) \leq n\}\\
\in& \mathcal{F}_n
\end{aligned}$$

- For $\theta \vee \tau$:

$$\begin{aligned}
\{\omega : \theta \vee \tau(\omega) \leq n\} =& \{\omega : \theta(\omega) \leq n \text{ and } \tau(\omega) \leq n\}\\
=& \{\omega : \theta(\omega) \leq n\} \cap \{\omega : \tau(\omega) \leq n\}\\
\in& \mathcal{F}_n
\end{aligned}$$

- For $\theta + \tau$:

$$\begin{aligned}
\{\omega : \theta + \tau(\omega) \leq n\} =& \{\omega : \theta(\omega) + \tau(\omega) \leq n\}\\
=& \bigcup_{k=0}^{n}\{\omega : \theta(\omega) = k \text{ and } \tau(\omega) = n - k\}\\
\in& \mathcal{F}_n
\end{aligned}$$

Exercise 3 5.1.15

3.(a)

- For $k = 1$:

$$\mathbb{P}(\tau > r) = \mathbb{P}(\tau > 0 + r|\mathcal{F}_0) \leq 1 - \varepsilon$$

- If for k we have $\mathbb{P}\left(\tau > kr\right) \leq (1-\varepsilon)^k$, then for $k+1$:

$$\begin{aligned}\mathbb{P}\left(\tau > (k+1)r\right) =& \mathbb{E}\left[\mathbb{E}\left[\mathbf{1}(\tau > kr + r)|\mathcal{F}_{kr}\right]\right]\\ =& \mathbb{E}\left[\mathbb{E}\left[\mathbf{1}(\tau > kr + r)\big(\mathbf{1}(\tau > kr) + \mathbf{1}(\tau \leq kr)\big)|\mathcal{F}_{kr}\right]\right]\\ =& \mathbb{E}\left[\mathbf{1}(\tau > kr)\mathbb{E}\left[\mathbf{1}(\tau > kr + r)|\mathcal{F}_{kr}\right]\right]\\ \leq& (1-\varepsilon)^k(1-\varepsilon) = (1-\varepsilon)^{k+1}\end{aligned}$$

Now we finish the proof by induction.

3.(b)

We have

$$\mathbb{E}\left[\tau\right] = \sum_{t=0}^{\infty}\mathbb{P}\left(\tau \geq t\right) \leq \sum_{t=0}^{\infty}(1-\varepsilon)^{\lfloor t/r\rfloor} \leq \sum_{s=0}^{\infty} r(1-\varepsilon)^s < \infty$$

Exercise 4 5.1.24

4.(a)

We have

$$\begin{aligned}\mathbb{E}\left[S_n^2|\mathcal{F}_{n-1}\right] =& \mathbb{E}\left[(\sum_{i=1}^{n}\xi_i)^2|\mathcal{F}_{n-1}\right]\\ =& (\sum_{i=1}^{n-1}\xi_i)^2 + 2\sum_{i=1}^{n-1}\xi_i\mathbb{E}\left[\xi_n|\mathcal{F}_{n-1}\right] + \mathbb{E}\left[\xi_n^2|\mathcal{F}_{n-1}\right]\\ =& S_{n-1}^2 + \mathbb{E}\left[\xi_n^2|\mathcal{F}_{n-1}\right]\\ \geq& S_{n-1}^2\end{aligned}$$

thus S_n^2 is a submartingale.

Then we have

$$\begin{aligned}\mathbb{E}\left[S_n^2 - s_n^2|\mathcal{F}_{n-1}\right] =& S_{n-1}^2 + \mathbb{E}\left[\xi_n^2|\mathcal{F}_{n-1}\right] - s_{n-1}^2 - \mathbb{E}\left[\xi_i^2|\mathcal{F}_{n-1}\right]\\ =& S_{n-1}^2 - s_{n-1}^2\end{aligned}$$

thus $S_n^2 - s_n^2$ is a martingale.

4.(b)

We have

$$\begin{aligned}\mathbb{E}\left[e^{S_n}|\mathcal{F}_{n-1}\right] =& \mathbb{E}\left[e^{S_{n-1}+\xi_n}|\mathcal{F}_{n-1}\right]\\ =& e^{S_{n-1}}\mathbb{E}\left[e^{\xi_n}|\mathcal{F}_{n-1}\right]\\ \geq& e^{S_{n-1}}e^{\mathbb{E}[\xi_n|\mathcal{F}_{n-1}]}\\ =& e^{S_{n-1}}\end{aligned}$$

thus e^{S_n} is a submartingale.

Then we have

$$\mathbb{E}\left[e^{S_n}/\prod_{i=1}^{n}\mathbb{E}\left[e^{\xi_i}\right]|\mathcal{F}_{n-1}\right]=e^{S_{n-1}}\mathbb{E}\left[e^{\xi_n}|\mathcal{F}_{n-1}\right]/\prod_{i=1}^{n}\mathbb{E}\left[e^{\xi_n}\right]=e^{S_{n-1}}/\prod_{i=1}^{n-1}\mathbb{E}\left[e^{\xi_n}\right]$$

thus $e^{S_n}/\prod_{i=1}^{n}\mathbb{E}\left[e^{\xi_i}\right] := e^{S_n}/m_n$ is a martingale.

Exercise 5 5.1.26

We have

$$\mathbb{E}\left[f(S_n)|F_{n-1}\right]=\mathbb{E}\left[f(S_{n-1}+\xi_n)|F_{n-1}\right]=\frac{1}{|B(0,1)|}\int_{B(S_{n-1},1)}f(x)\,\mathrm{d}x\leq f(S_{n-1})$$

thus $f(S_n)$ is a supermartingale.

Exercise 6 5.1.35

6.(a)

We verify the three conditions of σ-algebra:

- $\Omega\in\mathcal{F}$: we see that

$$\Omega\cap\{\omega:\tau(\omega)\leq n\}=\{\omega:\tau(\omega)\leq n\}\in\mathcal{F}_n$$

- Closure under complement: we see that if $A\cap\{\omega:\tau(\omega)\leq n\}\in\mathcal{F}_n$, then

$$A^{\complement}\cap\{\omega:\tau(\omega)\leq n\}=\{\omega:\tau(\omega)\leq n\}\backslash A\cap\{\omega:\tau(\omega)\leq n\}\in\mathcal{F}_n$$

- Closure under countable union: we see that if $A_i\cap\{\omega:\tau(\omega)\leq n\}\in\mathcal{F}_n$, then

$$\bigcup_{i=1}^{\infty}A_i\cap\{\omega:\tau(\omega)\leq n\}=\bigcup_{i=1}^{\infty}\big(A_i\cap\{\omega:\tau(\omega)\leq n\}\big)\in\mathcal{F}_n$$

Thus $\mathcal{F}_\tau$ is a σ-algebra.

Then if $\tau(\omega)\equiv n$ then:

- If $A\in\mathcal{F}_n$: we have

$$A\cap\{\omega:\tau(\omega)\leq n\}=A\in\mathcal{F}_\tau$$

- If $A\notin\mathcal{F}_n$: we have

$$A\cap\{\omega:\tau(\omega)\leq n\}=\emptyset\in\mathcal{F}_\tau$$

which means that in this case $\mathcal{F}_\tau=\mathcal{F}_n$.

6.(b)

$X_\tau \in m\mathcal{F}_\tau$ is trivial from the definition of $\mathcal{F}_\tau$, and thus $\sigma(\tau) \subseteq \mathcal{F}_\tau$. For any k we have for $\{\omega : X_k \mathbf{1}_{\tau=k}\}$ that:

$$\{\omega : X_k \mathbf{1}_{\tau=k} < t\} \cap \{\omega : \tau(\omega) \leq n\} = \{\omega : X_k(\omega) < t\} \cap \{\omega : \tau(\omega) = k\} \cap \{\omega : \tau(\omega) \leq n\}$$

for which if $n \geq k$, $\in \mathcal{F}_n$, and if $n < k$, the above is empty and thus $\in \mathcal{F}_n$, thus $X_k \mathbf{1}_{\tau=k} \in m\mathcal{F}_\tau$.

6.(c)

We have

$$\mathbb{E}\left[Y_\tau \mathbf{1}_{\tau=k} | \mathcal{F}_\tau\right] = \mathbf{1}_{\tau=k} \mathbb{E}\left[Y_k | \mathcal{F}_\tau\right] = \mathbf{1}_{\tau=k} \mathbb{E}\left[Y_k | \mathcal{F}_k\right]$$

because as we previously proved that when $\tau = n$, $\mathcal{F}_\tau = \mathcal{F}_n$.

6.(d)

Since $\theta \leq \tau$, we have

$$\{\omega : \tau(\theta) \leq n\} \subseteq \{\omega : \theta(\omega) \leq n\} \in \mathcal{F}_n,$$

then if some A satisfies $A \cap \{\omega : \theta(\omega) \leq n\} \in \mathcal{F}_n$, then

$$A \cap \{\omega : \tau(\omega) \leq n\} = A \cap \{\omega : \theta(\omega) \leq n\} \cap \{\omega : \tau(\omega) \leq n\} \in \mathcal{F}_n$$

thus we have $\mathcal{F}_\theta \subseteq \mathcal{F}_\tau$.

Exercise 7 5.2.8

Define $\tau_x = \inf_{n \geq 0}\{n : X_n \leq -x\}$, then we have Denote A_n as follows:

$$A_n = \{\omega : \tau_x(\omega) \leq n\} = \{\omega : \max_{k=0}^{n} X_k(\omega) \leq -x\}$$

then we have $X_{\tau_x} \leq -x$.

Since X_n is a sub-martingale, we have

$$\begin{aligned}
\mathbb{E}\left[X_0\right] \leq & \mathbb{E}\left[X_{n\wedge\tau}\right] \\
= & \mathbb{E}\left[X_n \mathbf{1}_{n<\tau}\right] + \mathbb{E}\left[X_\tau \mathbf{1}_{n\geq\tau}\right] \\
\leq & \mathbb{E}\left[X_n \mathbf{1}_{A_n^\complement}\right] - x\mathbb{P}\left(A_n\right)
\end{aligned}$$

We thus have

$$\begin{aligned}
\mathbb{P}\left(\min_{k=0}^{n} X_k \leq -x\right) = & \mathbb{P}\left(A_n\right) \\
\leq & x^{-1}(\mathbb{E}\left[X_n \mathbf{1}_{A_n^\complement}\right] - \mathbb{E}\left[X_0\right]) \\
\leq & x^{-1}(\mathbb{E}\left[(X_n)_+\right] - \mathbb{E}\left[X_0\right])
\end{aligned}$$

Exercise 8 5.2.11

8.(a)

If Y_n is a submartingale, then we have Y_n^p is a submartingale for $p > 1$, then we have

$$\mathbb{P}\left(\max_{k=0}^{n} Y_k \geq y\right) = \mathbb{P}\left(\max_{k=0}^{n} Y_k^p \geq y^p\right) \leq y^{-p}\mathbb{E}\left[(Y_n^p)_+\right] = y^{-p}\mathbb{E}\left[(Y_n)_+^p\right]$$

8.(b)

If Y_n is a martinagle, then we have Y_n^p is a submartingale for $p > 1$, then we have

$$\mathbb{P}\left(\max_{k=0}^{n} Y_k \geq y\right) \leq y^{-p}\mathbb{E}\left[(Y_n)_+^p\right] \leq y^{-p}\mathbb{E}\left[|Y_n|^p\right]$$

8.(c)

If Y_n is a supermartingale, then we have $(Y_n + c)^2$ is a submartingale for $c > 0$, then if $Y_0 = 0$, we have

$$\begin{aligned}\mathbb{P}\left(\max_{k=0}^{n} Y_k \geq y\right) &= \mathbb{P}\left(\max_{k=0}^{n} (Y_k + c)^2 \geq (y+c)^2\right) \\ &\leq (y+c)^{-2}\mathbb{E}\left[((Y_n + c)^2)_+\right] \\ &\leq \frac{\mathbb{E}\left[Y_n^2\right] + c^2}{(y+c)^2}\end{aligned}$$

optimize over c and we have optimal value of $c = \mathbb{E}\left[Y_n^2\right]/y$, then we have

$$\mathbb{P}\left(\max_{k=0}^{n} Y_k \geq y\right) \leq \frac{\mathbb{E}\left[Y_n^2\right]}{\mathbb{E}\left[Y_n^2\right] + y^2}.$$

Exercise 9 5.2.12

9.(a)

Since X_n is a supermartingale, we have

$$\mathbb{E}\left[X_0\right] = \mathbb{E}\left[X_{n\wedge 0}\right] \geq \mathbb{E}\left[X_{n\wedge\tau}\right] = \mathbb{E}\left[X_n \mathbf{1}_{n<\tau}\right] + \mathbb{E}\left[X_\tau \mathbf{1}_{\tau\leq n}\right] \geq \mathbb{E}\left[X_\tau \mathbf{1}_{\tau\leq n}\right]$$

9.(b)

If X_n is a supermartingale, we have

$$\begin{aligned}\mathbb{E}\left[X_0\right] &\geq \mathbb{E}\left[X_\tau \mathbf{1}_{\tau\leq n}\right] \geq x\mathbb{P}\left(\sup_k X_k \geq x\right) \\ \Rightarrow \mathbb{P}\left(\sup_k X_k \geq x\right) &\leq x^{-1}\mathbb{E}\left[X_0\right]\end{aligned}$$

Exercise 10 5.2.15

We have the following two lemma:

- For any $x, y > 0$:

$$x(\log y)_+ \leq e^{-1}y + x(\log x)_+$$

- If some X, Y satisfies $\mathbb{P}\left(Y \geq y\right) \leq y^{-1}\mathbb{E}\left[X\mathbf{1}_{Y\geq y}\right]$, then

$$\mathbb{E}\left[Y\right] \leq 1 + \mathbb{E}\left[X(\log Y)_+\right]$$

Substitute $Y = \max_{k\leq n} X_k$, $X = X_n$ for which we notice that the concentration condition is satisfied, then we have

$$\begin{aligned}
\mathbb{E}\left[\max_{k\leq n} X_k\right] \leq& 1 + \mathbb{E}\left[X_n(\log \max_{k\leq n} X_k)_+\right] \\
\leq& 1 + e^{-1}\mathbb{E}\left[\max_{k\leq n} X_k\right] + \mathbb{E}\left[X_n(\log X_n)_+\right] \\
\Rightarrow \mathbb{E}\left[\max_{k\leq n} X_k\right] \leq& (1-e^{-1})^{-1}\left[1 + \mathbb{E}\left[X_n(\log X_n)_+\right]\right]
\end{aligned}$$