Stat461 - 2023 Fall

HW1

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Exercise 1

1.(a)

We use the following equivalent expression for sub-Gaussian and sub-exponential random variables:

$$\text{Sub-Gaussian: } \exists\theta \geq 0,\ s.t. \mathbb{E}\left[X^{2k}\right] \leq \frac{(2k)!}{2^k k!}\theta^{2k},\ \forall k \in \mathbb{N}^+,$$
$$\text{Sub-Exponential: } \sup_{k\in\mathbb{N}^+}\left[\frac{\mathbb{E}\left[X^k\right]}{k!}\right]^{1/k} < \infty.$$

The proof is as follows:

$\Rightarrow$ If X is sub-Gaussian with such above-mentioned θ, then we have:

$$\begin{aligned}
\left[\frac{\mathbb{E}\left[(X^2)^k\right]}{k!}\right]^{1/k} &\leq \left[\frac{(2k)!}{k!k!}\frac{\theta^{2k}}{2^k}\right]^{1/k}\\
&= \left[\frac{(2k)!}{(2^k k!)\cdot(2^k k!)}\right]^{1/k}\cdot 2\theta\\
&=2\theta\left[\prod_{i=1}^k \frac{2i}{2i}\prod_{i=1}^k\frac{2i-1}{2i}\right]^{1/k}\\
&\leq 2\theta < \infty.
\end{aligned}$$

Thus proved that X^2 is sub-exponential.

$\Leftarrow$ If X^2 is sub-exponential, then take some γ satisfying the suprema condition, we have:

$$\begin{aligned}
\left[\frac{\mathbb{E}\left[(X^2)^k\right]}{k!}\right]^{1/k} &\leq\gamma\\
\Rightarrow \mathbb{E}\left[X^{2k}\right] &\leq\gamma^k k! \leq \frac{(2k)!}{2^k k!}(\sqrt{\gamma})^{2k}, \quad \forall k\in\mathbb{N}^+.
\end{aligned}$$

Thus proved that X^2 is sub-Gaussian with $\theta \leq \sqrt{\gamma}$.

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1.(b)

We use the following equivalent expression for sub-Gaussian and sub-exponential random variables:[2]

$$\text{Sub-Gaussian: } \exists K_2 > 0,\ s.t. \forall p \geq 1,\ \mathbb{E}\left[|X - \mathbb{E}[X]|^p\right]^{1/p} \leq K_2\sqrt{p}$$
$$\text{Sub-Exponential: } \exists K_1 > 0,\ s.t. \forall p \geq 1,\ \mathbb{E}\left[|X - \mathbb{E}[X]|^p\right]^{1/p} \leq K_1 p$$

Using the equivalentce, the proof is straightforward. Assume the K_2 parameter for X, Y are $K_{2,X}$ and $K_{2,Y}$ respectively and WLOG assume X, Y are both mean-zero. Using the convexity of $\|\exp\cdot\|_p := \mathbb{E}\left[|e^{\cdot}|^p\right]^{1/p}$, we have:

$$\begin{aligned}
\mathbb{E}\left[|XY|^p\right]^{1/p} =& \left\| e^{\frac{\log\left|X^2\right| + \log\left|Y^2\right|}{2}} \right\|_p \\
\leq& \frac{\left\| e^{\log\left|X^2\right|} \right\|_p + \left\| e^{\log\left|Y^2\right|} \right\|_p}{2} \\
=& \frac{1}{2}\mathbb{E}\left[|X|^{2p}\right]^{1/p} + \frac{1}{2}\mathbb{E}\left[|Y|^{2p}\right]^{1/p} \\
\leq& \frac{1}{2}(K_{2,X}\sqrt{p})^2 + \frac{1}{2}(K_{2,Y}\sqrt{p})^2 \\
=& \frac{K_{2,X}^2 + K_{2,Y}^2}{2} p, \qquad \forall p \geq 1
\end{aligned}$$

Exercise 2

Denote the polynomial side bound as

$$M := \inf_{k\in\mathbb{N}} \frac{\mathbb{E}\left[X^k\right]}{\delta^k}$$

Using the taylor expansion of $x \mapsto e^x$ at 0, we have

$$\begin{aligned}
\mathbb{E}\left[e^{\lambda}X\right] =& \sum_{k=0}^{\infty} \frac{\lambda^k}{k!}\mathbb{E}\left[X^k\right] \\
\geq& \sum_{k=0}^{\infty} \frac{\lambda^k}{k!}\delta^k M \\
=& M e^{\lambda\delta}
\end{aligned}$$

[2] which are straightforward to prove using the definition of sub-Gaussian and sub-exponential random variables. Taking sub-gaussian as an example, we have:

$$\begin{aligned}
\mathbb{E}\left[|X|^p\right] =& \int_0^\infty \mathbb{P}\left(|X|^p > t\right) \,\mathrm{d}t \\
=& \int_0^\infty \mathbb{P}\left(|X| > t\right) p t^{p-1} \,\mathrm{d}t \\
\leq& \int_0^\infty 2e^{-t^2/2\sigma^2} p t^{p-1} \,\mathrm{d}t \\
=& p\Gamma(p/2)\sigma^p 2^{p/2} \\
\leq& p(\sqrt{p}\sigma)^p
\end{aligned}$$

Thus we have:

$$\frac{\mathbb{E}\left[e^{\lambda}X\right]}{e^{\lambda\delta}} \geq M = \inf_{k\in\mathbb{N}} \frac{\mathbb{E}\left[X^k\right]}{\delta^k}, \quad \forall \lambda > 0$$

thus proved the result.

Exercise 3

Consider a random variable $X \sim \text{Binom}(n, 1/2)$, which has Moment Generating Function $M_X(\lambda) = (e^{\lambda}/2 + 1/2)^n$. Using Chernoff bound we have:

$$\begin{aligned}\frac{1}{2^n}\sum_{j=1}^{k}\binom{n}{j} =& \mathbb{P}\left(X \geq n-k\right) \\ &\leq \inf_{\lambda>0}\left\{n\log\frac{e^{\lambda}+1}{2} - \lambda(n-k)\right\} := f(\lambda)\end{aligned}$$

we optimize the bound w.r.t. λ. R.H.S. takes minimum at $\lambda = \log\dfrac{n-k}{k}$, yielding the upper bound:

$$\log\sum_{j=1}^{k}\binom{n}{j} - n\log 2 \leq -n\log 2 + n\log\frac{n}{k} - (n-k)\log\frac{n-k}{k}$$

Now it suffces to prove the following inequality:

$$n\log\frac{n}{k} - (n-k)\log\frac{n-k}{k} \leq k\log\frac{ne}{k}$$

taking $\alpha = \dfrac{k}{n} \in (0,1]$, it suffices to verify the following inequality:

$$f(\alpha) := \log\frac{1}{1-\alpha} + \alpha\log\frac{1-\alpha}{\alpha} - \alpha\log\frac{e}{\alpha} \leq 0$$

which can be easily verified that $f'(\alpha) \leq 0$, and notice that $f(0) = 0$, thus proved the inequality. And we have

$$\sum_{j=1}^{k}\binom{n}{j} \leq \left(\frac{en}{k}\right)^k$$

Exercise 4

4.(a)

We have

$$\begin{aligned}var(X) =& \mathbb{E}\left[(X - \mathbb{E}\left[X\right])^2\right] \\ =& \int_0^{\infty} \mathbb{P}\left((X-\mathbb{E}\left[X\right])^2 > t\right)\,\mathrm{d}t \\ \leq& \int_0^{\infty} c_1 e^{-c_2 t}\,\mathrm{d}t \\ =& \frac{c_1}{c_2}\end{aligned}$$

4.(b)

We can use example that $X = 1, 2, 3, 4$ equiprobably, for which median can take any number in $[2, 3]$

4.(c)

Using Chebyshev's inequality, we have:

$$\mathbb{P}\left(|X-\mu| \geq t\right) \leq \frac{\sigma^2}{t^2}$$

taking $t = \sqrt{2}\sigma$ we can obtain that $|m_X - \mu| \leq \sqrt{2}\sigma$. Using the fact we have:

- For $t \geq \sqrt{2}\sigma$:

$$\begin{aligned}\mathbb{P}\left(|X-m_X| \geq t\right) \leq & \mathbb{P}\left(|X-\mu| + |\mu - m_X| \geq t\right) \\ = & \mathbb{P}\left(|X-\mu| \geq t - |\mu - m_X|\right) \\ \leq & c_1 e^{-c_2(t-\sqrt{2}\sigma)^2} \leq \tilde{c}_3 e^{-c_4 t^2}\end{aligned}$$

- For $0 < t < \sqrt{2}\sigma$:

$$\mathbb{P}\left(|X-m_X| \geq t\right) \leq 1$$

To combine the two case, we just need to consider resetting $\tilde{c}_3$ s.t.

$$\tilde{c}_3 e^{-c_4^{>} t^2}\Big|_{t=\sqrt{2}\sigma} = \tilde{c}_3 e^{-2\sigma^2 c_4} \geq 1$$

i.e. we can take

$$c_3 = \max\{\tilde{c}_3, e^{2\sigma^2 c_4}\}, \quad c_4 = c_2$$

4.(d)

To prove the converse statement, the idea is exactly the same by using $|\mu - m_X| \leq \sqrt{2}\sigma$:

- For $t \geq \sqrt{2}\sigma$:

$$\begin{aligned}\mathbb{P}\left(|X-\mu| \geq t\right) \leq & \mathbb{P}\left(|X-m_X| + |\mu - m_X| \geq t\right) \\ = & \mathbb{P}\left(|X-m_X| \geq t - |\mu - m_X|\right) \\ \leq & c_3 e^{-c_4(t-\sqrt{2}\sigma)^2} \leq \tilde{c}_1 e^{-c_2 t^2}\end{aligned}$$

- For $0 < t < \sqrt{2}\sigma$:

$$\mathbb{P}\left(|X-m_X| \geq t\right) \leq 1$$

To combine the two case, we just need to consider resetting $\tilde{c}_1$ s.t.

$$\tilde{c}_1 e^{-c_2^{>} t^2}\Big|_{t=\sqrt{2}\sigma} = \tilde{c}_1 e^{-2\sigma^2 c_2} \geq 1$$

i.e. we can take

$$c_1 = \max\{\tilde{c}_1, e^{2\sigma^2 c_2}\}, \quad c_2 = c_4$$

Note: A better parameter $c_\cdot$ is possible if we treat the "boundary" between the two cases more carefully.

Exercise 5

5.(a)

Denote $\tilde{X}_i = X_i/\sqrt{2}\sigma$, which should satisfy tail bound

$$\mathbb{P}\left(\left|\tilde{X}_i\right| \geq t\right) \leq 2e^{-t^2}$$

we have $\forall q \geq 1$:

$$\begin{aligned}
\mathbb{E}\left[\left|\tilde{X}_i\right|^q\right] =& \int_0^\infty \mathbb{P}\left(\left|\tilde{X}_i\right|^q > t\right) \mathrm{d}t \\
=& \int_0^\infty \mathbb{P}\left(\left|\tilde{X}_i\right| > t\right) qt^{q-1} \mathrm{d}t \\
\leq& \int_0^\infty 2e^{-t^2} qt^{q-1} \mathrm{d}t \\
=& q\Gamma(q/2) \leq q(\frac{q}{2})^{q/2}
\end{aligned}$$

thus we have

$$\mathbb{E}\left[|X_i|^q\right] \leq (\sqrt{2}\sigma)^q q(\frac{q}{2})^{q/2} = q(\sigma^2 q)^{q/2}$$

Then using the concavity of $x \mapsto x^{1/q}$, we have:

$$\begin{aligned}
\mathbb{E}\left[\|X\|_q\right] \leq& (\mathbb{E}\left[\sum_{i=1}^n |X_i|^q\right])^{1/q} \\
=& (\sum_{i=1}^n \mathbb{E}\left[|X_i|^q\right])^{1/q} \\
\leq& \left(nq(\sigma^2 q)^{q/2}\right)^{1/q} \\
=& (nq)^{1/q}(\sigma^2 q)^{1/2} \\
=& q^{1/q} n^{1/q} \sigma\sqrt{q}
\end{aligned}$$

Further note that $x \mapsto x^{1/x}$ has maximum at $x = e$ of maximum value $1.44\cdots < 4$, we have:

$$\mathbb{E}\left[\|X\|_q\right] \leq 4n^{1/q}\sigma\sqrt{q}$$

5.(b)

Note that we have relation:

$$\max|X_i| \leq \|X_i\|_q, \quad \forall q \geq 1$$

we only need to optimize the above bound for $\mathbb{E}\left[\|X\|_q\right] \leq 4n^{1/n}\sigma\sqrt{q}$, which takes minimum at $q = 2\log n$, yielding a bound of

$$\begin{aligned}
\mathbb{E}\left[\max|X_i|\right] \leq & \inf_{q\geq 1}\mathbb{E}\left[\|X\|_q\right] \\
\leq & \inf_{q\geq 1} 4n^{1/q}\sigma\sqrt{q} \\
= & 4\sigma\sqrt{2e\log n} \\
\leq & 4e\sigma\sqrt{\log n}
\end{aligned}$$