

Stat450 - 2024 Fall

HW1

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Exercise 1

1.(a)

We use the following equivalent expression for sub-Gaussian and sub-exponential random variables:

$$\text{Sub-Gaussian: } \exists \theta \geq 0, \text{ s.t. } \mathbb{E} [X^{2k}] \leq \frac{(2k)!}{2^k k!} \theta^{2k}, \forall k \in \mathbb{N}^+,$$

$$\text{Sub-Exponential: } \sup_{k \in \mathbb{N}^+} \left[\frac{\mathbb{E} [X^k]}{k!} \right]^{1/k} < \infty.$$

The proof is as follows:

\Rightarrow If X is sub-Gaussian with such above-mentioned θ , then we have:

$$\begin{aligned} \left[\frac{\mathbb{E} [(X^2)^k]}{k!} \right]^{1/k} &\leq \left[\frac{(2k)!}{k! k!} \frac{\theta^{2k}}{2^k} \right]^{1/k} \\ &= \left[\frac{(2k)!}{(2^k k!) \cdot (2^k k!)} \right]^{1/k} \cdot 2\theta \\ &= 2\theta \left[\prod_{i=1}^k \frac{2i}{2i} \prod_{i=1}^k \frac{2i-1}{2i} \right]^{1/k} \\ &\leq 2\theta < \infty. \end{aligned}$$

Thus proved that X^2 is sub-exponential.

\Leftarrow If X^2 is sub-exponential, then take some γ satisfying the suprema condition, we have:

$$\begin{aligned} \left[\frac{\mathbb{E} [(X^2)^k]}{k!} \right]^{1/k} &\leq \gamma \\ \Rightarrow \mathbb{E} [X^{2k}] &\leq \gamma^k k! \leq \frac{(2k)!}{2^k k!} (\sqrt{\gamma})^{2k}, \quad \forall k \in \mathbb{N}^+. \end{aligned}$$

Thus proved that X^2 is sub-Gaussian with $\theta \leq \sqrt{\gamma}$.

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1.(b)

We use the following equivalent expression for sub-Gaussian and sub-exponential random variables:²

$$\text{Sub-Gaussian: } \exists K_2 > 0, \text{ s.t. } \forall p \geq 1, \mathbb{E}[|X - \mathbb{E}[X]|^p]^{1/p} \leq K_2 \sqrt{p}$$

$$\text{Sub-Exponential: } \exists K_1 > 0, \text{ s.t. } \forall p \geq 1, \mathbb{E}[|X - \mathbb{E}[X]|^p]^{1/p} \leq K_1 p$$

Using the equivalentce, the proof is straightforward. Assume the K_2 parameter for X, Y are $K_{2,X}$ and $K_{2,Y}$ respectively and WLOG assume X, Y are both mean-zero. Using the convexity of $\|\exp \cdot\|_p := \mathbb{E}[e^\cdot]^p]^{1/p}$, we have:

$$\begin{aligned} \mathbb{E}[|XY|^p]^{1/p} &= \left\| e^{\frac{\log|X^2| + \log|Y^2|}{2}} \right\|_p \\ &\leq \frac{\left\| e^{\log|X^2|} \right\|_p + \left\| e^{\log|Y^2|} \right\|_p}{2} \\ &= \frac{1}{2} \mathbb{E}[|X|^{2p}]^{1/p} + \frac{1}{2} \mathbb{E}[|Y|^{2p}]^{1/p} \\ &\leq \frac{1}{2} (K_{2,X} \sqrt{p})^2 + \frac{1}{2} (K_{2,Y} \sqrt{p})^2 \\ &= \frac{K_{2,X}^2 + K_{2,Y}^2}{2} p, \quad \forall p \geq 1 \end{aligned}$$

Exercise 2

Denote the polynomial side bound as

$$M := \inf_{k \in \mathbb{N}} \frac{\mathbb{E}[X^k]}{\delta^k}$$

Using the taylor expansion of $x \mapsto e^x$ at 0, we have

$$\begin{aligned} \mathbb{E}[e^\lambda X] &= \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} \mathbb{E}[X^k] \\ &\geq \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} \delta^k M \\ &= M e^{\lambda \delta} \end{aligned}$$

²which are straightforward to prove using the definition of sub-Gaussian and sub-exponential random variables. Taking sub-gaussian as an example, we have:

$$\begin{aligned} \mathbb{E}[|X|^p] &= \int_0^\infty \mathbb{P}(|X|^p > t) dt \\ &= \int_0^\infty \mathbb{P}(|X| > t) p t^{p-1} dt \\ &\leq \int_0^\infty 2e^{-t^2/2\sigma^2} p t^{p-1} dt \\ &= p \Gamma(p/2) \sigma^p 2^{p/2} \\ &\leq p(\sqrt{p}\sigma)^p \end{aligned}$$

Thus we have:

$$\frac{\mathbb{E}[e^{\lambda}X]}{e^{\lambda\delta}} \geq M = \inf_{k \in \mathbb{N}} \frac{\mathbb{E}[X^k]}{\delta^k}, \quad \forall \lambda > 0$$

thus proved the result.

Exercise 3

Consider a random variable $X \sim \text{Binom}(n, 1/2)$, which has Moment Generating Function $M_X(\lambda) = (e^\lambda/2 + 1/2)^n$. Using Chernoff bound we have:

$$\begin{aligned} \frac{1}{2^n} \sum_{j=1}^k \binom{n}{j} &= \mathbb{P}(X \geq n - k) \\ &\leq \inf_{\lambda > 0} \left\{ n \log \frac{e^\lambda + 1}{2} - \lambda(n - k) \right\} := f(\lambda) \end{aligned}$$

we optimize the bound w.r.t. λ . R.H.S. takes minimum at $\lambda = \log \frac{n - k}{k}$, yielding the upper bound:

$$\log \sum_{j=1}^k \binom{n}{j} - n \log 2 \leq -n \log 2 + n \log \frac{n}{k} - (n - k) \log \frac{n - k}{k}$$

Now it suffices to prove the following inequality:

$$n \log \frac{n}{k} - (n - k) \log \frac{n - k}{k} \leq k \log \frac{ne}{k}$$

taking $\alpha = \frac{k}{n} \in (0, 1]$, it suffices to verify the following inequality:

$$f(\alpha) := \log \frac{1}{1 - \alpha} + \alpha \log \frac{1 - \alpha}{\alpha} - \alpha \log \frac{e}{\alpha} \leq 0$$

which can be easily verified that $f'(\alpha) \leq 0$, and notice that $f(0) = 0$, thus proved the inequality. And we have

$$\sum_{j=1}^k \binom{n}{j} \leq \left(\frac{en}{k} \right)^k$$

Exercise 4

4.(a)

We have

$$\begin{aligned} \text{var}(X) &= \mathbb{E}[(X - \mathbb{E}[X])^2] \\ &= \int_0^\infty \mathbb{P}((X - \mathbb{E}[X])^2 > t) \, dt \\ &\leq \int_0^\infty c_1 e^{-c_2 t} \, dt \\ &= \frac{c_1}{c_2} \end{aligned}$$

4.(b)

We can use example that $X = 1, 2, 3, 4$ equiprobably, for which median can take any number in $[2, 3]$

4.(c)

Using Chebyshev's inequality, we have:

$$\mathbb{P}(|X - \mu| \geq t) \leq \frac{\sigma^2}{t^2}$$

taking $t = \sqrt{2}\sigma$ we can obtain that $|m_X - \mu| \leq \sqrt{2}\sigma$. Using the fact we have:

- For $t \geq \sqrt{2}\sigma$:

$$\begin{aligned} \mathbb{P}(|X - m_X| \geq t) &\leq \mathbb{P}(|X - \mu| + |\mu - m_X| \geq t) \\ &= \mathbb{P}(|X - \mu| \geq t - |\mu - m_X|) \\ &\leq c_1 e^{-c_2(t - \sqrt{2}\sigma)^2} \leq \tilde{c}_3 e^{-c_4 t^2} \end{aligned}$$

- For $0 < t < \sqrt{2}\sigma$:

$$\mathbb{P}(|X - m_X| \geq t) \leq 1$$

To combine the two case, we just need to consider resetting \tilde{c}_3 s.t.

$$\tilde{c}_3 e^{-c_4 t^2} \Big|_{t=\sqrt{2}\sigma} = \tilde{c}_3 e^{-2\sigma^2 c_4} \geq 1$$

i.e. we can take

$$c_3 = \max\{\tilde{c}_3, e^{2\sigma^2 c_4}\}, \quad c_4 = c_2$$

4.(d)

To prove the converse statement, the idea is exactly the same by using $|\mu - m_X| \leq \sqrt{2}\sigma$:

- For $t \geq \sqrt{2}\sigma$:

$$\begin{aligned} \mathbb{P}(|X - \mu| \geq t) &\leq \mathbb{P}(|X - m_X| + |\mu - m_X| \geq t) \\ &= \mathbb{P}(|X - m_X| \geq t - |\mu - m_X|) \\ &\leq c_3 e^{-c_4(t - \sqrt{2}\sigma)^2} \leq \tilde{c}_1 e^{-c_2 t^2} \end{aligned}$$

- For $0 < t < \sqrt{2}\sigma$:

$$\mathbb{P}(|X - m_X| \geq t) \leq 1$$

To combine the two case, we just need to consider resetting \tilde{c}_1 s.t.

$$\tilde{c}_1 e^{-c_2 t^2} \Big|_{t=\sqrt{2}\sigma} = \tilde{c}_1 e^{-2\sigma^2 c_2} \geq 1$$

i.e. we can take

$$c_1 = \max\{\tilde{c}_1, e^{2\sigma^2 c_2}\}, \quad c_2 = c_4$$

Note: A better parameter c . is possible if we treat the "boundary" between the two cases more carefully.

Exercise 5

5.(a)

Denote $\tilde{X}_i = X_i/\sqrt{2}\sigma$, which should satisfy tail bound

$$\mathbb{P}\left(\left|\tilde{X}_i\right| \geq t\right) \leq 2e^{-t^2}$$

we have $\forall q \geq 1$:

$$\begin{aligned} \mathbb{E}\left[\left|\tilde{X}_i\right|^q\right] &= \int_0^\infty \mathbb{P}\left(\left|\tilde{X}_i\right|^q > t\right) dt \\ &= \int_0^\infty \mathbb{P}\left(\left|\tilde{X}_i\right| > t\right) qt^{q-1} dt \\ &\leq \int_0^\infty 2e^{-t^2} qt^{q-1} dt \\ &= q\Gamma(q/2) \leq q\left(\frac{q}{2}\right)^{q/2} \end{aligned}$$

thus we have

$$\mathbb{E}\left[\left|X_i\right|^q\right] \leq (\sqrt{2}\sigma)^q q\left(\frac{q}{2}\right)^{q/2} = q(\sigma^2 q)^{q/2}$$

Then using the concavity of $x \mapsto x^{1/q}$, we have:

$$\begin{aligned} \mathbb{E}\left[\|X\|_q\right] &\leq \left(\mathbb{E}\left[\sum_{i=1}^n |X_i|^q\right]\right)^{1/q} \\ &= \left(\sum_{i=1}^n \mathbb{E}\left[|X_i|^q\right]\right)^{1/q} \\ &\leq \left(nq(\sigma^2 q)^{q/2}\right)^{1/q} \\ &= (nq)^{1/q} (\sigma^2 q)^{1/2} \\ &= q^{1/q} n^{1/q} \sigma \sqrt{q} \end{aligned}$$

Further note that $x \mapsto x^{1/x}$ has maximum at $x = e$ of maximum value $1.44 \cdots < 4$, we have:

$$\mathbb{E}\left[\|X\|_q\right] \leq 4n^{1/q} \sigma \sqrt{q}$$

5.(b)

Note that we have relation:

$$\max |X_i| \leq \|X_i\|_q, \quad \forall q \geq 1$$

we only need to optimize the above bound for $\mathbb{E} [\|X\|_q] \leq 4n^{1/n}\sigma\sqrt{q}$, which takes minimum at $q = 2\log n$, yielding a bound of

$$\begin{aligned}\mathbb{E} [\max |X_i|] &\leq \inf_{q \geq 1} \mathbb{E} [\|X\|_q] \\ &\leq \inf_{q \geq 1} 4n^{1/q}\sigma\sqrt{q} \\ &= 4\sigma\sqrt{2e\log n} \\ &\leq 4e\sigma\sqrt{\log n}\end{aligned}$$