IEMS 402 Statistical Learning - 2025 Winter
HW1

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Exercise 1 Design of Loss Function

1.(a)

Consider decomposing $m(X)$ as follows:

$$m(X) := \mathbb{E}\left[Y|X=x\right] + \delta(X),$$

i.e. with $\delta(X)$ being the deviation of the true conditional expectation from the model. Then the expected ℓ_2 error can be written as:

$$\begin{aligned}
\mathbb{E}\left[(Y-m(X))^2\right] =& \mathbb{E}\left[(Y-\mathbb{E}\left[Y|X=x\right]-\delta(X))^2\right] \\
=& \mathbb{E}\left[(Y-\mathbb{E}\left[Y|X=x\right])^2\right] + \mathbb{E}\left[\delta^2(X)\right] - 2\mathbb{E}\left[(Y-\mathbb{E}\left[Y|X=x\right])\delta(X)\right] \\
\overset{(i)}{=}& \mathbb{E}\left[(Y-\mathbb{E}\left[Y|X=x\right])^2\right] + \mathbb{E}\left[\delta^2(X)\right] \\
\overset{(ii)}{\geq}& \mathbb{E}\left[(Y-\mathbb{E}\left[Y|X=x\right])^2\right].
\end{aligned}$$

Thus proved that the expected ℓ_2 error is minimized by the conditional expectation $\mathbb{E}\left[Y|X=x\right]$. Here in the proof, (i) is due to the fact that $\mathbb{E}_{Y|X=x}\left[Y-\mathbb{E}\left[Y|X=x\right]\big|X=x\right]=0$, and (ii) is due to the non-negativity of $\mathbb{E}\left[\delta^2(X)\right]$, and equality holds if and only if $\delta(X)=0$ almost surely.

1.(b)

The expected ℓ_1 error can be written as:

$$\mathbb{E}\left[|Y-m(X)|\right] = \mathbb{E}_X\left[\int_{Y|X=x} |Y-m(x)| \,\mathrm{d}F(y|X=x)\right]$$

taking variation with respect to $m(X)$, we have (here δ refers to the variation operator):

$$\begin{aligned}
\delta\mathbb{E}\left[|Y-m(X)|\right] =& \mathbb{E}_X\left[\int_{Y|X=x} \delta m(x)\cdot \mathrm{sgn}(Y-m(x))\,\mathrm{d}F(y|X=x)\right] \\
=& \mathbb{E}_X\left[\delta m(x)\int_{Y|X=x} \mathrm{sgn}(Y-m(x))\,\mathrm{d}F(y|X=x)\right]
\end{aligned}$$

To minimize the expected ℓ_1 error, we requre the variation taking value of zero, i.e.:

$$0=\delta\mathbb{E}\left[|Y-m(X)|\right] = \mathbb{E}_X\left[\delta m(x)\int_{Y|X=x} \mathrm{sgn}(Y-m(x))\,\mathrm{d}F(y|X=x)\right], \quad \forall \delta m(\,\cdot\,)$$

which requires choosing $m(\,\cdot\,)$ s.t. $\int_{Y|X=x}\mathrm{sgn}(Y-m(x))\,\mathrm{d}F(y|X=x) = \mathbb{E}_{Y|X=x}\left[\mathrm{sgn}(Y-m(x))|X=x\right] = 0$ almost surely. This is equivalent to using the conditional median as the prediction function $m(x) = \mathrm{median}(Y|X=x)$.

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1.(c)

Similarly we write the following differentiation w.r.t. β

$$\frac{\partial}{\partial\beta}\mathbb{E}\left[(Y-\beta' X)^2\right] = -\,\mathbb{E}\left[-2X(Y-\beta' X)\right]$$

and set it to zero, we have:

$$\begin{aligned} 0 =& -\,\mathbb{E}\left[-2X(Y-\beta' X)\right] \\ &\Rightarrow \beta_* = \mathbb{E}\left[XX'\right]^{-1}\mathbb{E}\left[XY\right] \end{aligned}$$

1.(d)

We consider the following function:

$$s_\alpha(y,\hat{y}) := \begin{cases} \alpha, & \text{if } y-\hat{y} > 0 \\ \alpha - 1, & \text{if } y-\hat{y} < 0 \end{cases}$$

and notice that $s_\alpha(y,\hat{y}) = \dfrac{\partial}{\partial y}\alpha\cdot\mathrm{sgn}(y-\hat{y})$[2]. Then we can write the deviation of expected ℓ_α error as:

$$\delta\mathbb{E}\left[\rho_\alpha(y,m(x))\right] =\mathbb{E}_X\left[\delta m(x)\int_{Y|X=x} s_\alpha(y,m(x))\,\mathrm{d}F(y|X=x)\right]$$

minimizing the ρ_α loss function requires the variation to be zero for any δm, i.e.:

$$0 =\mathbb{E}_{Y|X=x}\left[s_\alpha(y,m(x))|X=x\right] \Rightarrow m(x) = q_\alpha(x)$$

where $q_\alpha(x)$ is the α-quantile of Y given $X=x$.

Exercise 2 Central Limit Theorem

2.(a)

First by SLLN we definitely have $\bar{X}\xrightarrow{\text{a.s.}}\mathbb{E}\left[X\right]=\mu$.

Then we re-write the expression of s_n^2 as:

$$\begin{aligned} s_n^2 =& \frac{1}{n}\sum_{i=1}^n (X_i-\bar{X})^2 \\ =& \frac{1}{n}\sum_{i=1}^n X_i^2 - \bar{X}^2 \end{aligned}$$

[2]Omitting the discontinuity at $y-\hat{y}=0$, which won't be a big problem if we have continuous and strictly increasing loss function.

Then we notice that:

$$\begin{aligned}
&\text{by LLN: } \frac{1}{n}\sum_{i=1}^{n} X_i^2 \xrightarrow{p} \mathbb{E}\left[X^2\right] \\
&\text{by LLN and continuous mapping theorem: } \bar{X}^2 \xrightarrow{p} \left(\mathbb{E}\left[X\right]\right)^2
\end{aligned}$$

thus we have by slutsky's theorem:

$$s_n^2 \xrightarrow{\mathrm{d}} \mathbb{E}\left[X^2\right] - \left(\mathbb{E}\left[X\right]\right)^2 = \sigma^2$$

2.(b)

Note that $s_n^2 \xrightarrow{\mathrm{d}} \sigma^2$ which is a constant, thus we also have $s_n \xrightarrow{\mathrm{p}} \sigma$. Then by Slutsky's theorem and CLT, we have the following:

$$\begin{aligned}
&\frac{\sqrt{n}(\bar{X}_n - \mu)}{\sigma} \xrightarrow{\mathrm{d}} N(0,1) \\
&\frac{s_n}{\sigma} \xrightarrow{\mathrm{p}} 1 \\
\Rightarrow &\frac{\sqrt{n}(\bar{X}_n - \mu)}{s_n} \xrightarrow{\mathrm{d}} N(0,1)
\end{aligned}$$

Exercise 3 Curse of Dimensionality: Asymptotic scaling of nearest neighbor distances

3.(a)

$$\begin{aligned}
\mathbb{P}\left(\left\|x_{i(X_0)} - x_0\right\| > \delta\right) =& \mathbb{P}\left(\bigcap_{i=1}^{n}\{\|x_i - x_0\|_2 > \delta\}\right) \\
=& \int \mathrm{d}P_{x_0} \int \mathrm{d}P_{x_1^n} \prod_{i=1}^{n} \mathbf{1}_{\|x_i - x_0\|_2 > \delta} \\
=& \int \left(1 - P\left(B_d(x,\delta)\right)\right)^n \mathrm{d}P(x)
\end{aligned}$$

3.(b)

We can construct the partition as follows: at each dimention, construct cutting points $\{-kr, (-k+1)r, \ldots, (k-1)r, kr\}$ where k chosen s.t. $k = \lceil \frac{R}{\delta/d} \rceil \leq 2Rd/\delta$ and $r = \delta/d$. And each $U_\cdot$ is constructed by the combination of the cutting points. Then we have number of partition

$$N(\delta) = (2k+1)^d \leq \frac{8(Rd)^d}{\delta^d} = \frac{c}{\delta^d}$$

in this way, each "block" of the partition is at most a hypercube with side length r, and diameter diam $= r\sqrt{d} < \delta$.

3.(c)

Since the partition $U_1^{N(\delta)}$ has diameter at most δ, we consider THE block $U_\imath$ that contains x for each given x. Then we have:

$$U_\imath \subseteq B_d(x,\delta) \Rightarrow P\left(U_\imath\right) \leq P\left(B_d(x,\delta)\right)$$

thus we have:

$$\begin{aligned}
\mathbb{P}\left(\left\|x_{i(X_0)} - x_0\right\| > \delta\right) =& \int \left(1 - P\left(B_d(x,\delta)\right)\right)^n \mathrm{d}P(x)\\
\leq& \sum_{i=1}^{N(\delta)} \int_{U_i} \left(1 - P\left(P\left(U_i\right)\right)\right)^n \mathrm{d}P(x)\\
=& \sum_{i=1}^{N(\delta)} \left(1 - P\left(P\left(U_i\right)\right)\right)^n P\left(U_i\right)\\
\overset{(i)}{\leq}& \frac{c}{en\delta^d}.
\end{aligned}$$

Thus finished the proof. Here in the proof, (i) is due to the fact that $x \mapsto x(1-x)^n$ reaches maximum at $x = 1/(n+1)$, with maximum value

$$\frac{1}{n+1}(1 - \frac{1}{n+1})^n = \frac{1}{n}\left(1 - \frac{1}{n+1}\right)^{n+1} \leq \frac{1}{en}.$$

3.(d)

With the probabilistic bound, we note that to maintain the bound at $O(1)$, we should choose $\delta \asymp n^{-1/d}$ (so that $c/en\delta^d = O(1)$). Which indicates that

$$\mathbb{P}\left(\left\|x_{i(X_0)} - x_0\right\| \lesssim n^{-1/d}\right) \geq 1 - C$$

i.e. with certain minimal probability, the nearest neighbor distance is at most $\lesssim n^{-1/d}$.

Exercise 4

4.(a)

Note that $f_\theta(x) = 0$ is a hyper plane in $\mathbb{R}^d$, the distance from $x^{(i)}$ to which is

$$\begin{aligned}
\text{distance} =& \frac{\left|\theta' x^{(i)} + \theta_0\right|}{\|\theta\|}\\
=& \begin{cases} \dfrac{\left|\theta' x^{(i)} + \theta_0\right|}{\|\theta\|}, & \text{if } \theta' x^{(i)} + \theta_0 > 0\\ -\dfrac{\left|\theta' x^{(i)} + \theta_0\right|}{\|\theta\|}, & \text{if } \theta' x^{(i)} + \theta_0 < 0 \end{cases}
\end{aligned}$$

further for hard margin SVM, $\theta' x^{(i)} + \theta_0$ has the same sign as $y^{(i)}$, thus we have:

$$\text{distance} = \frac{y^{(i)}(\theta' x^{(i)} + \theta_0)}{\|\theta\|} = \gamma^(i)$$

4.(b)

Optimization problem for hard margin SVM can be written as:

$$
\begin{aligned}
&\arg\min_{\theta,\theta_0} \frac{1}{2}\left\|\theta\right\|^2 \\
&w.r.t. y^{(i)}(\theta' x^{(i)} + \theta_0) \geq 1
\end{aligned}
$$

and note that the decision boundary is determined by the (θ, θ_0), which has an extra degree of freedom w.r.t. scale transformation. We cancel this degree of freedom by setting a constraint $\|\theta\| = \dfrac{1}{M}$.

The Lagrangian can be written as:

$$
\mathcal{L}(\theta, \theta_0, \alpha_1^n, \kappa) = \frac{1}{2}\left\|\theta\right\|^2 - \sum_{i=1}^{n} \alpha_i \left(y^{(i)}(\theta' x^{(i)} + \theta_0) - 1 \right) - \kappa \left(\|\theta\| - \frac{1}{M} \right)
$$

and the optimization problem can be solved by minimizing the Lagrangian w.r.t. θ, θ_0 and maximizing w.r.t. α_1^n.

$$
\begin{aligned}
\frac{\partial \mathcal{L}}{\partial \theta} &= \theta - \sum_{i=1}^{n} \alpha_i y^{(i)} x^{(i)} = 0 \\
\frac{\partial \mathcal{L}}{\partial \theta_0} &= -\sum_{i=1}^{n} \alpha_i y^{(i)} = 0 \\
\frac{\partial \mathcal{L}}{\partial \alpha_i} &= y^{(i)}(\theta' x^{(i)} + \theta_0) - 1 = 0 \\
\frac{\partial \mathcal{L}}{\partial \kappa} &= \|\theta\| - \frac{1}{M} = 0
\end{aligned}
$$