

# A Brief Summary of Statistics Course

## 统计学课程知识总结

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## 前言

The statistics course summary actually started from the exam cheatsheets during my first year in minor (2020 fall). At first it only contained some notion explanation / important results. Later I began to add mathematical deductions, intuition of methods, useful external links, etc.  $\LaTeX$  is used to compile so that I could include cross reference, citation, index in the note so that I could look something up in case of forgetting. Also I treat it as an opportunity to practice  $\LaTeX$ . I made it publicly available in 2022 fall and from then on I've been trying to enrich early chapters, and correct / improve contents. I hope that this summary note could benefit students like me (e.g. statistics minor students in Tsinghua Univ.). Also I am considering to produce a ZH translation, which might be more convenient for beginners / people not used to English. Also it could act as a ZH-EN contrast. Now the translation of some chapters are already in progress (but slow lol). Please contact me if you are also interested.

Since it's first written as a reminder for statistics, I may not contain some basic knowledge like calculus, linear algebra, etc. And usually I just write it in a physics student perspective. Besides, for consistency within file and familiarity for myself, I might not use some commonly used notations (An example may be the size of designed matrix in . I use  $n \times (p + 1)$  while some others use  $n \times p$ ). I will consider adding notation declarations later ...

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## Chapter. I 概率论部分

Instructor: Wanlu Deng

### Section 1.1 Some Important Distributions

$X$	$p_X(k)/f_X(x)$	$\mathbb{E}$	$var$	MGF
Bern( $p$ )		$p$	$pq$	$q + pe^s$
$B(n, p)$	$C_n^k p^k (1-p)^{n-k}$	$np$	$npq$	$(q + pe^s)^n$
Geo( $p$ )	$(1-p)^{k-1} p$	$\frac{1}{p}$	$\frac{q}{p^2}$	$\frac{pe^s}{1 - qe^s}$
$H(n, M, N)$	$\frac{C_M^k C_{N-M}^{n-k}}{C_N^n}$	$n \frac{M}{N}$	$\frac{nM(N-n)(N-M)}{N^2(n-1)}$	
$P(\lambda)$	$\frac{\lambda^k}{k!} e^{-\lambda}$	$\lambda$	$\lambda$	$e^{\lambda(e^s-1)}$
$U(a, b)$	$\frac{1}{b-a}$	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$	$\frac{e^{sb} - e^{sa}}{(b-a)^s}$
$N(\mu, \sigma^2)$	$\frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$	$\mu$	$\sigma^2$	$e^{\frac{\sigma^2 s^2}{2} + \mu s}$
$\epsilon(\lambda)$	$\lambda e^{-\lambda x}$	$\frac{1}{\lambda}$	$\frac{1}{\lambda^2}$	$\frac{\lambda}{\lambda-s}$
$\Gamma(\alpha, \lambda)$	$\frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x}$	$\frac{\alpha}{\lambda}$	$\frac{\alpha}{\lambda^2}$	$\left(\frac{\lambda}{\lambda-s}\right)^\alpha$
$B(\alpha, \beta)$	$\frac{1}{B(\alpha, \beta)} x^{\alpha-1} (1-x)^{\beta-1}$	$\frac{\alpha}{\alpha+\beta}$	$\frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$	
$\chi_n^2$	$\frac{1}{2^{\frac{n}{2}} \Gamma(\frac{n}{2})} x^{\frac{n}{2}-1} e^{-\frac{x}{2}}$	$n$	$2n$	$(1-2s)^{-n/2}$
$t_\nu$	$\frac{\Gamma(\frac{\nu+1}{2})}{\sqrt{\nu\pi}\Gamma(\frac{\nu}{2})} (1 + \frac{x^2}{\nu})^{-\frac{\nu+1}{2}}$	$0$	$\frac{\nu}{\nu-2}$	
$F_{m,n}$	$\frac{\Gamma(\frac{m+n}{2})}{\Gamma(\frac{m}{2})\Gamma(\frac{n}{2})} m^{\frac{m}{2}-1} n^{\frac{n}{2}-1} x^{\frac{m}{2}-1}$	$\frac{n}{m+n}$	$\frac{2n^2(m+n-2)}{m(n-2)^2(n-4)}$	

Definition of PGF, MGF, CF see [section. 1.5](#).

More Properties of  $\chi^2, t, F$  see [section. 1.8.2](#).

Relation between distributions and more properties see <http://www.math.wm.edu/~leemis/chart/UDR/UDR.html>. Distribution support in R. see <https://CRAN.R-project.org/view=Distributions>

Use the following command for all distributions supported in R. `stats::.`

```
1 ?Distributions
```

### Section 1.2 Probability and Probability Model

What is **Probability**? A ‘belief’ in ‘what would happen’.

#### 1.2.1 Sample Space and $\sigma$ -Field

□ **Experiment and Sample Space**

Def. sample space  $\Omega$ : The set of all possible outcomes of one particular **experiment**. Conducting the experiment would result in a result/sample point  $\omega$  in sample space  $\Omega$ . These results should be mutually exclusive, e.g. Tossing two coins simultaneously, the sample space is the set of all possible results

$$\Omega = \{(0, 0), (0, 1), (1, 0), (1, 1)\}, \quad \omega \in \Omega \quad (1.1)$$

On the sample space, the ‘belief’ in results happening is measured by probability  $\mathbb{P}(\omega)$ ,  $\omega \in \Omega$

**Note:** Randomness comes from the random result  $\omega$  that an experiment generates.

#### □ Event

We may care about a combination of some results, say ‘at least one of the coin lands tails-up’. It’s like a kind of ‘structure’ on sample space describing how we put results together to form **Events**. The definition is a  $\sigma$ -field(or a  $\sigma$ -algebra)  $\mathcal{F}$  as a collection of some subsets of  $\Omega$ , with properties:

- $\Omega \in \mathcal{F}$
- if  $A \in \mathcal{F}$ , then  $A^c \in \mathcal{F}$
- if  $A_n \in \mathcal{F}$ , then  $\bigcup_{n=1}^{\infty} A_n \in \mathcal{F}$

And  $(\Omega, \mathcal{F})$  is a measurable space, on which we can select the events that we care about.

Events (and their properties) can be described in the language of set, e.g. for events  $A, B \in \mathcal{F}$

- $A = B$  means they are the same event
- $A \cup B$  means one of them happens
- $A \cap B$  or  $AB$  means both happen

And some more complex ones

- $A \cup B = B \cup A, A \cap B = B \cap A$
- $A \cup (B \cap C) = A \cup B \cap C, A \cap (B \cup C) = A \cap B \cup C$
- $A \cap (B \cup C) = (A \cap B) \cup (A \cap C), A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
- $A \cup B = A + A^c \cap B, A = A \cap B + A \cap B^c$

$$\Delta (A \cup B)^c = A^c \cap B^c, (A \cap B)^c = A^c \cup B^c$$

- $(\bigcup_{j=1}^{\infty} A_j)^c = \bigcap_{j=1}^{\infty} A_j^c$
- $(\bigcap_{j=1}^{\infty} A_j)^c = \bigcup_{j=1}^{\infty} A_j^c$

### 1.2.2 Axioms of Probability

$\mathbb{P}(\cdot) : \mathcal{F} \mapsto [0, 1]$  is the probability measure (or probability function) defined on  $(\Omega, \mathcal{F})$  describing the possibility that some event  $A \in \mathcal{F}$  happens. Definition of probability  $\mathbb{P}(A)$  in useful models:

$$\mathbb{P}(A) := \begin{cases} \frac{\#A}{\#\Omega} & \text{Classical Model} \\ \frac{m(A)}{m(\Omega)} & \text{Geometric Model} \end{cases} \quad (1.2)$$

Where  $m(\cdot)$  is some measure of events in continuous space, say integral in Euclidean Space  $\mathbb{R}^r$

$$m_{\mathbb{R}^r}(A) = \int_A dx_1 dx_2 \dots dx_r \quad (1.3)$$

#### □ Basic Axioms of Probability Measure $\mathbb{P}(\cdot)$

- Non-negativity

$$\mathbb{P}(A) \geq 0 \quad \forall A \in \Omega \quad (1.4)$$

- Normalization<sup>1</sup>

$$\mathbb{P}(\Omega) = 1 \quad (1.6)$$

- Countable Subadditivity

$$\mathbb{P}(A_1 \cup A_2 \cup \dots) = \mathbb{P}(A_1) + \mathbb{P}(A_2) + \dots, \quad (A_i \perp A_j \quad \forall i \neq j) \quad (1.7)$$

where ‘countable subadditivity’ means the events can be sequentially listed. e.g.  $[0, 1] = \bigcup_{x \in [0, 1]} \{x\}$  is not countable, intuition:

$$1 = \mathbb{P}([0, 1]) = \mathbb{P}\left(\bigcup_{x \in [0, 1]} \{x\}\right) \neq \sum_{x \in [0, 1]} \mathbb{P}(x) = 0 \quad (1.8)$$

Then  $(\Omega, \mathcal{F}, \mathbb{P})$  is probability space, where  $\Omega$  for experiment outcomes and randomness,  $\mathcal{F}$  for events and their algebra,  $\mathbb{P}$  for probability measure.

#### □ Properties of Probability:

- Addition Formula

$$\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B) \quad (1.9)$$

- Monotonicity

$$\mathbb{P}(A) \leq \mathbb{P}(B) \quad \text{for } A \subset B \quad (1.10)$$

- Finite Subadditivity (Boole Inequality)

$$\mathbb{P}\left(\bigcup_{i=1}^n A_i\right) \leq \sum_{i=1}^n \mathbb{P}(A_i) \quad (1.11)$$

- Countable Subadditivity ( $\sigma$ -Subadditivity)

$$\mathbb{P}\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \sum_{i=1}^{\infty} \mathbb{P}(A_i) \quad (1.12)$$

---

<sup>1</sup>Note: In other sections when dealing with not-yet-normalized distribution (say in Bayesian statistics), I usually use  $Z$  as the normalize constant, following the tradition in statistical physics where  $Z$  is the partition function.

$$\mathbb{P} = \frac{1}{Z} \tilde{\mathbb{P}}, \quad Z = \int \tilde{\mathbb{P}} \quad (1.5)$$

- Inclusion-Exclusion Formula (Jordan Formula)

$$\mathbb{P}\left(\bigcup_{i=1}^n A_i\right) = \sum_{1 \leq i \leq n} \mathbb{P}(A_i) - \sum_{1 \leq i < j \leq n} \mathbb{P}(A_i \cap A_j) \quad (1.13)$$

$$+ \sum_{1 \leq i < j < k \leq n} \mathbb{P}(A_i \cap A_j \cap A_k) - \dots \quad (1.14)$$

$$+ (-1)^{n-1} \mathbb{P}(A_1 \cap A_2 \cap \dots \cap A_n) \quad (1.15)$$

Or in condensed notation:

$$\mathbb{P}\left(\bigcup_{i=1}^n A_i\right) = \sum_{k=1}^n (-1)^{k-1} \sum_{1 \leq j_1 < j_2 < \dots < j_k \leq n} \mathbb{P}(A_{j_1} \cap A_{j_2} \cap \dots \cap A_{j_k}) \quad (1.16)$$

- Borel-Cantelli Lemma

$$\sum_{n=1}^{\infty} \mathbb{P}(A_n) < \infty \Rightarrow \mathbb{P}\left(\lim_{n \rightarrow \infty} \sup A_n\right) = 0 \quad (1.17)$$

$$\sum_{n=1}^{\infty} \mathbb{P}(A_n) = \infty \Rightarrow \mathbb{P}\left(\lim_{n \rightarrow \infty} \sup A_n\right) = 1 \quad \text{if } A_i \text{ independent} \quad (1.18)$$

### □ An Example

We have  $n$  different balls. Draw  $m$  times with replacement. What is the number of results regardless of order the balls drawn (e.g. {red, red, black} is the same as {red, black, red})?

The model is the same as we are ‘voting’ for  $n$  different balls, with total ballot ticket  $m$ . The  $m$  tickets are divided by  $n - 1$  plates (making them similar to ballot boxes), e.g. here’s a  $n = 4, m = 6$  vote corresponding to a result  $\omega \in \Omega$ :

$$\bullet || \bullet \bullet \bullet | \bullet \bullet \quad (1.19)$$

which the same as inserting plates sequentially and then cancel the order of plates:

$$\#\Omega = (m+1) * (m+2) \dots (m+n-1) / (n-1)! = \frac{(n+m-1)!}{m!(n-1)!} = \binom{n+m-1}{m} \quad (1.20)$$

(The idea of spacer plate is quite useful in dealing with some troublesome discrete cases, I think.)

表 1:  $\#\Omega$  of Sampling  $n$  balls  $m$  draw

	Replacement	
	With	Without
Ordered	$n^m$	$A_n^m$
Unordered	$\binom{n+m-1}{m}$	$\binom{n}{m}$

## 1.2.3 Conditional Probability

Motivation: To update the knowledge of probability measure.

Def. **Conditional Probability** of  $B$  given  $A$ :

$$\mathbb{P}(B|A) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(A)} \quad (1.21)$$

Actually it's a change of  $\sigma$ -field:  $\Omega \rightarrow B$

$$\mathbb{P}(B|A) = \frac{m(B)}{m(A)} \quad (1.22)$$

□ **Application of conditional probability:**

- Multiplication Formula

$$\mathbb{P}\left(\bigcap_{i=1}^n A_i\right) = \mathbb{P}(A_1) \prod_{i=2}^n \mathbb{P}(A_i | A_1 \cap A_2 \cap \dots \cap A_{i-1}) \quad (1.23)$$

- Total Probability Thm.

$$\mathbb{P}(B) = \sum_{i=1}^n \mathbb{P}(A_i) \mathbb{P}(B|A_i) \quad (1.24)$$

where  $\{A_i\}$  is a partition of  $\Omega$ :  $\Omega = \bigcup_i A_i$ ,  $A_i \cap A_j = \delta_{ij} \emptyset$

(Actually just  $B \subset \bigcup_i A_i$  is enough, similar for Bayes's rule)

- Bayes's Rule

$$\mathbb{P}(A_i|B) = \frac{\mathbb{P}(A_i) \mathbb{P}(B|A_i)}{\sum_{j=1}^n \mathbb{P}(A_j) \mathbb{P}(B|A_j)}, \quad 1 \leq i \leq n \quad (1.25)$$

where  $\{A_i\}$  is a partition of  $\Omega$ :  $\Omega = \bigcup_i A_i$ ,  $A_i \cap A_j = \delta_{ij} \emptyset$

## 1.2.4 Independency

Statistical Independency is defined as:

$$A \perp\!\!\!\perp B : \mathbb{P}(A \cap B) = \mathbb{P}(A) \mathbb{P}(B) \quad (1.26)$$

Properties

- Complement set and independency

$$A \perp\!\!\!\perp B \Leftrightarrow A^c \perp\!\!\!\perp B \quad (1.27)$$

- Independency of multiple events

$$A_1 \perp\!\!\!\perp A_2 \perp\!\!\!\perp \dots \perp\!\!\!\perp A_n \Leftrightarrow \mathbb{P}(A_{j_1} \cap A_{j_2} \cap \dots \cap A_{j_k}) = \mathbb{P}(A_{j_1}) \mathbb{P}(A_{j_2}) \dots \mathbb{P}(A_{j_k}) \quad (1.28)$$

$$\forall 1 \leq j_1 \leq j_2 \leq \dots \leq j_k \leq n \quad \forall k \leq n, \quad n < \infty \quad (1.29)$$

## Section 1.3 Random Variable and Distribution

Motivation: defining events is troublesome, and unhelpful to extract the key feature of events. A wise approach is to map samples & events to numbers  $\Omega \mapsto \mathbb{R}^r$ .



### 1.3.1 Random Variable

Def. Random Variable: a **function**/mapping  $X$  defined on sample space  $\Omega$ , from  $\Omega$  to some  $\mathcal{X} \in \mathbb{R}$ .

$$X(\omega) : \Omega \mapsto \mathcal{X} \in \mathbb{R} \quad (1.30)$$

**Note:** The mapping itself is non-random, the heart of randomness is still sample  $\omega$  experimented.

Naturally  $X$  induces a mapping of probability measure

$$F_X : \mathcal{X} \mapsto \Omega \mapsto \mathbb{P} \quad (1.31)$$

To describe the mapping of probability, def. Cumulative Distribution Function (CDF). (Here  $X(\omega)$  is still used to remind the origin of randomness, in most case we simply use  $X$ .)

$$F_X(x) = \mathbb{P}(X(\omega) \leq x) \quad (1.32)$$

• PMF:

PDF:

$$p_X(x) = F_X(x^+) - F_X(x^-) \quad (1.33) \quad f_X(x) = \frac{dF_X(x)}{dx} \quad (1.34)$$

• Right-Continuity of CDF: A physical perspective is that PMF could be written as<sup>2</sup>

$$p_X(x) = \sum_{\tilde{x} \in \mathcal{X}} \mathbb{P}(X = \tilde{x}) \delta(x - \tilde{x}) \quad (1.35)$$

where discrete  $X$  take values in  $\mathcal{X}$ . In this way for any infinitesimal interval containing  $x$ :  $\mathbb{I}_x \ni x$ , we have

$$F_X(x^+) - F_X(x^-) = \int_{\mathbb{I}_x} p_X(x) dx = \int_{\mathbb{I}_x} \sum_{\tilde{x} \in \mathcal{X}} \mathbb{P}(X = \tilde{x}) \delta(x - \tilde{x}) dx = \begin{cases} F_X(x^+) - F_X(x^-), & x \in \mathcal{X} \\ 0, & \text{others} \end{cases} \quad (1.36)$$

With such notation, in this note I sometimes ignore the difference between discrete cases / continuous cases.

• Representation of events: We could use random variable to express, say event  $A$  defined as

$$A := \{\omega : X(\omega) \leq x\} \quad (1.37)$$

• Indicator function:

$$\mathbb{I}_{x \in A}(x) = \begin{cases} 1 & x \in A \\ 0 & x \notin A \end{cases} \quad (1.38)$$

• Convolution

–  $W = X + Y$

$$f_W(w) = \int_{-\infty}^{\infty} f_X(x) f_Y(w - x) dx \quad (1.39)$$

–  $V = X - Y$

$$f_V(v) = \int_{-\infty}^{\infty} f_X(x) f_Y(x - v) dx \quad (1.40)$$

<sup>2</sup>Definition of Dirac  $\delta$  function see [section. 12.4.3](#).

–  $Z = XY$

$$f_Z(z) = \int_{-\infty}^{\infty} \frac{1}{|x|} f_X(x) f_Y\left(\frac{z}{x}\right) dx \quad (1.41)$$

Examples:

– Poisson<sup>3</sup>

$$P(\lambda_1) + P(\lambda_2) \sim P(\lambda_1 + \lambda_2) \quad (1.42)$$

– Binomial

$$B(n_1, p) + B(n_2, p) \sim B(n_1 + n_2, p) \quad (1.43)$$

– Gamma / Exponential

$$\Gamma(\alpha_1, \lambda) + \Gamma(\alpha_2, \lambda) \sim \Gamma(\alpha_1 + \alpha_2, \lambda) \quad (1.44)$$

with

$$\varepsilon(\lambda) = \Gamma(1, \lambda) \quad (1.45)$$

– More relations of distributions see <http://www.math.wm.edu/~leemis/chart/UDR/UDR.html>

– Relation between Poisson Process and Exponential and Uniform distribution see [section. 12.2.5](#).

• Order Statistics<sup>4</sup>

Def  $X_{(1)}, X_{(2)}, \dots, X_{(n)}$  as order statistics of  $\vec{X}$

$$g_{X_{(i)}} = n! \prod_i f(x_i) \quad \text{for } x_1 < x_2 < \dots < x_n \quad (1.46)$$

PDF of  $X_{(k)}$

$$g_k(x_k) = n C_{n-1}^{k-1} [F(x_k)]^{k-1} [1 - F(x_k)]^{n-k} f(x_k) \quad (1.47)$$

•  $p$ -fractile

$$\xi_p = F^{-1}(p) = \inf\{x | F(x) \geq p\} \quad (1.48)$$

### 1.3.2 Random Vector

A general case of random variable. Its definition is similar

$$\vec{X}(\omega) : \Omega \mapsto \mathcal{X} \in \mathbb{R}^n \quad (1.49)$$

a  $n$ -dimension Random Vector  $\vec{X} = (X_1, X_2, \dots, X_n)$  defined on  $(\Omega, \mathcal{F}, \mathbb{P})$ .

CDF  $F(x_1, \dots, x_n)$  defined on  $\mathbb{R}^n$ :

$$F(x_1, \dots, x_n) = \mathbb{P}(X_1 \leq x_1, \dots, X_n \leq x_n) \quad (1.50)$$

<sup>3</sup>More about Poisson Distribution / Poisson Process see [section. 12.1.4](#)

<sup>4</sup>A relative object is Rank statistics, see [section. 2.4.6](#).

Joint PDF of random vector:

$$f(x_1, \dots, x_n) = \frac{\partial^n F(x_1, \dots, x_n)}{\partial x_1 \dots \partial x_n} \quad (1.51)$$

$k$ -dimensional Marginal Distribution: For  $1 \leq k < n$  and index set  $S_k = \{i_1, \dots, i_k\}$ , distribution of  $\vec{X} = (X_{i_1}, X_{i_2}, \dots, X_{i_k})$

$$F_{S_k}(X_{i_1} \leq x_{i_1}, X_{i_2} \leq x_{i_2}, \dots, X_{i_k} \leq x_{i_k}) = \mathbb{P}(X_{i_1} \leq x_{i_1}, \dots, X_{i_k} \leq x_{i_k}; X_{i_{k+1}}, \dots, X_{i_n} \leq \infty) \quad (1.52)$$

Marginal distribution:

$$g_{S_k}(x_{i_1}, \dots, x_{i_k}) = \int_{\mathbb{R}^{n-k}} f(x_1, \dots, x_n) dx_{i_{k+1}} \dots dx_{i_n} = \frac{\partial^{n-k} F(x_1, \dots, x_n)}{\partial x_{i_{k+1}} \dots \partial x_{i_n}} \quad (1.53)$$

#### △ Function of r.v.

For  $\vec{X} = (X_1, X_2, \dots, X_n)$  with PDF  $f(\vec{X})$  and define

$$\vec{Y} = (Y_1, Y_2, \dots, Y_n) = (y_1(\vec{X}), y_2(\vec{X}), \dots, y_n(\vec{X})) \quad (1.54)$$

with inverse mapping

$$\vec{X} = (X_1, X_2, \dots, X_n) = (x_1(\vec{Y}), x_2(\vec{Y}), \dots, x_n(\vec{Y})) \quad (1.55)$$

then

$$g(\vec{Y}) = f(x_1(\vec{Y}), x_2(\vec{Y}), \dots, x_n(\vec{Y})) \left| \frac{\partial \vec{X}}{\partial \vec{Y}} \right| \mathbb{I}_{D_Y} \quad (1.56)$$

(Intuitively:  $g(\vec{Y})d\vec{Y} = d\mathbb{P} = f(\vec{X})d\vec{X}$ )

## Section 1.4 Expectation $\mathbb{E}$ , Variance $var$ and Covariance $cov$

Motivation: what would happen ‘on average’?

Expectation and Variance of common distributions see [section. 1.1](#).

### 1.4.1 Expection $\mathbb{E}(\cdot)$

Expectation of r.v.  $g(X)$  def.:

$$\mathbb{E}[g(X)] = \begin{cases} \int_{\Omega} g(x) f_X(x) dx = \int_{\Omega} g(x) dF(x) \\ \sum_{\Omega} g(x) f_X(x) \end{cases} \quad (1.57)$$

Sometimes when there are more than 1 variables, say  $x, y$ , we would use notation  $\mathbb{E}_X(g(X, Y))$  to specify the variable to avoid confusion.

**Note:** For discrete r.v. the expectation always exists, but for continuous & unbounded r.v. the expectation might diverge, rigorously speaking:

$$\mathbb{E}[X] \exists : \int_{\mathbb{R}} |x| f(x) dx < \infty \quad (1.58)$$

□ **Properties of Expectation  $E(\cdot)$ :**

- Linearity of Expectation

$$\mathbb{E}(aX + bY) = a\mathbb{E}(X) + b\mathbb{E}(Y) \quad (1.59)$$

- Conditional Expectation

$$\mathbb{E}(X|A) = \frac{\mathbb{E}(X\mathbb{I}_A)}{\mathbb{P}(A)} \quad (1.60)$$

Note: if take  $A$  as  $Y$  is also a r.v. then conditional expectation is actually a function of  $Y$

$$\xi(Y) = \mathbb{E}(X|Y) = \int x f_{X|Y}(x) dx \quad (1.61)$$

- Law of Total Expectation

$$\mathbb{E}_Y\{\mathbb{E}_X[g(X)|Y]\} = \mathbb{E}_X[g(X)] \quad (1.62)$$

- r.v.& Event

$$\mathbb{P}(A|X) = \mathbb{E}(\mathbb{I}_A|X) \Rightarrow \mathbb{E}[P(A|X)] = \mathbb{E}(\mathbb{I}_A) = \mathbb{P}(A) \quad (1.63)$$

- Conditional Expectation

$$\mathbb{E}[h(Y)g(X)|Y] = h(Y)\mathbb{E}[g(X)|Y] \quad (1.64)$$

### 1.4.2 Variance $var(\cdot)$

Variance of r.v.  $X$ :

$$var(X) = \mathbb{E}[(X - \mathbb{E}(X))^2] = \mathbb{E}(X^2) - (\mathbb{E}(X))^2 \quad (1.65)$$

(sometimes denoted as  $\sigma_X^2$ .)

Another definition comes from the MMSE estimation,

$$var(X) = \min_c \mathbb{E}[(X - c)^2] \quad (1.66)$$

its solution is  $c = \mathbb{E}[X]$ . See [section. 12.4.1](#) for more.

#### □ Properties:

- Linear combination of Variance

$$var(aX + b) = a^2 var(X) \quad (1.67)$$

- Conditional Variance

$$var(X|Y) = \mathbb{E}[X - \mathbb{E}(X|Y)]^2|Y \quad (1.68)$$

- Law of Total Variance

$$var(X) = \mathbb{E}[var(X|Y)] + var[\mathbb{E}(X|Y)] \quad (1.69)$$

Standard Deviation def. as :

$$\sigma_X = \sqrt{var(X)} \quad (1.70)$$

Then can construct **Standardization** of r.v.

$$X_{sd} = \frac{X - \mathbb{E}(X)}{\sqrt{var(X)}} \quad (1.71)$$

### 1.4.3 Covariance $cov(\cdot)$ and Correlation $corr(\cdot)$

Covariance of r.v.  $X$  and  $Y$ :

$$cov(X, Y) = \mathbb{E}[(X - \mu_X)(Y - \mu_Y)] = \mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y) \quad (1.72)$$

And Correlation Coefficient

$$\rho_{X,Y} = corr(X, Y) = \frac{cov(X, Y)}{\sqrt{var(X)var(Y)}} \quad (1.73)$$

Remark: correlation  $\nRightarrow$  cause and effect. Detail on causal effect topic see [Chapter. 11](#).

Properties:

- Bilinear of Covariance

$$cov(X + Y, Z) = cov(X, Z) + cov(Y, Z) \quad (1.74)$$

$$cov(X, Y + Z) = cov(X, Y) + cov(X, Z) \quad (1.75)$$

- Variance and Covariance

$$var(X + Y) = var(X) + var(Y) + 2cov(X, Y) \quad (1.76)$$

- Covariance Matrix

Def  $\Sigma = \mathbb{E}[(X - \mu)(X - \mu)^T] = \{\sigma_{ij}\}$  (where  $X$  should be considered as a column vector)

$$\Sigma = \begin{pmatrix} var(X_1) & cov(X_1, X_2) & \dots & cov(X_1, X_n) \\ cov(X_2, X_1) & var(X_2) & \dots & cov(X_2, X_n) \\ \vdots & \vdots & \ddots & \vdots \\ cov(X_n, X_1) & cov(X_n, X_2) & \dots & var(X_n) \end{pmatrix} \quad (1.77)$$

Attachment: Independence:

$$X_i \perp\!\!\!\perp X_j \Rightarrow \begin{cases} f(x_1, x_2, \dots, x_n) = \prod f(x_i) \\ F(x_1, x_2, \dots, x_n) = \prod F(x_i) \\ E(\prod X_i) = \prod E(X_i) \\ var(\sum X_i) = \sum var(X_i) \end{cases}, \quad n < \infty \quad (1.78)$$

## Section 1.5 PGF, MGF and C.F

Generating Function: Representation of  $\mathbb{P}$  in function space.  $\mathbb{P} \Leftrightarrow$  Generating Function.

### 1.5.1 Probability Generating Function

PGF: used for non-negative, integer  $X$ , which is the  $z$ -transform of  $p_X$

$$g(s) = \mathbb{E}(s^X) = \sum_{j=0}^{\infty} s^j \mathbb{P}(X = j), s \in [-1, 1] \quad (1.79)$$

□ Properties

- $\mathbb{P}(X = k) = \frac{g^{(k)}(0)}{k!}$
- $\mathbb{E}(X) = g^{(1)}(1)$
- $\text{var}(X) = g^{(2)}(1) + g^{(1)}(1) - [g^{(1)}(1)]^2$
- For  $X_1, X_2, \dots, X_n$  independent with  $g_i(s) = \mathbb{E}(s^{X_i})$ ,  $Y = \sum_{i=1}^n X_i$ , then

$$g_Y(s) = \prod_{i=1}^n g_i(s), s \in [-1, 1] \quad (1.80)$$

- For  $X_i$  i.i.d with  $\psi_i(s) = \psi(s) \equiv \mathbb{E}(s^{X_i})$ ,  $Y$  with  $G(s) \equiv \mathbb{E}(s^Y)$ ,  $W = X_1 + X_2 + \dots + X_Y$ , then

$$g_W(s) = G[\psi(s)] \quad (1.81)$$

- 2-Dimensional PGF of  $(X, Y)$

$$g(s, t) = \mathbb{E}(s^X t^Y) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \mathbb{P}_{(X,Y)}(X = i, Y = j) s^i t^j, \quad s, t \in [-1, 1] \quad (1.82)$$

## 1.5.2 Moment Generating Function

MGF: used for non-negative  $X$ , which is the Laplacian transformation of  $f_X$ .

$$M_X(s) = \mathbb{E}(e^{sX}) = \begin{cases} \sum_j e^{sx} \mathbb{P}(X = x_j) \\ \int_{-\infty}^{\infty} e^{sx} f_X(x) dx \end{cases} \quad (1.83)$$

Properties

- MGF of  $Y = aX + b$ :  $M_Y(s) = e^{sb} M(sa)$
- $\mathbb{E}(X^k) = M^{(k)}(0)$
- $\mathbb{P}(X = 0) = \lim_{s \rightarrow -\infty} M(s)$
- For  $X_1, X_2, \dots, X_n$  independent with  $M_{X_i}(s) = \mathbb{E}(e^{sX_i})$ ,  $Y = \sum_{i=1}^n X_i$ , then

$$M_Y(s) = \prod_{i=1}^n M_{X_i}(s) \quad (1.84)$$

## 1.5.3 Characteristic Function

C.F is actually the Fourier Transform of  $f_X$ .

$$\phi(t) = \mathbb{E}(e^{itX}) = \int_{-\infty}^{\infty} e^{itx} f_X(x) dx \quad (1.85)$$

Properties

- if  $E(|X|^k) < \infty$ , then

$$\phi^{(k)}(t) = i^k \mathbb{E}(X^k e^{itX}) \quad \phi^{(k)}(0) = i^k \mathbb{E}(X^k) \quad (1.86)$$

- For  $X_1, X_2, \dots, X_n$  independent with  $\phi_{X_i}(t) = \mathbb{E}(e^{itX_i})$ ,  $Y = \sum_{i=1}^n X_i$ , then

$$\phi_Y(t) = \prod_{i=1}^n \phi_{X_i}(t) \quad (1.87)$$

- Inverse (Fourier) Transform

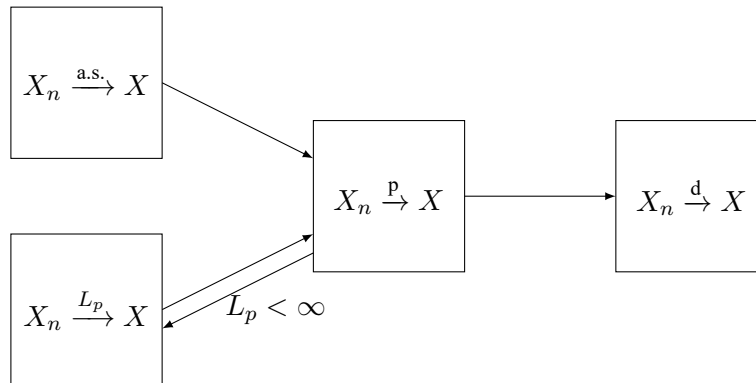
$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-itx} \phi(t) dt \quad (1.88)$$

## Section 1.6 Convergence and Limit Distribution

### 1.6.1 Convergence Mode

$$\left\{ \begin{array}{ll} \text{Convergence in Distribution} & X_n \xrightarrow{d} X : \lim_{n \rightarrow \infty} F_n(x) = F(x) \\ \text{Convergence in Probability} & X_n \xrightarrow{p} X : \lim_{n \rightarrow \infty} \mathbb{P}(|X_n - X| \geq \varepsilon) = 0, \forall \varepsilon > 0 \\ \text{Almost Sure Convergence} & X_n \xrightarrow{\text{a.s.}} X : \mathbb{P}(\lim_{n \rightarrow \infty} X_n = X) = 1 \\ L_p \text{ Convergence} & X_n \xrightarrow{L_p} X : \lim_{n \rightarrow \infty} \mathbb{E}(|X_n - X|^p) = 0 \end{array} \right. \quad (1.89)$$

Relations between convergence:



Note:  $L_2$  convergence is also denoted m.s. (mean squared) convergence  $\xrightarrow{\text{m.s.}}$ .

Useful Thm.:

- Continuous Mapping Thm.: For continuous function  $g(\cdot)$

1.  $X_n \xrightarrow{\text{a.s.}} X \Rightarrow g(X_n) \xrightarrow{\text{a.s.}} g(X)$
2.  $X_n \xrightarrow{p} X \Rightarrow g(X_n) \xrightarrow{p} g(X)$
3.  $X_n \xrightarrow{d} X \Rightarrow g(X_n) \xrightarrow{d} g(X)$

- Slutsky's Thm.: For  $X_n \xrightarrow{d} X, Y_n \xrightarrow{p} c$

1.  $X_n + Y_n \xrightarrow{d} X + c$

2.  $X_n Y_n \xrightarrow{d} cX$
3.  $X_n/Y_n \xrightarrow{d} X/c$

- Continuity Thm. for characteristic function:

$$\lim_{n \rightarrow \infty} \phi_n(t) = \varphi(t) \Leftrightarrow X_n \xrightarrow{d} X \quad (1.90)$$

## 1.6.2 Law of Large Number & Central Limit Theorem

- m.s. LLN: For  $X_i$  with  $\text{cov}(X_i, X_j) = 0$ , if  $i \neq j$ , and  $\mathbb{E}[X_i] = \mu < \infty$

$$\frac{1}{n} \sum X_i \xrightarrow{L_2} \mathbb{E}[X_1] \quad (1.91)$$

- WLLN: For  $X_i$  i.i.d.  $\sim f_X$ , with  $\mathbb{E}[X_i] = \mu < \infty$

$$\frac{1}{n} \sum X_i \xrightarrow{P} \mu \quad (1.92)$$

- SLLN: For  $X_i$  i.i.d.  $\sim f_X$ , with  $\mathbb{E}[X_i] = \mu < \infty$

$$\frac{1}{n} \sum X_i \xrightarrow{\text{a.s.}} \mu \quad (1.93)$$

- CLT: For  $X_i$  i.i.d.  $\sim f_X$ , with  $\mathbb{E}[X_i] = \mu < \infty$ ,  $\text{var}(X_i) = \sigma^2 < \infty$

$$\frac{\sqrt{n}(\bar{X} - \mu)}{\sigma} \xrightarrow{d} N(0, 1) \quad (1.94)$$

or in equivalent form

$$\frac{1}{\sigma\sqrt{n}} \sum (X_k - \mu) \xrightarrow{d} N(0, 1) \quad (1.95)$$

$$\bar{X} \xrightarrow{d} N\left(\mu, \frac{\sigma^2}{n}\right) \quad (1.96)$$

□ *Proof:* Denote the characteristic function of  $X \sim f_X(x)$  as  $\phi_X(t) := \mathbb{E}[e^{itX}]$ , with expectation  $\mu := \mathbb{E}[X]$  and variance  $\sigma^2 := \text{var}(X) = \mathbb{E}[X^2] - \mu^2$ .

Define  $Z = \frac{X - \mu}{\sigma}$ . The Taylor series of  $\phi_Z(t)$  at  $t = 0$  yields:

$$\phi_Z(t) = 1 - \frac{t^2}{2} + o(t^2)$$

The characteristic function of mean  $\bar{Z} := \frac{1}{n} \sum_{i=1}^n Z_i = \frac{1}{n} \sum_{i=1}^n \frac{X_i - \mu}{\sigma}$  w.r.t.  $X_i$  i.i.d.  $\sim f_X(x)$

$$\phi_{\bar{Z}}(t) = \mathbb{E}[e^{it\bar{Z}}] = \left[ \phi_Z\left(\frac{t}{n}\right) \right]^n = \left[ 1 - \frac{t^2}{2n^2} \right]^n$$

with  $n \rightarrow \infty$  limit:<sup>5</sup>

$$\lim_{n \rightarrow \infty} \phi_{\bar{Z}}(t) = \lim_{n \rightarrow \infty} \left[ 1 - \frac{t^2}{2n} \right]^n = e^{-\frac{t^2}{2}} \Rightarrow \bar{Z} = \frac{\bar{X} - \mu}{\sigma} \xrightarrow{d} N\left(0, \frac{1}{n}\right)$$

□

<sup>5</sup>Note: if use characteristic function of  $X_i$  directly, notice that

$$n \log \left( 1 + \frac{at}{n} - \frac{bt^2}{2n^2} \right) = at - (b + a^2) \frac{t^2}{2n} + \mathcal{O}\left(\frac{1}{n^2}\right)$$

using the Taylor series of  $\log(1 + \xi)$  at  $\xi = 0$ .



- de Moivre-Laplace Thm. is a special case of CLT at  $S_n \sim B(n, p)$

$$\mathbb{P}(k \leq S_n \leq m) \approx \Phi\left(\frac{m + 0.5 - np}{\sqrt{npq}}\right) - \Phi\left(\frac{k - 0.5 - np}{\sqrt{npq}}\right) \quad (1.97)$$

- Stirling Eqa. derived from CLT

$$\frac{\lambda^k}{k!} e^{-\lambda} \approx \frac{1}{\sqrt{\lambda} \sqrt{2\pi}} e^{-\frac{(k-\lambda)^2}{2\lambda}} \xrightarrow[\lambda=n]{k=n} n! \approx \sqrt{2\pi n} \left(\frac{n}{e}\right)^n \sim O\left(\left(\frac{n}{e}\right)^n\right) \quad (1.98)$$

## Section 1.7 Inequalities

- Cauchy-Schwarz Inequality

$$|\mathbb{E}(XY)| \leq \sqrt{\mathbb{E}(X^2)\mathbb{E}(Y^2)} \quad (1.99)$$

- Bonferroni Inequality

$$\mathbb{P}\left(\bigcup_{i=1}^n A_i\right) \geq \sum_{1 \leq i \leq n} \mathbb{P}(A_i) + \sum_{1 \leq i < j \leq n} \mathbb{P}(A_i \cap A_j) \quad (1.100)$$

- Markov Inequality

$$\mathbb{P}(|X| \geq \epsilon) \leq \frac{\mathbb{E}(|X|^\alpha)}{\epsilon^\alpha} \quad (1.101)$$

with  $\alpha = 1$ , and  $\epsilon$  selected as a multiple of  $\mathbb{E}[|X|]$ :

$$\mathbb{P}(|X| \geq m\mathbb{E}[|X|]) \leq \frac{1}{m} \quad (1.102)$$

- Chebyshev Inequality

$$\mathbb{P}(|X - \mathbb{E}(X)| \geq \epsilon) \leq \frac{\text{var}(X)}{\epsilon^2} \quad (1.103)$$

Chebyshev inequality is used to proof WLLN [equation. 1.92](#)

- Jensen Inequality: For convex function  $h(x)$ :<sup>6</sup>

$$\mathbb{E}[h(X)] \geq h(\mathbb{E}(X)) \quad (1.105)$$

Example of using Jensen Eqa. to proof some other inequalities:

- Non-negativity of Kullback-Leibler Divergence: For two distributions  $f(\cdot)$  and  $g(\cdot)$ , the K-L Divergence is defined as

$$\text{KL}(f\|g) := - \int f(x) \log \frac{g(x)}{f(x)} dx \quad (1.106)$$

Take  $h(\xi) := \log \xi$  a concave function for  $\xi \in (0, \infty)$  and  $Z := \frac{g(X)}{f(X)}$  with  $X \sim f(x)$ , then

$$\mathbb{E}(h(Z)) = \int_A (\log z) f_Z(z) dz = \int_A \left(\log \frac{g(x)}{f(x)}\right) f(x) dx \quad (1.107)$$

$$\leq h(\mathbb{E}(Z)) = \log \int_A z f_Z(z) dz = \log \int_A \frac{g(x)}{f(x)} f(x) dx = 0 \quad (1.108)$$

$$\Rightarrow - \int_A \log f(x) \frac{g(x)}{f(x)} dx \geq 0 \quad (1.109)$$

<sup>6</sup>Or equivalently for concave function  $\tilde{h}(x)$ :

$$\mathbb{E}[\tilde{h}(X)] \leq \tilde{h}(\mathbb{E}(X)) \quad (1.104)$$

- Cantelli Inequality

$$\mathbb{P}(X - \mathbb{E}[X] \geq \lambda) \leq \frac{\text{var}(X)}{\text{var}(X) + \lambda^2} \quad (1.110)$$

with  $\lambda = \sqrt{\text{var}(X)} := \sigma$ , we have

$$\begin{cases} \mathbb{P}(X \geq \mathbb{E}[X] + \sigma) \leq \frac{1}{2} \\ \mathbb{P}(X \leq \mathbb{E}[X] - \sigma) \leq \frac{1}{2} \end{cases} \quad (1.111)$$

i.e. difference between mean and median is upperly bounded by standard deviation

$$|\mathbb{E}[X] - \text{med}(X)| \leq \sigma \quad (1.112)$$

- Hoeffding Inequality: with independent r.v. sequence  $X_i \in [a_i, b_i]$ , and  $S_n := \sum_{i=1}^n X_i$

$$\mathbb{P}(|S_n - \mathbb{E}[S_n]| \geq \varepsilon) \leq 2 \exp \left[ -\frac{2\varepsilon^2}{\sum_{i=1}^n (b_i - a_i)^2} \right] \quad (1.113)$$

Or in equivalent form  $\varepsilon = nt$

$$\mathbb{P} \left( \frac{1}{n} \left| \sum_{i=1}^n (X_i - \mathbb{E}[X_i]) \right| \geq t \right) \leq 2 \exp \left[ -\frac{2n^2 t^2}{\sum_{i=1}^n (b_i - a_i)^2} \right]$$

For special case of  $[a_i, b_i] = [a, b]$ ,  $\forall i$ ,  $|[a, b]| := L$ ,

$$\mathbb{P} \left( \frac{1}{n} \left| \sum_{i=1}^n (X_i - \mathbb{E}[X_i]) \right| \geq t \right) \leq 2 \exp \left[ -\frac{2nt^2}{L^2} \right]$$

The proof needs the Hoeffding Lemma: for  $\mathbb{E}[Z] = 0$  and  $Z \in [a, b]$

$$\mathbb{E}[e^{tZ}] \leq \exp \left[ \frac{t^2(b-a)^2}{8} \right], \quad \forall t \quad (1.114)$$

- McDiarmid Inequality: with independent r.v. sequence  $X_i$ , and a function  $f(\cdot)$  with bounded difference  $c_i$ :

$$|f(X_1, X_2, \dots, X_{n+1}) - f(X_1, X_2, \dots, X_n)| \leq c_i$$

we have McDiarmid inequality

$$\mathbb{P}(|f(X_1, \dots, X_n) - \mathbb{E}[f(X_1, \dots, X_n)]| \geq nt) \leq 2 \exp \left[ -\frac{2n^2 t^2}{\sum_{i=1}^n c_i^2} \right]$$

## Section 1.8 Multivariate Normal Distribution

General Case and more discussion see [section. 4.2.1](#).

Distribution of Normal  $X \sim N(\mu, \sigma^2)$ :

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

For  $X_1, X_2, \dots, X_n$  independent and  $X_k \sim N(\mu_k, \sigma_k^2)$ ,  $k = 1, \dots, n$ ,  $T = \sum_{k=1}^n c_k X_k$ , ( $c_k$  const), then

$$T \sim N\left(\sum_{k=1}^n c_k \mu_k, \sum_{k=1}^n c_k^2 \sigma_k^2\right) \quad (1.115)$$

Deduction in some special cases:

- Given  $\mu_1 = \mu_2 = \dots = \mu_n = \mu$ ,  $\sigma_1^2 = \sigma_2^2 = \dots = \sigma_n^2 = \sigma^2$ , i.e.  $X_k$  i.i.d., then

$$T \sim N\left(\mu \sum_{k=1}^n c_k, \sigma^2 \sum_{k=1}^n c_k^2\right) \quad (1.116)$$

- Further take  $c_1 = c_2 = \dots = c_n = \frac{1}{n}$ , i.e.  $T = \sum_{k=1}^n X_k/n = \bar{X}$ , then

$$T = \bar{X} \sim N\left(\mu, \frac{\sigma^2}{n}\right) \quad (1.117)$$

### 1.8.1 Linear Transform

First consider  $\epsilon_1, \epsilon_2, \dots, \epsilon_m$  i.i.d.  $\sim N(0, 1)$ ,  $n \times 1$  const column vector  $\vec{\mu}$ ,  $n \times m$  const matrix  $\mathbf{B} = \{b_{ij}\}$ ,  
 def.  $X_i = \sum_{j=1}^m b_{ij}\epsilon_j$ , i.e.

$$\vec{X} = \begin{pmatrix} X_1 \\ X_2 \\ \vdots \\ X_n \end{pmatrix} = \begin{pmatrix} b_{11} & b_{12} & \dots & b_{1m} \\ b_{21} & b_{22} & \dots & b_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \dots & b_{nm} \end{pmatrix} \begin{pmatrix} \epsilon_1 \\ \epsilon_2 \\ \vdots \\ \epsilon_m \end{pmatrix} + \begin{pmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_n \end{pmatrix} = \mathbf{B}\vec{\epsilon} + \vec{\mu} \quad (1.118)$$

We have:  $\vec{X} \sim N(\vec{\mu}, \Sigma)$ , where  $\Sigma$ , as defined in [equation. 1.77](#) is

$$\Sigma = \mathbb{E}[(\vec{X} - \vec{\mu})(\vec{X} - \vec{\mu})^T] = \mathbf{B}\mathbf{B}^T = \begin{pmatrix} \text{var}(X_1) & \text{cov}(X_1, X_2) & \dots & \text{cov}(X_1, X_n) \\ \text{cov}(X_2, X_1) & \text{var}(X_2) & \dots & \text{cov}(X_2, X_n) \\ \vdots & \vdots & \ddots & \vdots \\ \text{cov}(X_n, X_1) & \text{cov}(X_n, X_2) & \dots & \text{var}(X_n) \end{pmatrix} = \{\sigma_{ij}\} \quad (1.119)$$

Furthur Consider  $\vec{Y} = (Y_1, \dots, Y_n)^T$ ,  $n \times n$  const square matrix  $\mathbf{A} = \{a_{ij}\}$  and def.  $\vec{Y} = \mathbf{A}\vec{X}$  i.e.

$$\begin{pmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_n \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \\ \vdots \\ X_n \end{pmatrix} \quad (1.120)$$

Then  $\vec{Y} \sim N(\mathbf{A}\vec{\mu}, \mathbf{A}\Sigma\mathbf{A}^T)$

Special case:  $X_1, \dots, X_n$  i.i.d.  $\sim N(\mu, \sigma^2)$ ,  $\vec{X} = (X_1, \dots, X_n)^T$ ,

$$\mathbb{E}(Y_i) = \mu \sum_{k=1}^n a_{ik} \quad (1.121)$$

$$\text{var}(Y_i) = \sigma^2 \sum_{k=1}^n a_{ik}^2 \quad (1.122)$$

$$\text{cov}(Y_i, Y_j) = \sigma^2 \sum_{k=1}^n a_{ik}a_{jk} \quad (1.123)$$

Specially when  $A = \{a_{ij}\}$  orthonormal, we have  $Y_1, \dots, Y_n$  independent

$$Y_i \sim N\left(\mu \sum_{k=1}^n a_{ik}, \sigma^2\right) \quad (1.124)$$

#### □ Definition of Jointly Gaussian/Normal

A random vector  $\vec{X}$  is called jointly Gaussian if and only if any (finite) linear combination of  $\vec{X}$  is still Gaussian (Normal)

$$\sum_{k=1}^m \alpha_k X_{i_k} \sim N(\cdot, \cdot), \forall \{\alpha_k\}_{k=1}^m, \forall \{i_k\}_{k=1}^m, \forall m \leq n \quad (1.125)$$

Counter Example:  $[X, Y]$  in which  $X \sim N(0, 1)$ ,  $Y = -X$  is not jointly Gaussian.

### 1.8.2 Distributions of Function of Normal Variable: $\chi^2$ , $t$ & $F$

Consider  $X_1, X_2, \dots, X_n$  i.i.d.  $\sim N(0, 1)$ ;  $Y, Y_1, Y_2, \dots, Y_m$  i.i.d.  $\sim N(0, 1)$

- $\chi^2$  Distribution: Def.  $\chi^2$  distribution with degree of freedom  $n$ :

$$\xi = \sum_{i=1}^n X_i^2 \sim \chi_n^2 \quad (1.126)$$

PDF of  $\chi_n^2$ :

$$g_n(x) = \frac{1}{2^{n/2} \Gamma(n/2)} x^{\frac{n}{2}-1} e^{-x/2} \mathbb{I}_{x>0} \quad (1.127)$$

Properties

- $\mathbb{E}$  and  $var$  of  $\xi \sim \chi_n^2$

$$\mathbb{E}(\xi) = n \quad var(\xi) = 2n \quad (1.128)$$

- For independent  $\xi_i \sim \chi_{n_i}^2$ ,  $i = 1, 2, \dots, k$ :

$$\xi_0 = \sum_{i=1}^k \xi_i \sim \chi_{n_1+\dots+n_k}^2 \quad (1.129)$$

- Denoted as  $\Gamma(\alpha, \lambda)$ :

$$\xi = \sum_{i=1}^n X_i^2 \sim \Gamma\left(\frac{n}{2}, \frac{1}{2}\right) = \chi_n^2 \quad (1.130)$$

- $t$  Distribution: Def.  $t$  distribution with degree of freedom  $n$ :

$$T = \frac{Y}{\sqrt{\frac{\sum_{i=1}^n X_i^2}{n}}} = \frac{Y}{\sqrt{\xi/n}} \sim t_n \quad (1.131)$$

(Usually take  $\nu$  instead of  $n$  as degree of freedom for  $t$  distribution)

PDF of  $t_\nu$ :

$$t_\nu(x) = \frac{\Gamma(\frac{\nu+1}{2})}{\Gamma(\frac{\nu}{2})\sqrt{\nu\pi}} \left(1 + \frac{x^2}{\nu}\right)^{-\frac{\nu+1}{2}} \quad (1.132)$$

Denote: Upper  $\alpha$ -fractile of  $t_\nu$ , satisfies  $\mathbb{P}(T \geq c) = \alpha$ :

$$t_{\nu,\alpha} = \arg_c \mathbb{P}(T \geq c) = \alpha, \quad T \sim t_\nu \quad (1.133)$$

(Similar for  $N$ ,  $\chi_n^2$  and  $F_{m,n}$  etc.)

- $F$  Distribution: Def.  $F$  distribution with degree of freedom  $m$  and  $n$ :

$$F = \frac{\sum_{i=1}^m Y_i^2/m}{\sum_{i=1}^n X_i^2/n} \sim F_{m,n} \quad (1.134)$$

PDF of  $F_{m,n}$ :

$$f_{m,n}(x) = \frac{\Gamma(\frac{m+n}{2})m^{\frac{m}{2}}n^{\frac{n}{2}}}{\Gamma(\frac{m}{2})\Gamma(\frac{n}{2})}x^{\frac{m}{2}-1}(mx+n)^{-\frac{m+n}{2}}\mathbb{I}_{x>0} \quad (1.135)$$

Properties

- If  $Z \sim F_{m,n}$ , then  $\frac{1}{Z} \sim F_{n,m}$ .
- If  $T \sim t_n$ , then  $T^2 \sim F_{1,n}$
- $F_{m,n,1-\alpha} = \frac{1}{F_{n,m,\alpha}}$

□ Some useful Lemma (used in statistic inference, see [section. 2.3.3](#)):

- For  $X_1, X_2, \dots, X_n$  independent with  $X_i \sim N(\mu_i, \sigma_i^2)$ , then

$$\sum_{i=1}^n \left( \frac{X_i - \mu_i}{\sigma_i} \right)^2 \sim \chi_n^2 \quad (1.136)$$

- For  $X_1, X_2, \dots, X_n$  i.i.d.  $\sim N(\mu, \sigma^2)$ , then

$$T = \frac{\sqrt{n}(\bar{X} - \mu)}{S} \sim t_{n-1} \quad (1.137)$$

For  $X_1, X_2, \dots, X_m$  i.i.d.  $\sim N(\mu_1, \sigma^2)$ ,  $Y_1, Y_2, \dots, Y_n$  i.i.d.  $\sim N(\mu_2, \sigma^2)$ ,

denote sample pooled variance  $S_\omega^2 = \frac{(m-1)S_1^2 + (n-1)S_2^2}{m+n-2}$ , then

$$T = \frac{(\bar{X} - \bar{Y}) - (\mu_1 - \mu_2)}{S_\omega} \cdot \sqrt{\frac{mn}{m+n}} \sim t_{m+n-2} \quad (1.138)$$

- For  $X_1, X_2, \dots, X_m$  i.i.d.  $\sim N(\mu, \sigma^2)$ ,  $Y_1, Y_2, \dots, Y_n$  i.i.d.  $\sim N(\mu_2, \sigma^2)$ , then

$$T = \frac{S_1^2 \sigma_2^2}{S_2^2 \sigma_1^2} \sim F_{m-1, n-1} \quad (1.139)$$

- For  $X_1, X_2, \dots, X_n$  i.i.d.  $\sim \epsilon(\lambda)$ , then

$$2\lambda n \bar{X} = 2\lambda \sum_{i=1}^n X_i \sim \chi_{2n}^2 \quad (1.140)$$

Remark: for  $X_i \sim \epsilon(\lambda) = \Gamma(1, \lambda) \Rightarrow 2\lambda \sum_{i=1}^n X_i \sim \Gamma(n, 1/2) = \chi_{2n}^2$ .

## Chapter. II 统计推断部分

Instructor: Jiangdian Wang

**Statistical Inference:** Given sample  $X = (x_1, x_2, \dots, x_n)$ , we want to estimate some features of the population. This part focus on parametric statistical inference, thus our task is to estimate/testing parameters.

### □ Example of statistical inference

- Sample item  $x_i$ , estimate its mean and variance
- Sample item  $x_i = (\vec{x}_i, y_i)$ , use multivariate linear model  $Y \sim \vec{X}'\beta + \beta_0$ , estimate slope & intercept  $\beta$  and variance  $\sigma^2$

### □ Two main tasks of Statistical Inference

- Parameter Estimation
  - Point Estimation: [section. 2.2](#)
  - Interval Estimation: [section. 2.3](#)
- Hypothesis Testing: [section. 2.4](#)

### ▷ R. Code

Example data `x`, `y`, `df` used in this section:

```
1 set.seed(42)
2 x <- rnorm(n = 50, mean = 2, sd = 2)
3 y <- 0.5*x + rnorm(n = 50, mean = 2.1, sd = 2.1)
4 df <- data.frame(x=x, y=y)
```

## Section 2.1 Statistical Model and Statistics

Random sample comes from population  $X$ . In parametric model case, we have population distribution family:

$$\mathcal{F} = \{f(x; \vec{\theta}) | \vec{\theta} \in \Theta\} \quad (2.1)$$

where parameter  $\vec{\theta}$  reflect some quantities of population (e.g. mean, variance, etc.), each  $\vec{\theta}$  corresponds to a distribution of population  $X$ .

Sample space: Def. as  $\mathcal{X} = \{\{x_1, x_2, \dots, x_n\}, \forall x_i\}$ , then  $\{X_i\} \in \mathcal{X}$  is random sample from population  $X \sim f(x; \vec{\theta})$ .

### 2.1.1 Statistics

Statistic(s): function of random sample  $\vec{T}(X_1, X_2, \dots, X_n)$ , **but not a function of parameter.**<sup>7</sup>

<sup>7</sup>Maybe to be more precise, the sample are drawn from the distribution  $f(x; \vec{\theta})$ , so naturally the data  $\{X_i\}$  is related to parameters. Here a better description would be ‘expression of statistics does not contain parameters explicitly’. And thus we could calculate the value to statistics as long as we have the sample data. Detail see [Sampling Distribution](#).

□ **Some useful statistics, e.g.**

- Sample mean (Consider  $X_i$  i.i.d.)

$$\bar{X} = \frac{1}{n} \sum_{i=1}^n X_i \quad (2.2)$$

- Sample variance

$$S^2 = \frac{1}{n-1} \sum_{i=1}^n (X_i - \bar{X})^2 \quad (2.3)$$

- Sample moments

– Origin moment

$$a_{n,k} = \frac{1}{n} \sum_{i=1}^n X_i^k \quad k = 1, 2, 3, \dots \quad (2.4)$$

– Center moment

$$m_{n,k} = \frac{1}{n} \sum_{i=1}^n (X_i - \bar{X})^k \quad k = 2, 3, 4, \dots \quad (2.5)$$

- Pearson's Correlation Coefficient  $r$

$$r_{X,Y} = \text{cov}(X, Y) = \frac{\sum_{i=1}^n (X_i - \bar{X})(Y_i - \bar{Y})}{\sqrt{\sum_{i=1}^n (X_i - \bar{X})^2} \sqrt{\sum_{i=1}^n (Y_i - \bar{Y})^2}} \quad (2.6)$$

Multivariate version see [equation. 4.23](#)

- Order statistics

$$(X_{(1)}, X_{(2)}, \dots, X_{(n)}), \text{ for } X_{(1)} \leq X_{(2)} \leq \dots \leq X_{(n)} \quad (2.7)$$

- Sample  $p$ -fractile

$$m_p = X_{(m)}, \quad m = \lfloor (n+1)p \rfloor \quad (2.8)$$

- Sample coefficient of variation

$$\hat{\nu} = \frac{S}{\bar{X}} \quad (2.9)$$

- Skewness and Kurtosis

$$\hat{g}_1 = \frac{m_{n,3}}{m_{n,2}^{3/2}} \quad \hat{g}_2 = \frac{m_{n,4}}{m_{n,2}^2} - 3 \quad (2.10)$$

▷ **R. Code**

R. code for some statistics

```
1 # mean function
2 mean(x)
3 mean(df)
4 # variance / covariance
5 var(x)
6 var(x, y)
7 var(df)
```

```

8 cov(x,y)
9 cov(df)
10 # correlation
11 cor(x,y)
12 cor(df)
13 cov2cor(cov(df))
14 # moments
15 library('moments')
16 moments::moment(df, order = ORDER_OF_M, central = FALSE, na.rm = FALSE)

```

### □ Properties

SamplingDistribution  $T$  is a function of random sample  $\vec{X} = \{X_i\}$ . Since the sample is drawn ‘at random’, then  $T(\vec{X})$  certainly also has its own distribution (say  $g_T(t)$ ) called **Sampling Distribution**.<sup>8</sup>

For  $X_i$  i.i.d. from  $X \sim f(x)$  with population mean  $\mu$  and variance  $\sigma^2$

- Calculation of sample variance  $S^2$

$$(n-1)S^2 = \sum_{i=1}^n x_i^2 - n\bar{x}^2 \quad (2.11)$$

- $\mathbb{E}$  and  $var$  of  $\bar{X}$  and  $S^2$

$$\mathbb{E}(\bar{X}) = \mu \quad var(\bar{X}) = \frac{\sigma^2}{n} \quad \mathbb{E}(S^2) = \sigma^2 \quad (2.12)$$

Further if  $X_i$  i.i.d. from  $X \sim N(\mu, \sigma^2)$  where  $\mu$  and  $\sigma^2$  unknown.

- Independence of  $\bar{X}$  and  $S^2$ <sup>9</sup>

$$\bar{X} \perp\!\!\!\perp S^2 \quad (2.13)$$

$$\text{– Distribution of } \bar{X} = \frac{1}{n} \sum_{i=1}^n X_i$$

$$\bar{X} \sim N\left(\mu, \frac{\sigma^2}{n}\right) \quad (2.14)$$

$$\text{– Distribution of } S^2 = \frac{1}{n-1} \sum_{i=1}^n (X_i - \bar{X})^2$$

$$\frac{(n-1)S^2}{\sigma^2} \sim \chi_{n-1}^2 \quad (2.15)$$

Comment: the independence here can explain the  $n-1$  degree of freedom of  $\chi_{n-1}^2$

<sup>8</sup>Now recap the statement that ‘statistic is not a function of parameter  $\theta$ ’: Recall that random variable (the sample is a set of r.v.)  $X$  is a mapping, so  $T: \Omega \mapsto \mathcal{X} \mapsto \mathbb{R}$  also. Parameter  $\theta$  does not involve in the mapping process, instead it influence the sample probability  $\mathbb{P}_\theta(\vec{X})$ , and thus the distribution of statistics  $g_T(t(\vec{X}); \theta)$ . Sometimes I use notation like  $T(\vec{X}; \theta)$  to remind me of this, but actually statistic should not contain  $\theta$  (at least in its expression).

<sup>9</sup>A brief proof is here <https://vincent19.github.io//texts/indepenencyXS/>



## 2.1.2 Exponential Family

Motivation: parametric statistical inference needs a (priorly assumed) distribution, e.g. Normal  $N(\mu, \sigma^2)$ , Poisson  $P(\lambda)$ , Gamma  $\Gamma(\alpha, \lambda)$ , etc. Exponential Family is a framework to represent them in the same form. Exponential Family can extract some key features of the distribution, and has some nice properties.

Def.  $\mathcal{F}_\Theta = \{f(x; \vec{\theta} | \vec{\theta} \in \Theta)\}$  is **Exponential Family** if  $f(x; \vec{\theta})$  has the form as

$$f(x; \vec{\theta}) = C(\vec{\theta})h(x) \exp \left[ \sum_{i=1}^k Q_i(\vec{\theta})T_i(x) \right] \quad \vec{\theta} \in \Theta \quad (2.16)$$

Or equivalently express  $c(\vec{\theta}) = \ln C(\vec{\theta})$ :

$$f(x; \vec{\theta}) = h(x) \exp \left[ \sum_{i=1}^k Q_i(\vec{\theta})T_i(x) + c(\vec{\theta}) \right] \quad \vec{\theta} \in \Theta \quad (2.17)$$

Canonical Form: Take  $Q_i(\vec{\theta}) = \varphi_i$ , then  $\vec{\varphi} = (\varphi_1, \varphi_2, \dots, \varphi_k) = (Q_1(\vec{\theta}), Q_2(\vec{\theta}), \dots, Q_k(\vec{\theta}))$  is a transform from  $\Theta \mapsto \Theta^*$ , s.t.  $\mathcal{F}$  has canonical form, i.e.

$$f(x; \vec{\varphi}) = C^*(\vec{\varphi})h(x) \exp \left[ \sum_{i=1}^k \varphi_i T_i(x) \right] \quad \vec{\varphi} \in \Theta^* \quad (2.18)$$

$\Theta^*$  is canonical parameter space.

### □ Examples of Exponential Family

- Normal Distribution  $X \sim N(\mu, \sigma^2)$

$$f(x; \mu, \sigma^2) = \frac{1}{\sqrt{2\pi}\sigma} \exp \left[ -\frac{(x - \mu)^2}{2\sigma^2} \right] = \frac{1}{\sqrt{2\pi}\sigma} \exp \left[ -\frac{x^2 - 2x\mu + \mu^2}{2\sigma^2} \right] \quad (2.19)$$

$$C(\theta) = \frac{1}{\sqrt{2\pi}\sigma} \quad (2.20)$$

$$Q_1(\theta) = -\frac{1}{2\sigma^2} \quad (2.21)$$

$$T_1(x) = x^2 \quad (2.22)$$

$$Q_2(\theta) = \frac{\mu}{\sigma^2} \quad (2.23)$$

$$T_2(x) = x \quad (2.24)$$

$$Q_3(\theta) = -\frac{\mu^2}{2\sigma^2} \quad (2.25)$$

$$T_3(x) = 1 \quad (2.26)$$

- Gamma Distribution  $X \sim \Gamma(\alpha, \lambda)$

$$f(x; \alpha, \lambda) = \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x} \quad (2.27)$$

$$C(\theta) = \frac{\lambda^\alpha}{\Gamma(\alpha)} \quad (2.28)$$

$$Q_1(\theta) = -\lambda \quad (2.29)$$

$$T_1(x) = x \quad (2.30)$$

$$Q_2(\theta) = \alpha - 1 \quad (2.31)$$

$$T_2(x) = \log x \quad (2.32)$$

- Binomial Distribution  $X \sim B(n, p)$

$$p(k; n, p) = \binom{n}{k} p^k (1-p)^{n-k} = \binom{n}{k} (1-p)^n \exp \left[ k \log \frac{p}{1-p} \right] \quad (2.33)$$

$$C(\theta) = (1-p)^n \quad (2.34)$$

$$h(x) = \binom{n}{k} \quad (2.35)$$

$$Q_1(\theta) = \log \frac{p}{1-p} \quad (2.36)$$

$$T_1(x) = k \quad (2.37)$$

### 2.1.3 Sufficient and Complete Statistics

(Note: For simplification, the following parts denote  $\vec{\theta}, \vec{T}, \dots$  as  $\theta, T, \dots$  etc.) Now say we are trying to estimate  $\theta$  by a statistic  $T(\vec{X})$ . We hope that  $T(\vec{X})$  contains ‘necessary/enough useful information’ when estimating  $\theta$ .<sup>10</sup>

- A **Sufficient Statistic**  $T(\vec{X})$  for  $\theta$  contains ‘*enough*’ information of sample when inferring  $\theta$ , knowing more would not help us get a better estimation. i.e. the (conditional) distribution of sample given  $T(\vec{X})$  is the same as that given the parameter.

$$f(\vec{X}; T(\vec{X})) = f(\vec{X}; T(\vec{X}), \theta) \quad (2.38)$$

Or,  $T(\vec{X})$  condensedly stores all information about  $\theta$  contained in sample  $\vec{X}$ .

Properties

- **Factorization Thm.**  $T(\vec{X})$  is sufficient **if and only if**  $f_{\vec{X}}(\vec{x}; \theta) = f(\vec{x}; \theta)$  can be written as

$$f(\vec{x}; \theta) = g(T(\vec{x}); \theta) h(\vec{x}) \quad (2.39)$$

- If  $T(\vec{X})$  sufficient, then  $T'(\vec{X}) = g[T(\vec{X})]$  also. (requires  $g$  single-valued and invertible)
- If  $T(\vec{X})$  sufficient, then  $[T, T_1]$  also.
- Sufficient statistic is **not** unique.
- Usually dimension of  $\vec{T}_\theta$  and  $\vec{\theta}$  equals.

- A **Complete Statistic**  $T(\vec{X})$  for  $\theta$  satisfies

$$\text{any } \phi(\cdot) \text{ with: } \mathbb{E} [\phi(T(\vec{X}))] = 0, \forall \theta, \text{ must have } \mathbb{P}(\phi(\vec{X}) = 0) = 1 \forall \theta \quad (2.40)$$

Explanation:  $T(\vec{X})$  as a function of sample, has its sampling distribution, say  $T \sim g_T(t)$ . ‘Complete’ is the description to the distribution family of  $T(\vec{X})$ :  $\{g_T(t(x; \theta)) : \theta \in \Theta\}$ . The above equation is rewritten as

$$\int \varphi(t) g_T(t) dt = 0 \forall \theta \xRightarrow{\text{compl stat}} \varphi(\cdot) = 0 \text{ a.s. } \forall \theta \quad (2.41)$$

Another perspective: Recall that  $\int \iota(u) j(u) du$  is a kind of inner product  $\langle \iota, j \rangle$ , the above statement is saying that: functional space of  $g_T$ , denoted  $\text{span}\{g_T(t); \forall \theta\}$  is a complete function space.

<sup>10</sup>Intuition: ‘Information’ might be described by ‘distribution of  $T(\vec{X})$  for different  $\theta$ ’. i.e. the distribution family  $\{g_T : \theta \in \Theta\}$  measures the performance of estimator.

Another statement for complete statistic is that

$$\varphi(T) \neq 0 \forall \theta \Rightarrow \mathbb{E}[\varphi(T(\vec{X}))] \neq 0 \quad (2.42)$$

Intuition: Not complete means  $\exists \phi(\cdot), \theta$  s.t.  $\mathbb{E}[\phi(T(\vec{X}))] = 0$ , and also  $\exists$  another function  $\tilde{\phi}(\cdot) = \phi(\cdot) + \text{const}$  so that  $\mathbb{E}[\tilde{\phi}(T(\vec{X}))]$  can be any const  $\rightarrow$  some information is **unnecessary**. So maybe complete means containing ‘*no extra*’ information, to a certain degree.

Properties

- If  $T(\vec{X})$  complete, then  $T'(\vec{X}) = g[T(\vec{X})]$  also.(requires  $g$  measurable)
- A complete statistic does not always exists.

► A **Minimal Sufficient Statistics**  $T(\vec{X})$  for  $\theta$  contains ‘just enough necessary’ information about  $\theta$ . Definition:

$$\forall \text{ sufficient statistic } \tilde{T}(\vec{X}), \exists q_{\tilde{T}}(\cdot), \text{ s.t. } T(\vec{X}) = q_{\tilde{T}}(\tilde{T}(\vec{X})) \quad (2.43)$$

Intuition:  $T(\vec{X})$  is a function of  $\tilde{T}(\vec{X})$  suggests that  $T$  contains no more information than  $\tilde{T}$ . And if sufficient statistic  $T$  can be function of all sufficient statistics, then  $T(\vec{X})$  contains ‘enough and minimal information’ about  $\theta$ .

Properties

- Sufficient & Complete  $\Rightarrow$  Minimal Sufficient ( $\neq$ )
- Sufficient as ‘enough’ + complete as ‘no extra’ = minimal sufficient as ‘just enough’.
- A minimal sufficient statistic does **not** always exists.

► An **Ancillary Statistic**  $S(\vec{X})$  is a statistic whose distribution does not depend on  $\theta$

**Basu Thm.:**  $\vec{X} = (X_1, X_2, \dots, X_n)$  is sample from  $\mathcal{F} = \{f(x; \theta), \theta \in \Theta\}$ .  $T(\vec{X})$  is a complete and minimal sufficient statistic,  $S(\vec{X})$  is ancillary statistic, then  $S(\vec{X}) \perp\!\!\!\perp T(\vec{X})$ . Intuitively  $S(\vec{X})$  contains no information about  $\theta$  and minimal sufficient  $T(\vec{X})$  contains all and necessary information about  $\theta$ .

► Exponential family: For  $\vec{X} = (X_1, X_2, \dots, X_n)$  from exponential family with canonical form, i.e.

$$f(\vec{x}; \theta) = C(\theta)h(\vec{x}) \exp \left[ \sum_{i=1}^k \theta_i T_i(\vec{x}) \right], \quad \theta \in \Theta \quad (2.44)$$

Then if  $\Theta \in \mathbb{R}^k$  interior point exists, then  $T(\vec{X}) = (T_1(\vec{X}), T_2(\vec{X}), \dots, T_k(\vec{X}))$  is sufficient & complete statistic.

## Section 2.2 Point Estimation

For parametric distribution family  $\mathcal{F} = \{f(x, \theta), \theta \in \Theta\}$ , random sample  $\vec{X} = (X_1, X_2, \dots, X_n)$  from  $\mathcal{F}$ .  $g(\theta)$  is a function defined on  $\Theta$ .

Mission: use sample  $\{X_i\}$  to estimate  $g(\theta)$ , called **Parameter Estimation**.

$$\text{Parameter Estimation} \begin{cases} \text{Point Estimation} & \checkmark \\ \text{Interval Estimation} \end{cases} \quad (2.45)$$

Point estimation: when estimating  $\theta$  or  $g(\theta)$ , denote the estimator (defined on sample space  $\mathcal{X}$ ) as

$$\hat{\theta}(\vec{X}) \xrightarrow{\text{estimates}} \theta \quad \text{or} \quad \hat{g}(\vec{X}) \xrightarrow{\text{estimates}} g(\theta) \quad (2.46)$$

Estimator is a statistic, with sampling distribution. In the following part we only give the expression for  $\hat{\theta}(\vec{X}) \xrightarrow{\text{estimates}} \theta$  ( $\hat{g}$  version is similar).

### 2.2.1 Optimal Criterion

Some nice properties of estimators (that we expect). They might not be satisfied simultaneously, e.g. we usually have to face trade-off between bias & precision.

- Unbiasedness

$$\mathbb{E}(\hat{\theta}) = \theta \quad (2.47)$$

Otherwise, say  $\hat{\theta}$  or  $\hat{g}$  is biased. Define **Bias**:

$$Bias(\hat{\theta}) := \mathbb{E}(\hat{\theta}) - \theta \quad (2.48)$$

in this way, an unbiased estimator is one with  $Bias(\hat{\theta}) = 0$

Asymptotic unbiasedness with  $n$  as sample size

$$\lim_{n \rightarrow \infty} \mathbb{E}(\hat{\theta}_n(\vec{X})) = \theta \quad (2.49)$$

- Efficiency: say  $\hat{\theta}_1(\vec{X})$  is more efficient than  $\hat{\theta}_2(\vec{X})$ , if

$$var(\hat{\theta}_1) \leq var(\hat{\theta}_2) \quad \forall \theta \in \Theta \quad (2.50)$$

Can we find a estimator with minimum variance / the most efficient? See [section. 2.2.4](#).

- Mean Squared Error (MSE): Most efficient in the sense with bias-variance trade-off. More about Minimum MSE estimation see [section. 12.4.1](#).

$$MSE = \mathbb{E}[(\hat{\theta} - \theta)^2] = \mathbb{E}[(\hat{\theta} - \mathbb{E}[\hat{\theta}] + \mathbb{E}[\hat{\theta}] - \theta)^2] = var(\hat{\theta}) + [Bias(\hat{\theta})]^2 \quad (2.51)$$

For unbiased estimator, i.e.  $Bias(\hat{\theta}) = 0$ , we have

$$MSE = \mathbb{E}[(\hat{\theta} - \theta)^2] = var(\hat{\theta}) \quad (2.52)$$

- (Weak) Consistency

$$\lim_{n \rightarrow \infty} \mathbb{P}(|\hat{\theta}_n(\vec{X}) - \theta| \geq \varepsilon) = 0 \quad \forall \varepsilon > 0 \quad (2.53)$$

- Asymptotic Normality

$$\hat{\theta}_n - \theta \xrightarrow{d} N(0, \sigma_{\hat{\theta}}^2) \quad (2.54)$$

## 2.2.2 Method of Moments

Review: Population moments & Sample moments

$$\alpha_k = \mathbb{E}(X^k) \quad \mu_k = \mathbb{E}[(X - \mathbb{E}(X))^k] \quad (2.55)$$

$$a_{n,k} = \hat{\alpha}_k = \frac{1}{n} \sum_{i=1}^n X_i^k \quad m_{n,k} = \hat{\mu}_k = \frac{1}{n} \sum_{i=1}^n (X_i - \bar{X})^k \quad (2.56)$$

Property:  $a_{n,k}$  is the unbiased estimator of  $\alpha_k$ . (while  $m_{n,k}$  usually biased for  $\mu_k$ )

For sample  $\vec{X} = (X_1, X_2, \dots, X_n)$  from  $\mathcal{F} = \{f(x; \theta, \theta \in \Theta)\}$ , unknown parameter (or its function)  $g(\theta)$  can be written as

$$g(\theta) = G(\alpha_1, \alpha_2, \dots, \alpha_k; \mu_2, \mu_3, \dots, \mu_l) \quad (2.57)$$

Then its **Moment Estimate**  $\hat{g}(\vec{X})$  is

$$\hat{g}(\vec{X}) = G(a_{n,1}, a_{n,2}, \dots, a_{n,k}; m_{n,2}, m_{n,3}, \dots, m_{n,l}) \quad (2.58)$$

Example: coefficient of variation  $\nu$  & skewness  $\beta_1$

$$\hat{\nu} = \frac{S}{\bar{X}} \quad \hat{\beta}_1 = \frac{m_{n,3}}{m_{n,2}^{3/2}} = \sqrt{n} \frac{\sum_{i=1}^n (X_i - \bar{X})^3}{\left[ \sum_{i=1}^n (X_i - \bar{X})^2 \right]^{\frac{3}{2}}} \quad (2.59)$$

□ **Note:**

- $G$  may not have explicit expression.
- Moment estimate may not be unique.
- If  $G = \sum_{i=1}^k c_i \alpha_i$  (linear combination of  $\alpha$ , without  $\mu$ ), then  $\hat{g}(\vec{X}) = \sum_{i=1}^k c_i a_{n,i}$  unbiased.

Usually  $\hat{g}(\vec{X})$  is asymptotically unbiased.

- For small sample, not so accurate.
- May not contain all the information about  $\theta$ , i.e. may not be sufficient statistic.
- Do not require a statistic model, as long as you can express  $G(\dots)$ .

## 2.2.3 Maximum Likelihood Estimation

For sample  $\vec{X} = (X_1, X_2, \dots, X_n)$  with distribution  $f(\vec{x}; \theta)$  from  $\mathcal{F} = \{f(x; \theta), \theta \in \Theta\}$ , def. **Likelihood Function**  $L(\theta; \vec{x})$ , defined on  $\Theta$  (as a function of  $\theta$ )

$$L(\theta; \vec{x}) = f(\vec{x}; \theta) \quad \theta \in \Theta, \vec{x} \in \mathcal{X} \quad (2.60)$$

for  $X_i$  i.i.d.  $\sim f(x; \theta)$  case:

$$L(\theta; \vec{x}) = \prod_{i=1}^n f(x_i; \theta) \quad (2.61)$$

Also def. log-likelihood function  $\ell(\theta; \vec{x}) = \ln L(\theta; \vec{x})$ .

A **Maximum Likelihood Estimator**  $\hat{\theta}(\vec{X})$  for  $\theta$  maximizes (or finds the upper bound) likelihood, or equivalently log-likelihood:

$$L(\hat{\theta}; \vec{x}) = \sup_{\theta \in \Theta} L(\theta; \vec{x}) \Leftrightarrow \ell(\hat{\theta}; \vec{x}) = \sup_{\theta \in \Theta} \ell(\theta; \vec{x}), \quad \vec{x} \in \mathcal{X} \quad (2.62)$$

#### □ Identification of MLE

- Differentiation: Fermat Lemma

$$\left. \frac{\partial L}{\partial \theta_i} \right|_{\theta=\hat{\theta}} = 0 \quad \left. \frac{\partial^2 L}{\partial \theta_i \partial \theta_j} \right|_{\theta=\hat{\theta}} \text{ negative definite} \quad \forall i, j = 1, 2, \dots, k \quad (2.63)$$

- Graphing method.
- Numerically compute maximum.

#### □ Properties

- **Not Always** unbiased, an example is variance estimator, where

$$\hat{\sigma}_{\text{MLE}}^2 = \frac{1}{n} \sum_{i=1}^n (x_i - \bar{x})^2 \quad (2.64)$$

- Invariance of MLE: If  $\hat{\theta}$  is MLE of  $\theta$ , then  $h(\hat{\theta})$  is MLE of  $h(\theta)$ , where  $h(\cdot)$  is an invertible function.
- MLE and Sufficiency:  $T = T(X_1, X_2, \dots, X_n)$  is a sufficient statistic of  $\theta$ , if MLE of  $\theta$  exists, say  $\hat{\theta}$ , then  $\hat{\theta}$  is a function of  $T$ , i.e.

$$\hat{\theta} = \hat{\theta}(\vec{X}) = \hat{\theta}^*(T(\vec{X})) \quad (2.65)$$

- Asymptotic Normality:

$$\sqrt{n}(\hat{\theta}_n - \theta) \xrightarrow{d} N(0, \sigma_\theta^2), \quad \sigma_\theta^2 = \frac{1}{\mathbb{E}_\theta \left[ \left[ \frac{\partial}{\partial \theta} \ln f(\vec{X}; \theta) \right]^2 \right]} \quad (2.66)$$

i.e.

$$\hat{\theta}_n \xrightarrow{d} N\left(\theta, \frac{\sigma_\theta^2}{n}\right) \quad (2.67)$$

We will later see that  $\sigma_\theta^2$  is the inversed Fisher Information.

$$\hat{\theta}_n \xrightarrow{d} N\left(\theta, \frac{I(\theta)^{-1}}{n}\right) \quad (2.68)$$

#### □ Comparison: MoM and MLE

- MoM do not require statistic model; MLE need to know PDF.
- MoM is more robust than MLE.

#### □ MLE in Exponential Family

For sample  $\vec{X} = (X_1, X_2, \dots, X_n)$  from canonical exponential family  $\mathcal{F} = \{f(x; \theta), \theta \in \Theta\}$

$$f(x; \theta) = C(\theta)h(x) \exp \left[ \sum_{i=1}^k \theta_i T_i(x) \right] \quad \theta = (\theta_1, \dots, \theta_k) \in \Theta \quad (2.69)$$

Likelihood function  $L(\theta, \vec{x}) = \prod_{j=1}^n f(x_j; \theta)$  and log-likelihood function  $l(\theta, \vec{x})$

$$L(\theta, \vec{x}) = C^n(\theta) \prod_{j=1}^n h(x_j) \exp \left[ \sum_{i=1}^k \theta_i \sum_{j=1}^n T_i(x_j) \right] \quad (2.70)$$

$$\ell(\theta, \vec{x}) = n \ln C(\theta) + \sum_{j=1}^n \ln h(x_j) + \sum_{i=1}^k \theta_i \sum_{j=1}^n T_i(x_j) \quad (2.71)$$

Solution of MLE: (Require  $\hat{\theta} \in \Theta$ )

$$\frac{n}{C(\theta)} \frac{\partial C(\theta)}{\partial \theta_i} \bigg|_{\theta=\hat{\theta}} = - \sum_{j=1}^n T_i(x_j), \quad i = 1, 2, \dots, k \quad (2.72)$$

## 2.2.4 Uniformly Minimum Variance Unbiased Estimator

Recall MSE: If  $\hat{g}(\vec{X})$  is an estimator of  $g(\theta)$ , then MSE

$$\text{MSE}(\hat{g}(\vec{X})) = \mathbb{E}[(\hat{g}(\vec{X}) - g(\theta))^2] = \text{var}(\hat{g}) + [\text{Bias}(\hat{g})]^2 \quad (2.73)$$

Note: Unbiased estimator (i.e.  $\text{Bias}(\hat{g}) = 0$ ) is not unique; and not always exists. But now anyway for UMVUE we only consider the case that unbiased estimators of  $g(\theta)$  exists, say  $\hat{g}(\vec{X})$ , then

$$\text{MSE}(\hat{g}(\vec{X})) = \text{var}(\hat{g}(\vec{X})) \quad (2.74)$$

If  $\forall$  unbiased estimate  $\tilde{g}(\vec{X})$ ,  $\hat{g}$  satisfies

$$\text{var}[\hat{g}(\vec{X})] \leq \text{var}[\tilde{g}(\vec{X})] \quad (2.75)$$

Then  $\hat{g}(\vec{X})$  is **Uniformly Minimum Variance Unbiased Estimator(UMVUE)** of  $g(\theta)$

□ **How to determine UMVUE? (Which is not an easy task)**

### 1. Zero Unbiased Estimate Method

Let  $\hat{g}(\vec{X})$  be an unbiased estimator  $\mathbb{E}[\hat{g}(\vec{X})] = g(\theta)$  with  $\text{var}(\hat{g}) < \infty$ . If  $\forall$  other unbiased estimator  $\hat{l}(\vec{X})$ ,  $\hat{g}$  holds that

$$\text{cov}(\hat{g}, \hat{l}) = \mathbb{E}(\hat{g} \cdot \hat{l}) = 0, \quad \forall \theta \in \Theta \quad (2.76)$$

Then  $\hat{g}$  is a UMVUE of  $g(\theta)$  (sufficient & necessary condition).

### 2. Sufficient and Complete Statistic Method

For  $T(\vec{X})$  sufficient statistic,  $\hat{g}(\vec{X})$  unbiased estimate of  $g(\theta)$ , then

$$h(T) = \mathbb{E}(\hat{g}(\vec{X})|T) \quad (2.77)$$

is an unbiased estimate of  $g(\theta)$  and  $\text{var}(h(T)) \leq \text{var}(\hat{g})$ .

Remark:

- A method to improve estimator.
- A UMVUE has to be a function of sufficient statistic.

**Lehmann-Scheffé Thm.:** For  $\vec{X} = (X_1, X_2, \dots, X_n)$  from population  $X \sim \mathcal{F} = \{f(x; \theta) : \theta \in \Theta\}$ .  $T(\vec{X})$  sufficient and complete, and  $\hat{g}(T(\vec{X}))$  be an unbiased estimator, then  $\hat{g}(T(\vec{X}))$  is the unique UMVUE.

Can be used to construct UMVUE: given  $T(\vec{X})$  sufficient and complete and some unbiased estimator  $\hat{g}'(\theta)$  then

$$\hat{g}(T) = \mathbb{E}(\hat{g}'|T) \quad (2.78)$$

is the unique UMVUE.

### 3. Cramer-Rao Inequality

Core idea: determine a lower bound of  $\text{var}(\hat{g})$ .

Consider  $\theta = \theta$  (One dimension parameter); For  $\{X_i\}$  i.i.d.  $f(x, \theta)$ : def.

- **Score function:** Reflects the steepness/slope of likelihood function.

$$S(\vec{x}; \theta) = \frac{\partial \ln f(\vec{x}; \theta)}{\partial \theta} = \frac{\partial \ell(\theta; \vec{x})}{\partial \theta} = \sum_{i=1}^n \frac{\partial \ln f(x_i; \theta)}{\partial \theta} \quad (2.79)$$

Property:<sup>11</sup>

$$\mathbb{E}[S(\vec{X}; \theta)] = 0 \quad (2.83)$$

- **Fisher Information:** Variance of  $S(\vec{x}; \theta)$ , reflects the accuracy to conduct estimation, i.e. reflects information of statistic model that sample brings.<sup>12</sup>

$$I(\theta) = \mathbb{E} \left[ \left( \frac{\partial \ln f(\vec{x}; \theta)}{\partial \theta} \right)^2 \right] = -\mathbb{E} \left[ \frac{\partial^2 \ln f(\vec{x}; \theta)}{\partial \theta^2} \right] \quad (2.90)$$

Consider  $\mathcal{F}$  satisfies some regularity conditions (in most cases, regularity conditions do hold), then the lower bound of  $\text{var}(\hat{g})$  satisfies **Cramer-Rao Inequality**:

$$\text{var}(\hat{g}(\vec{X})) \geq \frac{[g'(\theta)]^2}{nI(\theta)} \quad (2.91)$$

<sup>11</sup>Proof of  $\mathbb{E}(S(\vec{x}; \theta)) = 0$ :

$$\mathbb{E}(S|\theta) = \int f(\vec{x}; \theta) \frac{\partial \ln f(\vec{x}; \theta)}{\partial \theta} d\vec{x} \quad (2.80)$$

$$= \int f(\vec{x}; \theta) \frac{1}{f(\vec{x}; \theta)} \frac{\partial f(\vec{x}; \theta)}{\partial \theta} d\vec{x} \quad (2.81)$$

$$= \frac{\partial}{\partial \theta} \int f(\vec{x}; \theta) d\vec{x} = \frac{\partial}{\partial \theta} 1 = 0 \quad (2.82)$$

<sup>12</sup>Proof of  $I(\theta) = \mathbb{E} \left[ \left( \frac{\partial \ln f(\vec{x}; \theta)}{\partial \theta} \right)^2 \right] = -\mathbb{E} \left[ \frac{\partial^2 \ln f(\vec{x}; \theta)}{\partial \theta^2} \right]$ :

$$0 = \frac{\partial}{\partial \theta^T} \mathbb{E}(S|\theta) \quad (2.84)$$

$$= \int \frac{\partial}{\partial \theta^T} \left\{ \frac{\partial \ln f(\vec{x}; \theta)}{\partial \theta} f(\vec{x}; \theta) \right\} d\vec{x} \quad (2.85)$$

$$= \int \left\{ \frac{\partial^2 \ln f(\vec{x}; \theta)}{\partial \theta \partial \theta^T} f(\vec{x}; \theta) + \frac{\partial \ln f(\vec{x}; \theta)}{\partial \theta} \frac{\partial f(\vec{x}; \theta)}{\partial \theta^T} \right\} d\vec{x} \quad (2.86)$$

$$= \int \frac{\partial^2 \ln f(\vec{x}; \theta)}{\partial \theta \partial \theta^T} f(\vec{x}; \theta) d\vec{x} + \int \frac{\partial \ln f(\vec{x}; \theta)}{\partial \theta} \frac{\partial \ln f(\vec{x}; \theta)}{\partial \theta^T} f(\vec{x}; \theta) d\vec{x} \quad (2.87)$$

$$= \mathbb{E} \left( \frac{\partial^2 \ln f(\vec{x}; \theta)}{\partial \theta \partial \theta^T} \right) + \mathbb{E} \left( \frac{\partial \ln f(\vec{x}; \theta)}{\partial \theta} \frac{\partial \ln f(\vec{x}; \theta)}{\partial \theta^T} \right) \quad (2.88)$$

$$\Rightarrow \mathbb{E} \left( \frac{\partial^2 \ln f(\vec{x}; \theta)}{\partial \theta \partial \theta^T} \right) = -\mathbb{E} \left( \frac{\partial \ln f(\vec{x}; \theta)}{\partial \theta} \frac{\partial \ln f(\vec{x}; \theta)}{\partial \theta^T} \right) \quad (2.89)$$



Special case:  $g(\theta) = \theta$  then

$$\text{var}(\hat{\theta}) \geq \frac{1}{nI(\theta)} \quad (2.92)$$

note:

- C-R Inequality determine a lower bound, not the infimum(i.e.  $\text{UMVUE} \nRightarrow \text{var}(\hat{g}(\vec{X})) = \frac{[g'(\theta)]^2}{nI(\theta)}$ ).
- Take '=': Only some cases in Exponential family.
- **Efficiency**  $e_{\hat{g}}$ : How good the estimator is.

$$e_{\hat{g}(\vec{X})}(\theta) = \frac{[g'(\theta)]^2 / (nI(\theta))}{\text{var}(\hat{g}(\vec{X}))} \quad (2.93)$$

#### 4. Multi-Dimensional Cramer-Rao Inequality ReDef. Fisher Information:

$$\mathbf{I}(\theta) = \{I_{ij}(\theta)\} = \left\{ \mathbb{E} \left[ \left( \frac{\partial \ln f(\vec{x}; \theta)}{\partial \theta_i} \right) \left( \frac{\partial \ln f(\vec{x}; \theta)}{\partial \theta_j} \right) \right] \right\} \quad (2.94)$$

Then covariance matrix  $\Sigma(\theta)$  satisfies **Cramer-Rao Inequality**

$$\Sigma(\theta) \succeq (n\mathbf{I}(\theta))^{-1} \quad (2.95)$$

Note: ' $\succeq$ ' means ' $\geq$ ' holds for all diagonal elements, i.e.

$$\text{var}(\hat{\theta}_i) \geq \frac{I_{ii}^*(\theta)}{n}, \quad \forall i = 1, 2, \dots, k \quad (2.96)$$

### 2.2.5 MoM and MLE in Linear Regression

**Note:** More detailed knowledge see **Chapter. 3** Linear Regression Analysis.

#### □ Linear Regression Model (1-dimension case):

$$y_i = \beta_0 + \beta_1 x_i + \epsilon_i \quad (2.97)$$

where  $\beta_0, \beta_1$  are regression coefficient, and  $\epsilon_i$  are unknown random **error**.

Basic Assumptions (Guass-Markov Assumption):

$$\text{Zero-Mean: } \epsilon_i \text{ are i.i.d.} \quad (2.98)$$

$$\text{Homogeneity of Variance: } \mathbb{E}(\epsilon_i | x_i) = 0 \quad (2.99)$$

$$\text{Independent: } \text{var}(\epsilon_i) = \sigma^2 \quad (2.100)$$

Mission: use data  $\{(x_i, y_i)\}$  to estimate  $\beta_0, \beta_1$  (i.e. regression line), and error  $\epsilon_i$ .

#### 1. OLS (Ordinary Least Squares): Take $\beta_0, \beta_1$ so that MSE min, i.e. SSE min

$$(\hat{\beta}_0, \hat{\beta}_1) = \arg \min \sum_{i=1}^n (y_i - \beta_0 - \beta_1 x_i)^2 \quad (2.101)$$

(Express in Matrix Notation **equation. 2.119**, so that it can be generalized to multidimensional case) SSE can be expressed as the **Euclidean Distance** between  $\{y_i\}$  and  $\{\hat{\beta}_0 + \hat{\beta}_1 x_i\}$ , i.e.

$$\arg \min D(y, X\hat{\beta}) \quad (2.102)$$

i.e.  $\hat{\beta}$  is the Projection of  $y$  onto hyperplane  $X$ , then

$$(X\hat{\beta})^T(y - X\hat{\beta}) = 0 \Rightarrow \hat{\beta} = (X^T X)^{-1} X^T y \quad (2.103)$$

Solution for 1-D case:

$$\hat{\beta} = \begin{bmatrix} \hat{\beta}_0 \\ \hat{\beta}_1 \end{bmatrix} = \begin{bmatrix} \bar{y} - \hat{\beta}_1 \bar{x} \\ \frac{\sum_{i=1}^n (x_i - \bar{x})(y_i - \bar{y})}{\sum_{i=1}^n (x_i - \bar{x})^2} \end{bmatrix} \quad (2.104)$$

So get regression line:  $y = \hat{\beta}_0 + \hat{\beta}_1 x$

Def. Residuals

$$e_i = \hat{\epsilon}_i = y_i - \hat{y}_i = y_i - (\hat{\beta}_0 + \hat{\beta}_1 x_i) \quad (2.105)$$

Residuals can be used to estimate  $\epsilon_i$ :  $E[(\epsilon_i)^2] = \sigma^2$

$$\hat{\sigma}^2 = \frac{1}{n-2} \sum_{i=1}^n (y_i - \hat{\beta}_0 - \hat{\beta}_1 x_i)^2 \quad (2.106)$$

2. MoM: Consider r.v.  $\epsilon \sim f(\epsilon; x, y, \beta_0, \beta_1)$ , sample  $\{\epsilon_i | \epsilon_i = y_i - \beta_0 - \beta_1 x_i\}$ , then obviously

$$\bar{\epsilon} = \bar{y} - \beta_0 - \beta_1 \bar{x} \quad (2.107)$$

Take moment estimate of  $\epsilon$ , we have

$$\mathbb{E}(\epsilon_i) = 0 \quad \mathbb{E}(\epsilon_i x_i) = 0 \text{ (note that } \mathbb{E}(\epsilon|x) = 0) \quad (2.108)$$

$$\text{i.e.} \begin{cases} \frac{1}{n} \sum_{i=1}^n (y_i - \beta_0 - \beta_1 x_i) = 0 \\ \frac{1}{n} \sum_{i=1}^n x_i (y_i - \beta_0 - \beta_1 x_i) = 0 \end{cases} \quad (2.109)$$

Solution:

$$\begin{cases} \hat{\beta}_0 = \bar{y} - \beta_1 \bar{x} \\ \hat{\beta}_1 = \frac{\sum_{i=1}^n (x_i - \bar{x})(y_i - \bar{y})}{\sum_{i=1}^n (x_i - \bar{x})^2} \end{cases} \quad (2.110)$$

(the same as OLS estimation)

Moment estimate of  $\sigma^2$

$$\hat{\sigma}_n^2 = \frac{1}{n} \sum_{i=1}^n (y_i - \hat{\beta}_0 - \hat{\beta}_1 x_i)^2 \quad (2.111)$$

3. MLE: Assume  $\epsilon_i \sim N(0, \sigma^2)$ , then  $y_i | x_i \sim N(\beta_0 + \beta_1 x_i, \sigma^2)$ . Get likelihood function:

$$L(\beta_0, \beta_1, \sigma^2; x_1, \dots, x_n, y_1, \dots, y_n) = (2\pi\sigma^2)^{-\frac{n}{2}} \exp \left[ -\frac{\sum_{i=1}^n (y_i - \beta_0 - \beta_1 x_i)^2}{2\sigma^2} \right] \quad (2.112)$$

Log-likelihood:

$$\ell(\beta_0, \beta_1, \sigma^2; x_1, \dots, x_n, y_1, \dots, y_n) = -\frac{n}{2} \ln(2\pi\sigma^2) - \frac{1}{2\sigma^2} \sum_{i=1}^n (y_i - \beta_0 - \beta_1 x_i)^2 \quad (2.113)$$

MLE, use Fermat Lemma:

$$\begin{cases} \frac{\partial \ell}{\partial \beta_0} = 0 & \Rightarrow -\frac{1}{\sigma^2} \sum_{i=1}^n (y_i - \beta_0 - \beta_1 x_i) = 0 \\ \frac{\partial \ell}{\partial \beta_1} = 0 & \Rightarrow -\frac{1}{\sigma^2} \sum_{i=1}^n x_i (y_i - \beta_0 - \beta_1 x_i) = 0 \\ \frac{\partial \ell}{\partial \sigma^2} = 0 & \Rightarrow -\frac{n}{2\sigma^2} + \frac{1}{2(\sigma^2)^2} \sum_{i=1}^n (y_i - \beta_0 - \beta_1 x_i)^2 = 0 \end{cases} \quad (2.114)$$

Solution:

$$\hat{\beta}_0 = \bar{y} - \hat{\beta}_1 \bar{x} \quad (2.115)$$

$$\hat{\beta}_1 = \frac{\sum_{i=1}^n (x_i - \bar{x})(y_i - \bar{y})}{\sum_{i=1}^n (x_i - \bar{x})^2} \quad (2.116)$$

$$\hat{\sigma}_n^2 = \frac{1}{n} \sum_{i=1}^n (y_i - \hat{\beta}_0 - \hat{\beta}_1 x_i)^2 \quad (2.117)$$

#### □ Linear Regression Model (Multi-dimension case):

Detailed derivation see [section. 3.3](#)

$$y_i = \beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2} + \cdots + \beta_p x_{ip} + \epsilon_i \quad (2.118)$$

Denote:  $\vec{\beta} = (\beta_0, \beta_1, \dots, \beta_p)$ ,  $\vec{x}_i = (1, x_{i1}, x_{i2}, \dots, x_{ip})$ , then for each  $i$ :  $y_i = \vec{x}_i^T \vec{\beta} + \epsilon_i$

Further denote: Matrix form:

$$y = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} 1 & x_{11} & \cdots & x_{1p} \\ 1 & x_{21} & \cdots & x_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & x_{n1} & \cdots & x_{np} \end{pmatrix} \begin{pmatrix} \beta_0 \\ \beta_1 \\ \vdots \\ \beta_p \end{pmatrix} + \begin{pmatrix} \epsilon_1 \\ \epsilon_2 \\ \vdots \\ \epsilon_n \end{pmatrix} = X\vec{\beta} + \vec{\epsilon} \quad (2.119)$$

Basic Assumptions: Gauss-Markov Assumptions

- OLS unbiased

$$\mathbb{E}(\epsilon_i | x_i) = 0 \quad \mathbb{E}(y_i | x_i) = \vec{x}_i^T \vec{\beta} \quad (2.120)$$

- Homogeneity of  $\epsilon_i$

$$\text{var}(\epsilon_i) = \sigma^2 \quad (2.121)$$

- Independent of  $\epsilon$

- (For MLE)  $\epsilon_i$  i.i.d.  $\sim N(0, \sigma^2)$

Residuals:

$$e_i = \hat{\epsilon}_i = y_i - \hat{y}_i = y_i - \vec{x}_i^T \hat{\vec{\beta}} \quad (2.122)$$

Def. Error Sum of Squares (SSE)

$$\text{SSE} = \sum_{i=1}^n e_i^2 = \sum_{i=1}^n (y_i - \vec{x}_i^T \hat{\vec{\beta}})^2 \quad (2.123)$$

Estimator exists and unique: ( $\hat{\sigma}^2$  is after bias correction)

$$\begin{aligned}\hat{\beta} &= (X^T X)^{-1} X^T y \\ \hat{\sigma}_n^2 &= \frac{1}{n} \sum_{i=1}^n (y_i - x_i^T \hat{\beta})^2 \\ \hat{\sigma}^2 &= \frac{1}{n-p-1} \sum_{i=1}^n (y_i - x_i^T \hat{\beta})^2\end{aligned}\quad (2.124)$$

▷ **R. Code**

Example of linear regression model  $Y = \beta_0 + x\beta + \varepsilon$

```
1 lmfit <- lm(y~x, df)
2 summary(lmfit)
```

## 2.2.6 Kernel Density Estimation

Given random sample  $\{X_i\}$ . Def. Empirical CDF.

$$\hat{F}_n(x) = \frac{1}{n} \sum_{i=1}^n \mathbb{I}_{(-\infty, X_i]}(x) \quad (2.125)$$

Problem: Overfitting when getting  $\hat{f}$ . Solution: Using **Kernel Estimate**, replace  $I_{(-\infty, x]}(\cdot)$  with Kernel function  $K(\cdot)$ , then

$$\hat{f}_n(x) = \frac{F_n(x + h_n) - F_n(x - h_n)}{2h_n} = \frac{1}{nh_n} \sum_{i=1}^n K\left(\frac{x - X_i}{h_n}\right) \quad (2.126)$$

where  $h_n$  is **bandwidth**. Take proper kernel function  $K$  to get estimate of  $f$ .

Kernel density estimation can be considered as a convolution  $\otimes$  of sample  $\{X_i\}$  and kernel function  $K(\cdot)$ .

$$\hat{f}_K = \frac{1}{n} \sum_{i=1}^n \delta(x - X_i) \otimes K(x) \quad (2.127)$$

□ **Useful Kernel Functions**

$$K(x) := \begin{cases} \mathbb{I}_{[-\frac{1}{2}, \frac{1}{2}]}, & \text{Square Kernel} \\ (1 - |x|)\mathbb{I}_{[-1, 1]}, & \text{Triangle Kernel} \\ \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}, & \text{Gaussian Kernel} \\ \frac{1}{\pi(1+x^2)}, & \text{Cauchy Kernel} \\ \frac{1}{2\pi} \text{sinc}^2 \frac{x}{2} = \frac{1}{2\pi} \left( \frac{\sin x/2}{x/2} \right)^2 & \text{sinc Kernel} \end{cases} \quad (2.128)$$

▷ **R. Code**

Plot kernel density in R.

```
1 plot(density(x, kernel = KERNEL_TO_USE))
```

## Section 2.3 Interval Estimation

$$\text{Parameter Estimation} \begin{cases} \text{Point Estimation} \\ \text{Interval Estimation} \end{cases} \quad \checkmark \quad (2.129)$$

Interval Estimation: to estimate  $g(\theta)$ , give **two** estimators  $\hat{g}_1(\vec{X})$ ,  $\hat{g}_2(\vec{X})$  defined on  $\mathcal{X}$  as the two ends of interval (i.e. give an interval  $[\hat{g}_1(\vec{X}), \hat{g}_2(\vec{X})]$ ), then random interval  $[\hat{g}_1(\vec{X}), \hat{g}_2(\vec{X})]$  is an **Interval Estimation** of  $g(\theta)$ .

**Δ NOTE:** Here  $g(\theta)$  is the parameter, which is fixed, while confidence interval, as a function of data, is random. So all the probabilities discussed below are **Probability that the interval covers the true value**, rather than the true value falls in the interval. There is a huge difference.<sup>a</sup>

<sup>a</sup>A good example: Consider a bi-classification task into  $\uparrow$  or  $\downarrow$ . A confidence interval algorithm can randomly produce  $\{\uparrow, \downarrow\}$  19 times, and  $\emptyset$  1 time. This is still a 95% confidence interval algorithm (covers true label 19 in 20), but true label falls in  $\{\uparrow, \downarrow\}$  with pr 1, and in  $\emptyset$  with pr 0.

### 2.3.1 Confidence Interval

How to judge an interval estimation?

- Reliability

$$\mathbb{P}_{\hat{g}_1, \hat{g}_2}([\hat{g}_1, \hat{g}_2] \ni g(\theta)) \quad (2.130)$$

- Precision

$$\mathbb{E}(\hat{g}_2 - \hat{g}_1) \quad (2.131)$$

Trade off: (in most cases) Given a level of reliability, find an interval with the highest precision with reliability above the level.

□ **For a given  $0 < \alpha < 1$ , if**

$$\mathbb{P}(\hat{g}_1 \leq g(\theta) \leq \hat{g}_2) \geq 1 - \alpha \quad (2.132)$$

then  $[\hat{g}_1, \hat{g}_2]$  is a **Confidence Interval** for  $g(\theta)$ , with **Confidence Level**  $1 - \alpha$ .

**Confidence Coefficient:**

$$\inf_{\forall \theta \in \Theta} \mathbb{P}(\theta \in \text{CI}) \quad (2.133)$$

Other cases:

- **Confidence Limit:** (One-way) Upper/Lower Confidence Limit

$$\mathbb{P}(g \leq \hat{g}_U) \geq 1 - \alpha \quad (2.134)$$

$$\mathbb{P}(\hat{g}_L \leq g) \geq 1 - \alpha \quad (2.135)$$

- **Confidence Region:** For high dimensional parameters  $\vec{g} = (g_1, g_2, \dots, g_k)$

$$\mathbb{P}(\vec{g} \in S(\vec{X})) \geq 1 - \alpha \quad \forall \theta \in \Theta \quad (2.136)$$

Mission: Determine  $\hat{g}_1, \hat{g}_2$ .

### 2.3.2 Pivot Variable Method

Idea: Based on point estimation, construct a new variable and thus find the interval estimation.

Def. **Pivot Variable**  $T$ , satisfies:

- Expression of  $T$  contains  $\theta$  (thus  $T$  is not a statistic).
- Distribution of  $T$  independent of  $\theta$ .<sup>13</sup>

In different cases, construct different pivot variable, usually base on sufficient statistics and transform.

Knowing a proper pivot variable  $T = T(\hat{\varphi}, g(\theta)) \sim f$ , ( $f$  is some distribution independent of  $\theta$ ),  $\hat{\varphi}$  is a sufficient statistic), then we can take  $T$  satisfies:

$$\mathbb{P}(f_{1-\frac{\alpha}{2}} \leq T \leq f_{\frac{\alpha}{2}}) = 1 - \alpha \quad (2.137)$$

Construct the inverse mapping of  $T = T(\hat{\varphi}, g(\theta)) \Leftrightarrow g(\theta) = T^{-1}(T, \hat{\varphi})$ , we get

$$\mathbb{P}[T^{-1}(f_{1-\frac{\alpha}{2}}, \hat{\varphi}) \leq \hat{g} \leq T^{-1}(f_{\frac{\alpha}{2}}, \hat{\varphi})] = 1 - \alpha \quad (2.138)$$

Thus get a confidence interval for  $\theta$  with confidence coefficient  $1 - \alpha$ .

### 2.3.3 Confidence Interval for Common Distributions

Some important properties of  $\chi^2$ ,  $t$  and  $F$  see [section. 1.8.2](#).

1. Single normal population:  $\vec{X} = \{X_1, X_2, \dots, X_n\} \in \mathcal{X}$  i.i.d from Normal Distribution population  $N(\mu, \sigma^2)$ .

Denote sample mean and sample variance:

$$\bar{X} = \frac{1}{n} \sum_{i=1}^n X_i \quad S^2 = \frac{1}{n-1} \sum_{i=1}^n (X_i - \bar{X})^2 \quad S_\mu = \frac{1}{n} \sum_{i=1}^n (X_i - \mu)^2, (\text{for the case } \mu \text{ known}) \quad (2.139)$$

Estimating  $\mu$  &  $\sigma^2$ : construction of pivot variable under different circumstances:

Estimation	Pivot Variable	Confidence Interval
$\sigma^2$ known, estimate $\mu$	$T = \frac{\sqrt{n}(\bar{X} - \mu)}{\sigma} \sim N(0, 1)$	$\left[ \bar{X} - \frac{\sigma}{\sqrt{n}} N_{\frac{\alpha}{2}}, \bar{X} + \frac{\sigma}{\sqrt{n}} N_{\frac{\alpha}{2}} \right]$
$\sigma^2$ unknown, estimate $\mu$	$T = \frac{\sqrt{n}(\bar{X} - \mu)}{S} \sim t_{n-1}$	$\left[ \bar{X} - \frac{S}{\sqrt{n}} t_{n-1, \frac{\alpha}{2}}, \bar{X} + \frac{S}{\sqrt{n}} t_{n-1, \frac{\alpha}{2}} \right]$
$\mu$ known, estimate $\sigma^2$	$T = \frac{nS_\mu^2}{\sigma^2} \sim \chi_n^2$	$\left[ \frac{nS_\mu^2}{\chi_{n, \frac{\alpha}{2}}^2}, \frac{nS_\mu^2}{\chi_{n, 1-\frac{\alpha}{2}}^2} \right]$
$\mu$ unknown, estimate $\sigma^2$	$T = \frac{(n-1)S^2}{\sigma^2} \sim \chi_{n-1}^2$	$\left[ \frac{(n-1)S^2}{\chi_{n-1, \frac{\alpha}{2}}^2}, \frac{(n-1)S^2}{\chi_{n-1, 1-\frac{\alpha}{2}}^2} \right]$

<sup>13</sup>Comment:  $T(X, \theta)$  is both function of sample  $X$  an parameter in statistics model. Note that  $X$  also depends on  $\theta$ , but is fixed once we complete a sample.

2. Double normal population:  $\vec{X} = \{X_1, X_2, \dots, X_m\}$  i.i.d. from  $N(\mu_1, \sigma_1^2)$ ;  $\vec{Y} = \{Y_1, Y_2, \dots, Y_n\}$  i.i.d. from  $N(\mu_2, \sigma_2^2)$

Denote sample mean, sample variance and pooled sample variance:

$$\bar{X} = \frac{1}{m} \sum_{i=1}^m X_i \quad S_X^2 = \frac{1}{m-1} \sum_{i=1}^m (X_i - \bar{X})^2 \quad S_{\mu_1}^2 = \frac{1}{m} \sum_{i=1}^m (X_i - \mu_1)^2, (\mu_1 \text{ known}) \quad (2.140)$$

$$\bar{Y} = \frac{1}{n} \sum_{i=1}^n Y_i \quad S_Y^2 = \frac{1}{n-1} \sum_{i=1}^n (Y_i - \bar{Y})^2 \quad S_{\mu_2}^2 = \frac{1}{n} \sum_{i=1}^n (Y_i - \mu_2)^2, (\mu_2 \text{ known}) \quad (2.141)$$

$$S_\omega^2 = \frac{(m-1)S_X^2 + (n-1)S_Y^2}{m+n-2} \quad (2.142)$$

(a) Estimating  $\mu_1 - \mu_2$ :

When  $\sigma_1^2 \neq \sigma_2^2$  and unknown, estimate  $\mu_1 - \mu_2$ : Behrens-Fisher Problem, remains unsolved<sup>14</sup>, but we can deal with simplified cases.

Estimation	Pivot Variable	Confidence Interval
$\sigma_1^2$ & $\sigma_2^2$ known, estimate $\mu_1 - \mu_2$	$T = \frac{\bar{X} - \bar{Y} - (\mu_1 - \mu_2)}{\sqrt{\frac{\sigma_1^2}{m} + \frac{\sigma_2^2}{n}}} \sim N(0, 1)$	$\left[ \bar{X} - \bar{Y} - N_{\frac{\alpha}{2}} \sqrt{\frac{\sigma_1^2}{m} + \frac{\sigma_2^2}{n}}, \right. \\ \left. \bar{X} - \bar{Y} + N_{\frac{\alpha}{2}} \sqrt{\frac{\sigma_1^2}{m} + \frac{\sigma_2^2}{n}} \right]$
$\sigma_1^2 = \sigma_2^2$ unknown, estimate $\mu_1 - \mu_2$	$T = \frac{\bar{X} - \bar{Y} - (\mu_1 - \mu_2)}{S_\omega \sqrt{\frac{1}{m} + \frac{1}{n}}} \sim t_{m+n-2}$	$\left[ \bar{X} - \bar{Y} - S_\omega t_{m+n-2, \frac{\alpha}{2}} \sqrt{\frac{1}{m} + \frac{1}{n}}, \right. \\ \left. \bar{X} - \bar{Y} + S_\omega t_{m+n-2, \frac{\alpha}{2}} \sqrt{\frac{1}{m} + \frac{1}{n}} \right]$
Welch's $t$ -Interval (when $m, n$ large enough)	$T = \frac{\bar{X} - \bar{Y} - (\mu_1 - \mu_2)}{\sqrt{\frac{S_X^2}{m} + \frac{S_Y^2}{n}}} \xrightarrow{d} N(0, 1)$	$\left[ \bar{X} - \bar{Y} - N_{\frac{\alpha}{2}} \sqrt{\frac{S_X^2}{m} + \frac{S_Y^2}{n}}, \right. \\ \left. \bar{X} - \bar{Y} + N_{\frac{\alpha}{2}} \sqrt{\frac{S_X^2}{m} + \frac{S_Y^2}{n}} \right]$

(b) Estimating  $\frac{\sigma_1^2}{\sigma_2^2}$ :

Estimation	Pivot Variable	Confidence Interval
$\mu_1, \mu_2$ known, estimate $\frac{\sigma_1^2}{\sigma_2^2}$	$T = \frac{S_{\mu_2}^2 \sigma_1^2}{S_{\mu_1}^2 \sigma_2^2} \sim F_{n,m}$	$\left[ \frac{S_{\mu_1}^2}{S_{\mu_2}^2} \frac{1}{F_{m,n, \frac{\alpha}{2}}}, \frac{S_{\mu_1}^2}{S_{\mu_2}^2} \frac{1}{F_{m,n, 1-\frac{\alpha}{2}}} \right]$ or $\left[ \frac{S_{\mu_1}^2}{S_{\mu_2}^2} F_{m,n, \frac{\alpha}{2}}, \frac{S_{\mu_1}^2}{S_{\mu_2}^2} F_{m,n, \frac{\alpha}{2}} \right]$
$\mu_1, \mu_2$ unknown, estimate $\frac{\sigma_1^2}{\sigma_2^2}$	$T = \frac{S_Y^2 \sigma_1^2}{S_X^2 \sigma_2^2} \sim F_{n-1, m-1}$	$\left[ \frac{S_X^2}{S_Y^2} \frac{1}{F_{m-1, n-1, \frac{\alpha}{2}}}, \frac{S_X^2}{S_Y^2} \frac{1}{F_{m-1, n-1, 1-\frac{\alpha}{2}}} \right]$ or $\left[ \frac{S_X^2}{S_Y^2} \frac{1}{F_{m-1, n-1, \frac{\alpha}{2}}}, \frac{S_X^2}{S_Y^2} F_{n-1, m-1, \frac{\alpha}{2}} \right]$

<sup>14</sup> An approximation is Welch-Satterthwaite equation. Detail see [section. 14.1.2](#).

## 3. Non-normal population:

Estimation	Pivot Variable	Confidence Interval
Uniform Distribution: $\vec{X}$ i.i.d. from $U(0, \theta)$	$T = \frac{X_{(n)}}{\theta} \sim U(0, 1)$	$\left[ X_{(n)}, \frac{X_{(n)}}{\sqrt[n]{\alpha}} \right]$
Exponential Distribution: $\vec{X}$ i.i.d. from $\epsilon(\lambda)$	$T = 2n\lambda\bar{X} \sim \chi_{2n}^2$	$\left[ \frac{\chi_{2n, 1-\frac{\alpha}{2}}^2}{2n\bar{X}}, \frac{\chi_{2n, \frac{\alpha}{2}}^2}{2n\bar{X}} \right]$
Bernoulli Distribution: $\vec{X}$ i.i.d. from $B(1, \theta)$	$T = \frac{\sqrt{n}(\bar{X} - \theta)}{\sqrt{\bar{X}(1 - \bar{X})}} \xrightarrow{d} N(0, 1)$	$\left[ \bar{X} - N_{\frac{\alpha}{2}} \sqrt{\frac{\bar{X}(1 - \bar{X})}{n}}, \bar{X} + N_{\frac{\alpha}{2}} \sqrt{\frac{\bar{X}(1 - \bar{X})}{n}} \right]$
Poisson Distribution: $\vec{X}$ i.i.d. from $P(\lambda)$	$T = \frac{\sqrt{n}(\bar{X} - \lambda)}{\sqrt{\bar{X}}} \xrightarrow{d} N(0, 1)$	$\left[ \bar{X} - N_{\frac{\alpha}{2}} \sqrt{\frac{\bar{X}}{n}}, \bar{X} + N_{\frac{\alpha}{2}} \sqrt{\frac{\bar{X}}{n}} \right]$

4. General Case: Use asymptotic normality of MLE to construct CLT for large sample. MLE of  $\theta$  satisfies:

$$\sqrt{n}(\hat{\theta}^* - \theta) \xrightarrow{d} N(0, \frac{1}{I(\theta)}) \quad (2.143)$$

where  $\hat{\theta}^*$  is MLE of  $\theta$ . Replace  $\frac{1}{I(\theta)}$  by  $\sigma^2(\hat{\theta}^*)$ , then

$$T = \frac{\sqrt{n}(\hat{\theta}^* - \theta)}{\sigma(\hat{\theta}^*)} \xrightarrow{d} N(0, 1) \quad (2.144)$$

If  $I(\theta)$  is unknown, we can estimate it by sample:

$$\hat{I}(\theta) = \hat{\mathbb{E}} \left[ \left( \frac{\partial \ln f(\vec{x}; \theta)}{\partial \theta} \right)^2 \right] = \sum_{i=1}^n \left( \frac{\partial \ln f(x_i; \theta)}{\partial \hat{\theta}^*} \right)^2 \quad (2.145)$$

confidence interval:

$$\left[ \hat{\theta}^* - \frac{N_{\frac{\alpha}{2}}}{\sqrt{n}} \sigma(\hat{\theta}^*), \hat{\theta}^* + \frac{N_{\frac{\alpha}{2}}}{\sqrt{n}} \sigma(\hat{\theta}^*) \right] \quad (2.146)$$

## 2.3.4 Fisher Fiducial Argument\*

(Not complete yet)

Idea: When sample is known, we can get 'Fiducial Probability' of  $\theta$ , thus can find an interval estimation based on fiducial distribution. (Similar to the idea of MLE)

Remark: Fiducial probability (denoted as  $\tilde{\mathbb{P}}(\theta)$ ) is 'probability of parameter', in the case that sample is known.

**Fiducial probability is different from Probability.**

Thus get

$$\tilde{\mathbb{P}}(\hat{g}_1 \leq g(\theta) \leq \hat{g}_2) = 1 - \alpha \quad (2.147)$$



## Section 2.4 Hypothesis Testing

Hypothesis is a statement about the characteristic of population, e.g. distribution form, parameters, independency, etc.

Mission: Use sample to test the hypothesis, i.e. judge whether population has some characteristic.

### 2.4.1 Basic Concepts

Parametric hypothesis testing.

For random sample  $\vec{X} = (X_1, X_2, \dots, X_n) \in \mathcal{X}$  i.i.d. from  $\mathcal{F} = \{f(x; \theta); \theta \in \Theta\}$

- Null Hypothesis  $H_0$  & Alternative Hypothesis  $H_1$  (Sometimes denoted  $H_a$ ): Wonder whether a statement is true.

Def. **Null Hypothesis**:  $H_0 : \theta \in \Theta_0 \subset \Theta$ , **a statement that we try to reject based on sample**;  $H_1 : \theta \in \Theta_1 = \Theta/\Theta_0$  is **Alternative Hypothesis**.

□ **Note**: **Cannot** exchange  $H_0$  and  $H_1$ , because when the evidence is ambiguity, we have to accept  $H_0$ , regardless of what  $H_0$  is. So it is **very important** to pick the proper  $H_0$ <sup>15</sup>.

Thus Hypothesis Testing:

$$H_0 : \theta \in \Theta_0 \longleftrightarrow H_1 : \theta \in \Theta_1 \quad (2.148)$$

- Rejection Region  $R$  & Acceptance Region  $R^C$ : Judge whether to reject  $H_0$  from sample, Def. **Rejection Region**:

$$R \subset \mathcal{X}: \text{reject } H_0 \text{ if } \vec{X} \in R \quad (2.149)$$

Acceptance Region: accept  $H_0$  if  $\vec{X} \in R^C$

- Test Function: It's hard and unparctical to really dividing regions in  $\mathcal{X}$ . Instead the regions are usually described by some test function, basically it's like some indicator function.

- Continuous Case:

$$\varphi(\vec{X}) = \begin{cases} 1, & \vec{X} \in R \\ 0, & \vec{X} \in R^c \end{cases} \quad (2.150)$$

i.e.  $R = \{\vec{X} : \varphi(\vec{X}) = 1\}$ . Where  $R$  to be determined.

- Discrete Case: Randomized Test Function

$$\varphi(\vec{X}) = \begin{cases} 1, & \vec{X} \in R - \partial R \\ r, & \vec{X} \in \partial R \\ 0, & \vec{X} \in R^c \end{cases} \quad (2.151)$$

Where  $R$  and  $r$  to be determined.  $\partial R$  means the boundary of  $R$

<sup>15</sup>So when being uncertain about which to put on  $H_0$ , think about which one we are more intended to assume when evidence is ambiguous.

Examples:

- Clinical test, in which we should put 'being ill' in  $H_0$ , and 'all right' in  $H_1$ .
- Court trial, in which we should put 'innocent' in  $H_0$ , and 'guilty' in  $H_1$ .

△ Type I Error & Type II Error: Sample is random, possible to make a wrong judge.

- Type I Error (弃真):  $H_0$  is true but sample falls in  $R$ , thus  $H_0$  is rejected.

$$\mathbb{P}(\text{type I error}) = \mathbb{P}(\vec{X} \in R | H_0) = \alpha(\theta) \quad (2.152)$$

- Type II Error (取伪):  $H_0$  is wrong but sample falls in  $R^C$ , thus  $H_0$  is accepted.

$$\mathbb{P}(\text{type II error}) = \mathbb{P}(\vec{X} \notin R | H_1) = \beta(\theta) \quad (2.153)$$

		Judgement	
		Accept $H_0$	Reject $H_0$
Truth	$H_0$	✓	Type I Error
	$H_1$	Type II Error	✓

表 2: 'Confusion Matrix'

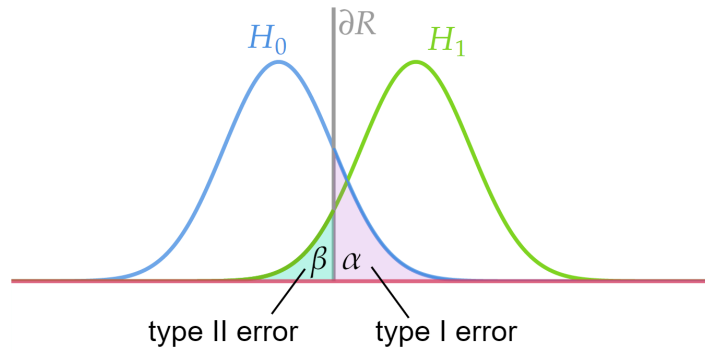


图 1: Illustration of type I&II error

It's impossible to make probability of Type I & II Error small simultaneously, how to pick a proper test  $\varphi(\vec{x})$ ?

□ **Neyman-Pearson Principle: First control  $\alpha \leq \alpha_0$ , then take min  $\beta$ .**

How to determine  $\alpha_0$ ? Depend on specific problem.<sup>16</sup>

△  $p$ -value: probability to get larger bias (or simply 'more extreme data') than observed  $\vec{x}_0$  if  $H_0$  as ground truth, and  $H_1$  as alternative.

e.g. For reject region defined with statistics  $R = \{\vec{X} | T(\vec{X}) \geq C\}$ ,  $p$ -value:

$$p_{H_0, H_1}(\vec{x}) = \mathbb{P}[T(\vec{X}) \geq t(\vec{x}_0) | H_0, H_1] \quad (2.154)$$

Remark: We believe that sample should reflect the property of model parameter, and  $p$ -value is that under  $H_0$ , the probability to get a **worse** result than  $\vec{x}$ . If the probability is small, then our assumption  $H_0$  might be invalid.

Rule: Reject  $H_0$  if  $p(\vec{x}_0) \leq \alpha_0$ .

**Note :**

<sup>16</sup>In most cases, take  $\alpha_0 = 0.05$ .

- $p$ -value is **different from**  $\alpha$  or type I error.  $p$ -value is generated before we make decision while  $\alpha$  arises after we decide how to make decisions. (But they do target the same result.)
- $p$ -value is calculated **after**  $H_0 \longleftrightarrow H_1$  pair is given. Avoid abusing the concept of  $p$ -value.
- Power Function: After  $H_0 : \theta \in \Theta_0$  is given, and we have determined the rejection region  $R$ , the probability that sample falls in  $R$ , i.e. reject  $H_0$  by sampling, as a function of ground truth  $\theta$ .

$$\pi(\theta) = \mathbb{P}(\vec{X}(\theta) \in R | H_0) = \begin{cases} \mathbb{P}(\text{type I error}), & \theta \in \Theta_0 \\ 1 - \mathbb{P}(\text{type II error}), & \theta \in \Theta_1 \end{cases} = \begin{cases} \alpha(\theta), & \theta \in \Theta_0 \\ 1 - \beta(\theta), & \theta \in \Theta_1 \end{cases} \quad (2.155)$$

Express as test function:

$$\pi(\theta) = \mathbb{E}[\varphi(\vec{X}) | \theta] \quad (2.156)$$

Power function is a measure of the goodness of test:  $\pi(\theta)$  should be small under  $H_0$ , and be large under  $H_1$  (and grows very fast at the boundary of  $H_0$  and  $H_1$ ).

#### □ General Steps of Hypothesis Testing:

1. Propose  $H_0$  &  $H_1$ .
2. Select a proper  $\alpha$  (to determine  $c$ ).
3. Determine  $R$  (usually in the form of a statistic, e.g.  $R = \{\vec{X} : T(\vec{X}) \geq c\}$ ).
4. Sampling, get sample (as well as  $t(\vec{x})$ ), then
  - compare with  $R$  and determine whether to reject/accept  $H_0$ , or
  - calculate  $p$ -value and determine whether to reject/accept  $H_0$

### 2.4.2 Hypothesis Testing of Common Distributions

For some common distribution populations, determine rejection region  $R$  under certain  $H_0$  with confidence coefficient  $\alpha$ .

Definition of necessary statistics see section [section. 2.3.3](#).

1. Single normal population:

Condition	$H_0$	$H_1$	Testing Statistic $T$	Rejection Region $R$
$\sigma^2$ known, test $\mu$	$\mu = \mu_0$ $\mu \leq \mu_0$ $\mu \geq \mu_0$	$\mu \neq \mu_0$ $\mu > \mu_0$ $\mu < \mu_0$	$T = \frac{\sqrt{n}(\bar{X} - \mu_0)}{\sigma} \sim N(0, 1)$	$ T  > N_{\frac{\alpha}{2}}$ $T > N_{\alpha}$ $T < -N_{\alpha}$
$\sigma^2$ unknown, test $\mu$	$\mu = \mu_0$ $\mu \leq \mu_0$ $\mu \geq \mu_0$	$\mu \neq \mu_0$ $\mu > \mu_0$ $\mu < \mu_0$	$T = \frac{\sqrt{n}(\bar{X} - \mu_0)}{S} \sim t_{n-1}$	$ T  > t_{n-1, \frac{\alpha}{2}}$ $T > t_{n-1, \alpha}$ $T < -t_{n-1, \alpha}$
$\mu$ known, test $\sigma^2$	$\sigma^2 = \sigma_0^2$ $\sigma^2 \leq \sigma_0^2$ $\sigma^2 \geq \sigma_0^2$	$\sigma^2 \neq \sigma_0^2$ $\sigma^2 > \sigma_0^2$ $\sigma^2 < \sigma_0^2$	$T = \frac{nS^2}{\sigma_0^2} \sim \chi_n^2$	$T < \chi_{n, 1-\frac{\alpha}{2}}^2 \cup T > \chi_{n, \frac{\alpha}{2}}^2$ $T > \chi_{n, \alpha}^2$ $T < \chi_{n, 1-\alpha}^2$
$\mu$ unknown, test $\sigma^2$	$\sigma^2 = \sigma_0^2$ $\sigma^2 \leq \sigma_0^2$ $\sigma^2 \geq \sigma_0^2$	$\sigma^2 \neq \sigma_0^2$ $\sigma^2 > \sigma_0^2$ $\sigma^2 < \sigma_0^2$	$T = \frac{(n-1)S^2}{\sigma_0^2} \sim \chi_{n-1}^2$	$T < \chi_{n-1, 1-\frac{\alpha}{2}}^2 \cup T > \chi_{n-1, \frac{\alpha}{2}}^2$ $T > \chi_{n-1, \alpha}^2$ $T < \chi_{n-1, 1-\alpha}^2$

## 2. Double normal population:

Condition	$H_0$	$H_1$	Testing Statistic $T$	Rejection Region $R$
$\sigma_1^2, \sigma_2^2$ known, test $\mu_1 - \mu_2$	$\mu_1 - \mu_2 = \mu_0$ $\mu_1 - \mu_2 \leq \mu_0$ $\mu_1 - \mu_2 \geq \mu_0$	$\mu_1 - \mu_2 \neq \mu_0$ $\mu_1 - \mu_2 > \mu_0$ $\mu_1 - \mu_2 < \mu_0$	$T = \frac{\bar{X} - \bar{Y} - \mu_0}{\sqrt{\frac{\sigma_1^2}{m} + \frac{\sigma_2^2}{n}}} \sim N(0, 1)$	$ T  > N_{\frac{\alpha}{2}}$ $T > N_{\alpha}$ $T < -N_{\alpha}$
$\sigma_1^2, \sigma_2^2$ unknown but same, test $\mu_1 - \mu_2$	$\mu_1 - \mu_2 = \mu_0$ $\mu_1 - \mu_2 \leq \mu_0$ $\mu_1 - \mu_2 \geq \mu_0$	$\mu_1 - \mu_2 \neq \mu_0$ $\mu_1 - \mu_2 > \mu_0$ $\mu_1 - \mu_2 < \mu_0$	$T = \frac{\bar{X} - \bar{Y} - \mu_0}{S_{\omega}} \sqrt{\frac{mn}{m+n}} \sim t_{m+n-2}$	$ T  > t_{m+n-2, \frac{\alpha}{2}}$ $T > t_{m+n-2, \alpha}$ $T < -t_{m+n-2, \alpha}$
$\mu_1, \mu_2$ known, test $\frac{\sigma_1^2}{\sigma_2^2}$	$\sigma_1^2 = \sigma_2^2$ $\sigma_1^2 \geq \sigma_2^2$ $\sigma_1^2 \leq \sigma_2^2$	$\sigma_1^2 \neq \sigma_2^2$ $\sigma_1^2 < \sigma_2^2$ $\sigma_1^2 > \sigma_2^2$	$T = \frac{S_{\mu_2}^2}{S_{\mu_1}^2} \sim F_{n,m}$	$T < F_{n,m, 1-\frac{\alpha}{2}}$ $\cup T > F_{n,m, \frac{\alpha}{2}}$ $T > F_{n,m, \alpha}$ $T < F_{n,m, 1-\alpha}$
$\mu_1, \mu_2$ unknown, test $\frac{\sigma_1^2}{\sigma_2^2}$	$\sigma_1^2 = \sigma_2^2$ $\sigma_1^2 \geq \sigma_2^2$ $\sigma_1^2 \leq \sigma_2^2$	$\sigma_1^2 \neq \sigma_2^2$ $\sigma_1^2 < \sigma_2^2$ $\sigma_1^2 > \sigma_2^2$	$T = \frac{S_2^2}{S_1^2} \sim F_{n-1, m-1}$	$T < F_{n-1, m-1, 1-\frac{\alpha}{2}}$ $\cup T > F_{n-1, m-1, \frac{\alpha}{2}}$ $T > F_{n-1, m-1, \alpha}$ $T < F_{n-1, m-1, 1-\alpha}$

$\sigma_1^2, \sigma_2^2$  unknown case see Welch Test in [section. 14.1.2](#).

## 3. None normal population:

## 4. More than two normal population: Analysis of Variance.

Condition	$H_0$	$H_1$	Testing Statistic $T$	Rejection Region $R$
$\vec{X}$ from $B(1, p)$ , test $p$	$p = p_0$	$p \neq p_0$	$T = \frac{\sqrt{n}(\bar{X} - p_0)}{\sqrt{p_0(1-p_0)}} \xrightarrow{d} N(0, 1)$	$ T  > N_{\frac{\alpha}{2}}$
$\vec{X}$ from $P(\lambda)$ , test $\lambda$	$\lambda = \lambda_0$	$\lambda \neq \lambda_0$	$T = \frac{\sqrt{n}(\bar{X} - \lambda_0)}{\sqrt{\lambda_0}} \xrightarrow{d} N(0, 1)$	$ T  > N_{\frac{\alpha}{2}}$

### 2.4.3 Likelihood Ratio Test

Idea: To test  $H_0 : \theta \in \Theta_0 \longleftrightarrow H_1 : \theta \in \Theta_1$  known  $\vec{x}$ , examine the likelihood function  $L(\theta; \vec{x})$  and **compare**  $L_{\theta \in \Theta_0}$  and  $L_{\theta \in \Theta_1}$  to see the likelihood that  $H_0$  is true.

Def. **Likelihood Ratio (LR)**:

$$\Lambda(\vec{x}) = \frac{\sup_{\theta \in \Theta_0} L(\theta; \vec{x})}{\sup_{\theta \in \Theta} L(\theta; \vec{x})} \quad (2.157)$$

Reject  $H_0$  if  $\Lambda(\vec{x}) < \Lambda_0$ . Or equivalently: Reject  $H_0$  if  $-2 \ln \Lambda(\vec{x}) > C (= -2 \ln \Lambda_0)$ .

where  $\Lambda_0$  (or equivalently  $C = -2 \ln \Lambda_0$ ) satisfies:

$$\mathbb{E}_{\Theta_0}[\varphi(\vec{X})] \leq \alpha, \quad \forall \theta \in \Theta_0 \quad (2.158)$$

LR and sufficient statistic:  $\Lambda(\vec{x})$  can be expressed as  $\Lambda(\vec{x}) = \Lambda^*(T(\vec{x}))$ , where  $T(\vec{X})$  is sufficient statistic. We usually denote  $\lambda = \log \Lambda$

□ **LRT for one-sample  $t$ -test:** For  $X_1, X_2, \dots, X_n$  i.i.d.  $\sim N(\mu, \sigma^2)$ , test

$$H_0 : \mu = \mu_0 \longleftrightarrow H_1 : \mu \neq \mu_0 \quad \text{when } \sigma^2 \text{ unknown}$$

Can prove:

$$\lambda^{2/n} = \frac{\sum_{i=1}^n (x_i - \bar{x})^2}{\sum_{i=1}^n (x_i - \mu_0)^2}$$

Denote  $T = \frac{\sqrt{n}(\bar{x} - \mu_0)}{S}$ , then LRT could be expressed in equivalent form

$$\lambda = \left(1 + \frac{T^2}{n-1}\right)^{-n/2}$$

The Multivariate case see **section. 4.2.4**, where  $T^2$  itself is the Hotelling's  $T^2$  statistic.

□ **Limiting Distribution of LRT: Wilks' Thm.**

If  $\dim \Theta = k > \dim \text{span}\{\Theta_0\} = s^{17}$ , then under  $H_0 : \theta \in \Theta_0$ :

$$-2\lambda = -2 \ln \Lambda(\vec{x}) \xrightarrow{d} \chi_{k-s}^2 \quad (2.159)$$

<sup>17</sup>Here 'dimension' refers to 'degree of freedom'.

### 2.4.4 Uniformly Most Powerful Test

Idea: Neyman-Pearson Principle: control  $\alpha$ , find  $\min \beta$ . i.e. control  $\alpha$ , find  $\max \pi(\theta)$

Def. **Uniformly Most Powerful Test** (UMP)  $\varphi_{\text{UMP}}$  with level of significance  $\alpha$  satisfies

$$\pi_{\text{UMP}}(\theta) \geq \pi(\theta), \forall \theta \in \Theta_1 \quad (2.160)$$

**Neyman-Pearson Lemma:** For  $\vec{X} = (X_1, X_2, \dots, X_n)$  i.i.d. from  $f(\vec{x}; \theta)$ .

Test hypothesis  $H_0 : \theta = \theta_0 \longleftrightarrow H_1 : \theta = \theta_1$ . Def. test function  $\varphi$  as:

$$\varphi(\vec{x}) = \begin{cases} 1, & \frac{f(\vec{x}; \theta_1)}{f(\vec{x}; \theta_0)} > C \\ r, & \frac{f(\vec{x}; \theta_1)}{f(\vec{x}; \theta_0)} = C \\ 0, & \frac{f(\vec{x}; \theta_1)}{f(\vec{x}; \theta_0)} < C \end{cases} \quad (2.161)$$

Then there exists  $C$  and  $r$  such that

- $\mathbb{E}[\varphi(\vec{x})|\theta_0] = \mathbb{P}\left(\frac{f(\vec{x}; \theta_1)}{f(\vec{x}; \theta_0)} > C\right) + r\mathbb{P}\left(\frac{f(\vec{x}; \theta_1)}{f(\vec{x}; \theta_0)} = C\right) = \alpha$
- This  $\varphi$  is UMP of level of significance  $\alpha$

Actually kind of 1-dimensional case of LRT.

Note: UMT exist for **simple**  $H_0, H_1$ , otherwise may not exist.

UMP and sufficient statistics: Test function  $\varphi(\vec{X})$  given by **equation. 2.161** is function of sufficient statistics  $T(\vec{X})$ , i.e.  $\varphi(\vec{X}) = \varphi^*(T(\vec{X}))$ .

UMP and Exponential Family: For sample  $\vec{X} = (X_1, X_2, \dots, X_n)$  from exponential family:

$$f(\vec{x}; \theta) = C(\theta)h(\vec{x}) \exp\{Q(\theta)T(\vec{x})\} \quad (2.162)$$

Test single hypothesis  $H_0 : \theta = \theta_0 \longleftrightarrow H_1 : \theta = \theta_1$ , (where  $\theta_0 < \theta_1$ ). If

- $\theta_0$  is inner point of  $\Theta$
- $Q(\theta)$  monotone increase with  $\theta$

Then UMP exists, in the form of:

$$\varphi(\vec{x}) = \begin{cases} 1, & T(\vec{x}) > C \\ r, & T(\vec{x}) = C \\ 0, & T(\vec{x}) < C \end{cases} \quad (2.163)$$

where  $C$  and  $r$  satisfies  $\mathbb{E}[\varphi(\vec{x})|\theta_0] = \alpha$ .

Note: or take  $Q(\theta)$  mono decreased, then in **equation. 2.163**, take opposite inequality operators.

#### □ General Steps of UMP:

1. Find a point  $\theta_0 \in \Theta_0$  and a point  $\theta_1 \in \Theta_1$ . (Note: **one** point)
2. Construct test function in the form of **equation. 2.161**, use  $\mathbb{E}[\varphi(\vec{x})|\theta_0] = \alpha$  to determine  $C$  and  $r$ .

3. Get  $R$  and  $\varphi(\vec{x})$ .
4. If  $\varphi$  does **not** depend on  $\theta_1$ , then  $H_1$  can be generalized to  $H_1 : \theta \in \Theta_1$ .
5. If  $\varphi$  satisfies  $\mathbb{E}_{\theta \in \Theta_0}(\varphi) \leq \alpha$ , then  $H_0$  can be generalized to  $H_0 : \theta \in \Theta_0$ .

## 2.4.5 Duality of Hypothesis Testing and Interval Estimation

- Thm.:  $\forall \theta_0 \in \Theta$  there exists hypothesis testing  $H_0 : \theta = \theta_0 \longleftrightarrow H_1 : \theta \neq \theta_0$  of level  $\alpha$  with rejection region  $R_{\theta_0}$ .  
Then

$$C(\vec{X}) = \{\theta : \vec{X} \in R_{\theta}^C\} \quad (2.164)$$

is a  $1 - \alpha$  confidence region for  $\theta$

- Thm.:  $C(\vec{X})$  is a  $1 - \alpha$  confidence region for  $\theta$ . Then  $\forall \theta_0 \in C(\vec{X})$ , the rejection region of hypothesis testing  $H_0 : \theta = \theta_0 \longleftrightarrow H_1 : \theta \neq \theta_0$  of level  $\alpha$  satisfies

$$R_{\theta_0}^C = \{\vec{X} : \theta_0 \in C(\vec{X})\} \quad (2.165)$$

□ **Idea:**

$$H_0 : \theta = \theta_0 \longleftrightarrow H_1 : \theta \neq \theta_0$$

$$\updownarrow$$

$$\mathbb{P}(R^C(\vec{X})|H_0) = \mathbb{P}(R^C(\vec{X})|\theta_0) = 1 - \alpha$$

$$\updownarrow$$

$$\text{Confidence Interval: } \theta_0 \in R^C(\vec{X})$$

Similar for Confidence Limit and One-Sided Testing.

▷ **R. Code**

The test function for one-way / two-way test. The function gives both interval estimation and hypothesis testing results.

```

1 # one-way
2 t.test(x, alternative = c("two.sided", "less", "greater"), mu = 0, conf.
   level = 0.95, ...)
3 # two-way
4 t.test(x, y, alternative = c("two.sided", "less", "greater"), mu = 0,
   paired = FALSE, var.equal = FALSE, conf.level = 0.95, ...)
5 t.test(df, ...)
```

where `paired = TRUE` for pairwise comparison requires  $|x| = |y|$ .

## 2.4.6 Introduction to Non-Parametric Hypothesis Testing

Motivation: Usually distribution form unknown, cannot use parametric hypothesis testing.

Useful Method:

- Sign Test: Used for paired comparison  $\vec{X} = (X_1, X_2, \dots, X_n), \vec{Y} = (Y_1, Y_2, \dots, Y_n)$ .

Take  $Z_i = Y_i - X_i$  i.i.d., denote  $\mathbb{E}(Z) = \mu$ . Test  $H_0 : \mu = 0 \longleftrightarrow H_1 : \mu \neq 0$ . Denote  $n_+ = \#(\text{positive } Z_i)$  and  $n_- = \#(\text{negative } Z_i)$ ,  $n_0 = n_+ + n_-$ . Then  $n_+ \sim B(n_0, \theta)$ , thus the test is  $H_0 : \theta = \frac{1}{2} \longleftrightarrow H_1 : \theta \neq \frac{1}{2}$

Then use Binomial Testing or large sample CLT Normal Testing on  $H_0$ .

Remark:

- Also can test  $H_0 : \theta \leq \frac{1}{2} \longleftrightarrow H_1 : \theta > \frac{1}{2}$
- Drawback: ignores magnitudes.

- Wilcoxon Signed Rank Sum Test: Improvement of Sign Test. Based on order statistics.

Order Statistics of  $Z_i$ :  $Z_{(1)} < Z_{(2)} < \dots < Z_{(n)}$ , where each  $Z_{(j)}$  corresponds to some  $Z_i$  as  $Z_i = Z_{(R_i)}$ , then  $R_i$  is the rank of  $Z_i$ .<sup>18</sup>

$$R_i = \sum_{j=1}^n \mathbb{I}_{Z_j < Z_i} + \frac{1}{2} \left( 1 + \sum_{j=1}^n \mathbb{I}_{Z_j = Z_i} \right)$$

Def.  $\vec{R} = (R_1, R_2, \dots, R_n)$  is **Rank Statistics** of  $(Z_1, Z_2, \dots, Z_n)$

$$R_i = \sum_{j=1}^n \mathbb{I}_{Z_j < Z_i} + \frac{1}{2} \left( 1 + \sum_{j=1}^n \mathbb{I}_{Z_j = Z_i} \right)$$

Def. **Sum of Wilcoxon Signed Rank**:

$$W^+ = \sum_{i=1}^{n_0} R_i \mathbb{I}_{Z_{R_i} > 0} \quad (2.166)$$

Distribution of  $W^+$  is complex.  $\mathbb{E}$  and  $var$  of  $W^+$  under  $H_0$ :

$$\mathbb{E}(W^+) = \frac{n_0(n_0 + 1)}{4} \quad var(W^+) = \frac{n_0(n_0 + 1)(2n_0 + 1)}{24} \quad (2.167)$$

Usually consider large sample CLT, construct normal approximation:

$$T = \frac{W^+ - \mathbb{E}(W^+)}{\sqrt{var(W^+)}} \xrightarrow{d} N(0, 1) \quad (2.168)$$

Rejection Region:  $R = \{|T| > N_{\frac{\alpha}{2}}\}$

- Wilcoxon Two-Sample Rank Sum Test: Used for two independent sample comparison.

Assume  $\vec{X} = (X_1, \dots, X_m)$  i.i.d.  $\sim f(x)$ ;  $\vec{Y} = (Y_1, \dots, Y_n)$  i.i.d.  $\sim f(x - \theta)$ , test  $H_0 : \theta = 0 \longleftrightarrow H_1 : \theta \neq 0$ .

Rank  $X_i$  and  $Y_i$  as:

$$Z_1 \leq Z_2 \leq \dots \leq Z_{m+n} \quad (2.169)$$

<sup>18</sup>If some  $X_i, X_j, \dots$  equal, then take same rank  $R = \text{mean}\{R_i, R_j, \dots\}$ .



in which denote rank of  $Y_i$  as  $R_i$ , and def. **Wilcoxon two-sample rank sum**:

$$W = \sum_{i=1}^n R_i \quad (2.170)$$

$\mathbb{E}$  and  $var$  of  $W$  under  $H_0$ :

$$\mathbb{E}(W) = \frac{n(m+n+1)}{2} \quad var(W) = \frac{mn(n+m+1)}{12} \quad (2.171)$$

Use large sample approximation, construct CLT:

$$T = \frac{W - \mathbb{E}(W)}{\sqrt{var(W)}} \xrightarrow{d} N(0, 1) \quad (2.172)$$

▷ **R. Code**

```
1 wilcox.test(x, y, alternative = c("two.sided", "less", "greater"), mu
   = 0, paired = FALSE)
```

- Goodness-of-Fit Test: For  $\vec{X} = (X_1, X_2, \dots, X_n)$  i.i.d. from some certain population  $X$ . Test  $H_0 : X \sim F(x)$ .

where  $F$  is theoretical distribution, can be either parametric or non-parametric.

Idea: Define some *quantity*  $D = D(X_1, \dots, X_n; F)$  to measure the difference between  $F$  and sample. And def. *Goodness-of-fit* when observed value of  $D$  (say  $d_0$ ) is given:

$$p(d_0) = \mathbb{P}(D \geq d_0 | H_0) \quad (2.173)$$

**Goodness-of-Fit Test:** Reject  $H_0$  if  $p(d_0) < \alpha$ .

Pearson  $\chi^2$  Test: Usually used for discrete case.

Test  $H_0 : \mathbb{P}(X_i = a_i) = p_i, i = 1, 2, \dots, r$ . Denote  $\#(X_j = a_i) = \nu_i$ , take  $D$  as:

$$K_n = K_n(X_1, \dots, X_n; F) = \sum_{i=1}^r \frac{(\nu_i - np_i)^2}{np_i} \quad (2.174)$$

Pearson Thm.: For  $K_n$  defined as **equation. 2.174**, then under  $H_0$ :

$$K_n \xrightarrow{d} \chi_{r-1-s}^2 \quad (2.175)$$

Here  $s$  is number of unknown parameter,  $r - 1 - s$  is the degree of freedom.

Note:

- $a_i$  must **not** depend on sample.
- For continuous case, construct division:

$$\mathbb{R} \rightarrow (-\infty, a_1, a_2, \dots, a_{r-1}, \infty = a_r) \quad (2.176)$$

and test  $H_0 : \mathbb{P}(X \in I_j) = p_j$

Criterion: Pick proper interval so that  $np_i$  and  $\nu_i$  both  $\geq 5$ .

- Contingency Table Independence & Homogeneity Test: Detailed knowledge and more complex application cases see [section. 7.2.4](#) and [section. 8.2.1](#)

– Independence Test:

Test a two-parameter sample and to see whether these two parameters(features) are independent. Denote  $Z = (X, Y)$  are some 'level' of sample,  $n_{ij}$  is number of sample with level  $(i, j)$

Contingency Table:

Y X	1	...	j	...	s	$\Sigma$
1	$n_{11}$	...	$n_{1j}$	...	$n_{1s}$	$n_{1\cdot}$
$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$
i	$n_{i1}$	...	$n_{ij}$	...	$n_{is}$	$n_{i\cdot}$
$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$
r	$n_{r1}$	...	$n_{rj}$	...	$n_{rs}$	$n_{r\cdot}$
$\Sigma$	$n_{\cdot 1}$	...	$n_{\cdot j}$	...	$n_{\cdot s}$	$n$

Test  $H_0$  :  $X$  &  $Y$  are independent. i.e.  $H_0 : P(X = i, Y = j) = P(X = i)P(Y = j) = p_{i\cdot}p_{\cdot j}$ .

Construct  $\chi^2$  test statistic:

$$K_n = \sum_{i=1}^r \sum_{j=1}^s \frac{[n_{ij} - n(\frac{n_{i\cdot}}{n})(\frac{n_{\cdot j}}{n})]^2}{n(\frac{n_{i\cdot}}{n})(\frac{n_{\cdot j}}{n})} = n \left( \sum_{i=1}^r \sum_{j=1}^s \frac{n_{ij}^2}{n_{i\cdot}n_{\cdot j}} - 1 \right) \quad (2.177)$$

Then under  $H_0$ ,  $K_n \xrightarrow{d} \chi_{rs-1-(r+s-2)}^2 = \chi_{(r-1)(s-1)}^2$

Reject  $H_0$  if  $p(k_0) = P(K_n \geq k_0) < \alpha$

– Homogeneity Test:

Test  $R$  groups of sample with category rank, to see whether these groups has similar rank distribution.

Category Group	Category 1	...	Category j	...	Category C	$\Sigma$
Group 1	$n_{11}$	...	$n_{1j}$	...	$n_{1C}$	$n_{1\cdot}$
$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$
Group i	$n_{i1}$	...	$n_{ij}$	...	$n_{iC}$	$n_{i\cdot}$
$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$
Group R	$n_{R1}$	...	$n_{Rj}$	...	$n_{RC}$	$n_{R\cdot}$
$\Sigma$	$n_{\cdot 1}$	...	$n_{\cdot j}$	...	$n_{\cdot C}$	$n$

Denote  $P(\text{Category } j | \text{Group } i) = p_{ij}$ . Test  $H_0 : p_{ij} = p_j, \forall 1 \leq i \leq R$ .

Construct  $\chi^2$  test statistic:

$$D = \sum_{i=1}^R \sum_{j=1}^C \frac{[n_{ij} - n(\frac{n_{i\cdot}}{n})(\frac{n_{\cdot j}}{n})]^2}{n(\frac{n_{i\cdot}}{n})(\frac{n_{\cdot j}}{n})} = n \left( \sum_{i=1}^R \sum_{j=1}^C \frac{n_{ij}^2}{n_{i\cdot}n_{\cdot j}} - 1 \right) \quad (2.178)$$

Then under  $H_0$ ,  $D \xrightarrow{d} \chi^2_{R(C-1)-(C-1)} = \chi^2_{(R-1)(C-1)}$

▷ **R. Code**

Contingency table test example:

```
1 table_df <- matrix(c(10,20,15,25), 2, 2)
2 chisq.test(table_df)
3 fisher.test(table_df)
```

- Test of Normality: normality is a good & useful assumption.

For  $\vec{Y} = (Y_1, Y_2, \dots, Y_n)$ ,

Test  $H_0$  : exists  $\mu$  &  $\sigma^2$  such that  $Y_i$  i.i.d.  $\sim N(\mu, \sigma^2)$ .

- Kolmogorov-Smirnov Test: Assume  $\vec{X}$  form population CDF  $F(x)$ , test  $H_0 : F(x) = F_0(x)$  (where can take  $F_0 = \Phi$  or some other known CDF).

use  $F_n(x)$  (as defined in [equation. 2.125](#)) as approx. to  $F(x)$ , test

$$D_n = \sum_{-\infty < x < +\infty} |F_n(x) - F_0(x)| \quad (2.179)$$

Reject  $H_0$  if  $D_n > c$

or use goodness-of-fit: denote observed value of  $D_n$  as  $d_n$ . Reject  $H_0$  if

$$p(d_n) = \mathbb{P}(D_n > d_n | H_0) < \alpha \quad (2.180)$$

- Shapiro-Wilk Test:

Test  $H_0$  : exists  $\mu$  &  $\sigma^2$  such that  $X_i$  i.i.d.  $\sim N(\mu, \sigma^2)$ .

Denote  $Y_{(i)} = \frac{X_{(i)} - \mu}{\sigma}$ ,  $m_i = \mathbb{E}(Y_{(i)})$

Under  $H_0$ ,  $(X_{(i)}, m_i)$  falls close to straight line. Test Statistic: Correlation

$$R^2 = \frac{(\sum_{i=1}^n (X_{(i)} - \bar{X})(m_i - \bar{m}))^2}{\sum_{i=1}^n (X_{(i)} - \bar{X})^2 \sum_{i=1}^n (m_i - \bar{m})^2} = \text{corr}(X_{(i)}, m_i) \quad (2.181)$$

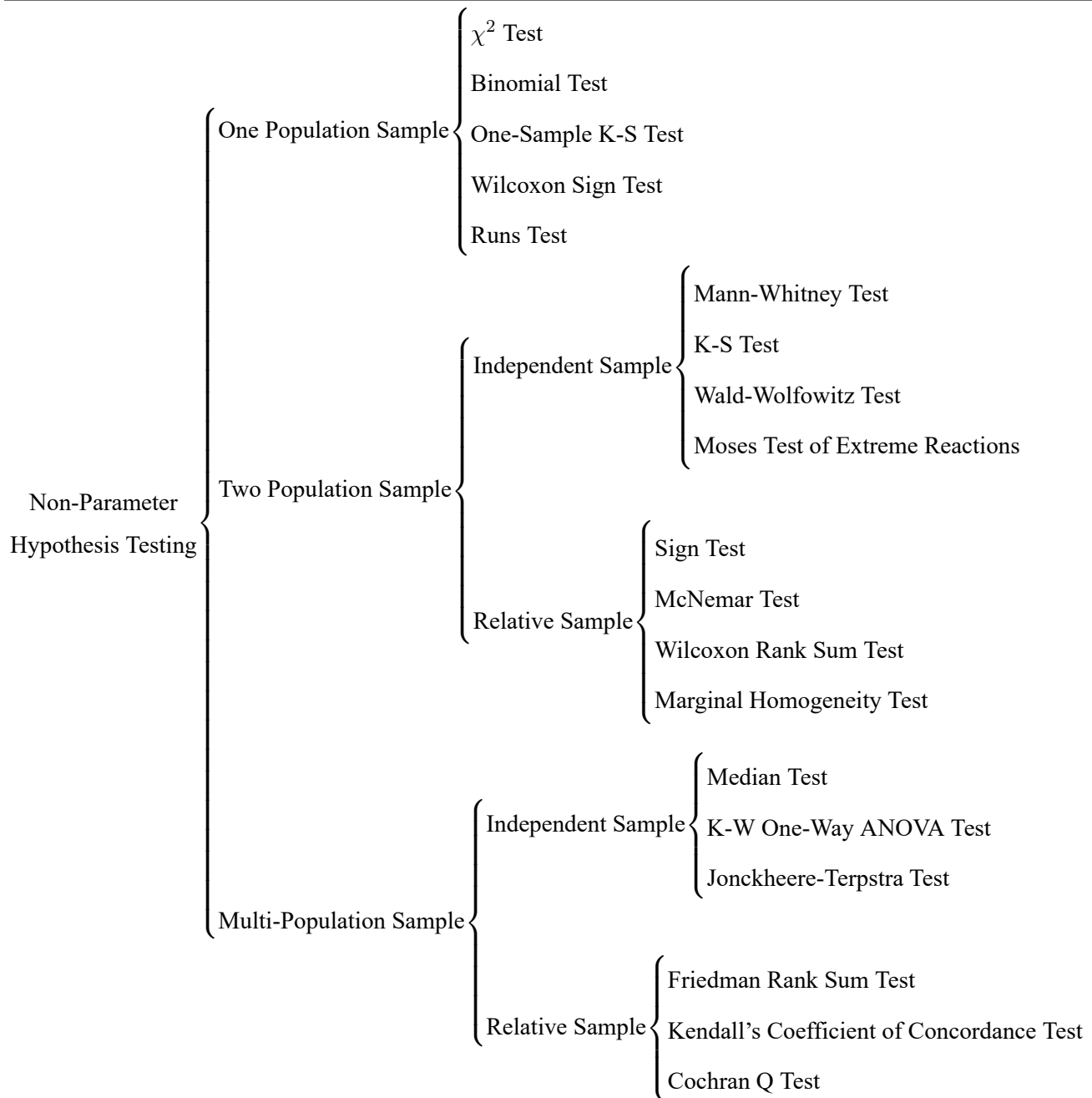
Reject  $H_0$  if  $R^2 < c$

Shapiro-Wilk correction:

$$W = \frac{\left( \sum_{i=1}^{\lfloor n/2 \rfloor} a_i (X_{(n+1-i)} - X_{(i)}) \right)^2}{\sum_{i=1}^n (X_{(i)} - \bar{X})^2} \quad (2.182)$$

▷ **R. Code**

```
1 shapiro.test(x)
```



## Chapter. III 线性回归分析部分

Instructor: Zaiying Zhou

### □ Steps in Regression Analysis

1. Statement of the problem;
2. Selection of potentially relevant **variables**;
3. Data collection;
4. Exploratory Data Analysis (**EDA**)
5. **Model** specification;
6. Choice of fitting method;
7. Model fitting;
8. Model validation and criticism;
9. Using the chosen model(s) for the solution of the posed problem;
10. **Explain** the result.

[R](#). Code for EDA

```
1 library('GGally')
2 head(df)
3 ggpairs(df)
4 str(df)
5 summary(df)
```

### □ Used Packages in [R](#).

```
1 library('ggplot2')
2 library('GGally')
3 library('car')
4 library('moments')
5 library('lmtest')
6 library('nortest')
7 library('MASS')
8 library('tseries')
9
10 source('package.r')
```

## Section 3.1 Regression Model

In regression model, we will observe pairs of variables, called 'cases'(样本点). A sample is  $(X_1; Y_1), \dots, (X_n; Y_n)$ , where  $X_i$  can be multivariate  $X_i = \vec{X}_i = (X_{i1}, X_{i2}, \dots, X_{ip})$ .

If  $X$  is continuous **numeric** variable, use Regression Model(s), else if  $X$  is discrete **factor** variable, use Factor Model(s).

▷ **R. Code**

Example data import:

```
1 df <- read.table('dataset/testdata.txt', header=FALSE, sep=',', col.names
  = c('y', 'x1', 'x2'))
```

### 3.1.1 Linear Regression Model

Regression Model focuses on how  $Y$  changes with continuous variables  $X \in \mathbb{R}$ . As a basic situation, we use **Linear Regression**, i.e.  $Y \sim X$  in linear relation.

#### □ Sample Geometry Notation (Full Version)

For most general case, in sample matrix notation:

$$Y = X\beta + \varepsilon \Leftrightarrow Y_j = X\beta_j + \varepsilon_j, \forall j = 1, 2, \dots, q \quad (3.1)$$

in Einstein Summation Convention:

$$Y_{ij} = X_{ij'}\beta_{j'j} + \varepsilon_{ij} \quad (3.2)$$

Why we need  $\varepsilon$  as 'random error term'?

- It represents the intrinsic random property of the model.
- Based on  $\varepsilon$ , we can take r.v. into our statistic model.

where

$$Y_{n \times q} = \begin{bmatrix} y_{11} & y_{12} & \dots & y_{1q} \\ y_{21} & y_{22} & \dots & y_{2q} \\ \vdots & \vdots & \ddots & \vdots \\ y_{n1} & y_{n2} & \dots & y_{nq} \end{bmatrix} = [y_1, y_2, \dots, y_q] \quad (3.3a)$$

$$X_{n \times (p+1)} = \begin{bmatrix} 1 & x_{11} & x_{12} & \dots & x_{1p} \\ 1 & x_{21} & x_{22} & \dots & x_{2p} \\ \vdots & \vdots & \ddots & \vdots & \\ 1 & x_{n1} & x_{n2} & \dots & x_{np} \end{bmatrix} = \begin{bmatrix} x'_1 \\ x'_2 \\ \vdots \\ x'_n \end{bmatrix} \quad (3.3b)$$

$$y_j = \begin{bmatrix} y_{1j} \\ y_{2j} \\ \vdots \\ y_{nj} \end{bmatrix} \quad (3.3a)$$

$$x_i = \begin{bmatrix} 1 \\ x_{i1} \\ \vdots \\ x_{ip} \end{bmatrix} \quad (3.3b)$$

$$\beta_{(p+1) \times q} = \begin{bmatrix} \beta_{01} & \beta_{02} & \dots & \beta_{0q} \\ \beta_{11} & \beta_{12} & \dots & \beta_{1q} \\ \beta_{21} & \beta_{22} & \dots & \beta_{2q} \\ \vdots & \vdots & \ddots & \vdots \\ \beta_{p1} & \beta_{p2} & \dots & \beta_{pq} \end{bmatrix} = [\beta_1, \beta_2, \dots, \beta_q] \quad \beta_j = \begin{bmatrix} \beta_{j0} \\ \beta_{j1} \\ \vdots \\ \beta_{jp} \end{bmatrix} \quad (3.3c)$$

$$\varepsilon_{n \times q} = \begin{bmatrix} \varepsilon_{11} & \varepsilon_{12} & \dots & \varepsilon_{1q} \\ \varepsilon_{21} & \varepsilon_{22} & \dots & \varepsilon_{2q} \\ \vdots & \vdots & \ddots & \vdots \\ \varepsilon_{n1} & \varepsilon_{n2} & \dots & \varepsilon_{nq} \end{bmatrix} = [\varepsilon_1, \varepsilon_2, \dots, \varepsilon_q] \quad \varepsilon_j = \begin{bmatrix} \varepsilon_{1j} \\ \varepsilon_{2j} \\ \vdots \\ \varepsilon_{nj} \end{bmatrix} \quad (3.3d)$$

with Gauss-Markov Assumption:

$$\text{Zero-Mean: } \mathbb{E}(\epsilon_i | X_i) = 0$$

$$\text{Homogeneity of Variance: } \text{var}(\epsilon_i) = \sigma^2 \quad (3.4)$$

$$\text{Independent: } \epsilon_i \text{ i.i.d. } \sim \varepsilon$$

and Normality Error Assumption:

$$\text{Normality: } \varepsilon_i \text{ i.i.d. } \sim N(0, \sigma^2) \quad (3.5)$$

Under matrix notation, model and assumptions [equation. 3.4](#)([equation. 3.5](#)) can be expressed in condensed notation:

$$Y_j = X\beta_j + \varepsilon_j \sim N_n(X\beta_j, \sigma_j^2 I_n), \quad j = 1, 2, \dots, q \quad (3.6)$$

**△ Note:** In this section we only focus on  $q = 1$ , i.e.

$$Y_{n \times 1} = X_{n \times (p+1)} \beta_{(p+1) \times 1} + \varepsilon_{n \times 1} \quad (3.7)$$

### 3.1.2 Factor Analysis Model

Regression Model focuses on continuous variables  $X \in \mathbb{R}$  while factor model focus on discrete variable. More specifically, the ‘value’ of  $X$  is just a label, not necessarily a ‘numeric value’.

Here only introduce one-way factor analysis,(single factor analysis) i.e.  $Y$  with only one factor with  $r$  levels:  $\text{fac} = 1, 2, \dots, r$ . Re-denote  $Y_{ij}$  = the observation outcome of the  $j^{\text{th}}$  item labelled the  $i^{\text{th}}$  level.

Model:

$$Y_{ij} = \mu + \tau_i + \varepsilon_{ij}, \quad i = 1, \dots, r, \quad j = 1, \dots, n_i \quad (3.8)$$

$$\text{w.r.t. } \sum_{i=1}^r \tau_i = 0 \quad (3.9)$$

where  $\mu$  is the average effect of all  $r$  factor levels,  $\tau_i$  is the level effect of the  $i^{\text{th}}$  factor level, and  $\varepsilon$  i.i.d.  $\sim N(0, \sigma^2)$  is noise error.

In matrix notation:

$$Y = \begin{bmatrix} y_{11} & \dots & y_{1n_1} & y_{21} & \dots & y_{2n_2} & \dots \end{bmatrix}^T \quad (3.10a)$$

$$X = \begin{bmatrix} 1 & 1 & 0 & \dots \\ 1 & 1 & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots \\ 1 & 0 & 1 & \dots \\ 1 & 0 & 1 & \dots \\ \vdots & \vdots & \vdots & \ddots \\ \vdots & \vdots & \vdots & \vdots \\ 1 & -1 & -1 & \dots \\ 1 & -1 & -1 & \dots \end{bmatrix} = \begin{bmatrix} \mathbf{1}_{n_1} & \mathbf{1}_{n_1} & 0 & \dots & 0 \\ \mathbf{1}_{n_2} & 0 & \mathbf{1}_{n_2} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{1}_{n_{r-1}} & 0 & 0 & \dots & \mathbf{1}_{n_{r-1}} \\ \mathbf{1}_{n_r} & -\mathbf{1}_{n_r} & -\mathbf{1}_{n_r} & \dots & -\mathbf{1}_{n_r} \end{bmatrix} \quad (3.10b)$$

$$\tau = \begin{bmatrix} \mu & \tau_1 & \tau_2 & \dots & \tau_{r-1} \end{bmatrix}^T \quad (3.10c)$$

$$\varepsilon = \begin{bmatrix} \varepsilon_{11} & \dots & \varepsilon_{1n_1} & \varepsilon_{21} & \dots & \varepsilon_{2n_2} & \dots \end{bmatrix}^T \quad (3.10d)$$

For more factor model e.g. two-way factor analysis with  $k$  denoting item and  $i, j$  denoting factor:

$$Y_{ijk} = \mu + \alpha_i + \beta_j + \varepsilon_{ijk} \quad (3.11)$$

cannot be simply expressed in matrix notation  $\rightarrow$  use index notation.

Assumption: Normal, Equal variance, independent

- One-way:  $Y_{j|i}$  i.i.d.  $\sim N(\mu + \tau_i, \sigma^2), \forall i$
- Two-way:  $Y_{k|ij}$  i.i.d.  $\sim N(\mu + \alpha_i + \beta_j, \sigma^2), \forall i, j$

## Section 3.2 Monivariate Linear Regression Model

First focus on the simplest monivariate case  $\vec{X}_i = X_i$ . Monivariate Linear Model<sup>19</sup> with Gauss-Markov assumption & Normal Error assumption:

$$Y_i = \beta_0 + \beta_1 X_i + \varepsilon_i, \varepsilon_i \text{ i.i.d. } \sim N(0, \sigma^2) \quad (3.12)$$

What does Linear Regression do? Try to estimate

- $\beta_0$  (intercept) ;
- $\beta_1$  (slope) ;
- $\sigma^2$  (variance of error).

(Thus Linear Regression is also a Statistics Inference process: deduce properties of model from data)

<sup>19</sup>Here in linear regression, we consider  $X_i$  only as real number, **without** randomness. So here  $Y_i$  can be considered as an r.v. with  $X_i$  as parameter, i.e.  $Y_i|_{X_i=x_i}$



### 3.2.1 The Ordinary Least Square Estimation

Aim: use  $(x_i, y_i)$  to estimate  $\beta_0, \beta_1, \sigma^2$ . The idea is to define a 'loss function' to reflect the 'distance' from sample point to estimation point.

Estimate Principle: <sup>20</sup>

- Ordinary Least Squares:

$$(\hat{\beta}_0, \hat{\beta}_1) = \arg \min \sum_{i=1}^n (y_i - \beta_0 - \beta_1 x_i)^2 \quad (3.13)$$

- MLE or MoM Estimation.

And get  $\hat{\beta}_1, \hat{\beta}_0$  as well as  $\hat{\sigma}^2$  (see [equation. 3.18](#):<sup>21</sup>

$$\begin{aligned} \hat{\beta}_1 &= \frac{\sum_{i=1}^n (x_i - \bar{x})(y_i - \bar{y})}{\sum_{i=1}^n (x_i - \bar{x})^2} \\ \hat{\beta}_0 &= \bar{y} - \hat{\beta}_1 \bar{x} \\ \hat{\sigma}^2 &= \frac{1}{n - p - 1} \sum_{i=1}^n (y_i - \hat{\beta}_0 - \hat{\beta}_1 x_i)^2 \end{aligned} \quad (3.15)$$

Def. **Residual**: distance from sample point to estimate point, to reflect how the sample points fit the model.

$$e_i = y_i - \hat{y}_i = \text{observed value of } \varepsilon_i \quad (3.16)$$

Note: under least square estimation, we have<sup>22</sup>

$$\sum e_i = 0 \quad \sum x_i e_i = 0 \quad (3.17)$$

Then use  $e_i$  to estimate  $\sigma^2$  (because it is  $\varepsilon_0$  that are i.i.d., not  $Y_i$ ), where  $(n - p - 1)$  is Degree of Freedom (df or dof)<sup>23</sup>

$$\begin{aligned} \hat{\sigma}_n^2 &= \frac{1}{n} \sum e_i^2 \quad (\text{use MLE or MoM}) \\ \hat{\sigma}^2 &= \frac{1}{n - p - 1} \sum e_i^2 = \frac{1}{n - 2} \sum e_i^2 \quad (\text{use OLS, unbiased}) \end{aligned} \quad (3.18)$$

**Degree of Freedom** of a Quadric Form:

- Intuitively: the number of independent variable;

<sup>20</sup>Detailed Definition and Derivation see [section. 2.2.5](#) or [section. 3.3](#).

<sup>21</sup>A memory trick: use  $\frac{Y}{\sqrt{s_Y}} = r_{XY} \frac{X}{\sqrt{s_X}}$  to get formular of  $Y \sim X$ :

$$\hat{\beta}_1 = r_{XY} \frac{\sqrt{s_Y}}{\sqrt{s_X}} = \frac{\sum (x_i - \bar{x})(y_i - \bar{y})}{\sum (x_i - \bar{x})^2} \quad (3.14)$$

<sup>22</sup>Intuitively, they each means ' $E(\varepsilon) = 0$ ' and ' $X \perp \varepsilon$ '.

<sup>23</sup>Generally, MLE and OLSE are different.

Comment from R.A.Fisher:  $\sum e_i^2$  should be divided by 'number of  $e_i^2$  that contribute to variance'. Here  $(n - p - 1)$  corresponds to 'degree of freedom' =  $(n - 2)$ ,  $p = 1$  corresponds to 'one' variable (see [section. 2.2.5](#), [equation. 2.124](#)), and corresponds to the two equations of  $e_i$ , [equation. 3.17](#)

- Rigorously: for quadric  $SS = x'Ax$ :

$$dof_{SS} = \text{rank}(A) \quad (3.19)$$

Which comes from Cochran's Theorem. A proof can be found here: <https://vincent19.github.io//texts/Cochran/>

#### ▷ R. Code

```
1 lmfit <- lm(formula,df)
2 summary(lmfit,cor=TRUE)
3 ggcoef(lmfit)
```

lmfit includes parameters `lmfit$coefficient` and `lmfit$residuals`

Example `lm()` output:

```
1 Call:
2 lm(formula = y ~ x, data = df)
3
4 Residuals:
5      Min       1Q   Median       3Q      Max
6 -16.1368  -6.1968  -0.5969   6.7607  23.4731
7
8 Coefficients:
9             Estimate Std. Error t value Pr(>|t|)
10 (Intercept) 156.3466     5.5123   28.36  <2e-16 ***
11 x           -1.1900     0.0902  -13.19  <2e-16 ***
12 ---
13 Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
14
15 Residual standard error: 8.173 on 58 degrees of freedom
16 Multiple R-squared:  0.7501,    Adjusted R-squared:  0.7458
17 F-statistic: 174.1 on 1 and 58 DF,  p-value: < 2.2e-16
```

### 3.2.2 Statistical Inference to $\beta_0, \beta_1, \sigma^2, e_i$

#### □ Sampling Distribution of $\hat{\beta}_1, \hat{\beta}_0$

Consider  $\hat{\beta}_1, \hat{\beta}_0$  as statistics of sample, then we can examine the sampling distribution of  $\hat{\beta}_1, \hat{\beta}_0$ . Their randomness comes from

$$Y_i = \beta_0 + \beta_1 X_i + \varepsilon_i \quad (3.20)$$

(The following part treats  $\hat{\beta}_1, \hat{\beta}_0$  as r.v., and note that  $X_i$  are **not** r.v.. And for convenience and conciseness, denote  $S_{XX} = \sum_{i=1}^n (X_i - \bar{X})^2$ )

$$\begin{aligned}\hat{\beta}_1 &= \beta_1 + \sum_{i=1}^n \frac{X_i - \bar{X}}{S_{XX}} \varepsilon_i \\ \hat{\beta}_0 &= \beta_0 + \sum_{i=1}^n \left( \frac{1}{n} - \frac{(X_i - \bar{X})\bar{X}}{S_{XX}} \right) \varepsilon_i\end{aligned}$$

Denote corresponding variance as  $\sigma_{\hat{\beta}_1}^2$  and  $\sigma_{\hat{\beta}_0}^2$ , using [equation. 1.116](#) to get:

$$\sigma_{\hat{\beta}_1}^2 = \frac{\sigma^2}{S_{XX}} \quad \sigma_{\hat{\beta}_0}^2 = \sigma^2 \left( \frac{1}{n} + \frac{\bar{X}^2}{S_{XX}} \right) \quad (3.21)$$

And under normal error assumption, distribution of  $\hat{\beta}_1, \hat{\beta}_0$  are

$$\begin{aligned}\hat{\beta}_1 &\sim N(\beta_1, \sigma_{\hat{\beta}_1}^2) = N\left(\beta_1, \frac{\sigma^2}{S_{XX}}\right) \\ \hat{\beta}_0 &\sim N(\beta_0, \sigma_{\hat{\beta}_0}^2) = N\left(\beta_0, \sigma^2 \left( \frac{1}{n} + \frac{\bar{X}^2}{S_{XX}} \right)\right)\end{aligned}$$

Based on sampling distribution of  $\hat{\beta}_1, \hat{\beta}_0$ , we can conduct statistical inference, including CI and HT.<sup>24</sup>

Note: In linear regression model, we usually focus more on  $\beta_1$ . And note that when 0 is **not** within the fitting range,  $\beta_0$  is not so important.<sup>25</sup>

□ **Sampling Distribution of  $e_i$**  Consider  $e_i$  as r.v. satisfies

$$e_i = Y_i - \hat{Y}_i = Y_i - \hat{\beta}_0 - \hat{\beta}_1 X_i \quad (3.22)$$

and get the expression of  $\hat{e}_i$

$$\hat{e}_i = \varepsilon_i - \sum_{k=1}^n \left( \frac{1}{n} + \frac{(X_i - \bar{X})^2}{S_{XX}} \right) \varepsilon_k \quad (3.23)$$

$$\mathbb{E}(e_i) = 0 \quad \sigma_{e_i}^2 = \sigma^2 \left( 1 - \frac{1}{n} - \frac{(X_i - \bar{X})^2}{S_{XX}} \right) \quad (3.24)$$

Under normal assumption:

$$\varepsilon_i \sim N\left(0, \sigma^2 \left( 1 - \frac{1}{n} - \frac{(X_i - \bar{X})^2}{S_{XX}} \right)\right) \quad (3.25)$$

Further we can get  $\hat{\sigma}^2 = \mathbb{E}\left(\frac{1}{n-2} \sum_{i=1}^n e_i^2\right)$  where  $e_i^2 \sim \sigma^2 \left( 1 - \frac{1}{n} - \frac{(X_i - \bar{X})^2}{S_{XX}} \right) \chi^2$

$$\hat{\sigma}^2 = \frac{1}{n-2} \sigma^2 \sum_{i=1}^n \left( 1 - \frac{1}{n} - \frac{(X_i - \bar{X})^2}{S_{XX}} \right) = \sigma^2 \quad (3.26)$$

More definition of refined residuals see [section. 3.4.3](#) in page [3.4.3](#).

□ **Why we choose OLS to get regression coefficients?**

Gauss-Markov Thm.: the OLS estimator has the lowest sampling variance within the class of linear unbiased estimators, i.e. OLS is the **Best Linear Unbiased Estimator(BLUE)**.<sup>26</sup>

<sup>24</sup>Detail see [section. 2.4](#), estimating/testing  $\hat{\beta}_1, \hat{\beta}_0$  usually corresponds to 'estimate  $\mu$ , with  $\sigma^2$  unknown'.

<sup>25</sup>Two reason:

- The estimation error of  $Y$  from  $\hat{\beta}_1$  increases with  $X_h - \bar{X}$ ;
- $\beta_1 = 0$  is important: decides whether linear model can be used.

<sup>26</sup>This Thm. does **not** require normal error assumption.

### 3.2.3 Prediction to $Y_h$

For a new  $X_h$  at which we wish to predict the corresponding  $Y_h$  (based on other known points  $\{(X_i, Y_i)\}_{i=1}^n$ ), denote the estimator of mean as  $\hat{\mu}_h$ :

$$\hat{\mu}_h = \hat{\beta}_1 X_h + \hat{\beta}_0 = \beta_1 X_h + \beta_0 + \sum_{i=1}^n \left( \frac{1}{n} + \frac{(X_i - \bar{X})(X_h - \bar{X})}{S_{XX}} \right) \varepsilon_i \quad (3.27)$$

Thus we can get<sup>27</sup>

$$\mathbb{E}(\hat{\mu}_h) = \beta_1 X_h + \beta_0 \quad \sigma_{\hat{\mu}_h}^2 = \left( \frac{1}{n} + \frac{(X_h - \bar{X})^2}{S_{XX}} \right) \sigma^2 \quad (3.28)$$

Under Normal assumption:

$$\hat{\mu}_h \sim N(\beta_1 X_h + \beta_0, \left( \frac{1}{n} + \frac{(X_h - \bar{X})^2}{S_{XX}} \right) \sigma^2) \quad (3.29)$$

Base on distribution we can give CI and HT.

□ **Interval: We can either consider ...**

- $\hat{Y}_h^{\text{conf}} = \hat{\mu}_h$ : a function of (estimated) parameter to get **confidence interval**: We can just use the above distribution of  $\hat{\mu}_h$

$$\hat{Y}_h^{\text{conf}} = \hat{\mu}_h \sim N(\beta_1 X_h + \beta_0, \left( \frac{1}{n} + \frac{(X_h - \bar{X})^2}{S_{XX}} \right) \sigma^2)$$

▷ **R. Code**

```
1 predict(lmfit, newdata = 40),
2 interval="confidence", level=0.95)
```

- $\hat{Y}_h^{\text{pred}} = \hat{\mu}_h + \varepsilon$ : a function of (estimated) parameter and an extra random term to get **prediction interval**.

In this case we have an extra randomness. Def. Prediction Error:  $Y_h$  itself is an  $Y$  of the linear model, i.e.

$Y_i = \hat{\beta}_0 + \hat{\beta}_1 X_h + \varepsilon_h$ , we can define **Prediction Error**:

$$d_h = Y_h - \hat{\mu}_h \quad (3.30)$$

with

$$\mathbb{E}(d_h) = 0 \quad \sigma_{d_h}^2 = \text{var}(Y_h - \hat{\mu}_h) = \left[ 1 + \frac{1}{n} + \frac{(X_h - \bar{X})^2}{S_{XX}} \right] \sigma^2 > \sigma_{\hat{\mu}_h}^2 \quad (3.31)$$

thus

$$\hat{Y}_h^{\text{conf}} = \hat{\mu}_h + \varepsilon_h \sim N(\beta_1 X_h + \beta_0, \left( 1 + \frac{1}{n} + \frac{(X_h - \bar{X})^2}{S_{XX}} \right) \sigma^2)$$

▷ **R. Code**

```
1 predict(lmfit, newdata = 40),
2 interval="prediction", level=0.95)
```

<sup>27</sup>So  $\sigma^2(\hat{\mu}_h)$  increases with  $X_h - \bar{X}$ . Intuitively it make sense, because  $(\bar{X}, \bar{Y})$  must falls on regression line.

Comment: in prediction error, we considered more random component, thus the CI is also larger.

### □ Simultaneous Confidence Band (SCB)

Confidence Band is **not** the CI at each point, but really a **band** for the **entire** regression line.<sup>28</sup>

Aim: Find lower and upper function  $L(x)$  and  $U(x)$  such that

$$\mathbb{P}[L(x) < (\beta_0 + \beta_1 x) < U(x), \forall x \in I_x] = 1 - \alpha \quad (3.32)$$

and get **Confidence Band**:

$$\{(x, y) | L(x) < y < U(x) | \forall x \in I_x\} \quad (3.33)$$

Where  $(L(x), U(x))$  can be derived as

$$(L(x), U(x)) = \hat{\mu}_x \pm s_{\hat{\mu}_x} W_{2, n-2, 1-\alpha} \quad \forall x \in I_x \quad (3.34)$$

Where  $W$  corresponds to  $W$  distribution:  $W_{m,n} = \sqrt{2F_{m,n}}$

Small sample case: Bonferroni correction.

### ▷ R. Code

```
1 library(ggplot2)
2 ggplot(df, aes(x, y)) + geom_point() + geom_smooth(method='lm', formula=y~x)
```

## 3.2.4 Analysis of Variance: Monovariate

ANalysis Of VAriance (ANOVA): One-sample  $t$  test  $\rightsquigarrow$  Two sample  $t$  test  $\rightsquigarrow$  More-sample: ANOVA

### □ Key Idea Of ANOVA: Test whether the mean of some groups are the same, i.e. $\mu_1 = \mu_2 = \dots = \mu_r$

In linear regression model, modified as testing  $\beta_1 == 0$ . Conduction: Take Partition of Total Sum of Square To Examine **Variation**. Because  $Y_i$  are not i.i.d. (different mean value  $X\beta$ ), so has different parts of variation from Regression Model/Error Term.

Measure of Variation: Sum of Square (SS) & Mean Sum of Square (MS).

MS: Divide each SS by corresponding *dof*. Definition of *dof* see equation. 3.19.

$$MS = \frac{SS}{dof} \quad (3.35)$$

- SST: Total Sum of Squares

$$SST = \sum_{i=1}^n (Y_i - \bar{Y})^2 \quad dof_{SST} = n - 1 \quad (3.36)$$

- SSR: Regression: Variation due to Regression Model (which is explained by regression line);<sup>29</sup>

$$SSR = \sum_{i=1}^n (\hat{Y}_i - \bar{Y})^2 \quad dof_{SSR} = 1 \quad (3.37)$$

<sup>28</sup>Why they are different? We require the confidence band have a **simultaneous** coverage probability. For the same band  $(L(x), U(x))$ ,  $P(\text{the whole line}) < P(\text{each point})$ , so Confidence Band is wider than  $\bigcup$  CIs to hold the same  $1 - \alpha$ .

Also, we will see that for linear model, the boundary of SCB forms hyperbola, which make sense considering its asymptotic line.

<sup>29</sup> $SSR = \hat{\beta}_1^2 \sum_{i=1}^n (X_i - \bar{X})^2$ , so  $dof_R = 1$

- SSError: Variation attributes to  $\varepsilon$  (which is reflected by residuals).

$$\text{SSE} = \sum_{i=1}^n (Y_i - \hat{Y}_i)^2 \quad \text{dof}_{\text{SSE}} = n - 2 \quad (3.38)$$

△ **IMPORTANT:** In some books

- SSRRegression  $\rightarrow$  SSExplained or SSModel;
- SSError  $\rightarrow$  SSResidual.

And Cause **Confusion!** In this summary we take the former.

Idea: take partition of SST. i.e.

$$Y_i - \bar{Y} = (Y_i - \hat{Y}_i) + (\hat{Y}_i - \bar{Y}) = e_i \quad (3.39)$$

And we can prove that

$$\text{SST} = \sum_{i=1}^n (Y_i - \bar{Y})^2 = \sum_{i=1}^n (\hat{Y}_i - \bar{Y})^2 + \sum_{i=1}^n (Y_i - \hat{Y}_i)^2 = \text{SSR} + \text{SSE} \quad (3.40)$$

That is: we **partition** SST into two parts, so that we can examine them seperately.

#### □ ANOVA Table

Source	dof	SS	MS	F-Statistic
SSRegression	1	$\sum_{i=1}^n (\hat{Y}_i - \bar{Y})^2$	$\text{SSR}/\text{dof}_R$	$\text{MSR}/\text{MSE}$
SSError	$n - 2$	$\sum_{i=1}^n (Y_i - \hat{Y}_i)^2$	$\text{SSE}/\text{dof}_E$	
SSTotal	$n - 1$	$\sum_{i=1}^n (Y_i - \bar{Y})^2$	$\text{SST}/\text{dof}_T$	

#### ▷ R. Code

```
1 anova(lmfit)
```

Properties:

$$\mathbb{E}(\text{MSE}) = \sigma^2 \quad \mathbb{E}(\text{MSR}) = \sigma^2 + \beta_1^2 S_{XX} \quad (3.41)$$

## Section 3.3 Multivariate Linear Regression Model

As a more general case of  $\vec{X}_i = (X_{i1}, X_{i2}, \dots, X_{ip})$ , Multivariate Linear Model is expressed as in **equation. 3.7**:

$$Y = X\beta + \varepsilon, \varepsilon \sim N_p(0, \sigma^2 I) \quad (3.42)$$

### 3.3.1 The Ordinary Least Estimation

To conduct OLS

$$\hat{\beta} = \arg \min_{\beta \in \mathbb{R}^{p+1}} (Y - X\beta)^T (Y - X\beta) \quad (3.43)$$

Here we introduce two approaches:

- Analytical: Take matrix differentiation (See [section. 4.1.2 equation. 4.40](#))

$$\begin{aligned} 0 &= \frac{\partial(Y - X\beta)^T(Y - X\beta)}{\partial\beta} = \frac{\partial}{\partial\beta}(Y^T Y - Y^T X\beta - \beta^T X^T Y + \beta^T X^T X\beta) \\ &= -X^T Y - X^T Y + (X^T X + X X^T)\beta = -2X^T(Y - X\beta) \end{aligned}$$

Thus we get OLS:

$$\hat{\beta} = (X'X)^{-1}X'Y \quad (3.44)$$

- Geometric/Algebraical: Use hyper-projection.

$$\hat{\beta} = \arg \min_{\beta \in \mathbb{R}^{p+1}} d(Y, X\beta) \quad (3.45)$$

i.e.  $\hat{\beta}$  is the (hyper-)projection of  $Y$  onto  $X$  (within Euclidean Space), naturally we have

$$(X\beta)^T(Y - X\beta) = 0 \Rightarrow \hat{\beta} = (X'X)^{-1}X'Y \quad (3.46)$$

□ **Matrix Notation of OLS Estimator:**

$$\hat{\beta} = (X'X)^{-1}X'Y \quad (3.47)$$

### 3.3.2 Statistical Inference to $\beta, \sigma^2, e$

Properties & Extrapolation

- Sampling Distribution of  $\hat{\beta}$ : (Here consider normal case  $Y \sim N(X\beta, \sigma^2 I_n)$ , and use [equation. 4.61](#))

$$\hat{\beta} = (X'X)^{-1}X'Y \sim N_p(\beta, \sigma^2(X'X)^{-1}) \quad (3.48)$$

Comment:  $cov(\beta_i, \beta_j)$  are generally not 0,  $\Rightarrow \beta_i, \beta_j$  dependent.

- Predicted Response & Hat Matrix  $H$ :

$$\hat{Y} = X\hat{\beta} = X(X'X)^{-1}X'Y \equiv HY = P_X Y \quad (3.49)$$

where **Hat Matrix**/Influence matrix/Projection matrix  $H = P_X = X(X'X)^{-1}X'$ , with properties

- Symmetric:  $H^T = H$ ;
- Idempotence:  $H^2 = H$
- Rank:  $\text{rk}(H) = \text{tr}(H) = \text{rk}(X)$
- $H$  and self-influence factor  $h_{ii}$ : Note the linearity of  $\hat{Y}$  on  $Y$

$$\hat{Y} = HY \Rightarrow H = \frac{\partial \hat{Y}}{\partial Y} \quad (3.50)$$

The diagonal elements of  $H$  is

$$h_{ii} = \frac{\partial \hat{y}_i}{\partial y_i} = X_i(X'X)^{-1}X_i' \quad (3.51)$$

Comment on  $h_{ii}$ :  $\text{var}(e_i) = \sigma^2(1 - h_{ii})$ , for  $h_{ii} \rightarrow 1$ , i.e. the regression line always pass  $y_i$ , thus it's 'influential'.

- Residual:

$$e = Y - \hat{Y} = (I - H)Y \sim N_n(0, \sigma^2(I - H)) \quad (3.52)$$

where  $I - H$  is the complementary projection of  $X$

Covariance Matrix of Residual:

$$\text{cov}(e) = \sigma^2(I - H) = \sigma^2 \begin{bmatrix} 1 - h_{11} & -h_{12} & \dots & -h_{1n} \\ -h_{21} & 1 - h_{22} & \dots & -h_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ -h_{n1} & -h_{n2} & \dots & 1 - h_{nn} \end{bmatrix} \quad (3.53)$$

- Estimator and Distribution of  $\sigma^2$ :

First use [equation. 4.62](#) to get <sup>30</sup>

$$\mathbb{E}(\text{SSE}) = \mathbb{E}(e'e) = \mathbb{E}(Y'(I - H)Y) = (X\beta)'(I - H)X\beta + \text{tr}((I - H)\sigma^2 I_n) = \sigma^2(n - p - 1) \quad (3.55)$$

$\text{dof}$  of Residual  $e$  (use definition [equation. 3.19](#)):

$$\text{dof}_e = \text{dof}_{(I-H)Y} = \text{rank}(I - H) = n - p - 1 \quad (3.56)$$

Thus the unbiased estimator of  $\sigma^2$  is

$$\hat{\sigma}^2 = \text{MSE} = \frac{e'e}{n - p - 1} = \frac{Y'(I - H)Y}{n - p - 1} \quad (3.57)$$

Distribution (under normal assumption):

$$\frac{(n - p - 1)\hat{\sigma}^2}{\sigma^2} \sim \chi_{n-p-1}^2 \quad (3.58)$$

- Gauss-Markov Thm.: OLS Estimator of  $\beta$  is the BLUE Estimator.

More hypothesis testing to  $\beta$  see [section. 4.2.4](#).

### 3.3.3 Prediction to $Y_h$

For a new  $\vec{X}_h$  at which we wish to **predict** the corresponding  $Y_h$  (based on other known point  $(X_i, Y_i)$ ), denote the estimator of mean as  $\hat{\mu}_h$ :

$$\hat{\mu}_h = X_h' \hat{\beta} = X_h'(X'X)^{-1}X'Y \quad (3.59)$$

thus we get

$$\mathbb{E}(\hat{\mu}_h) = X_h'\beta \quad \sigma_{\hat{\mu}_h}^2 = \sigma^2(1 + X_h'(X'X)^{-1}X_h) \quad (3.60)$$

under normal assumption:

$$\hat{\mu}_h \sim N(X_h'\beta, \sigma^2(1 + X_h'(X'X)^{-1}X_h)) \quad (3.61)$$

<sup>30</sup> Also we need the property of idempotent matrix

$$\lambda_i = 0 \text{ or } 1 \Rightarrow \text{tr}(H) = \text{rank}(H) = \sum_{i=1}^n \lambda_i = \#(\lambda = 1) \quad (3.54)$$



### 3.3.4 Analysis of Variance: Multivariate

Sampling Notation see [equation. 3.3](#), still consider  $(p+1)$ -dim  $(\mathbf{1}_n, X_i)$  v.s. 1-dim  $Y$ , and  $\beta = (\beta_0, \beta_1, \beta_2, \dots, \beta_p)$

- SST:

$$\text{SST} = (Y - \bar{Y}\mathbf{1}_n)'(Y - \bar{Y}\mathbf{1}_n) \quad \text{dof}_{\text{SST}} = n - 1 \quad (3.62)$$

- SSR:

$$\text{SSR} = (\hat{Y} - \bar{Y}\mathbf{1}_n)'(\hat{Y} - \bar{Y}\mathbf{1}_n) \quad \text{dof}_{\text{SSR}} = p \quad (3.63)$$

Denoted in hat matrix  $H$  and  $\mathcal{J}$  in [equation. 4.16](#)

$$\text{SSR} = Y'(H - \frac{1}{n}\mathcal{J})Y \quad (3.64)$$

- SSE:

$$\text{SSE} = (Y - \hat{Y})'(Y - \hat{Y}) \quad \text{dof}_{\text{SSE}} = n - p - 1 \quad (3.65)$$

Denoted in residual  $e$  and hat matrix  $H$ :

$$\text{SSE} = e'e = Y'(I - H)Y \quad (3.66)$$

More knowledge about multivariate ANOVA see [section. 3.4.5](#).

#### □ ANOVA Table

Source	dof	SS	MS	F-Statistic
SSRegression	$p$	$\sum_{i=1}^n (\hat{Y}_i - \bar{Y})^2$	$\text{SSR}/\text{dof}_R$	$\text{MSR}/\text{MSE}$
SSError	$n - p - 1$	$\sum_{i=1}^n (Y_i - \hat{Y}_i)^2$	$\text{SSE}/\text{dof}_E$	
SSTotal	$n - 1$	$\sum_{i=1}^n (Y_i - \bar{Y})^2$	$\text{SST}/\text{dof}_T$	

#### ▷ R. Code

```
1 anova(lmfit)
```

## Section 3.4 Diagnostics

To apply OLS, we need the basic Gauss-Markov Assumption [equation. 3.4](#); or we further need better properties of the model, e.g. take Normal Assumption.

Assumptions:

$$\text{Zero-Mean: } \mathbb{E}(\epsilon_i | X_i) = 0$$

$$\text{Homogeneity of Variance: } \text{var}(\epsilon_i) = \sigma^2 \quad (3.67)$$

$$\text{Independent: } \epsilon_i \text{ i.i.d. } \sim \varepsilon$$

$$\text{Normal: } \varepsilon \sim N(0, \sigma^2)$$

Or sum up as

$$Y \sim N_n(X\beta, \sigma^2 I_n) \quad (3.68)$$

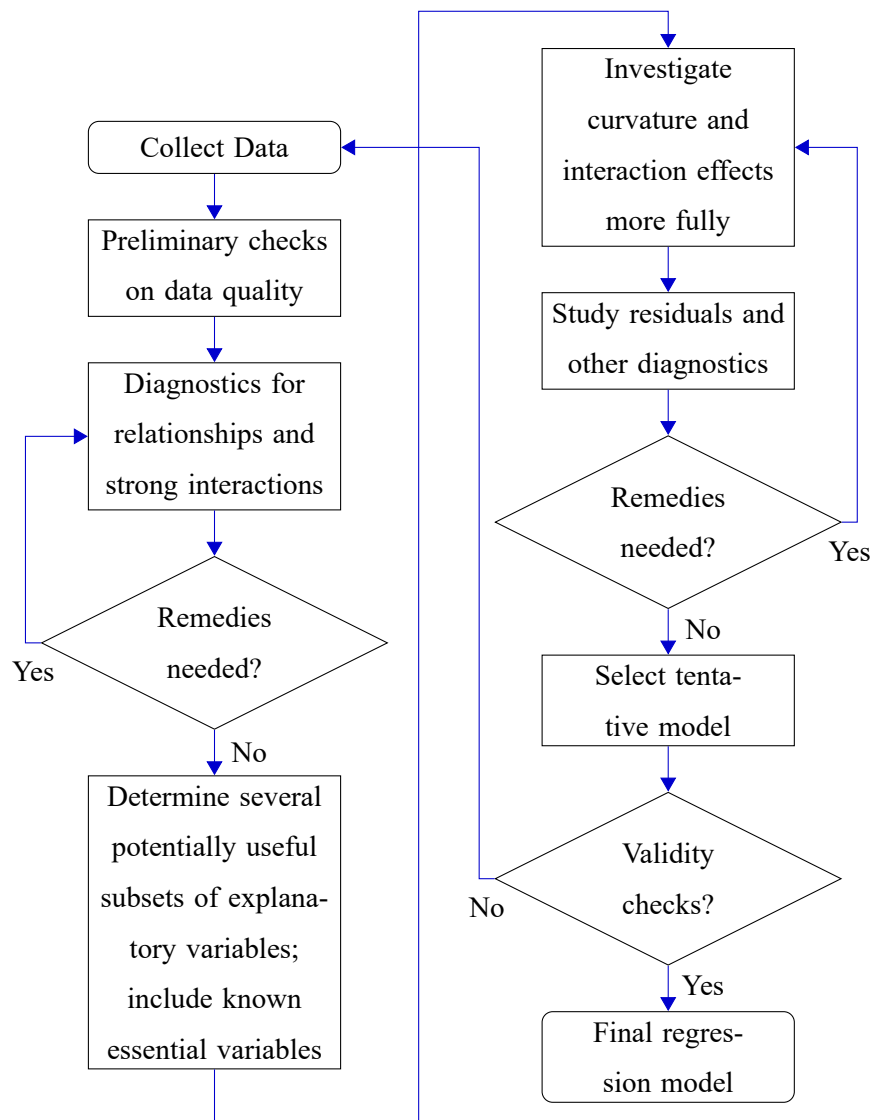


图 2: Diagnostics and Remedies for Regression Model

Thus we need to conduct Diagnostics and Remedies to

- examine whether these assumptions are satisfies;
- perform correction to regression method.

Preliminary Diagnostics:

▷ **R. Code**

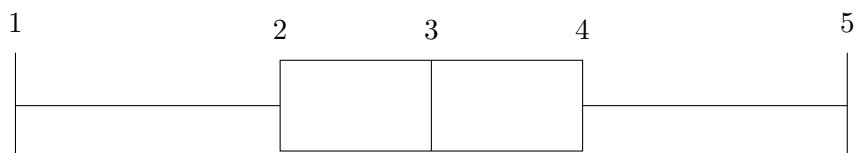
```
1 lmfit <- lm(y~x,lmfit)
2 par(mfrow = c(2, 2))
3 plot(lmfit)
4 par(mfrow = c(1, 1))
```

### 3.4.1 Useful Diagnostics Plots

- BoxPlot: to examine the similarity of shape of distribution.

Notation:

1. min point above 25% quartile−1.5IQR;
2. 25% quartile;
3. median;
4. 75% quartile;
5. max point below 75% quartile+1.5IQR.



- Histogram Plots: Frequency distribution (can deal with many-peak)
- Quartile-Quartile Plots: Examine the similarity between distribution.

For two CDF  $q = F(x)$  and  $q = G(x)$  (where  $q$  for quartile), with  $x = F^{-1}(q)$ ,  $x = G^{-1}(q)$ . And Plot  $F^{-1}(q)$ - $G^{-1}(q)$ .

Usually test normality, take  $G = \Phi$

- Partial Regression Plot: Test non-linearity/heterogeneous-variance.

For each  $X_i$  variable:

- Use other  $X_{(\wedge i)}$  to predict  $Y$ , get residual  $e_Y|X_{(\wedge i)}$ ;
- Use other  $X_{(\wedge i)}$  to predict  $X_i$ , get residual  $e_{X_i}|X_{(\wedge i)}$

Plot  $(e_Y|X_{(\wedge i)}) - (e_{X_i}|X_{(\wedge i)})$  as Added Variable Plot (AV Plot). Used for testing non-linearity/heterogeneous-variance.

## ▷ R. Code

```

1  boxplot(df$x)
2
3  hist(df$x)
4
5  hist(df$x, freq=FALSE)
6  lines(density(df$x))
7
8  stem(df$x)
9
10 qqnorm(df$x)
11 qqline(df$x, col='red')
12
13 library(car)
14 avPlots(lmfit)

```

3.4.2 Diagnostics to  $X$  Distribution

Considering the dependence of  $Y_i$  on  $X_i$ , to get a more reliable  $\hat{\beta}_1$ , we cannot just focus on the (marginal) distribution of  $Y_i$ , we would also need a better 'distribution' of  $X_i$

- Plots: BoxPlot/QQPlot
- 4 statistics(parameters);<sup>31</sup>

– Mean: Location;

$$\bar{X} = \frac{1}{n} \sum_{i=1}^n X_i \quad (3.69)$$

– Standard Deviation: Variability;

$$S^2 = \frac{1}{n-1} \sum_{i=1}^n (X_i - \bar{X})^2 \quad (3.70)$$

– Skewness: Lack of Symmetry;

$$\hat{g}_1 = \frac{m_{n,3}}{m_{n,2}^{3/2}} = \frac{\frac{1}{n} \sum_{i=1}^n (X_i - \bar{X})^3}{\left( \frac{1}{n} \sum_{i=1}^n (X_i - \bar{X})^2 \right)^{3/2}} \quad (3.71)$$

Adjusted Skewness (Least MSE):

$$\frac{\sqrt{n(n-1)}}{n-2} \hat{g}_1 \quad (3.72)$$

\*  $\hat{g}_1 > 0$ : Right skewness, longer right tail;

\*  $\hat{g}_1 < 0$ : Left skewness, longer left tail.

<sup>31</sup>See section. 2.1.1

Fisher-Pearson coefficient of skewness:  $\frac{3(\text{mean} - \text{median})}{\sigma}$ .

– Kurtosis: Heavy/Light Tailed.

$$\hat{g}_2 = \frac{m_{n,4}}{m_{n,2}^2} - 3 = \frac{\frac{1}{n} \sum_{i=1}^n (X_i - \bar{X})^4}{\left( \frac{1}{n} \sum_{i=1}^n (X_i - \bar{X})^2 \right)^2} - 3 \quad (3.73)$$

$\hat{g}_2 = 0 \Rightarrow$  similar to normal.

\*  $\hat{g}_2 > 0$ : Leptokurtic, heavy tail, slender;

\*  $\hat{g}_2 < 0$ : Platykurtic, light tail, broad.

Note: In expression of  $\hat{g}_1$  and  $\hat{g}_2$ , we already divide the variance. So Skewness and Kurtosis only reflect the difference from normal, but **not** related to variance.

Best tool to determine Kurtosis: **QQ-Plot**.

▷ **R. Code**

```
1 summary(df$x)
```

Other moments use package `moments`

- Bias: Inspect the design methodology
  - Selection Bias: Not completely random sampling;
  - Information Bias: Difference between 'designed' and 'get', e.g. no response;
  - Confounding: Exist another important variable, while the model actually focuses on a less important variable, or even reverse the causality.

### 3.4.3 Diagnostics to Residual

□ **Residual Reflects the properties of  $\varepsilon$**

- **Linearity** : use Residual Plot/AV Plot to Reflect the linearity and variance assumption.

▷ **R. Code**

```
1 lmfit <- lm(y~x,df)
2 scatter(df$x,lmfit$residuals)
3 abline(h=0)
4
5 library(car)
6 avPlots(lmfit)
```

- The Assumption of **Equal Variances** / Homoscedasticity (齐方差性):
  - **AV Plot**, e.g. test the  $R^2$  of  $(e_Y|X_{(\wedge i)})-(e_{X_i}|X_{(\wedge i)})$  relation.

– Bartlett's test:

Idea: divide the sample into groups  $g$ , and get each MSE

$$\text{MSE}_g = \frac{1}{n_g} \sum_{i=1}^{n_g} (Y_{gi} - \hat{Y}_g)^2 \quad (3.74)$$

and take statistic

$$S = - \frac{(N - g) \ln \left[ \sum_{g=1}^G \frac{n_g}{N - n_g} \text{MSE}_g \right] - \sum_{g=1}^G (n_g) \ln \frac{n_g - 1}{N - n_g} \text{MSE}_g}{1 + \frac{1}{3(G - 1)} \left( \sum_{g=1}^G \frac{1}{n_g} - \frac{1}{N - G} \right)} \sim \chi_{G-1}^2 \quad (3.75)$$

to conduct test.

Note: **sensitive** to normal assumption, not robust. Used when normal assumption is satisfied.

– Levene's test: Divide the sample into  $G$  groups. Denote **mean** of residual within each group as  $\bar{e}_g$ , and in each group compute

$$d_{ig} = |e_{ig} - \bar{e}_g| \Rightarrow \bar{d}_g = \frac{1}{n_g} \sum_{j=1}^{n_g} d_{ig} \quad (3.76)$$

Then conduct ANOVA to  $d_{ig}$ .

If  $G = 2$ : 2-sample  $t$ -test,

$$T = \frac{\bar{d}_1 - \bar{d}_2}{s \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}} \xrightarrow{d} t_{n-2} \quad s^2 = \frac{\sum (d_{i1} - \bar{d}_1)^2 + \sum (d_{i2} - \bar{d}_2)^2}{n - 2} \quad (3.77)$$

– Brown-Forsythe's Test (Modified Levene's test): For skewed sample, take the **mean** as **median**, more robust.

★ Breusch-Pagan Test:

Assume variance of  $\varepsilon_i$  dependent on  $X_i$  as  $m^{\text{th}}$  polynomial:

$$\sigma_i^2 = \alpha_0 + \sum_{k=1}^m \alpha_k X_i^k \quad (3.78)$$

and test

$$H_0 : \alpha_k = 0 \forall k = 1, 2, \dots, m \longleftrightarrow H_1 \quad (3.79)$$

Method: First conduct OLS to get regression line  $\hat{l}_1$  and residuals  $e_i$  and SSE, and conduct regression of  $e_i^2$  over  $X_i$  to get another regression line  $\hat{l}_2$  and corresponding SSR\*.

Then statistic

$$S = \frac{\text{SSR}^*/2}{(\text{SSE}/n)^2} \xrightarrow{d} \chi_m^2 \quad (3.80)$$

▷ R. Code

Example for  $G = 2$ :

```
1 group <-factor(rep(c(1,2),length.out=length(df$x),
2     each=(ceiling(length(df$x)/2))))
3
```

```

4 bartlett.test(lmfit$residuals~group, group)
5
6 library(car)
7 leveneTest(lmfit$residuals~group, group, center=mean)
8 leveneTest(lmfit$residuals~group, group, center=median)
9
10 library(lmtest)
11 bptest(lmfit)

```

• The Assumption of **Normality** :

In most case we use S-W Test( $n < 2000$ ) and K-S Test( $n > 2000$ ):

– QQ-plot of ordered residuals.

★ Shapiro-Wilk Test (Most Powerful)<sup>32</sup>: To test  $H_0 : \exists \sigma^2, s.t. \varepsilon \sim N_n(0, \sigma^2 I_n)$ , denote

$$m_i = \mathbb{E}\left(\frac{\varepsilon_{(i)}}{\sigma}\right) \quad (3.81)$$

then under  $H_0, \varepsilon_{(i)} \sim m_i \rightarrow$  linear, thus test correlation

$$R^2 = \frac{(\sum_{i=1}^n (e_{(i)} - \bar{e})(m_i - \bar{m}))^2}{\sum_{i=1}^n (e_i - \bar{e})^2 \sum_{i=1}^n (m_i - \bar{m})^2} = \text{corr}(e_{(i)}, m_i) \quad (3.82)$$

– Kolmogorov-Smirnov Test:

$$D_n = \sum_e |F_n(e) - \Phi(e)| \quad (3.83)$$

– Cramér-von Mises Test:

$$T = n \int_{-\infty}^{\infty} (F_n(e) - \Phi(e))^2 d\Phi(e) \quad (3.84)$$

– Anderson-Darling Test:

$$A^2 = n \int_{-\infty}^{\infty} (F_n(e) - \Phi(e))^2 \frac{1}{\Phi(e)(1 - \Phi(e))} d\Phi(e) \quad (3.85)$$

– Jarque-Bera Test , using skewness  $\hat{g}_1$  and kurtosis  $\hat{g}_2$  of  $\vec{e}$

$$JB = \frac{n}{6} (\hat{g}_1^2 + \frac{1}{4} \hat{g}_2^2) \xrightarrow{d} \chi_2^2 \quad (3.86)$$

▷ **R. Code**

```

1 qqnorm(lmfit$residuals)
2 qqline(lmfit$residuals)
3
4 qqp <- qqnorm(lmfit$residuals)
5 cor(qqp$x, qqp$y)
6

```

<sup>32</sup>Detail of S-W Test and K-S Test see **Test of Normality** in **section. 2.4.6**

```

7 shapiro.test(lmfit$residuals)
8
9 ks.test(jitter(lmfit$residuals), pnorm, mean(lmfit$residuals), sd(
    lmfit$residuals))
10
11 library(nortest)
12 cvm.test(lmfit$residuals)
13
14 ad.test(lmfit$residuals)
15
16 library(tseries)
17 jarque.bera.test(lmfit$residuals)

```

• The Assumption of **Independence** :

– Durbin-Watson Test:

$$d = \frac{\sum_{j=2}^n (e_j - e_{j-1})^2}{\sum_{j=1}^n e_j^2} \quad (3.87)$$

$d \in (1.5, 2.5)$  is fine.

– Ljung-Box Test:

$$Q = n(n+2) \sum_{k=1}^n \frac{\hat{\rho}_k^2}{n-k} \quad (3.88)$$

▷ **R. Code**

```
1 dwtest(lmfit)
```

### 3.4.4 Diagnostics to Influentials

An intuitive explanation to extreme values:

- Outliers: Extreme case for  $Y$ ;
- High Leverage: Extreme case for  $X$ ;
- Influentials: Cases that influence the regression line.

□ **Influentials = Outliers  $\cap$  High Leverage**

In [section. 3.3](#), we got the  $\hat{\beta}$  as  $\hat{\beta} = (X'X)^{-1}X'Y = HY$  and got  $\hat{Y}$  as

$$\hat{Y} = X\hat{\beta} = X(X'X)^{-1}X'Y = \hat{H}Y \quad (3.89)$$

where hat matrix  $H \equiv X(X'X)^{-1}X' = \frac{\partial \hat{Y}}{\partial Y}$



Also we got statistical inference to  $\beta, \sigma^2, e$

$$\hat{\beta} = (X'X)^{-1}X'Y \sim N(\beta, \sigma^2(X'X)^{-1}) \quad (3.90)$$

$$e = Y - \hat{Y} = (I - H)Y \sim N(0, \sigma^2(I - H)) \quad (3.91)$$

$$\hat{\sigma}^2 = \text{MSE} = \frac{e'e}{n - p - 1} = \frac{Y'(I - H)Y}{n - p - 1} \quad (3.92)$$

$$\frac{(n - p - 1)\hat{\sigma}^2}{\sigma^2} \sim \chi_{n-p-1}^2 \quad (3.93)$$

The diagonal elements of  $\hat{H}$  is self-sensitivity  $h_{ii}$

$$h_{ii} = X_i'(X'X)^{-1}X_i \quad (3.94)$$

#### □ Some refined residuals to help conduct Diagnostics:

- Standardized Residual:

$$e_{sdi} = \frac{e_i}{\sigma_{e_i}} = \frac{e_i}{\sigma\sqrt{1 - h_{ii}}} \quad (3.95)$$

- (Internally) Studentized Residual: replace  $\sigma$  with  $s = \hat{\sigma}$

$$r_i = \frac{e_i}{\hat{\sigma}\sqrt{1 - h_{ii}}} = \frac{e_i}{\sqrt{\text{MSE}}\sqrt{1 - h_{ii}}} \sim t_{n-p-1} \quad (3.96)$$

- Deleted Residual:<sup>33</sup>

$$d_i = Y_i - \hat{Y}_{i(\wedge i)} = \frac{e_i}{1 - h_{ii}} \quad (3.99)$$

<sup>33</sup>□ Proof:

Lemma:  $(A + B)^{-1} = A^{-1} - \frac{1}{1 + \text{tr}(BA^{-1})}A^{-1}BA^{-1}$ , where  $\text{rk}(B) = 1$ .

$$\hat{\beta}_{(\wedge i)} = (X'_{(\wedge i)}X_{(\wedge i)})^{-1}X'_{(\wedge i)}Y_{(\wedge i)} \quad (3.97)$$

Using the above lemma: (here for aesthetic purpose, treat  $X_i$  as row vector)

$$\begin{aligned} (X'_{(\wedge i)}X_{(\wedge i)})^{-1} &= (X'X - X'_iX_i)^{-1} \\ &= (X'X)^{-1} + \frac{1}{1 - \text{tr}[X'_iX_i(X'X)^{-1}]}(X'X)^{-1}X'_iX_i(X'X)^{-1} \\ &= (X'X)^{-1} + \frac{1}{1 - h_{ii}}(X'X)^{-1}X'_iX_i(X'X)^{-1} \\ X_{(\wedge i)}Y_{(\wedge i)} &= X'Y - X'_iY_i \end{aligned}$$

then calculate  $\hat{\beta}_{(\wedge i)}$ :

$$\begin{aligned} \hat{\beta}_{(\wedge i)} &= (X'_{(\wedge i)}X_{(\wedge i)})^{-1}X'_{(\wedge i)}Y_{(\wedge i)} \\ &= \left[ (X'X)^{-1} + \frac{(X'X)^{-1}X'_iX_i(X'X)^{-1}}{1 - h_{ii}} \right] (X'Y - X'_iY_i) \\ &= \hat{\beta} + \frac{(X'X)^{-1}X'_iX_i(X'X)^{-1}X'Y}{1 - h_{ii}} - (X'X)^{-1}X'_iY_i - \frac{(X'X)^{-1}X'_iX_i(X'X)^{-1}X'_iY_i}{1 - h_{ii}} \\ &= \hat{\beta} + \frac{(X'X)^{-1}X'_i\hat{Y}_i}{1 - h_{ii}} - \frac{(X'X)^{-1}X'_iY_i(1 - h_{ii})}{1 - h_{ii}} - \frac{(X'X)^{-1}X'_iY_i}{1 - h_{ii}} h_{ii} \\ &= \hat{\beta} + \frac{(X'X)^{-1}X'_i}{1 - h_{ii}}(\hat{Y}_i - Y_i) \\ \Rightarrow \hat{\beta} - \hat{\beta}_{(\wedge i)} &= (X'X)^{-1}X'_i \frac{e_i}{1 - h_{ii}} \end{aligned} \quad (3.98)$$

where  $\hat{Y}_{i(\wedge i)}$  is predicted  $Y$  value at  $X_i$  obtained from the regression of dataset with the  $i$  case  $(X_i, Y_i)$  removed:

$$\hat{\beta}_{(\wedge i)} = (X'_{(\wedge i)} X_{(\wedge i)})^{-1} X'_{(\wedge i)} Y_{(\wedge i)} \quad \hat{Y}_{i(\wedge i)} = X'_i \hat{\beta}_{(\wedge i)} \quad (3.100)$$

- (Externally) Studentized Residual: To avoid self-influence, take deleted residual in [equation. 3.96](#)

$$t_i = \frac{d_i}{s^2(d_i)} = \frac{e_i}{\hat{\sigma}_{(\wedge i)} \sqrt{1 - h_{ii}}} = \frac{e_i}{\sqrt{\text{MSE}_{(\wedge i)}} \sqrt{1 - h_{ii}}} \sim t_{n-p-2} \quad (3.101)$$

Relation between MSE and  $\text{MSE}_{(\wedge i)}$ :

$$(n - p - 1)\text{MSE} = (n - p - 2)\text{MSE}_{(\wedge i)} + \frac{e_i^2}{1 - h_{ii}} \quad (3.102)$$

which also gives the relation between  $t_i$  and  $r_i$ :

$$t_i = r_i \left( \frac{n - p - 2}{n - p - 1 - r_i^2} \right)^{1/2} \Leftrightarrow r_i = t_i \left( \frac{n - p - 1}{n - p - 2 + t_i^2} \right)^{1/2}$$

- Diagnostics to **Outlier**: use external studentized residual for  $t$ -test with Bonferroni adjustment. Declare the  $i^{\text{th}}$  case an outlier if:

$$|t_i| > t_{\alpha/2n, n-p-2} \quad (3.103)$$

- Diagnostics to **Leverage**: use hat matrix  $H$ /self-sensitivity  $h_{ii}$ .

$$\sum_{i=1}^n h_{ii} = \text{tr}(H) = p + 1 \Rightarrow \bar{h} = \frac{p + 1}{n} \quad (3.104)$$

Declare the  $i^{\text{th}}$  case a leverage if:

$$h_{ii} > \kappa \bar{h} = \kappa \frac{p + 1}{n} \quad (3.105)$$

where usually take  $\kappa = 2$  or  $3$ .

- Diagnostics to **Influential**: Studentized DIFFerence caused to FITted values (DIFFITS)

DIFFIT:

$$\text{DIFFIT}_i = \hat{Y}_i - \hat{Y}_{i(\wedge i)} = e_i \frac{h_{ii}}{1 - h_{ii}} \quad (3.106)$$

DIFFITS:

---

Then

$$\begin{aligned} Y_i - \hat{Y}_{i(\wedge i)} &= Y_i - \hat{Y}_i + \hat{Y}_i - \hat{Y}_{i(\wedge i)} \\ &= e_i + X_i(\hat{\beta} - \hat{\beta}_{(\wedge i)}) \\ &= e_i + X_i(X'X)^{-1}X'_i \frac{e_i}{1 - h_{ii}} \\ &= \frac{e_i}{1 - h_{ii}} \end{aligned}$$

□

$$\text{DIFFITS}_i = \frac{\text{DIFFIT}_i}{s(\hat{Y}_i)} = t_i \sqrt{\frac{h_{ii}}{1 - h_{ii}}} \quad (3.107)$$

Declare the  $i^{\text{th}}$  case an influential if:

$$\begin{cases} \text{DIFFITS}_i > 1 & \text{small/medium data} \\ \text{DIFFITS}_i > 2\sqrt{\frac{p+1}{n}} & \text{large data} \end{cases} \quad (3.108)$$

- Diagnostics to **Influential**: Cook's Distance, by quantifying the 'influence' to  $\hat{\beta}$ .

Using [equation. 3.48](#)([equation. 3.58](#)) we could construct the following Cook's Distance<sup>34</sup>

$$D_i = \frac{\|X(\hat{\beta} - \hat{\beta}_{(\wedge i)})\|^2}{(p+1)\hat{\sigma}^2} = \frac{e_i^2}{(p+1)\hat{\sigma}^2} \frac{h_{ii}}{(1-h_{ii})^2} \quad \frac{1-h_{ii}}{h_{ii}} D_i \sim F_{p+1, n-p-1} \quad (3.109)$$

Comment:

$$D_i = \frac{e_i^2}{(p+1)\hat{\sigma}^2} \left[ \frac{h_{ii}}{(1-h_{ii})^2} \right] = \frac{1}{p+1} \frac{h_{ii}}{1-h_{ii}} \times r_i^2 \quad (3.110)$$

where  $\frac{1}{p+1} \frac{h_{ii}}{1-h_{ii}}$  corresponds to hige leverage, and  $r_i^2$  corresponds to outliers, multiply to get influentials.

Declare the  $i^{\text{th}}$  case an influential if

$$D_i > \frac{4}{n} \quad (3.111)$$

Or conduct  $F$ -test using the distribution of  $D_i$ , with  $\alpha \sim 20\%$ .

- Diagnostics to **Influential**: Studentized DiFference in BETA estimates (DFBETAS). Use [equation. 3.48](#), define

$$\text{var}(\hat{\beta}_k) = \sigma^2 (X'X)_{kk}^{-1} := \sigma^2 c_{kk} \quad (3.112)$$

And studentize difference in  $\hat{\beta}$  with  $i^{\text{th}}$  case removed:  $\hat{\beta}_k - \hat{\beta}_{k(\wedge i)}$

$$\text{DFBETAS}_{k(\wedge i)} = \frac{\hat{\beta}_k - \hat{\beta}_{k(\wedge i)}}{\sqrt{\text{MSE}_{(\wedge i)} c_{kk}}}, \quad k = 1, 2, \dots, p \quad (3.113)$$

Declare the  $i^{\text{th}}$  case an influential if

$$\begin{cases} \text{DFBETAS}_i > 1 & \text{small/medium data} \\ \text{DFBETAS}_i > \frac{2}{\sqrt{n}} & \text{large data} \end{cases} \quad (3.114)$$

#### ▷ R. Code

```
1 rstudent(lmfit)
2 library(car)
3 outlierTest(lmfit)
4
5 hatvalues(lmfit)
```

<sup>34</sup>Proof uses [equation. 3.98](#).

```

6
7 cooks.distance(lmfit)
8 plot(lmfit, which=4)
9
10 dfbetas(lmfit)

```

Leverage and Mahalanobis Distance:

Mahalanobis Distance between  $X$  and  $Y$  as defined in [equation. 4.28](#)

$$d_M(\vec{x}) = \sqrt{(\vec{x} - \vec{\mu})^T S^{-1} (\vec{x} - \vec{\mu})} \quad (3.115)$$

And we can proof  $d_M$  of a case item  $X_{i.} = (1, X_{i1}, X_{i2}, \dots, X_{ip})$  is<sup>35</sup>

$$d_M^2(X_{i.}) = (n-1)(h_{ii} - \frac{1}{n}) \quad (3.116)$$

here  $S = \frac{S}{(p+1) \times (p+1)}$ . Note that L.H.S.  $\geq 0$ , thus it's also an evidence that  $h_{ii} \geq \frac{1}{n}$

### 3.4.5 Extra Sum Of Square

Def. Extra SS: the part of SSE explained by a new  $X_2$  when adding to model  $Y \sim X_1$ :

$$SSR(X_2|X_1) = SSE(X_1) - SSE(X_1, X_2) = SSR(X_1, X_2) - SSR(X_1) \quad (3.117)$$

where  $SS(\cdot)$  represents the SS when the model contains variable  $\cdot$ .<sup>36</sup>

(The following part use model  $(Y, X_1, X_2)$  as example.)

We could use extra SS to examine the proper regression model: examine the F value and  $\Pr(>F)$  in the output.

▷ **R. Code**

```
1 lm(y~x1+x2+x1:x2) %>% anova
```

**Note:** Three types of SS

Term	Type I SS <sup>37</sup>	Type II SS	Type III SS
$X_1$	$SSR(X_1)$	$SSR(X_1 X_2)$	$SSR(X_1 X_2, X_1X_2)$
$X_2$	$SSR(X_2 X_1)$	$SSR(X_2 X_1)$	$SSR(X_2 X_1, X_1X_2)$
$X_1X_2$	$SSR(X_1X_2 X_1, X_2)$	Assume no interaction term	$SSR(X_1X_2 X_1, X_2)$
Language.Function	<a href="#">R. anova</a>	python.	SPSS, SAS, <a href="#">R. lm</a>

To get Type II and III anova, use `Anova(lmfit, type='III')` in 'car' package.

<sup>35</sup>Proof hint: use lemma

$$(A + B)^{-1} = A^{-1} - \frac{A^{-1}BA^{-1}}{1 + \text{tr}(B^{-1}A)}, \quad \text{rank}(B) = 1$$

and note that  $X_{:,1} = \mathbf{1}_n$

<sup>36</sup> $SSE(1) = SST$ , where 1 corresponds to intercept.

**Hierarchical Principle:** the interaction term  $X_1X_2$  should always come in **after** marginal term  $X_1$  and  $X_2$ .

▷ **R. Code**

```
1 libaray('car')
2 Anova(lmfit,type='II')
3 Anova(lmfit,type='III')
```

### 3.4.6 Hypotheses Testing to Slope

Main focus: whether the linear relation exist:

$$H_0 : \beta_1 = \beta_2 = \dots = \beta_p = 0 \longleftrightarrow H_1 : \exists \beta_i \neq 0, i = 1, 2, \dots, p \quad (3.118)$$

As for general case  $H_0 : \frac{C}{q \times (p+1)} \beta - \frac{t}{(p+1) \times 1} = 0$ , use **General Linear Test  $F$** .

- ANOVA  $F$ -Test:

We can examine

$$F = \frac{\text{MSR}}{\text{MSE}} \sim F_{p, n-p-1} \quad (3.119)$$

- General Linear Test (GLT)

First we introduce the examine models:

- Full model: Include all variable/parameters to be examined, with  $p$  variables.

$$Y = X\beta + \varepsilon \quad (3.120)$$

And define  $\text{SSE}_F$  with  $\text{dof}_F = n - p - 1$  under Full Model.

- Reduced model: Apply the Null Hypothesis to Full Model, with  $\tilde{p}$  variables

$$Y_i = \tilde{X}\tilde{\beta} + \varepsilon \quad (3.121)$$

And define  $\text{SSE}_R$  with  $\text{dof}_R = n - \tilde{p} - 1$  under Reduced Model.

Then conduct test to the difference between Full model and Reduced model through  $\text{SSE}_F$  and  $\text{SSE}_R$ .

- One dimensional case:  $H_0 : \beta_1 = 0$

Examine

$$F = \frac{(\text{SSE}_R - \text{SSE}_F)/(\text{dof}_R - \text{dof}_F)}{\text{SSE}_F/\text{dof}_F} \sim F_{1, n-2} \quad (3.122)$$

▷ **R. Code**

```
1 fullmodel <- lmfit
2 nullmodel <- lm(y ~ 1, df)
3 anova(nullmodel, fullmodel)
```

- General case: Test  $H_0 : \begin{matrix} C \\ q \times (p+1) \end{matrix} \begin{matrix} \beta - t \\ (p+1) \times 1 \end{matrix} = 0$ , construct  $F$  statistics as

$$F = \frac{(C\hat{\beta} - t)' [C(X'X)^{-1}C']^{-1} (C\hat{\beta} - t)}{q\hat{\sigma}^2} \sim F_{q, n-q-1} \quad (3.123)$$

- $r$  and Different  $R^2$ :

- Pearson's  $r$ :

$$r_{Y, \hat{Y}} = \text{cov}(Y, \hat{Y}) = \frac{\sum_{i=1}^n (Y_i - \bar{Y})(\hat{Y}_i - \bar{Y})}{\sqrt{\sum_{i=1}^n (Y_i - \bar{Y})^2} \sqrt{\sum_{i=1}^n (\hat{Y}_i - \bar{Y})^2}} = \frac{\sqrt{\sum_{i=1}^n (\hat{Y}_i - \bar{Y})^2}}{\sqrt{\sum_{i=1}^n (Y_i - \bar{Y})^2}} \quad (3.124)$$

- Coefficient of Multiple Determination  $R^2$ :

$$R^2 = \frac{\text{SSR}}{\text{SST}} = 1 - \frac{\text{SSE}}{\text{SST}} \quad (3.125)$$

- Adjusted  $R^2$ :

$$R_a^2 = 1 - \frac{\text{MSE}}{\text{MST}} = 1 - \frac{n-1}{n-p-1} \frac{\text{SSE}}{\text{SST}} \quad (3.126)$$

Relation between  $r$  and  $R^2$ : Under Simple Linear Model, we have

$$R^2 = r^2 \quad (3.127)$$

Relation between  $R^2$  and  $F$ -Statistic:

$$F = \frac{R^2}{1-R^2} \frac{n-p-1}{n-1} \sim F_{n-1, n-p-1} \quad (3.128)$$

Hypothesis testing for  $r$ :

$$t = \frac{r}{\sqrt{1-r^2}} \sqrt{n-2} \sim t_{n-p-1} \quad (3.129)$$

▷ R. Code

```
1 cor.test(df$x, df$y)
```

- Coefficient of Partial Determination  $R_{Y_k|X_k}^2$  and Coefficient of Multiple Determination  $R^2$ : CMD reflects the interpretability of the model, to examine the interpretability of each variable, use coef. partial determination

$$R_{Y_{X_k|X_1, \dots, X_{k-1}, X_{k+1}, \dots, X_p}}^2 = R_{Y_{X_k|\wedge X_k}}^2 = \frac{\text{SSR}(X_k|X_1, \dots, X_{k-1}, X_{k+1}, \dots, X_p)}{\text{SSE}(X_1, \dots, X_{k-1}, X_{k+1}, \dots, X_p)} = \frac{\text{SSR}(X_k|\wedge X_k)}{\text{SSE}(\wedge X_k)} \quad (3.130)$$

Note: Coef. Partial determination can also be used for  $X_i, X_j$ :  $R_{X_i X_j|\wedge X_i, X_j}^2$

Sometimes we use  $\eta_k^2 = R_{Y_{X_k|\wedge k}}^2 = R_{Y_{k.\wedge k}}^2$

- Coefficient of Partial Correlation  $\eta_k$ : Measures the strength of linear relation,  $\pm$  sign depend on posi./nega. correlation.

$$\eta_k = \pm \sqrt{\eta_k^2} \quad (3.131)$$

▷ R. Code

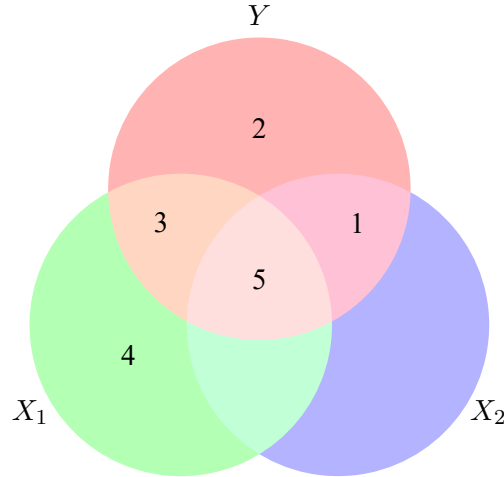
```

1 library('heplots')
2 etasq(lmfit)

```

### 3.4.7 Diagnostics to Multi-collinearity

- Venn Diagram for Multi-Linear Regression: Used to show the interpretability of variables.



Explanation of each region:

- 1/3: Variation in  $Y$  uniquely attributes to  $X_2/X_1$ ;
- 2: Variation in  $Y$  that cannot be explained by regression to  $X_1, X_2$ , corresponds to  $\varepsilon$ ;
- 5: Cross term of  $X_1, X_2$ , **cannot** verify the orientation, corresponds to **Multi-collinearity**.

In the presence of multi-collinearity, i.e.  $X$  is column singular ( $\frac{S_5}{S_1 \text{ or } S_3}$  large), the regression parameter

$$\hat{\beta} = (X'X)^{-1}X'Y \quad (3.132)$$

Issue of multi-collinearity:

- Statistically: ‘better’ prediction, worse interpretability;
- Numerically: Calculation of  $(X'X)^{-1}$  becomes unstable/ill-posed/NAN.

#### □ Use Variance Inflation Factor (VIF) to detect multi-collinearity.

First construct  $R_k^2, k = 1, 2, \dots, p$ : Regress  $X_k$  against other  $p - 1$  variable  $X_i$ s and get corresponding  $R_k^2$ , and

$$\text{VIF}_k = (1 - R_k^2)^{-1} \quad (3.133)$$

$$\overline{\text{VIF}} = \frac{1}{p} \sum_{k=1}^p \text{VIF}_k \quad (3.134)$$

If  $\|\text{VIF}_i\|_{\infty} > 10$  or  $\overline{\text{VIF}} > 1$ , then we identify an excessive multi-collinearity.<sup>38</sup>

<sup>38</sup>Why  $\text{VIF}_k = \frac{1}{R_k^2}$  is called ‘variance inflation factor’? We can prove that

$$\text{var}(\hat{\beta}_k) = \frac{\sigma^2}{(n-1)S_{x_k}^2} \cdot \frac{1}{R_k^2} = \frac{\sigma^2}{(n-1)S_{x_k}^2} \cdot \text{VIF}_k \quad (3.135)$$

## ▷ R. Code

```
1 library('car')
2 vif(lmfit)
```

## 3.4.8 Diagnostics to Model Variable Selection

In Multi-variate regression, proper explanatory variables form a subset of all available variables.

Aim: Avoid over-fitting, get a simple explanatory model.

Comment: If we consider the model with all  $p_{\max}$  variables as full, unbiased model, then model selection is a kind of **Bias-Variance Trade-Off**.

□ Model Validation:  $k$ -Fold Cross Validation(CV):

1. Separate the dataset size  $n$  into  $k$  parts;
2. pick the  $i^{\text{th}}$  part as test set  $y_i$ , and the other  $k - 1$  part as train set  $y_{\wedge i}$  (to conduct regression, etc); then conduct prediction of model  $y_{\wedge i}$  to part  $y_i$  and get  $\text{MSE}_i$ ;
3. Take average of  $\text{MSE}_i$  as the measure of validity.

## □ Evaluation Criteria

Useful model validation approach: To check a model with  $p - 1$  variable (this part  $p$  for 1+# variable)

- Traditional way: Test  $r$ ,  $R^2$ ,  $R_a^2$ ,  $p$ -value, etc.
- Mallows's  $C_p$ : For a model with  $p$  variable:

$$\hat{Y}^p = X_p(X_p'X_p)^{-1}X_p'Y = H_pY \quad (3.136)$$

Denote:

$$\mathbb{E}(\hat{Y}^p) = H_p E(Y) \equiv H_p \mu \quad \text{var}(\hat{Y}^p) = H_p \sigma^2 I_n H_p' = \sigma^2 H_p \quad (3.137)$$

Recall the MSE expansion of bias-variance trade-off in [equation. 2.51](#)<sup>39</sup>

<sup>39</sup>Derivation:

- Bias part: (Here use [equation. 4.62](#) in 3<sup>rd</sup> line; use [equation. 3.55](#) in 4<sup>th</sup> line.)

$$\begin{aligned} \sum_{i=1}^n [e(\hat{Y}_i^p - \mu_i)]^2 &= \mu'(H_p - I)'(H_p - I)\mu \\ &= \mu(I - H_p)\mu' \\ &= E(Y'(I - H_p)Y) - \text{tr}[(I - H_p)\sigma^2] \\ &= E(\text{SSE}(p)) - (n - p)\sigma^2 \end{aligned}$$

- Variance part:

$$\sum_{i=1}^n \text{var}(\hat{Y}_i^p) = \text{tr}(\text{var}(\hat{Y}^p)) = \sigma^2 \text{tr}(H_p) = p\sigma^2$$

Then

$$\frac{\sum_{i=1}^n \mathbb{E}[(\hat{Y}_i^p - \mu_i)^2]}{\sigma^2} = \frac{\mathbb{E}(\text{SSE}(p))}{\sigma^2} - (n - 2p) \quad (3.138)$$



$$\begin{aligned}\sum_{i=1}^n \mathbb{E}[(\hat{Y}_i^p - \mu_i)^2] &= \sum_{i=1}^n \mathbb{E}[(\hat{Y}_i^p - \mu_i)^2] + \sum_{i=1}^n \text{var}(\hat{Y}_i^p) \\ &\Rightarrow \mathbb{E}(\text{SSE}(p)) - (n - 2p)\sigma^2\end{aligned}$$

Sum Squared Prediction Error (SSPE):

$$\Gamma_0 \equiv \frac{\sum_{i=1}^n \mathbb{E}[(\hat{Y}_i^p - \mu_i)^2]}{\sigma^2} = \frac{\mathbb{E}(\text{SSE}(p))}{\sigma^2} - (n - 2p) \quad (3.139)$$

And construct Mallows's  $C_p$ : Estimation of  $\Gamma_p$

$$C_p = \hat{\Gamma}_p = \frac{\mathbb{E}(\text{SSE}(p))}{\hat{\sigma}^2} - (n - 2p) \quad (3.140)$$

where  $\text{SSE}(p) = Y'(I - H_p)Y$ .

When the model is unbiased, then  $\mathbb{E}(\text{SSE}(p)) \rightarrow n - p$ , use  $C_p$ - $p$  plot to pick proper  $p$ :

- $C_p \approx p$ : Model unbiased, then choose model with smaller  $C_p$ ;
- $C_p \gg p$ : Significant biased, miss some important predictors;
- $C_p \ll p$ : Overfitting.

- Akaike Information Criterion (AIC): Euivalent to Mallows's  $C_p$  for gaussian regression model.

$$\text{AIC}(p) = -2 \log(\hat{L}) + 2p \quad (3.141)$$

where  $\hat{L}$  is the maximum likelihood, for linear regression case

$$\text{AIC}(p) = n \log \left( \frac{\text{SSE}(p)}{n} \right) + 2p \quad (3.142)$$

Select the model that minimizes  $\text{AIC}(p)$ .

- Bayesian Information Criterion (BIC)/Schwarz's Bayesian Criterion (SBC):

$$\text{BIC}(p) = -2 \log(\hat{L}) + p \log n \quad (3.143)$$

where  $\hat{L}$  is the maximum likelihood, for linear regression case

$$\text{BIC}(p) = n \log \left( \frac{\text{SSE}(p)}{n} \right) + p \log n \quad (3.144)$$

Select the model that minimizes  $\text{BIC}(p)$ .

- PRESS Creterion (Predictive Residual Error Sum of Squares): A kind of within-model cross validation

$$\text{PRESS}(p) = \sum_{i=1}^n (Y_i - \hat{Y}_{i(\wedge i)})^2 \quad (3.145)$$

where

$$\hat{Y}_{i(\wedge i)} = (1, X_{i1}, \dots, X_{ip}) \hat{\beta}_{(\wedge i)}$$

$$\hat{\beta}_{(\wedge i)} = (X'_{(\wedge i)} X_{(\wedge i)})^{-1} X'_{(\wedge i)} Y_{(\wedge i)}$$

where  $\hat{\beta}_{(\wedge i)}$  as in `EqaEstimatorWithWedgeX`, is the estimated  $\beta$  with  $(X_i, Y_i)$  removed from  $X$ .<sup>40</sup>

Select the model that minimizes  $\text{PRESS}(p)$ .

#### ▷ R. Code

```
1 library('leaps')
2 predictor <- df[,c('...', '...', ...)]
3 response <- df[,...]
4 leapSet <- leaps(x=predictor, y=response, nbest = ...)
5 # method=c('Cp', 'adjr2', 'r2')
6 leapSet$which[which.min(leapSet$Cp),]
```

nbest for NUMBER\_OF\_BEST\_MODELS

## Section 3.5 Remedies

### 3.5.1 Variable Transformation

The goal of Transformation:

- Stabilize Variance;
- Improve Normality;
- Simplify the Model.

#### □ Transformation Methods:

- Variance Stabilizing Transformations: For  $E(Y_X) = \mu_X$ ,  $\text{var}(Y_X) = h(\mu_X)$ , take transformation  $f(Y)$  such that  $\text{var}(f(Y)) = \text{const}$ , satisfies

$$f(\mu) = \int \frac{c d\mu}{\sqrt{h(\mu)}} \quad (3.147)$$

Examples:

$$h(\mu) = \mu^2 \Rightarrow f(\mu) = \ln \mu$$

$$h(\mu) = \mu^{2\nu} \Rightarrow f(\mu) = \mu^{1-\nu}$$

- Box-Cox Transformation: Take

$$Y^* = \frac{Y^\lambda - 1}{\lambda} \quad (3.148)$$

<sup>40</sup> A useful thm.: Deleted Residual

$$d_i := Y_i - \hat{Y}_{i(\wedge i)} = \frac{e_i}{1 - h_{ii}} \quad (3.146)$$

Examples:

$$\begin{aligned}\lambda = 1 &\Rightarrow Y^* \sim Y \\ \lambda = 0.5 &\Rightarrow Y^* \sim \sqrt{Y} \\ \lambda = 0 &\Rightarrow Y^* \sim \ln Y \\ \lambda = -1 &\Rightarrow Y^* \sim 1/Y\end{aligned}$$

And conduct regression to model

$$Y^* = \beta_0 + \beta_1 X + \varepsilon_i \quad (3.149)$$

Likelihood Function

$$L(\beta, \sigma^2; \lambda) = \frac{1}{(2\pi\sigma^2)^{n/2}} \exp\left(-\frac{1}{2\sigma^2} \sum_{i=1}^n (Y_i^* - \beta_0 - \beta_1 X_i)^2\right) J\left(\frac{\partial Y^*}{\partial Y}\right) \quad (3.150)$$

where the Jacobi Matrix denoted in Geometric Mean  $\text{GM}(Y) = \prod_{i=1}^n Y_i^{1/n}$

$$J\left(\frac{\partial Y^*}{\partial Y}\right) = \prod_{i=1}^n Y_i^{\lambda-1} = \text{GM}(Y)^{n(\lambda-1)} \quad (3.151)$$

MLE Estimator:

$$\begin{aligned}\hat{\beta}^* &= (X'X)^{-1} X'Y^* \\ \hat{\sigma}_n^2 &= \frac{1}{n} \text{SSE}^* \\ \text{SSE}^* &= \sum_{i=1}^n (Y_i^* - \hat{Y}^*)^2\end{aligned}$$

And when  $\beta, \sigma^2$  take MLE estimator,  $L(\beta, \sigma^2; \lambda)$  can be regarded a function of  $\lambda$ :

$$\ln L(\beta, \sigma^2; \lambda) = l(\lambda) = -\frac{n}{2} \ln \frac{\hat{\sigma}_n^2}{\text{GM}(Y)^{2(\lambda-1)}} + \text{const} \quad (3.152)$$

For simplification, denote  $Z = Y^* / J^{1/n}$  and get

$$\ell(\lambda) = -n \ln \sigma_{nZ}^2 + \text{const} \quad (3.153)$$

where

$$Z_i^* = \begin{cases} \frac{Y_i^\lambda - 1}{\lambda} \frac{1}{\prod_{k=1}^n Y_k^{\frac{\lambda-1}{n}}}, & \lambda \neq 0 \\ \ln Y_i \prod_{k=1}^n Y_k^{\frac{1}{n}}, & \lambda = 0 \end{cases} \quad (3.154)$$

Plot  $l(\lambda)$ - $\lambda$  to determine a proper  $\lambda$  and transform  $Y^* = \frac{Y^\lambda - 1}{\lambda}$ :

- Selected  $\lambda$  should be closed to  $\lambda_{\arg \max l}$ , at least within CI<sup>41</sup>

$$\{\lambda | l(\lambda) \geq l(\lambda_{\arg \max l}) - \frac{1}{2} \chi_{1,1-\alpha}^2\} \quad (3.155)$$

<sup>41</sup>Here CI can be derived using Wilk's Thm.

- Should pick a  $\lambda$  which is **Interpretable**. e.g. If  $\lambda = 1$  is within range  $[0.94, 1.08]$ , then take  $\lambda = 1$  (does not transform).

▷ **R. Code**

```
1 library(MASS)
2 bctrans <- boxcox(y~x, df, lambda = seq(-1.5, 1.5, length = 15))
3 bctrans$x[which.max(bctrans$y)]
```

Note: we can transform on  $X$  or  $Y$  or simultaneously to get better regression model.

### 3.5.2 Weighted Least Squares Regression

To deal with heterogeneous variance, use Weighted Least Squares (WLS) instead of OLS: Minimizing

$$\sum_{i=1}^n e_i^2 \longrightarrow \sum_{i=1}^n w_i e_i^2 \quad (3.156)$$

And e.g. take weight for each case as

$$w_i = \frac{1}{\sigma_i^2} \quad (3.157)$$

Solution:

$$\hat{\beta}_W = (X'WX)^{-1}X'WY \quad (3.158)$$

▷ **R. Code**

```
1 Wlmfit <- lm(y~x, weights=WEIGHT_VECTOR, data=df)
```

### 3.5.3 Remedies for Model Variable Selection & More Regression Model

Several Algorithm to search for best variable set:

- Exhaustive Search and **Test**: Used for  $p \leq \sim 30$
- Greedy Search: Get a locally optimal solution.
  - Forward Selection: Start with  $p = 0$ , add one variable each times and conduct  $t/F/p$ -value test until a presupposed certain limit.
  - Backward Elimination: Start with  $p_{\max}$ , eliminate one variable each times and conduct  $t/F/p$ -value test until a presupposed certain limit.
  - Stepwise Regression: Alternate forward selection & backward elimination until no add/elimination.
- Penalized Optimization:

Recall: OLS regression model: Minimize SSE<sup>42</sup>

$$\hat{\beta} = \arg \min \|Y - X\beta\|_2^2 \quad (3.159)$$

Idea: Add a penalty term in SSE, such that SSE increases with # of variables/value of variables.

<sup>42</sup>Here expressed in  $\ell_p$  norm, definition see sec.4.1.2, **Norm**

- LASSO (Least Absolute Shrinkage and Selection Operator)

Penalty term:  $\lambda \|\beta\|_1$ , where  $\lambda$  is a proper penalty parameter.

$$\hat{\beta} = \arg \min (\|Y - X\beta\|_2^2 + \lambda \|\beta\|_1) \quad (3.160)$$

or equivalently expressed as <sup>43</sup>

$$\hat{\beta} = \arg \min \|Y - X\beta\|_2^2, \text{ with } \|\beta\|_1 \leq s \quad (3.161)$$

where  $s$  is a parameter corresponding to  $\lambda$ . Select a proper value of  $\lambda$  (or equivalently  $s$ ) for expected model:

Some  $\hat{\beta}_i$  would be exactly 0.

- Ridge Regression/Tikhonov Regularization:

Penalty term:  $\lambda \|\beta\|_2^2$ , where  $\lambda$  is penalty parameter.

$$\hat{\beta} = \arg \min (\|Y - X\beta\|_2^2 + \lambda \|\beta\|_2^2) \quad (3.162)$$

or equivalently expressed as

$$\hat{\beta} = \arg \min \|Y - X\beta\|_2^2, \text{ s.t. } \|\beta\|_2^2 \leq s \quad (3.163)$$

Select a proper value of  $\lambda$  (or equivalently  $s$ ) for expected model. Generally Ridge regression **cannot** conduct variable selection, but usually used to avoid non-invertible  $X'X$ , or used to retain important but collinear variable.

Solution of Ridge regression: <sup>44</sup>

$$\hat{\beta}_\lambda^{\text{Ridge}} = (X'X + \lambda I)^{-1} X'Y \quad (3.164)$$

- Mixed Model: Elastic Net

$$\hat{\beta} = \arg \min (\|Y - X\beta\|_2^2 + \lambda_1 \|\beta\|_1 + \lambda_2 \|\beta\|_2^2) \quad (3.165)$$

or equivalent form:

$$\begin{aligned} \hat{\beta} = \arg \min_{\beta} & \|Y - X\beta\|^2 \\ \text{s.t.} & \frac{\lambda_1}{\lambda_1 + \lambda_2} \|\beta\|_1 + \frac{\lambda_2}{\lambda_1 + \lambda_2} \|\beta\|_2^2 \leq s \end{aligned}$$

picking proper hyper-parameter  $(s, \lambda = \frac{\lambda_2}{\lambda_1 + \lambda_2})$

<sup>43</sup>Constrained optimization theory intro see [section. 5.1.4](#).

<sup>44</sup>Why Ridge regression can also fix the problem of colinearity, i.e. non-full rank  $XX'$ :

Assume the SVD decomposition of  $X$ :  $X = U\Sigma V'$ , then

$$\begin{aligned} X'X + \lambda I &= V\Sigma U'U\Sigma V' + \lambda I \\ &= V \begin{bmatrix} \sigma_1^2 + \lambda & 0 & \dots & 0 \\ 0 & \sigma_2^2 + \lambda & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \sigma_{p+1}^2 + \lambda \end{bmatrix} V' \end{aligned}$$

then for  $\lambda > 0$ , we can get a positive-definite matrix  $X'X + \lambda I$

## ▷ R. Code

```

1 library('MASS')
2 Rfit <- lm.ridge(y~x,lambda=seq(0,0.1,0.001),data=df)
3 summary(Rfit)
4 whichLambda <- which.min(Rfit$GCV)
5 coef(fits)[whichLambda,]
6
7 library('lars')
8 Lfit <- lars(x,y,type='lasso')
9 summary(Lfit)
10 whichCp <- which.min(Lfit$Cp)
11 Lfit$Cp[whichCp]
12 Lfit$beta[whichCp,]

```

- Non-parametric Regression Model: Add smooth/penalty function.

Example: `loess` (Locally Regression), `lowess` (Locally Weighted ScatterPlot Smoother), Regression Tree.

- Other Regression Model: 、

- Standardized Regression Model For regression model  $Y_i = \beta_0 + \sum_{j=1}^p X_{ij}\beta_j + \varepsilon_i$ ,  $i = 1, 2, \dots, n$ , conduct Standardization (with an extra const  $1/\sqrt{n-1}$ ) to  $Y$  and  $X$ .

$$Y_i^* = \frac{1}{\sqrt{n-1}} \frac{Y_i - \bar{Y}}{s_Y} \quad X_{ij}^* = \frac{1}{\sqrt{n-1}} \frac{X_{ij} - \bar{X}_i}{s_{X_i}} \quad \varepsilon_i^* = \frac{1}{\sqrt{n-1}} \frac{\varepsilon_i - \bar{\varepsilon}}{s_Y} \quad (3.166)$$

And the regreesion model for standardized data:

$$Y_i^* = 0 + \sum_{j=1}^n X_{ij}^* \beta_j^* + \varepsilon_i^* \quad (3.167)$$

with

$$\beta_j^* = \frac{\beta_j s_{X_j}}{s_Y} \quad (3.168)$$

Note: set the const as  $\sqrt{n-1}$  so that

$$r_{X^*X^*} = X^{*T}X^* \quad r_{Y^*X^*} = X^{*T}Y^* \quad (3.169)$$

## ▷ R. Code

```

1 scaledf <- data.frame(scale(df))
2 scalelmfit <- lm(~,scaledf)
3 summary(scalelmfit)

```

- Polynomial Regression Model

## ▷ R. Code

```

1 polfit <- lm(y~x+I(x^2),df)
2 polfit <- lm(y~polym(x1,x2,degree=),df)

```

– Interaction Model Example:

$$Y = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + \beta_3 X_1 X_2 + \varepsilon \quad (3.170)$$

Re-write as

$$Y = \beta_0 + (\beta_1 + \beta_3 X_2) X_1 + \beta_2 X_2 + \varepsilon$$

$$Y = \beta_0 + \beta_1 X_2 + (\beta_2 + \beta_3 X_1) X_2 + \varepsilon$$

test the regression coefficient dependence on another variable.

## Section 3.6 Factor Analysis of Variance

Here are some basic introduction to factor model. For more knowledge see [Chapter. 8](#) and [Chapter. 14](#)

### 3.6.1 Single Factor Model

Single factor, or one-way analysis of variance focuses on continuous  $Y \sim$  categorical  $X$  (numeric-factor). Regression goal is the mean response of each category  $\pi_i$ : whether & how much they are different.

Basic assumptions: Normal within each categories, Equal variance, independent

Model: See [equation. 3.10](#) expression for single factor model

$$Y_{ij} = \mu + \tau_i + \varepsilon_{ij}, \quad \varepsilon_{ij} \sim N(0, \sigma^2) \quad (3.171)$$

where  $\tau_i$  for group effect,  $\mu_i = \mu + \tau_i$  for factor effect. Originally only  $\mu_i$  are estimatable.

▷ **R. Code**

`lm()` in **R**. uses cell means model, returns  $\mu_i = \mu + \tau_i$  for each categories.

```
1 facfit <- lm(y~x,df) # where x should be as.factor() type
```

□ **Statistical Inference to Individual  $\mu, \tau_i$**

Note: Initially we have  $r + 2$  variable  $(\mu, \tau_{i=1}^r, \sigma^2) \Rightarrow$  estimator not unique. So we use a constraint to cancel the extra degree of freedom.

$$\sum_{i=1}^r c_i \tau_i = 0$$

usually we take  $c_i = 1$  or  $c_i = n_i$

- Factor effect solution for  $c_i = 1$ , i.e.  $\sum_{i=1}^r \tau_i = 0$  (used more).

$$\hat{\mu} = \frac{1}{r} \sum_{i=1}^r \bar{Y}_i = \frac{1}{r} \sum_{i=1}^r \sum_{j=1}^{n_i} \frac{Y_{ij}}{n_i}$$

$$\hat{\tau}_i = \bar{Y}_i - \hat{\mu}$$

- Factor effect solution for  $c_i = n_i$ , i.e.  $\sum_{i=1}^r n_i \tau_i = 0$

$$\hat{\mu} = \bar{Y} = \frac{1}{n_T} \sum_{i,j} Y_{ij}$$

$$\hat{\tau}_i = \bar{Y}_i - \bar{Y} = \frac{1}{n_i} \sum_{j=1}^{n_i} Y_{ij} - \hat{\mu}$$

### □ One-Way ANOVA

ANOVA table in the form of  $r = p + 1$  multivariate ANOVA in page 73

Source	dof	SS	MS	F-Statistic
SSRegression	$r - 1$	$\sum_{i=1}^r (\hat{Y}_i - \bar{Y})^2$	$SSR/dof_R$	$MSR/MSE$
SSError	$n_T - r$	$\sum_{i=1}^r \sum_{j=1}^{n_i} (Y_{ij} - \hat{Y}_i)^2$	$SSE/dof_E$	
SSTotal	$n_T - 1$	$\sum_{i=1}^r \sum_{j=1}^{n_i} (Y_{ij} - \bar{Y})^2$	$SST/dof_T$	

Use MSE as estimator of  $\sigma^2$ :

$$\hat{\sigma}^2 = \frac{1}{n_T - r} \sum_{i=1}^r \sum_{j=1}^{n_i} (Y_{ij} - \hat{Y}_i)^2 = \frac{1}{n_T - r} \left[ \sum_{i=1}^r \sum_{j=1}^{n_i} Y_{ij}^2 - \sum_{i=1}^r \frac{\bar{Y}_i^2}{n_i} \right] \quad (3.172)$$

Also  $F$ -statistics for  $H_0 : \tau_1 = \tau_2 = \dots = \tau_r = 0$

$$F = MSR/MSE = \frac{SSR/(r-1)}{SSE/(n_T - r)} \sim F_{r-1, n_T-r}, \text{ under } H_0 \quad (3.173)$$

### □ Statistical Inference to Difference

We usually focus on ‘difference’ between factor effects, general form

$$\phi = \sum_{i=1}^r \xi_i \tau_i, \quad \sum_{i=1}^r \xi_i = 0 \quad (3.174)$$

where  $\phi$  with  $\sum_{i=1}^r \xi_i = 0$  is called a **contrast**. Assume there are  $m$  estimator  $\phi_k$ ,  $k = 1, 2, \dots, m$

$$\phi_{m \times 1} = \begin{bmatrix} \phi_1 \\ \phi_2 \\ \vdots \\ \phi_m \end{bmatrix} = \begin{matrix} \xi \\ m \times r \end{matrix} \begin{matrix} \tau \\ r \times 1 \end{matrix} = \begin{bmatrix} \xi_{11} & \xi_{12} & \dots & \xi_{1r} \\ \xi_{21} & \xi_{22} & \dots & \xi_{2r} \\ \vdots & \vdots & \ddots & \vdots \\ \xi_{m1} & \xi_{m2} & \dots & \xi_{mr} \end{bmatrix} \begin{bmatrix} \tau_1 \\ \tau_2 \\ \vdots \\ \tau_r \end{bmatrix} \quad (3.175)$$

- Distribution of  $\phi_k = \sum_{i=1}^r \xi_{ki} \tau_i$ , with  $\sum_{i=1}^r \xi_i = 0$ :

$$\phi_k = \sum_{i=1}^r \xi_{ki} \tau_i \sim N\left(\sum_{i=1}^r \xi_{ki} \bar{Y}_i, \sigma^2 \sum_{i=1}^r \frac{\xi_{ik}^2}{n_i}\right) \quad (3.176)$$

Or use transform of multivariate normal in equation. 4.61

$$\phi \sim N_m(\xi \tau, \sigma^2 \xi \xi') \quad (3.177)$$



- Sampling Distribution of  $\hat{\phi}_k$ :

$$\hat{\phi}_k \sim N\left(\sum_{i=1}^r \xi_{ki} \bar{Y}_i, \sigma^2 \sum_{i=1}^r \frac{\xi_{ki}^2}{n_i}\right) \quad (3.178)$$

- Bonferroni's Confidence Region for  $\phi$ , using result in [equation. 4.105](#)

$$R(\phi) = \bigotimes_{k=1}^m \left( \sum_{i=1}^r \xi_{ki} \bar{Y}_i \pm \hat{\sigma} t_{n_T-r, \frac{\alpha}{2m}} \sqrt{\sum_{i=1}^r \frac{\xi_{ki}^2}{n_i}} \right) \quad (3.179)$$

- Scheffè's Confidence Region for  $\phi$ :

$$R(\phi) = \sum_{i=1}^r \xi_i \bar{Y}_i \pm \hat{\sigma} \sqrt{(r-1) F_{r-1, n_T-r, \alpha}} \sqrt{\sum_{i=1}^r \frac{\xi_i^2}{n_i}} \quad (3.180)$$

- Tukey's Confidence Region for  $\phi$ , under condition  $n_1 = \dots = n_r = n$ : focus on estimating  $\tau_i - \tau_j$

– Def.: studentized range distribution: for  $Z_1, \dots, Z_n$  i.i.d.  $\sim N(0, 1)$ ,  $mW^2 \sim \chi_m^2$ , then

$$q = \frac{\max Z_i - \min Z_i}{W} \sim q_{n,m} \quad (3.181)$$

Then confidence interval for  $\phi = \tau_i - \tau_j$

$$R(\phi) = \bar{Y}_i - \bar{Y}_j \pm q_{r, n_T-r, \alpha} \frac{\hat{\sigma}}{\sqrt{n}} \quad (3.182)$$

General case:  $\phi = \sum_{i=1}^r \xi_i \tau_i$

$$R(\phi) = \sum_{i=1}^r \xi_i \bar{Y}_i \pm q_{r, n_T-r, \alpha} \frac{\hat{\sigma}}{2\sqrt{n}} \sum_{i=1}^r |\xi_i| \quad (3.183)$$

Comment: Scheffè is more conservative, i.e. shorter. If confidence interval does not include 0, we can say they are significantly different. More about theory behind these confidence region see [section. 14.1.3-Multiple Comparison](#)

#### ▷ R. Code

```
1 library('agricolae')
2 facaov <- aov(y~0+x,df)
3
4 LSD.test(facaov, trt='design', group=FALSE, console=TRUE)
5
6 scheffe.test(facaov, trt='design', group=FALSE, console=TRUE)
7
8 TukeyHSD(facaov, conf.level=0.95)
```

use `plot()` to view interval estimation

### 3.6.2 Double Factor Model

Double factor, or two-way analysis of variance, categories  $\pi_{ij}$ :

$$Y_{ijk} = \mu + \alpha_i + \beta_j + e_{ijk} \quad (3.184)$$

LS estimator with  $\sum_{i=1}^a \alpha_i = 0$ ,  $\sum_{j=1}^b \beta_j = 0$ :

$$\begin{aligned} \hat{\mu} &= \frac{1}{ab} \sum_{i=1}^a \sum_{j=1}^b \sum_{k=1}^{n_{ij}} \frac{Y_{ijk}}{n_{ij}} \\ \hat{\alpha}_i &= \frac{1}{b} \sum_{j=1}^b \sum_{k=1}^{n_{ij}} \frac{Y_{ijk}}{n_{ij}} - \hat{\mu} \\ \hat{\beta}_j &= \frac{1}{a} \sum_{i=1}^a \sum_{k=1}^{n_{ij}} \frac{Y_{ijk}}{n_{ij}} - \hat{\mu} \end{aligned}$$

MSE estimator of  $\sigma^2$ :

$$\hat{\sigma}^2 = \frac{1}{n_T - ab} \sum_{i=1}^a \sum_{j=1}^b \sum_{k=1}^{n_{ij}} (Y_{ijk} - \bar{Y}_{ij})^2 = \frac{1}{n_T - ab} \left[ \sum_{i=1}^a \sum_{j=1}^b \sum_{k=1}^{n_{ij}} Y_{ijk}^2 - \sum_{i=1}^a \sum_{j=1}^b \frac{\bar{Y}_{ij}^2}{n_{ij}} \right] \quad (3.185)$$

More about theory of Factor model or Dealing with complicated cases see [Chapter. 8](#) and [Chapter. 14](#).

## Section 3.7 Generalized Linear Model

Recall: Linear model with normal assumption can be expressed as :

$$Y_i \sim N(\mu_i, \sigma_i^2) = N(x_i' \beta, \sigma_i^2) \quad (3.186)$$

Question: How to generalize the simple linear model?

- Generalize the distribution
- Generalize the dependent mode

### □ Distribution Generalize: Scaled Exponential Family

For different range and feature of  $Y$  we can use different distribution for regression. We usually use Exponential Family distribution  $f(y; \vec{\theta}, \vec{\phi})$  as in [equation. 2.18](#), with some constraint on subfunctions for better distribution properties, written as linear scaled exponential family:

$$f(y; \vec{\theta}, \vec{\phi}) = \exp \left\{ \frac{y' \theta - b(\theta)}{a(\phi)} + c(y, \phi) \right\} \quad (3.187)$$

where  $\vec{\theta}$  is the canonical parameter for location and  $\vec{\phi}$  for scale(usually we take  $a(\phi) \propto \phi$ ).

Properties of  $f(y; \theta, \phi)$ :

- Expectation

$$\mu \equiv \mathbb{E}(Y) = \int y f(y) dy = \int \left( a(\phi) \frac{\partial}{\partial \theta} + \frac{db(\theta)}{d\theta} \right) f(y) dy = b'(\vec{\theta}) \quad (3.188)$$

• Variance

$$\sigma^2 \equiv \text{var}(Y) = \int yy^T f(y) dy - \mathbb{E}(Y)\mathbb{E}(Y)^T \quad (3.189)$$

$$= \int \left( \frac{\partial^2}{\partial \theta \partial \theta^T} + (b'(\theta)y + yb'(\theta)) - b'(\theta)b'(\theta)^T + a(\phi) \frac{d^2 b(\theta)}{d\theta d\theta^T} \right) f(y) dy - \mathbb{E}(Y)\mathbb{E}(Y)^T \quad (3.190)$$

$$= a(\phi) \frac{d^2 b(\vec{\theta})}{d\theta d\theta^T} = a(\phi) b''(\vec{\theta}) \quad (3.191)$$

• Examples: Normal, Binomial, Poisson

– Normal  $f(y) = \frac{1}{\sqrt{2\pi}|\Sigma|} \exp\left(-\frac{1}{2}(y - \mu)' \Sigma^{-1}(y - \mu)\right)$  with  $\Sigma = \sigma^2 I$ :

$$f(y) = \exp\left(\frac{y'\mu - \frac{1}{2}\mu'\mu}{\sigma^2} - \frac{y'y}{2\sigma^2} - \frac{1}{2} \ln(2\pi\sigma^2)\right) \quad (3.192)$$

Compare with [equation. 3.187](#),  $\theta = \mu$ :  $b(\theta) = \frac{1}{2}\mu'\mu$ ,  $a(\phi) = \sigma^2$

$$* \mathbb{E}(Y) = b'(\theta) = \mu$$

$$* \text{var}(Y) = a(\phi)b''(\theta) = \sigma^2$$

– Binomial  $\mathbb{P}(y) = \binom{n}{y} \pi^y (1 - \pi)^{n-y} \sim B(n, \pi)$ :

$$f(y) = \exp\left(y \ln\left(\frac{\pi}{1 - \pi}\right) + n \ln(1 - \pi) + \ln\left(\binom{n}{y}\right)\right) \quad (3.193)$$

Compare with [equation. 3.187](#),  $\theta = \ln\left(\frac{\pi}{1 - \pi}\right) \Leftrightarrow \pi = \frac{1}{1 + e^{-\theta}}$ :  $b(\theta) = -n \ln(1 - \pi) = -n \ln \frac{1}{1 + e^\theta}$ ,  $a(\phi) = 1$

$$* \mathbb{E}(Y) = b'(\theta) = n \ln \frac{1}{1 + e^{-\theta}} = n\pi$$

$$* \text{var}(Y) = a(\phi)b''(\theta) = n\pi(1 - \pi)$$

– Poisson  $\mathbb{P}(y) = \frac{\lambda^y}{y!} e^{-\lambda} \sim P(\lambda)$ :

$$f(y) = \exp(y \ln \lambda - \lambda - \ln y!) \quad (3.194)$$

Compare with [equation. 3.187](#),  $\theta = \ln \lambda \Leftrightarrow \lambda = e^\theta$ :  $v(\theta) = \lambda = e^\theta$ ,  $a(\phi) = 1$

$$* \mathbb{E}(Y) = b'(\theta) = \lambda$$

$$* \text{var}(Y) = a(\phi)b''(\theta) = \lambda$$

□ **Dependent Mode Generalize: Link Function**

Note that  $Y_i \sim N(\mu_i, \sigma_i^2) = N(x'_i \beta, \sigma_i^2)$  contains the dependency of  $\mu_i$  on  $x'_i \beta$  thus we can further generalize the regression model as  $\mu_i = x'_i \beta$ , here  $\mu_i$  stands for  $\mathbb{E}(Y)$  as in [equation. 3.188](#). However for different distributions,  $\mu = \mathbb{E}(Y)$  have specific range, e.g.  $\mu \in [0, n]$  for  $B(n, p)$ , while  $x'_i \beta \in \mathbb{R}$ , thus use a **link function**:  $I_\mu \rightarrow I_{x'_i \beta}$  to adjust the range:

$$x'_i \beta = g(\mu_i) \Leftrightarrow \mu_i = g^{-1}(x'_i \beta) \quad (3.195)$$

Note: Link function should be monodrome & differentiable such that  $g^{-1}$  exists. And here  $x'_i \beta$  term still exist (because it's still generalized linear model), thus we denote  $\eta := x'_i \beta$  as a linear predictor/classifier

$$\eta := x'_i \beta \quad (3.196)$$

Regression Model:

$$\eta_i = g(\mu_i) \Leftrightarrow \mu_i = g^{-1}(\eta_i) \quad (3.197)$$

#### □ Useful Generalized Linear Model:

Important Question: how to choose proper generalization ‘pair’: Distribution & Link Function pair?

Idea: Use the expectation transform:

$$\text{Distribution: } \mu = \mathbb{E}(Y) = b'(\theta)$$

$$\text{Link Function: } \mu = g^{-1}(x'\beta)$$

Thus

$$g^{-1}(x'\beta) = b'(\theta) \Rightarrow \eta = x'\beta = g(b'(\theta)) \quad (3.198)$$

For model simplification, we can choose  $g(\cdot), b(\cdot)$  such that

$$g(b'(\cdot)) = \text{Id}(\cdot) \Leftrightarrow g^{-1}(\cdot) = b'(\cdot) \quad (3.199)$$

such condition is called **Canonical Link** of generalized linear model, such choice of link function makes  $x'\beta$  the canonical parameter in model.

$$\theta = \eta = x'\beta = g(\mu) \Leftrightarrow g^{-1}(\theta) = g^{-1}(\eta) = g^{-1}(x'\beta) = \mu = \mathbb{E}(Y) \quad (3.200)$$

- Simple linear model:  $N(\mu, \sigma^2), g(\cdot) = \text{Id}(\cdot)$

$$\mu_i = \eta_i \quad (3.201)$$

- Logistic Model:  $B(n, \pi), g(x) = \text{logit}(x) = \ln \frac{x}{1-x} \Leftrightarrow g^{-1}(y) = \text{logistic}(y) = \frac{1}{1+e^{-y}}$

$$n\pi_i = \mu_i = g^{-1}(\eta_i) \quad (3.202)$$

- Poisson Model:  $P(\lambda), g(\cdot) = \ln(\cdot) \Leftrightarrow g^{-1}(\cdot) = \exp(\cdot)$

$$\lambda_i = \mu_i = g^{-1}(\eta_i) \quad (3.203)$$

#### □ Solution of Generalized Linear Model

Using the distribution of  $Y_i$  dependent on  $x'_i\beta$ , we can use MLE maximizing to solve  $\beta$ . Algorithm for such maximizing task is called Iterative Re-weighted Least Squares, more specifically when using Newton-Raphson Method, this method is called Fisher's Scoring Method. Detail see [section. 5.4.3](#).

## Chapter. IV 多元统计分析部分

Instructor: Dong Li & Tianying Wang

### Section 4.1 Multivariate Data

In this section, we consider a **Multivariate Statistic Model**. Sample comes from  $p$  dimension multivariate population  $f(x_1, x_2, \dots, x_p)$ .

**Notation** : In this section, we still denote random variable in upper case and observed value in lower case, specially express random vector in bold font. **But** in this section we usually omit the vector symbol  $\vec{\cdot}$ . e.g. random vector with  $n$  **variable** is denoted as  $\mathbf{X} = (X_1, X_2, \dots, X_p)$ ; sample of size  $n$  from the multivariate population is a  $n \times p$  matrix  $\{x_{ij}\}$ , each sample item (a row in sample matrix) is denoted as  $x'_i$  or  $x_i^T$ .<sup>45</sup>

#### 4.1.1 Matrix Representation

- Random Variable Representation
- Sample Representation
- Statistics Representation
- Sample Statistics Properties

##### □ Random Variable Representation:

- Random Vector: For a  $p \times 1$  random vector  $\vec{X} = (X_1, X_2, \dots, X_p)^T$ , denote (Marginal) expectation and variance, and covariance, correlation coefficient between  $X_i, X_j$  as follows:<sup>46</sup>

$$\mu_i = \mathbb{E}(X_i) \quad (4.1)$$

$$\sigma_{ii} = \sigma_i^2 = \mathbb{E}(X_i - \mu_i)^2 \quad (4.2)$$

$$\sigma_{ij} = \mathbb{E}[(X_i - \mu_i)(X_j - \mu_j)] \quad (4.3)$$

$$\rho_{ij} = \frac{\sigma_{ij}}{\sqrt{\sigma_{ii}}\sqrt{\sigma_{jj}}} = \frac{\sigma_{ij}}{\sigma_i\sigma_j} \quad (4.4)$$

and we have covariance matrix (as defined in [section. 1.4.3, equation. 1.77](#))

$$\Sigma = \mathbb{E}[(X - \mu)(X - \mu)^T] = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \dots & \sigma_{1p} \\ \sigma_{21} & \sigma_{22} & \dots & \sigma_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{1p} & \sigma_{p2} & \dots & \sigma_{pp} \end{bmatrix} \quad (4.5)$$

and Standard Deviation Matrix

$$V^{1/2} = \text{diag}\{\sqrt{\sigma_{ii}}\} \quad (4.6)$$

<sup>45</sup>Here sample item (or sample case)  $x_i = [x_{i1}, x_{i2}, \dots, x_{ip}]^T$  is a column vector.

<sup>46</sup>An intuition to avoid confusion of  $\sigma_{..}$ : two subscripts means quadratic.

Based on  $\vec{X} = (X_1, X_2, \dots, X_p)$ , consider the linear combination:  $Y = c'X = c_1X_1 + c_2X_2 + \dots + c_pX_p$

$$\mathbb{E}(y) = c'\mu \quad \text{var}(Y) = c'\Sigma c \quad (4.7)$$

and  $Z_i = \sum_{j=1}^p c_{ij}X_j$  (i.e.  $Z = CX$ ):

$$\mu_Z = \mathbb{E}(Z) = C\mu_X \quad \Sigma_Z = C\Sigma_X C^T \quad (4.8)$$

and Correlation Matrix<sup>47</sup>

$$\varrho = \begin{bmatrix} \rho_{11} & \rho_{12} & \dots & \rho_{1p} \\ \rho_{21} & \rho_{22} & \dots & \rho_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ \rho_{1p} & \rho_{p2} & \dots & \rho_{pp} \end{bmatrix} = V^{-1/2}\Sigma V^{-1/2} \quad (4.11)$$

- Random Matrix: Definition and basic properties of r.v. see [section. 1.3](#). Now extend the definition to matrix  $X = \{X_{ij}\}$ .

$$X = \{X_{ij}\} = \begin{bmatrix} X_{11} & X_{12} & \dots & X_{1p} \\ X_{21} & X_{22} & \dots & X_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ X_{1n} & X_{n2} & \dots & X_{np} \end{bmatrix} \quad (4.12)$$

And we can further define  $\mathbb{E}(X) = \{\mathbb{E}(X_{ij})\}$ . For any const matrix  $A, B$  we have

$$\mathbb{E}(AXB) = A\mathbb{E}(X)B \quad (4.13)$$

Some more complex parameter can be expressed in language of tensors.

#### □ Sample Representation (for random vector):

Sample of  $n$  items from population characterized by  $p$  variables

$$\begin{array}{c} \text{var 1} \quad \text{var 2} \quad \dots \quad \text{var } j \quad \dots \quad \text{var } p \\ \text{item 1} \left( \begin{array}{cccccc} x_{11} & x_{12} & \dots & x_{1j} & \dots & x_{1p} \\ \text{item 2} & x_{21} & x_{22} & \dots & x_{2j} & \dots & x_{2p} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ \text{item } i & x_{i1} & x_{i2} & \dots & x_{ij} & \dots & x_{ip} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ \text{item } n & x_{n1} & x_{n2} & \dots & x_{nj} & \dots & x_{np} \end{array} \right) \end{array}$$

<sup>47</sup>Here the correlation matrix is the matrix of Pearson's Correlation Coefficients. Another frequently use correlation matrix called Cross Correlation Matrix is

$$\text{cross}(X, Y) = \mathbb{E}[X'Y] \quad (4.9)$$

and cross correlation matrix with  $Y = X$ :

$$\text{cross}(X, X) = \mathbb{E}[X'X] \quad (4.10)$$

Or represented in condense notation:

$$X = \{x_{ij}\} = \begin{bmatrix} x_1^T \\ x_2^T \\ \vdots \\ x_n^T \end{bmatrix} = \begin{bmatrix} x_{11} & x_{12} & \dots & x_{1p} \\ x_{21} & x_{22} & \dots & x_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ x_{n1} & x_{n2} & \dots & x_{np} \end{bmatrix} = \begin{bmatrix} x_{\cdot 1} & x_{\cdot 2} & \dots & x_{\cdot p} \end{bmatrix} \quad (4.14)$$

#### □ Statistics Representation

- Unit 1 vector:

$$\mathbf{1}_k = \underbrace{(1, 1, \dots, 1)}_{k \text{ 1 in total}}^T \quad (4.15)$$

Unit 1 matrix: (Sometimes I also use notation  $\mathcal{I}_n$ )

$$\mathcal{I}_n = \{1\}_{n \times n} = \begin{bmatrix} 1 & 1 & \dots & 1 \\ 1 & 1 & \dots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \dots & 1 \end{bmatrix}_{n \times n} \quad (4.16)$$

- Sample mean of the  $j^{\text{th}}$  variable:

$$\bar{x}_j = \frac{x_{1j} + x_{2j} + \dots + x_{nj}}{n} = \frac{\mathbf{1}_n' x_{\cdot j}}{n}, \quad j = 1, 2, \dots, p \quad (4.17)$$

- Deviation of measurement of the  $j^{\text{th}}$  variable:

$$d_j = \begin{bmatrix} x_{1j} - \bar{x}_j \\ x_{2j} - \bar{x}_j \\ \vdots \\ x_{nj} - \bar{x}_j \end{bmatrix} = x_{\cdot j} - \bar{x}_j \mathbf{1}_n = (I - \frac{1}{n} \mathcal{I}_n) x_{\cdot j}, \quad j = 1, 2, \dots, p \quad (4.18)$$

- Covariance Matrix:

– Variance of  $x_{\cdot j}$ :

$$s_{jj} = s_j^2 = \frac{1}{n} d_j' d_j = \frac{1}{n} \sum_{k=1}^n (x_{kj} - \bar{x}_j)^2, \quad i = 1, 2, \dots, p \quad (4.19)$$

$$= x_{\cdot j}' (I - \frac{1}{n} \mathcal{I}_n) x_{\cdot j}, \quad j = 1, 2, \dots, p \quad (4.20)$$

– Covariance between  $x_i$  and  $x_j$ :

$$s_{ij} = \frac{1}{n} d_i' d_j = \frac{1}{n} \sum_{k=1}^n (x_{ki} - \bar{x}_i)(x_{kj} - \bar{x}_j), \quad i, j = 1, 2, \dots, p \quad (4.21)$$

$$= x_{\cdot i}' (I - \frac{1}{n} \mathcal{I}_n) x_{\cdot j}, \quad i, j = 1, 2, \dots, p \quad (4.22)$$

– Pearson's Correlation Coefficient between  $x_i$  and  $x_j$ :

$$r_{ij} = \frac{s_{ij}}{\sqrt{s_{ii}} \sqrt{s_{jj}}} = \frac{\sum_{k=1}^n (x_{ki} - \bar{x}_i)(x_{kj} - \bar{x}_j)}{\sqrt{\sum_{k=1}^n (x_{ki} - \bar{x}_i)^2} \sqrt{\sum_{k=1}^n (x_{kj} - \bar{x}_j)^2}}, \quad i, j = 1, 2, \dots, p \quad (4.23)$$

In condense notation, define Covariance Matrix from sample of size  $n$ :

$$S_n = \begin{bmatrix} s_{11} & s_{12} & \cdots & s_{1p} \\ s_{21} & s_{22} & \cdots & s_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ s_{1p} & s_{p2} & \cdots & s_{pp} \end{bmatrix} \quad (4.24)$$

and sample Correlation Coefficient Matrix:

$$R_n = \begin{bmatrix} r_{11} & r_{12} & \cdots & r_{1p} \\ r_{21} & r_{22} & \cdots & r_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ r_{1p} & r_{p2} & \cdots & r_{pp} \end{bmatrix} \quad (4.25)$$

- Generalized sample variance:  $|S_n| = \lambda_1 \lambda_2 \cdots \lambda_p$ , where  $\lambda_i$  are eigenvalues of  $S_n$ .
- ‘Statistical Distance’ between vectors: to measure the difference between two vectors  $x = (x_1, x_2, \dots, x_p)$  and  $y = (y_1, y_2, \dots, y_p)$ .

– Euclidean Distance:

$$d_E(x, y) = \sqrt{(x - y)^T (x - y)} \quad (4.26)$$

– **Mahalanobis Distance**: Scale invariant distance, and include information about relativity position:

$$d_M(x, y) = \sqrt{(x - y)' S^{-1} (x - y)} \quad (4.27)$$

Remark: Mahalanobis distance is actually the normalized Euclidean distance in principal component space. So we can actually define the Mahalanobis distance for one sample case  $\vec{x} = (x_1, x_2, \dots, x_p)$  from distribution of  $(\vec{\mu}, \Sigma)$

$$d_M(\vec{x}) = \sqrt{(\vec{x} - \vec{\mu})^T \Sigma^{-1} (\vec{x} - \vec{\mu})} \quad (4.28)$$

Note: the hyper-surface  $d_M(\vec{x}) = \text{const}$  forms a ellipsoid.

#### □ Sample Statistics Properties

Consider taking an  $n$  cases sample from r.v. population  $\vec{X} = (X_1, X_2, \dots, X_p)$ , population mean  $\mu$  and covariance matrix  $\Sigma$ . Basic statistics are sample mean and sample variance

$$\bar{X} = \frac{1}{n} X' \mathbf{1}_n, \quad S_n = \frac{1}{n} \left( X - \frac{1}{n} \mathcal{I}_n X \right)' \left( X - \frac{1}{n} \mathcal{I}_n X \right) = \frac{1}{n} X' \left( I - \frac{1}{n} \mathcal{I}_n \right) X \quad (4.29)$$

Properties:

$$\mathbb{E}[\bar{X}] = \mu \quad \text{cov}(\bar{X}) = \frac{1}{n} \Sigma \quad \mathbb{E}[S_n] = \frac{n-1}{n} \Sigma \quad (4.30)$$



### 4.1.2 Review: Some Matrix Notation & Lemma

- Orthonormality: For square matrix  $P$  satisfies:

$$x_i^T x_j = \delta_{ij} \quad (4.31)$$

where  $x_i, x_j$  are columns of  $P$ .

- Eigenvalue and Eigenvector: For square matrix  $A$ , its eigenvalues  $\lambda_i$  and corresponding eigenvectors  $e_i$  satisfies:

$$Ae_i = \lambda_i e_i, \forall i = 1, 2, \dots, p \quad (4.32)$$

Denote  $P = [e_1, e_2, \dots, e_p]$ , which is an orthonormal matrix. And denote  $\Lambda = \text{diag}\{\lambda_1, \lambda_2, \dots, \lambda_p\}$ .

$$A = \sum_{i=1}^p \lambda_i e_i e_i^T = P \Lambda P^T = P \Lambda P^{-1} \quad (4.33)$$

is called the Spectral Decomposition of  $A$

- Square root matrix: Def. as

$$A^{1/2} = \sum_{i=1}^p \sqrt{\lambda_i} e_i e_i^T = P \Lambda^{1/2} P^T \quad (4.34)$$

Properties:

- $A^{1/2} A^{1/2} = A$ ;
- $A^{-1/2} = (A^{1/2})^{-1} = P \Lambda^{-1/2} P^T$ ;
- $\text{tr}(A) = \sum_{i=1}^n \lambda_i$ ;
- $|A| = \prod_{i=1}^n \lambda_i$ .

- (Symmetric) Positive Definite Matrix: Say  $A$  a Positive Definite Matrix if

$$x^T A x > 0, \forall x \in \mathbb{R}^p \quad (4.35)$$

where  $x^T A x$  is called a Quadric Form.

Properties:

- Use the Spectral Decomposition of  $A$ , we can write the Quadric Form as

$$x^T A x = x^T P \Lambda P^T x = y^T \Lambda y = \sum_{i=1}^p \lambda_i y_i^2 = \sum_{i=1}^p (\sqrt{\lambda_i} y_i)^2 \quad (4.36)$$

- Eigenvalues  $\lambda_i > 0, \forall i = 1, 2, \dots, p$
- $A$  can be written as product of symmetric matrix:  $A = Q^T Q$  ( $Q$  is symmetric);

Positive Semi-definite matrix is one with  $\lambda_i \geq 0$

- Trace of Matrix: For  $p \times p$  square matrix  $A$

$$\text{tr}(A) = \sum_{i=1}^p a_{ii} \quad (4.37)$$

Properties:

- $tr(AB) = tr(BA)$ ;
- $x'Ax = tr(x'Ax) = tr(Axx')$
- $tr(A) = \sum_i \lambda_i$

- Matrix Partition: partition square matrix  $A$  as  $p \times p$

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \quad \begin{matrix} q_1 \times q_1 & q_1 \times q_2 \\ q_2 \times q_1 & q_2 \times q_2 \end{matrix} \quad (4.38)$$

where  $p = q_1 + q_2$

Property:

$$|A| = |A_{22}| |A_{11} - A_{12}A_{22}^{-1}A_{21}| = |A_{11}| |A_{22} - A_{21}A_{11}^{-1}A_{12}| \quad (4.39)$$

- Matrix Differentiation

Calculus Notations: Take derivative of  $y = (y_1, y_2, \dots, y_q)^T$  over  $x = (x_1, x_2, \dots, x_p)^T$ ; or similarly of matrix  $A$  over scalar, etc.

We use 'Denominator-layout', which means the result follows the shape of denominator:

$$\frac{\partial y}{\partial x} := \frac{\partial y^T}{\partial x} = \begin{bmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} & \cdots & \frac{\partial y_1}{\partial x_p} \\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} & \cdots & \frac{\partial y_2}{\partial x_p} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial y_q}{\partial x_1} & \frac{\partial y_q}{\partial x_2} & \cdots & \frac{\partial y_q}{\partial x_p} \end{bmatrix} \Leftrightarrow \left( \frac{\partial y}{\partial x} \right)_{ij} = \frac{\partial y_j}{\partial x_i} \quad (4.40)$$

Properties (under denominator-layout):<sup>48</sup>

- $\frac{\partial}{\partial x} Ax = A^T$ ;
- $\frac{\partial}{\partial x} x^T A = A$ ;
- $\frac{\partial}{\partial x} x^T x = 2x$ ;

<sup>48</sup>More matrix diffrenciation equation see book [9] P49. Or can be easily derivated using Einstein sumation notation.

An example:

$$\begin{aligned} \frac{\partial |A|}{\partial A} &= \frac{\partial}{\partial A_{ij}} \text{Ex}_{i_1, i_2, \dots, i_n} A_{1i_1} A_{2i_2} \dots A_{ni_n} \\ &= \sum_{k=1}^n \text{Ex}_{i_1, \dots, (\wedge i_k), \dots, i_n} \delta_{ki} \delta_{i_k j} A_{1i_1} \dots (\wedge A_{ki_k}) \dots A_{ni_n} \times (-1)^{(n-k)+(n-i_k)} \\ &= (-1)^{i+j} \text{Ex}_{i_1, \dots, (\wedge j), \dots, i_n} A_{1i_1} \dots (\wedge A_{ij}) \dots A_{ni_n} \\ &= |A| A^{-1} \end{aligned}$$

$$- \frac{\partial}{\partial x} x^T A x = A x + A^T x;$$

$$- \frac{\partial}{\partial x} \log(x^T A x) = \frac{2Ax}{x^T A x};$$

$$- \frac{\partial |A|}{\partial A} = |A| A^{-1};$$

$$- \frac{\partial \text{tr}(AB)}{\partial A} = B^T;$$

$$- \frac{\partial \text{tr}(A^{-1}B)}{\partial A} = -A^{-1} B^T A^{-1}$$

- Kronecker Product: For matrix  $A = \{a_{ij}\}_{m \times n}$ ,  $B = \{b_{ij}\}_{p \times q}$ . Their Kronecker product

$$A \otimes B = \begin{bmatrix} a_{11}B & a_{12}B & \dots & a_{1n}B \\ a_{21}B & a_{22}B & \dots & a_{2n}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}B & a_{m2}B & \dots & a_{mn}B \end{bmatrix} \quad (4.41)$$

- Norm:

- Vector Norm: for vector  $x, y \in \mathbb{C}^m$ , norm  $\|\cdot\|$  is a function  $\mathbb{C}^m \rightarrow \mathbb{R}$ , with:

$$\text{Semi-definiteness: } \|x\| \geq 0, = \text{ for } x = 0 \quad (4.42)$$

$$\text{Absolute homogeneity: } \|kx\| = |k|\|x\|, k \in \mathbb{C} \quad (4.43)$$

$$\text{Triangle inequality: } \|x\| + \|y\| \geq \|x + y\| \quad (4.44)$$

the  $\ell_p$ -norm of  $x$  is

$$\|x\|_p = \left( \sum_{i=1}^n |x_i|^p \right)^{1/p} \quad (4.45)$$

Useful norm:

\*  $\ell_0$ -norm: # of none-0 elements in  $x$ ,<sup>49</sup>

\*  $\ell_1$ -norm:  $\|x\|_1 = \sum_{i=1}^n |x_i|$ ;

\*  $\ell_2$ -norm/Euclidean norm:  $\|x\|_2 = \sqrt{\sum_{i=1}^n x_i^2}$ ;

\*  $\ell_\infty$ -norm:  $\max |x_i|$ .

- Matrix Norm: for matrix  $A, B \in \mathbb{C}^{m \times n}$ , norm  $\|\cdot\|$  is a function  $\mathbb{C}^{m \times n} \rightarrow \mathbb{R}$ , with:

$$\text{Semi-definiteness: } \|A\| \geq 0, = \text{ for } x = 0 \quad (4.46)$$

$$\text{Absolute homogeneity: } \|kA\| = |k|\|A\|, k \in \mathbb{C} \quad (4.47)$$

$$\text{Triangle inequality: } \|A\| + \|B\| \geq \|A + B\| \quad (4.48)$$

<sup>49</sup>Note: actually triangle inequality is not satisfied for  $\|\cdot\|_0$

further for  $m = n$ , i.e.  $A, B \in \mathbb{C}^{m \times m}$ , usually append

$$\text{Sub-multiplicative: } \|A\|\|B\| \geq \|AB\| \quad (4.49)$$

$$\text{Hermite: } \|A\| = \|A^*\| \quad (4.50)$$

Matrix norm induced by vector norm:

$$\|A\| = \max \frac{\|Ax\|}{\|x\|} \quad (4.51)$$

e.g.  $\ell_p$  induced matrix norm:

$$* \ell_1\text{-norm: } \|A\|_1 = \max_{1 \leq j \leq n} \sum_{i=1}^m |A_{ij}|$$

$$* \ell_2\text{-norm/Euclidean norm: } \|A\|_2 = \sigma_{\max}(A);$$

$$* \ell_\infty\text{-norm: } \|A\|_\infty = \max_{1 \leq i \leq m} \sum_{j=1}^n |A_{ij}|.$$

Non-induced matrix norm, e.g.

$$* \text{Frobenius norm: } \|A\|_F = \left( \sum_{i=1}^m \sum_{j=1}^n |A_{ij}|^2 \right)^{1/2} = \sqrt{\text{tr}(A^*A)}$$

$$* \text{Weighted Frobenius norm: } \|A\|_W = \|W^{-1/2}AW^{-1/2}\|_F \text{ (or some textbooks uses } \|W^{1/2}AW^{1/2}\|_F)$$

$$* \text{Max norm: } \|A\|_{\max} = \max_{i,j} |A_{ij}|$$

- Sherman-Morrison Formula:

$$(A + u^T v)^{-1} = A^{-1} - \frac{A^{-1}u^T v A^{-1}}{1 + v^T A^{-1}u} \quad (4.52)$$

Or in matrix form:

$$(A + B)^{-1} = A^{-1} - \frac{A^{-1}BA^{-1}}{1 + \text{tr}(A^{-1}B)}, \quad \text{rank}(B) = 1 \quad (4.53)$$

Application instances see <https://vincent19.github.io/texts/MahalanobisAndLeverage/> and <https://vincent19.github.io/texts/DeletedResidual/>.

### 4.1.3 Useful Inequalities

- Cauchy-Schwartz Inequality:

Let  $b, d$  any  $p \times 1$  vectors.

$$(b'd)^2 \leq (b'b)(d'd) \quad (4.54)$$

- Extended Cauchy-Schwartz Inequality:

Let  $B$  be a positive definite matrix.

$$(b'd)^2 \leq (b'Bb)(d'B^{-1}d) \quad (4.55)$$

- Maximization Lemma:

$d$  be a given vector, for any non-zero vector  $x$ ,

$$\frac{(x'd)^2}{x'Bx} \leq d'B^{-1}d \quad (4.56)$$

Take Maximum when  $x = cB^{-1}d$ .

## Section 4.2 Statistical Inference to Multivariate Population

Statistics model: a  $n$  cases sample  $\mathbf{X}_1, \mathbf{X}_2, \dots, \mathbf{X}_n$ , where each  $\mathbf{X}_i$  i.i.d. from a multivariate population (usually consider a multi-normal). i.e.

$$\mathbf{X} = \begin{bmatrix} X_{11} & X_{12} & \dots & X_{1p} \\ X_{21} & X_{22} & \dots & X_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ X_{1n} & X_{n2} & \dots & X_{np} \end{bmatrix} = \begin{bmatrix} \mathbf{X}'_1 \\ \mathbf{X}'_2 \\ \vdots \\ \mathbf{X}'_n \end{bmatrix} \quad (4.57)$$

### 4.2.1 Multivariate Normal Distribution

Univariate Normal Distribution:  $N(\mu, \sigma^2)$

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) \quad (4.58)$$

Multivariate Normal Distribution:  $X \sim N_p(\vec{\mu}, \Sigma)$ <sup>50</sup>

$$f_{\mathbf{X}}(\vec{x}) = \frac{1}{(2\pi)^{p/2} |\Sigma|^{1/2}} \exp\left(-\frac{(\vec{x} - \vec{\mu})' \Sigma^{-1} (\vec{x} - \vec{\mu})}{2}\right) \quad (4.59)$$

Note: Here in the exp, the  $(\vec{x} - \vec{\mu})' \Sigma^{-1} (\vec{x} - \vec{\mu})$  is the Mahalanobis Distance  $d_M$  defined in [equation. 4.28](#)

Remark: A  $n$ -dimension multivariate normal has  $\frac{p(p+1)}{2}$  free parameters. Thus for a very high dimension, contains too many free parameters to be determined!

Properties: Consider  $X \sim N_p(\mu, \Sigma)$

- Linear Transform:

- For a  $p \times 1$  vector  $a$ :

$$X \sim N_p(\mu, \Sigma) \Leftrightarrow a'X \sim N(a'\mu, a'\Sigma a), \forall a \in \mathbb{R}^p \quad (4.60)$$

(Proof: use characteristic function.)

- For a  $q \times p$  const matrix  $A$ :

$$AX + a \sim N_q(A\mu + a, A\Sigma A') \quad (4.61)$$

- For a  $p \times p$  square matrix  $A$ :

$$\mathbb{E}(X'AX) = \mu' A \mu + \text{tr}(A\Sigma) \quad (4.62)$$

- Conditional Distribution: Take partition of  $X \sim N\left(\begin{smallmatrix} \mu \\ \mu_2 \end{smallmatrix}, \begin{smallmatrix} \Sigma \\ \Sigma_{21} \end{smallmatrix}\right)$  into  $X_1$  and  $X_2$ , where  $q_1 + q_2 = p$ . Write in matrix form:

$$\begin{matrix} X \\ p \times 1 \end{matrix} = \begin{bmatrix} X_1 \\ q_1 \times 1 \\ X_2 \\ q_2 \times 1 \end{bmatrix} \quad \begin{matrix} \mu \\ p \times 1 \end{matrix} = \begin{bmatrix} \mu_1 \\ q_1 \times 1 \\ \mu_2 \\ q_2 \times 1 \end{bmatrix} \quad \begin{matrix} \Sigma \\ p \times p \end{matrix} = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ q_1 \times q_1 & q_1 \times q_2 \\ \Sigma_{21} & \Sigma_{22} \\ q_2 \times q_1 & q_2 \times q_2 \end{bmatrix} \quad (4.63)$$

<sup>50</sup>Detailed derivation see [section. 1.8](#)

i.e.

$$X = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}_{p \times 1} \sim N_{q_1+q_2} \left( \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix}_{q_1+q_2 \times 1}, \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix}_{(q_1+q_2) \times (q_1+q_2)} \right) \quad (4.64)$$

Independence:  $X_1 \parallel X_2 \Leftrightarrow \Sigma_{21} = \Sigma_{12}^T = 0$

And the conditional distribution  $X_1|X_2 = x_2$  is given by<sup>51</sup>

$$X_1|X_2=x_2 \sim N_p(\mu_1 + \Sigma_{12}\Sigma_{22}^{-1}(x_2 - \mu_2), \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}) \quad (4.66)$$

#### • Multivariate Normal & $\chi^2$

Let  $X \sim N_p(\mu, \Sigma)$ , then

$$(X - \mu)^T \Sigma^{-1} (X - \mu) \sim \chi_p^2 \quad (4.67)$$

### 4.2.2 MLE of Multivariate Normal

Under the notation in [equation. 4.57](#), i.e. each sample case  $\mathbf{X}_i$  i.i.d.  $\sim N_p(\mu, \Sigma)$ , we can get the joint PDF of  $\mathbf{X}$ :

$$f_{\mathbf{X}_1, \dots, \mathbf{X}_n; \mu, \Sigma}(x_1, \dots, x_n) = \frac{1}{(2\pi)^{np/2} |\Sigma|^{n/2}} \exp \left( - \sum_{i=1}^n \frac{(x_i - \mu)' \Sigma^{-1} (x_i - \mu)}{2} \right) \quad (4.68)$$

and at the same time get likelihood function<sup>52</sup>:

$$L(\mu, \Sigma; x_1, \dots, x_n) = \frac{1}{(2\pi)^{np/2} |\Sigma|^{n/2}} \exp \left[ -\frac{1}{2} \text{tr} \left( \Sigma^{-1} \left( \sum_{i=1}^n (x_i - \bar{x})(x_i - \bar{x})' + n(\bar{x} - \mu)(\bar{x} - \mu)' \right) \right) \right] \quad (4.70)$$

And we can get the MLE of  $\mu$  and  $\Sigma$  as follows<sup>53</sup>:

$$\hat{\mu} = \frac{1}{n} \sum_{i=1}^n x_i = \bar{x} \quad (4.71)$$

$$\hat{\Sigma} = \frac{1}{n} \sum_{i=1}^n (x_i - \bar{x})(x_i - \bar{x})' = \frac{n-1}{n} S \quad (4.72)$$

**Δ Note:** In this section,  $S$  is used to denote  $\hat{\Sigma}$ , which is different from that in [section. 2.1.1](#) ( $S^2$  for  $\hat{\Sigma}$ )

And we can further construct MLE of function of  $\mu, \Sigma$  (use invariance property of MLE), for example

$$|\hat{\Sigma}| = |\hat{\Sigma}| \quad (4.73)$$

Note:  $(\hat{\mu}, \hat{\Sigma})$  is sufficient statistic of multi-normal population.

<sup>51</sup>In [equation. 4.61](#), take

$$A = \begin{bmatrix} I_{q \times q} & -\Sigma_{12}\Sigma_{22}^{-1} \\ 0_{(p-q) \times q} & I_{(p-q) \times (p-q)} \end{bmatrix} \quad (4.65)$$

<sup>52</sup>Here we need to use the property of trace

$$x'Ax = \text{tr}(x'Ax) = \text{tr}(Ax'x) \quad (4.69)$$

<sup>53</sup>Detailed proof see 'Applied Multivariate Statistical Analysis' P130

### 4.2.3 Sampling distribution of $\bar{X}$ and $S$

$\hat{\mu} = \bar{X}$  and  $\hat{\Sigma} = \frac{n-1}{n}S$  are statistics, with sampling distribution.

#### □ Sampling distribution of $\bar{X}$

Similar to monovariate case:

$$\bar{X} \sim N_p(\mu, \frac{1}{n}\Sigma) \quad (4.74)$$

#### □ Sampling distribution of $S^2$

- Monovariate case: Consider  $(X_1, X_2, \dots, X_n)$  i.i.d.  $\sim N(\mu, \sigma^2)$

Then

$$\frac{(n-1)S}{\sigma^2} \sim \chi_{n-1}^2 \quad (4.75)$$

- Multivariate case: Consider  $(\mathbf{X}_1, \mathbf{X}_2, \dots, \mathbf{X}_n)$  i.i.d.  $\sim N_p(\mu, \Sigma)$

Then

$$(n-1)S \sim W_p(n-1, \Sigma) \quad (4.76)$$

Where  $W_p(n-1, \Sigma)$  is Wishart Distribution, details as follows:

For r.v.  $Z_1, Z_2, \dots, Z_m$  i.i.d.  $\sim N_p(0, \Sigma)$ , def  $p$  dimensional **Wishart Distribution** with dof  $m$  as  $W_p(m, \Sigma)$ .<sup>54</sup>

$$W_p = \sum_{i=1}^n Z_i Z_i' \quad (4.77)$$

PDF of  $W_p(m, \Sigma)$ :

$$f_W(w; p, m, \Sigma) = \frac{|w|^{\frac{m-p-1}{2}} \exp\left(-\frac{1}{2}tr(\Sigma^{-1}w)\right)}{2^{\frac{mp}{2}} |\Sigma|^{-1/2} \pi^{\frac{p(p-1)}{4}} \prod_{i=1}^p \Gamma\left(\frac{m-i+1}{2}\right)} \quad (4.78)$$

C.F.

$$\phi(T) = |I_p - 2i\Sigma T|^{-\frac{m}{2}} \quad (4.79)$$

Properties:

- For independent  $A_1 \sim W_p(m_1, \Sigma)$  and  $A_2 \sim W_p(m_2, \Sigma)$ , then

$$A_1 + A_2 \sim W_p(m_1 + m_2, \Sigma) \quad (4.80)$$

- For  $A \sim W_p(m, \Sigma)$ , then

$$CAC' \sim W_p(m, C\Sigma C') \quad (4.81)$$

- Wishart distribution is the matrix generization of  $\chi_n^2$ . When  $p = 1$ ,  $\Sigma = \sigma^2 = 1$ ,  $W_p(m, \Sigma)$  naturally reduce to  $\chi_m^2$ .

$$\chi_n^2 = W_1(n, 1) \quad (4.82)$$

▷ **R. Code**

Distribution functions are in package `MCMCpack`, or use `rWishart()` function.

<sup>54</sup> $W_p(m, \Sigma)$  is a distribution defined on  $p \times p$  matrix space.

□ **Large sample  $\bar{X}$  and  $S$**

- $\sqrt{n}(\bar{X} - \mu) \xrightarrow{d} N_p(0, \Sigma)$ ;
- $n(\bar{X} - \mu)'S^{-1}(\bar{X} - \mu) \xrightarrow{d} \chi_p^2$

#### 4.2.4 Hypothesis Testing for Normal Population

• **One-Population Hypothesis Testing:**

Conduct hypothesis testing to  $\mu$ :

$$H_0 : \mu = \mu_0 \longleftrightarrow H_1 : \mu \neq \mu_0 \quad (4.83)$$

□ **Hotelling's  $T^2$  test**

– One-Dimensional case:  $t$ -test

$$T = \frac{\sqrt{n}(\bar{X} - \mu_0)}{S} \sim t_{n-1} \quad (4.84)$$

i.e.

$$T^2 = [\sqrt{n}(\bar{X} - \mu_0)]S^{-1}[\sqrt{n}(\bar{X} - \mu_0)] \sim t_{n-1}^2 = F_{1,n-1} \quad (4.85)$$

– Multi-Dimensional case: Hotelling's  $T^2$

$$T^2 = [\sqrt{n}(\bar{X} - \mu_0)']S^{-1}[\sqrt{n}(\bar{X} - \mu_0)] \sim \frac{p}{n-p}(n-1)F_{p,n-p} \quad (4.86)$$

And we can get the distribution of **Hotelling's  $T^2$** :

$$\frac{n-p}{p} \frac{T^2}{n-1} \sim F_{p,n-p} \quad (4.87)$$

Rejection Rule:

$$T^2 > \frac{p(n-1)}{n-p} F_{p,n-p,\alpha} \quad (4.88)$$

Property:

Invariant for  $X$  transform: For  $Y = CX + d$ , then

$$T_Y^2 = n(\bar{X} - \mu_0)'S^{-1}(\bar{X} - \mu_0) = T_X^2 \quad (4.89)$$

□ **LRT of  $\hat{\mu}$**

Monovariate case see [section. 2.4.3](#).

LRT uses the statistic:

$$\Lambda = \frac{\max_{H_0} L(\mu_0, \Sigma)}{\max_{H_0 \cup H_1} L(\mu, \Sigma)} = (1 + \frac{T^2}{n-1})^{-n/2} \quad (4.90)$$

where  $T^2 = n(\bar{x} - \mu_0)'S^{-1}(\bar{x} - \mu_0)$

• **Two-Population Hypothesis Testing:**

Conduct hypothesis testing to  $\delta = \mu_1 - \mu_2$ :

$$H_0 : \delta = \delta_0 \longleftrightarrow H_1 : \delta \neq \delta_0 \quad (4.91)$$



Notation: The two sample of size  $n_1, n_2$ , each denoted as

$$X_{1,ij} \quad X_{2,ij} \quad (4.92)$$

with mean  $\mu_1, \mu_2$  and covariance matrix  $\Sigma_1, \Sigma_2$

– Paired Samples:  $n_1 = n_2$

For two paires samples  $\{X_{1,ij}\}, X_{2,ij}$ , take subtraction as

$$D_{ij} = X_{1,ij} - X_{2,ij} \quad (4.93)$$

denote  $\bar{D} = \frac{1}{n} \sum_{j=1}^n D_j$ ,  $S_D^2 = \frac{1}{n-1} \sum_{j=1}^n (D_j - \bar{D})(D_j - \bar{D})'$

and conduct test to

$$H_0 : \bar{D} = \delta_0 \longleftrightarrow H_1 : \bar{D} \neq \delta_0 \quad (4.94)$$

And the folloeing steps are as in One-population testing, test

$$T^2 = n(\bar{D} - \delta_0)'(S_D^2)^{-1}(\bar{D} - \delta_0) \sim \frac{(n-1)p}{n-p} F_{p,n-p} \quad (4.95)$$

– Under Equal Unknown Variance:  $\Sigma_1 = \Sigma_2$

$$\bar{X}_1 = \frac{1}{n_1} \sum_{j=1}^{n_1} X_{1,j} \quad \bar{X}_2 = \frac{1}{n_2} \sum_{j=1}^{n_2} X_{2,j} \quad (4.96)$$

$$S_1 = \frac{1}{n_1-1} \sum_{j=1}^{n_1} (X_{1,j} - \bar{X}_1)(X_{1,j} - \bar{X}_1)' \quad S_2 = \frac{1}{n_2-1} \sum_{j=1}^{n_2} (X_{2,j} - \bar{X}_2)(X_{2,j} - \bar{X}_2)' \quad (4.97)$$

And denote pooled variance

$$S_{\text{pooled}} = \frac{1}{n_1 + n_2 - 2} ((n_1 - 1)S_1 + (n_2 - 1)S_2) \sim \frac{W_p(n_1 + n_2 - 2, \Sigma)}{n_1 + n_2 - 2} \quad (4.98)$$

Under  $H_0$ , we have

$$T^2 = \frac{1}{\frac{1}{n_1} + \frac{1}{n_2}} (\bar{X}_1 - \bar{X}_2 - \delta_0)' S_{\text{pooled}}^{-1} (\bar{X}_1 - \bar{X}_2 - \delta_0) \sim \frac{p(n_1 + n_2 - 2)}{n_1 + n_2 - p - 1} F_{p, n_1 + n_2 - p - 1} \quad (4.99)$$

#### 4.2.5 Confidence Region

Estimate the confidence region for  $\mu$  of  $X \sim N_p(\mu, \Sigma)$ , Monovariate case see [section. 2.3.3](#)

• Confidence Region:

Also use Hotelling's  $T^2$

$$\frac{n-p}{p} \frac{T^2}{n-1} \sim F_{p, n-p} \quad (4.100)$$

And take  $100(1 - \alpha)\%$  confidence region of  $\mu$  as

$$R(x) = \{x | T(x)^2 \leq c^2\} \quad c^2 = \frac{p}{n-p} (n-1) F_{p, n-p, \frac{\alpha}{2}} \quad (4.101)$$

The shape of  $R(x)$  is an ellipsoid.

- Individual Converage Interval

Use the decomposition of  $S$  as a positive definite matrix  $S = A^T A$ , where  $A$  is some  $p \times p$  matrix, then

$$T^2 = [\sqrt{n}(\bar{X} - \mu_0)]' S^{-1} [\sqrt{n}(\bar{X} - \mu_0)] = [A^{-1'} \sqrt{n}(\bar{X} - \mu_0)]' [A^{-1'} \sqrt{n}(\bar{X} - \mu_0)] \quad (4.102)$$

Thus denote  $Z = A^{-1'}(X - \mu_0) \sim N_p(0, A^{-1'} \Sigma A^{-1})$ , the  $T^2$  estimator of  $Z$  would be

$$T_Z^2 = [\sqrt{n}\bar{Z}]' S_Z^{-1} [\sqrt{n}\bar{Z}] = n\bar{Z}' \bar{Z} = \frac{1}{n} \sum_{i=1}^n \bar{Z}_i^2 \sim F_{p, n-p} \quad (4.103)$$

As a simplified case, we can take the **Individual Converage Interval** of  $Z_i$ , which is

$$\frac{\sqrt{n}Z_i}{s_{Z_i}} \sim t_{n-1} \quad (4.104)$$

And we can take the Confidence Region<sup>55</sup> as

$$R(z) = \bigotimes_{i=1}^n (\bar{Z}_i \pm s_{Z_i} t_{n-1, \frac{\beta}{2}}) \quad (4.105)$$

where  $\beta$  taken with Bonferroni correction

$$1 - p\beta = 1 - \alpha \quad (4.106)$$

Note: Consider that

$$P(\text{all } Z_i \text{ in CI}_i) \geq 1 - m\beta = 1 - \alpha \quad (4.107)$$

So the real CR for  $\mu$  should be larger.

The shape of  $R(x)$  is an oblique cubold.

## 4.2.6 Large Sample Multivariate Inference

Basic point:

$$\bar{X} \xrightarrow{d} \mu \quad S \xrightarrow{d} \Sigma \quad (4.108)$$

- One-sample Mean:

$$n(\bar{X} - \mu) S^{-1} (\bar{X} - \mu) \xrightarrow{d} \chi_p^2 \quad (4.109)$$

- Unequal Variance Two-sample Mean:

$$\bar{X}_1 - \bar{X}_2 \xrightarrow{d} N\left(\mu_1 - \mu_2, \frac{1}{n_1} \Sigma_1 + \frac{1}{n_2} \Sigma_2\right) \quad \frac{1}{n_1} S_1 + \frac{1}{n_2} S_2 \xrightarrow{d} \frac{1}{n_1} \Sigma_1 + \frac{1}{n_2} \Sigma_2 \quad (4.110)$$

Test:

$$T^2 = [(\bar{X}_1 - \bar{X}_2) - (\mu_1 - \mu_2)]' \left( \frac{1}{n_1} S_1 + \frac{1}{n_2} S_2 \right)^{-1} [(\bar{X}_1 - \bar{X}_2) - (\mu_1 - \mu_2)] \xrightarrow{d} \chi_p^2 \quad (4.111)$$

<sup>55</sup>The confidence region of  $Z$  can be transformed to that of  $X$  using  $\hat{Z} = A^{-1'}(\hat{X} - \bar{X})$ .

## Section 4.3 Principal Component Analysis

PCA and next subsection FA focus on data dimension reduction. Why?

### □ ‘Curse of Dimensionality’

- Difficulty in computation complexity: Many algorithms has complexity  $O(n^2)$  or more, high dimension  $n$  cause high complexity.
- Hughes Phenomenon: As the number of feature dimension increases, the classifier’s performance increases as well until an optimal dimension. Adding more features based on the same size as the training set will then degrade the classifier’s performance. <sup>a</sup>

<sup>a</sup>Example: Volumn of unit sphere in  $n$ -dim space

$$V_n = \pi^{n/2} \frac{1}{\Gamma(1 + n/2)} \rightarrow \left(\frac{2\pi e}{n}\right)^{n/2} \rightarrow 0 \quad (4.112)$$

i.e. data will naturally become ‘sparse’ in high dimension data  $\rightarrow$  difficult to extract information.

Key Idea of PCA: Find the components most powerful in explaining variance. (Similar to the idea of ANOVA)

### 4.3.1 Population Principal Component

For population  $\vec{X} = (X_1, X_2, \dots, X_p) \sim (\mu, \Sigma)_p$ , conduct spectrum decomposition to  $\Sigma$  such that

$$\Sigma P = P \Lambda \quad P = [e_1, e_2, \dots, e_p] \quad \Lambda = \text{diag}\{\lambda_1, \lambda_2, \dots, \lambda_p\}, \lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_p \quad (4.113)$$

where  $(\lambda_i, e_i)$  is the  $i^{\text{th}}$  eigenvalue-eigenvector pair of  $\Sigma$ , large  $\lambda_i$  suggests  $X$  is more ‘extended’ in  $e_i$  direction (large variance).

Then the **Principal Components**  $Y = \{Y_i\}$

$$Y = P'X \sim (P'\mu, P'\Sigma P)_p = (P'\mu, \Lambda) \quad (4.114)$$

$$\begin{cases} Y_1 = e_1'X \sim (e_1'\mu, \lambda_1) \\ \vdots \\ Y_p = e_p'X \sim (e_p'\mu, \lambda_p) \end{cases} \quad (4.115)$$

Properties & Definitions:

- Trace of cov. matrix:

$$\sum_{i=1}^p \sigma_{ii} = \sum_{i=1}^p \text{var}(X_i) = \sum_{i=1}^p \text{var}(Y_i) = \sum_{i=1}^p \lambda_i \quad (4.116)$$

- *corr* between  $Y_i, X_j$ :

$$\rho_{Y_i, X_j} = \frac{\text{cov}(Y_i, X_j)}{\sqrt{\lambda_i} \sqrt{\sigma_{jj}}} = \frac{(e_i)_j \sqrt{\lambda_i}}{\sqrt{\sigma_{jj}}} \quad (4.117)$$

- Factor Loading:

$$\text{FL}_{ij} = (e_i)_j \sqrt{\lambda_i} \quad (4.118)$$

- PC Score:

$$\text{PC Score}_i = Y_i = e_i'X \text{ or } Y_i = e_i'(X - \mu) \quad (4.119)$$

In practice, we pick the first several  $m$  PC such that

$$\sum_{i=1}^m \frac{\lambda_i}{\sum_{k=1}^p \lambda_k} \text{ large enough} \quad (4.120)$$

Note: Another important point for PCA is the **interpretability** of principal components.

A continuous version of PCA in stochastic process is Karhunen-Loève Expansion in ??.

#### □ Standardized Principal Component

To cancel out the influence due to scale, we can also obtain standardized PC from  $Z = (V)^{-1/2}(X_\mu)$ , where  $V$  is standard deviation matrix as def. in [equation. 4.6](#).

And we have  $\vec{Z} = (Z_1, Z_2, \dots, Z_p) \sim N_p(0, V^{-1/2}\Sigma V^{-1/2}) = N_p(0, \rho)$ . Then obtain  $(\lambda_i, e_i)$  pairs<sup>56</sup> from  $\rho$  to form PC.

$$\rho P = P\Lambda \quad P = [e_1, e_2, \dots, e_p] \quad \Lambda = \text{diag}\{\lambda_1, \lambda_2, \dots, \lambda_p\}, \lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_p \quad (4.121)$$

Then the Principal Components  $W = \{W_i\}$

$$W = P'Z \sim (0, P'\rho P)_p = (0, \Lambda) \quad (4.122)$$

$$\begin{cases} W_1 = e_1'Z \sim (0, \lambda_1) \\ \vdots \\ W_p = e_p'Z \sim (0, \lambda_p) \end{cases} \quad (4.123)$$

Properties:

- Trace of cov. matrix:

$$\sum_{i=1}^p \text{var}(Z_i) = \sum_{i=1}^p \text{var}(W_i) = \sum_{i=1}^p \lambda_i = p \quad (4.124)$$

- *corr* between  $Y_i, X_j$ :

$$\rho_{W_i, Z_j} = (e_i)_j \sqrt{\lambda_i} \quad (4.125)$$

### 4.3.2 Sample Principal Component

For sample matrix  $X$  denoted in [equation. 4.57](#), with cov. matrix  $S$  in [equation. 4.24](#). Then conduct the above spectrum decomposition to  $S$  to get sample PCs.

$$\hat{Y} = \hat{P}\hat{\Lambda}\hat{P}' \quad \hat{P} = [\hat{e}_1, \hat{e}_2, \dots, \hat{e}_p] \quad \hat{\Lambda} = \text{diag}\{\hat{\lambda}_1, \hat{\lambda}_2, \dots, \hat{\lambda}_p\}, \hat{\lambda}_1 \geq \hat{\lambda}_2 \geq \dots \geq \hat{\lambda}_p \quad (4.126)$$

Properties and Definitions

- Trace of cov. matrix:

$$\sum_{i=1}^p s_{ii} = \sum_{i=1}^p \hat{\lambda}_i \quad (4.127)$$

- Sample corr & factor load:

$$\rho(\hat{y}_i, x_j) = \frac{(\hat{e}_i)_j \sqrt{\hat{\lambda}_j}}{\sqrt{s_{jj}}} \quad (4.128)$$

<sup>56</sup>The eigenvalue-eigenvector pairs obtained from  $\rho$  is generally **different** from  $\Sigma$ .

### □ Large Sample & Normal PCA

Under normal assumption or large sample case, i.e.

$$X \sim N_p(\mu, \Sigma) \text{ or } X \xrightarrow{d} N_p(\mu, \Sigma) \quad (4.129)$$

We can examine the (asymptotic) distribution of  $(\hat{\lambda}_1, \hat{\lambda}_2, \dots, \hat{\lambda}_p)$  and  $(\hat{e}_1, \hat{e}_2, \dots, \hat{e}_p)$ :

- $\hat{\lambda}$ :

$$\sqrt{n}(\hat{\lambda} - \lambda) \sim N_p(0, 2\Lambda^2) \quad (4.130)$$

- $\hat{e}_i$ :

$$\sqrt{n}(\hat{e}_i - e_i) \sim N_p(0, E_i), \quad E_i = \lambda_i \sum_{k \neq i} \frac{\lambda_k}{(\lambda_k - \lambda_i)^2} e_k e_k' \quad (4.131)$$

- Independence:

$$\hat{\lambda}_i \perp\!\!\!\perp \hat{e}_i \quad (4.132)$$

## Section 4.4 Factor Analysis

Key idea of FA: For a model with  $p$  variable  $X = (X_1, X_2, \dots, X_p) \sim (\mu, \Sigma)_p$  (especially when  $p$  large and  $X_i$  interrelated), there would be some internal, latent **factors**  $F$  behind  $X$  determining the model structure.<sup>57</sup>

### 4.4.1 Orthogonal Factor Model

$$X - \mu = \underset{p \times 1}{L} \underset{p \times m \ m \times 1}{F} + \underset{p \times 1}{\varepsilon}, \quad m < p \quad (4.133)$$

where  $L$  is the const **loading matrix**;  $F$  is r.v. **factor**; and  $\varepsilon$  is r.v. error.

$$X = \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_p \end{bmatrix} \quad L = \begin{bmatrix} \ell_{11} & \ell_{12} & \dots & \ell_{1m} \\ \ell_{21} & \ell_{22} & \dots & \ell_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ \ell_{p1} & \ell_{p2} & \dots & \ell_{pm} \end{bmatrix} \quad F = \begin{bmatrix} F_1 \\ F_2 \\ \vdots \\ F_m \end{bmatrix} \quad \varepsilon = \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \vdots \\ \varepsilon_p \end{bmatrix} \quad (4.134)$$

Note: Intuitively, we cannot estimate  $(m + p)$  (unobservable) r.v. from  $p$  r.v., so we need the following assumptions on  $F$  and  $\varepsilon$

$$\begin{aligned} \mathbb{E}(F) &= 0 & \text{cov}(F) &= I_m \\ \mathbb{E}(\varepsilon) &= 0 & \text{cov}(\varepsilon) &= \Psi = \text{diag}\{\psi_1, \psi_2, \dots, \psi_p\} \\ \varepsilon &\perp\!\!\!\perp F & \Leftrightarrow \text{cov}(F, \varepsilon) &= 0 \end{aligned} \quad (4.135)$$

Derived Conclusions:

- Representation of  $\Sigma$ :

$$\text{cov}(X) = \Sigma = LL' + \Psi \quad (4.136)$$

<sup>57</sup> As the most simplified case, here only consider  $X$  linear dependent on  $F$ .

– Diagonal Elements:

$$\text{var}(X_i) = \sum_{k=1}^m \ell_{ik}^2 + \psi_i = h_i^2 + \psi_i \quad (4.137)$$

where  $h_i^2$  is Communality,  $\psi_i$  is Specific variance.

– NonDiagonal Elements:

$$\text{cov}(X_i, X_j) = \sum_{k=1}^m \ell_{ik} \ell_{jk} \quad (4.138)$$

• relation bet.  $X$  and  $F$ :

$$\text{cov}(X, F) = L \quad (4.139)$$

□ **Factor Rotation** For any orthonormal rotation/reflection matrix  $T_{m \times m}$ ,  $\tilde{L} = LT$  satisfies the same factor model (with a different  $\tilde{F}$ ):

$$\begin{aligned} X &= LF + \varepsilon = LTT'F + \varepsilon = \tilde{L}\tilde{F} + \varepsilon & \tilde{L} &= LT, \tilde{F} = T'F \\ \Sigma &= LL' + \Psi = \tilde{L}\tilde{L}' + \Psi \end{aligned}$$

Comment: Factor rotation reflects the arbitrariness of selection of  $L$ , allowing us to choose an **interpretable**  $L$  for FA model.

#### 4.4.2 Principal Component Approach

Origin: when  $m = p$ , factor decomposition reduces to spectrum(PC) decomposition. (At the same time  $\Psi$  can be taken 0.)

$$\begin{aligned} X &= LF + \varepsilon = PY \Rightarrow \Psi = 0 \\ \Sigma &= LL' + \Psi = P\Lambda P' \Rightarrow L = P\Lambda^{1/2} \end{aligned} \quad (4.140)$$

Then take the first  $m$  eigenvectors to form  $L$ , and use  $\psi_i = \sigma_{ii} - \sum_{k=1}^m \ell_{ik}^2$  as an approximation.

$$\Sigma = LL' + \Psi \quad L = [\sqrt{\lambda_1}e_1, \sqrt{\lambda_2}e_2, \dots, \sqrt{\lambda_m}e_m] \quad \Psi = \text{diag}\{\psi_i\} \quad (4.141)$$

#### □ Sample Factor Decomposition

From sample cov. matrix  $S$  and eigenvalue-eigenvector pairs  $(\hat{\lambda}_i, e_i)$ , pick the first  $m$  pairs to form  $L = \{\ell_{ij}\}$ :

$$\hat{L} = \{\hat{\ell}_{ij}\} = [\sqrt{\hat{\lambda}_1}\hat{e}_1, \sqrt{\hat{\lambda}_2}\hat{e}_2, \dots, \sqrt{\hat{\lambda}_m}\hat{e}_m] \quad \hat{\Psi} = \text{diag}\{s_{ii} - \sum_{k=1}^m \hat{\ell}_{ik}^2\} \quad (4.142)$$

• Selection of  $m$ : Construct Residual Matrix

$$\hat{E} = S - (\hat{L}\hat{L}' + \hat{\Psi}) \quad (4.143)$$

Residual matrix is trace 0, pick  $m$  such that

$$\text{Sum of All Elements in } \hat{E} < \sum_{k=m+1}^p \hat{\lambda}_k^2 \text{ small enough} \quad (4.144)$$

### 4.4.3 MLE Method

Assumption: Factor  $F$  and error  $\varepsilon$  are normal. (Then also  $X \sim N_p(\mu, \Sigma)$  is normal)

$$F \sim N_m(0, I_m) \quad \varepsilon \sim N_p(0, \Psi) \quad X \sim N_p(\mu, \Sigma) \quad (4.145)$$

Likelihood Function:

$$L(\mu, \Sigma) = (2\pi)^{-np/2} |\Sigma|^{-n/2} \exp \left( -\frac{1}{2} \text{tr} \left[ \Sigma^{-1} \left( \sum_{k=1}^n (x_k - \bar{x})(x_k - \bar{x})' + n(\bar{x} - \mu)(\bar{x} - \mu)' \right) \right] \right) \quad (4.146)$$

Maximize  $L$  to get  $\hat{L}$  and  $\hat{\Psi}$ , usually for convenient (and to counteract the arbitrariness of factor rotation) we further assume

$$L' \Psi^{-1} L = \Xi \text{ (diagonal matrix)} \quad (4.147)$$

- Estimtor of communality variance  $h_i^2$ :

$$\hat{h}_i^2 = \sum_{k=1}^m \hat{l}_{ik}^2 \quad (4.148)$$

## Section 4.5 Canonical Correlation Analysis

Key idea of CCA: For a model with two multivariate population  $X^{(1)} = (X_1^{(1)}, X_2^{(1)}, \dots, X_p^{(1)})$ ,  $X^{(2)} = (X_1^{(2)}, X_2^{(2)}, \dots, X_q^{(2)})$  with covariance

$$\Sigma_{(p+q) \times (p+q)} = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix} \quad (4.149)$$

find a few condensed variable to measure their similarity.

### 4.5.1 Canonical Variate Pair

By using the linear combination, we can construct a pair of vector  $\begin{matrix} a \\ p \times 1 \end{matrix}$  and  $\begin{matrix} b \\ q \times 1 \end{matrix}$  such that  $\text{corr}(a'X^{(1)}, b'X^{(2)})$  large, i.e.

$$\{a, b\} = \arg \max_{a, b \neq 0} \frac{a' \Sigma_{12} b}{\sqrt{a' \Sigma_{11} a} \sqrt{b' \Sigma_{22} b}} \quad (4.150)$$

where  $U_1 = a'X^{(1)}$ ,  $V_1 = b'X^{(2)}$  with  $\text{var}(U_1) = \text{var}(V_1) = 1$  are the **(first) canonical variate pair**, and  $\rho_1^* = \text{corr}(U_1, V_1)$  is the **(first) canonical correlation**.

Similarly, the  $k^{\text{th}}$  canonical pair  $(U_k, V_k)$  satisfy the same criterion as [equation. 4.150](#) but with  $a_k \in \text{span}\{a_1, \dots, a_{k-1}\}^\perp$ ,  $b_k \in \text{span}\{b_1, \dots, b_{k-1}\}^\perp$ ,  $k \leq \min\{p, q\}$ .

Result:  $U_k, V_k$  can be expressed as

$$U_k = a_k' X^{(1)} = e_k' \Sigma_{11}^{-1/2} X^{(1)} \quad V_k = b_k' X^{(2)} = f_k' \Sigma_{22}^{-1/2} X^{(2)} \quad (4.151)$$

where  $e_k$  is the  $k^{\text{th}}$  eigen vector of  $\Sigma_{11}^{-1/2} \Sigma_{12} \Sigma_{22}^{-1} \Sigma_{21} \Sigma_{11}^{-1/2}$ ,  $f_k$  is the  $k^{\text{th}}$  eigenvector of  $\Sigma_{22}^{-1/2} \Sigma_{21} \Sigma_{11}^{-1} \Sigma_{12} \Sigma_{22}^{-1/2}$ .  $e_k$  and  $f_k$  satisfies:

$$f_k = \frac{1}{\rho_k^*} \Sigma_{22}^{-1/2} \Sigma_{21} \Sigma_{11}^{-1/2} e_k \quad e_k = \frac{1}{\rho_k^*} \Sigma_{11}^{-1/2} \Sigma_{12} \Sigma_{22}^{-1/2} f_k \quad (4.152)$$

### 4.5.2 Canonical Correlation based on Standardized Variables

Using standardized variable of  $X$ :

$$Z_k^{(\nu)} = \frac{X_k^{(\nu)} - \mu_k^{(\nu)}}{\sqrt{\sigma_{kk}^{(\nu)}}}, \quad k = 1, 2, \dots, p \text{ or } q, \nu = 1, 2 \quad (4.153)$$

with covariance

$$\rho_{(p+q) \times (p+q)} = V^{-1/2} \Sigma V^{-1/2} = \begin{bmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{bmatrix} \quad (4.154)$$

And similarly, the CCA pair is

$$U_k = a'_k Z^{(1)} = e'_k \rho_{11}^{-1/2} Z^{(1)} \quad V_k = b'_k Z^{(2)} = f'_k \rho_{22}^{-1/2} Z^{(2)} \quad (4.155)$$

with  $e_k$  is the  $k^{\text{th}}$  eigenvector of  $\rho_{11}^{-1/2} \rho_{12} \rho_{22}^{-1} \rho_{21} \rho_{11}^{-1/2}$ ,  $f_k$  is the  $k^{\text{th}}$  eigenvector of  $\rho_{22}^{-1/2} \rho_{21} \rho_{11}^{-1} \rho_{12} \rho_{22}^{-1/2}$ , and

$$f_k = \frac{1}{\rho_k^*} \rho_{22}^{-1/2} \rho_{21} \rho_{11}^{-1/2} e_k \quad e_k = \frac{1}{\rho_k^*} \rho_{11}^{-1/2} \rho_{12} \rho_{22}^{-1/2} f_k \quad (4.156)$$

### 4.5.3 Sample Canonical Correlation

Replacement:

$$\Sigma \longrightarrow S \quad \rho \longrightarrow R \quad (4.157)$$

to get

$$\hat{U} = \hat{A}x^{(1)} \quad \hat{V} = \hat{B}x^{(2)} \quad (4.158)$$

and we can use  $\hat{U}, \hat{V}, \hat{A}, \hat{B}$  to express  $S_{12}$  as

$$S_{12} = \hat{A}^{-1} \begin{bmatrix} \hat{\rho}_1^* & 0 & \dots & 0 \\ 0 & \hat{\rho}_2^* & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \hat{\rho}_p^* \end{bmatrix} (\hat{B}^{-1})' \quad (4.159)$$

When applying CCA, we pick the first  $r$  canonical variable, thus some information is lost. But we hope the first  $r$  canonical variables can contain enough information of  $X^{(1)}$  and  $X^{(2)}$ .

Determine of  $r$ : consider the error if approximation by expressing

$$\hat{A}^{-1} = [\alpha_1, \alpha_2, \dots, \alpha_p] \quad \hat{B}^{-1} = [\beta_1, \beta_2, \dots, \beta_p] \quad (4.160)$$

and

$$S_{12} = \sum_{i=1}^p \hat{\rho}_i^* \alpha_i \beta_i' \quad (4.161)$$

$$S_{11} = \hat{A}^{-1} (\hat{A}^{-1})' = \sum_{i=1}^p \alpha_i \alpha_i' \quad (4.162)$$

$$S_{22} = \hat{B}^{-1} (\hat{B}^{-1})' = \sum_{i=1}^p \beta_i \beta_i' \quad (4.163)$$

Total sample variance explained by the first  $r$  canonical variables:

$$\frac{\sum_{i=1}^r \alpha_i' \alpha_i}{\text{tr}(S_{11})} \quad \frac{\sum_{i=1}^r \beta_i' \beta_i}{\text{tr}(S_{22})} \quad (4.164)$$



## Section 4.6 Discriminant Analysis

Key idea of DA: for  $X$  with an extra column labeling the classification, we want to determine a rule to assign new objects. More specifically, determine the classification region  $R_i$  for each class  $\pi_i$ .

### 4.6.1 Classification Criterion

- Two-category classification case: Each row of  $X$  is labeled in  $\pi_1$  or  $\pi_2$ , for two-category, only one of  $R_1, R_2$  is needed.

Some basic concept in classification model:

- Prior Possibility  $p_i, i = 1, 2$ ;
- Penalty for misclassification  $c(i|j), i, j = 1, 2$ : cost if a  $\pi_j$  object is classified in  $R_i$ .
- Conditional Probability  $\mathbb{P}(i|j), i, j = 1, 2$ : probability that a  $\pi_j$  object falls in region  $R_i$

#### □ Determination Criterion:

- Expected Cost of Misclassification (ECM) Criterion: Minimizing ECM,

$$\text{ECM} = c(2|1)\mathbb{P}(2|1)p_1 + c(1|2)\mathbb{P}(1|2)p_2 \quad (4.165)$$

For two-category problem,  $R_1, R_2$  can be determined as

$$R_1 = \frac{f_{\pi_1}(x)}{f_{\pi_2}(x)} \geq \frac{c(1|2)p_2}{c(2|1)p_1} \quad (4.166)$$

$$R_2 = \mathbb{C}_{R_x}^{R_1} = \arg \frac{f_{\pi_1}(x)}{f_{\pi_2}(x)} < \frac{c(1|2)p_2}{c(2|1)p_1} \quad (4.167)$$

- Total Probability of Misclassification (TPM) Criterion: Minimizing TPM,

$$\text{TPM} = \mathbb{P}(\text{misclass}) = \mathbb{P}(2|1)p_1 + \mathbb{P}(1|2)p_2 \quad (4.168)$$

actually  $\arg \min \text{TPM} = \arg \min_{c(1|2)=c(2|1)} \text{ECM}$

- Posterior Probability Criterion: Maximize posterior probability  $P(\pi_i|x_0)$ ,

$$\mathbb{P}(X \in \pi_i | X = x_0) = \frac{p_i f_{\pi_i}(x_0)}{p_1 f_{\pi_1}(x_0) + p_2 f_{\pi_2}(x_0)}, i = 1, 2 \quad (4.169)$$

Also equivalent to ECM for  $c(1|2) = c(2|1)$

- Here only introduce ECM:  $\{R_i\} = \arg \min \text{ECM}$

$$\text{ECM}(i) = \sum_{j \neq i} c(j|i)\mathbb{P}(j|i) \quad (4.170)$$

$$\text{ECM} = \sum_{i=1}^g p_i \text{ECM}_i = \sum_{i=1}^g \sum_{j \neq i} c(j|i)p(j|i)p_i \quad (4.171)$$

### 4.6.2 Linear & Quadratic Discriminant Analysis

Now take two-category ECM criterion as example. An estimation to  $\mathbb{P}(1|2)$ ,  $\mathbb{P}(2|1)$ , i.e. to  $f_{\pi_1}$ ,  $f_{\pi_2}$  is needed.

Assumption: for  $\pi_1 : X \sim N(\mu_1, \Sigma_1), \pi_2 : X \sim N(\mu_2, \Sigma_2)$ , further for

- $\Sigma_1 = \Sigma_2 = \Sigma$ : Linear Discriminant Analysis (LDA).

$$f_{\pi_i}(x) = \frac{1}{(2\pi)^{p/2}|\Sigma|^{1/2}} \exp\left(-\frac{1}{2}(x - \mu_i)' \Sigma^{-1}(x - \mu_i)\right), i = 1, 2 \quad (4.172)$$

then

$$R_1 = \arg_{x \in R} (\mu_1 - \mu_2)' \Sigma^{-1} x - \frac{1}{2}(\mu_1 - \mu_2)' \Sigma^{-1}(\mu_1 - \mu_2) \geq \ln\left(\frac{c(1|2)}{c(2|1)} \frac{p_2}{p_1}\right) \quad (4.173)$$

$$R_2 = \arg_{x \in R} (\mu_1 - \mu_2)' \Sigma^{-1} x - \frac{1}{2}(\mu_1 - \mu_2)' \Sigma^{-1}(\mu_1 - \mu_2) < \ln\left(\frac{c(1|2)}{c(2|1)} \frac{p_2}{p_1}\right) \quad (4.174)$$

Note that L.H.S. is a linear combination of  $x$ , thus called LinearDA.

Sample estimation to  $\Sigma$ : use pooled variance in [equation. 4.98](#).

- $\Sigma_1 \neq \Sigma_2$ : Quadratic Discriminant Analysis (QDA).

$$f_{\pi_i}(x) = \frac{1}{(2\pi)^{p/2}|\Sigma_i|^{1/2}} \exp\left(-\frac{1}{2}(x - \mu_i)' \Sigma_i^{-1}(x - \mu_i)\right), i = 1, 2 \quad (4.175)$$

then

$$R_1 = -\frac{1}{2}x'(\Sigma_1^{-1} - \Sigma_2^{-1})x + (\mu_1' \Sigma_1^{-1} - \mu_2' \Sigma_2^{-1})x - \frac{1}{2} \ln\left(\frac{|\Sigma_1|}{|\Sigma_2|}\right) + \frac{1}{2}(\mu_1' \Sigma_1^{-1} \mu_1 - \mu_2' \Sigma_2^{-1} \mu_2) \geq \ln\left(\frac{c(1|2)}{c(2|1)} \frac{p_2}{p_1}\right) \quad (4.176)$$

$$R_2 = -\frac{1}{2}x'(\Sigma_1^{-1} - \Sigma_2^{-1})x + (\mu_1' \Sigma_1^{-1} - \mu_2' \Sigma_2^{-1})x - \frac{1}{2} \ln\left(\frac{|\Sigma_1|}{|\Sigma_2|}\right) + \frac{1}{2}(\mu_1' \Sigma_1^{-1} \mu_1 - \mu_2' \Sigma_2^{-1} \mu_2) < \ln\left(\frac{c(1|2)}{c(2|1)} \frac{p_2}{p_1}\right) \quad (4.177)$$

Note that L.H.S. is a quadric form of  $x$ , thus called QuadraticDA.

- Two extension: allow more flexible estimation to variance:
  - $\hat{\Sigma}_i(\alpha) = \alpha \hat{\Sigma}_i + (1 - \alpha) \hat{\Sigma}$ , shrink between QDA and LDA;
  - $\hat{\Sigma}_i(\gamma) = \gamma \hat{\Sigma} + (1 - \gamma) \hat{\sigma}^2 I$ , shrink toward scalar cov.

### 4.6.3 Fisher's Discriminant Analysis

Project  $X$  onto some hyperplane and conduct low-dimensional classification.

Project  $x$  onto some hyperplane by  $y = a'x$ , then we maximize  $\psi = \frac{\text{mean of treatment}^2}{\text{variance}}$ <sup>58</sup>. i.e.

$$\psi = \frac{\sum_{i=1}^g (\mu_{iY} - \mu_Y)^2}{\sigma_Y^2} = \frac{a' \left( \sum_{i=1}^g (\mu_i - \mu)(\mu_i - \mu)' \right) a}{a' \Sigma a} = \frac{a' B_\mu a}{a' \Sigma a} \quad (4.181)$$

Result:  $a$  is the largest eigen vector of  $W^{-1}B$ .

Relation between FDA and LDA: in FDA, take the first  $\xi$  eigenvectors to conduct classification, thus loses more information. But when  $\xi = g - 1$ , FDA  $\equiv$  LDA.<sup>59</sup>

#### 4.6.4 Evaluation of Discriminant Model

##### □ Judging Index:

- Total Probability of Misclassification (TPM):

$$\text{TPM} = p_1 \mathbb{P}(2|1) + p_2 \mathbb{P}(1|2) = p_1 \int_{R_2} f_{\pi_1}(x) dx + p_2 \int_{R_1} f_{\pi_2}(x) dx \quad (4.182)$$

- Apparent Error Rate (APER): used with cross validation (CV). The fraction of misclassification in training set.

### Section 4.7 Clustering Analysis

Key idea of CA: Group a collection of data according to similarity and relation of objects.

#### 4.7.1 Agglomerative Clustering Algorithm

##### □ Clustering Algorithm

Hierarchical clustering: start with individual points and combine them to form groups.

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##### Algorithm Hierarchical Clustering

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1. All  $k = n$  points are individual clusters;

2. In each iteration step  $k$ :

- (a) Use a distance/dissimilarity matrix  $D_{k \times k}$  to express distances between clusters; the 'distance' between clusters is diversified, choice of which see the following part;

---

<sup>58</sup>MANOVA Model: For  $g$  groups with same  $\Sigma$ , consider an MANOVA model:  $X_{ij} = \mu + \tau_i + e_{ij}$ . Then MANOVA table gives Sum of Squares and cross Products (SSP):

$$\text{Treatment: } B = \sum_{i=1}^g n_i (\bar{x}_i - \bar{x})(\bar{x}_i - \bar{x})' \quad (4.178)$$

$$\text{Residual: } W = \sum_{i=1}^g \sum_{j=1}^{n_i} (x_{ij} - \bar{x}_i)(x_{ij} - \bar{x}_i)' \quad (4.179)$$

$$\text{Total: } T = B + W = \sum_{i=1}^g \sum_{j=1}^{n_i} (x_{ij} - \bar{x})(x_{ij} - \bar{x})' \quad (4.180)$$

use  $B$  and  $W$  to measure the variance of sample.

<sup>59</sup>Because  $a$  is eigenvector of  $W^{-1}B$ , while  $\text{rk}(B) = g - 1$ , thus there are  $g - 1$  non-zero eigenvalues at most.

---

(b) merge the closest pair of clusters(or points) to form a larger cluster, and now number of clusters

(c)  $k = k - 1$ ;

3. Only  $k = 1$  cluster is left

4. Choose a proper threshold of distance to determine  $K$

□ **Choice of between-cluster distance: To express distance between two clusters  $A$  and  $B$ ,**

- Choice of distance functional  $D(\cdot, \cdot)$ :

- Euclidean Distance  $D_E$ ;
- Mahalanobis Distance  $D_M$ ;
- Jaccard Distance  $D_J = 1 - \frac{|A \cap B|}{|A \cup B|}$ ;
- etc.

- Location choice of cluster:

- Complete link:  $\max D(a \in A, b \in B)$ ;
- Single link:  $\min D(a \in A, b \in B)$ ;
- Centroid distance:  $D(A \text{ centroid}, B \text{ centroid})$ ;
- Group average:  $\langle D(a \in A, b \in B) \rangle$

- Note: pros-and-cons of agglomerative clustering algorithm

- No assumptions for final  $k$  needed;
- Intuitive display of relations;
- Large computational requirement:  $\sim O(n^3)$ ;
- Sensitive to noise and outliers.

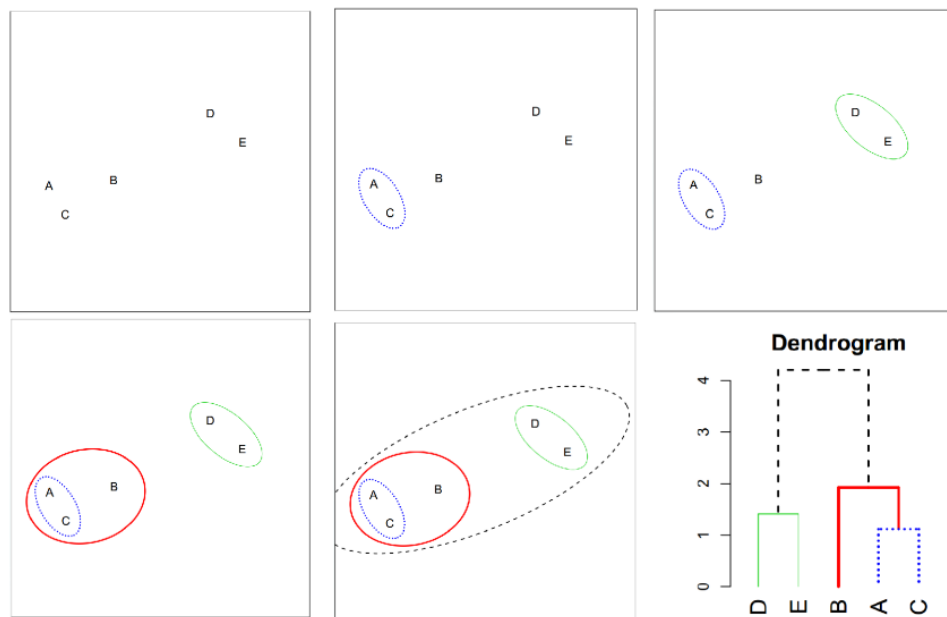


图 3: Illustration of Hierarchical Clustering

### 4.7.2 $K$ -Means Clustering Algorithm

Assume we have a preset number  $K$  of clusters, we can use  $K$ -means clustering.

---

#### Algorithm $K$ -Means Clustering

---

1. Choose/Preset number of clusters  $K$ ;
  2. Select  $K$  points as initial centroids, useful methods:
    - Randomly select;
    - Use Centroid of agglomerative algorithm;
    - Successively pick the farthest point from others.
  3. In each iteration of centroids:
    - (a) For all points  $i$ , calculate its distance from the  $l^{\text{th}}$  centroid  $D(i, l)$
    - (b) Classify each  $i$  point to the nearest centroid cluster;
    - (c) Re-calculate the centroid of new  $K$  clusters;
  4. Repeat until convergence. (Convergence criterion can be e.g.  $\langle \sum_i D(i \in g_l, l) \rangle \rightarrow \text{const}$ )
- 

Note: prosandcons of  $K$ -Means clustering algorithm

- Efficient:  $\sim O(n)$ ;
  - Sensitive to outliers;
  - Ineffective for non-convex shapes.
-

### 4.7.3 Gaussian Mixture Model with Expectation Maximization Algorithm

The Gaussian Mixture Model (GMM) for clustering assumes  $X$  is generated from a mixed distribution of  $K$  normal, i.e.  $X$  has probability  $\pi_l$  to be generated from corresponding normal  $N(\mu_l, \Sigma_l)$ :

$$X \sim \sum_{l=1}^K \pi_l N(\mu_l, \Sigma_l) = \sum_{l=1}^K \pi_l N(\theta_l), \quad \sum_{l=1}^K \pi_l = 1, \pi_l \geq 0. \quad (4.183)$$

Use its likelihood function  $L(\theta; x)$  and maximize posterior probability by  $\frac{\partial \ell}{\partial \theta}$ :

$$L(\{\pi_l\}, \{\theta_l\}; x) = \prod_{i=1}^N \sum_{l=1}^K \pi_l \frac{1}{(2\pi)^{p/2} |\Sigma_l|^{1/2}} \exp \left( -\frac{1}{2} (x_i - \mu_l)' \Sigma_l^{-1} (x_i - \mu_l) \right) \quad (4.184)$$

E-M Algorithm uses the ELBO maximizing method, detail see [section 5.5](#). For simplification express  $\theta \equiv \{\cup \pi_l, \cup \mu_l, \cup \Sigma_l\}$ . The maximizing function  $Q(\theta|\theta^{(t)})$  for GMM model and corresponding iteration:

$$\theta^{(t+1)} = \arg \max_{\theta} Q(\theta|\theta^{(t)}) = \arg \max_{\theta} \sum_{i=1}^N \sum_{l=1}^K \gamma_{il}^{(t)} \log \pi_l \phi(x_i | \mu_l, \Sigma_l), \quad \gamma_{il}^{(t)} \equiv \frac{\pi_l^{(t)} \phi(x_i | \mu_l^{(t)}, \Sigma_l^{(t)})}{\sum_{j=1}^K \pi_j^{(t)} \phi(x_i | \mu_j^{(t)}, \Sigma_j^{(t)})} \quad (4.185)$$

Lagrange Multiplier: Extreme value  $\arg \max_{\theta} Q(\theta|\theta^{(t)})$  with constraint  $\sum_{l=1}^K \pi_l = 1$  requires

$$\frac{\partial Q(\theta|\theta^{(t)})}{\partial \mu_l} = 0 \quad \frac{\partial Q(\theta|\theta^{(t)})}{\partial \Sigma_l^{-1}} = 0 \quad \frac{\partial Q(\theta|\theta^{(t)}) + \lambda(\sum_{j=1}^K \pi_j - 1)}{\partial \pi_j} = 0, \quad \forall l = 1, 2, \dots, K \quad (4.186)$$

Result:

$$\begin{cases} \mu_l^{(t+1)} = \frac{\sum_{i=1}^N \gamma_{il}^{(t)} x_i}{\sum_{i=1}^N \gamma_{il}^{(t)}} \\ \Sigma_l^{(t+1)} = \frac{\sum_{i=1}^N \gamma_{il}^{(t)} (x_i - \mu_l)(x_i - \mu_l)'}{\sum_{i=1}^N \gamma_{il}^{(t)}} \\ \pi_l^{(t+1)} = \frac{1}{N} \sum_{i=1}^N \gamma_{il}^{(t)} \end{cases} \quad (4.187)$$

$$\gamma_{il}^{(t)} \equiv \frac{\pi_l^{(t)} \phi(x_i | \mu_l^{(t)}, \Sigma_l^{(t)})}{\sum_{j=1}^K \pi_j^{(t)} \phi(x_i | \mu_j^{(t)}, \Sigma_j^{(t)})} \quad (4.188)$$

where  $\gamma_{il}$  is the posterior probability that the  $i^{\text{th}}$  object belongs to the  $l^{\text{th}}$  group.

The above constraint equations are difficult to solve, use iteration algorithm:

---

**Algorithm** EM-Algorithm for Gaussian Mixture Model

---

1. Use e.g.  $K$ -means method to set an initial estimation as  $(\hat{\mu}_l^{(0)}, \hat{\Sigma}_l^{(0)})$ ,  $\hat{\pi}_l^{(0)} = 1/K$ ;
  2. Repeat Expectation & Maximization:
-

(a)  $E_{\text{expectation}}$ -Step: Compute posterior of latent variable on each point;

$$\hat{\gamma}_{il}^{(t)} = \frac{\pi_l^{(t)} \phi(x_i | \mu_l^{(t)}, \Sigma_l^{(t)})}{\sum_{j=1}^K \pi_j^{(t)} \phi(x_i | \mu_j^{(t)}, \Sigma_j^{(t)})}, \quad 1 \leq i \leq N, 1 \leq l \leq K \quad (4.189)$$

(b)  $M_{\text{aximize}}$ -Step: Re-calculate parameters  $\{\mu_l, \Sigma_l, \pi_l\}$  by [equation. 4.187](#).

3. Repeat until convergence.

Note: EM method for Gaussian Mixture Model is a greedy algorithm  $\rightarrow$  local maximum.

#### 4.7.4 DBSCAN & OPTICS Density Clustering Algorithm

**DBSCAN** algorithm (Density-Based Spatial Clustering of Application with Noise) is a kind of density clustering algorithm. **OPTICS** algorithm (Ordering Point To Identify the Cluster Structure) is its improved version.

□ **DBSCAN Algorithm** Key (preset) index in DBSCAN:

- Eps  $\varepsilon$ : Radius of neighbourhood of a point;
- MinPts  $M$ : Minimum number of points to be indentified as cluster core point, usually choose  $M \geq \dim + 1$ ;
- (Also, a distance norm is needed, e.g. Euclidean  $D$ ).

Notation:

- $\varepsilon$  neighbourhood of point  $x_i$ :

$$\mathcal{N}_\varepsilon(x_i) \equiv \{y \in \mathbb{R}^n : 0 < D(y, x) < \varepsilon\} \quad (4.190)$$

- ‘Density’ (is actually an integer):

$$\rho_\varepsilon(x_i) \equiv \#x_j \in \mathcal{N}_\varepsilon(x_i) \quad (4.191)$$

- Three types of Points:  $X_c, X_{bd}, X_{noi}$ .

- Core Point: label an  $x_i$  as core point if

$$\rho_\varepsilon(x_i) \geq M \quad (4.192)$$

Denote the set of core point as  $X_c$ , and set of non-core point as  $X_{nc}$

- Border Point: label an  $x_j \in X_{nc}$  as border point if

$$\exists(x_i \in X_c) \in \mathcal{N}_\varepsilon(x_j) \& x_j \in X_{nc} \quad (4.193)$$

Denote the set of border point as  $X_{bd}$

- Noise Point: the set of noise point is

$$X_{noi} \equiv \mathbb{C}_X^{X_c \cup X_{bd}} \quad (4.194)$$

- Point Relations: DDR, DR, DC

- Directly Density Reachable: For  $x_i, x_j \in X$ , if  $x_i \in X_c$ ,  $x_j \in \mathcal{N}_\varepsilon(x_i)$ , then say  $x_j$  is DDR from  $x_i$ ;
- Density Reachable: For point chain  $x_{i_1}, x_{i_2}, \dots, x_{i_m}$ ,  $m \geq 2$ . If  $x_{i_{\kappa+1}}$  is DDR from  $x_{i_\kappa}$ ,  $\forall 1 \leq \kappa \leq m-1$ , then say  $x_{i_m}$  is DR from  $x_{i_1}$ .

- Density Connected: For point  $x_{i_1}, x_{i_2}, x_{i_3}$ , if  $x_{i_2}$  and  $x_{i_3}$  are both DR from  $x_{i_1}$ , then say  $x_{i_2}$  and  $x_{i_3}$  are DC.

Note: DR is not symmetric for  $x_{i_1}$  and  $x_{i_m}$ ; while DC is.

DBSCAN algorithm classify all points that are Density Connected to each other into a cluster  $C \subset X$ , i.e.

$$\text{Maximality: } x \in C \&\& y \text{ DR from } x \Rightarrow y \in C \quad (4.195)$$

$$\text{Connectivity: } x, y \in C \Rightarrow x, y \text{ DC.} \quad (4.196)$$

Pros and cons of DBSCAN:

- Insensitive to noise;
- Based on density, with no constraint on the shape of cluster;
- Suitable for clusters with uniformly densed data, otherwise difficult to choose proper Eps  $\varepsilon$ ;
- Complexity  $\sim O(n^2)$ , at least  $O(n \log n)$ .

#### □ OPTICS Algorithm

OPTICS is based on DBSCAN and shares most of the basic concepts and ideas. Further define the following distance (preset  $\varepsilon$  and  $M$ ):

- Core Distance: For  $x_i \in X_c$ , the smallest distance allowing  $x_i$  to become core point.

$$CD(x_i) = D(x_i, N_{\varepsilon}^M(x_i)), \rho_{\varepsilon}(x_i) \geq M \quad (4.197)$$

where  $N_{\varepsilon}^M(x_i)$  is the  $M^{\text{th}}$  closest point from  $x_i$ ;

- Reachability Distance: For  $y \in X, x_i \in X_c \subset X$ ,

$$RD(y, x_i) = \max\{CD(x_i, D(y, x_i))\} \quad (4.198)$$

Or equivalently

$$RD(y, x_i) = \arg \min_{\rho_d(x_i) \geq M, y \in N_d(x_i)} d \quad (4.199)$$

Algorithm flow:

---

#### Algorithm OPTICS

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1. Construct  $X_c$  based on preset  $M, \varepsilon$ ;
  2. Pick an ‘unprocessed’ point  $x_{n_i} \in X_c$  and calculate  $RD(x_j, x_{n_i}), \forall \text{ ‘unprocessed’ } x_j \in \mathcal{N}_{\varepsilon}(x_{n_i}) \cap X_c$ . Pick the  $x_j \in X_c$  with smallest RD and label as  $x_{n_{i+1}}$  processed;
  3. Repeat step 2 until all points are processed. Output  $\{x_{n_i}\} = (x_{n_1}, x_{n_2}, \dots, x_{n_{|X_c|}})$ . Each  $x_{n_i}$  is attached with a  $CD(x_{n_i})$  and a  $r(x_{n_i}) := RD(x_{n_{i-1}}, x_{n_i})$ <sup>60</sup>.
- 

Then break the ordering sequence  $n_i$  according to  $r(x_{n_i})$ , .e.g. break  $n_i$  if  $r(x_{n_i}) \geq \tilde{\varepsilon}$

Comment: OPTICS is more stable than DBSCAN, capable of dealing with multi-density clustering.

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<sup>60</sup>For  $i = 1$ , just define as 0



## Chapter. V 统计计算与软件部分

Instructor: Zaiying Zhou

### Section 5.1 Algorithm Theory Introduction

#### 5.1.1 Finite Precision Computation

An arbitrary real number  $r \in \mathbb{R}$  is represented as (the nearest adjacent) float number  $v_r$ . A float is basically stored as (example take 32-bit float): 1 bit **Sign** + 8 bit **Exponent** + 23 bit **Mantissa**.

$$v = (-1)^S \times 2^{E-127} \times \left( 1 + \sum_{i=1}^{23} (M_i \times 2^{-i}) \right) \quad (5.1)$$

Further, extreme value of  $(M, E)$  is used for some ‘special value’: denormalized number, NaN, inf, etc.

- Denormalized number: to fill the gap  $[0, \pm 2^{-126}](E = 1)$ , for  $E = 0$  extremely small number, definition use

$$v_{\text{denormalized}} = (-1)^S \times 2^{1-127} \times \left( \textcolor{red}{0} + \sum_{i=1}^{23} (M_i \times 2^{-i}) \right) \quad (5.2)$$

i.e. for  $E = 0$ , range  $[2^{-127}, 2^{-126})_{\text{nor}} \rightarrow [0, 2^{-126})_{\text{denor}}$ .

- NaN: ( $E = 255, M \neq 0$ )
- inf: ( $E = 255, M = 0$ )

	$E = 0$	$0 < E < E_{\max}$	$E = E_{\max}$
$M = 0$	$\pm 0$	$v_{\text{normalized}}$	$\pm \infty$
$M \neq 0$	$v_{\text{denormalized}}$		NaN

表 3: Normalized Number

Use  $v_r$  to represent  $r$ : approximation  $r \sim v_r$ , the round-off error of  $r$ :

- Absolute rounding error:

$$\varepsilon = |r - v_r| \quad (5.3)$$

- Relative rounding error:

$$\varepsilon_{\text{machine}} = \frac{|r - v_r|}{|r|} = \text{const} \quad (5.4)$$

Note that for large  $|r|$ , the adjacency between floats  $|r - v_r| = |r| \varepsilon_{\text{machine}}$  might be large, even cause some integer missing.

□ **Representation and arithmetic of floating-point number follows IEEE-754 standard**

- For 32-bit float (single precision float): 1 bit **Sign** + 8 bit **Exponent** + 23 bit **Mantissa**.  $\varepsilon_{\text{machine}} = 0.5 \times 2^{-23} = 2^{-24}$

$$v = (-1)^S \times 2^{E-127} \times \left( 1 + \sum_{i=1}^{23} (M_i \times 2^{-i}) \right) \in [-3.4 \times 10^{38}, 3.4 \times 10^{38}] \quad (5.5)$$

- For 64-bit float (double precision float): 1 bit **Sign** + 11 bit **Exponent** + 52 bit **Mantissa**.  $\varepsilon_{\text{machine}} = 0.5 \times 2^{-52} = 2^{-53}$

$$v = (-1)^S \times 2^{E-1023} \times \left(1 + \sum_{i=1}^{52} (M_i \times 2^{-i})\right) \in [-1.79 \times 10^{308}, 1.79 \times 10^{308}] \quad (5.6)$$

Key point for algorithm design: avoid plus/minus of numbers of significantly large magnitude difference.

### 5.1.2 Stability & Accuracy

- Forward/Backward Error:

For a algorithm design  $\tilde{f}$  of a problem  $f$ , with input  $x$ . Denote:

- Expected output:  $y \equiv f(x)$
- Algorithm output:  $\tilde{y} \equiv \tilde{f}(x)$
- Forward Error:  $\Delta_F = \tilde{f}(x) - f(x)$
- Backward Error:  $\Delta_B = \arg \min_{f(\tilde{x})=\tilde{f}(x)} |\tilde{x} - x|$

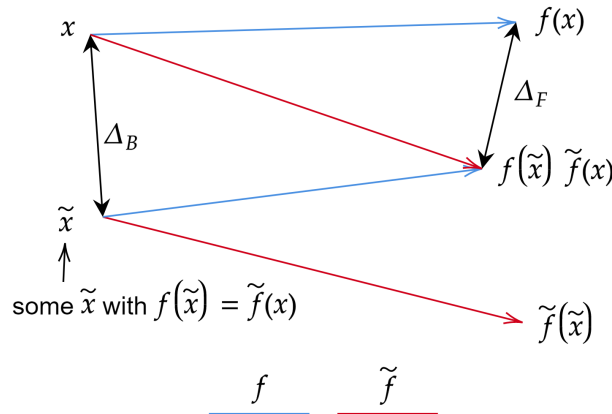


图 4: Illustration of Forward/Backward Error

- (Forward) Stability: An algorithm  $\tilde{f}$  is stable if

$$\frac{\|\tilde{f}(x) - f(\tilde{x})\|}{\|f(\tilde{x})\|} = O(\varepsilon_{\text{machine}}), \forall \frac{\|\tilde{x} - x\|}{\|x\|} = O(\varepsilon_{\text{machine}}) \quad (5.7)$$

- Condition Number of problem  $f$ :

- Absolute condition number:

$$\hat{\kappa}(x) = \lim_{\varepsilon \rightarrow 0} \sup_{\|\delta x\| < \varepsilon} \frac{\|\delta f(x)\|}{\|\delta x\|} = \left\| \frac{\partial f}{\partial x} \right\| \quad (5.8)$$

- Relative condition number:

$$\kappa(x) = \lim_{\varepsilon \rightarrow 0} \sup_{\|\delta x\| < \varepsilon} \frac{\|\delta f\|/\|f\|}{\|\delta x\|/\|x\|} \quad (5.9)$$

(Relative) Condition Number of Matrix  $A_{m \times m}$ :

–  $f(x) \equiv Ax$ :

$$\kappa = \|A\| \frac{\|x\|}{\|Ax\|} \leq \|A\| \|A^{-1}\| \quad (5.10)$$

–  $f(b) \equiv \text{solving } Ax = b$

$$\kappa = \|A^{-1}\| \frac{\|b\|}{\|x\|} \leq \|A^{-1}\| \|A\| \quad (5.11)$$

Thus for matrix  $A$ , denote

$$\kappa(A) \equiv \|A\| \|A^{-1}\| \quad (5.12)$$

– For  $\ell_2$  norm  $\|\cdot\|_2$ :  $\kappa(A) = \frac{\sigma_1}{\sigma_m}$ <sup>61</sup>

### 5.1.3 Iteration Algorithm

Iteration methods are used especially for problems without analytical solution, to obtain a numerical solution.

Iteration method: for problem  $f$  with solution  $x^*$  design an iteration function  $g: X \rightarrow X$  so that

$$\lim_{n \rightarrow \infty} g^{\{n\}}(x) = \lim_{n \rightarrow \infty} \underbrace{g(g(\dots g(g(x)) \dots))}_n = x^* \quad (5.13)$$

then get solution by setting initial input value  $x^{(0)}$  and calculate  $x^{(t+1)} = g(x^{(t)})$  repeatedly until convergence as approximate solution.

#### □ Three Steps for Iteration:

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#### Algorithm General Steps for Iteration

---

1. Starting: set  $x^{(0)}$ , more trials to initial value is recommended
2. Updating:  $x^{(t+1)} = g(x^{(t)})$ ,  $\forall t = 0, 1, 2, \dots$
3. Stopping: when to stop, can choose various stopping criterion, e.g.

- Absolute convergence criterion

$$|x^{(t+1)} - x^{(t)}| < \varepsilon \quad (5.14)$$

- Relative convergence criterion

$$\frac{|x^{(t+1)} - x^{(t)}|}{|x^{(t)}|} < \phi \quad (5.15)$$

- Relative convergence criterion (2), avoid  $x^{(t)} = 0$

$$\frac{|x^{(t+1)} - x^{(t)}|}{|x^{(t)}| + \xi} < \phi \quad (5.16)$$

---

<sup>61</sup> Knowledge about matrix norm see [section. 4.1.2](#)

### □ Convergence Order and Convergence Rate

For each iteration value  $x^{(t)}$ , define iteration error as  $\varepsilon^{(t)} \equiv x^{(t)} - x^*$ . Then an iteration method  $\lim_{t \rightarrow \infty} \varepsilon^{(t)} = 0$  has convergence order  $\alpha$  and convergence rate  $c$  as:

$$\lim_{t \rightarrow \infty} \frac{|\varepsilon^{(t+1)}|}{|\varepsilon^{(t)}|^\alpha} = c \quad (5.17)$$

A large  $\alpha$  and small  $c$  declare a quick convergence. (Large  $\alpha$  is needed more)

Comment: Actually convergence rate and order are generally dependent on specific problem, so we usually estimate  $\alpha, c$  using some approximation/scaling to represent a generally case.

## 5.1.4 Constrained Optimize Theory

### □ Primal Problem

For optimize problem in convex set  $\mathcal{X}$

$$\arg \min_{x \in \mathcal{X}} f(x) \quad (P)$$

$$s.t. \quad g_i(x) \leq 0, \quad i = 1, 2, \dots, k \quad (5.18)$$

$$h_j(x) = 0, \quad j = 1, 2, \dots, l \quad (5.19)$$

which is called the **primal problem** for optimization.

The **generalized Lagrange function** for primal problem defined as

$$\mathcal{L}(x, \kappa, \lambda) \equiv f(x) + \sum_{i=1}^k \kappa_i g_i(x) + \sum_{j=1}^l \lambda_j h_j(x) \quad (5.20)$$

$$w.r.t. \quad \kappa_i \geq 0, \quad i = 1, 2, \dots, k$$

and we could further define a function of  $x$ :

$$\theta_P(x) \equiv \max_{\kappa, \lambda: \kappa_i \geq 0} \mathcal{L}(x, \kappa, \lambda) = \begin{cases} f(x) & \text{constraint } g, h \text{ satisfied} \\ +\infty & \text{constraint unsatisfied} \end{cases} \quad (5.21)$$

which means we can give the solution value of primal problem (P) simply by minimizing  $\theta_P(x)$ , minimum denoted  $p^*$

$$p^* \equiv \min_x \theta_P(x) = \min_x \max_{\kappa, \lambda: \kappa_i \geq 0} \mathcal{L}(x, \kappa, \lambda) \quad (5.22)$$

### □ Dual Problem

Similar to primal problem, we can define a function of  $\kappa, \lambda$ :

$$\theta_D(\kappa, \lambda) \equiv \min_x \mathcal{L}(x, \kappa, \lambda) \quad (5.23)$$

and similarly get the **dual problem** of primal, value denoted  $d^*$

$$d^* \equiv \max_{\kappa, \lambda: \kappa_i \geq 0} \theta_D(\kappa, \lambda) = \max_{\kappa, \lambda: \kappa_i \geq 0} \min_x \mathcal{L}(x, \kappa, \lambda) \quad (5.24)$$

it is obvious that

$$d^* = \max_{\kappa, \lambda: \kappa \geq 0} \min_x \mathcal{L}(x, \kappa, \lambda) \leq \min_x \max_{\kappa, \lambda: \kappa_i \geq 0} \mathcal{L}(x, \kappa, \lambda) = p^* \quad (5.25)$$

#### □ Karush-Kuhn-Tucker Condition (KKT Condition)

KKT condition to allow  $d^* = p^*$  at  $(x^*, \kappa^*, \lambda^*)$ : in the case that

- $f(x)$  and  $g_i(x)$  are convex
- $h_j(x)$  in the form of affine function  $A_j x + b$
- $g_i(x)$  are feasible constraints

then KKT  $\Leftrightarrow p^* = d^* = \mathcal{L}(x^*, \kappa^*, \lambda^*)$ .

the KKT conditions are:

$$\begin{aligned} \nabla_x \mathcal{L}(x^*, \kappa^*, \lambda^*) &= 0 \\ \kappa_i^* g_i(x^*) &= 0 & i = 1, 2, \dots, k \\ g_i(x^*) &\leq 0 & i = 1, 2, \dots, k \\ \kappa_i &\geq 0 & i = 1, 2, \dots, k \\ \lambda_j(x^*) &= 0 & j = 1, 2, \dots, l \end{aligned} \quad (5.26)$$

## Section 5.2 Algebraic Problem in Statistics

Considering the data structure and algorithm implement, many fundamental problems in statistics are basically algebraic problem, e.g.

- Matrix multiplication:

$$y = Ax, \text{ solve } y \quad (5.27)$$

- Linear equation solution:

$$b = Ax = \sum_{i=1}^n x_i a_i, \text{ solve } x \quad (5.28)$$

- OLS solution:

$$\hat{\beta} = (X'X)^{-1}XY \quad (5.29)$$

Generally speaking matrix  $A$  can be constructed in an arbitrary form, so an algorithm implementation needs **matrix composition** so that we have a better form to handle.

### 5.2.1 Matrix Operation

- Inverse Matrix: Inverse matrix of  $A = [a_1, \dots, a_m]$  satisfies

$$A^{-1}A = AA^{-1} = I \quad (5.30)$$

then  $Ax = b \Leftrightarrow x = A^{-1}b$

Or generally speaking, solve inverse matrix  $A^{-1} = [\alpha_1, \dots, \alpha_m]$  is solving linear equations

$$A\alpha_i = e_i \quad (5.31)$$

In the view of column space transform,  $A$  and  $A^{-1}$  are mappings between space  $\text{span}\{e_1, \dots, e_m\}$  and  $\text{span}\{a_1, \dots, a_m\}$ , i.e.

$$\text{span}\{e_1, \dots, e_m\} \xrightleftharpoons[A^{-1}b]{Ax} \text{span}\{a_1, \dots, a_m\} \quad (5.32)$$

• Unitary Matrix: Further for unitary  $A$ , denoted as  $Q$  with  $QQ^* = I$ , is an orthonormal transformation.

- $|Q| = 1$  for rotation,  $|Q| = -1$  for reflection.
- $\lambda_Q = \pm 1$
- Geometric structure preserved, e.g. inner product and norm.

• Projection:

- Basic definition of projector  $P_X$ : idempotent matrix, project onto hyperplane  $X$

$$P_X^2 = P_X \quad (5.33)$$

- Complementary projector  $I - P_X$ : onto the complementary space of  $X$

$$(I - P)^2 = I - P \quad (5.34)$$

- Orthogonal Projection: Projector such that  $Pv \perp (I - P)v$ . Thm.:  $Pv \perp (I - P)v \Leftrightarrow P^* = P$

Derivation: Projection of vector  $v$  on hyperplane  $X$  satisfies (denoted as  $Xp$ )

$$0 = \langle Xp, Xp - v \rangle = p^* X^* (Xp - v) \Rightarrow p = (X^* X)^{-1} X^* v \Rightarrow Xp = X(X^* X)^{-1} X^* v = P_X v \quad (5.35)$$

More Properties of orthogonal projector see [section. 3.3.2](#).

- Orthogonal projector onto vector  $q$ :

$$P_q = q(q^* q)^{-1} q^* = \frac{qq^*}{\|q\|_2^2} \quad (5.36)$$

## 5.2.2 Projection and Least Square Problem

Recall: Linear model  $Y = X\beta + \varepsilon$ , basically solving linear equation  $Y = X\beta$ , however generally  $Y \notin \text{span}(X)$ , then we use OLS method to reach an estimation of  $\beta$ :

$$\hat{\beta} = \arg \min_{\beta} \|Y - X\beta\|^2 \quad (5.37)$$

where for  $\|\cdot\| = \ell_2$ -norm,  $X\hat{\beta}$  is the projection of  $X\beta$  onto hyperplane  $X$ :

$$X\hat{\beta} = X(X^* X)^{-1} X^* Y \equiv HY = P_X Y \quad (5.38)$$

For non-full rank  $A = X^* X$ : use pseudoinverse  $A^+ = (A^* A)^{-1} A^*$

□ **Task of OLS (Linear Model):** Solve  $\hat{\beta} = (X^* X)^{-1} X^* Y$ , or equivalently solve  $X^* X \hat{\beta} = X^* Y$

Note: size of matrix denoted  $X = \begin{smallmatrix} X \\ m \times n \end{smallmatrix}$

- Cholesky decomposition algorithm: computation complexity  $\sim mn^2 + \frac{n^3}{3}$

1. Use Cholesky decomposition for  $X^*X$ :

$$A^*A = R^*R \Rightarrow R^*R\hat{\beta} = X^*Y \quad (5.39)$$

2. Solve  $\xi = \arg\{R^*\xi = X^*Y\}$ :

$$R^*R\hat{\beta} = X^*Y = R^*\xi \Rightarrow R\hat{\beta} = \xi \quad (5.40)$$

3. Solve  $R\hat{\beta} = \xi$  to get  $\hat{\beta}$

- QR decomposition algorithm: computation complexity  $\sim 2mn^2 - \frac{2}{3}n^3$

1. Use e.g. Householder Reflection algorithm to compute  $X = QR$

2. use the orthonormal property of  $Q$ :

$$X^*X\hat{\beta} = X^*Y \Rightarrow R^*Q^*QR\hat{\beta} = R^*R\hat{\beta} = R^*Q^*Y \Rightarrow R\hat{\beta} = Q^*Y \quad (5.41)$$

3. Solve  $R\hat{\beta} = Q^*Y$  to get  $\hat{\beta}$

- SVD algorithm: computation complexity  $\sim 2mn^2 + 11n^3$

1. Compute SVD of  $X$ :  $X = U\Sigma V^*$

$$X^*X\hat{\beta} = X^*Y \Rightarrow V\Sigma^2V^*\hat{\beta} = V\Sigma U^*Y \Rightarrow \Sigma V^*\hat{\beta} = U^*Y \quad (5.42)$$

2. Solve  $\hat{\beta} = V\Sigma^{-1}U^*Y$  to get  $\hat{\beta}$

Algorithm comparison & trade-off: faster  $\rightsquigarrow$  less stable.

### 5.2.3 Gaussian LU Decomposition & Cholesky Decomposition

#### □ Gaussian Elimination Algorithm

Gaussian Elimination decomposes matrix  $A$  as lower triangular matrix  $\times$  upper triangular matrix

$$A_{m \times m} = L_{m \times m} U_{m \times m} = \begin{bmatrix} * & & & \\ * & * & & \\ \vdots & \vdots & \ddots & \\ * & * & \dots & * \end{bmatrix} \begin{bmatrix} * & * & \dots & * \\ & * & \dots & * \\ & & \ddots & \vdots \\ & & & * \end{bmatrix} \quad (5.43)$$

Conducted by continuously row transformation of  $A$ :

$$L_{m-1} \dots L_2 L_1 A = L^{-1} A = U \quad (5.44)$$

where each  $L_i$  corresponds to a gauss elimination operation such that  $[L_i(L_{i-1} \dots L_2 L_1 A)]_{i+1:m,i} = 0$ , with  $[L_i(L_{i-1} \dots L_2 L_1 A)]_{1:i,:}$  fixed.  $L_i$  has the form as

$$L_i = I - l_i e_i^*, \quad l_i = [0, \dots, l_{i+1,i}, \dots, l_{m,i}]^T \quad l_{j,i} = A_{ji}/A_{ii} \quad (5.45)$$

Then we have  $L = L_1^{-1} L_2^{-1} \dots L_{m-1}^{-1} U$ , with  $U = L_{m-1} \dots L_2 L_1 A$

If some pivot element  $(L_{i-1} \dots L_1 A)_{ii} = 0$ , use a row transformation  $P_i$  such that  $(P_i L_{i-1} \dots L_1 A)_{ii} \neq 0$ , thus  $LU$  decomposition is expanded as

$$L_{m-1} P_{m-1} \dots L_2 P_2 L_1 P_1 A = U \quad (5.46)$$

Good properties of  $L_i = I - l_i e_i^*$ : enable a quick algorithm implement of  $LU$  decomposition:

- Inverse of  $L_i$ :

$$L_i^{-1} = (I - l_i e_i^*)^{-1} = I + l_i e_i^* \quad (5.47)$$

- Multiplication of  $L_i^{-1}$ :

$$L_i^{-1} L_{i+1}^{-1} = (I + l_i e_i^*)(I + l_{i+1} e_{i+1}^*) = I + l_i e_i^* + l_{i+1} e_{i+1}^* \quad (5.48)$$

- Interchangeability of  $P_i$  and  $L_j$ :

$$L_{m-1} P_{m-1} \dots L_2 P_2 L_1 P_1 = (\tilde{L}_{m-1} \dots \tilde{L}_2 \tilde{L}_1) (P_{m-1} \dots P_2 P_1), \quad \tilde{L}_i = P_{m-1} \dots P_{i+1} L_i P_{i+1}^{-1} \dots P_{m-1}^{-1} \quad (5.49)$$

where note that  $P_k$  only exchange row/column  $k$  and  $\kappa > k$ , thus  $\tilde{L}_i$  is still left triangular.

Thus get expression of  $LU$  decomposition  $PA = LU$ :

$$PA = LU \quad \begin{cases} P = P_{m-1} \dots P_2 P_1 \\ L = (\tilde{L}_{m-1} \dots \tilde{L}_2 \tilde{L}_1)^{-1} \\ \tilde{L}_i = P_{m-1} \dots P_{i+1} L_i P_{i+1} \dots P_{m-1} \\ U = L_{m-1} P_m \dots L_2 P_2 L_1 P_1 A \end{cases} \quad (5.50)$$

Complexity of Gaussian Elimination:

$$\text{flops}_{\text{GE}} = \sum_{i=1}^{m-1} \sum_{k=i+1}^m 2(m-i+1) \sim \frac{2}{3} m^3 \quad (5.51)$$

## □ Cholesky Decomposition

Hermitian positive-definite matrix  $A$  can  $LU$  decompose as

$$A = LU = R^* R \quad (5.52)$$

Algorithm: write  $A$  in partitioned matrix then conduct symmetric row/column transformation

$$A = \begin{bmatrix} 1 & w_1^* \\ w_1 & K \end{bmatrix} \quad (5.53)$$

$$= \begin{bmatrix} 1 & 0 \\ w_1 & I \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & K - w_1 w_1^* \end{bmatrix} \begin{bmatrix} 1 & w_1^* \\ 0 & I \end{bmatrix} \quad (5.54)$$

$$= R_1^* K_1 R_1 \quad (5.55)$$



Note that  $K_1$  is still hermite positive-definite, we can repeat the above process

$$K_1 = \begin{bmatrix} 1 & 0 \\ 0 & K - w_1 w_1^* \end{bmatrix} \quad (5.56)$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & w_2 & I \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & K - w_1 w_1^* - w_2 w_2^* \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & w_2^* \\ 0 & 0 & I \end{bmatrix} \quad (5.57)$$

$$= R_2^* K_2 R_2 \quad (5.58)$$

repeat untill  $K_m = I$ :  $A = (R_m R_{m-1} \dots R_1)^* I (R_m R_{m-1} \dots R_1) = R^* R$

Complexity of Cholesky Decomposition:

$$\text{flops}_{\text{CD}} = \sum_{i=1}^m \sum_{k=i}^m 2(m-k+1) + 1 \sim \frac{1}{3}m^3 \quad (5.59)$$

#### 5.2.4 QR Decomposition: Gram-Schmidt/Householder/Givens Method

QR Decomposition: Orthogonal Triangularization of matrix  $A$

$$A = \begin{matrix} m \times n \\ \end{matrix} = \begin{matrix} m \times n \\ \end{matrix} Q \begin{matrix} n \times n \\ \end{matrix} R \quad (5.60)$$

$$A = \begin{bmatrix} a_1 & \dots & a_n \end{bmatrix} = QR = \begin{bmatrix} q_1 & \dots & q_n \end{bmatrix} \begin{bmatrix} r_{11} & r_{12} & \dots & r_{1n} \\ & r_{22} & \dots & r_{2n} \\ & & \ddots & \vdots \\ & & & r_{nn} \end{bmatrix} \quad (5.61)$$

Every  $A \in \mathbb{C}^{m \times n}$  ( $m \geq n$ ) has QR decomposition, specially:

- Full decomposition exists
- Reduced decomposition with  $r_{ii} > 0$  is unique.

Here introduce 3 kinds of algorithm:

- **Gram-Schmidt Orthogonalization**:  $\sim O(2mn^2)$ , sequentially orthogonalizes the columns of  $A$ , traditional way
- ★ **Householder Reflection**:  $\sim O(2mn^2 - \frac{2}{3}n^3)$ , most commonly used, stable for ill & dense matrix
- **Givens Rotation**:  $\sim O(3mn^2 - n^3)$ , used for sparse matrix, e.g. Hessenberg matrix

##### □ (Classical) Gram-Schmidt Orthogonalization

Key idea: project  $a_i$  onto  $\text{span}\{q_1, \dots, q_{i-1}\}^\perp$  as  $q_i$ , with  $q_1$  initialized as  $\hat{a}_1$ , projection coefficient  $r_{ij}$  forms  $R$ .

For each projection, the projector matrix is

$$P_i = I - \sum_{k=1}^{i-1} q_k q_k^* \quad (5.62)$$

expression of  $q_i$  and  $r_{ij}$ :

$$r_{ij} = \begin{cases} q_i^* a_j & i \neq j \\ \|a_i - \sum_{k=1}^{i-1} r_{ki} q_k\| & i = j \end{cases} \quad q_i = \frac{a_i - \sum_{k=1}^{i-1} r_{ki} q_k}{r_{ii}} = \frac{a_i - \sum_{k=1}^{i-1} q_k q_k^* a_i}{\|a_i - \sum_{k=1}^{i-1} q_k q_k^* a_i\|} \quad (5.63)$$

Note: Algorithm implementation of  $q_i$  is  $(q_{<i}) \rightarrow r_{k<i,i} \rightarrow q_i \& r_{ii} \rightarrow (q_{>i})$

#### □ Modified Gram-Schmidt Orthogonalization Algorithm

In [equation. 5.62](#), projection of G-S orthogonalization for each  $a_i$  is conducted ‘simultaneously’, while modified G-S decomposition is conducted step by step.

$$P_i = I - \sum_{k=1}^{i-1} q_k q_k^* = \prod_{k=1}^{i-1} (I - q_k q_k^*) \quad (5.64)$$

Decomposition result are the same, but modified algorithm is more stable for numerical computation, avoid problem of recursive  $q_i$ .

#### ▷ R. Code

Algorithm of CGS/MGS

```

1  GS <- function(A,MGS=FALSE){
2    stopifnot(is.matrix(A))
3    m <- dim(A)[[1]]
4    n <- dim(A)[[2]]
5    v=matrix(0,nrow = m,ncol=m)
6    r=matrix(0,nrow = m,ncol=n)
7    q=matrix(0,nrow = m,ncol=m)
8    for(j in 1:n){
9      v[,j] <- A[,j]
10     if(j>1){
11       for(i in 1:(j-1)){
12         r[i,j] <- sum(q[,i]*ifelse(MGS,v[,j],A[,j]))#对MGS取v, CGS取a
13         v[,j] <- v[,j]-r[i,j]*q[,i]
14       }
15       r[j,j] <- sqrt(sum(v[,j]^2))
16       q[,j] <- v[,j]/r[j,j]
17     }
18     return(list(q,r))
19 }

```

#### □ Householder Reflection

Key idea: Reflect  $A_{i:m,i}$  onto  $e_1 \in \mathbb{C}^{m-i+1}$  as a vector of the same length  $\|A_{i:m,i}\|e_1 \in \mathbb{C}^{m-i+1}$  (later we denote

the  $l^{\text{th}}$  unit vector  $e_l \in \mathbb{C}^{m-i+1} \equiv e_{m-i+1,l}$ , reflector  $F_i$  in  $\mathbb{C}^{(m-i+1) \times (m-i+1)}$  and auxiliary vector  $v_i$ :<sup>62</sup>

$$\mathbb{C}^{(m-i+1) \times (m-i+1)} \ni F_i = I_{m-i+1} - 2 \frac{vv^*}{\|v\|_2^2} \quad v = \text{sgn}(A_{i,i}) \|A_{i:m,i}\| e_{m-i+1,1} + A_{i:m,i} \quad (5.65)$$

where  $\text{sgn}(\cdot)$  corresponds to reflection onto  $\hat{e}$  or  $-\hat{e}$ . Reflector on  $A \in \mathbb{C}^{m \times n}$ :

$$Q_i = \begin{bmatrix} I_{i-1} & 0 \\ 0 & F_i \end{bmatrix} \quad (5.66)$$

and  $QR$  calculated by (note that  $F^2 = I_{m-i+1}$ )

$$R = Q_n \dots Q_2 Q_1 A \quad Q = Q_1 Q_2 \dots Q_n \quad (5.67)$$

Householder Reflection is more stable than Gram-Schmidt Orthogonalization

Error of Householder Reflection  $A = \tilde{Q}\tilde{R} + E$ , residual is controlled by  $\|E\| \leq \|A\|O(\varepsilon_{\text{machine}})$

Mainly caused by stability and accuracy of orthogonal matrix  $\tilde{Q}$ .

#### ▷ R. Code

R. uses Householder Reflection to conduct  $QR$  decomposition.

```
1 A.qr <- qr(A)
2 Q <- qr.Q(A.qr)
3 R <- qr.R(A.qr)
```

#### □ Givens Rotation

Key idea: use rotation

$$Rx = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} x_i \\ x_j \end{bmatrix} = \begin{bmatrix} \sqrt{x_i^2 + x_j^2} \\ 0 \end{bmatrix} \Leftrightarrow \begin{cases} \cos \theta = \frac{x_i}{\sqrt{x_i^2 + x_j^2}} \\ \sin \theta = \frac{x_j}{\sqrt{x_i^2 + x_j^2}} \end{cases} \quad (5.68)$$

act on  $A_{i-1:i,j:n}$  so that  $A_{i,j} = 0$ , each time use two rows to create 1 zero. Slow, used for special sparse matrix.

### 5.2.5 Eigenvalue Decomposition

For square matrix  $A \in \mathbb{C}^{m \times m}$ , its eigenvector is the vector  $x_i$  whose direction (subspace) is invariant under transform operator  $A$ .

$$Ax_i = \lambda_i x_i \quad (5.69)$$

Properties:

- Determinant and trace of  $A$ :

$$\det(A) = \prod_{i=1}^m \lambda_i \quad \text{tr}(A) = \sum_{i=1}^m \lambda_i \quad (5.70)$$

<sup>62</sup>Here  $\text{sgn}(\cdot)$  for reflecting toward  $-e_1/e_1$ .

- $x_i$  for special kinds of  $A$ : if  $\text{span}\{q_i\} = \mathbb{C}^m$ , then (generally  $X$  is **not** orthogonal)

$$AX = X\Lambda \Rightarrow A = X\Lambda X^{-1} \quad (5.71)$$

Further for  $AA^* = A^*A$  (Normal Matrix 规范矩阵. Include: hermitian  $A = A^*$ , skew hermitian  $A = -A^*$ , unitary  $A^{-1} = A^*$ , circulant matrix<sup>63</sup>, and such  $A + kI$ ), orthonormality of  $x_i \rightarrow q_i$ :

$$\langle q_i, q_j \rangle = \delta_{ij} \quad (5.73)$$

Eigenvalue Decomposition/Spectrum Decomposition,  $X \rightarrow Q$ :

$$AQ = Q\Lambda \Rightarrow A = Q\Lambda Q^{-1} = Q\Lambda Q^* \quad (5.74)$$

- Eigenvalue decomposition and positive definite matrix (Gershgorin circle thm.),  $\lambda_i$  falls in neighbourhood of  $a_{ii}$ :

$$D(\lambda_i, a_{ii}) < \sum_{j=1, j \neq i}^m |a_{ij}| \quad (5.75)$$

- Rayleigh quotient:

$$\max R(A, q) \equiv \max \frac{q^* A q}{q^* q} = \lambda_1 \quad (5.76)$$

Eigenvector Algorithm: Power method to find leading eigen pair.

for independent eigenvectors  $x_i$  and an arbitrary vector  $\xi = \sum_{i=1}^m c_i x_i$ :

$$A^k \xi = A^k \sum_{i=1}^m c_i x_i = \sum_{i=1}^m c_i \lambda_i^k x_i = c_1 \lambda_1^k \left[ x_1 + \sum_{i=2}^n \frac{c_i}{c_1} \left( \frac{\lambda_i}{\lambda_1} \right)^k x_i \right] \rightarrow c_1 \lambda_1^k x_1 \quad (5.77)$$

---

### Algorithm Basic Eigen Decomposition

---

1. pick a random  $q_0$
  2. compute normalized  $\frac{Aq_i}{\|Aq_i\|} = q_{i+1}$
  3. repeat until  $\|q_{i-1} - q_{i-2}\| < \varepsilon_{\text{preset}}$
  4.  $q_i$  as the eigenvector,  $q_i^T A q_i \approx q_i^T \lambda_1 q_i = \lambda_1$
- 

This algorithm requires  $|\lambda_1| > |\lambda_2| \geq \dots$  for quick convergence.

---

<sup>63</sup>Circulant Matrix, or similarly Latin Square.

$$C = \begin{bmatrix} c_0 & c_1 & c_2 & c_3 \\ c_3 & c_0 & c_1 & c_2 \\ c_2 & c_3 & c_0 & c_1 \\ c_1 & c_2 & c_3 & c_0 \end{bmatrix} \quad (5.72)$$

## 5.2.6 SVD Decomposition

### □ SVD (Singular Value Decomposition) Form:

- Reduced Form:

$$A_{m \times n} = U_{m \times n} \Sigma_{n \times n} V_{n \times n}^* \quad (5.78)$$

$$A = \begin{bmatrix} a_1 & \dots & a_n \end{bmatrix} = U \Sigma V^* = \begin{bmatrix} u_1 & \dots & u_n \end{bmatrix} \begin{bmatrix} \sigma_1 & & & \\ & \sigma_2 & & \\ & & \ddots & \\ & & & \sigma_n \end{bmatrix} \begin{bmatrix} v_1^* \\ v_2^* \\ \vdots \\ v_n^* \end{bmatrix} \quad (5.79)$$

- Full Form:

$$A_{m \times n} = U_{m \times m} \Sigma_{m \times n} V_{n \times n}^* \quad (5.80)$$

$$A = \begin{bmatrix} a_1 & \dots & a_n \end{bmatrix} = U \Sigma V^* = \begin{bmatrix} u_1 & u_2 & \dots & u_m \end{bmatrix} \begin{bmatrix} \sigma_1 & & & & \\ & \sigma_2 & & & \\ & & \ddots & & \\ & & & \sigma_n & \\ 0 & 0 & 0 & 0 & \\ \vdots & \vdots & \vdots & \vdots & \end{bmatrix} \begin{bmatrix} v_1^* \\ v_2^* \\ \vdots \\ v_n^* \end{bmatrix} \quad (5.81)$$

Existence and uniqueness of SVD:

- Every  $A \in \mathbb{C}^{m \times n}$  has SVD with  $\{\sigma_i\}$  unique

$$A = U \Sigma V^* = \sum_{i=1}^n \sigma_i u_i v_i^* \quad (5.82)$$

- if  $A$  is squared, then  $U, V$  determined
- if  $A \in \mathbb{R}^{m \times n}$ , then  $U, V \in \mathbb{R}$

### □ SVD Expression

$U, V$  are eigenvectors of  $AA^*, A^*A$  respectively

$$A^*A = V \Sigma^2 V^* \quad AA^* = U \Sigma^2 U^* \quad u_j = \frac{A v_j}{\sigma_j} \quad \sigma_j = \sqrt{\lambda_{A^*A}} = \sqrt{\lambda_{AA^*}} \quad (5.83)$$

### □ Properties of SVD:

- rank of  $A$ :  $r = \text{rk}\left(\begin{smallmatrix} A \\ m \times n \end{smallmatrix}\right) = \# \text{ non-zero } \sigma_i$

- Space of  $A$ :

$$\mathcal{R}(A) = \text{span}\{u_1, \dots, u_r\} \quad \mathcal{C}(A) = \text{span}\{v_1, \dots, v_r\} \quad \mathcal{N}(A) = \text{span}\{v_{r+1}, \dots, v_n\} \quad (5.84)$$

- Norm:

- Euclidean Norm:  $\|A\|_2 = \sigma_1$
- Frobenius Norm:  $\|A\|_F = \sqrt{\sum_{i=1}^r \sigma_i^2}$
- Nuclear Norm:  $\|A\|_* = \sum_{i=1}^r \sigma_i$

- Square matrix:

- if  $A = A^*$ , then  $\sigma_j = |\lambda_A|_j$
- $\det(A) = \prod_{i=1}^m \sigma_i$

- Low-rank Approximation of  $A$  using SVD:

$$A_k = \sum_{i=1}^k \sigma_i u_i v_i^* = A - \sum_{j=k+1}^r \sigma_j u_j v_j^* \quad (5.85)$$

is the ‘nearest’ rank  $k$  matrix from  $A$

$$\min_{\text{rk}(\Xi)=k} \|A - \Xi\|_2 = \|A - A_k\|_2 = \sigma_{k+1} \quad (5.86)$$

□ When  $A$  is positive definite, SVD and ED get the same result.

$$A = Q\Lambda Q^* \Rightarrow A = Q\text{sgn}(\Lambda)|\Lambda|Q^* = U\Sigma Q^* = U\Sigma V^* \quad (5.87)$$

## 5.2.7 Schur Decomposition

Unitary Triangularization of matrix  $A$  (always exists in  $\mathbb{C}^{m \times m}$ ):

$$A = QTQ^*, \quad Q \text{ unitary, } T \text{ upper-triangular} \quad (5.88)$$

for  $A \in \mathbb{R}^{m \times m}$ :  $T$  is quasi-triangular,  $\text{diag}$  of  $T$  is  $\text{Re}(\lambda_i)$

## Section 5.3 Numeric Optimization Algorithm I

Algorithm Optimization in Statistics: e.g.

- MLE Maximization
- Linear/Logistics Regression: Minimizing error
- Clustering: minimizing within-cluster distance & maximizing between-cluster distance
- Box-Cox  $\lambda$  determining
- Machine Learning Model training, minimizing loss function

### □ Duality of Optimization and Rooting:

- Optimization: e.g. minimizing function  $g(x)$ :

$$\arg \min g(x) \Leftrightarrow \arg \{\nabla g(x) = 0\} \quad (5.89)$$

- Rooting: extract root  $f(x) = 0$ :

$$\arg \{f(x) = 0\} \Leftrightarrow \arg \min f(x)^T f(x) \quad (5.90)$$

More specific example: expand function to 2<sup>nd</sup> as  $g(x) \approx \frac{1}{2}x^T A x - bx + c$  (differentiation of quadric  $x^T A x$  see [section. 4.1.2](#))

$$\arg \min \frac{1}{2}x^T A x - bx + c \Leftrightarrow \arg \left\{ \frac{A + A^T}{2} x = b \right\} \quad (5.91)$$

△ i.e. for optimizing task  $\arg \min g(x)$ , we can either minimizing  $g(x)$ , or rooting  $f(x) \equiv \nabla g(x)$

### □ Algorithm Design Aim:

- Robustness: can be applied on various problems
- Accuracy: reach solution with great precision, at the same time insensitive to machine error
- Efficiency: computer time/storage not required

### □ Iteration in Optimization Problem

Usually iteration is used in optimizing problem, by approximate solution  $x^*$  step by step.

- Bracketing method means the solution  $x^*$  is always within some iteration interval  $I^{(t)} = [x_{\text{left}}, x^*, x_{\text{right}}]$ , use convergence condition  $m(I^{(t)}) < \varepsilon$  to obtain solution.
- Open method: Not necessarily  $x^* \in I^{(t)}$ , but convergence using  $d(x^{(t)}, x^{(t-1)}) < \varepsilon$ . Usually faster than bracketing, but less stable, and sensitive to initial value.
- Hybrid Method: Mixture of bracketing and open according to iteration step feature

### □ Content

- **Golden Section & Fibonacci Section Search:** Bracketing method direct search for minimizer;
- **Bisection Search:** Bracketing method direct search for root
- **Interpolation Method:** Include either bracketing/open method, approximate function to obtain root/minimizer
  - **Regula Falsi:** Bracketing linear interpolation for rooting
  - **Secant Interpolation:** Open linear interpolation for rooting
  - **Parabolic Interpolation:** Open parabolic interpolation for minimizing
  - **Inverse Parabolic Interpolation (IQI):** Open interpolation for rooting
- **Hybrid Method:** Combination of bracketing method and open interpolation method for rooting. Include Dekker's and Brent's, most used method

- **Dekker's Method**: Hybrid of bisection and secant interpolation method for rooting
- ★ **Brent's Method**: Hybrid of bisection, secant interpolation and IQI for rooting
- **Fixed Point Iteration Method**: Open method for rooting, including univariate and multivariate linear case.
  - Univariate Fixed Point Iteration
  - **Jacobi Method**
  - **Gauss Seidel Method**
  - **Successive Over-Relaxation Method**
- ★ **Nelder-Mead Method/Simplex Method**: Open method for minimizer based on simplex iteration

□ **Default Methods in R.**

- `optim(VEC_OF_INI_VAR, FUN)`: Nelder-Mead Simplex search method, use `method=c('Nelder-Mead', 'BFGS', 'L-BFGS-B', 'CG', 'SANN', 'Brent')` to choose different methods
- `uniroot(FUN, INTERVAL)`: Brent's Method;
- `optimize(FUN, INTERVAL)`: Golden Section+Parabolic Interpolation.

### 5.3.1 Golden Section/Fibonacci Section Search

Problem: minimizing univariate function  $g(x)$ , within a pre-estimated interval  $[x_1^{(0)}, x_4^{(0)}]$ . For  $f$  that is undifferentiable/complicated to compute, this method is often used.

Basic idea: within a unimodal interval  $I^{(0)} = [x_1^{(0)}, x_4^{(0)}]$  of  $f(x)$ , pick two symmetric points  $x_2^{(0)}, x_3^{(0)}$  in  $I_0$  so that

$$x_2^{(0)} - x_1^{(0)} = x_4^{(0)} - x_3^{(0)} = (1 - r^{(0)})(x_4^{(0)} - x_1^{(0)}) \quad r^{(t)} > 1/2 \quad (5.92)$$

then extreme point should falls in one of  $[x_1^{(t)}, x_3^{(t)}]$  or  $[x_2^{(t)}, x_4^{(t)}]$ , iteration the interval by comparing  $g(x_2)$  and  $g(x_3)$ : use one of them as the next interval. And for less computation, we hope that one of  $g(x_2^{(t)})$  or  $g(x_3^{(t)})$  can be used in step  $t + 1$  as  $g(x_3^{(t+1)})$  or  $g(x_2^{(t+1)})$ , i.e.

$$\text{if } g(x_2^{(t)}) > g(x_3^{(t)}) : [x_1^{(t+1)}, x_2^{(t+1)}, x_4^{(t+1)}] := [x_2^{(t)}, x_3^{(t)}, x_4^{(t)}] \quad (5.93)$$

$$\text{if } g(x_2^{(t)}) \leq g(x_3^{(t)}) : [x_1^{(t+1)}, x_3^{(t+1)}, x_4^{(t+1)}] := [x_1^{(t)}, x_2^{(t)}, x_3^{(t)}] \quad (5.94)$$

also use **equation. 5.92**, we have (here use  $g(x_2^{(t)}) > g(x_3^{(t)})$  case for derivation)

$$r^{(t+1)} = \frac{x_4^{(t)} - x_2^{(t)}}{x_4^{(t)} - x_1^{(t)}} = \frac{x_4^{(t+1)} - x_2^{(t+1)}}{x_4^{(t+1)} - x_1^{(t+1)}} = \frac{x_4^{(t)} - x_3^{(t)}}{x_4^{(t)} - x_2^{(t)}} = \frac{1 - r^{(t)}}{r^{(t)}} \quad (5.95)$$

---

**Algorithm** *Golden Section/Fibonacci Section Search*

---

1. Initialize  $I^{(0)} = [x_1^{(0)}, x_4^{(0)}]$  with  $x^* \in I^0$
  2. For each step  $x^{(t)}$ :
    - (a) Calculate  $r^{(t)}$ , and then  $g(x_2^{(t)}), g(x_3^{(t)})$
-



(b) compare  $g(x_2^{(t)})$  and  $g(x_3^{(t)})$ , and update interval

$$\text{if } g(x_2^{(t)}) > g(x_3^{(t)}) : [x_1^{(t+1)}, x_2^{(t+1)}, x_4^{(t+1)}] \equiv [x_2^{(t)}, x_3^{(t)}, x_4^{(t)}] \quad (5.96)$$

$$\text{if } g(x_2^{(t)}) \leq g(x_3^{(t)}) : [x_1^{(t+1)}, x_3^{(t+1)}, x_4^{(t+1)}] \equiv [x_1^{(t)}, x_2^{(t)}, x_3^{(t)}] \quad (5.97)$$

3. Repeat until convergence  $m(I^{(t)}) < \varepsilon$

Choice of  $r^{(t)}$ : for algorithm robustness and avoid ill sequence, we will usually use some special  $r^{(t)}$ :

- Golden Section Search: use  $r^{(t)} = r = \text{const}$ , such  $r$  should satisfies

$$r = \frac{1-r}{r} \Rightarrow r = \frac{\sqrt{5}-1}{2} = \frac{1}{\phi} \approx 0.618 \quad (5.98)$$

Convergence at

$$m(I^{(t)}) = r^t m(I^{(0)}) < \varepsilon \quad (5.99)$$

- Fibonacci Section Search: choose for  $t = 0$  as  $r^{(0)} = \frac{F_{n-1}}{F_n}$ , where  $\{F_n\}$  is Fibonacci sequence, then

$$r^{(0)} = \frac{F_{n-1}}{F_n} \quad (5.100)$$

$$r^{(1)} = \frac{1-r^{(0)}}{r^{(0)}} = \frac{F_{n-2}}{F_{n-1}} \quad (5.101)$$

$$r^{(2)} = \frac{1-r^{(1)}}{r^{(1)}} = \frac{F_{n-3}}{F_{n-2}} \quad (5.102)$$

$$\vdots \quad (5.103)$$

$$r^{(t)} = \frac{F_{n-t-1}}{F_{n-t}} \quad (5.104)$$

$$\vdots \quad (5.105)$$

$$r^{(n-3)} = \frac{F_2}{F_3} = \frac{1}{2} \text{ (the last step of iteration)} \quad (5.106)$$

To determine the preset  $n$ , first use convergence condition

$$m(I^{(n-2)}) = \prod_{i=0}^{n-3} r^{(i)} m(I^{(0)}) = \frac{F_2}{F_n} m(I^{(0)}) < \varepsilon \Rightarrow \begin{cases} F_n > \frac{m(I^{(0)})}{\varepsilon} \\ F_{n-1} < \frac{m(I^{(0)})}{\varepsilon} \end{cases} \quad (5.107)$$

then conduct iteration, using  $r^{(t)} = \frac{F_{n-t-1}}{F_{n-t}}$ .

Basically the two methods have similar background, noticing that the eigen equation of Fibonacci sequence is  $x^2 = x + 1$ , and  $\lim_{n \rightarrow \infty} \frac{F_{n-1}}{F_n} = \frac{\sqrt{5}-1}{2} = \frac{1}{\phi}$ <sup>64</sup>

<sup>64</sup>General Formula of Fibonacci sequence:

$$F_n = \frac{1}{\sqrt{5}} \left( (\phi)^n - (-\frac{1}{\phi})^n \right) \quad (5.108)$$

Can be proven: Golden section need one more iteration call than Fibonacci section:

$$n_{\text{GS}} = n_{\text{Fib}} + 1 \quad (5.109)$$

Convergence order  $\alpha = 1$ , rate  $c = \frac{1}{\phi}$

### 5.3.2 Bisection Search Method

Problem: rooting univariate function  $f(x)$ , with a pre-estimated interval  $I^{(0)} = [x_1^{(0)}, x_2^{(0)}]$ , with  $f(x_1^{(0)})f(x_2^{(0)}) < 0$

Idea: Intermediate value thm.: for continuous  $f : [a, b] \rightarrow \mathbb{R}$ ,  $f(a)f(b) < 0 \Rightarrow \exists x^*, s.t. f(x^*) = 0$ .

---

**Algorithm** *Bisection Search*

---

1. Initialize  $I^{(0)} = [x_1^{(0)}, x_2^{(0)}]$  satisfying  $f(x_1^{(0)})f(x_2^{(0)}) < 0$

2. In each iteration  $x^{(t)}$ :

(a) compute midpoint function value

$$x_m^{(t)} = \frac{1}{2} (x_1^{(t)} + x_2^{(t)}) \quad (5.110)$$

(b) update interval according sign of  $f(x_m^{(t)})$ :

$$I^{(t+1)} = [x_1^{(t+1)}, x_2^{(t+1)}] := \begin{cases} [x_1^{(t)}, x_m^{(t)}], & f(x_1^{(t)})f(x_m^{(t)}) < 0 \\ [x_m^{(t)}, x_2^{(t)}], & f(x_m^{(t)})f(x_2^{(t)}) < 0 \end{cases} \quad (5.111)$$

3. Repeat until convergence  $m(I^{(t)}) < \varepsilon$

---

Convergence order  $\alpha = 1$ , rate  $c = \frac{1}{2}$ .

### 5.3.3 Interpolation Methods: Linear/Quadratic/Lagrange Interpolation

Interpolation is an approximation to function, thus can get approximation to solution. Interpolation can be used for both minimizing or root finding.

□ **Regula Falsi/Linear Interpolation (Bracketing) for Root Finding:**

Idea of regula falsi linear interpolation: at root  $x^*$  of  $f(x)$ :

$$f(x) \approx \left. \frac{df}{dx} \right|_{x^*} (x - x^*) \quad (5.112)$$

Iterate by repeatedly constructing linear interpolation/secant and use the root as an approximation to  $x^*$ .

---

**Algorithm** *Regula Falsi Interpolation*

---

1. Initialize interval  $I^{(0)} = [x_1^{(0)}, x_2^{(0)}]$  with  $f(x_1^{(0)})f(x_2^{(0)}) < 0$

2. In each iteration  $x^{(t)}$ :

---

- (a) Compute linear interpolation of  $(x_1^{(t)}, f(x_1^{(t)}))$ ,  $(x_2^{(t)}, f(x_2^{(t)}))$ , and compute the root of the straight line

$$x_r^{(t)} = \frac{x_1^{(t)} f(x_2^{(t)}) - x_2^{(t)} f(x_1^{(t)})}{f(x_2^{(t)}) - f(x_1^{(t)})} \quad (5.113)$$

compute  $f(x_r^{(t)})$

- (b) update interval according to sign of  $f(x_r^{(t)})$ :

$$I^{(t+1)} = [x_1^{(t+1)}, x_2^{(t+1)}] := \begin{cases} [x_1^{(t)}, x_r^{(t)}], & f(x_1^{(t)})f(x_r^{(t)}) < 0 \\ [x_r^{(t)}, x_2^{(t)}], & f(x_r^{(t)})f(x_2^{(t)}) < 0 \end{cases} \quad (5.114)$$

3. Repeat until convergence  $m(I^{(t)}) < \varepsilon$

Note: for enough steps of iteration  $t_\xi$ , the iteration would be short enough such that  $\text{sgn}(f''(x)) = \text{const}$ , in which case one of  $x_1^{(t)}$  or  $x_2^{(t)}$  would remain fixed for  $t > t_\xi$ .<sup>65</sup>

Convergence order  $\alpha = 1$ , rate  $c = -\frac{f''(x^*)}{2f'(x^*)}(x^* - x_{\text{fixed}})$ . Note that sign dependency of  $x_{\text{fixed}}$  on  $f''(x)$  and  $f'(x)$  ensures  $c > 0$ .

#### □ Secant Interpolation/Linear Interpolation (Open) for Root Finding

Instead of limiting  $x^* \in [x_1^{(t)}, x_2^{(t)}]$  (bracketing) by ensuring  $f(x_1^{(t)})f(x_2^{(t)}) < 0$ , we can also remove the restrict, i.e. just use the latest two points to construct secant.

#### Algorithm Secant Interpolation

1. Initialize two points  $x^{(-1)}, x^{(0)}$ . (interval not necessarily include potential root  $\hat{x}^*$ )
2. In each iteration  $x^{(t)}$ :
  - (a) Compute linear interpolation of  $(x^{(t-1)}, f(x^{(t-1)}))$ ,  $(x^{(t)}, f(x^{(t)}))$
  - (b) Use the root to update  $x^{(t+1)}$ :

$$x^{(t+1)} = \frac{x^{(t-1)} f(x^{(t)}) - x^{(t)} f(x^{(t-1)})}{f(x^{(t)}) - f(x^{(t-1)})} \quad (5.115)$$

3. Repeat until convergence, .e.g.  $|x^{(t)} - x^{(t-1)}| < \varepsilon$

Comment: For interval small enough such that  $\text{sgn}(f''(x)) = \text{const}$  and  $f(x^{(t)})f(x^{(t-1)}) < 0$ , this method might goes back to bracketing linear interpolation.

Convergence order  $\alpha \approx 1.618$ .

#### □ Parabolic Interpolation for Minimizing

Idea of parabolic interpolation: at extreme point  $x^*$ , function  $f$  has taylor series

$$g(x) \approx g(x^*) + \frac{1}{2} \frac{d^2 g}{dx^2} \Big|_{x^*} (x - x^*)^2 \quad (5.116)$$

we can iteration by repeatedly construct parabola to approximate  $f(x^*) + \frac{1}{2} \frac{d^2 f}{dx^2} \Big|_{x^*} (x - x^*)^2$  and use the extreme point of the parabola.

<sup>65</sup>For  $f''(x)f'(x) > 0$ ,  $x_2$  fixed;  $f''(x)f'(x) < 0$ ,  $x_1$  fixed.

**Algorithm** *Parabolic Interpolation*

1. First initialize three point  $(x_1^{(0)}, x_2^{(0)}, x_3^{(0)})$ ,
2. In each iteration  $x^{(t)}$ :
  - (a) Use  $(x_1^{(t)}, x_2^{(t)}, x_3^{(t)})$  to compute corresponding  $f(x)$ , then use quadric fitting to obtain parabola  $\Gamma^{(t)}$
  - (b) Replace  $\max\{x_1^{(t)}, x_2^{(t)}, x_3^{(t)}\}$  by extreme point of  $\Gamma^{(t)}$  to update as  $(x_1^{(t+1)}, x_2^{(t+1)}, x_3^{(t+1)})$
3. Repeat until convergence.

Convergence order  $\alpha \approx 1.3247$ .

**□ Lagrange Polynomial Interpolation**

Lagrange Polynomial is a function base set: Given  $n+1$  point  $(x_0, y_0), \dots, (x_n, y_n)$  ( $n \geq 1$ ), Lagrange polynomial:

$$\ell_i = \prod_{j=1, j \neq i}^n \frac{x - x_j}{x_i - x_j}, \quad i = 0, 1, \dots, n \quad (5.117)$$

And Lagrange interpolation function:  $L(x) = \sum_{i=0}^n y_i \ell_i$

$n = 1$  for linear interpolation,  $n = 2$  for parabolic interpolation.

**□ Inverse Parabolic Interpolation (IQI): Open interpolation for rooting**

Note that general parabola  $y = \frac{1}{2}ax^2 + bx + c$  might have 0 or 2 root simultaneously, thus use inverse quadric function  $x = \frac{1}{2}ay^2 + by + c$ , i.e. inverse quadric interpolation.

**Algorithm** *Inverse Parabolic Interpolation*

1. First initialize three point  $C^{(0)} = (x^{(-2)}, x^{(-1)}, x^{(0)})$
2. In each iteration  $x^{(t)}$ :
  - (a) Use  $C^{(t)} = (x^{(t-2)}, x_2^{(t-1)}, x_3^{(t)})$  to compute IQI function, and get root

$$s = \sum_{\text{cycle } x^{(t-2)}, x^{(t-1)}, x^{(t)}} \frac{x^{(t-2)} f(x^{(t-1)}) f(x^{(t)})}{[f(x^{(t-2)}) - f(x^{(t-1)})] [f(x^{(t-2)}) - f(x^{(t)})]} \quad (5.118)$$

- (b) Update points

$$C^{(t+1)} = (x^{(t-1)}, x^{(t)}, x^{(t+1)}) = (x^{(t-1)}, x^{(t)}, s) \quad (5.119)$$

3. Repeat until convergence  $|x^{(t)} - x^{(t-1)}| < \varepsilon$

**5.3.4 Hybrid Method: Dekker's/Brent's****□ Dekker's Method**

Dekker's method is a hybrid of open linear interpolation and bisection, in each step, use one of interpolation/bisection according to iteration condition to achieve both quick convergence and stability.

**Algorithm** Dekker's Method

1. Initialize three point  $a^{(0)}, b^{(0)}, b^{(-1)} = a^{(0)}$ , where interval between  $a^{(0)}, b^{(0)}$  should include potential root  $\hat{x}^*$ , i.e.  $f(a^{(0)})f(b^{(0)}) < 0$
2. In each iteration  $x^{(t)}$ :
  - (a)  $a^{(t)}, b^{(t)}$  is labelled as follows: label ensure  $|f(a^{(t)})| \geq |f(b^{(t)})|$ , thus  $b^{(t)}$  is the estimate of root, while  $a^{(t)}$  is the 'contrapoint' of  $b^{(t)}$
  - (b) compute root  $s$  of linear interpolation of  $(a^{(t)}, f(a^{(t)})), (b^{(t)}, f(b^{(t)}))$ , and compare with midpoint  $m = \frac{a^{(t)} + b^{(t)}}{2}$ 

$$\tilde{b}^{(t+1)} = \begin{cases} s = \frac{a^{(t)}f(b^{(t)}) - b^{(t)}f(a^{(t)})}{f(b^{(t)}) - f(a^{(t)})}, & s \in [m, b^{(t)}] \text{ (or } [b^{(t)}, m]) \\ m = \frac{a^{(t)} + b^{(t)}}{2}, & s \notin [m, b^{(t)}] \text{ (or } [b^{(t)}, m]) \end{cases} \quad (5.120)$$
  - (c) Then update  $\tilde{a}^{(t+1)}$  as one of  $a^{(t)}$  and  $b^{(t)}$ , such that  $f(\tilde{a}^{(t+1)})f(\tilde{b}^{(t)}) < 0$ , then relabel  $\tilde{a}^{(t+1)}, \tilde{b}^{(t)}$  to  $a^{(t+1)}, b^{(t+1)}$  according to  $|f(a^{(t+1)})| > |f(b^{(t+1)})|$
3. Repeat until convergence  $|b^{(t)} - b^{(t-1)}| < \varepsilon$

Comment: In step 3, the choice between bisection and open interpolation take advantage of quick convergence of open method, also ensure stability by using bisection for ill secant root  $s$ . However for interval small enough, this method might also goes back to bracketing linear interpolation, then  $b^{(t)}$  convergence very slow.

**□ Brent's Method**

Brent's Method is an improvement of Dekker's Method:

- Avoid convergence problem of  $b^{(t)}$  in the case of bracketing linear interpolation by checking  $|b^{(t)} - b^{(t-1)}| > \delta$  before linear interpolation, otherwise use bisection
- Further adding IQI interpolation if  $a^{(t)}, b^{(t)}, b^{(t-1)}$  are distinct for quicker convergence, root for IQI:

$$s' = \sum_{\text{cycle } a^{(t)}, b^{(t)}, b^{(t-1)}} \frac{a^{(t)}f(b^{(t)})f(b^{(t-1)})}{[f(a^{(t)}) - f(b^{(t)})][f(a^{(t)}) - f(b^{(t-1)})]} \quad (5.121)$$

**▷ R. Code**

```
1 uniroot()
```

**5.3.5 Fixed Point Iteration: Univariate**

Idea: Contraction mapping thm.: for function  $f : X \rightarrow X$  satisfying

$$d(f(x), f(y)) \leq \beta d(x, y), \beta < 1 \quad (5.122)$$

then such  $f$  has a unique fixed point  $x^*$  such that  $f(x^*) = x^*$ , and convergence is ensured:

$$d(f^{\{n\}}(x), x^*) \leq \frac{\beta^n}{1 - \beta} d(f(x), x^*) \quad (5.123)$$

For univariate function, requires  $|f'(x)| < 1$  (at least at  $x$  near  $x^*$ )

To minimize  $f(x)$ , i.e. find root of  $f'(x) = g(x)$ , i.e. find fixed point of  $G(x) \equiv \alpha f'(x) + x = x$ , requires  $|G'(x)| = |\alpha f''(x) + 1| < 1$ .

Note: We can also use inverse function of  $\alpha f'(x) + x$ , and further use  $G_1(x) = rG(x) + (1-r)x$  to find fixed point.

Iteration: use  $\hat{x}^* = x^{(n)} = G^{\{n\}}(x) = \underbrace{G(G(\dots G(G(x)) \dots))}_n$ , until  $|x^{(n)} - x^{(n-1)}| < \varepsilon$

Basically, fixed point iteration is the same as parallel chord method: use the root of  $y - g(x^{(t)}) = -\frac{1}{\alpha}(x - x^{(t)})$  as  $x^{(t+1)}$ .

Convergence order is  $\alpha$  in  $G(x) = \alpha f'(x)_x$

### 5.3.6 Fixed Point Iteration: Multivariate Linear

For solution of  $Ax = b$  using fixed point iteration, where  $AA^* = A^*A$  (normal matrix), requires:

$$\rho(A) = \max |\lambda| < 1 \quad (5.124)$$

- Jacobi Method: Decompose  $A = D + E$ , where  $D$  is diagonal part

$$A_{n \times n} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix} = D + E = \begin{bmatrix} a_{11} & 0 & \dots & 0 \\ 0 & a_{22} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_{nn} \end{bmatrix} + \begin{bmatrix} 0 & a_{12} & \dots & a_{1n} \\ a_{21} & 0 & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & 0 \end{bmatrix} \quad (5.125)$$

Then fixed point iteration: using  $(D + E)x = b \Rightarrow x^{(t+1)} = D^{-1}(b - Ex^{(t)})$

$$x_i^{(t+1)} = \frac{1}{a_{ii}} \left( b_i - \sum_{j \neq i} a_{ij} x_j^{(t)} \right) \quad (5.126)$$

- Gauss-Seidel Method: Decompose  $A = L + U$

$$A_{n \times n} = L + U = \begin{bmatrix} a_{11} & 0 & \dots & 0 \\ a_{21} & a_{22} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix} + \begin{bmatrix} 0 & a_{12} & \dots & a_{1n} \\ 0 & 0 & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{bmatrix} \quad (5.127)$$

Then fixed point iteration: using  $(L + U)x = b \Rightarrow Lx^{(t+1)} = (b - Ux^{(t)})$ , iteration:

$$x_i^{(t+1)} = \frac{1}{a_{ii}} \left( b_i - \sum_{j=1}^{i-1} a_{ij} x_j^{(t+1)} - \sum_{j=i+1}^n a_{ij} x_j^{(t)} \right) \quad (5.128)$$

- Successive Over-Relaxation Method (SOR Method): Decompose  $A = D + L + U$

$$A_{n \times n} = D + L + U = \begin{bmatrix} a_{11} & 0 & \dots & 0 \\ 0 & a_{22} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_{nn} \end{bmatrix} + \begin{bmatrix} 0 & 0 & \dots & 0 \\ a_{21} & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & 0 \end{bmatrix} + \begin{bmatrix} 0 & a_{12} & \dots & a_{1n} \\ 0 & 0 & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{bmatrix} \quad (5.129)$$

Then fixed point iteration: using  $\omega(D + L + U)x = \omega b \Rightarrow (D + \omega L)x = \omega b - [\omega U + (\omega - 1)D]x$ , move non-diagonal elements to R.H.S.

$$x_i^{(t+1)} = (1 - \omega)x_i^{(t)} + \frac{\omega}{a_{ii}} \left( b_i - \sum_{j=1}^{i-1} a_{ij}x_j^{(t+1)} - \sum_{j=i+1}^n a_{ij}x_j^{(t)} \right), \quad \omega \in (0, 2) \quad (5.130)$$

Comment: SOR iteration step is the  $\omega$  weighted average of  $x^{(t)}$  and Gauss-Seidel iteration.

### 5.3.7 Nelder-Mead Method

For multivariate function  $g(x)$ , with  $x \in \mathbb{R}^n$ , usually use Nelder-Mead Method, or Simplex Search Method. Simplex is a generalization of triangle/tetrahedron to any arbitrary dimension, and Nelder-Mead method is conducted by iterating simplex.

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#### Algorithm Nelder-Mead Method

---

1. First initialize simplex  $C^{(0)}$  by preset  $p_0$  and  $\vec{\lambda}$ :

$$C^{(0)} = \{x_0^{(0)}, x_1^{(0)}, \dots, x_n^{(0)}\}, x_0^{(0)} = p_0, x_i^{(0)} = p_0 + \lambda_i \hat{e}_i \quad (5.131)$$

2. In each iteration  $x^{(t)}$ :

- (a) First sort  $\{x_i^{(t)}\}$  according to  $g(x_i^{(t)})$  as

$$g(x_{(0)}^{(t)}) \leq g(x_{(1)}^{(t)}) \leq \dots \leq g(x_{(n)}^{(t)}) \quad (5.132)$$

- (b) Compute centroid of  $\mathcal{C}_{C^{(t)}}^{x_{(n)}^{(t)}} = \{x_{(0)}^{(t)}, x_{(1)}^{(t)}, \dots, x_{(n-1)}^{(t)}\}$

$$x_g^{(t)} = \frac{1}{n} \sum_{i=0}^{n-1} x_{(i)}^{(t)} \quad (5.133)$$

And compute the reflection point of  $x_{(n)}^{(t)}$ :

$$x_r^{(t)} := x_g^{(t)} + (x_g^{(t)} - x_{(n)}^{(t)}) \quad (5.134)$$

- (c) Compute  $g_{(0)}^{(t)} = g(x_{(0)}^{(t)})$ ,  $g_{(n-1)}^{(t)} = g(x_{(n-1)}^{(t)})$ ,  $g_{(n)}^{(t)} = g(x_{(n)}^{(t)})$ ,  $g_r^{(t)} = g(x_r^{(t)})$  and compare:

- $g_r^{(t)} < g_{(0)}^{(t)}$ : reflection point  $x_r^{(t)}$  is a good trial for minimizing, further try a farther point

$$x_{2r}^{(t)} := x_g^{(t)} + 2(x_g^{(t)} - x_{(n)}^{(t)}) \quad (5.135)$$

then iteration according to  $g_{2r}^{(t)}$ :

$$C^{(t+1)} = \{x_0^{(t+1)}, x_1^{(t+1)}, \dots, x_n^{(t+1)}\} \equiv \begin{cases} \{x_{(0)}^{(t)}, x_{(1)}^{(t)}, \dots, x_{(n-1)}^{(t)}, x_{2r}^{(t)}\}, & g_{2r}^{(t)} < g_r^{(t)} \\ \{x_{(0)}^{(t)}, x_{(1)}^{(t)}, \dots, x_{(n-1)}^{(t)}, x_r^{(t)}\}, & g_{2r}^{(t)} \geq g_r^{(t)} \end{cases} \quad (5.136)$$

- $g_{(0)}^{(t)} \leq g_r^{(t)} < g_{(n-1)}^{(t)}$ : better simplex but not necessarily the best, just use

$$C^{(t+1)} = \{x_0^{(t+1)}, x_1^{(t+1)}, \dots, x_n^{(t+1)}\} \equiv \{x_{(0)}^{(t)}, x_{(1)}^{(t)}, \dots, x_{(n-1)}^{(t)}, x_r^{(t)}\} \quad (5.137)$$


---

- $g_{(n-1)}^{(t)} \leq g_r^{(t)} : x_r^{(t)}$  might not optimize the simplex, conduct shrinkage:

$$x_s^{(t)} := \begin{cases} x_g^{(t)} + 0.5(x_g^{(t)} - x_{(n)}^{(t)}), & g_{(n-1)}^{(t)} \leq g_r^{(t)} < g_{(n)}^{(t)} \\ x_g^{(t)} - 0.5(x_g^{(t)} - x_{(n)}^{(t)}), & g_{(n)}^{(t)} \leq g_r^{(t)} \end{cases} \quad (5.138)$$

if  $g_s^{(t)} \leq g_{(n)}^{(t)}$ , suggesting a successful shrinkage, use  $g_s^{(t)}$  for iteration

$$C^{(t+1)} = \{x_0^{(t+1)}, x_1^{(t+1)}, \dots, x_n^{(t+1)}\} \equiv \{x_{(0)}^{(t)}, x_{(1)}^{(t)}, \dots, x_{(n-1)}^{(t)}, x_s^{(t)}\}, g_s^{(t)} \leq g_{(n)}^{(t)} \quad (5.139)$$

otherwise we have to update the whole simplex:

$$x_0^{(t+1)} = x_0^{(t)}, x_i^{(t+1)} = x_0^{(t)} + \frac{1}{2}(x_i^{(t)} - x_0^{(t)}) \quad (5.140)$$

## Section 5.4 Numeric Optimization Algorithm II

To minimize some arbitrary function  $f(x)$ , the idea of gradient iteration method is to update  $x^{(t)}$  based on (minus) gradient  $-\nabla f(x)$ , with some modification on direction  $p^{(t)} = T(-\nabla f(x))$  and step length  $\alpha^{(t)}$

$$x^{(t+1)} = x^{(t)} + \alpha^{(t)} T(-\nabla f(x^{(t)})) = x^{(t)} + \alpha^{(t)} p^{(t)} \quad (5.141)$$

- Modifying Direction  $p^{(t)}$ :
  - **Gradient Descent**:  $p^{(t)} = -\nabla f(x^{(t)})$
  - **Newton-Raphson Method**: use Hessian matrix  $p^{(t)} = -[H(x^{(t)})]^{-1} \nabla f(x^{(t)})$
  - **Fisher Scoring Method**: for statistics problem, use fisher information  $I(x^{(t)}) = -E_Y(H(x^{(t)}))$ ,  $p^{(t)} = I(x^{(t)})^{-1} \nabla f(x^{(t)})$
  - **Quasi-Newton Method**: usually use secant condition to approximate Hessian  $\hat{H}^{(t)} = M^{(t)}$  or  $\hat{H}^{-1(t)} = B^{(t)}$ , with various updating SR-1/DFP/BFGS/L-BFGS/Broyden Class
  - **Steepest Descent**: general form based on various norm choice.
  - **Stochastic Gradient Descent (SGD)**: modification for large sample
  - **Conjugate Gradient Method**: Use the ‘perpendicular’ property of conjugate vector for quick updating of  $p_k$
- Modifying Step-Length / Learning Rate  $\alpha^{(t)}$ :
  - **Fixed step-length**:  $\alpha^{(t)} = \alpha$
  - **Backtracking line search**:  $\alpha^{(t)} = \frac{\alpha}{2^{n^{(t)}}}$
  - **Exact line search**:  $\alpha^{(t)} = \arg \min_{\alpha} f(x^{(t)} + \alpha p^{(t)})$
  - **Trust Region Method**: use Hessian matrix  $H(x^{(t)})$ , but restrict direction & step-length with trust region  $\|\alpha^{(t)} p^{(t)}\| \leq \Delta^{(t)}$



### 5.4.1 Gradient Descent Method

The simplest choice for  $T(\cdot)$  is identity  $p^{(t)} = -\nabla f(x^{(t)})$ , because negative gradient direction is the (local) descent direction. Iteration:

$$x^{(t+1)} = x^{(t)} - \alpha^{(t)} \nabla f(x^{(t)}) \quad (5.142)$$

Note: for such gradient method, step-length should be carefully specified, use proper fixed step-length or backtracking/exact line search.

Convergence order  $\alpha_{\text{conv}} = 1$ .

### 5.4.2 Newton-Raphson Method

Idea: For minimizing problem  $x^* = \arg \min f(x)$ <sup>66</sup>, using iteration method with an initial value  $x^{(0)}$ , we hope to find iteration step  $x^{(t+1)} - x^{(t)}$  such that  $x^{(t+1)}$  can approach  $x^*$  quickly. We can try to use the Taylor series at  $x^{(t)}$  to  $O(x^2)$  and try the minimizer of the quadric function:

$$f(x) \approx \tilde{f}_{x^{(t)}}(x) = f(x^{(t)}) + (x - x^{(t)})^T \nabla f(x)|_{x^{(t)}} + \frac{1}{2} (x - x^{(t)})^T \nabla \nabla f(x)|_{x^{(t)}} (x - x^{(t)}) \quad (5.143)$$

minimizer  $\frac{\partial \tilde{f}(x)}{\partial x} = 0$ :

$$\frac{\partial \tilde{f}}{\partial x} = \nabla \tilde{f}(x)|_{x^{(t)}} + \nabla \nabla \tilde{f}(x)|_{x^{(t)}} (x - x^{(t)}) = 0 \Rightarrow x^{(t+1)} - x^{(t)} = \left( \nabla \nabla \tilde{f}(x) \right)^{-1} \nabla \tilde{f}(x)|_{x^{(t)}} \quad (5.144)$$

Use the above solution as the iteration step:

$$x^{(t+1)} = x^{(t)} - \left[ H^{(t)} \right]^{-1} \nabla f(x^{(t)}) \quad (5.145)$$

where  $H^{(t)}$  is the Hessian matrix  $H^{(t)} \equiv \frac{\partial^2 f(x)}{\partial x \partial x^T} \Big|_{x^{(t)}}$

Convergence order  $\alpha_{\text{conv}} = 2$ .

#### □ Main difficulties of Newton-Raphson method:

- Calculation of  $H(f(x^{(t)}))^{-1}$ , a task of second derivative + matrix inverse.
- As an open method, Newton-Raphson method is unstable and sensitive to initial value: more initial trials suggested
- Positive/Negative Definition of Hessian  $\frac{\partial^2 f}{\partial x \partial x^T}$  is not guaranteed, while positive/negative definition would lead to local minimum/maximum respectively, i.e. descent not guaranteed.

### 5.4.3 Fisher's Scoring Method in MLE

For MLE optimizing problem in statistics using Newton-Raphson Method, we can use properties of log-likelihood  $l(\theta; \vec{x})$  to help overcome the difficulty of calculating  $H^{-1}$ . This method is called Fisher's Scoring Method/Iteratively Re-weighted Least Squares (IRLS).

Notation: for simplification, the following part uses  $\nabla f(x) := f'(x)$  (a vector),  $\nabla \nabla f(x) := f''(x)$  (a matrix)

<sup>66</sup>Here uses different notation from previous part to avoid confusion of  $g(x)$  as link function.

### □ MLE Maximizing $\Leftrightarrow$ minus of MLE Minimizing

MLE maximizing problem:

$$\theta^* = \arg \max l(\theta; \vec{x}) = \arg \max \ln \prod_{x_i} f(x_i; \theta) \quad (5.146)$$

Newton-Raphson iteration gives

$$\theta^{(t+1)} = \theta^{(t)} - l''(\theta^{(t)}; x)^{-1} l'(\theta^{(t)}; x) \quad (5.147)$$

Note that here  $l'(\theta)$  is Score Function (equation. 2.79), and  $l''(\theta)$  is relative to Fisher Information (equation. 2.90).<sup>67</sup>

Note that Fisher Information is the **expectation** of  $-l''(\theta)$   $:= J(\theta)$ , the idea of Fisher scoring method is the estimate  $l''(x)$  using Fisher information:

$$\theta^{(t+1)} = \theta^{(t)} - l''(\theta^{(t)}; x)^{-1} l'(\theta^{(t)}; x) \longrightarrow \theta^{(t+1)} = \theta^{(t)} + I(\theta^{(t)})^{-1} l'(\theta^{(t)}; x) \quad (5.150)$$

How does Fisher Scoring improve Newton-Raphson method?

- Note that  $l(\theta; \vec{x}) = \sum_{i=1}^n l(\theta; x_i) \Rightarrow l''(\theta; \vec{x}) = \sum_{i=1}^n l''(\theta; x_i)$ , need much more computation for large  $n$ , while Fisher Information is a reasonable ‘average’ of  $l''(\theta, x_i)$  and total Information is just the sum of each  $I_i$

$$I(\theta) = nI_1(\theta) = n\mathbb{E}_{\xi}\left(\frac{\partial^2 l(\theta; \xi)}{\partial \theta \partial \theta^T}\right) \quad (5.151)$$

- Fisher Information  $I(\theta)$  is always positive definite, thus improve stability.

### □ More Specific Case: Scaled Exponential Family $f(y; \vec{\theta}, \phi) = \exp\left(\frac{y'\theta - b(\theta)}{a(\phi)} + c(y; \phi)\right)$

where  $\theta$  is the canonical parameter, declaring location.

This form of exponential family distribution possesses some good properties (when approaching expectation and variance), and is one of the basic distribution assumption in Generalized Linear Model, which is an important MLE task. Detail about GLM and scaled exponential family see section. 3.7.

Further note that here we demand  $\theta$  as canonical parameter, which is not necessarily the parameter  $\mu$  we use. Assume  $\theta$  as function of  $\mu$  as  $\theta = g(\mu)$ .<sup>68</sup>

Properties:

- Log-likelihood:

$$l(\theta, \phi; y) = \frac{y'\theta - b(\theta)}{a(\phi)} + c(y; \phi) \quad (5.152)$$

<sup>67</sup>Detail see section. 2.2.3 & section. 2.2.4, page 39

- Score Function

$$S(\theta; \vec{x}) = \frac{\partial \ln f(\vec{x}; \theta)}{\partial \theta} = \frac{\partial l(\theta; \vec{x})}{\partial \theta} = \sum_{i=1}^n \frac{\partial \ln f(x_i; \theta)}{\partial \theta} \quad (5.148)$$

- Fisher Information

$$I(\theta) = \mathbb{E}\left[\frac{\partial \ln f(\vec{x}; \theta)}{\partial \theta} \frac{\partial \ln f(\vec{x}; \theta)}{\partial \theta^T}\right] = \mathbb{E}\left[-\frac{\partial^2 \ln f(\vec{x}; \theta)}{\partial \theta \partial \theta^T}\right] \quad (5.149)$$

<sup>68</sup>Here use notation in GLM, where  $\theta = \eta = g(\mu)$ .

- Expectation: [equation. 3.188](#)

$$\mathbb{E}(Y) = b'(\theta) \quad (5.153)$$

- Variance: [equation. 3.189](#)

$$\text{var}(Y) = a(\phi)b''(\theta) \quad (5.154)$$

- Score function:

$$S(\theta; y) = \frac{\partial l(\theta, \phi; y)}{\partial \theta} = \frac{y - b'(\theta)}{a(\phi)} \quad (5.155)$$

$$S(\mu; y) = \frac{\partial l}{\partial \theta} \frac{dg(\mu)}{d\mu} = \frac{y - b'(g(\mu))}{a(\phi)} g'(\mu) \quad (5.156)$$

- $J(\theta)$  or  $J(\mu)$ :

$$J(\theta) = -l''(\theta) = \frac{b''(\theta)}{a(\phi)} \quad (5.157)$$

$$J(\mu) = -\frac{\partial^2 l(\mu)}{\partial \mu \partial \mu^T} = \frac{\partial g}{\partial \mu} b''(g(\mu)) \frac{\partial g}{\partial \mu^T} + \left( \frac{\partial}{\partial \mu} \otimes \frac{\partial}{\partial \mu^T} g \right) (b'(g(\mu)) - y) \quad (5.158)$$

- Fisher Information:

$$I(\theta) = \mathbb{E}(J(\theta)) = \frac{b''(\theta)}{a(\phi)} \quad (5.159)$$

$$I(\mu) = \mathbb{E}(J(\mu)) = \frac{1}{a(\phi)} \frac{\partial g}{\partial \mu} b''(g(\mu)) \frac{\partial g}{\partial \mu^T} \quad (5.160)$$

#### □ Fisher Scoring and GLM: Iterative Re-weighted Least Square (IRLS)

Recall in GLM in [section. 3.7](#)

$$\mu_i \sim g^{-1}(x_i' \beta) \text{ or } g(\mu_i) \sim x_i' \beta \quad (5.161)$$

where minimizing task is

$$\hat{\beta} = \arg \max \sum_i l(\mu; x_i, y_i) = \arg \max \sum_i l(\beta; x_i, y_i) \quad (5.162)$$

where  $l(\mu; x, y)$  satisfies  $y_i \sim f(\mu_{y_i} = g(-1(x_i' \beta)))$ . Use  $\mathbb{E}(Y) = b'(\theta)$  we have

$$\mu = \mathbb{E}(Y) = g^{-1}(\eta) = g^{-1}(x' \beta) = b'(\theta) \quad (5.163)$$

Note that in GLM model we should have chosen canonical link [equation. 3.199](#) such that  $g^{-1} = b'$ , then

$$\theta = \eta = x' \beta = g(\mu) \longleftrightarrow g^{-1}(\theta) = g^{-1}(\eta) = g^{-1}(x' \beta) = \mu = E(Y) \quad (5.164)$$

i.e. we could get: (Here  $Y$  and  $X$  for sample matrix notation  $\vec{y}$  and  $\mathbf{X}$ )

$$S(\beta; Y) = \frac{\partial l(\beta)}{\partial \beta} = \frac{X^T Y - X^T g^{-1}(X\beta)}{a(\phi)} \quad (5.165)$$

$$I(\beta) = \frac{1}{a(\phi)} \frac{\partial g}{\partial \beta} b''(\theta) \frac{\partial g}{\partial \beta^T} = \frac{1}{a(\phi)} X' W X \quad (5.166)$$

$$W(\theta) := b''(\theta) = \left. \frac{\partial g^{-1}(\theta)}{\partial \theta} \right|_{\theta=X\beta} = \frac{\text{var}(Y)}{a(\phi)} \quad (5.167)$$

Then we can use above result to modify Newton-Raphson Algorithm as

$$\beta^{(t+1)} = \beta^{(t)} + I(\beta^{(t)})^{-1} S(\beta) = \beta^{(t)} + (X' W^{(t)} X)^{-1} X' (Y - g^{-1}(X\beta^{(t)})) \quad (5.168)$$

where

$$W^{(t)} = b''(\xi) \big|_{\xi=X\beta^{(t)}} \quad (5.169)$$

$$g^{-1}(\xi) = b'(\xi) \quad (5.170)$$

Further comment: iteration can be written

$$\beta^{(t+1)} = (X' W^{(t)} X)^{-1} X' W^{(t)} \left( X\beta^{(t)} + W^{-1(t)} (Y - g^{-1}(X\beta^{(t)})) \right) \quad (5.171)$$

where  $Z = X\beta^{(t)} + W^{-1}(Y - g^{-1}(X\beta^{(t)}))$  can be expressed as the taylor series of  $Z = g(Y)$  at  $\hat{Y} = g^{-1}(X\beta)$ :

$$Z = g(Y) \approx g(g^{-1}(X\beta)) + \frac{\partial g}{\partial \mu} (Y - g^{-1}(X\beta)) \quad (5.172)$$

$$= X\beta + W^{-1}(Y - g^{-1}(X\beta)) \quad (5.173)$$

i.e. each step of iteration is a weighted generalized linear regression  $Z \approx g(Y) \sim X\beta$

#### □ Useful choise of General Linear Model and MLE iteration

Note: for conciseness, the following part would use the most commonly used parameter, and canonical variable

$$\theta = \eta = x'\beta$$

Regression data:  $(y_i, x_i), i = 1, 2, \dots, n$

- Simple Linear Regression: Normal Distribution

$$Y_i \sim N(x'_i \beta, \sigma^2) \quad f(y; \mu, \sigma^2) = \exp \left\{ \frac{y\mu - \frac{1}{2}\mu^2}{\sigma^2} - \frac{y^2}{2\sigma^2} - \frac{1}{2} \ln(2\pi\sigma^2) \right\} \quad (5.174)$$

– Link function:

$$g(y) = y \Leftrightarrow g^{-1}(x'\beta) = x'\beta \quad (5.175)$$

– Canonical variable  $\theta = x'\beta = \mu$  and its function

$$b(\theta) = \frac{1}{2}\theta^2 \quad a(\sigma^2) = \sigma^2 \quad (5.176)$$

$$\mathbb{E}(Y) = b'(\theta) = \theta \quad (5.177)$$

$$\text{var}(Y) = a(\phi)b''(\theta) = \sigma^2 \quad (5.178)$$

– Log-likelihood:

$$l(\beta, \sigma^2; y, x) = \frac{yx'\beta - \frac{1}{2}\beta'xx'\beta}{\sigma^2} - \frac{y^2}{2\sigma^2} - \frac{1}{2}\ln(2\pi\sigma^2) \quad (5.179)$$

– Gauss-Raphson Iteration:

$$\frac{\partial l}{\partial \beta} = \frac{1}{\sigma^2} \sum_{i=1}^n (x_i(y_i - x_i'\beta)) \quad (5.180)$$

$$\frac{\partial l}{\partial \beta \partial \beta^T} = -\frac{1}{\sigma^2} \sum_{i=1}^n x_i x_i' \quad (5.181)$$

iteration step:

$$\beta^{(t+1)} = \beta^{(t)} + (X'X)^{-1}X'(Y - X\beta) \quad (5.182)$$

– Fisher' Scoring Iteration:

$$W(\beta) = b''(\mu) = I_{p+1} \quad (5.183)$$

$$I(\beta) = \frac{1}{a(\sigma^2)} X'W X = \frac{1}{\sigma^2} X'X \quad (5.184)$$

iteration step: the same as G-R method

$$\beta^{(t+1)} = \beta^{(t)} + (X'X)^{-1}X'(Y - X\beta) \quad (5.185)$$

• Logistic Regression: Binomial Distribution

$$Y_i \sim B(n_0, \text{logistic}(x'\beta)) \quad f(y; n_0, \pi) = \exp \left\{ y \ln \frac{\pi}{1-\pi} + n_0 \ln(1-\pi) + \ln \binom{n_0}{y} \right\} \quad (5.186)$$

– Link function:

$$g(y) = \ln \frac{y}{1-y} = \text{logit}(y) \Leftrightarrow g^{-1}(x'\beta) = \frac{1}{1+e^{-x'\beta}} = \text{logistic}(x'\beta) \quad (5.187)$$

– Canonical variable  $\theta = x'\beta = \text{logit}(\pi)$

$$b(\theta) = n_0 \ln(1-\pi) = n_0 \ln \frac{1}{1+e^\theta} \quad a(\phi) = 1 \quad (5.188)$$

$$\mathbb{E}(Y) = b'(\theta) = n_0 \frac{1}{1+e^{-\theta}} = n_0 \pi \quad (5.189)$$

$$\text{var}(Y) = a(\phi)b''(\theta) = n_0 \frac{e^{-\theta}}{(1+e^{-\theta})^2} = n_0 \pi(1-\pi) \quad (5.190)$$

– Log-likelihood:

$$l(n_0, \beta; y, x) = yx'\beta + n_0 \ln(1 - g^{-1}(x'\beta)) + \ln \binom{n_0}{y} \quad (5.191)$$

– Gauss-Raphson Iteration:

$$\frac{\partial l}{\partial \beta} = \sum_{i=1}^n x_i (y_i - n_0 \text{logistic}(x_i'\beta)) \quad (5.192)$$

$$\frac{\partial l}{\partial \beta \partial \beta^T} = \sum_{i=1}^n x_i x_i' \frac{n_0 e^{-x_i'\beta}}{(1+e^{-x_i'\beta})^2} = \sum_{i=1}^n x_i x_i' n_0 g^{-1}(x_i'\beta)(1 - g^{-1}(x_i'\beta)) \quad (5.193)$$

iteration step:

$$\beta(t+1) = \beta^{(t)} - \left( \frac{\partial l}{\partial \beta \partial \beta^T} \right)^{-1} \frac{\partial l}{\partial \beta} \bigg|_{\beta^{(t)}} \quad (5.194)$$

– Fisher's Scoring Iteration:

$$W(\beta) = \frac{\text{var}(Y)}{a(\phi)} = n_0 g^{-1}(X\beta)(1 - g^{-1}(X\beta)) \quad (5.195)$$

$$I(\beta) = X'WX = X' \text{diag}\{n_0 g^{-1}(x'_i\beta)(1 - g^{-1}(x'_i\beta))\}X \quad (5.196)$$

• Poisson Regression: Poisson Distribution

$$Y_i \sim P(e^{x'_i\beta}) \quad f(y; \lambda) = \exp\{y \ln \lambda - \lambda - \ln(y!)\} \quad (5.197)$$

– Link function:

$$g(y) = \ln y \Leftrightarrow g^{-1}(x'\beta) = e^{x'\beta} \quad (5.198)$$

– Canonical variable  $\theta = x'\beta = \ln \lambda$

$$b(\theta) = \lambda = e^\theta \quad a(\phi) = 1 \quad (5.199)$$

$$\mathbb{E}(Y) = b'(\beta) = e^\theta = \lambda \quad (5.200)$$

$$\text{var}(Y) = a(\phi)b''(\theta) = e^\theta = \lambda \quad (5.201)$$

#### 5.4.4 Linear Modification to Step Length

In minimizing methods, the key idea is usually approximate original  $g(x)$  with some  $\tilde{g}(x)$ , and the idea of restricting step length is to avoid severe deviation of  $\tilde{g}$  from  $g$ , the most direct method is to adjust step length on a given direction:

$$x^{(t+1)} = x^{(t)} + \alpha^{(t)} p^{(t)} \quad (5.202)$$

we should choose proper scale of  $\alpha^{(t)}$  adapted to the craggedness of  $g(x)$  for better convergence. In machine learning,  $\alpha^{(t)}$  also refers to learning rate.

- Fixed step-length: Fix  $\alpha^{(t)} = \alpha$  (usually  $\alpha = 1$ )
- Backtracking: Starting from e.g.  $\alpha_0^{(t)} = 1$  and calculate corresponding  $g(x^{(t)} + \alpha_i^{(t)} p^{(t)})$ , update  $\alpha_{i+1}^{(t)} = \alpha_i^{(t)}/2$  until  $g(x^{(t)} + \alpha_i^{(t)} p^{(t)}) < g(x^{(t)})$ , i.e.

$$\alpha^{(t)} = \max \frac{\alpha_0}{2^n}, \text{ s.t. } g(x^{(t)} + \alpha_i^{(t)} p^{(t)}) < g(x^{(t)}) \quad (5.203)$$

- Exact line search:

$$\alpha^{(t)} = \arg \min_{\alpha} g(x^{(t)} + \alpha p^{(t)}) \quad (5.204)$$

Special case for quadric form

Properties:

- For exact line search, contiguous direction step are perpendicular, i.e.

$$\left. \frac{\partial f(x^{(t)} + \alpha p^{(t)})}{\partial \alpha} \right|_{\alpha^{(t)}} = 0 = \nabla f^T|_{x^{(t+1)}} p^{(t)} \Rightarrow p^{(t+1)} \perp p^{(t)} \quad (5.205)$$

–  $\alpha^{(t)}$  in special case for quadric form  $f(x) = \frac{1}{2}x^T Ax - b^T x + c$ , denote ‘residual’  $r^{(t)} \equiv Ax^{(t)} - b = \nabla f(x^{(t)})$

$$\alpha^{(t)} = \arg \min f(x^{(t)} + \alpha p) = -\frac{p^T (Ax^{(t)} - b)}{p^T A p} = -\frac{p^T r^{(t)}}{p^T A p} \quad (5.206)$$

$$\text{for } p = -\nabla f^{(t)} = -r^{(t)} \quad \frac{r^{(t)T} r^{(t)}}{r^{(t)T} A r^{(t)}} \quad (5.207)$$

More general modification based on quadric form see [section. 5.4.7](#), Trust Region Method.

### 5.4.5 Quasi Newton Method

One of the main difficulty of Newton-Raphson method is calculation of Hessian  $H(x^{(t)})$  (as well as its inverse). We can use some estimation method for  $M^{(t)} \equiv \hat{H}^{(t)}$ , for equivalently for  $B^{(t)} \equiv [\hat{H}^{(t)}]^{-1}$ <sup>69</sup>

Updating:

$$x^{(t+1)} = x^{(t)} - \alpha^{(t)} [M^{(t)}]^{-1} \nabla f(x^{(t)}) = x^{(t)} - \alpha^{(t)} B^{(t)} \nabla f(x^{(t)}) \quad (5.208)$$

#### □ Discrete Newton Method

Numerical finite differential for  $M^{(t)}$ :

$$M_{ij}^{(t)} = \frac{f'_i(x^{(t)} + h_{ij}^{(t)} \hat{e}_j) - f'_i(x^{(t)})}{h_{ij}^{(t)}} \quad (5.209)$$

This basic numeric method for Hessian has heavy calculation burden, and cannot ensure positive definition of Hessian, **Not** recommended.

#### □ Quasi Newton Method: SR1, DFP, BFGS, L-BFGS, Broyden Class

Instead of ‘recalculating’  $M^{(t+1)}$  (or  $B^{(t+1)}$ ) in each step, we can ‘update’  $M^{(t+1)}$  based on known  $M^{(t)}$ ,  $x^{(t+1)}$ ,  $x^{(t)}$ ,  $\nabla f^{(t+1)}$ ,  $\nabla f^{(t)}$ . And Update of  $x^{(t+1)}$  as

$$x^{(t+2)} = x^{(t+1)} - [M^{(t+1)}]^{-1} \nabla f^{(t+1)} \quad (5.210)$$

Calculation of second derivative is avoided. Note that in  $M^{(t+1)}$ , in total  $n^2$  elements are needed, thus we usually has some basic assumptions/conditions for  $M^{(t+1)}$  which should be inherited in iteration

- Symmetry:

$$M^{(t+1)} = (M^{(t+1)})^T \Leftrightarrow B^{(t+1)} = (B^{(t+1)})^T \quad (5.211)$$

- Secant Condition/Quasi-Newton Condition:

Define

$$y^{(t)} \equiv \nabla f^{(t+1)} - \nabla f^{(t)} \quad s^{(t)} \equiv x^{(t+1)} - x^{(t)} \quad (5.212)$$

Secant condition:

$$y^{(t)} = M^{(t+1)} s^{(t)} \Leftrightarrow s^{(t)} = B^{(t+1)} y^{(t)} \quad (5.213)$$

<sup>69</sup>Notation different from lecture note. Here always use  $H$  for Hessian  $H \equiv \nabla \nabla f$

- Curvature Condition/Strong Convex Condition (on function property)

$$\langle s^{(t)}, y^{(t)} \rangle \geq \xi > 0 \quad (5.214)$$

With these two constraint, degree of freedom of  $M^{(t+1)}$  is reduced to  $\frac{n(n-1)}{2}$

In the following part in this subsection, we will usually ignore the superscript  $\cdot^{(t)}$  or use subscript  $\cdot_t$  if necessary.

- SR-1 Method/Davidon Update: Rank-1 updated

$$M_{(t+1)} = M_{(t)} + \frac{(y - M_{(t)}s)(y - M_{(t)}s)^T}{(y - M_{(t)}s)^T s} \quad (5.215)$$

$$B_{(t+1)} = B_{(t)} + \frac{(s - B_{(t)}y)(s - B_{(t)}y)^T}{(s - B_{(t)}y)^T y} \quad (5.216)$$

Note: SR-1 update cannot guaranteed the positive definition of  $M_{(t+1)}$  and  $B_{(t+1)}$ . But this method can be used together with **Trust Region method** to avoid the disadvantage.

- DFP Method & BFGS Method:

Idea: We want to pick the Hessian  $M$  nearest to  $M_{(t)}$ , with constraints above, i.e.

$$M_{(t+1)} = \arg \min_M \|M - M_{(t)}\| \quad s.t. M = M^T, y = Ms \quad (5.217)$$

where norm  $\|\cdot\|$  can take different form, each giving a corresponding quasi-Newton update. Here we take weighted frobenius norm

$$\|A\|_W = \|W^{-1/2}AW^{-1/2}\|_F \quad y = Ws \quad (5.218)$$

Note: Here we take any  $W$  with secant condition for a scale-invariant norm (because  $W$  would also looks like some ‘hessian’<sup>70</sup>)

Solution<sup>71</sup>:

---

<sup>70</sup>One of possible form of  $W$  can take

$$W = \int_0^1 \nabla \nabla f(x_{(t)} + \tau s) d\tau \quad (5.219)$$

<sup>71</sup>Solution of minimizing problem using Lagrange multiplier: Note that weighted frobenius norm is

$$\|M - M_t\|_W^2 = \text{tr} \left( W^{-1/2}(M - M_{(t)})W^{-1}(M - M_{(t)})W^{-1/2} \right) \quad (5.220)$$

with constraints  $M = M^T, y = Ms$ , given  $y = Ws, M_{(t)} = M_{(t)}^T$ . Minimizing Lagrange function taken as

$$\Xi(M, \lambda, \Lambda) = \text{tr} \left( W^{-1/2}(M - M_{(t)})W^{-1}(M - M_{(t)})W^{-1/2} \right) - 4\lambda^T(Ms - y) - 4\text{tr} \left( \Lambda(M - M^T) \right) \quad (5.221)$$

$$\arg \min \Xi(M, \lambda, \Lambda) \Rightarrow \begin{cases} \frac{\partial \Xi}{\partial M} = 2W^{-1}(H - H_{(t)})^T W^{-1} - 4\lambda s^T - 4(\Lambda^T - \Lambda) = 0 \\ \frac{\partial \Xi}{\partial \lambda} = Ms - y = 0 \\ \frac{\partial \Xi}{\partial \Lambda} = M^T - M = 0 \end{cases} \quad (5.222)$$



$$M_{(t+1)} = \left( I - \frac{ys^T}{y^T s} \right) M_{(t)} \left( I - \frac{sy^T}{y^T s} \right) + \frac{yy^T}{y^T s} \quad (5.225)$$

And inverse using Sherman-Morrison formula  $(A + u^T v)^{-1} = A^{-1} - \frac{A^{-1} u v^T A^{-1}}{1 + v^T A^{-1} u}$  (Note: tough calculation, haven't tried) is the DFP updating:

$$B_{(t+1)} = M_{(t+1)}^{-1} = B_{(t)} - \frac{B_{(t)} y y^T B_{(t)}}{y^T B_{(t)} y} + \frac{s s^T}{y^T s} \quad (\text{DFP})$$

★ Similarly, using dual minimizing problem

$$B_{(t+1)} = \arg \min_B \|B - B_{(t)}\|_{W^{-1}} \quad s.t. B = B^T, s = By \quad (5.226)$$

Solution:

$$B_{(t+1)} = \left( I - \frac{sy^T}{y^T s} \right) B_{(t)} \left( I - \frac{ys^T}{y^T s} \right) + \frac{s s^T}{y^T s} \quad (\text{BFGS})$$

Also we can inverse to get estimation of hessian in BFGS updating:

$$M_{(t+1)} = M_{(t)} - \frac{M_{(t)} s s^T M_{(t)}}{s^T M_{(t)} s} + \frac{y y^T}{y^T s} \quad (5.227)$$

Note that our final goal is to evaluate  $B_{(t+1)}$  to get step direction

$$p_{(t+1)} = -B_{(t+1)} \nabla f_{(t+1)} \quad (5.228)$$

$$B_{(t+1)} = \begin{cases} B_{(t)} + \frac{(s - B_{(t)}y)(s - B_{(t)}y)^T}{(s - B_{(t)}y)^T y} & (\text{SR1}) \\ B_{(t)} - \frac{B_{(t)} y y^T B_{(t)}}{y^T B_{(t)} y} + \frac{s s^T}{y^T s} & (\text{DFP}) \\ \left( I - \frac{sy^T}{y^T s} \right) B_{(t)} \left( I - \frac{ys^T}{y^T s} \right) + \frac{s s^T}{y^T s} & (\text{BFGS}) \end{cases} \quad (5.229)$$

Comment: DFP updating and BFGS updating can both update  $(M^{(t+1)}, B^{(t+1)})$  from  $(M^{(t)}, B^{(t)})$ , with symmetry condition and secant condition. But such updating has  $\text{dof} = \frac{n(n-1)}{2} > 0$ , thus we can have multiple choice of updating, in which DFP and BFGS get such update from minimizing weight norm. In practical terms, **BFGS is usually more suitable than DFP in general optimization problem.**

A guess from their minimizing problem: BFGS is more 'direct' by minimizing  $\|B - B^{(t)}\|_{W^{-1}}$ , without the inverse of minimizer matrix as in DFP.

#### □ More methods based on DFP and BFGS:

Solve: first eliminate  $\Lambda - \Lambda^T$  into  $\lambda s^T - s \lambda^T$ , then eliminate  $\lambda s^T$ , finally eliminate  $s \lambda^T$ , solution:

$$M_{(t+1)} = M_{(t)} + \frac{y - M_{(t)} s}{y^T s} y^T + \frac{y}{y^T s} \left( \frac{s^T M_{(t)} s y^T}{y^T s y^T s} - \frac{s^T M_{(t)}}{y^T s} \right) \quad (5.223)$$

$$= \left( I - \frac{ys^T}{y^T s} \right) M_{(t)} \left( I - \frac{sy^T}{y^T s} \right) + \frac{y y^T}{y^T s} \quad (5.224)$$

- Broyden Class: linear combination of DFP and BFGS

$$B_{(t+1)} = M_{(t+1)}^{-1} \quad M_{(t+1)} = (1 - \phi_{(t)})M_{(t+1)}^{\text{BFGS}} + \phi_{(t)}M_{(t+1)}^{\text{DFP}} \quad (5.230)$$

Set:  $\phi = 1$  for DFP,  $\phi = 0$  for BFGS,  $\phi = \frac{s^T y}{s^T y - s^T M_{(t)} s}$  for SR-1

- L-BFGS Method: For high dimension  $n = \dim(x) \gg 1$ , storage of  $M_{(t)}$  or  $B_{(t)}$  take  $\sim n^2$ , which could be unacceptable. Thus instead of storing  $B_{(t)}$ ,  $y_{(t)}$  and  $s_{(t)}$ , we can store  $y_{(t_i)}$ ,  $s_{(t_i)} \forall t_i < t$ , or at least as more  $t_i$  as possible.

### 5.4.6 Steepest Descent\*

Steepest Descent Method

### 5.4.7 Trust Region Method

Approximation quadric form  $\tilde{f}$  at iteration  $x^{(t)}$

$$\tilde{f}_{x^{(t)}}(x) = f(x^{(t)}) + (x - x^{(t)})^T \nabla f^{(t)} + \frac{1}{2} (x - x^{(t)})^T M^{(t)} (x - x^{(t)}) \quad (5.231)$$

Trust Region: within  $\|x - x^{(t)}\| \leq \Delta^{(t)}$ ,  $\tilde{f}_{x^{(t)}}$  is similar enough to  $g$ , and we minimize  $\tilde{f}_{x^{(t)}}$  within trust region.

#### □ Iteration:

- Preset parameters:

$$\text{Region Radius : } \Delta^{(t)} > 0 \quad (5.232)$$

$$\text{TR step quality measure : } \eta_\nu (= 0.9), \eta_s < \eta_\nu (= 0.1) \quad (5.233)$$

$$\text{region update : } \gamma_i \geq 1 (= 2), \gamma_d (= 0.5) \quad (5.234)$$

and approximation function (usually use quadric form)

$$\tilde{f}_{x^{(t)}}(x) \left( = f(x^{(t)}) + (x - x^{(t)})^T \nabla f^{(t)} + \frac{1}{2} (x - x^{(t)})^T M^{(t)} (x - x^{(t)}) \right) \quad (5.235)$$

- In each iteration step  $(t)$ , solve constraint minimizing problem

$$x_{\text{cm}} = \arg \min_x \tilde{f}_{x^{(t)}}(x), \quad \text{s.t. } \|x - x^{(t)}\| \leq \Delta^{(t)} \quad (5.236)$$

and the quality of reduction:  $\rho^{(t)}$ :

$$\rho^{(t)} = \frac{f^{(t)} - f(x_{\text{cm}})}{f^{(t)} - \tilde{f}(x_{\text{cm}})} \quad (5.237)$$

- Update  $x^{(t+1)}$  and  $\Delta^{(t+1)}$  based on quality  $\rho^{(t)}$

$$\begin{cases} x^{(t+1)} = x_{\text{cm}}, \Delta^{(t+1)} = \gamma_i \Delta^{(t)} & \rho^{(t)} \geq \eta_\nu \\ x^{(t+1)} = x_{\text{cm}}, \Delta^{(t+1)} = \Delta^{(t)} & \eta_s \leq \rho^{(t)} < \eta_\nu \\ x^{(t+1)} = x^{(t)}, \Delta^{(t+1)} = \gamma_d \Delta^{(t)} & \rho^{(t)} < \eta_s \end{cases} \quad (5.238)$$

### 5.4.8 Conjugate Gradient Method

Note that in Gauss-Raphson method, our iteration step was obtained by minimizing the Taylor series to  $O(x^2)$  in [equation. 5.143](#),

$$f(x) \approx \tilde{f}_{x^{(t)}}(x) = f(x^{(t)}) + (x - x^{(t)})^T \nabla f(x)|_{x^{(t)}} + \frac{1}{2}(x - x^{(t)})^T \nabla \nabla f(x)|_{x^{(t)}} (x - x^{(t)}) \quad (5.239)$$

or, as a more specific problem: get  $x^*$  by minimizing function

$$x^* = \arg \min_x f(x) = \frac{1}{2}x^T A x - b^T x + c \quad (5.240)$$

which has analytical solution  $Ax^* = b$ , and we could solve this equation using algebraic methods in [section. 5.2](#). Here Conjugate Gradient Methods uses iteration method to solve it, which can be used in Newton-Raphson/Fisher Scoring etc. to help find  $\hat{H}^{(t)} p^{(t)} = -\nabla f^{(t)}$ . Or use some modified conjugate gradient method directly on  $f(x)$ .

#### □ Conjugate vectors of $A$

Note: Here we assume  $A$  is symmetric positive definite (SPD). SPD of  $A$  allows us to define an inner product based on  $A$ :

$$\langle \xi_i, \xi_j \rangle_A = \xi_i^T A \xi_j \quad (5.241)$$

and conjugate vectors of  $A$  are vector set that are ‘orthonormal’ in the sense of  $\langle \cdot, \cdot \rangle_A$ :

$$\xi_i^T A \xi_j = \delta_{ij}, \forall \xi_i, \xi_j \in \text{CV set} \quad (5.242)$$

Further if  $A$  is full-rank and conjugate vector set has  $n$  independent vector, it can span the whole space  $\text{span}\{\xi_1, \xi_2, \dots, \xi_n\} = \mathbb{R}^n$ , thus we can expand any vector  $x - x^{(0)}$  on  $\{\xi_i\}$ :

$$x = x^{(0)} + \sum_{i=1}^n c_i \xi_i \quad (5.243)$$

and express  $f(x)$  as function of  $c_i$ , using orthonormal condition  $\xi_i^T A \xi_j = \delta_{ij}$

$$f(x) = \frac{1}{2}x^T A x - b^T x + c \quad (5.244)$$

$$= \frac{1}{2}(x^{(0)} + \sum_{i=1}^n c_i \xi_i)^T A (x^{(0)} + \sum_{i=1}^n c_i \xi_i) - b^T (x^{(0)} + \sum_{i=1}^n c_i \xi_i) + c \quad (5.245)$$

$$= \frac{1}{2} \sum_{i=1}^n c_i^2 + \sum_{i=1}^n c_i (Ax^{(0)} - b)^T \xi_i + f(x^{(0)}) \quad (5.246)$$

$$= \left( \sum_{i=1}^n \frac{1}{2} c_i^2 + c_i (Ax^{(0)} - b)^T \xi_i \right) + f(x^{(0)}) \quad (5.247)$$

i.e. we can minimize the quadric form by minimizing on each direction separately.

#### □ Conjugate Direction Construction

General procedue: Using a linear-independent vector set  $\{\nu_i\}$  and use a process similar to Gauss-Elimination to get  $\{\xi_i\}$ :

$$\xi_k = \nu_k - \sum_{i=1}^{k-1} \frac{\xi_i^T A \nu_k}{\xi_i^T A \xi_i} \xi_i \quad (5.248)$$

$$= \left( I - \sum_{i=1}^{k-1} \frac{\xi_i \xi_i^T A}{\xi_i^T A \xi_i} \right) \nu_k = \prod_{i=1}^{k-1} \left( I - \frac{\xi_i \xi_i^T A}{\xi_i^T A \xi_i} \right) \nu_k \quad (5.249)$$

Note that here we only use the condition  $\xi_i^T A \xi_j = \delta_{ij}$ , and  $\{\nu_i\}$  is arbitrary. To avoid the storage spend of  $O(n^2)$ , we could choose special way in descent such that conjugate perpendicular information of  $\xi_{i < k}$  are automatically ‘stored’ in  $\xi_k$ , and we would only need storage  $O(n)$ :

- Conjugate Gradient for Quadric Form: In each descent steps  $k$

$$x_{k+1} = x_k + \alpha_k \xi_k, \quad \xi_k = \nu_k - \sum_{i=1}^{k-1} \frac{\xi_i^T A \nu_k}{\xi_i^T A \xi_i} \xi_i \quad (5.250)$$

choose  $\alpha_k$  by **exact line search**  $\alpha_k = -\frac{r_k^T \xi_k}{\xi_k^T A \xi_k} = -\frac{\nabla f_k^T \xi_k}{\xi_k^T A \xi_k}$ ,  $r_k = Ax_k - b = \nabla f_k$ , and  $\nu_k = -\nabla f(x_k)$ <sup>72</sup>, and using the fomula:

$$\alpha_i A \xi_i = A(x_{i+1} - x_i) = r_{i+1} - r_i = \nabla f_{i+1} - \nabla f_i \quad (5.253)$$

and the orthogonality of  $\xi$ ,  $\nabla f$ ,  $\xi_k$  can be expressed as

$$\xi_k = -\nabla f(x_k) - \sum_{i=1}^{k-1} \frac{(\nabla f_{i+1} - \nabla f_i)^T \nabla f_k}{(\nabla f_{i+1} - \nabla f_i)^T \xi_i} \xi_i \quad (5.254)$$

$$= -\nabla f(x_k) + \frac{(\nabla f_k - \nabla f_{k-1})^T \nabla f_k}{\nabla f_{k-1}^T \nabla f_{k-1}} \xi_{k-1} \quad (\text{PR})$$

$$= -\nabla f(x_k) + \frac{\|\nabla f_k\|^2}{\|\nabla f_{k-1}\|^2} \xi_{k-1} \quad (\text{FR})$$

For general minimizing problem, we can either use conjugate gradient just for solving  $\hat{H}^{(t)} p^{(t)} = -\nabla f^{(t)}$  in each step  $(t)$ , or more directly use the following conjugate method directly on the general  $f(x)$ : take different  $\alpha$  and coefficient of vector as modification to the non-quadric part of  $f$

$$x_{k+1}^{(t)} = x_k^{(t)} + \alpha_k^{(t)} p_k^{(t)} \quad (5.255)$$

$$p_k^{(t)} = -\nabla f(x_k^{(t)}) + \beta_k^{(t)} p_{k-1}^{(t)} \quad (5.256)$$

where:

- General Form of Conjugate Gradient: In each sub-step  $(t)k$ , replace  $A_k^{(t)} = A^{(t)}$  by  $\nabla \nabla f(x_k^{(t)})$ , i.e.

$$\alpha_k^{(t)} = -\frac{\nabla f(x_k^{(t)})^T p_k^{(t)}}{p_k^{(t)T} \nabla \nabla f(x_k^{(t)}) p_k^{(t)}} \quad (5.257)$$

$$\beta_k^{(t)} = \frac{\nabla f(x_k^{(t)})^T \nabla \nabla f(x_k^{(t)}) p_k^{(t)}}{p_{k-1}^{(t)T} \nabla \nabla f(x_k^{(t)}) p_{k-1}^{(t)}} \quad (5.258)$$

$$k = 1, 2, \dots, n \quad (5.259)$$

<sup>72</sup>Such that  $\nu_k = -\nabla f_k \perp \text{span}\{\xi_1, \dots, \xi_{k-1}\} = \text{span}\{\nu_1, \dots, \nu_{k-1}\}$ , then.

$$\nu_k = -\nabla f_k \perp \nu_i, \forall i < k \quad (5.251)$$

$$\nu_k = -\nabla f_k \perp \xi_i, \forall i < k \quad (5.252)$$

- Fletcher-Reeves Method:

$$\alpha_k^{(t)} = \arg \min_{\alpha} f \left( x_k^{(t)} + \alpha p_k^{(t)} \right) \quad (5.260)$$

$$\beta_k^{(t)} = \frac{\|\nabla f(x_k^{(t)})\|^2}{\|\nabla f(x_{k-1}^{(t)})\|^2} \quad (5.261)$$

$$k = 1, 2, \dots, n \quad (5.262)$$

- Polak-Ribière Method:

$$\alpha_k^{(t)} = \arg \min_{\alpha} f \left( x_k^{(t)} + \alpha p^{(t)} \right) \quad (5.263)$$

$$\beta_k^{(t)} = \frac{\left( \nabla f(x_k^{(t)}) - \nabla f(x_{k-1}^{(t)}) \right)^T \nabla f(x_k^{(t)})}{\|\nabla f(x_{k-1}^{(t)})\|^2} \quad (5.264)$$

$$k = 1, 2, \dots, n \quad (5.265)$$

## Section 5.5 Expectation Maximization Algorithm

Motivation: use MLE to estimate some model parameter  $\theta$  for model  $\{x_i\}$  i.i.d.  $\sim f(x|\theta)$ . Difficulty: for complex model

Main application: Probability Generative Model, observed value  $x_i$  is generated from distribution  $f(x|z_i, \theta_i, \theta_z)$  dependent on **unobserved** random  $z \sim g(z|\theta_z)$  (usually  $z$  is discrete, denoted  $z_\nu = z_\alpha, \dots, z_\gamma$ ). Where we know the form of  $f(x, z|\theta_{z_\nu}, \theta_z)$ , but form of  $f(x|\theta_{z_\nu}, \theta)$  might be hard to solve, thus we use an iterative method to deal with the latent variable  $z$  so that we can use the known form  $f(x, z|\theta_{z_\nu}, \theta)$ .

### 5.5.1 Requisite Knowledge

- Kullback-Leibler Divergence: measures the difference of distribution  $p(x)$  from distribution  $q(x)$

$$\text{KL}(q||p) \equiv - \int q(x) \log \frac{p(x)}{q(x)} dx \quad (5.266)$$

Note: non-exchange for  $p, q$ .

- Jensen Inequality: For **concave** function  $h(x)$  and random variable  $X \sim f$

$$\mathbb{E}_f(h(X)) \leq h(\mathbb{E}_f(X)) \quad (5.267)$$

Then we have the property of non-negativity of  $\text{KL}(q||p)$ :

$$\text{KL}(q||p) \geq 0, \forall p(x), \quad = \text{ for } p(x) = q(x) \quad (5.268)$$

A brief proof see [section. 1.7](#).

### 5.5.2 Derivation

Notation:  $\theta = (\theta_{z\nu}, \theta_z)$ , sample  $X = (x_1, x_2, \dots, x_N)$ . Expectation of function of random variable  $h(Y)$  on distribution  $q(y)$  as  $E_{(q_y)}(h(Y))$ .

Target: MLE of  $l(\theta|X) \equiv \sum_{i=1}^N \log f(x_i|\theta)$ . i.e. get  $\theta^* = \arg \max_{\theta} l(\theta|X)$ .

#### □ Key Formula

But due to the intractability of  $f(x|\theta)$ , we have to expand to the full form  $f(x, z|\theta)$ , and use a mathematic trick of  $E_q(\cdot)$ , where  $q(z)$  is any arbitrary distribution of  $z$ .

$$f(x|\theta) = f(x, z|\theta)f(z|x, \theta) \Rightarrow \quad (5.269)$$

$$\Rightarrow \log f(x|\theta) = E_{q(z)}(\log f(x|\theta)) = E_{q(z)}(\log f(x, z|\theta)f(z|x, \theta)) \quad (5.270)$$

$$= \int q(z) \log f(x, z|\theta) f(z|x, \theta) dz \quad (5.271)$$

$$= \int q(z) \log \frac{f(x, z|\theta)}{q(z)} dz + \text{KL}(q||f(z|x, \theta)) \quad (5.272)$$

$$\geq \int q(z) \log \frac{f(x, z|\theta)}{q(z)} dz, \quad \forall x = x_1, x_2, \dots, x_N \quad (5.273)$$

where  $\int q(z) \log \frac{f(x, z|\theta)}{q(z)} dz$  is also called ELBO (Evidence Lower Bound) of  $\log f(x|\theta)$ . And we could similarly get the ELBO of log-likelihood:

$$l(\theta|X) = \sum_{i=1}^N \log f(x_i|\theta) \geq \sum_{i=1}^N \int_z q_i(z) \log \frac{f(x_i, z|\theta)}{q_i(z)} dz \equiv \text{ELBO}(q, \theta), \quad q = \{q_i\} \quad (5.274)$$

i.e. ELBO provides a lower bound estimate for  $l(\theta|X)$ , thus we can instead maximize  $\text{ELBO}(q, \theta)$ , using coordinate ascent is the Maximization-Maximization Algorithm.<sup>73</sup>

$$q \text{ Maximum : } q^{(t+1)} = \arg \max_{q(z)} \text{ELBO}(q, \theta^{(t)}) = p(z|x, \theta^{(t)}) \quad (5.275)$$

$$\theta \text{ Maximum : } \theta^{(t+1)} = \arg \max_{\theta} \text{ELBO}(q^{(t+1)}, \theta) \quad (5.276)$$

Further if we take can derive and use the form of  $p(z|x, \theta)$  (sometimes this posterior is also intractable), then  $\theta$  maximization step becomes

$$\theta^{(t+1)} = \arg \max_{\theta} \text{ELBO} \left( p(z|x, \theta^{(t)}), \theta \right) = \sum_{i=1}^N \int_z p(z|x_i, \theta^{(t)}) \log \frac{f(x_i, z|\theta)}{p(z|x_i, \theta^{(t)})} dz \quad (5.277)$$

$$= \arg \max_{\theta} \sum_{i=1}^N \int_z p(z|x_i, \theta^{(t)}) \log f(x_i, z|\theta) dz \equiv Q(\theta|\theta^{(t)}) \quad (5.278)$$

$$= \arg \max_{\theta} \sum_{i=1}^N \int_z p(z|x_i, \theta^{(t)}) \log f(x_i, z|\theta) dz \quad (5.279)$$

and naturally  $q$  maximization Step becomes computing  $Q(\theta|\theta^{(t)}) = \sum_{i=1}^N \int_z p(z|x_i, \theta^{(t)}) \log f(x_i, z|\theta) dz$ , i.e. the Expectation of  $f(x_i, z|\theta)$  on the posterior  $p(z|x_i, \theta^{(t)})$ , gather as Expectation-Maximization Algorithm:

<sup>73</sup>where one of the 'coordinate' is the function space  $q(z)$

**Algorithm** *Expectation-Maximization*

$$E_{\text{xpectation-Step}} : Q(\theta|\theta^{(t)}) = \sum_{i=1}^N \int_z p(z|x_i, \theta^{(t)}) \log f(x_i, z|\theta) dz = \sum_{i=1}^N E_{p(z|x_i, \theta^{(t)})} [\log f(x_i, z|\theta)] \quad (5.280)$$

$$M_{\text{aximization-Step}} : \theta^{(t+1)} = \arg \max_{\theta} Q(\theta|\theta^{(t)}) = \arg \max_{\theta} \sum_{i=1}^N \int_z p(z|x_i, \theta^{(t)}) \log f(x_i, z|\theta) dz \quad (5.281)$$

E-M Algorithm can guarantee ascent of ELBO, and finally can ensure convergence (at least to a local maximum).

An application of E-M Algorithm is Gaussian Mixture Model for Clustering, detail see [section. 4.7.3](#).

### □ Limitation and Improvement

- Note that for generative model, we used a set of latent variable  $z$ , further we need an  $\int_z dx$  in  $Q(\theta|\theta^{(t)})$ , thus E-M requires low-dimensionality of  $z$  (e.g. in GMM,  $z$  is one-dimensional).
- Slow convergence near extreme point, use acceleration improvement, e.g. Louis acceleration.
- In  $q$ -Maximization step, the form of  $q$  might be intractable (i.e.  $p(z|x, \theta)$  intractable). For such function extreme value problem, use VEM (Variational Expectation Maximization) / VBEM (Variational Bayesian Expectation Maximization)

## Section 5.6 Statistical Simulation

Statistic model inference problem can be solved using simulation, i.e. Monte-Carlo simulation. We can use the model-based random numbers to analyze model.

- Simulation is well-adapted, especially for high-dimensional problems
- Low-precision, usually  $sd \sim O(\frac{1}{\sqrt{N}})$ .
- Simulation method is also usually used for validation of model reliability.

### ▷ R. Code

Remember to set random generator seed before simulation.

```
1 set.seed(INI_NUM)
```

### 5.6.1 Random Number Generation

Motivation: In many simulation models, we need to generate sets of random number with some distribution, **however** they are not totally ‘random’ because of repeatability need.

Idea: use a ‘seed’ to generate pseudo random number, where within each seed, numbers are random. The random number sequence can be repeated by setting the same seed.

### □ Linear Congruential Method for $U(0, 1)$

Linear Congruential Method (LCM) is the most commonly used method for generating uniform distribution  $U(0, 1)$ , which is the basic for more complex distribution.

**Algorithm** *Linear Congruential for Uniform Distribution*

1. Set seed  $X_0$  and pick proper  $a, c, m$  for LCM

2. Repeat for iterative  $i$ :

(a) compute  $X_{i+1}$

$$X_{i+1} \equiv aX_i + c \pmod{m} \quad (5.282)$$

(b) Random number normalized to  $(0, 1)$

$$R_{i+1} = \frac{X_{i+1}}{m} \quad (5.283)$$

Choice of  $a, c, m$ : LCM sequence has period  $m$  thus  $m$  should large, and choice of  $a, c$  should avoid early period value, and let  $R_i$  distribute uniformly in  $(0, 1)$ . Useful choice:

	$a$	$c$	$m$
Lehmer's	23		$10^8 + 1$
RANDU	$2^{16} + 3$	1	$2^{31}$
IBM	16807		$2^{31} - 1$

### □ Improvement of LCG

Key problem is the periodically structure of generated  $X_i$ , i.e. when some  $X_i$  return to  $X_0$ , then the following  $X_{i+i} = X_i$  will repeat. Idea: modify the generation rule, e.g. use groups  $m$  of LCM  $X_{im}$  with different period  $P_m$  to generate  $R_i$ . Example: L' Ecuyer-CMRG Algorithm.

$$X_i = \left( \sum_{j=1}^m (-1)^{j+1} X_{ij} \right) \pmod{m} \quad R_i = \begin{cases} \frac{X_i}{m} & X_i > 0 \\ \frac{X_i}{m} + 1 & X_i < 0 \\ 1 - \frac{1}{m} & X_i = 0 \end{cases} \quad (5.284)$$

Note (Guess): why we want  $X_i \in (0, 1)$  rather than  $[0, 1]$ .  $(0, 1)$  is homeomorphous with  $\mathbb{R}$ , which would be convenient for generate more distribution on  $\mathbb{R}$ .

More improvement: use general form

$$X_{i+1} = g(X_i, X_{i-1}, \dots) \pmod{m} \quad (5.285)$$

where in  $g(\dots)$  use more  $X_j$ , or take different function form.

### □ Random Variate Generation

Further for any arbitrary distribution generation, which is 'variate' of uniform distribution<sup>74</sup>

Target: generate random number sequence with some distribution ( $f(x)$  or  $F(x)$  known). Denote random number sequence with  $U(0, 1)$  distribution as  $U_i$

<sup>74</sup>关于 variate 的中译, 笔者想到一个有趣的翻译是国际象棋术语“变例” variation, 原指一类开局方法的衍生分支, 这里或许可以指其他分布随机数可由均匀分布随机数衍生而来这一特点。



- Quantile Method/Inverse Transform Method: For distributions with traceable CDF  $F(x) \in (0, 1)$ .

$$X_i = F_X^{-1}(U_i) \quad (5.286)$$

□ *Proof:*

$$\mathbb{P}(x < X < x + dx) = \mathbb{P}(F(x) < U < F(x + dx)) \quad (5.287)$$

$$= \frac{\partial F(x)}{\partial x} dx = f(x) dx \quad (5.288)$$

□

- Acceptance-Rejection Method: For  $F(x)$  intractable, only  $f(x)$  known,

First decompose  $f(x)$  as

$$f(x) = \frac{p(x)g(x)}{\int p(x)g(x) dx} \quad (5.289)$$

where  $g(x)$  is some distribution that we can generate,  $p(x) \in [0, 1]$ . Then  $X_{n_k}$  sequence  $\sim f(x)$  can be generated as follows:

1. Propose a  $x_k \sim g(x)$  and a  $u_k \sim U(0, 1)$
2. Decide whether accept/reject  $x_k$  to be  $X_{n_k}$ :

$$\begin{cases} p(x_k) \geq u_k & \text{Reject} \\ p(x_k) < u_k & \text{Accept} \end{cases} \quad (5.290)$$

□ *Proof:*

$$\mathbb{P}(\text{Accept}|x) = \frac{f(x)/g(x)}{\int p(\xi)g(\xi) d\xi} \Rightarrow \mathbb{P}(\text{Accept}) = \int_x \mathbb{P}(\text{Accept}|x)g(x) dx = \frac{1}{\int p(\xi)g(\xi) d\xi} \quad (5.291)$$

Using Bayesian Rule:

$$\mathbb{P}(x_k|\text{Accept}) = \frac{\mathbb{P}(\text{Accept}|x_k)g(x_k)}{\mathbb{P}(\text{Accept})} = f(X_{n_k}) \quad (5.292)$$

□

Intuitively view: figure  $f(x)$  lies under  $\tilde{c}g(x)$  (where  $\tilde{c} = \sup p(x)$ ). If for each  $x$  we accept it with probability  $\frac{f(x)}{\tilde{c}g(x)}$ , then figure  $\tilde{c}g(x)$  is ‘cut down’ into  $f(x)$ ,  $\int p(x)g(x) dx$  acts as the normalize constant, which corresponds to ‘accept rate’ controlling generate frequency.

We should choose a proper  $g(x)$  which is similar to  $f(x)$ , so that  $p(x)$  is close to 1 and  $\int p(x)g(x) dx$  could be large and the algorithm is efficient.

Note on Acceptance-Rejection Sampling:

- Inefficient for high-dimensional problem: we have limited choices of sample-able distribution  $g(\vec{x})$ , and usually the accept rate  $\int p(x)g(x) dx$  is low.
- Thick tail of  $g(x)$  matters, to control the behaviour of  $p(x)$ .  

$$x \rightarrow \infty$$

▷ **R. Code**

Use the following command for all distributions supported in R. `stats::.` More distributions based on packages see <https://CRAN.R-project.org/view=Distributions>

```
1 ?Distributions
```

**5.6.2 Markov Chain Monte Carlo Method**

Markov Chain Monte Carlo (MCMC) aims at solving integration and simulation problems by sampling from some distribution. MCMC can deal with complex distribution in high dimensional, an example is Gibbs distribution

$$\mathbb{P}(s) = \frac{e^{-\beta E(s)}}{\sum_{\sigma} e^{-\beta E(\sigma)}}, s \in \text{phase space} \quad (5.293)$$

In this case, partition function is almost impossible to calculate, what we could obtain is just the unnormalized distribution.

□ **Markov chain**

Denote phase space  $\mathcal{X} \ni x$ . We could design such a process  $X_t$  to **transit from one state to another**, i.e. a conditional probability

$$\mathbb{P}(X_{t+1} = x | X_0 = x_0, X_1 = x_1, \dots, X_t = x_t) \quad (5.294)$$

a markov process is a memoryless one in which future only depends on the current one step, i.e.

$$\mathbb{P}(X_{t+1} = x | X_0 = x_0, X_1 = x_1, \dots, X_t = x_t) = \mathbb{P}(X_{t+1} = x | X_t = x_t) \quad (5.295)$$

for discrete version  $x \in \{1, 2, \dots\}$ , we could denote it into a discrete-time stochastic process that is time-homogeneous

$$p_{ij} := \mathbb{P}(X_{t+1} = j | X_t = i), \quad \sum_j p_{ij} = 1 \quad (5.296)$$

which could be denoted in matrix form  $P = \{p_{ij}\}$

Further,  $n$ -step transition denoted

$$p_{ij}^{(n)} := \mathbb{P}(X_{t+n} = j | X_t = i) \quad (5.297)$$

$$= \sum_k p_{ij}^{(n-1)} p_{kj} \quad (5.298)$$

$$= \sum_{k_1, k_2, \dots, k_{n-1}} p_{ik_1} p_{k_1 k_2} \cdots p_{k_{n-1} j} \quad (5.299)$$

$$= P^n \quad (5.300)$$

**Stationary Distribution** / equilibrium distribution / invariant distribution  $\pi_{\infty}$  of a markov satisfies

$$\pi_{\infty} = \pi_{\infty} P = \pi_{\infty} P^n \quad (5.301)$$

Convergence and Ergodic Thm.: An ergodic markov chain converges to a unique stationary distribution  $\pi_{\infty}$ <sup>75</sup>

$$\pi_{\infty} = \pi_0 \lim_{n \rightarrow \infty} P^n \quad (5.304)$$

<sup>75</sup>Ergodic = Irreducible + Aperiodic. Denote  $i \rightsquigarrow j$  if  $\exists n \text{ s.t. } \mathbb{P}(X_n = j | X_0 = i) \neq 0$

where  $\pi_0$  is an arbitrary initial distribution.

**Detailed Balance Condition** of stationart distribution  $\pi$ :<sup>76</sup>

$$\pi(i)p_{ij} = \pi(j)p_{ji}, \quad \forall i, j \quad (5.305)$$

is a sufficient condition for stationary distribution. Proof see [section. 12.1.2](#).

MCMC aims at designing a proper chain  $p_{ij}$ , starting from some arbitrary state  $\pi_0$ , and after some (large enough) transitions  $t$  we would expect  $\pi_{t+n} \rightarrow \pi_\infty$ ,  $n = 1, 2, \dots$

#### □ MCMC Algorithms for Unnormalized distribution

To sample from an unnormalized distribution  $\tilde{p}$ , i.e.  $p = \frac{\tilde{p}(x)}{\int \tilde{p}(\xi) d\xi}$ , but normalizer  $Z = \int \tilde{p}(\xi) d\xi$  is impossible to calculate, we could only get relative probability ratio of states.

- Metropolis-Hastings Algorithm:

---

#### Algorithm MCMC

---

1. A pre-selected conditional distribution  $q(\cdot|x)$  is used as **proposal distribution**. In each step  $t$ , a new state is proposed as

$$Y \sim q(\cdot|X_t), \text{ i.e. } \mathbb{P}(Y = y|X_t) = q(y|X_t) \quad (5.306)$$

2. Acceptance ratio  $\alpha_{Y|X_t}$  is the probability to accept the proposal as the new state

$$\alpha(Y|X_t) = \min \left\{ 1, \frac{\tilde{p}(Y)q(X_t|Y)}{\tilde{p}(X_t)q(Y|X_t)} \right\} \quad (5.307)$$

3. Increment of  $t \rightarrow t + 1$  if accept.
- 

Comment:

- Detailed balanced condition of M-H Algorithm

$$p(x)p_{xy} = p(x)q(y|x)\alpha(y|x) \quad (5.308)$$

$$= p(x)q(y|x) \min \left\{ 1, \frac{p(y)q(x|y)}{p(x)q(y|x)} \right\} \quad (5.309)$$

$$= \min \{ p(x)q(y|x), p(y)q(x|y) \} \quad (5.310)$$

$$= p(y)q(x|y) \min \left\{ \frac{p(x)q(y|x)}{p(y)q(x|y)}, 1 \right\} \quad (5.311)$$

$$= p(y)p_{yx} \quad (5.312)$$

- 
- Irreducible:

$$i \rightsquigarrow j, j \rightsquigarrow i, \quad \forall i, j \quad (5.302)$$

All states of irreducible chain have the same period  $T_i = T$ .

- Aperiodic: if one of the state is aperiodic  $T = 1$ , then all states are, where

$$\text{Period } T_i = \gcd\{n : \mathbb{P}(X_n = i|X_0 = i) > 0\} \quad (5.303)$$

---

<sup>76</sup>Detailed balance condition has a similar correspondence in Quantum Mechanics, in which  $\pi(i)$  is the state density at  $i$ , and  $p_{ij}$  is the transition probability.

- i.e.  $p_{xy} = q(y|x)\alpha(y|x)$  is the transition matrix to generate the stationary distribution as  $\pi_\infty = p(x)$
- Choice of proposal  $q(\cdot|x)$  is flexible, but should be properly chosen for higher acceptance to increase efficiency.

### 5.6.3 Numerical Integration With Simulation

Motivation: In Bayesian statistics we usually use the following expression to calculate some posterior:

$$f(z|x) = \frac{f(x|z)f(z)}{\int_z f(x|z)f(z) dz} \quad (5.313)$$

Key difficulty is calculation of the normalize integration  $\int_z f(x|z)f(z) dz = E_{f(z)}[f(x|z)]$ , where  $f(z)$  is the prior of  $z$ . Usually such integration needs numeric calculation. Statistical simulation using sampling is one of the methods.

Target: calculate integration

$$I(h) = \int_{x \in \mathcal{X}} h(x) dx \quad (5.314)$$

- Hit-and-Miss Method: if  $\mathcal{X} \otimes h(x)$  is bounded in e.g.  $[a, b] \otimes [0, M]$ . We can generate uniform distribution  $(x, y)$  in the region, and count # points under  $h(x)$ , proportion of accept denoted  $\hat{p}$ , then

$$\hat{I} = M(b - a)\hat{p} \quad (5.315)$$

such estimation is guaranteed by CLT:

$$\hat{I}_H \xrightarrow{d} N\left(I, \frac{[M(b-a)]^2 p(1-p)}{N}\right) \quad (5.316)$$

- Mean Value Method: generate uniform distribution in e.g.  $\mathcal{X} = [a, b]$ , and calculate function value at each sample item  $h(u_i)$ , estimator

$$\hat{I} = \frac{N}{a-b} \sum_{i=1}^N h(u_i), \quad w.r.t. u_i \sim U(a, b) \quad (5.317)$$

with CLT:

$$\hat{I}_M \xrightarrow{d} N\left(I, \frac{(b-a)^2 var(h(U))}{N}\right) \quad (5.318)$$

Note:  $var(\hat{I}_H) > var(\hat{I}_M)$ . Intuitively, more points are used in mean value method, thus is more precise.

Random simulation has good performance for high-dimensional case by avoiding curse of dimensionality.

#### □ Importance Sampling Estimator

Improvement of mean value estimator: Note that in mean value with uniform distribution, variance

$$var(\hat{I}_M) = \frac{(b-a)^2}{N} var(h(U)) \quad (5.319)$$

could be large if  $h(x)$  varies dramatically. To avoid the disadvantage, we could use some other distribution of  $x_i \sim p(x)$ , the integration

$$I = \int_{x \in \mathcal{X}} h(x) dx = \int_{x \in \mathcal{X}} \frac{h(x)}{p(x)} p(x) dx = \mathbb{E}_{p(x)} \left[ \frac{h(x)}{p(x)} \right] \quad (5.320)$$

Estimator use

$$\hat{I}_{g(x)} = \frac{1}{N} \sum_{i=1}^N \frac{h(x_i)}{g(x_i)}, \quad w.r.t. x_i \sim g(x) \quad (5.321)$$

Variance

$$var(\hat{I}_{g(x)}) = \frac{1}{N} var\left(\frac{h(X)}{g(X)}\right) \quad (5.322)$$

i.e. if  $\frac{h(x)}{g(x)} \approx \text{const}$ , the estimator can be more precise.

- An application of importance sampling: estimating expectation of function of r.v.  $E_{f(z)}(\phi(z))$ , where r.v. with  $f(z)$  distribution is hard to generate. We can generate another random number series  $x_i \sim q(x)$ :

$$I(\phi, h) = \int \phi(z) f(z) dz = \int \phi(x) \frac{f(x)}{q(x)} q(x) dx = \mathbb{E}_{q(x)} \left( \phi(x) \frac{f(x)}{q(x)} \right) \quad (5.323)$$

$$= \int \phi(x) W(x) q(x) dx, \quad W(x) \equiv \frac{f(x)}{q(x)} \quad (5.324)$$

Use Estimator:

$$\hat{I} = \frac{1}{N} \sum_{i=1}^N \phi(x_i) W(x_i) \quad (5.325)$$

As a worse case where we only have an unnormalized  $\tilde{f}(x)$ , with the normalize integration  $f(x) = \frac{\tilde{f}(x)}{\int \tilde{f}(\xi) d\xi} = \frac{1}{c} \tilde{f}(x)$  incomputable. Use property of weight  $\tilde{W}(x) \equiv \frac{\tilde{f}(x)}{g(x)}$ :

$$\int \tilde{W}(x) g(x) dx = \int \tilde{f}(\xi) d\xi = c \Rightarrow \hat{c} = \frac{1}{N} \sum_{i=1}^N \tilde{W}(x_i) \quad (5.326)$$

Estimator:

$$\hat{I} = \frac{\sum_{i=1}^N \phi(x_i) \tilde{W}(x_i)}{\sum_{i=1}^N \tilde{W}(x_i)}, \quad \tilde{W}(x_i) = \frac{\tilde{f}(x_i)}{g(x_i)} \quad (5.327)$$

Comment: Idea of importance sampling estimation is to put more point at where  $h(x)$  has large function value to get better fit of integration, i.e. smaller variance.

## 5.6.4 Bootstrap

In statistic inference for distribution  $x \sim f(x; \theta)$ ,  $\theta \in \Theta$ , we want to estimate some statistic  $\phi$  by estimator  $\hat{\phi}$ , including e.g. mean  $E(\hat{\phi})$ , standard error  $SE = \sqrt{var(\hat{\phi})}$ . In [section. 2.3](#) we used pivot variable method to estimate statistics: parametric method, model required. Difficlutly: strange distribution/strange statistics  $\rightarrow$  use non-parametric method, e.g. bootstrap method.

### □ Bootstrap Method

Conduct bootstrap given sample  $X = (X_1, X_2, \dots, X_N)$ ,  $X_i$  i.i.d.  $\sim f(x; \theta)$ .

1. Use sample  $X$  to estimate population distribution as  $\hat{f}(x)$ . e.g. empirical CDF.

2. Repeatedly sample from  $\hat{f}(x)$  to get  $B$  samples of size  $n$ :

$$X^{(b)} = (X_1^{(b)}, X_2^{(b)}, \dots, X_n^{(b)}), \quad b = 1, 2, \dots, B \quad (5.328)$$

3. For each sample  $X^{(b)}$  estimate a statistic  $\hat{\phi}^{(b)}$

4.  $\{\hat{\phi}^{(b)}\}$ ,  $b = 1, 2, \dots, B$  is the distribution estimation of  $\hat{\phi}$  based on  $\hat{f}(x)$ , i.e. sample of statistics. We could use this sample of statistics to estimate e.g.  $SE(\hat{\phi})$ , or get interval estimation of  $\hat{\phi}$ .

$$\hat{\phi}_{\text{boot}} = \frac{1}{B} \sum_{b=1}^B \hat{\phi}^{(b)} \quad (5.329)$$

#### □ Bias Correction

The above estimator is the unbiased estimator for  $\hat{\phi}$ . However in the sense of minimizing MSE, usually  $\tilde{\phi} \equiv \hat{\phi} - \hat{bias}(\phi)$  is a better estimator. Bias  $b = \hat{\phi} - \phi$  can be estimated as

$$\hat{b} = \hat{\phi}_{\text{boot}} - \hat{\phi} \quad (5.330)$$

where  $\hat{\phi}$  is calculated by using the original sample  $X$ . And MSE estimator is:

$$\tilde{\phi} = \hat{\phi} - \hat{b} = 2\hat{\phi} - \hat{\phi}_{\text{boot}} = 2\hat{\phi} - \frac{1}{B} \sum_{b=1}^B \hat{\phi}^{(b)} \quad (5.331)$$

Instructor: Sheng Yu

## □ Road to Data Scientist

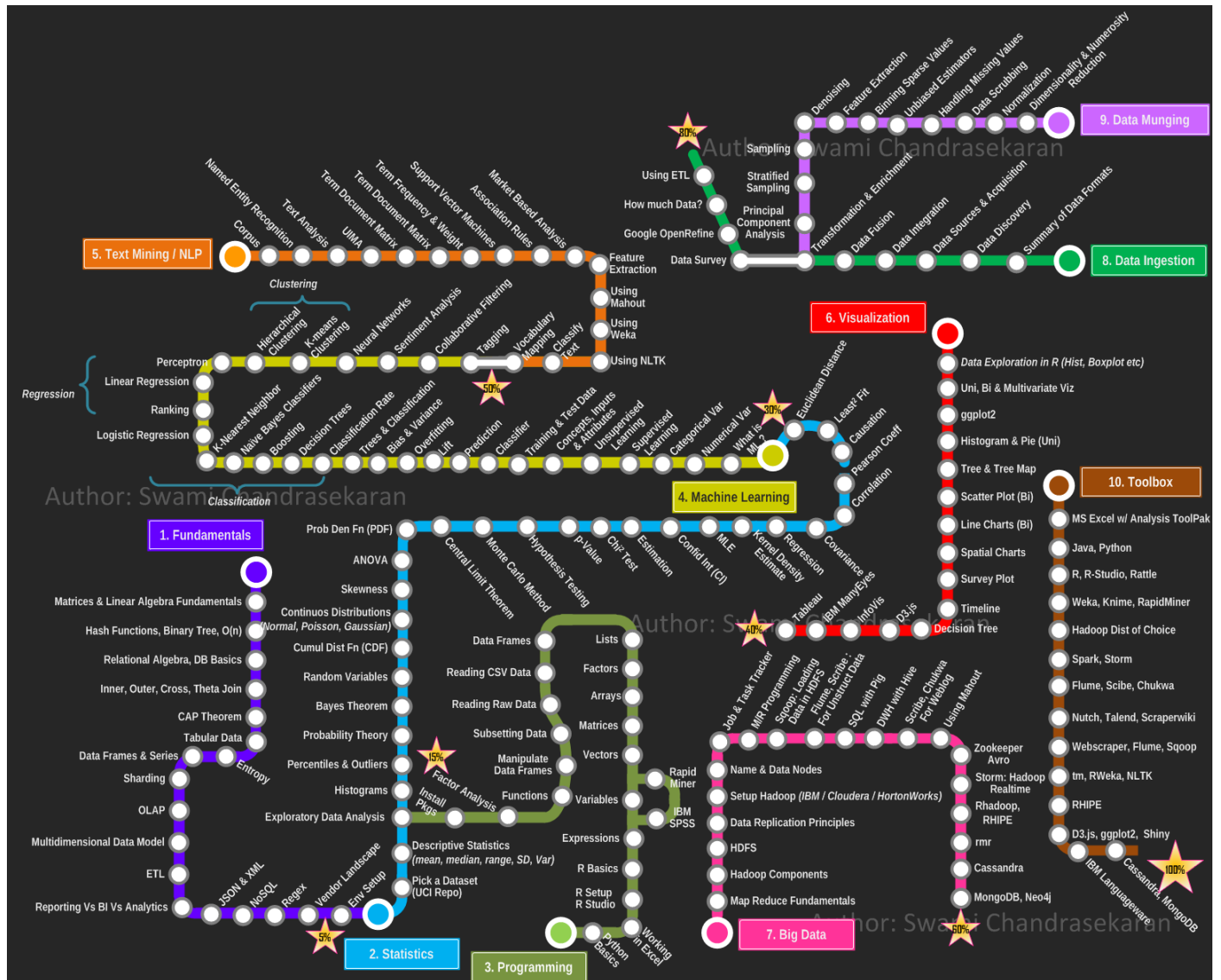


图 5: Road to Data Scientist

- Difference in programming philosophy: R for data analysis and python for data processing
- Difference in operating domain: R for statistical programming while python for general programming.

## Section 6.1 Basic R. Manipulation

### 6.1.1 Installation and Maintenance of R.

#### □ Installing and Updating

R.: update by delete old version and install new version.

- In CRAN (The Comprehensive R Archive Network): <https://cran.r-project.org>
- In Mirror@TUNA: <https://mirrors.tuna.tsinghua.edu.cn/CRAN>

RStudio: <https://www.rstudio.com>

#### □ Running R. command :

- In R. GUI;
- In R. command line terminal;
- R. CMD BATCH;
- Rscript;
  - Use > to redirect output(overwrite);
  - Use >> to append output.

#### □ R. package library: packages are collection of R. functions (as well as test data and sample code).

- `.libPaths()` show package library location<sup>77</sup> ;
- `library('PACKAGE_NAME1', 'PACKAGE_NAME2', ...)` load packages.
- `install.packages('PACKAGE_NAME1', 'PACKAGE_NAME2', ...)` install package from CRAN/mirrors;
- `installed.packages()` show all installed packages;
- `update.packages(checkBuilt = TRUE, ask = FALSE)` update installed packages;

#### □ Working directory manipulation:

- `getwd()` get current working directory;
- `setwd('TARGET_PATH')` set working directory (as an existing path).
- `dir()` show current directory.

#### □ Recommended R. Project Organization : working directory organized like

- `data/` folder for structured original dataset;
- `result/` folder for output result;
- `presentation/` folder for result representing slides/reports/etc.;
- `.r` project file  $\times n$ .

#### □ Looking for Help/Example of function:

---

<sup>77</sup>Unlike in C or python where `.` is an operator, `.` in R. is just a common character, without special meaning.

This feature can be used in naming self-defined functions: use `.FUN_NAME1` for within-project function while `FUN_NAME2` for external interface.



- ?FUN\_NAME();
- `help('FUN_NAME')`;

### 6.1.2 Data Structure and Basic Manipulation in R.

#### □ Atomic Classes

- Character: 'abc';
- Integer: 3L;
- Numeric: 2.4;
- Logical: TRUE, FALSE, T, F;
- Special types: NA, NaN, NULL, Inf

#### □ Operators

- Numerical Operators: +, -, \*(multiply by column), /, %\*(matrix multiply), ^, %%(remainder operate);
- Logical Operators: ==, etc.; & and | for common operator, && and || for comparing the first element;
- Round a numeric:
  - as.integer(), round towards 0
  - trunc()
  - ceiling()
  - floor()
  - round(NUMBER\_TO\_ROUND, digits = DIGITS)

#### □ Type Conversion

- First need to meet the need of

Key Criterion: when converting mixed type in to the same type, use the type with more compatibility.

- Logical → Numeric:

#### □ Data Structure

- **Atomic Vector** : Column vector is the **basic** data structure in R. (scalar is length=1 vector).

Only data of the same class can be held in one vector.

Initialization:

- Ordinary way:

```
* c(1,2,3), c(T,FALSE,TRUE), c('a',NA,'b')
```

```
* vector(mode = MODE,length = LENGTH)
```

```
* logical(LENGTH) return FALSE vector
```

where c() for 'combine';

c() combines all 'vector-like objects' into one vector, e.g. c(c(1,2,3),c(1,2))>> c(1,2,3,1,2).

- Sequence vector:

```
* 1:3.5>> c(1,2,3), 3:1>> c(3,2,1)
* seq(from, to, by, length.out), length.out for total vector length;
* rep(SEQ_TO_REP, times, length.out, each), used in k-fold cross validation labelling.
```

Operations:

- between vectors of different length SHORT and LONG: First `SHORT <- rep(SHORT, length.out=length(LONG))`. Then operate SHORT and LONG.  
e.g. `c(1,2) + c(1,2,3)>> c(1,2,1) + c(1,2,3)>> c(2,4,4)`
- Element access: `a[i]`, *i* starts from 1

**Vectorized Operation:** All operation in R. are based on vector, and vectorized operation is Parallel Arithmetic, which is **much faster** than loop such as **for**

**Δ** Consider using vectorized operation when writing code for **Speed!** Detail see [section. 6.1.4](#).

- **Factor** : A special kind of ‘vector’ in R., used to label discrete categorical data.<sup>78</sup>

Initialization:

`factor(FACTOR_SEQ, levels = FACTOR_LEVEL, labels = ...)`, `FACTOR_LEVEL` is the ‘rank’ of each factor, `labels` is the ‘tag’ of levels.

A quick way to factorize a numeric vector *x* by interval division:

```
cut_number(x, NUM_OF_LEVELS)
```

- **Matrix** : Only data of the same class can be held in one matrix.

Initilaization: `matrix(DATA_SEQ, nrow, ncol, byrow = FALSE, dimnames = NULL)`. Default `byrow = FALSE` because matrix data is stored as combination of column vectors.

If `length(DATA_SEQ) < nrow*ncol`, then `DATA_SEQ` is repeated with `length.out=nrow*ncol`.

Operation:

- Common operators `+-*/^` etc. operate in column-by-column mode (vectorized operation).
- Binding matrix: `cbind` for `[A,B]` and `rbind` for `[A;B]`
- Transpose: `t()`
- Matrix multiplication: `%*%`
- Inverse matrix: `solve()` (The essence of inversion is solving linear equations)
- Diagonal matrix:
  - \* `diag(VECTOR)` returns a matrix `diag{VECTOR}`
  - \* `diag(MATRIX)` returns the diagonal element vector
- Element access: `a[i,j]`, `a$OBJECT_NAME`
- Dimension: `dim()`, `nrow()`, `ncol()`
- Rank: `qr(MATRIX)$rank`

<sup>78</sup>Factor vector is stored as integer vector.

- **List** : A pack containing various datatype, generally also a kind of vector (but not atomic vector)

Initialization: `list(OBJECT1, OBJECT2, ...)`

Element access: `a[[i]]`, `a$OBJ_NAME`

- `data.frame`: 'Mixture' of matrix and list. `data.frame` is actually a kind of list (with some constraint), organized in the shape of matrix (but allowing different datatype for different columns, each column is a list object).

Each column of `data.frame` has name: `names(DATA_FRAME)`, `colnames(DATA_FRAME)`

Element access: `a[i, j]`, `a[[i]]`, `a$COL_NAME`

## □ Data Read & Write

- Common R&W: `read.` / `write.`

– `read.table(FILE_NAME, header = FALSE, sep, colClasses, stringAsFactors = FALSE)`

★ `read.csv()` basically the same as `read.table`

★ `write.table(DF, FILE_NAME, sep, row.names=FALSE)`

– `readxl::read_xlsx(FILE_NAME, sheet = SHEET_NUM, range = 'RANGE')`

Some relative arguments:

- `quote=""`, use `'` to quote/identify string, set `quote=''` to avoid misread strings such as 'Levene's Test'
- `encoding='UTF-8'`, char encoding system, used especially for dataset containing CJK char.
- `nrows=LINE_NUM` read first LINE\_NUM lines

- Large Data Read & Write:

- preset `colClasses`

```
1 temp.dat <- read.table(FILE_NAME, nrows = 100)
2 classes <- sapply(temp.dat, class)
3 dat <- read.table(FILE_NAME, colClasses = classes)
```

- `readr::read_delim(FILE_NAME, delim=SEP)` can speed up

- Text Write: `sink(FILE_NAME, append=FALSE)`, write output into a file, the same as `>` in terminal.
- .RData Binary Format Read & Write: RW in .RData format, fast to load.

– `save(DF, file = FILENAME)`

– `load(FILE_NAME)`

## 6.1.3 Functions and Control Flow

### □ Program Speed:

`system.time({COMMAND})`

### □ Function Call

- `FUN_NAME(ARGUs)`

- `do.call('FUN_NAME', LIST_OF_ARGS)`, look for a function naming FUN\_NAME in R. and call.
- `a % NEW_OPRTR % b` to call self-defined binary operator.
- `'*'` etc. used in `apply(FUN = '*')`
- R. allows auto-completion to ARGS, e.g. `rep(0, length.out = 10) » rep(0, length = 10)`

## □ Function Definition

### ▷ R. Code

Basic function definition in R.

```
1 FUNC_NAME <- function(ARG1 = ARG1_VALUE , ARG2, ...) {
2     FUNCTION_BODY
3 }
```

More key elements in `function{}`

- `return(RETURN_OBJ)` at the end of function, without `return()`, output the last line
- `stopifnot(COND1, COND2, ...)` at the beginning of function, used to test ARG class
- `stop(ERROR_MSG)` output error message
- `...` as a special argument
  - Pass `...` to another func in this function
  - Handle arbitrary number of input
- Function can be defined within function
- Function is a kind of variable → used in `apply`, `sapply` etc. for vectorized programming.
- Anonymous function: used in `sapply(X, FUN=function(){STATs})` for quick definition
- FUNC\_NAME can be used for new-defined binary operator as `'%NEW_OPRTR%' <- function()`

## □ Flow Control

- `if` and `else if`, example:

```
1 if(COND1) {
2     STATEMENT
3 } else if(COND2) {
4     STATEMENT
5 } else {
6     STATEMENT
7 }
```

- `ifelse(COND, IF_YES_STAT, IF_NO_STAT)` a vectorized version of `for` + `if else`.
- `for`: Loop in R. is **Extremely Slow**, avoid loop, use **vectorized operation**.

```

1 for(VAR in SEQ) {
2     STATEMENT
3 }

```

- `switch`(TEST\_EXPR, CASE1= RETN1, CASE2= RETN2, ...)

### 6.1.4 Vectorized Operation

- `apply()` function series:

- `apply`(MAT, MARGIN, FUN) for matrix apply, MARGIN=1 for each row, 2 for each column

Example:

```

1 apply(matrix(c(1,2,3,4),2,2), 1, sum) >> c(4,6)
2 apply(matrix(c(1,2,3,4),2,2), 2, sum) >> c(3,7)

```

- `lapply`(LIST, FUN) for list/data.frame, apply FUN on each list elements, list returned
- ★ `sapply`(X, FUN) for list/data.frame apply+simply, vector/matrix/list returned
- `tapply`(X, INDEX, FUN): for each index, use FUN respectively.
- `mapply`(FUN, ARGU\_OF\_FUN), use argument name to label ARGU\_OF\_FUN, or causes bad readability.

Example:

```

1 mapply(function(x,y,z,k){(x+k)^(y+z)}, x = , y = , z = , k = )

```

- `Vfunc <- Vectorize(FUNC_NAME)`: define vectorize version of function.
- `with()` and `within()`:
  - `with`(DF, `aggregate`(PART, by, FUN))
  - `with`(DF, STATE), `within`(DF, STATE), `within` allows new column append
- `outer`(VEC1, VEC2, FUN): A Two-variate extension of `mapply()`, output wedge of two vectors.
- `ifelse`(COND, YES\_STAT, NO\_STAT), vectorization supported.

### 6.1.5 Subsetting

- By position: `x[RANGE]`
  - `x[4]`
  - `x[-4]`: x without the 4<sup>th</sup> item (which is different from python, where selects the reciprocal 4<sup>th</sup> element).
  - `x[2:4]`
  - `x[c(1,2,5)]`
- By name: `x[, 'COL_NAMES']`, `x[, 'COL_NAME1': 'COL_NAME2']`
- By condition: basically, `x[LOGI_VEC]`

- `x[x==10]`
- `x[x %in% c(1,3,4)]`, linear search, not based on hash algorithm<sup>79</sup>.

Usually used for conditional selection of data.frame

- Subsetting for data.frame and list: `x[[RANGE]]`

Simplified / Preserved subsetting: whether preserved datatype, e.g. `df → df` (preserved) v.s. `df → vector` (simplified).

DataType	Simplified	Preserved
vector		<code>x[[1]]</code> / <code>x[1]</code>
list	<code>x[[1]]</code>	<code>x[1]</code>
factor	<code>x[1:4, drop=T]</code>	<code>x[1:4]</code>
matrix	<code>x[, 1]</code>	<code>x[, 1, drop=F]</code>
data.frame	<code>x[, 1], x[[1]]</code>	<code>x[, 1, drop=F], x[1]</code>

表 4: Simplified/Preserved subsetting

- Other subsetting:

- `%in%`
- `unique()`, return with each element appears only one times
- `duplicated()`, `TRUE` when appear the  $n > 1$  times
- `which(x==4)`, return position of matched element
- `which.min()`, `which.max`, `min()`, `max()`
- `grep(REGEX, X, value)`, search for elements with REGEX pattern: `value=F` returns position, `value=T` returns elements, `grep1(REGEX, X)` returns logical vector
- `match(TO_BE_MATCHED, TARGET)`, returns the index of elements of TO\_BE\_MATCHED in TARGET

▷ **R. Code**

Example:

```

1 vec1 <- c('a', 'a', 'b', 'b', 'd', 'd', 'b')
2 vec2 <- c('d', 'a', 'b')
3 match(vec1, vec2)
4 > [1] 2 2 3 3 1 1 3

```

- `subset(X, ...)`, ... a series of select criterion. **not** allowed: `subset(X, ...) <-`

- Use subsetting to sample: `DATA[sample(1:nrow(DATA), NUM_OF_SAMPLE, replace), ]`, `replace=T` for with replacement

<sup>79</sup>If really needed, use `env()` to reset environment.

### 6.1.6 Data Manipulation With dplyr. And tidyr.

dplyr and tidyr are two useful package for data cleaning & manipulation. Use package tidyverse include both of them.

tidyverse for tidyuniverse, includes dplyr, tidyr, readr, ggplot2, stringr, etc.

□ **%>%: pipe in tidyverse, so that functions in tidyverse with format FUNC(DF, ...) can pass on DF results along the pipeline.**

Some examples see [section. 8.1.5](#).

□ **dplyr Package.**

- Cheat Sheet: <https://nyu-cdsc.github.io/learningr/assets/data-transformation.pdf>
- `select(DF, ...)`, where ... can use column index/name range as in subsetting, or some helper function for advanced subsetting:
  - matching position:
    - \* `everything()`
    - \* `last_col()`
  - matching column name:
    - \* `start_with('PATTERN'), end_with('PATTERN'), contains('PATTERN')`
    - \* `match('REGEX')`, column name with REGEX pattern
    - \* `num_range('x', 1:4)` select column name c('x1', 'x2', 'x3', 'x4')
    - \* `any_of(CHR_VEC)` select column from CHR\_VEC
  - `where(FUN)`, select those FUN(COL\_NAME) returns TRUE
- `filter(DATA, CONDS)`, select elements with CONDS conditions
- `arrange(DATA, COL)`, sort by COL, `arrange(DATA, desc(COL))` for descending order
- `mutate(DATA, ...)`, append new columns according to ... definition; `transmute()` drops original columns.
 

... definition can use advanced window function:

  - `lead(COL), lag(COL)`, e.g. `lead(COL)[i]=COL[i+1]`, can use `...=COL-lead(COL)` for differnetial
  - `dense_rank(COL)`, `percent_rank(COL)` rank number
  - `ntile(COL, N)` break into N groups labeling 1:N
  - `cume_dist(COL)`, `cummean(COL)`, `cumsum(COL)`, `cummax(COL)`, `cummin(COL)`, etc. cumulative value
- `summarise(data, ...)`, ... for summarise function.
- Row selection:
  - `slice(DF, ROW_RANGE)`
  - `distinct(DF)` remove duplicated rows
  - `sample_frac(DF, FRAC, replace)`, sample FRAC fraction from DF
  - `sample_n(DF, N, replace)`, sample N cases from DF

– `top_n(DF, AMOUNT, RANK_COL)` select AMOUNT top ranking by RANK\_COL cases

- Data combining see slides.

#### □ tidy Package

- Cheat Sheet: [https://leadousset.github.io/intro-to-R/cheatsheet\\_tidy.pdf](https://leadousset.github.io/intro-to-R/cheatsheet_tidy.pdf)
- `gather(DF, key='KEY_NAME', value='VALUE_NAME', ..., na.rm)`, melt a data.frame.  
e.g. `gather(df, 'KEY', 'VALUE', c('COL1', 'COL2', 'COL3'))` transfers ... as:

					ID	KEY	VALUE
					1	COL1	$a_1$
					2	COL1	$a_2$
					⋮	⋮	⋮
ID	COL1	COL2	COL3		1	COL2	$b_1$
1	$a_1$	$b_1$	$c_1$	→	2	COL2	$b_2$
2	$a_2$	$b_2$	$c_2$		⋮	⋮	
⋮	⋮	⋮	⋮		1	COL3	$c_1$
					2	COL3	$c_2$
					⋮	⋮	

- `spread(DF, key='KEY_NAME', value='VALUE_NAME')`, inverse of `gather()`
- `separate(DF, COL, into=SET_VEC, sep='REGEX')`, separate COL into columns with name in SET\_VEC, sep according to sep
- `unite(DF, COL, SET_VEC, sep='')` inverse of `separate()`

## Section 6.2 Text Processing & Text Mining

- Data cleaning
- Data manipulation
- Information extraction: mode identifying/relation extraction
- Text mining: analyzing token distribution, ignore word order
- NLP: concept identifying based on sentence; ultimate goal: ‘understand’ sentence meaning.

Tools for Text processing:

- R.: suitable for easy task
- python.: best
- java: strong, but not suitable for deep learning
- c++: fast, inadequate package
- Notepad++ / Vim / VSCode, etc.



## 6.2.1 Basic Text Manipulation With stringr.

### □ R. base & stringr package:

The prior one is used more often

- Cheat Sheet: <http://edrub.in/CheatSheets/cheatSheetStringr.pdf>
- `str_length(String)`, `nchar(String)`
- `paste(..., collapse=NULL)`, `str_c(...)`, both are vectorized operation

Argument:

- `sep`: sep between each ... corresponding elements, with `collapse=NULL`, return a char vector
- `collapse`: sep when combining `collapse=NULL` vector elements, `NULL` for not combining
- Special character: `\t` tab, `\r` & `\n` & `\r\n` new line, `\xad` '-' at end on line for word-connecting
- `str_split(String, pattern='REGEX')/strsplit()`, split string at REGEX pattern fitted, list returned
- `str_sub(String, start, end)`, `substr()`. The `start` char to `end` char of string, use negative index as in python.
- Can be used to replace: `str_sub(...)<- REP_STR`
- `str_locate_all('STRING', pattern='REGEX')/str_match_all('STRING', pattern='REGEX')`  
`grep(pattern='REGEX', x='STRING', value=T)`, search for elements with REGEX pattern: `str_locate_all()` or `value=F` returns position, `str_match_all()` or `value=T` returns elements.
- `str_replace_all('STRING', pattern='REGEX', replacement='REP')` `grepl(REGEX, X)` returns logical vector, include or not. `str_extract_all('STRING', pattern='REGEX')`
- `gsub(pattern='REGEX', replacement='REP', x='STRING')`, replace REGEX field with REP
- `str_trim(..., side = )`, trim extra white space at `side='both'/'left'/'right'`

## 6.2.2 Regular Expression

Regular expression is a text pattern/mode. abbr. regex/regexp. Regex is supported in most common language, same syntax used.

Tutorial: <https://www.runoob.com/regexp/regexp-tutorial.html>

### □ Key Elements

- Literal: common char, e.g. a. Include most char on keyboard. Upper/Lower case sensitive.
- Metacharacters: `\^$.|?*( ) [] {}`, use e.g. `\.` to escape meaning.

Note: when typing regex in programming language, sometimes use `\\.`: `\\.`  $\xrightarrow{\text{language interpreter}}$  `\.`  $\xrightarrow{\text{regex interpreter}}$  identifying .

- Character Class: `[]`, identify one of elements in `[]`. `^` within `[]` for  $\mathbb{C}$ .
  - e.g. `gr[ae]y` identifies `grey` and `gray`.

- e.g. `[0-9]` numbers, `[a-zA-Z]` letter
- e.g. `q[^\x]` matches `question`, not matches `qquestion`, not matches `Iraq`

character class shorthand

ShortHand	Meaning	Equivalent REGEX
<code>\d</code>	numeric digit	<code>[0-9]</code>
<code>\D</code>	Not numeric digit	<code>[^\d]</code>
<code>\w</code>	a word character	<code>[a-zA-Z0-9_]</code>
<code>\s</code>	white space	<code>[\t\r\n\f]</code>

- Wildcard（通配符）：. matches any single character except line break `\r`, `\n`
- Anchor（词边界/定位符）：match ‘word boundary’ (not the space at the start/end of string).  
`^` string start, `$` string end, `\b` word boundary, `\B` not-a-word-boundary position
- Repetition/Quantifier: here X for some regex pattern like CHAR, `[]` etc.

Greedy	★ Reluctant	Possessive	Freq of Occurrence
<code>X?</code>	<code>X??</code>	<code>X?+</code>	0, 1
<code>X+</code>	<code>X+?</code>	<code>X++</code>	$\geq 1$
<code>X*</code>	<code>X*?</code>	<code>X*+</code>	0, $> 1$
<code>X{n}</code>	<code>X{n}?</code>	<code>X{n}+</code>	$n$
<code>X{n,}</code>	<code>X{n,}?</code>	<code>X{n,}+</code>	$\geq n$
<code>X{n,m}</code>	<code>X{n,m}?</code>	<code>X{n,m}+</code>	$[n, m]$

Example: Search ‘foo’ in ‘xfoooooxxxfoo’:

- Greedy: ‘xfoooooxxxfoo’ found at index 0-13
- ★ Reluctant: ‘xfoo’ found at index 1-4, ‘xxxxxxxfoo’ found at index 4-13
- Possessive: no match found (not usually used)

Example: regex match ‘aaaa’

–

- Alternation & Grouping & Back Reference: `XA|XB` identify XA or XB, use grouping `()` to set boundary of XA,XB.

Use `\n` for back reference the  $n^{\text{th}}$  group. ▷ [R. Code](#)

Example: search for immediate repeat word in a sentence

```
1 (\b[a-zA-Z]+\b) \1
```

- Lookaround:

- LookAhead: (?<=X)q
- LookBehind: q(?=X)

### 6.2.3 Web Scraping

Basic elements of web page:

- HTML (HyperText Markup Language): structure and content of page
- CSS (Cascading Style Sheet): page style.
- JavaScript: functionality, interaction

Basic html document format:

▷ R. Code

```

1 <!DOCTYPE html> # an html document
2 <html> # html page begin
3 <head> # head elements declare
4 <meta charset="utf-8">
5 <title> TITLE OF WEB PAGE </title>
6 </head>
7 <body> # html body begin
8
9 <h1> HEADING 1 </h1>
10 <p class='TEST_TEXT'> PARAGRAPH 1 </p>
11
12 </body>
13 </html>

```

We can use elements like <p> or `class` to extract page information.

□ Web Scraping with `rvest`.

- `pge <- read_html('URL')`: page read

Proxy set: `Sys.setenv(https_proxy='http://127.0.0.1:7890')`

- `pge %>% html_elements(css='.CSS_CLASS_NAME')%>% html_text()`: basic scraping. use `SelectGadget` tool for help finding proper css label.

## Section 6.3 Graphic in R.

### 6.3.1 R::base Plotting

Plot function in `R::base`:

```

1 plot(X,Y) # scatter/line plot of Y-X

```

```

2 plot(FUNC_OBJ, from = , to = ) # function plot ranging in c(from, to)
3 plot(FACTOR) # barplot of factors
4 plot(FACTOR, Y) # boxplot of numeric v.s. levels of factor
5 plot(DATA.FRAME) # correlation plot
6 plot(ANY_PLOTTABLE_OBJ) # plot any plottable object

```

- Plot saving: first open a plotting device, then make plot and close the device

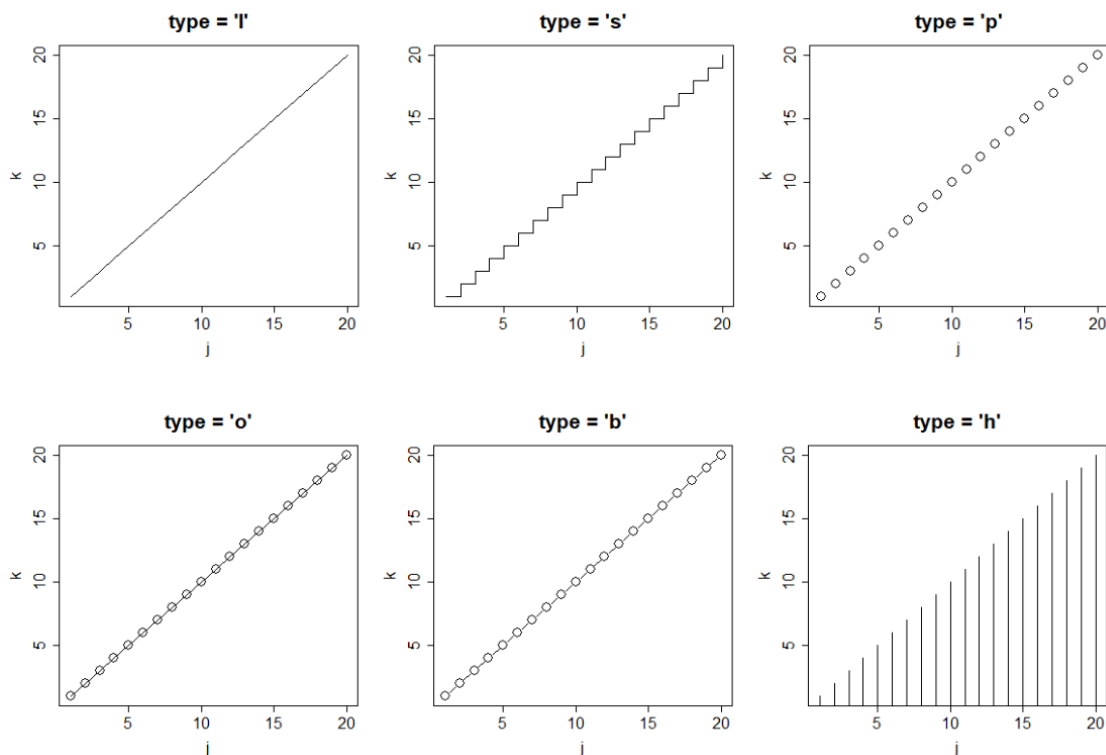
```

1 pdf("PLOT_FILE_NAME.pdf", FIG_HEIGHT, FIG_WIDTH)
2 plot(PLOT_PARAM)
3 dev.off()

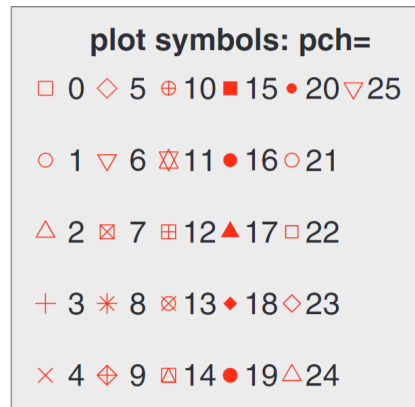
```

- `plot()` plotting parameters:

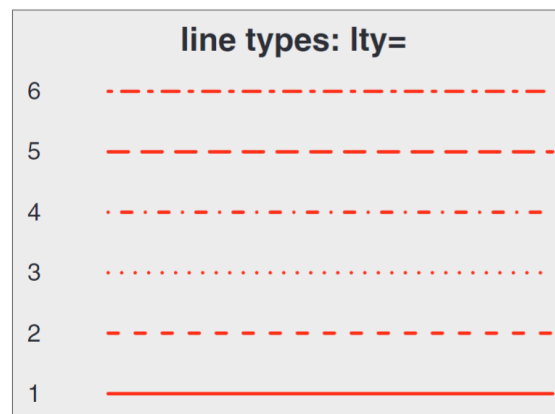
- `main` = string for title; or use `title('TITLE')` as the next command
- `sub` = string for subtitle;
- `xlab` = , `ylab` = string axis labels;
- adding  $\text{L}^{\text{A}}\text{T}_{\text{E}}\text{X}$  expression as text: use `main = expression(PLOTMATH_EXPRESSION)`, use `?plotmath` to look for possible symbols
- `xlim` = , `ylim` = axis range, e.g. use `xlim = c(0,100)`
- `type` = value taken in `c('p', 'l', 'b', 'o', 's', 'h')` for plot type



- `pch` = point character, value taken in `0:25` for default point characters listed below, or use (vector of) character to specify, e.g. `pch = c(' ')`



- `lty` = **line type**, value taken in 1:6 (0 for not shown)



- `cex` = **character expansion**, relative size with 1 as baseline and default.

Some derivative function to control size of other plotting elements:

- \* `cex.axis` = relative size of axis node text
- \* `cex.lab` = relative size of labels
- \* `cex.main` = relative size of title
- \* `cex.sub` = relative size of subtitle

- `lwd` = **line width**, relative width of line with 1 as baseline and default

- `col` = **color** of elements in plot, value examples for color white:

- \* Index: `col = 1` predefined color in R.
- \* Color name: `col = 'white'`, use `colors()` to see all available color names
- \* Hexadecimal code: `col = '#FFFFFF'`
- \* RGB code: `col = rgb(1,1,1)`, `col = rgb(255,255,255, maxColorValue = 255)`
- \* HSV code: `col = hsv(0,0,1)`

`col` = can accept vector for various colors, or accept some function for continuous colors:

- \* Discrete color: `col = c('red', 'blue')`, or use `col = df$GROUP` to color different groups
- \* Continuous color function: `rainbow(NUM_OF_COLORS)`, `heat.colors()`, `terrain.colors()`, `topo.colors()`, `cm.colors()`

Some derivative function to control color of other plotting elements:

- \* `col.axis` = color of axis node text
- \* `col.lab` = color of labels
- \* `col.main` = color of title
- \* `col.sub` = color of subtitle
- \* `bg` = color of background

– `font` = **font** used in plot, with 1 = plain, 2 = bold, 3 = italic, 4 = bold italic

Some derivative function to control font of other plotting elements:

- \* `font.axis` = font of axis node text
- \* `font.lab` = font of labels
- \* `font.main` = font of title
- \* `font.sub` = font of subtitle
- \* `ps` = baseline font point size, i.e. text size = `ps*cex`
- \* `family` = extra text type, value taken in `c('serif', 'sans', 'mono')` etc. use `names(pdfFonts())` to see possible font families

– `bty` = **box type** of the box surrounding the figure. Value taken in `c('o', '7', 'L', 'U', 'C', 'n')`

– `las` = relation btw. **l**able and **a**xis. Value taken `c(0,1,2,3)`.

- `axis()` parameters for axis settings: after using `xaxt = 'n'` or `yaxt = 'n'` to remove corresponding axis when executing `plot()`, other variation of axis could be made by using `axis()`

– `axis(1)` for creating *x* axis, `axis(2)` for creating *y* axis. Here we would use *x* axis in the following parts.

– `axis(1, at = )` to specify ticks.

– `plot(las = )` to specify rotation of ticks, value taken in `c('Parallel', 'Horizontal', 'Perpendicular', 'Vertical')`

– `plot(xlim = c( , ), ylim = c( , ))` for axis limits

– `plot(log = )` for log transform on axis, value taken in `c('x', 'y', 'xy')`.

- `legend()` parameters:

– `x` = position of legend, value taken in `c("top", "bottom", "topleft", "topright", "bottomleft", "bottomright")`

– `inset` =

– other parameters are set following the setting in `plot`. An example:

```
1 legend("bottomright", legend = c("red", "green"), lty = c(2,4),
      lwd = 3, col = c("red", "green"))
```

- `text(X_COOR, Y_COOR, labels = TEXT)` parameters for adding text in figure. An application is `text(df$X, df$Y, labels = df$Z)` to label each point.

– `pos` = **p**osition of text around the coordinate point, value taken in `c(1,2,3,4)`

- `lines()` to put an extra line on existing figure (device). Parameters are similarly set as `plot()`

- `par()` to set global **parameters**. An example to put 3 different figure in the same device:

```

1 opar <- par(no.readonly = TRUE) # copy original setting
2 par(mfrow = c(1,3))
3 plot()
4 plot()
5 plot()
6 par(opar)

```

## □ More Charts

- `barplot(counts, horiz, besides, ...)` for bar plot. Data should be first prepared by `counts <- table(Y_TO_COUNT)`.
- `hist(x, breaks, freq, ...)` for histogram.
- `plot(density(df, kernel = ), ...)` for density plot.
- `boxplot(x, ...)` for box plot. use `boxplot(x ~ GROUP, data = , ...)` to plot grouped boxplot
- `dotchart(x, labels, groups, ...)` to compare x value for categories

## 6.3.2 R::ggplot2 Plotting

**ggplot2: Grammar of Graphics plot** (2nd edi). It provides a convenient way to produce fancy plots. Reference see <https://ggplot2.tidyverse.org/reference/>

Basic steps for ggplot2:

1. Specify data and arsthetic mapping
2. Adding 'layers' with `geom_`
3. Adding labels

An example:

```

1 ggplot(data=mtcars, aes(x=wt, y=mpg)) +
2   geom_point(pch=17, color="blue", size=2) +
3   geom_smooth(method="lm", color="red", linetype=2) +
4   labs(title="Automobile Data", x="Weight", y="Miles Per Gallon")

```

Elements in ggplot2:

- `aes()` to specify **aesthetic** mapping, e.g. `aes(x = , y = , col = , ...)`. Used in `ggplot()` as global setting, in `geom_()` as local override (different `geom_()` may need different local settings). Examples:

```

1 aes(x = mpg ^ 2, y = wt / cyl, col = am)
2 #> Aesthetic mapping:
3 #> * x -> mpg^2
4 #> * y -> wt/cyl

```

```
5 | #> * color -> am
```

- `geom_layer` to specify statistical figure you want. Some useful plot:

<code>geom_()</code> Function	Charts	Options
<code>geom_bar()</code>	bar plot	color, fill, alpha
<code>geom_boxplot()</code>	box plot	color, fill, alpha, notch, width
<code>geom_density()</code>	density plot	color, fill, alpha, linetype
<code>geom_histogram()</code>	histogram	color, fill, alpha, linetype, binwidth
<code>geom_hline()</code>	horizontal line	color, alpha, linetype, size
<code>geom_vline()</code>	vertical line	color, alpha, linetype, size
<code>geom_line()</code>	line graph	color, alpha, linetype, size
<code>geom_point()</code>	scatter plot	color, alpha, shape, size
<code>geom_smooth()</code>	fitted line	method, formula, color, fill, linetype, size
<code>geom_violin()</code>	violin plot	color, fill, alpha, linetype
<code>geom_text()</code>	text annotation	see function help

- `labs(title, x, y)` to specify labels and title
- `facet_grid()` and `facet_wrap()` to plot multiple plot, with factor levels as categories, parameters:
  - `facets` = facet variable. For `facet_wrap()` use `~VAR1` (one variable); `facet_grid()` use `~VAR1` or `VAR1~.` or `VAR1~VAR2` (allow two variable)
  - `nrow` = , `ncol` = grid shape
  - `shrink` = whether adjust ticks, set `TRUE` or `FALSE`
  - `drop` = whether drop levels with censored data, set `TRUE` or `FALSE`
- `theme()` to set fonts, backgrounds, gridlines, etc.

There are some pre-defined theme: `theme_grey()`, `theme_bw()`, `theme_linedraw()`, `theme_light()`, `theme_dark()`, `theme_minimal()`, `theme_classic()`, `theme_void()`, `theme_test()`.

Detailed elements in a plot is adjust by passing `element_()`:

- `element_line()` set some line element
- `element_rect()` set some rectangular element
- `element_text()` set some text element

Some useful command:

- `plot.title` = `element_text(hjust = 0.5)` adjust position of title to mid. Other similar parameters: `plot.background`, `plot.title.position`, `plot.subtitle`, `plot.caption`, `plot.caption.position`, `plot.tag`, `plot.tag.position`, `plot.margin`



- `panel.background = element_rect(fill = 'white', color = 'blue')` adjust figure background and border. Other similar parameters: `panel.grid.major/minor.x/y`
- `aspect.ratio = height:width`
- `legend.position = 'none'` to remove automatic legend
- `ggsave('FILE_NAME', PLOT, WID, HEI)`, or use `ggsave('FILE_NAME')` to save the active device.

## Chapter. VII 可靠性数据与生存分析部分

Instructor: Jiangdian Wang

Key focus of reliability data and survival analysis: Study the ‘survival time’  $T$  before some ‘failure event’. Basically the research problem is the distribution of  $T$ , including topics on descriptive statistics, estimation and hypothesis testing. Further for actual cases,  $T$  might be censored, i.e. the observe time is not exact; and we may also wonder the influence of covariants  $z$ .

### Section 7.1 Reliability Data

The main feature of reliability data is **censoring**, to be distinguished from the exact numbers in usual statistical inference. Censor means we cannot observe the exact **event time**  $T$ . Instead, a **censoring time**  $C$  is observed, together with a censoring type, e.g.

$$\text{Right Censoring: } T_{\text{actual}} > C \quad (7.1)$$

$$\text{Left Censoring: } T_{\text{actual}} < C \quad (7.2)$$

$$\text{Interval Censoring: } C_l < T_{\text{actual}} < C_r \quad (7.3)$$

$$\dots \quad (7.4)$$

#### 7.1.1 Right Censor Data and Representation

In most parts of this course we focus on right censor data, i.e. dataset contains both event time  $T$  and right censor time  $T^+$ :

$$\text{Event Time: } T_1, \dots, T_{n_1} \quad (7.5)$$

$$\text{Right Censor: } T_1^+, \dots, T_{n_r}^+ \quad (7.6)$$

$$(7.7)$$

Or we could use an indicator  $\delta$  to express whether a time is event (1) or right censored (0):

$$(T_i, \delta_i; z_i), i = 1, 2, \dots, n_1 + n_r \quad (7.8)$$

where  $z_i$  for covariants.

Usually we assume that event and censor are independent  $T \perp\!\!\!\perp C$

#### 7.1.2 Life Table Data

Life table collect survival data at ordinal, uniformly-spaced time points, where each row contains # items at risk, # events, . . .

## Section 7.2 Survival Model and Statistical Inference

### 7.2.1 Survival Function and Hazard

Key focus of survival analysis problem is the distribution of  $T$  (note that in actual cases we need to make use of both event time  $T_i$  and censored data  $T_i^+$  to estimate the distribution of  $T$ ). The distribution feature can be described in various approaches: PDF  $f(t)$ , CDF  $F(t)$ , Survival Function  $S(t)$ , Hazard Function  $\lambda(t)$ , Cumulative Hazard Function  $\Lambda(t)$ :

- Continuous Case:  $t \in \mathbb{R}^+$

- Survival Function  $S(t)$ :

$$S(t) \equiv 1 - F(t) = \int_t^\infty f(\tau) d\tau, \quad f(t) = -\frac{dS(t)}{dt} \quad (7.9)$$

- Hazard Function  $\lambda(t)$  (or in some materials denoted  $h(t)$ ): mortality at  $t$ :

$$\lambda(t) = \lim_{h \rightarrow 0} \frac{\mathbb{P}[t \leq T < t+h | T \geq t]}{h} = \frac{f(t)}{S(t)} = -\frac{d \log S(t)}{dt} \quad (7.10)$$

- Cumulative Hazard Function  $\Lambda(t)$  (or in some materials denoted  $H(t)$ ):

$$\Lambda(t) = \int_0^t \lambda(\tau) d\tau = -\log S(t) \quad (7.11)$$

$$S(t) = e^{-\Lambda(t)} = e^{-\int_0^t \lambda(\tau) d\tau} \quad (7.12)$$

- Discrete Case:  $t \in \{t_1, t_2, \dots, t_n\}$

- PMF:  $p(t)$  is defined on

$$t \in \mathcal{T}, \quad p(t) \in (\mathcal{T} \rightarrow [0, 1]^n) \quad (7.13)$$

- Survival Function: Note that CDF  $F(t)$  is right continuous, then  $S(t) = 1 - F(t)$  is left continuous:

$$S(t) = \mathbb{P}(T > t) = \sum_{t_i > t} p(t_i), \quad p(t_i) = S(t_{i-1}) - S(t_i) = \lambda(t_i)S(t_{i-1}) \quad (7.14)$$

Decomposition of survival function into hazard production

$$\begin{aligned} S(t) &= \mathbb{P}(T > t) = \mathbb{P}(T > t \cap T > t_j), \quad \forall t_j < t \\ &= \mathbb{P}(T > t | T > t_j) \cdot \mathbb{P}(T > t_j) \\ &= \mathbb{P}(T > t | T > t_j) \cdot \mathbb{P}(T > t_j | T > t_{j-1}) \cdot \mathbb{P}(T > t_{j-1}) \\ &= \mathbb{P}(T > t | T > t_j) \cdot \frac{S(t_j)}{S(t_{j-1})} \cdot \mathbb{P}(T > t_{j-1}) \\ &= \prod_{0 < t_j \leq t} \frac{S(t_j)}{S(t_{j-1})} \\ &= \prod_{0 < t_j \leq t} [1 - \lambda(t_j)] \end{aligned} \quad (7.15)$$

- Hazard Function  $\lambda(t)$ :

$$\lambda(t_i) = \mathbb{P}(T = t_i | T \geq t_i) = \frac{p(t_i)}{S(t_{i-1})} = 1 - \frac{S(t_i)}{S(t_{i-1})} \quad (7.16)$$

### □ Properties of survival function and hazard function & More concepts and definition

- Mean Survival Time:

$$\mu \equiv \mathbb{E}(T) = \begin{cases} \int_0^\infty \tau f(\tau) d\tau = \int_0^\infty S(\tau) d\tau \\ \sum_{i=1}^n t_i p(t_i) \end{cases} \quad (7.17)$$

- Mean Residual Life Time (mrl):

$$\text{mrl}(t) = \mathbb{E}[T - t | T \geq t] = \frac{\int_t^\infty S(\tau) d\tau}{S(t)} \quad (7.18)$$

- Considering that  $T > 0$  and  $\lim_{t \rightarrow \infty} F(t) \rightarrow 0$ ,  $S(t)$  has following properties

$$S(0) = 1 \quad S(\infty) = 0 \quad (7.19)$$

- For independent survival time  $T_1, T_2$ , define  $T = \min\{T_1, T_2\}$ , then

$$\lambda_T(t) = \lambda_1(t) + \lambda_2(t) \quad (7.20)$$

- Hazard Rate: for two survival r.v.  $T_1, T_2$ , the hazard rate at  $t$

$$\text{hazard ratio}(t) = \frac{\lambda_1(t)}{\lambda_2(t)} \quad (7.21)$$

### 7.2.2 Parametric Statistical Inference to Survival Function

Usually the parametric inference is based on a hypothetical distribution, then we conduct estimation using the parametric distribution, or conduct hypothesis testing on parameter(s).

#### □ Common Survival Distribution Prior

In parametric model, there are some commonly used distribution models

- Exponential  $T \sim \varepsilon(\lambda)$

$$f(t) = \lambda e^{-\lambda t} \quad (7.22)$$

$$F(t) = 1 - e^{-\lambda t} \quad (7.23)$$

$$S(t) = e^{-\lambda t} \quad (7.24)$$

$$\lambda(t) = - \frac{d \log S(t)}{dt} = \lambda \quad (7.25)$$

$$H(t) = \lambda t \quad (7.26)$$

$$\mathbb{E}(T) = \frac{1}{\lambda} \quad (7.27)$$

$$\text{var}(T) = \frac{1}{\lambda^2} \quad (7.28)$$

- Weibull  $T \sim W(p, \lambda) = [\varepsilon(\lambda^p)]^{1/p}$ , degrade to exponential  $\varepsilon(\lambda)$  when  $p = 1$ <sup>80</sup>

$$f(t) = p\lambda^p t^{p-1} e^{-(\lambda t)^p} \quad (7.29)$$

$$F(t) = 1 - e^{-(\lambda t)^p} \quad (7.30)$$

$$S(t) = e^{-(\lambda t)^p} \quad (7.31)$$

$$\lambda(t) = p\lambda^p t^{p-1} \quad (7.32)$$

$$H(t) = (\lambda t)^p \quad (7.33)$$

$$\mathbb{E}(T) = \frac{1}{\lambda} \Gamma(1 + \frac{1}{p}) \quad (7.34)$$

$$\text{var}(T) = \frac{1}{\lambda^2} \left[ \Gamma(1 + \frac{2}{p}) - (\Gamma(1 + \frac{1}{p}))^2 \right] \quad (7.35)$$

$$t_{0.5} = \left[ \frac{\log 2}{\lambda^p} \right]^{1/p} \quad (7.36)$$

- Gamma  $T \sim \Gamma(\alpha, \lambda)$ . Degrade to exponential  $\varepsilon(\lambda)$  when  $\alpha = 1$ , to  $2\lambda T \sim \chi_{2\alpha}^2$  when  $2\alpha \in \mathbb{N}$

$$F(t) = \frac{\lambda^\alpha}{\Gamma(\alpha)} t^{\alpha-1} e^{-\lambda t} \quad (7.37)$$

$$\mathbb{E}(T) = \frac{\alpha}{\lambda} \quad (7.38)$$

$$\text{var}(T) = \frac{\alpha}{\lambda^2} \quad (7.39)$$

- Log-Normal  $T \sim \text{LN}(\mu, \sigma^2) = \exp [N(\mu, \sigma^2)]$ .

$$f(t) = \frac{\phi\left(\frac{\log(t) - \mu}{\sigma}\right)}{t\sigma} \quad (7.40)$$

$$F(t) = \Phi\left(\frac{\log(t) - \mu}{\sigma}\right) \quad (7.41)$$

$$S(t) = 1 - \Phi\left(\frac{\log(t) - \mu}{\sigma}\right) \quad (7.42)$$

$$\mathbb{E}(T) = e^{\mu + \frac{\sigma^2}{2}} \quad (7.43)$$

$$\text{var}(T) = e^{2\mu + \sigma^2} (e^{\sigma^2} - 1) \quad (7.44)$$

- Generalized Gamma  $T \sim \text{GG}(\alpha, p, \lambda)$ , degrade to Weibull when  $\alpha = p$ , to Gamma when  $p = 1$

$$f(t) = p\lambda(\lambda t)^{\alpha-1} e^{-(\lambda t)^p} \Big/ \Gamma\left(\frac{\alpha}{p}\right) \quad (7.45)$$

$$\mathbb{E}(T) = \frac{\Gamma((\alpha + 1)/p)}{\lambda \Gamma(\alpha/p)} \quad (7.46)$$

## □ Likelihood for Censored Data

When dealing with censored data, we put a basic assumption that  $T \parallel C$  so that we can consider their distribution

<sup>80</sup>Weibull distribution could also be parameterized as  $W(p, \gamma)$ , where  $\gamma = 1/\lambda$  is the scale factor.

separately:

$$\text{General Notation: } \begin{cases} f(t) \\ F(t) \\ S(t) \\ \lambda(t) \end{cases} \quad T : \begin{cases} f_T(t) \\ F_T(t) \\ S_T(t) \\ \lambda_T(t) \end{cases} \quad C : \begin{cases} f_C(t) \\ F_C(t) \\ S_C(t) \\ \lambda_C(t) \end{cases} \quad (7.47)$$

Probability that we observe either  $T$  or  $C$ , or equivalently observe  $(\tilde{T}, \delta)$ :

$$\mathbb{P}(\tilde{T}, \delta) = \begin{cases} f_T(t)S_C(t), & \text{case event} \\ f_C(t)S_T(t), & \text{case censor} \end{cases} \quad (7.48)$$

$$= [f_T(t)S_C(t)]^\delta [f_C(t)S_T(t)]^{1-\delta} \quad (7.49)$$

$$\propto f_T(t)^\delta S_T(t)^{1-\delta} = \lambda_T(t)^\delta S_T(t) \quad (7.50)$$

Likelihood for estimating survival  $S(t)$  can be taken as the part of  $T$ :

$$L(\theta; \tilde{t}, \delta) = \prod_{i=1}^n f_T(\tilde{t}_i; \theta)^{\delta_i} S_T(\tilde{t}_i; \theta)^{1-\delta_i} = \prod_{i=1}^n \lambda_T(\tilde{t}_i; \theta)^{\delta_i} S_T(\tilde{t}_i; \theta) \quad (7.51)$$

$$= \prod_{e \in \mathcal{E}} f(\tilde{t}_e; \theta) \prod_{r \in \mathcal{R}} S(\tilde{t}_r; \theta) \quad (7.52)$$

where  $\mathcal{E}$  denotes indices of event data,  $\mathcal{R}$  for indices of right censored data. This form can be generalized to more kinds of censoring, e.g. left censor  $\mathcal{L}$ , interval censor  $\mathcal{I} = \{(t_{i,l}, t_{i,r})\}_{i=1}^{n_{\mathcal{I}}}$ :

$$L(\theta; \tilde{t}, \delta) = \prod_{e \in \mathcal{E}} f(\tilde{t}_e; \theta) \prod_{r \in \mathcal{R}} S(\tilde{t}_r; \theta) \prod_{l \in \mathcal{L}} [1 - S(\tilde{t}_l; \theta)] \prod_{(t_{i,l}, t_{i,r}) \in \mathcal{I}} [S(\tilde{t}_{i,l}; \theta) - S(\tilde{t}_{i,r}; \theta)] \quad (7.53)$$

then use proper methods to maximize the Likelihood / conduct hypothesis testing. Following are some knowledge recap for inference concerning likelihood:

#### □ Likelihood Function

Knowledge on likelihood function see [section. 2.2.4](#). Some recap:

$$\text{Likelihood: } L(\theta; X_1, X_2, \dots, X_n) = \prod_{i=1}^n f(X_i; \theta) \quad (7.54)$$

$$\text{Log-Likelihood: } \ell(\theta; X_1, X_2, \dots, X_n) = \sum_{i=1}^n \log \{f(X_i; \theta)\} \quad (7.55)$$

$$\text{Score: } U(\theta; X_1, X_2, \dots, X_n) = \frac{\partial \ell(\theta; X_1, X_2, \dots, X_n)}{\partial \theta} = \sum_{i=1}^n \frac{\partial \log \{f(X_i; \theta)\}}{\partial \theta} \quad (7.56)$$

$$\text{Fisher Information: } I(\theta) = -\mathbb{E} \left[ \frac{\partial^2 \log f(\vec{X}; \theta)}{\partial \theta \partial \theta^T} \right] = -n \mathbb{E}_{\vec{X}} \left[ \frac{\partial^2 \log f(X_i; \theta)}{\partial \theta \partial \theta^T} \right] = n \bar{I}(\theta) \quad (7.57)$$

$$\bar{I} = I_i(\theta) = -\mathbb{E} \left[ \frac{\partial^2 \log f(X_i; \theta)}{\partial \theta \partial \theta^T} \right] \quad (7.58)$$

$$\text{Observed Information: } I_n(\theta) = J(\theta) = -\sum_{i=1}^n \frac{\partial^2 \log f(X_i; \theta)}{\partial \theta \partial \theta^T} \quad (7.59)$$

Note: Fisher is an expectation of function of r.v., not random.

Properties:

$$\mathbb{E}_{\vec{X}} [U(\theta; \vec{X})] = 0 \quad (7.60)$$

$$I(\theta) = -\mathbb{E}_{\vec{X}} \left[ \frac{\partial^2 \log f(\vec{X}; \theta)}{\partial \theta \partial \theta^T} \right] \quad (7.61)$$

$$= \mathbb{E}_{\vec{X}} \left[ \frac{\partial \log f(\vec{X}; \theta)}{\partial \theta} \frac{\partial \log f(\vec{X}; \theta)}{\partial \theta^T} \right] = \mathbb{E}_{\vec{X}} [U(\theta; \vec{X}) U(\theta; \vec{X})^T] \quad (7.62)$$

$$\text{var}_{\vec{X}} [U(\theta; \vec{X})] = \mathbb{E}_{\vec{X}} \left[ \left( U(\theta; \vec{X}) - \mathbb{E}_{\vec{X}} [U(\theta; \vec{X})] \right) \left( U(\theta; \vec{X}) - \mathbb{E}_{\vec{X}} [U(\theta; \vec{X})] \right)^T \right] \quad (7.63)$$

$$= \mathbb{E}_{\vec{X}} [U(\theta; \vec{X}) U(\theta; \vec{X})^T] = I(\theta) \quad (7.64)$$

$$(7.65)$$

By CLT, considering  $U$  as a function of r.v.: (for a given  $\theta$  and the data generated from the distribution with **this** parameter  $\theta$ , i.e.  $U(\theta) = U(\theta; \vec{X}(\theta))$ )

$$\sqrt{n} \{ \bar{U}(\theta) - \mathbb{E}(U(\theta)) \} = \frac{1}{\sqrt{n}} U(\theta) \xrightarrow{d} N(0, \frac{I(\theta)}{n}) \quad (7.66)$$

and by taking MLE estimation  $\hat{\theta}^{MLE} \xrightarrow{p} \theta^*$ , we can estimate the distribution (Note that MLE Estimator requires  $U(\theta) = 0$ )

$$J(\hat{\theta})^{-1/2} \left( U(\hat{\theta}) - \mathbb{E}(U(\hat{\theta})) \right) = J(\hat{\theta})^{-1/2} U(\hat{\theta}) \xrightarrow{d} N(0, 1) \quad (7.67)$$

#### □ Statistical Inference on Parameter $\theta$

Statistical Inference concerning  $\theta$  can be conducting using the above functions of  $\theta$

- **Score Test:** Use the distribution of score function directly: we can construct

$$J(\theta_0)^{-1/2} U(\theta_0; \vec{X}(\theta_0)) \xrightarrow[\mathcal{H}_0]{\mathcal{L}} N(0, 1) \quad (7.68)$$

explanation: under  $H_0 : \theta = \theta_0$ , we should have  $\hat{\theta} \rightarrow \theta = \theta_0$ , which would lead to

$$J(\theta_0)^{-1/2} U(\theta_0; \vec{X}(\theta_0)) \xrightarrow{d} N(0, 1) \quad (7.69)$$

however if  $\hat{\theta} \rightarrow \theta \neq \theta_0$ , then

$$\mathbb{E} [U(\theta_0; \vec{X}(\theta))] \neq 0 \quad (7.70)$$

which would lead to a different distribution, thus we can test the assumption  $H_0 : \theta = \theta_0$  using **equation. 7.68**.

Conduct hypothesis testing utilizing the fractiles of  $N(0, 1)$

- **Wald Test:** Use the Taylor Series of  $U(\theta)$  to the 1<sup>st</sup> order

$$U(\theta) \approx -J(\hat{\theta})(\theta - \hat{\theta}) \Rightarrow J(\hat{\theta})^{1/2}(\hat{\theta} - \theta) \approx J(\hat{\theta})^{-1/2} U(\hat{\theta}) \xrightarrow{d} N(0, 1) \quad (7.71)$$

i.e.

$$\hat{\theta} \xrightarrow{d} N(\theta, J(\hat{\theta})^{-1}) \quad (7.72)$$

which can be utilized to construct testing statistics/interval estimations.

- **Likelihood Ratio Test:** Use the Taylor Series of  $\ell(\theta)$  to the 2<sup>nd</sup> order, and take  $\hat{\theta} = \hat{\theta}^{MLE}$  so that  $\ell'(\hat{\theta}) = 0$

$$\ell(\theta) \approx \ell(\hat{\theta}) - \frac{1}{2}(\theta - \hat{\theta})^T J(\theta)(\theta - \hat{\theta}) \Rightarrow 2(\ell(\hat{\theta}) - \ell(\theta)) \approx (\theta - \hat{\theta})^T J(\theta)(\theta - \hat{\theta}) \xrightarrow{d} \chi_p^2 \quad (7.73)$$

where  $p$  is the dimension of  $\theta$

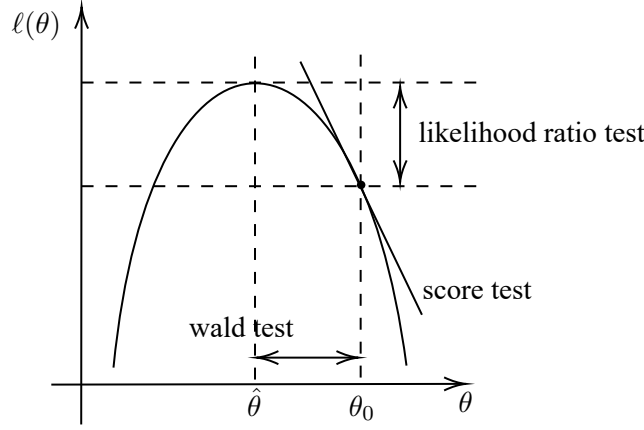


图 6: Illustration of Tests on  $\ell(\theta)$ - $\theta$  Plot

### 7.2.3 Non-Parametric Estimation to Survival Function

In this part we only focus on right censor data  $(\tilde{T}_i, \delta_i)$ ,  $\delta_i = 0$  for right censoring.

#### □ Kaplan-Meier Estimator

Idea of KM Estimator: Separate time into segments by censor/event time  $t_i$ , and decompose survival function into products of hazard within segments, using [equation. 7.15](#) which is:

$$\hat{S}(t) = \hat{\mathbb{P}}(T > t) = \prod_{t_i \leq t} \hat{\mathbb{P}}(T > t_i | T > t_{i-1}) \quad (7.74)$$

$$= \hat{\mathbb{P}}(T > t | T > t_i) \prod_{t_i \leq t} \left[ 1 - \hat{\mathbb{P}}(t_i \leq T < t_{i+1} | T > t_{i-1}) \right] \quad (7.75)$$

$$= (1 - \hat{\lambda}(t_i)) \prod_{t_i \leq t} (1 - \hat{\lambda}(t_{i-1})) \quad (7.76)$$

$$= \prod_{t_i \leq t} [1 - \hat{\lambda}(t_i)] \quad (7.77)$$

where  $\hat{\lambda}(t_i)$  are relatively easy to estimate with censoring considered.  $r_i$  for # at risk: not censored/event till  $t_i$ ,  $d_i$  for # event (death). We can model  $\hat{\lambda}_i$  as

$$d_i | r_i \sim B(r_i, \lambda_i) \xrightarrow{d} N(r_i \lambda_i, r_i \lambda_i (1 - \lambda_i)) \quad (7.78)$$

and obtain the MLE estimation of  $\hat{\lambda}_i | r_i, d_i$ <sup>81</sup>

<sup>81</sup>Here we use the  $\Delta$  method for estimating the variance of function of r.v.: if  $X \sim f(\mu, \sigma^2)$ :

$$g(X) \approx g(\mu) + g'(\mu)(X - \mu) \Rightarrow \text{var}(g(X)) \approx [g'(\mu)]^2 \text{var}(X) \leftarrow [g'(X)]^2 \text{var}(X) \quad (7.79)$$



$$\hat{\lambda}_i = \frac{d_i}{r_i} \quad (7.80)$$

$$\text{var}(\hat{\lambda}_i) = \text{var}\left(\frac{d_i}{r_i}\right) = \frac{\hat{\lambda}_i(1 - \hat{\lambda}_i)}{r_i} \quad (7.81)$$

$$\hat{S}(t) = \prod_{t_i \leq t} [1 - \hat{\lambda}(t_i)] = \prod_{t_i \leq t} \left[1 - \frac{d_i}{r_i}\right] \quad (\text{KM Estimator})$$

$$\text{var}(\hat{S}(t)) = \text{var} \left\{ \exp \left[ \log \hat{S}(t) \right] \right\} \quad (7.82)$$

$$\approx [\hat{S}(t)]^2 \text{var} \left[ \log \hat{S}(t) \right] \quad (7.83)$$

$$= [\hat{S}(t)]^2 \sum_{t_i \leq t} \text{var} \left[ \log(1 - \hat{\lambda}_i) \right] \quad (7.84)$$

$$= [\hat{S}(t)]^2 \sum_{t_i \leq t} \frac{1}{(1 - \hat{\lambda}_i)^2} \text{var}(\hat{\lambda}_i) \quad (7.85)$$

$$= [\hat{S}(t)]^2 \sum_{t_i \leq t} \frac{d_i}{r_i(r_i - d_i)} \quad (\text{Greenwood' Formula})$$

$$= [\hat{S}(t)]^2 \text{var}(\hat{\Lambda}(t)) \quad (7.86)$$

Interval Estimation of  $\hat{S}(t)$  can be conducted using pointwise interval/confidence band:

- Plain pointwise approach:

$$\hat{S}(t) \pm N_{1-\frac{\alpha}{2}} \sigma[\hat{S}(t)] \quad (7.87)$$

- Log-Log pointwise approach (R. default): using  $\hat{L}(t) = \log \left[ -\log \hat{S}(t) \right] = \log \left[ \hat{\Lambda}(t) \right]$

$$\hat{S}(t) \times e^{\pm N_{1-\frac{\alpha}{2}} \sigma(\hat{L}(t))} \quad (7.88)$$

where

$$\sigma(\hat{L}(t)) = \sqrt{\frac{1}{[\log \hat{S}(t)]^2} \sum_{t_i \leq t} \frac{d_i}{r_i(r_i - d_i)}} \quad (7.89)$$

- EP confidence band approach
- HW confidence band approach

Estimator of mean survival time:

$$\hat{\mu}_\tau = \int_0^\tau \hat{S}(t) dt \quad (7.90)$$

$$\text{var}(\hat{\mu}_\tau) = \sum_{t_i} \left[ \int_{t_i}^\tau \hat{S}(t) dt \right]^2 \frac{d_i}{r_i(r_i - d_i)} \quad (7.91)$$

#### □ Nelson-Aalen Estimator

Idea of NA Estimator: estimate  $\hat{\Lambda}(t)$  first, then obtain Fleming-Harrington Estimator  $\hat{S}_{FH}(t) = e^{-\hat{\Lambda}(t)}$ :

$$\hat{\Lambda}(t) = \sum_{t_i \leq t} \hat{\lambda}(t_i) = \sum_{t_i \leq t} \frac{d_i}{r_i} \quad (7.92)$$

$$\text{var}(\hat{\Lambda}(t)) = \sum_{t_i \leq t} \frac{d_i(r_i - d_i)}{r_i^2(r_i - 1)} \quad (7.93)$$

$$\hat{S}_{FH}(t) = \exp \left[ -\hat{\Lambda}(t) \right] \quad (7.94)$$

### □ Survival Function of Life Table

A key difference of survival data of life table is that we cannot know the exact event time/censor time, locating in  $[t_{i-1}, t_i)$ , in this case we usually estimate  $d_i, r_i$  using

$$d'_i = d_i \quad (7.95)$$

$$r'_i = r_i - \frac{c_i}{2} \quad (7.96)$$

where  $c_i$  is # censor in  $[t_i, t_{i+1})$ ,  $r_i$  is # censoring at the beginning of interval, i.e.  $t_{i-1}$ . And construct KM/NA estimator:

$$\hat{S}_{KM}(t) = \prod_{t_i \leq t} \left(1 - \frac{d_i}{r'_i}\right) \quad (7.97)$$

$$var(\hat{S}_{KM}(t)) = [\hat{S}_{KM}(t)]^2 \sum_{t_i \leq t} \frac{d_i}{r'_i(r'_i - d_i)} \quad (7.98)$$

$$\hat{\lambda}(t_{\text{mid } i}) = \frac{\hat{f}(t_i)}{\hat{S}(t_i)} = \frac{d_i}{(t_i - t_{i-1})(r'_i - \frac{d_i}{2})} = \frac{d_i}{(t_i - t_{i-1})(r_i - \frac{c_i + d_i}{2})} \quad (7.99)$$

where mid  $i$  means the mid point of  $[t_{i-1}, t_i)$ , i.e.  $\frac{t_{i-1} + t_i}{2}$

## 7.2.4 Hypothesis Testing to Group Comparison

Key focus: how to judge the difference between two survival function  $S_1(t), S_2(t)$ , or even when there are more than two groups.

### □ Mantel-Haenszel Logrank Test <sup>82</sup>

Idea of logrank test: adapt contingency table to censor table

表 5:  $2 \times 2$  contingency table

Group	Event $\delta$		Total
	Yes(1)	No(0)	
0	$d_0$	$r_0 - d_0$	$r_0$
1	$d_1$	$r_1 - d_1$	$r_1$
Total	$d$	$r - d$	$r$

- Recap: Pearson's  $\chi^2$  test: assign  $n$  sample into  $k$  groups, and conduct test on  $p_i, i = 1, 2, \dots, k$ , denote that  $v_i$  samples are assigned to the  $i^{\text{th}}$  groups, then

$$K_n = \sum_{i=1}^k \frac{(v_i - np_i)^2}{np_i} \xrightarrow{d} \chi^2_{df} \quad (7.100)$$

In the above example,  $df = k - 1$ . In  $2 \times 2$  contingency table,  $df = 1$  because we assume  $d, r, r_0, r_1$  are fixed.

<sup>82</sup>Note: Log means 'time record' here, rather than logarithm.

Pearson's  $\chi^2$  statistics for  $2 \times 2$  contingency table:

$$\chi_P^2 = \sum_{4 \text{ grids}} \frac{(\text{obs} - \text{expe})^2}{\text{expe}} \quad (7.101)$$

$$= \frac{(d_0 - r_0 \frac{d}{r})^2}{r_0 \frac{d}{r}} + \text{etc} \quad (7.102)$$

$$= \frac{[d_0 - r_0 \frac{d}{r}]^2}{r_0 r_1 d(r-d)/r^3} \sim \chi_1^2 \quad (7.103)$$

- Recap: Mental-Haenszel test, based on the Hypergeometric distribution that

$$d_0 \sim H(r_0, d, r) \Rightarrow \begin{cases} \mathbb{E}(d_0) = r_0 \frac{d}{r} \\ \text{var}(d_0) = \frac{r_0 r_1 d(r-d)}{r^2(r_0-1)} \end{cases}, \quad d_1, r_0 - d_0, r_1 - d_1 \text{ similar} \quad (7.104)$$

and construct

$$\chi_{MH}^2 = \frac{(\sum_{4 \text{ grids}} \text{obs} - \mathbb{E}(\text{obs}))^2}{\sum_{4 \text{ grids}} \text{var}(\text{obs})} = \frac{[d_0 - r_0 \frac{d}{r}]^2}{\frac{r_0 r_1 d(r-d)}{r^2(r-1)}} \sim \chi_1^2 \quad (7.105)$$

$\chi_{MH}^2$  and  $\chi_P^2$  are equal for large  $r$ .

$$\chi_{MH}^2 = \frac{r-1}{r} \chi_P^2 \quad (7.106)$$

- Cochran-Mantel-Haenszel log-rank test

For survival data  $t_1, t_2, \dots, t_K$ , we can construct a contingency table  $\mathcal{C}_i$  at each time, and test on the  $K \times 2 \times 2$  contingency table sequence:

表 6:  $2 \times 2$  contingency table,  $j = 1, 2, \dots, K$

Group	Event $\delta$		Total
	Yes(1)	No(0)	
0	$d_{0j}$	$r_{0j} - d_{0j}$	$r_{0j}$
1	$d_{1j}$	$r_{1j} - d_{1j}$	$r_{1j}$
Total	$d_j$	$r_j - d_j$	$r_j$

and get the CMH statistics for testing  $H_0 : \theta_{t_1} = \theta_{t_2} = \dots = \theta_{t_K} = 1, \theta$  for odds ratio between group 0/1.

$$\chi_{CMH}^2 = \frac{\left[ \sum_{j=1}^K (d_{0j} - r_{0j} \frac{d_j}{r_j}) \right]^2}{\sum_{j=1}^K \frac{r_{0j} r_{1j} d_j (r_j - d_j)}{r_j^2 (r_j - 1)}} \sim \chi_1^2 \quad (7.107)$$

where the  $K$  contingency tables are treated independent, but they are still ordinal because  $r_j$  contains information of history  $d_{t_i < t_j}, c_{t_i < t_j}$

Properties & Special Cases & Extension of CMH logrank test:

- No tied death  $d_j = 1$ :

$$\chi_{CMH}^2 = \frac{\left[ \sum_{j=1}^K (d_{0j} - r_{0j} \frac{d_j}{r_j}) \right]^2}{\sum_{j=1}^K \frac{r_{0j} r_{1j} d_j (r_j - d_j)}{r_j^2 (r_j - 1)}} = \frac{\left[ \sum_{j=1}^K (d_{0j} - r_{0j} \frac{d_j}{r_j}) \right]^2}{\sum_{j=1}^K \frac{r_{0j} r_{1j}}{r_j^2}} \sim \chi_1^2, \quad d_{0j} \in \{0, 1\} \quad (7.108)$$

- Intuition of  $\text{obs} - \mathbb{E}(\text{obs})$ :

$$\text{obs} - \mathbb{E}(\text{obs}) \approx d_{0j} - d_j \frac{r_{0j}}{r_j} \quad (7.109)$$

$$= \frac{r_{0j} r_{1j}}{r_j} (\hat{\lambda}_{0j} - \hat{\lambda}_{1j}) \quad (7.110)$$

- Attach weight  $w_i \geq 0, i = 1, 2, \dots, K$  to  $\mathcal{C}_i$ :

$$\chi_{CMH,w}^2 = \frac{\left[ \sum_{j=1}^K w_j (d_{0j} - r_{0j} \frac{d_j}{r_j}) \right]^2}{\sum_{j=1}^K w_j^2 \frac{r_{0j} r_{1j} d_j (r_j - d_j)}{r_j^2 (r_j - 1)}} \sim \chi_1^2 \quad (7.111)$$

bu choosing different kinds of weight  $\vec{w}$  we could get variants of CMH test.

- $w_i = 1$  for log-rank test. Focus more on difference at large  $t$
- $w_i = r_i$  for generalized Wilcoxon rank sum test. Focus more on difference at small  $t$ .

Note: weighted log-rank test should be used when **no cross** btw.  $S_1(t)$  and  $S_2(t)$ . Kink-of-Weight to choose depends on  $H_1$ .

#### □ Generalized Wilcoxon Rank Sum Test

- Wilcoxon Two-Sample Rank Sum Test: Knowledge of Wilcoxon two-sample rank sum test see [section. 2.4.6](#). Recap: to test the distribution difference of  $\vec{X} = (X_1, X_2, \dots, X_n)$  and  $\vec{Y} = (Y_1, Y_2, \dots, Y_m)$ , we mix them together and rank as  $\vec{Z} = (Z_{(1)}, Z_{(2)}, \dots, Z_{(m+n)})$ . Rank of  $X_i$ :

$$R_i \equiv \text{rank}(X_i) \text{ in } \vec{Z}, \quad i = 1, 2, \dots, n \quad (7.112)$$

$$R \equiv \sum_{i=1}^n R_i \quad (7.113)$$

A rank sum statistic to test:

$$\frac{R - \mathbb{E}(R)}{\sqrt{\text{var}(R)}} \sim N(0, 1) \quad (7.114)$$

$$\begin{cases} \mathbb{E}(R) = \frac{n(m+n+1)}{2} \\ \text{var}(R) = \frac{mn(m+n+1)}{12} \end{cases} \quad (7.115)$$

Rank sum statistic can be written in a Mann-Whitney form that can be generalized:

$$U_{ij} = U(X_i, Y_j) \equiv \begin{cases} +1 & , \text{case } X_i > Y_j \\ 0 & , \text{case } X_i = Y_j \\ -1 & , \text{case } X_i < Y_j \end{cases}, \quad U = \sum_{i,j}^{n,m} U_{ij} \quad (7.116)$$

$$R = \frac{n(m+n+1)}{2} + \frac{U}{2} \quad (7.117)$$

- Mann-Whitney-Wilcoxon rank sum test for censored data:

Notation: we still mix  $X = \{(\tilde{t}_{1i}, \delta_{1i})\}_{i=1}^n$  and  $Y = \{(\tilde{t}_{2j}, \delta_{2j})\}_{j=1}^m$  to get:

$$Z_{\text{mix}} = \{(\tilde{t}_i, \delta_i)\}_{i=1}^{m+n} \quad (7.118)$$

and the Mann-Whitney form for  $Z_{\text{mix}}$ :

$$U_{ij} = U(Z_i, Z_j) \equiv \begin{cases} +1 & , \text{case } \tilde{t}_i > \tilde{t}_j, \delta_j = 1 \\ 0 & , \text{case } \tilde{t}_i = \tilde{t}_j \text{ or } \delta_j = 0, \\ -1 & , \text{case } \tilde{t}_i < \tilde{t}_j, \delta_j = 1 \end{cases} \quad i = 1, 2, \dots, m+n, j = 1, 2, \dots, m+n. \quad (7.119)$$

and the Extended Wilcoxon rank sum statistic:

$$W = \sum_{i \text{ if } Z_i \in X}^{m+n} \sum_{j=1}^{m+n} U_{ij} \quad (7.120)$$

Under  $H_0 : X \sim Y$ , distribution features

$$\mathbb{E}(W) = 0 \quad (7.121)$$

$$\text{var}(W) = \frac{mn}{(m+n)(m+n-1)} \sum_{i=1}^{m+n} \left( \sum_{j=1}^{m+n} U_{ij} \right)^2 \quad (7.122)$$

– choose  $w_i = r_i$  in weighted log-rank test, and nominator becomes

$$\sum_{j=1}^K r_j (d_{0j} - r_{0j} \frac{d_j}{r_j}) = \sum_{j=1}^K [(r_{1j} - d_{1j})d_{0j} - (r_{0j} - d_{0j})d_{1j}] \quad (7.123)$$

$$= \sum_{j=1}^K [\#_{Y>t_j} \times \#_{X=t_j} - \#_{X>t_j} \times \#_{Y=t_j}] \quad (7.124)$$

$$= \#_{Y>X} - \#_{Y<X} \quad (7.125)$$

$$= -W \quad (7.126)$$

in which  $\chi_{w_i=r_i, CMH}^2$  test is the same as generalized Wilcoxon rank sum test.

## Section 7.3 Survival Model with Covariants

To research on the dependence of  $T$  with regard to covariants  $z$ . Survival data with covariants:  $X = (\tilde{t}_i, \delta_i, z_i)$

### 7.3.1 Cox's Proportion Hazard Model

Basic assumption on dependence form:  $T$  hazard part and covariants part are Separatable:

$$\lambda(t|z) = \lambda_0(t)g(z) \Leftrightarrow S(t|z) = [S_0(t)]^{g(z)}, \quad S_0(t) = e^{-\int_0^t \lambda_0(\tau) d\tau} \quad (7.127)$$

further a linear form  $g(z) = \beta^T z$  is used;

$$\lambda(t|z) = \lambda_0(t) \exp [\beta^T z] \quad (7.128)$$

Basic Assumptions of Cox's PH Model:

- constant regression coefficient  $\beta$ ;
- linear dependent of covariants  $\beta' z$ ;

- exponential link function  $e^{\cdot}$

in this proportion hazard model, the ratio of hazard only depend on  $\beta$ :

$$\log \left\{ \frac{\lambda_{z_i}(t)}{\lambda_0(t)} \right\} = \beta^T z_i \parallel t \quad (7.129)$$

The unknown components are  $\lambda_0(t), \beta$ , where the  $\lambda_0(t)$  lies in the  $dim \rightarrow \infty$  space, and causes difficulty in conducting inference. Solution: decompose full likelihood into two parts, in which one of them, **Partial Likelihood** is only function of  $\beta$ :

$$L(\beta, \lambda_0(\cdot); X) = \prod_i \left[ \left( \lambda_0(t_i) e^{\beta^T z_i} \right)_i^\delta \left( e^{-\int_0^{t_i} \lambda_0(\tau) d\tau} \right)^{e^{\beta^T z_i}} \right] \quad (7.130)$$

$$= L_{PH}(\beta; X) L_{res}(\beta, \lambda_0; X) \quad (7.131)$$

and we could focus on  $L_{PH}$  for further inference.

Note: the feasibility of partial likelihood comes from the form of proportion hazard.

#### □ Partial Likelihood without Tie

*Derivation:* First we assert  $t_i$  in ascending order and without tie:  $t_1 < t_2 < \dots < t_n$ , and we use an discrete estimated form of  $\lambda_0(t_i) = \lambda_i$

$$\int_0^{t_i} \lambda_0(\tau) d\tau \approx \sum_{j=1}^i \lambda_j \quad (7.132)$$

then we could use a trick to reformulate  $\ell(\beta, \lambda_1, \dots, \lambda_n; X)$  as<sup>83</sup>

$$\ell(\beta, \lambda_1, \dots, \lambda_n) = \sum_{i=1}^n \left\{ \delta_i (\log \lambda_i + \beta^T z_i) - \sum_{j=1}^i \lambda_j e^{\beta^T z_j} \right\} \quad (7.134)$$

$$= \sum_{i=1}^n \left\{ \delta_i (\log \lambda_i + \beta^T z_i) - \lambda_i \sum_{j=i}^n e^{\beta^T z_j} \right\} \quad (7.135)$$

and use MLE with regard to  $\lambda_i$  to get an estimate to  $\lambda_i$ :

$$\frac{\partial \ell(\beta, \lambda_1, \dots, \lambda_n)}{\partial \lambda_i} = 0 \Rightarrow \lambda_i(\beta) = \frac{\delta_i}{\sum_{j=1}^n e^{\beta^T z_j}} \quad \forall i \quad (7.136)$$

<sup>83</sup> Illustration for  $\sum_{i=1}^n \lambda_i \sum_{j=i}^n e^{\beta^T z_j} = \sum_{i=1}^n \sum_{j=1}^i \lambda_j e^{\beta^T z_i}$  (Abel's Lemma for Summation by Parts)

$$\begin{pmatrix} \lambda_1 e^{\beta^T z_1} & & & & \\ \lambda_1 e^{\beta^T z_2} & \lambda_2 e^{\beta^T z_2} & & & \\ \lambda_1 e^{\beta^T z_3} & \lambda_2 e^{\beta^T z_3} & \lambda_3 e^{\beta^T z_3} & & \\ \vdots & \vdots & \vdots & \ddots & \\ \lambda_1 e^{\beta^T z_n} & \lambda_2 e^{\beta^T z_n} & \lambda_3 e^{\beta^T z_n} & \dots & \lambda_n e^{\beta^T z_n} \end{pmatrix} \begin{matrix} \leftarrow \sum_{j=1}^1 \lambda_j e^{\beta^T z_1} \\ \leftarrow \sum_{j=1}^2 \lambda_j e^{\beta^T z_2} \\ \leftarrow \sum_{j=1}^3 \lambda_j e^{\beta^T z_3} \\ \leftarrow \vdots \\ \leftarrow \sum_{j=1}^n \lambda_j e^{\beta^T z_n} \end{matrix} \quad (7.133)$$

$$\begin{matrix} \uparrow & \uparrow & \uparrow & \uparrow & \uparrow \\ \lambda_1 \sum_{j=1}^n e^{\beta^T z_j} & \lambda_2 \sum_{j=2}^n e^{\beta^T z_j} & \lambda_3 \sum_{j=3}^n e^{\beta^T z_j} & \dots & \lambda_n \sum_{j=n}^n e^{\beta^T z_j} \end{matrix} \quad \sum_{i=1}^n \lambda_i \sum_{j=i}^n e^{\beta^T z_j} = \sum_{i=1}^n \sum_{j=1}^i \lambda_j e^{\beta^T z_i}$$

then we could get the partial likelihood

$$L(\beta, \lambda_1(\beta), \dots, \lambda_n(\beta)) = \prod_{i=1}^n \lambda_i(\beta)^{\delta_i} e^{\delta_i \beta' z_i} e^{-\sum_{j=1}^n e^{\beta' z_j}} \quad (7.137)$$

$$= e^{-\sum_i \delta_i} \prod_{i=1}^n \left( \frac{e^{\beta' z_i}}{\sum_{j:t_j \geq t_i} e^{\beta' z_j}} \right)^{\delta_i} \quad (7.138)$$

$$PL(\beta) \equiv \prod_{i=1}^n \left( \frac{e^{\beta' z_i}}{\sum_{j:t_j \geq t_i} e^{\beta' z_j}} \right)^{\delta_i} \quad (7.139)$$

$$P\ell = \sum_{i=1}^n \delta_i \left[ \beta' z_i - \log \left( \sum_{j:t_j \geq t_i} e^{\beta' z_j} \right) \right] \quad (7.140)$$

$$U(\beta) = \sum_{i=1}^n \delta_i \left[ z_i - \frac{\sum_{j:t_j \geq t_i} z_j e^{\beta' z_j}}{\sum_{j:t_j \geq t_i} e^{\beta' z_j}} \right] \quad (7.141)$$

$$J(\beta) = \sum_{i=1}^n \delta_i \left[ \sum_{j:t_j \geq t_i} \frac{e^{\beta' z_j}}{\sum_{l:t_l \geq t_j} e^{\beta' z_l}} \left( z_j - \frac{\sum_{l:t_l \geq t_j} z_l e^{\beta' z_l}}{\sum_{l:t_l \geq t_j} e^{\beta' z_l}} \right)^2 \right] \quad (7.142)$$

The above statistics can be use for further inference.

$$J(\beta_0)^{-1/2} U(\beta_0) \xrightarrow{d} N(0, 1) \quad (7.143)$$

$$(\hat{\beta} - \beta_0) \xrightarrow{d} N(0, J(\hat{\beta})^{-1}) \quad (7.144)$$

$$2(\ell(\hat{\beta}) - \ell(\beta)) \xrightarrow{d} \chi_p^2 \quad (7.145)$$

#### □ Modification for Partial Likelihood with Tie

There are various modification for tied data case. In PL without tie, the  $\frac{e^{\beta' z_i}}{\sum_{j:t_j \geq t_i} e^{\beta' z_j}}$  term are usually changed to adapt for the case of. Intuition:

$$\frac{e^{\beta' z_i}}{\sum_{j:t_j \geq t_i} e^{\beta' z_j}} = \frac{\lambda(t_i | z_i)}{\sum_{j:t_j \geq t_i} \lambda(t_j | z_j)} \approx \mathbb{P} \left( i^{\text{th event}} | \text{out of } \#\{j : t_j \geq t_i\} \right) \quad (7.146)$$

Notation:  $\mathcal{R}_i$  for all datapoints at risk at time  $t_i$ ,  $\mathcal{D}_i$  for event cases at time  $t_i$ ,  $\mathcal{D}_i \subset \mathcal{R}_i$

- Cox's modification: 索引 Cox's Modification

$$\mathbb{P} \left( \mathcal{D}_i \text{ events} \middle| |\mathcal{D}_i| \text{ out of } \#\{j : t_j \geq t_i\} \right) = \frac{e^{\sum_{l \in \mathcal{D}_i} \beta' z_l}}{\sum_{\text{all possible } |\mathcal{D}_j|=|\mathcal{D}_i|} e^{\sum_{l \in \mathcal{D}_j} \beta' z_l}} \quad (7.147)$$

drawback:  $\sim O(|\mathcal{D}_i|!)$  complexity

$$PL(\beta) = \prod_{i=1}^n \left\{ \frac{e^{\sum_{l \in \mathcal{D}_i} \beta' z_l}}{\sum_{\text{all possible } |\mathcal{D}_j|=|\mathcal{D}_i|} e^{\sum_{l \in \mathcal{D}_j} \beta' z_l}} \right\} \quad (7.148)$$

- Breslow's approximation:

$$\mathbb{P} \left( \mathcal{D}_i \text{ event} \middle| |\mathcal{D}_i| \text{ out of } \#\{j : t_j \geq t_i\} \right) \approx \frac{e^{\sum_{l \in \mathcal{D}_i} \beta' z_l}}{(\sum_{l \in \mathcal{R}_i} e^{\beta' z_l})^{|\mathcal{D}_i|}} \quad (7.149)$$

or directly write the PL as

$$PL(\beta) = \prod_{i=1}^n \left\{ \prod_{j \in \mathcal{D}_i} \frac{e^{\beta' z_j}}{\sum_{l \in \mathcal{R}_i} e^{\beta' z_l}} \right\} \quad (7.150)$$

- Efron's approximation: usually better than Breslow's, default method in `coxph()`

$$PL(\beta) = \prod_{i=1}^n \left\{ \frac{e^{\sum_{l \in \mathcal{D}_i} \beta' z_l}}{\prod_{j=1}^{|\mathcal{D}_i|} \left( \sum_{l \in \mathcal{R}_i} e^{\beta' z_l} - \frac{j-1}{|\mathcal{D}_i|} \sum_{l \in \mathcal{D}_i} e^{\beta' z_l} \right)} \right\} \quad (7.151)$$

#### □ Extension for Time-Dependent Variable

Model:

$$\begin{cases} \lambda(t) = \lambda_0(t) e^{\beta' z(t)} \\ \lambda(t) = \lambda_0(t) e^{\beta(t)' z} \end{cases} \quad (7.152)$$

#### □ Diagnostic Methods for PH Assumption

- log-log plots: for categorical  $z_1, z_2$ , use relation

$$\log [-\log S(t, z_1)] - \log [-\log S(t, z_2)] = \beta'(z_1 - z_2) \perp\!\!\!\perp t \quad (7.153)$$

Plot of  $\log [\log \hat{S}(t, z)]$  should be 'parallel' curves.

- Check the coherence bet. observed data v.s. expected data.
- Goodness-of-fit using Schoenfeld residuals 索引 Schoenfeld Residual

$$\hat{r}_i = z_i - \sum_{j \in \mathcal{R}_i} z_k \cdot p(\hat{\beta}, z_k) = z_i \bar{z}_i \quad (7.154)$$

$$p(\beta, z_k) := \frac{e^{\beta' z_k}}{\sum_{j \in \mathcal{R}_k} e^{\beta' z_j}} \quad (7.155)$$

- (Generalized) Cox-Snell Residual for overall goodness-of-fit:

Recall: for r.v.  $T \sim f(t)$ ,  $S(t) = \int_t^\infty f(\tau) d\tau$ . function of r.v. has distribution:

$$S(T) \sim U(0, 1) \Rightarrow \Lambda(T) \sim \varepsilon(1) \quad (7.156)$$

define Cox-Snell Residual:

$$\hat{\Lambda}(z_i) = -\log \hat{S}(z_i) \quad (7.157)$$

the set  $\{\hat{\Lambda}(z_i)\}$  could be viewed as a sample from  $\varepsilon(1)$ , we could test on the distribution, e.g. plot the cumulative hazard **of residual** v.s. residual to check  $\Lambda(e) = e$ .

- Delta-Beta Residual for influential: for  $\beta = (\beta_0 = 1, \beta_1, \dots)$ , define

$$\hat{\Delta}_{ij} = \hat{\beta}_j - \hat{\beta}_{j(\wedge i)} \quad (7.158)$$

where  $\wedge i$  for estimator with the  $i^{\text{th}}$  subject removed. Plot the scatter plot of  $\hat{\Delta}_{ij}$  to locate influential.

#### □ Experiment Design for Log-rank Test under PH Assumption

Question: how many events are needed for the testing  $H_0 : \beta = 0 \rightsquigarrow H_a : \beta = \beta_a$ ?



Using log-rank statistics [equation. 7.108](#) in  $z$ -test form, under condition 1. no ties  $d_j = 0, 1, 2$ .  $\beta_a$  is small enough for taylor expansion:<sup>84</sup>

$$T_{CMH} = \frac{\sum_{j=1}^K \left( d_{0j} - r_{0j} \frac{d_j}{r_j} \right)}{\sqrt{\sum_{j=1}^K \frac{r_{0j} r_{1j} d_j (r_j - d_j)}{r_j^2 (r_j - 1)}}} \xrightarrow{d} N(\beta_a \sqrt{d\theta(1-\theta)}, 1) \quad (7.160)$$

where  $d = \sum_{j=1}^K d_j$ ,  $\theta$  is the prevalence of group 1.

Power of the test: denote  $\gamma$  for probability of type II error

$$\mathbb{P}(T_{CMH} > N_{\alpha/2} | H_a) = 1 - \gamma \Rightarrow \mu := \beta_a \sqrt{d\theta(1-\theta)} \approx N_{\alpha/2} + N_\gamma \quad (7.161)$$

Minimum number of events:

$$d = \frac{(N_{\alpha/2} + N_\gamma)^2}{\beta_a^2 \theta(1-\theta)} \quad (7.162)$$

### 7.3.2 Accelerated Failure Time Model

Basic form of AFT Model (Accelerated Failure Time Model) for categorical covariants:

$$S(t; z = 1) = S(\gamma t; z = 2) \Leftrightarrow \mathbb{P}(T_1 > t) = \mathbb{P}(T_2 > \gamma t) \quad (7.163)$$

Usually we attach some assumptions on function form of  $S(t, z)$ , usually take (parameter denoted  $\alpha$ ):

- Exponential:

$$S(t) = e^{-\lambda t}, \quad \lambda(t) = \lambda \quad (7.164)$$

$$\Rightarrow t = -\frac{1}{\lambda} \log S(t) \quad (7.165)$$

$$\Rightarrow \gamma := e^{\alpha' z} = \frac{1}{\lambda} = e^{-\beta' z} \quad (7.166)$$

i.e. Exponential AFT model in which  $\gamma = e^{\alpha' z}$  is equivalent to PH model with  $\lambda = e^{\beta' z}$ , and  $\beta = -\alpha$

- Weibull:

$$S(t) = e^{-\lambda t^p}, \quad \lambda(t) = \lambda p t^{p-1} \quad (7.167)$$

$$\Rightarrow t = -\frac{1}{\lambda^{1/p}} \log S(t) \quad (7.168)$$

$$\Rightarrow \gamma := e^{\alpha' z} = \frac{1}{\lambda^{1/p}} = e^{-\beta' z/p} \quad (7.169)$$

i.e. Weibull AFT model with  $\gamma = e^{\alpha' z}$  is equivalent to PH model with  $\lambda = e^{\beta' z}$ , and  $\beta = -\alpha p$

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<sup>84</sup>Proof key:

$$d_{0j} \sim B(p_{0j}), \quad p_{0j} = \frac{r_{0j} \lambda_0}{r_{0j} \lambda_0 + r_{1j} \lambda_0 e^{\beta_a}} \quad (7.159)$$

and at small  $\beta_a$ , take approximation  $\theta \approx r_{1j}/r_j$

- General Case: In different groups  $z$ , survival time

$$T_i = T_0 e^{\alpha' z_i + \varepsilon_i / p}, \quad \varepsilon_i \sim \varepsilon(1) \quad (7.170)$$

$$S_i(t) = \mathbb{P}(T_i \geq t) \quad (7.171)$$

$$= \mathbb{P}\left(\log T_0 + \alpha' z_i + \frac{\varepsilon_i}{p} \geq \log t\right) \quad (7.172)$$

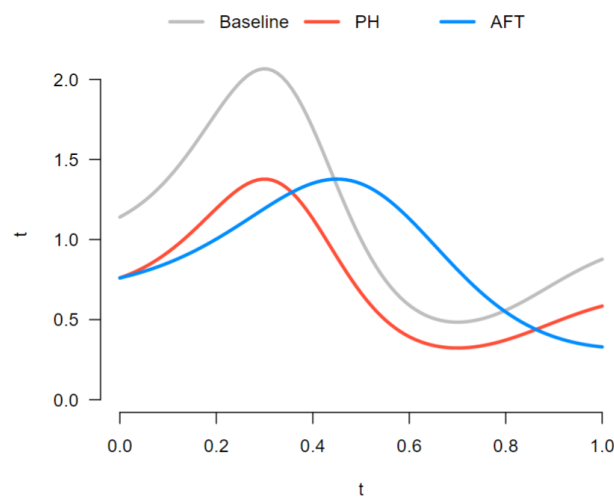
$$= S_{\varepsilon(1)}(p(\log t - \log T_0 - \alpha' z_i)) \quad (7.173)$$

#### □ AFT Model and PH Model

An intuition for parameters in AFT model and PH model:

$$\text{PH} : \lambda_i(t) = \lambda_0(t) e^{\eta_i}$$

$$\text{AFT} : \lambda_i(t) = \lambda_0(e^{-\eta_i} t) e^{-\eta_i}$$



Usually AFT model depends on a parametric model, while PH model only depends on the PH assumption.

#### ▷ R. Code

An example:

```
1 coxph(formula = Surv(start, stop, event) ~ rx + number + size + factor(
  enum), data = bladder2)
```

## Chapter. VIII 生物统计学概论部分

Instructor: Tianying Wang

Biostatistics is discipline to apply statistical methods to biological problems, including medicine, biology experiment, public health, etc. This section would focus on basic quantative skills to be used in advanced biostatistics research.

### Section 8.1 Factor Model and ANOVA

A major question in biostatistics is to study the difference between groups, i.e. explanatory variable  $X$  is categorical. A ‘way’ to conduct grouping is called a **factor**, e.g.  $\{\alpha_i\}$  where each  $i$  corresponds to a **level** of the factor.

To compare groups, e.g. to determine whether there is significant difference between  $Y$  of each group, ANOVA is used. The key thought is to analyze difference value and variance and see whether the difference is large enough to ‘exceed’ variance.

#### □ Factor Notation

Response  $Y$  is denoted by its subscript to declare its group and index in this group, e.g.  $Y_{ijkl}$  indicates it is the  $l^{\text{th}}$  sample in group  $(i, j, k)$

#### 8.1.1 Single Factor Model and One-Way ANOVA

##### □ Cell Means Model

$$Y_{ij} = \mu_i + \varepsilon_{ij}, \quad \varepsilon_{ij} \text{ i.i.d. } \sim N(0, \sigma^2)$$

Estimation target:  $\mu_1, \dots, \mu_r, \sigma^2$

Hypothesis testing  $H_0: \mu_1 = \dots = \mu_r = \mu$ , v.s.  $H_1$ : at least 1  $\mu_i$  is different.

Estimation:

$$\hat{\mu}_i = \bar{Y}_{i.} = \frac{1}{n_i} \sum_{j=1}^{n_i} Y_{ij} \quad (8.1)$$

$$s_i^2 = \frac{1}{n_i - 1} \sum_{j=1}^{n_i} (Y_{ij} - \bar{Y}_{i.})^2 \quad (8.2)$$

$$s^2 = \frac{\sum_{i=1}^r (n_i - 1) s_i^2}{\sum_{i=1}^r (n_i - 1)} = \frac{\sum_{i=1}^r (n_i - 1) s_i^2}{n_T - r} \quad (8.3)$$

Key of ANOVA: Decomposition of variation SS:

$$SST = \sum_{i=1}^r \sum_{j=1}^{n_i} (Y_{ij} - \bar{Y}_{..})^2 = \sum_{i=1}^r \sum_{j=1}^{n_i} (Y_{ij} - \bar{Y}_{i.} + \bar{Y}_{i.} - \bar{Y}_{..})^2 \quad (8.4)$$

$$= \sum_{i=1}^r \sum_{j=1}^{n_i} (Y_{ij} - \bar{Y}_{i.})^2 + \sum_{i=1}^r (\bar{Y}_{i.} - \bar{Y}_{..})^2 \quad (8.5)$$

$$= SSE + SSR \quad (8.6)$$

##### □ Effect Model

$$Y_{ij} = \mu + \alpha_i + \varepsilon_{ij} \quad \varepsilon_{ij} \text{ i.i.d. } \sim N(0, \sigma^2) \quad (8.7)$$

Estimation target:  $\mu, \alpha_1, \dots, \alpha_r, \sigma^2$ , w.r.t.  $\sum_{i=1}^r \alpha_i = 0$ .

Hypothesis testing:  $H_0 : \alpha_1 = \dots = \alpha_r = 0$ , v.s.  $H_1 : \text{at least 1 } \alpha_i \neq 0$

Estimation:

$$\hat{\mu} = \frac{1}{r} \sum_{i=1}^r \sum_{j=1}^{n_i} \frac{Y_{ij}}{n_i} \quad (8.8)$$

$$\hat{\alpha}_i = \bar{Y}_{i\cdot} - \hat{\mu} = \frac{1}{n_i} \sum_{j=1}^{n_i} Y_{ij} - \hat{\mu} \quad (8.9)$$

$$s_i^2 = \frac{1}{n_i - 1} \sum_{j=1}^{n_i} (Y_{ij} - \bar{Y}_{i\cdot})^2 \quad (8.10)$$

$$s^2 = \frac{\sum_{i=1}^r (n_i - 1) s_i^2}{n_T - r} \quad (8.11)$$

### 8.1.2 Fixed Effect and Random Effect

When divided into groups/naturally assigned in groups, we need to specify whether the factor levels are specially chosen (fixed effect) or randomly chosen from a ‘population of levels’ (random effect).

- Fixed Effect: whether there is a difference between / estimating the value of mean value  $\mu_i$  of each specific levels
- Random Effect: whether the overall behaviour of  $\mu_i$  comes from a ‘random distribution’

Comment on fixed / random in actual model building and statistical inference:

- whether a factor is fixed or random should be determined by how the data are obtained and the research problem to be studied, i.e. determining fixed / random model does **not** come from mathematics.
- for effect of interaction term, say  $(\alpha\beta)_{ij}$  as the interaction effect of factor  $\alpha_i$  and  $\beta_j$ , then  $(\alpha\beta)_{ij}$  would be random once one of  $\alpha_i$  or  $\beta_j$  is random.

Here use a one-way factor model as example:

□ **Fixed Effect:**

$$Y_{ij} = \mu + \alpha_i + \varepsilon_{ij} \quad \varepsilon_{ij} \text{ i.i.d. } \sim N(0, \sigma^2) \quad (8.12)$$

Estimation target:  $\mu, \alpha_1, \dots, \alpha_r, \sigma^2$ , w.r.t.  $\sum_{i=1}^r \alpha_i = 0$ .

Hypothesis testing:  $H_0 : \alpha_1 = \dots = \alpha_r = 0$ , v.s.  $H_1 : \text{at least 1 } \alpha_i \neq 0$

Estimation (the same):

$$\hat{\mu} = \frac{1}{r} \sum_{i=1}^r \sum_{j=1}^{n_i} \frac{Y_{ij}}{n_i} \quad (8.13)$$

$$\hat{\alpha}_i = \bar{Y}_{i\cdot} - \hat{\mu} = \frac{1}{n_i} \sum_{j=1}^{n_i} Y_{ij} - \hat{\mu} \quad (8.14)$$

$$s_i^2 = \frac{1}{n_i - 1} \sum_{j=1}^{n_i} (Y_{ij} - \bar{Y}_{i.})^2 \quad (8.15)$$

$$s^2 = \frac{\sum_{i=1}^r (n_i - 1) s_i^2}{n_T - r} \quad (8.16)$$

ANOVA table:

Source of Var	SS	dof	MS	$\mathbb{E}(\text{MS})$
$\alpha_i$	$\text{SS}\alpha = \sum_{i=1}^r n_i (\bar{Y}_{i.} - \bar{Y}_{..})^2$	$r - 1$	$\frac{\text{SS}\alpha}{r - 1}$	$\sigma^2 + \frac{\sum_{i=1}^r n_i \alpha_i^2}{r - 1}$
$\sigma^2$	$\text{SSE} = \sum_{i=1}^r \sum_{j=1}^{n_i} (Y_{ij} - \bar{Y}_{i.})^2$	$n_T - r$	$\frac{\text{SSE}}{n_T - r}$	$\sigma^2$

$F$  statistics for  $H_0 : \alpha_1 = \dots = \alpha_r = 0$ :

$$F = \frac{\text{MS}\alpha}{\text{MSE}} \sim F_{r-1, n_T-r} \quad (8.17)$$

#### □ Random Effect

$$Y_{ij} = \mu + \alpha_i + \varepsilon_{ij} \quad \alpha_i \text{ i.i.d. } \sim N(0, \sigma_\alpha^2), \quad \varepsilon_{ij} \text{ i.i.d. } \sim N(0, \sigma^2) \quad (8.18)$$

Estimation target:  $\mu, \sigma_\alpha^2, \sigma^2$

Hypothesis testing  $H_0 : \sigma_\alpha^2 = 0$ , v.s.  $H_1 : \sigma_\alpha^2 \neq 0$

Estimation (the same):

$$\hat{\mu} = \frac{1}{r} \sum_{i=1}^r \sum_{j=1}^{n_i} \frac{Y_{ij}}{n_i} \quad (8.19)$$

$$\hat{\alpha}_i = \bar{Y}_{i.} - \hat{\mu} = \frac{1}{n_i} \sum_{j=1}^{n_i} Y_{ij} - \hat{\mu} \quad (8.20)$$

$$s_i^2 = \frac{1}{n_i - 1} \sum_{j=1}^{n_i} (Y_{ij} - \bar{Y}_{i.})^2 \quad (8.21)$$

$$s^2 = \frac{\sum_{i=1}^r (n_i - 1) s_i^2}{n_T - r} \quad (8.22)$$

ANOVA table:

Source of Var	SS	dof	MS	$\mathbb{E}(\text{MS})$
$\alpha_i$	$\text{SS}\alpha = \sum_{i=1}^r n_i (\bar{Y}_{i.} - \bar{Y}_{..})^2$	$r - 1$	$\frac{\text{SS}\alpha}{r - 1}$	$\sigma^2 + n\sigma_\alpha^2$
$\sigma^2$	$\text{SSE} = \sum_{i=1}^r \sum_{j=1}^{n_i} (Y_{ij} - \bar{Y}_{i.})^2$	$n_T - r$	$\frac{\text{SSE}}{n_T - r}$	$\sigma^2$

$F$  statistics for  $H_0 : \alpha_1 = \dots = \alpha_r = 0$ :

$$F = \frac{\text{MS}\alpha}{\text{MSE}} \sim F_{r-1, n_T-r} \quad (8.23)$$

### 8.1.3 Two Factor Model and Two-Way ANOVA

Two factor model with interaction term:

$$Y_{ijk} = \mu + \alpha_i + \beta_j + (\alpha\beta)_{ij} + \varepsilon_{ijk}$$

$$Y_{ijk} - \bar{Y}_{...} = (\bar{Y}_{i..} - \bar{Y}_{...}) + (\bar{Y}_{.j.} - \bar{Y}_{...}) \quad (8.24)$$

$$+ (\bar{Y}_{ij.} - \bar{Y}_{i..} - \bar{Y}_{.j.} + \bar{Y}_{...}) + (Y_{ijk} - \bar{Y}_{ij.}) \quad (8.25)$$

$$\alpha_i + \beta_j + (\alpha\beta)_{ij} + \varepsilon_{ijk} = ((\mu + \alpha_i) - \mu) + ((\mu + \beta_j) - \mu) \quad (8.26)$$

$$+ ((\mu + \alpha_i + \beta_j + (\alpha\beta)_{ij}) - (\mu + \alpha_i) - (\mu + \beta_j) + \mu) + (\varepsilon_{ijk}) \quad (8.27)$$

Here for convenience and clarity, when applying model with more factors, we use terms like  $(\alpha\beta)_{ij}$  to avoid confusion of too many symbols.

### 8.1.4 General Case for Factor Model

e.g. three factors model

$$Y_{ijkl} = \mu + \alpha_i + \beta_j + \gamma_k + (\alpha\beta)_{ij} + (\alpha\gamma)_{ik} + (\beta\gamma)_{jk} + (\alpha\beta\gamma)_{ijk} + \varepsilon_{ijkl}$$

#### □ Montgomery's Method for Restricted Model

Montgomery describe a useful trick to form the ANOVA table and to find corresponding  $\mathbb{E}(MS)$  (EMS), and finally help construct proper  $F^*$  statistics. Here an explicit example of three factor (1F+2R) model is provided to illustrate the procedure.

Model we use here as example:

$$Y_{ijkl} = \mu + \alpha_i + \beta_j + \gamma_k + (\alpha\beta)_{ij} + (\alpha\gamma)_{ik} + (\beta\gamma)_{jk} + (\alpha\beta\gamma)_{ijk} + \varepsilon_{ijkl} \quad (8.28)$$

$$i = 1, 2, \dots, a \quad (8.29)$$

$$j = 1, 2, \dots, b \quad (8.30)$$

$$k = 1, 2, \dots, c \quad (8.31)$$

$$l = 1, 2, \dots, n \quad (8.32)$$

where  $a$  is for fixed effect,  $b$  and  $c$  are for random effect.

model parameter:

$$\theta = \{\mu, \alpha_i^{i=1, \dots, a}, \sigma_\beta^2, \sigma_\gamma^2, \sigma_{\alpha\beta}^2, \sigma_{\alpha\gamma}^2, \sigma_{\beta\gamma}^2, \sigma_{\alpha\beta\gamma}^2, \sigma^2\} \quad (8.33)$$

1. Prepare the framework of the EMS table, including:

- column: list groups, and their **random/fixed**, and their **number of levels**.
- row: terms in the model
- **error term** written as  $\varepsilon_{(ijk)l}$ , i.e. random term index excluded from the bracket.

Random/Fix	F	R	R	R	
# level	<i>a</i>	<i>b</i>	<i>c</i>	<i>n</i>	
Index	<i>i</i>	<i>j</i>	<i>k</i>	<i>l</i>	E (MS)
$\alpha_i$					
$\beta_j$					
$\gamma_k$					
$(\alpha\beta)_{ij}$					
$(\alpha\gamma)_{ik}$					
$(\beta\gamma)_{jk}$					
$(\alpha\beta\gamma)_{ijk}$					
$\varepsilon_{(ijk)l}$					

2. For each row, copy the number of observations under each column subscripts, if the column subscript does not appear in the index subscripts of the term. e.g.  $(\alpha\beta)_{ij}$  does not contain,  $k, l$  so fill in the grid  $((\alpha\beta)_{ij}, k)$  with  $c$ , and fill  $((\alpha\beta)_{ij}, l)$  with  $n$ .

Random/Fix	F	R	R	R	
# level	<i>a</i>	<i>b</i>	<i>c</i>	<i>n</i>	
Index	<i>i</i>	<i>j</i>	<i>k</i>	<i>l</i>	E (MS)
$\alpha_i$		<i>b</i>	<i>c</i>	<i>n</i>	
$\beta_j$	<i>a</i>		<i>c</i>	<i>n</i>	
$\gamma_k$	<i>a</i>	<i>b</i>		<i>n</i>	
$(\alpha\beta)_{ij}$			<i>c</i>	<i>n</i>	
$(\alpha\gamma)_{ik}$		<i>b</i>		<i>n</i>	
$(\beta\gamma)_{jk}$	<i>a</i>			<i>n</i>	
$(\alpha\beta\gamma)_{ijk}$				<i>n</i>	
$\varepsilon_{(ijk)l}$					

3. **1** is filled in the row of error term  $(\varepsilon_{(ijk)l}, \cdot)$

Random/Fix	F	R	R	R	
# level	$a$	$b$	$c$	$n$	
Index	$i$	$j$	$k$	$l$	$\mathbb{E}(\text{MS})$
$\alpha_i$		$b$	$c$	$n$	
$\beta_j$	$a$		$c$	$n$	
$\gamma_k$	$a$	$b$		$n$	
$(\alpha\beta)_{ij}$			$c$	$n$	
$(\alpha\gamma)_{ik}$		$b$		$n$	
$(\beta\gamma)_{jk}$	$a$			$n$	
$(\alpha\beta\gamma)_{ijk}$				$n$	
$\varepsilon_{(ijk)l}$	1	1	1	1	

4. for remaining grids, fill 1 if the column is Fixed, or 0 if the column is Random

Random/Fix	F	R	R	R	
# level	$a$	$b$	$c$	$n$	
Index	$i$	$j$	$k$	$l$	$\mathbb{E}(\text{MS})$
$\alpha_i$	0	$b$	$c$	$n$	
$\beta_j$	$a$	1	$c$	$n$	
$\gamma_k$	$a$	$b$	1	$n$	
$(\alpha\beta)_{ij}$	0	1	$c$	$n$	
$(\alpha\gamma)_{ik}$	0	$b$	1	$n$	
$(\beta\gamma)_{jk}$	$a$	1	1	$n$	
$(\alpha\beta\gamma)_{ijk}$	0	1	1	$n$	
$\varepsilon_{(ijk)l}$	1	1	1	1	

5. Now the L.H.S. of the table is finished. To get the  $\mathbb{E}(\text{MS})$ , we will need the coefficients in front of the variance term<sup>85</sup>. The approach is as follows: use the fourth row  $(\alpha\beta)_{ij}$  as example:

\* (e.g. focus on row  $(\alpha\beta)_{ij}$ )

(a) ignore columns with the same indexes, here it would be column  $i$  and  $j$

(b) select rows with the same or more extra indexes, here it would be row  $(\alpha\beta)_{ij}$ ,  $(\alpha\beta\gamma)_{ijk}$ ,  $\varepsilon_{(ijk)l}$

(c) now the grids to be used are colored brown

(d) for each row, multiply all used grids to form the corresponding coefficient (of the variance of this row), here it would be

$$\mathbb{E}(\text{MS}_{(\alpha\beta)}) = c \times n\sigma_{\alpha\beta}^2 + 1 \times n\sigma_{\alpha\beta\gamma}^2 + 1 \times 1\sigma^2 = \sigma^2 + cn\sigma_{\alpha\beta}^2 + n\sigma_{\alpha\beta\gamma}^2 \quad (8.34)$$

<sup>85</sup>Note the variance term is what we already know: for fixed effect it would be  $\frac{\sum_i \alpha_i^2}{a-1}$ , for random effect it would be  $\sigma_\beta^2$



Random/Fix	F	R	R	R	
# level	$a$	$b$	$c$	$n$	
Index	$i$	$j$	$k$	$l$	$\mathbb{E}(\text{MS})$
$\alpha_i$	0	$b$	$c$	$n$	$\sigma^2 + cn\sigma_{\alpha\beta}^2 + bn\sigma_{\alpha\gamma}^2 + n\sigma_{\alpha\beta\gamma}^2 + bcn\frac{\sum_i \alpha_i^2}{a-1}$
$\beta_j$	$a$	1	$c$	$n$	$\sigma^2 + an\sigma_{\beta\gamma}^2 + acn\sigma_{\beta}^2$
$\gamma_k$	$a$	$b$	1	$n$	$\sigma^2 + an\sigma_{\beta\gamma}^2 + abn\sigma_{\gamma}^2$
$(\alpha\beta)_{ij}$	0	1	$c$	$n$	$\sigma^2 + cn\sigma_{\alpha\beta}^2 + n\sigma_{\alpha\beta\gamma}^2$
$(\alpha\gamma)_{ik}$	0	$b$	1	$n$	$\sigma^2 + bn\sigma_{\alpha\gamma}^2 + n\sigma_{\alpha\beta\gamma}^2$
$(\beta\gamma)_{jk}$	$a$	1	1	$n$	$\sigma^2 + an\sigma_{\beta\gamma}^2$
$(\alpha\beta\gamma)_{ijk}$	0	1	$c$	$n$	$\sigma^2 + n\sigma_{\alpha\beta\gamma}^2$
$\varepsilon_{(ijk)l}$	1	1	$c$	$n$	$\sigma^2$

6. Now we can use  $\mathbb{E}(\text{MS})$  to construct corresponding  $F^*$ . e.g. to test  $H_0 : \alpha_1 = \alpha_2 = \dots = \alpha_a$ , test:

$$\mathbb{E}(\text{MS}_{\alpha}) = \sigma^2 + cn\sigma_{\alpha\beta}^2 + bn\sigma_{\alpha\gamma}^2 + n\sigma_{\alpha\beta\gamma}^2 + bcn\frac{\sum_i \alpha_i^2}{a-1} \quad (8.35)$$

$$\mathbb{E}(\text{MS}_{\alpha\beta} + \text{MS}_{\alpha\gamma} - \text{MS}_{\alpha\beta\gamma}) = \sigma^2 + cn\sigma_{\alpha\beta}^2 + bn\sigma_{\alpha\gamma}^2 + n\sigma_{\alpha\beta\gamma}^2 \quad (8.36)$$

$$F_{\alpha_i}^* = \frac{\text{MS}_{\alpha} + \text{MS}_{\alpha\beta\gamma}}{\text{MS}_{\alpha\beta} + \text{MS}_{\alpha\gamma}} \sim F_{(a-1)+(a-1)(b-1)(c-1), (a-1)(b-1)+(a-1)(c-1)} \quad (8.37)$$

### 8.1.5 Diagnosis

Some useful diagnosis to check assumptions:

- Levene's Test for homogeneity of variance: ▷ [R. Code](#)

```
1 dat %>% group_by(cat_1) %>% rstatix::levene_test(y ~ group)
```

- Shapiro-Wilk Test for Normality: ▷ [R. Code](#)

```
1 dat %>% group_by(cat_1) %>% rstatix::shapiro_test(y)
```

- Outlier test: ▷ [R. Code](#)

```
1 dat %>% group_by(cat_1) %>% rstatix::identify_outliers(y)
```

### 8.1.6 Miscellaneous Topics

Some miscellanea in design of experiment and about some advanced models:

#### □ Crossed and Nested Factors

In multi-factor studies, we may not be able to go through all possible factor settings.

- Crossed factor: all level combinations are covered in the experiment.
- Nested factor: the levels of one factor are unique to a particular level of another factor.

### □ Longitudinal Study

When discrete **time** is used as factors, say  $\tau_t^{t=\{t_1, \dots, t_T\}}$  in  $Y_{ijt}$  where  $i$  for treatment,  $j$  for individuals, we may notice that response  $Y_{ijt}$  is effected by individual baseline, in such case we cannot use the ordinary factor model to study the difference of trent. Instead we would use **longitudinal study** to construct model and study the trend.. e.g.

$$Y_{ijt} = \mu + \alpha_i + \beta_{j(i)} + \tau_t + \varepsilon_{ijt} \quad (8.38)$$

where  $\beta_{j(i)}$  stands for indivudual difference (say, with assumption  $\beta_{j(i)} \sim N(0, \sigma_\beta^2)$ )

## Section 8.2 Statistical Inference on Contingency Table

Contingency table is an easy way to display categorical variables, an example:

表 7: A  $2 \times 2$  contingency table

Variable Y	Variable Z		Total
	D	D <sup>c</sup>	
E	$n_{11}$	$n_{12}$	$n_{1\cdot}$
E <sup>c</sup>	$n_{21}$	$n_{22}$	$n_{2\cdot}$
Total	$n_{\cdot 1}$	$n_{\cdot 2}$	$n_{\cdot\cdot}$

### 8.2.1 Quantities and Statistics from Contingency Table

#### □ Prospective Study and Retrospective Study

Contingency table itself is symmetric w.r.t.  $Y, Z$ , but in experimental design we usually first specify and divide groups, and then conduct experiment (prospective) or conduct survey (retrospective), which would cause different conditional probability. An example in studying the effect of medicine

- Prospective Study: say,  $Y = E/E^c$  for drug/placebo group is assigned before experiment, and then  $Z = D/D^c$  for medicine effect is studied after treatment.

In this case  $n_{1\cdot}, n_{2\cdot}$  are pre-determined fixed number.

Such design is a well-controlled experiment to study the effct, but sometimes faced with problem concerning survival analysis, see [Chapter. 7](#) for detail. And for some problems like, e.g.  $Z$  is related to rare disease, this method is **low-efficient**.

- Retrospective Study: say, some  $Z = D/D^c$  for medicine effect patients are selected, and then their history of taking drug or not is collected.

In this case  $n_{\cdot 1}, n_{\cdot 2}$  are pre-determined fixed number.

This method is quick and convenient to conduct study, but usually we cannot control the exposure status  $Y$  accurately (because they are collected by, e.g. questionnaire)

Statistics and tests should be selected based on the data collection design (prospective/retrospective) because of different probability condition.

## □ Statistics and Estimation

With respective probabilities in two groups  $E$ ,  $E^c$  denoted as

$$p_1 = \mathbb{P}(D|E), \quad p_2 = \mathbb{P}(D|E^c) \quad (8.39)$$

we usually focus on the ‘difference’ between group  $E$  and  $E^c$ , there are some quantities to help measure the group difference:

$$\text{Risk difference: } \Delta = p_1 - p_2 \quad (8.40)$$

$$\text{Relative risk: } \phi = p_1/p_2 \quad (8.41)$$

$$\text{Odds ratio: } \theta = \frac{p_1/(1-p_1)}{p_2/(1-p_2)} \quad (8.42)$$

Their estimation:

- Respective probability  $p_1, p_2$ :

$$\text{Prospective: } \begin{cases} \hat{p}_1 = \frac{n_{11}}{n_{1.}} \\ \hat{p}_2 = \frac{n_{21}}{n_{2.}} \end{cases} \quad (8.43)$$

$$\text{Retrospective: } \begin{cases} \hat{p}_1 = \frac{\rho \frac{n_{11}}{n_{.1}}}{\rho \frac{n_{11}}{n_{.1}} + (1-\rho) \frac{n_{12}}{n_{.2}}} \\ \hat{p}_2 = \frac{\rho \frac{n_{21}}{n_{.1}}}{\rho \frac{n_{21}}{n_{.1}} + (1-\rho) \frac{n_{22}}{n_{.2}}} \end{cases} \quad (8.44)$$

$$\text{where } \rho \text{ is the prevalence btw } D, D^c \text{ in natural condition} \quad (8.45)$$

- Relative Risk  $\phi$ :

$$\text{Prospective: } \hat{\phi} = \frac{n_{11}/n_{1.}}{n_{21}/n_{2.}} \quad (8.46)$$

$$\text{Retrospective: } \hat{\phi} = \frac{\hat{p}_1}{\hat{p}_2} \quad (8.47)$$

- Odds Ratio  $\theta$ :

$$\text{Prospective\&Retrospective: } \hat{\theta} = \frac{n_{11}n_{22}}{n_{21}n_{12}} \quad (8.48)$$

which is the same in either cases.

variance of  $\hat{\theta}$ : estimated at  $(n_{11}, n_{12}, n_{21}, n_{22}) \sim \text{Multinomial}(n_{..}, \pi_{11}, \pi_{12}, \pi_{21}, \pi_{22})$ :

$$\text{var}(\log \hat{\theta}) = \frac{1}{n_{11}} + \frac{1}{n_{12}} + \frac{1}{n_{21}} + \frac{1}{n_{22}} \quad (8.49)$$

## □ Hypothesis Testing

The mostly used hypothesis is the dependence assumption:  $p_1 = p_2$ , or more generally speaking for  $m \times n$  table:

$$H_0 : \pi_{ij} = \pi_{i.}\pi_{.j}, \quad \forall i, j \quad (8.50)$$

Denote  $O_{ij} = n_{ij}$  as the **Observed** value,  $E_{ij} = n_{..}\pi_{ij}$  as the **Expected** value.<sup>86</sup> Expected value is calculated for the model used, under null hypothesis  $H_0$ . Example for independence test  $\pi_{ij} = \pi_{i.}\pi_{.j}$ :

$$\hat{\pi}_{ij} = \hat{\pi}_{i.}\hat{\pi}_{.j} = \frac{n_{i.}}{n_{..}} \frac{n_{.j}}{n_{..}} \Rightarrow E_{ij} = n_{..}\hat{\pi}_{ij} = \frac{n_{i.}n_{.j}}{n_{..}} \quad (8.57)$$

Statistics:

- **Pearson's  $\chi^2$  Test:**

$$\chi_P^2 = \sum_{i=1}^I \sum_{j=1}^J \frac{(O_{ij} - E_{ij})^2}{E_{ij}} \xrightarrow{\mathcal{L}} \chi_{(I-1)(J-1)}^2 \quad (8.58)$$

- **Likelihood Ratio Test:**

$$G^2 = -2 \log(\Lambda) = 2 \sum_{i=1}^I \sum_{j=1}^J O_{ij} \log \frac{O_{ij}}{E_{ij}} \xrightarrow{\mathcal{L}} \chi_{(I-1)(J-1)}^2 \quad (8.59)$$

Some other useful tests:

- McNemar test on  $\pi_{12} = \pi_{21}$  for matched pairs:

$$z^2 = \frac{(n_{12} - n_{21})^2}{n_{12} + n_{21}} \xrightarrow{\mathcal{L}} \chi_1^2 \quad (8.60)$$

## Section 8.3 Clinical Trial Design

## Section 8.4 GWAS

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<sup>86</sup>  $E_{ij}$  is calculated based on data and the model you choose, thus can be applied to more complex cases, e.g. Hardy-Weinberg proportions with  $X^a$  gene frequency  $p$

$$\mathbb{P}(X^a X^a; \text{Female}) = p^2 \quad (8.51)$$

$$\mathbb{P}(X^A X^a; \text{Female}) = 2p(1-p) \quad (8.52)$$

$$\mathbb{P}(X^A X^A; \text{Female}) = (1-p)^2 \quad (8.53)$$

$$\mathbb{P}(X^a Y; \text{Male}) = p \quad (8.54)$$

$$\mathbb{P}(X^A Y; \text{Male}) = (1-p) \quad (8.55)$$

In such complex case, parameter should be estimated using e.g. MLE estimation. And then calculate  $E_{ij}$ s

$$L(p) = [p^2]^{O_{a,F}} [1-p^2]^{O_{A,F}} [p]^{O_{a,M}} [1-p]^{O_{A,M}} \quad (8.56)$$

## Chapter. IX 统计学习导论部分

Instructor: Sheng Yu

In this course, some key formulations/theorem in machine learning are deduced, together with core principles illustrated.

### □ What is Machine Learning?

Machine learning is a field of computer science that uses statistical techniques to give computer systems the ability to "learn" with data, without being explicitly programmed.

Examples of Machine Learning:

- Linear/Logistic Regression (Linear Model)
- Decision Tree
- Support Vector Machine
- Clustering
- Bayesian Network
- Neural Network
- Conditional Random Field
- etc.

This section will cover some of the methods above in a machine learning perspective.

## Section 9.1 Linear Model

Linear model is the basic model in statistics, see [Chapter. 3](#).

### 9.1.1 Linear Model in Machine Learning Perspective

In machine learning field, key feature of linear model is its affine form of variable dependence:

$$Y = f(X) + \varepsilon = \tilde{f}_\beta(X'\beta) + \varepsilon$$

where usually  $X = (1, X_1, X_2, \dots, X_p)$ ,  $\beta = (\beta_0, \beta_1, \beta_2, \dots, \beta_p)$ .<sup>87</sup> Some example of linear model:

- Linear Regression:

$$Y = \beta_0 + \beta_1 x_1 + \dots + \beta_p x_p + \varepsilon = X'\beta + \varepsilon$$

- General Linear Model:

$$Y \sim f(\theta(X'\beta))$$

in this framework,

---

<sup>87</sup>Some materials use  $X = (X_1, X_2, \dots, X_p)$ ,  $\beta = (\beta_1, \beta_2, \dots, \beta_p)$ , and the affine dependence is  $\tilde{f}(\beta_0 + X'\beta)$

- Linear regression:

$$Y \sim N(X'\beta, \sigma^2)$$

- Logistic regression:

$$Y \sim \text{Bernoulli}(\text{logistic}(X'\beta))$$

### 9.1.2 Linear Regression

Linear Regression:

$$Y = \beta_0 + \beta_1 x_1 + \dots + \beta_p x_p + \varepsilon = X'\beta + \varepsilon, \quad \varepsilon \sim N(0, \sigma^2)$$

usually use Squared Error Loss to estimate  $(\beta, \sigma^2)$

$$\mathcal{L}(Y, \hat{f}(X)) = (Y - \hat{f}(X))^2 = (Y - X\hat{\beta})^2$$

LSE estimator (where  $Y$  and  $X$  imply corresponding sample vector/matrix), more detail see [section. 3.3](#):

$$\frac{\partial \mathcal{L}}{\partial \beta} = 0 \Rightarrow \hat{\beta} = (X'X)^{-1}X'Y$$

- Predict:

$$\hat{Y} = X\hat{\beta} = X(X'X)^{-1}X'Y$$

- Hat Matrix:

$$H = P_X \equiv X(X'X)^{-1}X'$$

idempotent and symmetry

$$H^2 = H, \quad H = H'$$

- Properties of  $\hat{\beta}$ ,  $\hat{\sigma}^2$ :<sup>88</sup>

$$\text{cov}(\hat{\beta}) = \text{cov}((X'X)^{-1}X'(X\beta + \varepsilon)) = (X'X)^{-1}\sigma^2 \quad (9.1)$$

$$\text{var}(\hat{\beta}_j) = \frac{\sigma^2}{(n-1)S_{x_j}^2} \cdot \text{VIF}_j \quad (9.2)$$

$$\text{cov}(e) = \text{cov}(Y - \hat{Y}) = (I - H)\sigma^2 \quad (9.3)$$

$$\text{var}(\hat{\sigma}^2) = \text{var}(\text{MSE}) = \frac{Y'(I - H)Y}{n - (p + 1)} \quad (9.4)$$

### 9.1.3 Normalization Methods

In machine learning topic we would focus more on model generalization ability, so that the model can perform better on reality problems. In linear regression, we usually use normalization methods.

Basically linear model uses SE loss:

$$\mathcal{L} = \sum_{i=1}^n (y_i - \beta_0 - \beta'x_i)^2 = \sum_{i=1}^n (y_i - x_i'\beta)^2$$

we can put various normalize term (penalty) in loss or put constraint on  $\beta$ : (these two methods are equivalent in many cases)

<sup>88</sup>Definition of  $\text{VIF}_j$  see [section. 3.4.7](#)

- Ridge Regression/ $\ell_2$  Penalty/Tikhonov Regularization:<sup>89</sup>

$$\hat{\beta}^{\text{ridge}} = \arg \min_{\beta} \sum_{i=1}^n (y_i - x'_i \beta)^2 + \lambda \|\beta\|_2^2$$

or equivalent form

$$\hat{\beta}^{\text{ridge}} = \arg \min_{\beta} \sum_{i=1}^n (y_i - x'_i \beta)^2 \quad (9.5)$$

$$s.t. \|\beta\|_2^2 \leq s \quad (9.6)$$

in either case,  $\lambda$  or  $s$  is hyper-parameter.

Ridge regression has closed form solution

$$\hat{\beta}^{\text{ridge}} = (X'X + \lambda I)^{-1} X'Y$$

Intuitively speaking, ridge regression help shrink  $\hat{\beta}$  by a non-zero factor.

- LASSO/ $\ell_1$  Penalty:

$$\hat{\beta}^{\text{LASSO}} = \arg \min_{\beta} \sum_{i=1}^n (y_i - x'_i \beta)^2 + \lambda \|\beta\|_1$$

or equivalent form

$$\hat{\beta}^{\text{LASSO}} = \arg \min_{\beta} \sum_{i=1}^n (y_i - x'_i \beta)^2 \quad (9.7)$$

$$s.t. \|\beta\|_1 \leq s \quad (9.8)$$

LASSO help shrink significantly large coefficients and truncate small coefficients.

- Generalized  $\ell_p$  norm penalty:

$$\hat{\beta}^{\text{ridge}} = \arg \min_{\beta} \sum_{i=1}^n (y_i - x'_i \beta)^2 + \lambda \|\beta\|_2^2$$

or equivalent form

$$\hat{\beta}^{\text{ridge}} = \arg \min_{\beta} \sum_{i=1}^n (y_i - x'_i \beta)^2 \quad (9.9)$$

$$s.t. \|\beta\|_2^2 \leq s \quad (9.10)$$

- Elastic Net:

$$\hat{\beta} = \arg \min_{\beta} \|Y - X\beta\|_2^2 + \lambda_1 \|\beta\|_1 + \lambda_2 \|\beta\|_2^2$$

equivalent form:

---

<sup>89</sup>Recall for  $\ell_p$  norm: for  $n$ -dim vector  $\vec{v} = (v_1, v_2, \dots, v_n)$

$$\|v\|_p = \left( \sum_{i=1}^m |v_i|^p \right)^{1/p}$$

$$\hat{\beta} = \arg \min_{\beta} \|Y - X\beta\|^2 \quad (9.11)$$

$$s.t. \frac{\lambda_1}{\lambda_1 + \lambda_2} \|\beta\|_1 + \frac{\lambda_2}{\lambda_1 + \lambda_2} \|\beta\|_2^2 \leq s \quad (9.12)$$

picking proper hyper-parameter ( $s, \lambda = \frac{\lambda_2}{\lambda_1 + \lambda_2}$ )

A note on elastic net: the boundary of elastic net  $\lambda_1 \|\beta\|_1 + \lambda_2 \|\beta\|_2^2 = \text{const}$  is between  $\ell_1$  boundary and  $\ell_2$  boundary. Both the variable selection feature of  $\ell_1$  and the differentiable feature of  $\ell_2$  are partially maintained.

- Adaptive LASSO:

$$\hat{\beta} = \arg \min_{\beta} \sum_{i=1}^n (y_i - x_i' \beta)^2 + \lambda \sum_{j=1}^p \frac{|\beta_j|}{|\hat{\beta}_j^{\text{OLS}}|}$$

- Non-negative Garrote method.
- SCAD

## Section 9.2 Basic Classification Model

Denote: Dataset  $\mathcal{D} = \{(x_i, y_i), i = 1, 2, \dots, N\}$ ,  $x_i = [x_{i1}, x_{i2}, \dots, x_{ip}]$ , with response  $y_i \in \mathcal{C} = \{c_1, c_2, \dots, c_K\}$  as a  $K$ -classification problem. When  $K = |\mathcal{C}| = 2$  for binary classification, in this case we usually denote  $\mathcal{C}_{01} = \{0, 1\}$ .

Target is to predict/classify  $Y$  from  $X$

$$\hat{Y} = \hat{f}(X) \rightsquigarrow Y \quad (9.13)$$

### 9.2.1 Classification Metrics

- Accuracy

$$\mathbb{P}(\hat{Y} = Y) \hat{\rightarrow} \frac{\sum_{i=1}^N \mathbb{I}(\hat{y}_i = y_i)}{N} \quad (9.14)$$

- Error Rate/ Misclassification Rate

$$\mathbb{P}(\hat{Y} \neq Y) \hat{\rightarrow} \frac{\sum_{i=1}^N \mathbb{I}(\hat{y}_i \neq y_i)}{N} \quad (9.15)$$

- prevalence for binary classification

$$\mathbb{P}(Y = 1) \hat{\rightarrow} \frac{\sum_{i=1}^N y_i}{N} \quad (9.16)$$

### □ Confusion Matrix and Metrics for Binary Classification

表 8: Confusion matrix for binary classification

Ground Truth $Y$	Predicted Value $\hat{Y}$	
	1	0
1	$n_{11}$	$n_{10}$
0	$n_{01}$	$n_{00}$



Metrics:

- True Positive Rate (TPR)/ Sensitivity/ Recall:

$$\mathbb{P}(\hat{Y} = 1|Y = 1) \hat{\rightarrow} \frac{\sum_{i=1}^N \mathbb{I}(\hat{y}_i = 1) \cdot \mathbb{I}(y_i = 1)}{\sum_{i=1}^N \mathbb{I}(y_i = 1)} = \frac{n_{11}}{n_{11} + n_{10}} \quad (9.17)$$

- False Positive Rate (FPR):

$$\mathbb{P}(\hat{Y} = 1|Y = 0) \hat{\rightarrow} \frac{\sum_{i=1}^N \mathbb{I}(\hat{y}_i = 1) \cdot \mathbb{I}(y_i = 0)}{\sum_{i=1}^N \mathbb{I}(y_i = 0)} = \frac{n_{01}}{n_{01} + n_{00}} \quad (9.18)$$

- True Negative Rate (TNR)/ Specific (SPC):

$$\mathbb{P}(\hat{Y} = 0|Y = 0) \hat{\rightarrow} \frac{\sum_{i=1}^N \mathbb{I}(\hat{y}_i = 0) \cdot \mathbb{I}(y_i = 0)}{\sum_{i=1}^N \mathbb{I}(y_i = 0)} = \frac{n_{00}}{n_{01} + n_{00}} \quad (9.19)$$

- False Negative Rate (FNR):

$$\mathbb{P}(\hat{Y} = 0|Y = 1) \hat{\rightarrow} \frac{\sum_{i=1}^N \mathbb{I}(\hat{y}_i = 0) \cdot \mathbb{I}(y_i = 1)}{\sum_{i=1}^N \mathbb{I}(y_i = 1)} = \frac{n_{10}}{n_{11} + n_{10}} \quad (9.20)$$

•

- Positive Predictive Value (PPV)/ Precision:

$$\mathbb{P}(Y = 1|\hat{Y} = 1) \hat{\rightarrow} \frac{\sum_{i=1}^N \mathbb{I}(\hat{y}_i = 1) \cdot \mathbb{I}(y_i = 1)}{\sum_{i=1}^N \mathbb{I}(\hat{y}_i = 1)} = \frac{n_{11}}{n_{11} + n_{01}} \quad (9.21)$$

- False Discovery Rate (FDR):

$$\mathbb{P}(Y = 0|\hat{Y} = 1) \hat{\rightarrow} \frac{\sum_{i=1}^N \mathbb{I}(\hat{y}_i = 1) \cdot \mathbb{I}(y_i = 0)}{\sum_{i=1}^N \mathbb{I}(\hat{y}_i = 1)} = \frac{n_{01}}{n_{11} + n_{01}} \quad (9.22)$$

- Negative Predictive Value (NPV):

$$\mathbb{P}(Y = 0|\hat{Y} = 0) \hat{\rightarrow} \frac{\sum_{i=1}^N \mathbb{I}(\hat{y}_i = 0) \cdot \mathbb{I}(y_i = 0)}{\sum_{i=1}^N \mathbb{I}(\hat{y}_i = 0)} = \frac{n_{00}}{n_{10} + n_{00}} \quad (9.23)$$

- False Omission Rate (FOR):

$$\mathbb{P}(Y = 1|\hat{Y} = 0) \hat{\rightarrow} \frac{\sum_{i=1}^N \mathbb{I}(\hat{y}_i = 0) \cdot \mathbb{I}(y_i = 1)}{\sum_{i=1}^N \mathbb{I}(\hat{y}_i = 0)} = \frac{n_{10}}{n_{10} + n_{00}} \quad (9.24)$$

$F_1$  Score:

$$F_1 = 2 \frac{\text{precision} \cdot \text{recall}}{\text{precision} + \text{recall}} \quad (9.25)$$

Receive Operating Characteristic Curve (ROC Curve) is used to examining performance of a model with threshold  $s$ :

$$\hat{Y} = \begin{cases} 1, & \text{case } \hat{f}(X) > s \\ 0, & \text{case } \hat{f}(X) \leq s \end{cases} \quad (9.26)$$

for each  $s$ , the model gives a corresponding  $\text{TPR}(s)$  (recall) and  $\text{FPR}(s)$ , all  $(\text{TPR}(s), \text{FPR}(s))$  forms the ROC curve.

Area Under ROC Curve (AUC) is used also as a measure of model performance.

## 9.2.2 Cross-Validation

In general process of train & validate, we split the data into train set and validation set, which causes insufficient usage of data.  $k$ -fold Cross-validation (CV) is proposed to overcome the problem.

1. Divide  $\mathcal{D}$  into  $k$  folds
2. For each time  $i = 1, 2, \dots, k$ , pick the  $i^{\text{th}}$  fold as validation set, others as train set, train the model and calculate the metric  $m_i$
3. Average over all folds is used as final performance

$$m = \frac{\sum_{i=1}^k m_i}{k} \quad (9.27)$$

CV could help ease the problem of overfitting.

## 9.2.3 Bayes Optimal Classifier

Due to the randomness of class distribution, no classifier could reach 100% accuracy, but there is a optimal classifier (if we really know the underlying distribution) to minimize the expected loss:

$$\mathbb{E}(\mathcal{L}) = \mathbb{E}_X \left( \sum_{k=1}^K \mathcal{L}(k, \hat{y}(X)) \right) \cdot \|Y = k|X\| \quad (9.28)$$

$$\Rightarrow \hat{y}(x)_{\text{optimal}} = \arg \min_j \mathcal{L}(k, j) \cdot \mathbb{P}(Y = k|X = x) \quad (9.29)$$

$$0/1 \text{ loss} = \arg \max_j \mathbb{P}(Y = j|X = x) \quad (9.30)$$

which is the Bayes Optimal Classifier  $\hat{y}(x)_{\text{optimal}}$ , its error rate is Bayes optimal rate.

## 9.2.4 $k$ -Nearest Neighbours Approach

The  $k$ -nearest neighbours (KNN) fit with threshold  $s$ :

$$\hat{f}(x) = \frac{1}{k} \sum_{i: x_i \in \mathcal{N}_k(x)} y_i \quad (9.31)$$

$$\hat{Y} = \begin{cases} 1, & \text{case } \hat{f}(X) > s \\ 0, & \text{case } \hat{f}(X) \leq s \end{cases} \quad (9.32)$$

where  $\mathcal{N}_k(x)$  is the nearest  $k$  datapoints of  $x$ , various distance measure  $\|\cdot\|$  could be used.  $k$ -NN method is faced with the problem of curse of dimensionality (see [section. 4.3](#)) in high dimension case. Calculation cost is at  $O(N)$ .

## 9.2.5 Density Based Classification

An intuition: samples from the same class  $k$  should be clustered, we use some distribution to represent it as  $f_k(x)$ . Bayes optimal criterion with prior  $\pi_k$ :

$$\hat{y}(x) = \arg \max_k \mathbb{P}(Y = k|X = x) = \arg \max_k \frac{f_k(x)\pi_k}{\sum_{l=1}^K f_l(x)\pi_l} = \arg \max_k f_k(x)\pi_k \quad (9.33)$$

## □ Discriminant Analysis

Detail about discriminant analysis could be found in section [section. 4.6](#). Here are some recaps:

Discriminant analysis assume a gaussian distribution

$$f_k(x) = \frac{1}{(2\pi)^{p/2} |\Sigma_k|^{1/2}} \exp \left\{ -\frac{1}{2} (x - \mu_k)' \Sigma_k^{-1} (x - \mu_k) \right\} \quad (9.34)$$

- Linear Discriminant Analysis (LDA): Assume  $\Sigma_k = \Sigma, \forall k$

$$\log \frac{\mathbb{P}(k|x)}{\mathbb{P}(l|x)} = \log \frac{f_k(x) \pi_k}{f_l(x) \pi_l} \quad (9.35)$$

$$= \log \frac{\pi_k}{\pi_l} - \frac{1}{2} (\mu_k + \mu_l)' \Sigma^{-1} (\mu_k - \mu_l) + x' \Sigma^{-1} (\mu_k - \mu_l) \quad (9.36)$$

Classification function:

$$\hat{y}(x) = \arg \max_k \delta_k(x) = \arg \max_k \log \hat{\pi}_k + x' \hat{\Sigma}^{-1} \hat{\mu}_k - \frac{1}{2} \hat{\mu}_k' \hat{\Sigma}^{-1} \hat{\mu}_k \quad (9.37)$$

$$\hat{\pi}_k = \frac{N_k}{N} \quad (9.38)$$

$$\hat{\mu}_k = \frac{\sum_{i:y_i=k} x_i}{N_k} \quad (9.39)$$

$$\hat{\Sigma} = \frac{\sum_{k=1}^K \sum_{i:y_i=k} (x_i - \hat{\mu}_k)(x_i - \hat{\mu}_k)'}{N - K} \quad (9.40)$$

- Quadratic Discriminant Analysis (QDA): Allow different  $\Sigma_k$ , Classification function:

$$\hat{y}(x) = \arg \max_k \delta_k(x) = \arg \max_k \log \hat{\pi}_k - \frac{1}{2} \log |\hat{\Sigma}_k| - \frac{1}{2} (x - \hat{\mu}_k)' \hat{\Sigma}_k^{-1} (x - \hat{\mu}_k) \quad (9.41)$$

$$\hat{\pi}_k = \frac{N_k}{N} \quad (9.42)$$

$$\hat{\mu}_k = \frac{\sum_{i:y_i=k} x_i}{N_k} \quad (9.43)$$

$$\hat{\Sigma}_k = \frac{\sum_{i:y_i=k} (x_i - \hat{\mu}_k)(x_i - \hat{\mu}_k)'}{N_k - 1} \quad (9.44)$$

## □ Naïve Bayes Classifier

Distribution is estimated as (which is a naïve decomposition)

$$f_k(\vec{x}) = f_k(x_1) f_k(x_2) \dots f_k(x_p) \quad (9.45)$$

Classification function:

$$\hat{y}(x) = \arg \max_k \hat{\pi}_k \prod_{i=1}^p \hat{f}_k(x_i) = \arg \max_k \sum_{i=1}^p \pi_k \log \hat{f}_k(x_i) \quad (9.46)$$

## 9.2.6 Logistic Regression

Logistic Regression calculates  $\mathbb{P}(Y|X)$  directly. Detail theory see [section. 3.7](#). Here are some recaps:

$$y|x \sim \text{Binom} \left( 1, \frac{e^{x'\beta}}{1 + e^{x'\beta}} \right) \quad (9.47)$$

$$\mathbb{P}(Y = 1|X = x) = \frac{e^{x'\beta}}{1 + e^{x'\beta}} := \text{logit}(x'\beta) \quad (9.48)$$

Classify with threshold  $s$ .

### □ Multiple Classification

$$\mathbb{P}(Y = k|X = x) = \frac{e^{x'\beta_k}}{1 + \sum_{l=1}^{K-1} e^{x'\beta_l}}, \quad k = 1, 2, \dots, K-1 \quad (9.49)$$

$$\mathbb{P}(Y = K|X = x) = \frac{1}{1 + \sum_{l=1}^{K-1} e^{x'\beta_l}} \quad (9.50)$$

Comment on Logistic Regression:

- Classification core  $x'\beta$  is linear, so logistic regression is still a linear classifier.
- Classification parameter  $\beta$ s are usually obtained using MLE. Detail see [section. 5.4.3](#).

$$\beta^{(t+1)} = \beta^{(t)} - \left( \frac{\partial^2 \ell(\beta)}{\partial \beta \partial \beta'} \right) \frac{\partial \ell(\beta)}{\partial \beta} \quad (9.51)$$

$$= \beta^{(t)} + (X'WX)^{-1}X'(Y - \text{logit}(X, \beta^{(t)})), \quad W = \text{diag} \left\{ \text{logit}(X, \beta^{(t)}) \odot (1 - \text{logit}(X, \beta^{(t)})) \right\} \quad (9.52)$$

### ▷ R. Code

```
1 library(glmnet)
2 glmnet(x, y, family="binomial") # two-class
3 glmnet(x, y, family="multinomial") # multi-class
4 glmnet(x, y, family="binomial", alpha, lambda) # with penalty
```

### □ Logistic Regression as Loss-Penalization Method

Logistic Regression with  $\ell_2$  norm regularized term is

$$\arg \min_{\beta} \sum_{i=1}^N \log \mathbb{P}(Y \neq y_i | X = x_i; \beta) + \frac{\lambda}{2} \|\beta\|^2 \quad (9.53)$$

$$= \arg \max_{\beta} \sum_{i=1}^N \log[1 + e^{y_i f(x_i)}] + \frac{\lambda}{2} \|\beta\|^2, \quad y_i \in \{+1, -1\} \quad (9.54)$$

where  $f(\cdot)$  is classification function,  $\beta_0 + x'\beta$  for linear classification.

## Section 9.3 Support Vector Machine

Support vector machine (SVM) classifier was one of the most successful classification model in 2010±, mainly because of the kernel trick method in extending feature space.

First we will consider the linear classification case, i.e. dataset  $\mathcal{D} = \{(\vec{x}_i, y_i), i = 1, 2, \dots, N\}$  are divided by a linear boundary  $x'\beta + \beta_0$ , where label  $y_i \in \{1, -1\}$ .

### 9.3.1 Derivation of Basic Optimize Problem

#### □ Hard Margin SVM

The intuition of SVM is to determine the classification boundary by ensuring all the points are ‘far away enough’ from the boundary.

$$\begin{aligned} \arg \max_{\beta, \beta_0, M} \quad & M \\ \text{s.t.} \quad & \frac{1}{\|\beta\|} y_i (x'_i \beta + \beta_0) \geq M \quad i = 1, 2, \dots, N \end{aligned}$$

where  $M$  for ‘Margin’, which indicates the distance of point from boundary. L.H.S. of inequality is the distance from  $x_i$  to boundary.<sup>90</sup>

However note that the *dof* of this problem is 1, i.e. all  $(\beta_0, \beta) \propto (\beta_0^*, \beta^*)$  give the same result. We could omit this *dof* by putting an extra constraint, here a convenient one is used:  $\|\beta\| = \frac{1}{M}$ . i.e.

$$\begin{aligned} \arg \min_{\beta, \beta_0: M=1/\|\beta\|} \quad & \frac{1}{2} \|\beta\|^2 \\ \text{s.t.} \quad & y_i (x'_i \beta + \beta_0) \geq 1 \quad i = 1, 2, \dots, N \end{aligned}$$

### □ Soft Margin SVM

To tackle the case when  $y_i (x'_i \beta + \beta_0) \geq 1$  cannot always be satisfied, use soft margin by inducing a ‘slack variable’  $\xi_i$  for each point, indicating the proportion of distance that the point enters the margin, see figure. 7

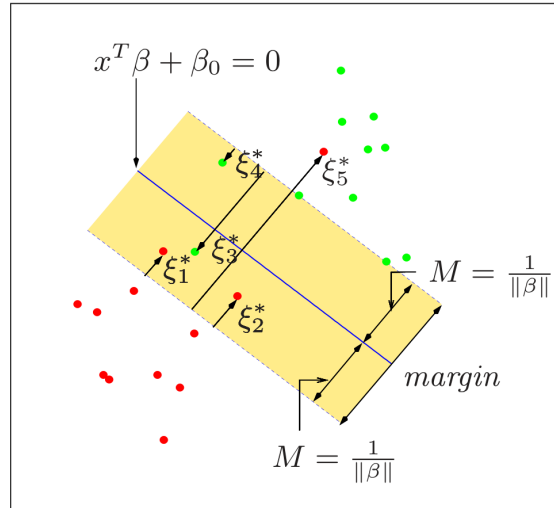


图 7: Support Vector Machine Illustration

<sup>90</sup>□ *Proof:* denote some point on  $x' \beta + \beta_0 = 0$  as  $x_\perp$  (i.e.  $x'_\perp \beta + \beta_0 = 0$ ), then the distance of  $x$  to boundary is the projection of  $x - x_\perp$  on unit normal vector  $\frac{\beta}{\|\beta\|}$ :

$$d = \left| (x - x^*)' \frac{\beta}{\|\beta\|} \right| = \frac{1}{\|\beta\|} |x' \beta + \beta_0|$$

further because  $y_i$  varies at different sides of boundary:

$$y_i = 1 : x' \beta + \beta_0 > 0 \quad (9.55)$$

$$y_i = -1 : x' \beta + \beta_0 < 0 \quad (9.56)$$

we can replace the  $|\cdot|$  using label:

$$d = \frac{1}{\|\beta\|} y(x' \beta + \beta_0)$$

Primal  $\theta_P$ :

$$\begin{aligned} \arg \min_{\beta, \beta_0: M=1/\|\beta\|} \quad & \frac{1}{2}\|\beta\|^2 + C \sum_{i=1}^N \xi_i \\ \text{s.t.} \quad & y_i(x'_i\beta + \beta_0) \geq 1 - \xi_i \quad i = 1, 2, \dots, N \\ & \xi_i \geq 0 \quad i = 1, 2, \dots, N \end{aligned}$$

write the generalized lagrange function as defined in [equation. 5.20](#):

$$\mathcal{L}(\beta, \beta_0, \xi_i; \alpha, \mu) = \frac{1}{2}\|\beta\|^2 + C \sum_{i=1}^N \xi_i + \sum_{i=1}^N \alpha_i [1 - \xi_i - y_i(x'_i\beta + \beta_0)] - \sum_{i=1}^N \mu_i \xi_i \quad (9.57)$$

$$\text{s.t.} \quad \alpha_i \geq 0, \quad \mu_i \geq 0, \quad i = 1, 2, \dots, N \quad (9.58)$$

dual problem is given when  $\frac{\partial \mathcal{L}}{\partial \beta, \beta_0, \xi_i} = 0$ :

$$\frac{\partial \mathcal{L}}{\partial \beta} = 0 : \hat{\beta} = \sum_{i=1}^N \alpha_i y_i x_i \quad (9.59)$$

$$\frac{\partial \mathcal{L}}{\partial \beta_0} = 0 : \sum_{i=1}^N \alpha_i y_i = 0 \quad (9.60)$$

$$\frac{\partial \mathcal{L}}{\partial \xi_i} = 0 : C = \alpha_i + \mu_i, \quad i = 1, 2, \dots, N \quad (9.61)$$

Dual  $\theta_D$ :

$$\theta_D(\alpha, \mu) = \min_{\beta, \beta_0, \xi_i} \mathcal{L} = -\frac{1}{2} \sum_{i=1}^N \sum_{j=1}^N \alpha_i \alpha_j y_i y_j x'_i x_j + \sum_{i=1}^N \alpha_i \quad (9.62)$$

$$\text{s.t.} \quad 0 \leq \alpha_i \leq C \quad (9.63)$$

$$\sum_{i=1}^N \alpha_i y_i = 0 \quad (9.64)$$

we can maximize  $\theta_D$  to obtain  $\hat{\alpha}_i, \hat{\mu}_i = C - \hat{\alpha}_i$ . And  $(\hat{\beta}, \hat{\beta}_0, \xi_i)$  are given utilizing KKT condition for  $d^* =$

$$\max_{\alpha, \mu} \theta_D = \min_{\beta, \beta_0, \xi_i} \theta_P = p^*:$$

$$\hat{\alpha}_i [1 - \hat{\xi}_i - y_i(x'_i \hat{\beta} + \hat{\beta}_0)] = 0 \quad (9.65)$$

$$(C - \hat{\alpha}_i) \hat{\xi}_i = 0 \quad (9.66)$$

$$1 - \hat{\xi}_i - y_i(x'_i \hat{\beta} + \hat{\beta}_0) \leq 0 \quad (9.67)$$

$$0 \leq \hat{\alpha}_i \leq C \quad (9.68)$$

$$\hat{\xi}_i \geq 0 \quad (9.69)$$

$$\hat{\beta} = \sum_{i=1}^N \hat{\alpha}_i y_i x_i \quad (9.70)$$

discussion on different cases of  $\alpha_i, \xi_i$ :

$$\hat{\alpha}_i = 0 : \hat{\xi}_i = 0 \quad (9.71)$$

$$\hat{\alpha}_i = C : y_i(x'_i \hat{\beta} + \hat{\beta}_0) = 1 - \hat{\xi}_i \quad (9.72)$$

$$0 < \hat{\alpha}_i < C : \hat{\xi}_i = 0, y_i(x'_i \hat{\beta} + \hat{\beta}_0) = 1 \quad (9.73)$$

where all points  $\mathcal{I}^{\text{sv}} := \{i^{\text{sv}} | 0 < \hat{\alpha}_{i^{\text{sv}}} < C, \hat{\xi}_{i^{\text{sv}}} = 0\}$  are called ‘**support vector**’, that can be used to determine  $\beta_0$ :

$$\hat{\beta} = \sum_{i=1}^N \hat{\alpha}_i y_i x_i = \sum_{i \in \mathcal{I}^{\text{sv}}} \hat{\alpha}_i y_i x_i \quad (9.74)$$

$$\hat{\beta}_0 = y_{i^{\text{sv}}} - x'_{i^{\text{sv}}} \hat{\beta} \quad (9.75)$$

### 9.3.2 Support Vector Machine as Loss-Penalization Method

SVM Primal can be express in equivalent form with  $f(x_i)$  as prediction function, e.g.  $f(x_i) = \beta_0 + x'_i \beta$  for linear SVM:

$$\begin{cases} \xi_i \geq 0 \\ \xi_i \geq 1 - y_i f(x_i) \end{cases} \Rightarrow \xi_i \geq \max\{0, 1 - y_i f(x_i)\} = [1 - y_i f(x_i)]_+ \quad (9.76)$$

in which  $[\cdot]_+ \equiv \max\{0, \cdot\}$  is hinge loss:

$$\arg \min_{\beta, \beta_0} \sum_{i=1}^N [1 - y_i f(x_i)]_+ + \frac{\lambda}{2} \|\beta\|^2, \quad \lambda = \frac{1}{C}, \quad f(x_i) = \beta_0 + x'_i \beta$$

which is naturally in an  $\arg \min_f \sum_{i=1}^N \mathcal{L}(x_i, y_i, f(x_i)) + \frac{\lambda}{2} \mathcal{P}(f(\cdot))$  Loss+Penalty form.

## Section 9.4 Feature Expansion and Kernel Methods

Motivation: Map the data point  $x \in \mathcal{X}$  (e.g.  $= \mathbb{R}^p$ ) to another feature space  $\mathcal{F}$  (e.g.  $= \mathbb{R}^M$ ) (not necessarily a linear transform, usually  $M > p$ , or just proper to describe the features). The mapping function lies in a Hilbert space  $\mathcal{H}$  of function:

$$h(\cdot) = (h_1(\cdot), h_2(\cdot), \dots, h_M(\cdot))' \in \mathcal{H} : \mathcal{X} \rightarrow \mathcal{F}$$

and we can construct model in feature space.

### 9.4.1 Reproducing Kernel Hilbert Space and The Representer Theorem

Based on the idea of feature space, make a step forward: the key focus of model is actually ‘measuring space structure by similarity between points’ rather than having to define a feature space. i.e. describe similarity by a bi-linear **Kernel Function**

$$K(x, x') \in \mathcal{X} \times \mathcal{X} \rightarrow \mathbb{R}$$

In intuition for Kernel is an ‘inner product kernel’. The Kernel corresponds a kind of inner product structure on  $\mathcal{H}$ , the kernel should satisfies the following properties:

1. Positive Semi-Definition:

$$\iint K(x, y) g(x) g(y) dx dy \geq 0, \quad \forall g(\cdot) \quad (9.77)$$

or an equivalent form:

$$\sum_{i,j=1}^n K(x_i, x_j) a_i a_j \geq 0, \quad \forall \{x_i\}_{i=1}^n, \{a_i\}_{i=1}^n, \quad \forall n \in \mathbb{Z}^+ \quad (9.78)$$

2. symmetry:

$$K(x, y) = K(y, x) \quad (9.79)$$

Eigenvalue  $\gamma_i$  and eigen function  $\phi_i(x)$  of Kernel:

$$\int_x K(x, y)\phi_i(y) dy = \gamma_i\phi_i(x) \quad (9.80)$$

In Hilbert space, the eigen functions are orthonormal:

$$\langle \phi_i, \phi_j \rangle = \int_x \phi_i(x)\phi_j(x) dx = \delta_{ij} \quad (9.81)$$

And Kernel  $K(x, y)$  could be represented from its eigen value and eigen function:

$$K(x, y) = \sum_i \gamma_i \phi_i(x)\phi_i(y) \quad (9.82)$$

which is Mercer's Thm.: Semi-positive definite symmetric kernel could be expressed as an inner product form. Such a form is also called the kernel trick because it usually avoid calculating inner product in high dimensional space.

□ **Reproducing Kernel Hilbert Space (RKHS)** Now use set  $\{\phi_i\}$  as the orthonormal base to form a Hilbert space  $\mathcal{H}_K = \text{span}\{\phi_i\}$  i.e. any function  $f \in \mathcal{H}_K$  could be expressed as expansion

$$\mu(x) = \sum_i \mu_i \phi_i(x) \quad (9.83)$$

The inner product defined for this Hilbert space is<sup>91</sup>

$$\left\langle \sum_i \mu_i \phi_i(x), \sum_i \nu_i \phi_i(x) \right\rangle_{\mathcal{H}_K} = \sum_i \frac{\mu_i \nu_i}{\gamma_i} \quad (9.84)$$

and norm induced by inner product

$$\|f\|_{\mathcal{H}_K} = \sum_i \frac{f_i^2}{\gamma_i}, \quad f(x) = \sum_i f_i \phi_i(x) \quad (9.85)$$

Note: when  $x$  is fixed,  $f_x(y) = K(x, y)$  is a function of  $y$ , and vice versa. Use the above expansion and inner product:

$$K(x, y) = \sum_i \gamma_i \phi_i(x)\phi_i(y) \quad (9.86)$$

$$= \sum_i \sqrt{\gamma_i} \phi_i(x) \sqrt{\gamma_i} \phi_i(y) \quad (9.87)$$

$$= \sum_i \frac{(\gamma_i \phi_i(x))(\gamma_i \phi_i(y))}{\gamma_i} \quad (9.88)$$

$$= \left\langle \sum_i \gamma_i \phi_i(x) \phi_i(\xi), \sum_i \gamma_i \phi_i(y) \phi_i(\xi) \right\rangle_{\mathcal{H}_K} \quad (9.89)$$

$$= \langle K(x, \xi), K(\xi, y) \rangle_{\mathcal{H}_K} \quad (9.90)$$

which is the reproducing property of Kernel  $K(\cdot, \cdot)$  and its corresponding Hilbert space  $\mathcal{H}_K$

<sup>91</sup>Hilbert space is complete linear space with inner product defined.



### □ Representer Thm. for RKHS

With Kernel and its corresponding RKHS defined, we could write a optimization problem as loss+penalty form:

$$\arg \min_{f \in \mathcal{H}_K} \sum_{i=1}^N \mathcal{L}(y_i, f(x_i)) + \frac{\lambda}{2} \|f\|_{\mathcal{H}_K}^2 \quad (9.91)$$

Representer Thm.: Solution to above optimization has a **finite** form

$$\hat{f}(x) = \sum_{i=1}^N \hat{\alpha}_i K(x, x_i) \quad (9.92)$$

i.e. we can optimize over  $\{\hat{\alpha}_i\}_{i=1}^N$ , instead of optimizing over  $\{f_i\}_{i=1}^\infty$ .

norm of  $\hat{f}$  is represented as

$$\|\hat{f}\|_{\mathcal{H}_K}^2 = \left\langle \sum_{i=1}^N \hat{\alpha}_i K(x, x_i), \sum_{i=1}^N \hat{\alpha}_i K(x, x_i) \right\rangle_{\mathcal{H}_K} \quad (9.93)$$

$$= \sum_{i=1}^N \sum_{j=1}^N \hat{\alpha}_i \hat{\alpha}_j K(x_i, x_j) \quad (9.94)$$

Optimization problem **equation. 9.91** is parameterized by  $\{\hat{\alpha}_i\}_{i=1}^N$ :

$$\arg \min_{\{\hat{\alpha}_i\}_{i=1}^N \in \mathbb{R}^N} \sum_{i=1}^N \mathcal{L}(y_i, \sum_{j=1}^N \hat{\alpha}_j K(x_i, x_j)) + \frac{\lambda}{2} \sum_{i=1}^N \sum_{j=1}^N \hat{\alpha}_i \hat{\alpha}_j K(x_i, x_j) \quad (9.95)$$

Or written in matrix form  $y = (y_1, y_2, \dots, y_N)$ ,  $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_N)$ ,  $K = \{K(x_i, x_j)\}_{i,j=1}^N$ :

$$\arg \min_{\alpha \in \mathbb{R}^N} \sum_{i=1}^N \mathcal{L}(y_i, K\alpha) + \frac{\lambda}{2} \alpha' K \alpha \quad (9.96)$$

Classification criterion

$$\hat{f}(x) = \sum_{i=1}^N \hat{\alpha}_i K(x, x_i) \quad (9.97)$$

## 9.4.2 Useful Kernel

Some useful Kernel for numeric vector  $x$ :

- Linear Kernel (identity):

$$K(x, y) := \langle x, y \rangle \quad (9.98)$$

- $d^{\text{th}}$  Degree Polynomial Kernel

$$K(x, y) := (1 + \langle x, y \rangle)^d \quad (9.99)$$

- Radical Base Function Kernel:

$$K(x, y) := \exp \left[ -\frac{\|x - y\|^2}{\sigma^2} \right] \quad (9.100)$$

- Sigmoid Kernel:

$$K(x, y) = \tanh(1 + \langle x, y \rangle) \quad (9.101)$$

Note that **equation. 9.95** includes Kernel  $K(\cdot, \cdot)$  only, thus Kernel trick could be applied to various scenarios once we could define a proper Kernel. e.g. Substring Kernel for string sequence.

### 9.4.3 Kernel Support Vector Machine

Replace the inner produce term in Dual problem of SVM [equation. 9.62](#) into Kernel function to obtain Kernel SVM:

$$\arg \max_{\alpha} \sum_{i=1}^N \alpha_i - \frac{1}{2} \sum_{i=1}^N \sum_{j=1}^N \alpha_i \alpha_j y_i y_j K(x_i, x_j) \quad (9.102)$$

$$s.t. 0 \leq \alpha_i \leq C, \quad \sum_{i=1}^N \alpha_i y_i = 0 \quad (9.103)$$

$$\hat{f}(x) = \sum_{i \in \mathcal{I}^{\text{sv}}} \alpha_i y_i K(x, x_i) \quad (9.104)$$

Or use the loss+penalization primal form of SVM:

$$\arg \min_{\hat{\alpha}} \sum_{i=1}^N \left[ 1 - y_i \sum_{j=1}^N \hat{\alpha}_j K(x_i, x_j) \right]_+ + \frac{\lambda}{2} \sum_{i=1}^N \sum_{j=1}^N \hat{\alpha}_i \hat{\alpha}_j K(x_i, x_j), \quad \lambda = \frac{1}{C} \quad (9.105)$$

$$\hat{y}(x) = \begin{cases} 1 & , \hat{f}(x) = \sum_{i=1}^N \hat{\alpha}_i K(x, x_i) \geq s \\ -1 & , \hat{f}(x) = \sum_{i=1}^N \hat{\alpha}_i K(x, x_i) < s \end{cases} \quad (9.106)$$

Note: Here  $\alpha_i$  and  $\hat{\alpha}_i$  are not the same set of number, but the optimization problems are the same (if  $\{K(x_i, x_j)\}$  is non-singular).

### 9.4.4 SMO Algorithm for Kernel SVM

Idea: reduction from  $N$ -dim optimization to many 2-dim optimization.

### 9.4.5 Kernel Regression

#### □ Kernel Regression with Squared Error Loss

Recall linear regression with RS loss

$$\arg \min_{\beta_0, \beta} \sum_{i=1}^N [y_i - \beta_0 - x_i' \beta]^2 + \frac{\lambda}{2} \|\beta\|_2^2 \quad (9.107)$$

replace linear classification  $f(x) = \beta_0 + x' \beta$  into Kernel  $K\alpha$ :

$$\arg \min_{\hat{\alpha}} (y - K\hat{\alpha})'(y - K\hat{\alpha}) + \frac{\lambda}{2} \hat{\alpha}' K \hat{\alpha} \quad (9.108)$$

Solution is similar to ridge regression form:

$$\hat{\alpha} = (K + \lambda I)^{-1} y \quad (9.109)$$

#### □ Kernel Logistic Regression

In logistic regression, the loss function is binomial deviance  $\log [1 + e^{-yf(x)}]$

$$\arg \min_{\hat{\alpha}} \sum_{i=1}^N \log [1 + e^{y_i \sum_{j=1}^N \hat{\alpha}_j K(x_i, x_j)}] + \frac{\lambda}{2} \sum_{i=1}^N \sum_{j=1}^N \hat{\alpha}_i \hat{\alpha}_j K(x_i, x_j) \quad (9.110)$$

$$\hat{y}(x) = \begin{cases} 1 & , \hat{f}(x) = \sum_{i=1}^N \hat{\alpha}_i K(x, x_i) \geq s \\ -1 & , \hat{f}(x) = \sum_{i=1}^N \hat{\alpha}_i K(x, x_i) < s \end{cases} \quad (9.111)$$

## Section 9.5 Clustering

Clustering is an important scenario of unsupervised learning  $\mathcal{D} = \{x_i\}_{i=1}^N$ , to cluster ‘similar’ data points into the same group.

### 9.5.1 Proximity Matrix

For separation concern, we should first define some metric to measure similarity between data

$$d_{ij} = D(x_i, x_j) \quad (9.112)$$

common usage of distance measure see [section. 4.7](#)

And form the proximity matrix  $D$ :

$$D = \{d_{ij}\}_{i,j=1}^N \quad (9.113)$$

Usually some clustering algorithm would claim some properties:

- Non-negative element and non-zero diagonal:

$$d_{ij} \geq 0, \forall i, j. \quad d_{ii} = 0, \forall i \quad (9.114)$$

- Symmetry

$$D = D^T \quad (9.115)$$

Overall dissimilarity:

$$\bar{D} = \frac{1}{N^2} \sum_{i=1}^N \sum_{j=1}^N D(x_i, x_j) \quad (9.116)$$

#### □ Optimizing Goal of Clustering

With similarity/dissimilarity defined, clustering target could be expressed as maximizing within cluster scatter/ minimizing between cluster scatter, with respect to clustering group  $C(\cdot)$

$$\arg \max_{C(\cdot)} \frac{1}{2} \sum_{k=1}^K \sum_{i:C(x_i)=k} \sum_{j:C(x_j)=k} D(x_i, x_j) \quad (9.117)$$

$$\arg \min_{C(\cdot)} \frac{1}{2} \sum_{k=1}^K \sum_{i:C(x_i)=k} \sum_{j:C(x_j) \neq k} D(x_i, x_j) \quad (9.118)$$

The two forms are equivalent due to a fixed sum:

$$\frac{1}{2} \sum_{k=1}^K \sum_{i:C(x_i)=k} \sum_{j:C(x_j)=k} D(x_i, x_j) + \frac{1}{2} \sum_{k=1}^K \sum_{i:C(x_i)=k} \sum_{j:C(x_j) \neq k} D(x_i, x_j) = \frac{1}{2} \sum_{i=1}^N \sum_{j=1}^N D(x_i, x_j) := T = \text{const} \quad (9.119)$$

Usually search for cluster assignment is based on iterative greedy descent search.

Some frequently used clustering methods are included in [section. 4.7](#)

- Hierarchical Method

- $K$ -Means
- EM-Gaussian Mixture Model
- DBSCAN & OPTICS Density Method

In this section, an extra model based on spectrum is introduced

### 9.5.2 Spectrum Clustering

Express the dataset as a Graph  $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{W})$ , where  $\mathcal{V} = \{v_i\}_{i=1}^N$  for vertex,  $\mathcal{E} = \{e_{ij}\}_{i,j=1}^N$ ,  $\mathcal{W} = \{w_{ij}\}_{i,j=1}^N$  for edges and weights. In this case cluster is a graph partition problem.

#### □ Graph Laplacian

Some definition:

- Degree of vertex:

$$d_i = \sum_{j=1}^N w_{ij} \quad (9.120)$$

- Degree matrix

$$D = \text{diag}\{d_1, d_2, \dots, d_N\} \quad (9.121)$$

- Unnormalized graph Laplacian:

$$L := D - W \quad (9.122)$$

is symmetric and semi-positive definite

$$\xi' L \xi = \sum_{i,j=1}^N w_{ij} (\xi_i - \xi_j)^2 \geq 0, \quad \forall \xi \in \mathbb{R}^N \quad (9.123)$$

Spectrum is based on studying the eigen vector and eigen value of  $L$ .

- For any graph Laplacian  $L_{m \times m}$ ,  $\mathbf{1}_m$  is a eigen vector with eigen value 0
- In the case that  $\mathcal{G}$  is **not** fully connected, with  $K$  subgraph  $\mathcal{G} = \{\mathcal{G}_1, \mathcal{G}_2, \dots, \mathcal{G}_K\}$ , i.e.  $W$  and  $L$  could be written in diagonal form (usually need some row/column transformation)

$$L = \begin{bmatrix} L_1 & 0 & \dots & 0 \\ 0 & L_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & L_K \end{bmatrix} \quad (9.124)$$

the multiplicity of eigen value 0 is  $K$ , with each eigen vector as

$$\mathbf{1}_{\mathcal{G}_k} = [\mathbb{I}(v_1 \in \mathcal{G}_k), \dots, \mathbb{I}(v_N \in \mathcal{G}_k)], \quad k = 1, 2, \dots, K \quad (9.125)$$

- In real world case, the graph could probably expressed as a small deviance from a graph with subgraph:

$$L = \begin{bmatrix} L_1 & 0 & \dots & 0 \\ 0 & L_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & L_K \end{bmatrix} + N \times N_{\delta} \quad (9.126)$$

where we would expect the smallest  $K$  eigen value  $0 = \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_K$  corresponds to the  $K$  cluster we want.

---

**Algorithm** *Spectral Clustering*


---

1. Compute  $L_{N \times N}$
2. Determine the  $K$  smallest eigen values  $0 = \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_K$  with eigen vector  $u_i, i = 1, 2, \dots, K$

$$U = [u_1, u_2, \dots, u_K] = \begin{bmatrix} u_{11} & u_{12} & \dots & u_{1K} \\ u_{21} & u_{22} & \dots & u_{2K} \\ \vdots & \vdots & \ddots & \vdots \\ u_{N1} & u_{N2} & \dots & u_{NK} \end{bmatrix} = [z_1, z_2, \dots, z_N]^T \quad (9.127)$$

$$z_i = [u_{i1}, u_{i2}, \dots, u_{iK}]^T, \quad i = 1, 2, \dots, N \quad (9.128)$$

3. Cluster  $\{z_i\}_{i=1}^N$  with e.g.  $K$ -Means.
- 

Choice of normalized graph Laplacian, would cause different cluster results:

- Ratio Cut  $L = I - D^{-1}W$

$$\arg \min_{\{\mathcal{G}_1, \dots, \mathcal{G}_K\}} \frac{1}{2} \sum_{i=1}^K \frac{\text{Bet}(\mathcal{G}_i, \mathcal{G}_i^c)}{|\mathcal{G}_i|} \quad (9.129)$$

- Normalized Cut  $L = I - D^{-1/2}WD^{-1/2}$

$$\arg \min_{\{\mathcal{G}_1, \dots, \mathcal{G}_K\}} \frac{1}{2} \sum_{i=1}^K \frac{\text{Bet}(\mathcal{G}_i, \mathcal{G}_i^c)}{\sum_{i \in \mathcal{G}_i} \sum_{j \in \mathcal{G}} d_{ij}} \quad (9.130)$$

## Section 9.6 Tree-Based Classification Model

Idea of tree: divide the space  $\mathcal{X}$  into grids  $R_m$  and assign prediction into the most frequent class

$$\hat{f}(x \in R_m) = \arg \max_k \sum_{x_i \in R_m} \mathbb{I}(y_i = k) \quad (9.131)$$

But such method is not practical in high dimensional due to curse of dimensionality. Nore practical method would be a greedy search, each step along one variable.

---

## 9.6.1 Tree-Based Classification

### □ Branch Growing Process

Grow branch on a node

---

#### Algorithm *Classification Tree*

---

In each branch growing on a node:

1. Look for a splitting variable  $x_j$  and split value  $s$ :

$$\arg \min_{j,s} [N_{\text{left}} \text{ImPu}(x_i \in R_{\text{left}}(j, s)) + N_{\text{right}} \text{ImPu}(x_i \in R_{\text{right}}(j, s))] \quad (9.132)$$

$$R_{\text{left}}(j, s) = \{x : x_j \leq s\}, \quad R_{\text{right}}(j, s) = \{x : x_j > s\} \quad (9.133)$$

useful impurity measure  $\text{ImPu}(\{x\})$  with  $p_k(X = \{x\})$  defined

$$p_k(X) = \frac{\sum_{x \in X} \mathbb{I}(C(x) = k)}{|X|} \quad (9.134)$$

- Misclassification rate

$$1 - \max_k p_k \quad (9.135)$$

- Gini impurity

$$\sum_{k=1}^K p_k(1 - p_k) = \sum_{k=1}^K \sum_{k' \neq k} p_k p_{k'} \quad (9.136)$$

Gini impurity with category weight  $W_{K \times K} = \{w_{kk'}\}_{k,k'=1}^K$

$$\sum_{k=1}^K \sum_{k' \neq k} w_{kk'} p_k p_{k'} \quad (9.137)$$

- Entropy

$$-\sum_{k=1}^K p_k \log p_k \quad (9.138)$$

2. usually the process ends when

$$|\text{node}| \leq \text{const}, \quad \forall \text{node} \quad (9.139)$$

3. Apply cost complexity pruning strategy

$$C_\alpha(T) = \sum_{m=1}^{|T|} N_m \text{ImPu}(R_m) + \alpha |T| \quad (9.140)$$

where  $T$  is tree,  $|T|$  for number of nodes in the tree.

---

Comment:

- Tree methods is well-interpreted, especially similar to a natural decision making process
-

- Handle non-linear classification pattern
- Unstable to data.

Performance of tree classification could be largely improved with bagging method and boosting method.

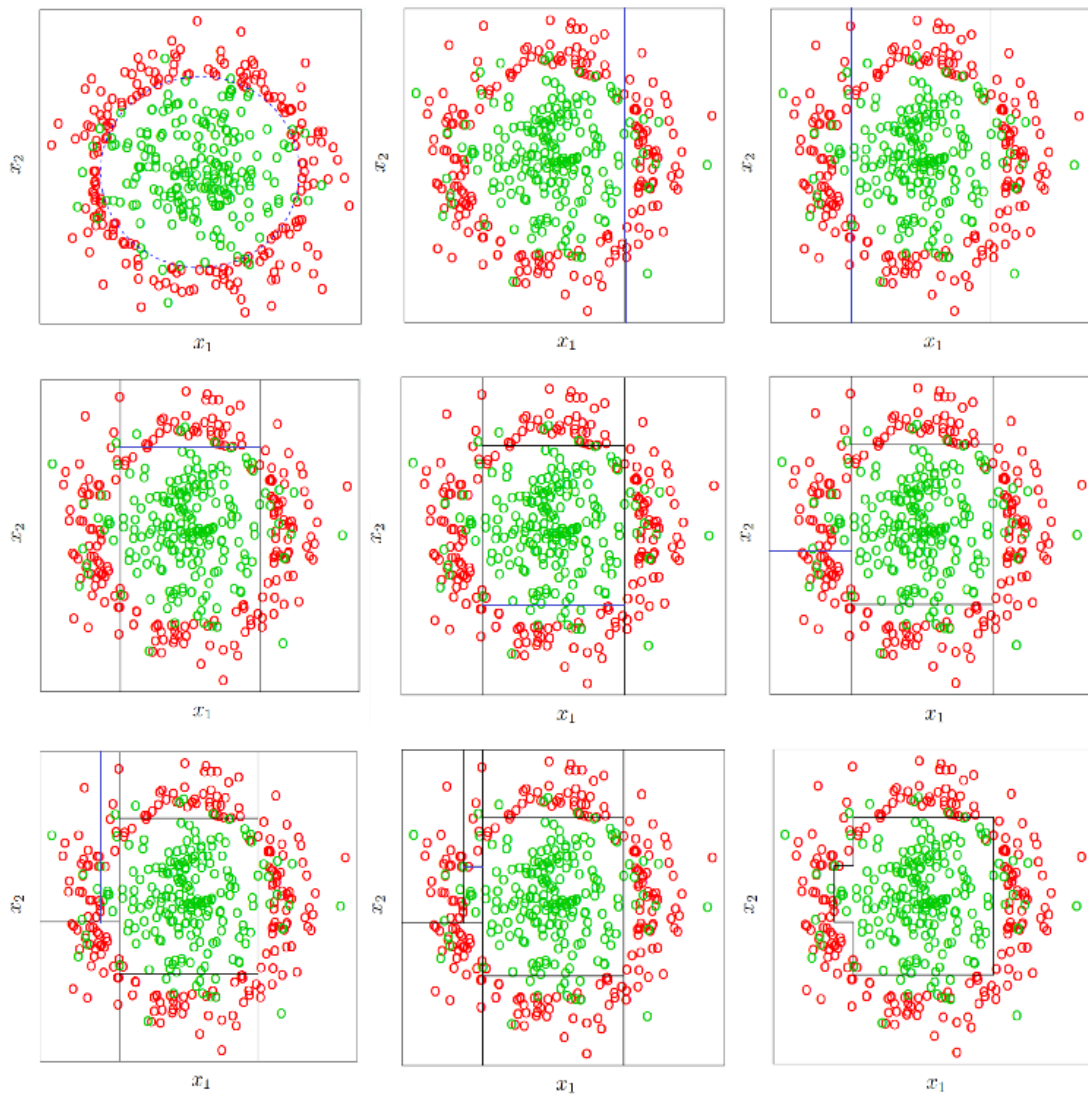


图 8:

## 9.6.2 Bagging and Boosting

### □ Bagging

Bagging is short for **B**ootstrap **A**ggregation. Idea: for  $B$  bootstrapped training data, the bootstrapping result

$$\hat{f}_{\text{boot}}(x) = \frac{1}{B} \sum_{b=1}^B \hat{f}_b(x) \text{ or } = \arg \max_k \sum_{b=1}^B \mathbb{I}(\hat{f}_b(x) = k) \quad (9.141)$$

### □ Random Forest

Random Forest aims at decorrelating trees to reduce variance when averaging trees.

---

**Algorithm** *Random Forest Bagging*

---

1. Generate  $B$  different bootstrapped training data. (*random 1* by bootstrap sampling)
2. For each sample, grow a tree. In each split of tree (i.e. a branch growth),  $q \approx \sqrt{p}$  variable components are randomly selected for classification. (*random 2* by randomizing components)
3. Take average or vote of all  $B$  trees as the final result

Comment: A prune is usually needed, cause variance is reduced by averaging.

## □ Boosting

Idea: Fitting result of previous trees could be used to modify following trees. The error rate of each tree would influence the vote weight when bagging the results.

### Algorithm Adaboost

1. Each observant is given weights  $w_i^{(0)} = \frac{1}{N}$ ,  $i = 1, 2, \dots, N$
2. For  $m = 1 : M$ ,  $M$  for loops of boosting:
  - (a) Grow a tree  $T^{(m)}(x)$  with weight  $w_i^{(m)}$
  - (b) Compute **error rate**

$$\text{err}^{(t)} := \frac{\sum_{i=1}^N w_i^{(m)} \mathbb{I}(y_i \neq T^{(m)}(x_i))}{\sum_{i=1}^N w_i^{(m)}} \quad (9.142)$$

and define

$$\alpha^{(m)} = \log \left[ (1 - \text{err}^{(m)}) / \text{err}^{(m)} \right] \quad (9.143)$$

- (c) Reset weights by

$$w_i^{(m+1)} = w_i^{(m)} \cdot \exp \left[ \alpha^{(m)} \mathbb{I}(y_i \neq T^{(m)}(x_i)) \right] \quad (9.144)$$

3. Output

$$\hat{f}(x) = \text{sgn} \left[ \sum_{m=1}^M \alpha^{(m)} T^{(m)}(x) \right] \quad (9.145)$$

## Section 9.7 Neural Network

### □ Linear Perceptron with Activate Function

Usually linear perceptron is used as a neuron in neutral network:

$$y = g(w_0 + w_1 x_1 + \dots + w_p x_p) = g(x'w), \quad x_0 \equiv 1 \quad (9.146)$$

where  $g(\cdot)$  is activate function. Such Perceptron could be optimized by gradient

Some useful activate function:



- Linear Threshold Unit (LTU)

$$g(\xi) = \begin{cases} 0, & \xi < 0 \\ 1, & \xi \geq 0 \end{cases} = \eta(\xi) \quad (9.147)$$

- Logistic Function

$$g(\xi) = \frac{1}{1 + e^{-\xi}} \quad (9.148)$$

- Hyperbolic Tangent Function

$$g(\xi) = \tanh \xi = \frac{e^{2\xi} - 1}{e^{2\xi} + 1} \quad (9.149)$$

- Rectified Linear Unit (ReLU)

$$g(\xi) = \begin{cases} 0, & \xi < 0 \\ \xi, & \xi \geq 0 \end{cases} \quad (9.150)$$

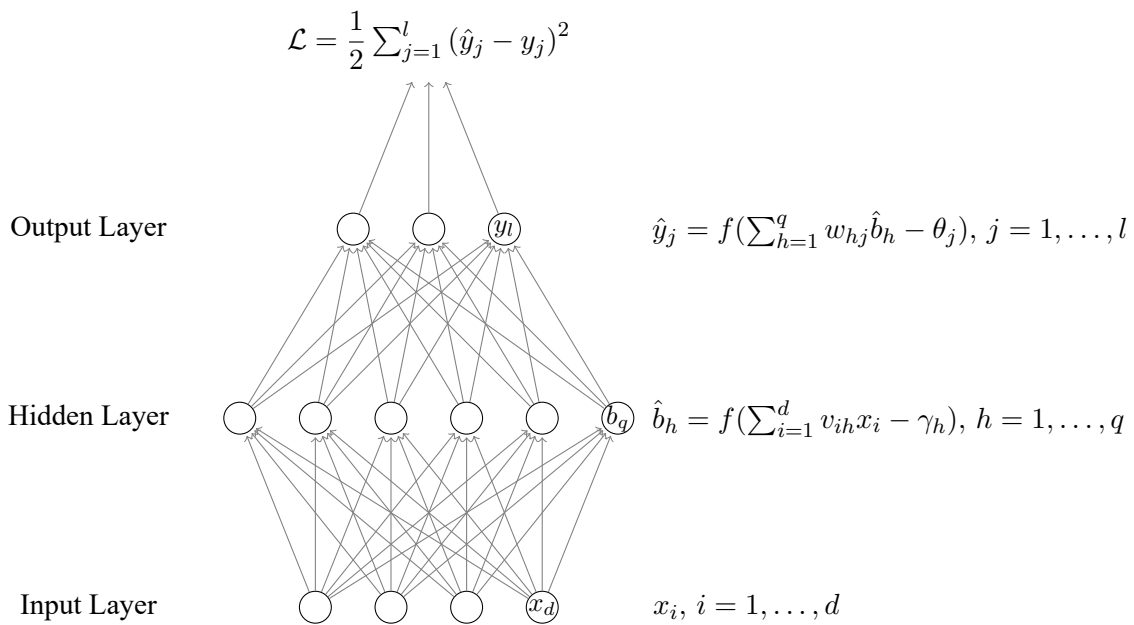


图 9: Structure of Feed-Forward Neural Network (1 Layer)

A MonoLayer perceptron with enough neurons (hidden units) could represent any continuous function. MultiLayer Perceptron (MLP) could even represent discontinuous functions.

### 9.7.1 Back Propagation

Perceptron system is usually optimized by back propagation (of gradient).

An example to optimize  $v_{ih}$ ,  $\gamma_h$  in **figure. 9**:

$$\frac{\partial \mathcal{L}}{\partial v_{ih}} = \sum_{j=1}^l \frac{\partial \mathcal{L}}{\partial \hat{y}_j} \frac{\partial \hat{y}_j}{\partial \hat{b}_h} \frac{\partial \hat{b}_h}{\partial v_{ih}} \quad (9.151)$$

$$= \sum_{j=1}^l \hat{y}_j (\hat{y}_j - y_j) \cdot \frac{\partial f(u)}{\partial u} \Big|_{u=\sum w_{hj} \hat{b}_h - \theta_j} w_{hj} \cdot \frac{\partial f(v)}{\partial v} \Big|_{v=\sum_{i=1}^d v_{ih} x_i - \gamma_h} x_i \quad (9.152)$$

$$\frac{\partial \mathcal{L}}{\partial \gamma_h} = \sum_{j=1}^l \frac{\partial \mathcal{L}}{\partial \hat{y}_j} \frac{\partial \hat{y}_j}{\partial \hat{b}_h} \frac{\partial \hat{b}_h}{\partial \gamma_h} \quad (9.153)$$

$$= \sum_{j=1}^l \hat{y}_j (\hat{y}_j - y_j) \cdot \frac{\partial f(u)}{\partial u} \Big|_{u=\sum w_{hj} \hat{b}_h - \theta_j} w_{hj} \cdot \frac{\partial f(v)}{\partial v} \Big|_{v=\sum_{i=1}^d v_{ih} x_i - \gamma_h} \cdot (-1) \quad (9.154)$$

## Chapter. X 应用时间序列部分

Instructor: Dong Li

### Section 10.1 Time Series Data and Model

#### 10.1.1 Time Series Data and Tasks

**Time Series** : a sequential r.v. indexed in time order.

$$\{Y_t\}, t \in \mathcal{T} \quad \mathcal{T} \text{ is index set} \quad (10.1)$$

and actual data of time series, i.e. times series data is called a **Realization** of time series, denoted<sup>92</sup>

$$\{y_t\}, t \in T \subset \mathcal{T}$$

e.g. in forecasting task,  $T$  encodes history. In this chapter we usually focus on easier case of arithmetic progression  $T = \{1, 2, \dots, N\}$ , or at least numeric ordinal sequence.

Time Series Analysis (TSA): Analysis on time series data to extract meaningful statistics/other characteristics. Task of TSA includes:

- Describing and Explanaing the machanism of time series
- Forecasting
- Guiding the intervention of Time Series

In this section several modelling/forecasting methods would be included.

#### 10.1.2 Time Series Model

There are plenty of useful modelling methods:

- Regression Model: View  $y$  as function of  $t$ , regression on some model  $y = f(t)$  with loss  $\mathcal{L}$ . e.g. linear regression

$$y = \beta_0 + \beta_1 t + \varepsilon, \quad \mathcal{L} = \sum_{t \in T} (y_t - \hat{y}_t)^2 \quad (10.2)$$

Modelling strategy is similar to that introduced in linear regression, see [Chapter. 3](#)

- STL Method: Seasonal and Trend decomposition using Loess. A decomposition of time series into ‘TS = Trend + Season + Random’, i.e.

$$Y_\tau = T_\tau + S_\tau + X_\tau \quad (10.3)$$

and we could model  $T_\tau, S_\tau, X_\tau$  separately. The focus is the modelling of random term  $X_t$ , which we expect to be ‘stationarily random’ through time. (Usually we model this part also by ARMA model)

- Exponential Smoothing Model: Use weighted average over history to predict future.
- ARIMA Model: The main focus of this chapter.

<sup>92</sup>A note on  $T \subset \mathcal{T}$ : actually  $T$  has to be discrete beacuse it is a sample of  $\mathcal{T}$ . while  $\mathcal{T}$  is not necessarily defined as discrete.

## Section 10.2 Stochastic Process and Statistics

### 10.2.1 Basic Knowledge of Stochastic Process

A stochastic process can be denoted:

$$\{X_t : t \in \mathcal{T}\} : \Omega \rightarrow \mathcal{T} \times \mathcal{E} \quad (10.4)$$

i.e. the random ‘variable’ of stochastic process is a function  $X(t) \in L^2(\mathcal{T})$

□ **Some important cases of stochastic process:**

- i.i.d. sequence:  $\varepsilon_t$  i.i.d.  $\sim \varepsilon$
- White Noise: uncorrelated for different subscript  $t$  in the sense of 2<sup>nd</sup> moment,  $\varepsilon_t \sim \text{WN}(\mu, \sigma^2)$ . where

$$\mathbb{E}(\varepsilon_t) = \mu \quad (10.5)$$

$$\text{cov}(\varepsilon_t, \varepsilon_s) = \sigma^2 \delta_{t,s} \quad (10.6)$$

Further we can append more constraints on WN:

- +  $\{\varepsilon_t\}$  independent: independent white noise  $\varepsilon_t \sim \text{IWN}(\mu, \sigma^2)$
- +  $\mu = 0$ : zero-mean white noise  $\varepsilon_t \sim \text{WN}(0, \sigma^2)$
- +  $\mu = 0, \sigma^2 = 1$ : standard white noise  $\varepsilon_t \sim \text{WN}(0, 1)$
- +  $\varepsilon \sim N(\mu, \sigma^2)$ : normal white noise.
- Martingale difference sequence (MDS): zero expectation given history information:  $\varepsilon_t \sim \text{MDS}$ , where

$$\mathbb{E}(|\varepsilon_t|) < \infty \quad (10.7)$$

$$\mathbb{E}(\varepsilon_t | \mathcal{F}_{t-1}) = 0 \quad (10.8)$$

where  $\mathcal{F}_\tau$  denotes the history until time  $\tau$ :

$$\mathcal{F}_\tau \equiv \sigma(\varepsilon_s, s \leq \tau) \{\varepsilon_s, \varepsilon_{s-1}, \varepsilon_{s-2}, \dots\} \quad (10.9)$$

Relation: i.i.d. > MDS > WN > Stationary

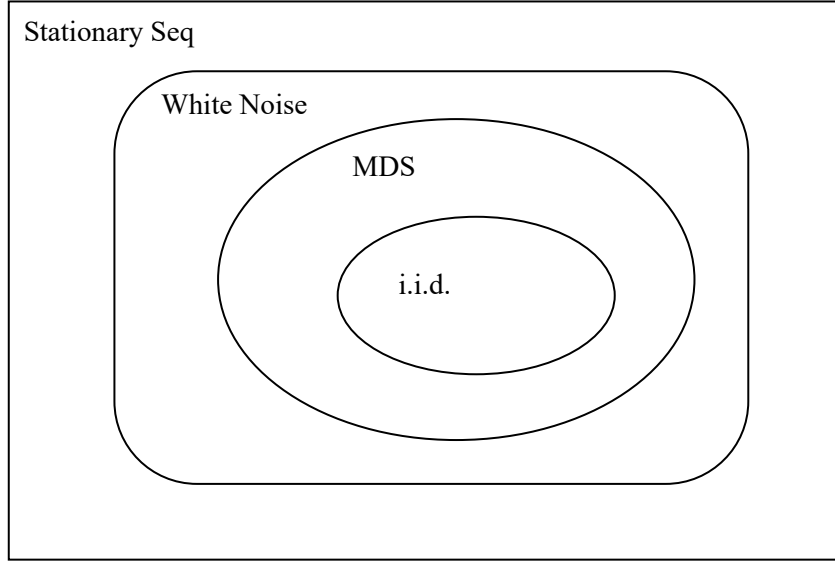


图 10: Relation bet. Sequences

### □ Measure of dependence within stochastic process

Given a stochastic process  $\{X_t : t \in \mathcal{T}\}$

- Mean Function

$$\text{Mean: } \mu_t \equiv \mathbb{E}(X_t), \quad \mathcal{T} \mapsto \mathbb{R} \quad (10.10)$$

- AutoCovariance Function (ACVF) and AutoCorrelation Function (ACF):

$$\text{ACVF: } \gamma_{t,s} \equiv \text{cov}(X_t, X_s), \quad \mathcal{T} \times \mathcal{T} \mapsto \mathbb{R} \quad (10.11)$$

$$\text{ACF: } \rho_{t,s} \equiv \text{corr}(X_t, X_s) = \frac{\gamma_{t,s}}{\sqrt{\gamma_{t,t}\gamma_{s,s}}}, \quad \mathcal{T} \times \mathcal{T} \mapsto [-1, 1] \quad (10.12)$$

- Stationarity: Stationarity is a measure that the ‘correlation structure of stochastic process looks the same’ at any time  $t$ , i.e. is stationary through time.

- Weakly Stationary (WS): given  $\mathbb{E}(X_t^2) < \infty$ , has const  $\mathbb{E}[]$  and  $\text{cov}$  (independent of time)

$$\mathbb{E}(X_t) = \mu_t = \mu \quad (10.13)$$

$$\text{cov}(X_t, X_{t+k}) = \gamma_{t,t+k} = \gamma_k \perp\!\!\!\perp t \quad (10.14)$$

- Strictly Stationary (SS): joint distribution invariant through time. For any given  $\{t_1, t_2, \dots, t_n\} \subset \mathcal{T}$

$$f_{X_{t_1}, X_{t_2}, \dots, X_{t_n}} = f_{X_{t_1+h}, X_{t_2+h}, \dots, X_{t_n+h}}, \quad \forall h \quad (10.15)$$

Some note on WS and SS:

- Generally speaking, WS and SS are not equivalent,  $\text{WS} \not\Rightarrow \text{SS}$  (note that SS does not put constraint on  $\mathbb{E}(X_t^2)$ )
- equivalent for gaussian stochastic process.
- ACF and ACVF of WS:

$$\gamma_{t,t+k} = \gamma_k = \gamma_{-k}, \quad \forall t \in \mathcal{T} \quad (10.16)$$

$$\rho_{t,t+k} = \rho_k = \frac{\gamma_k}{\gamma_0}, \quad \forall t \in \mathcal{T} \quad (10.17)$$

Notation of ACVF matrix:

$$\Gamma_k = \{\gamma_{i-j}\}_{i,j=1}^k = \begin{bmatrix} \gamma_0 & \gamma_1 & \gamma_2 & \cdots & \gamma_{k-2} & \gamma_{k-1} \\ \gamma_1 & \gamma_0 & \gamma_1 & \cdots & \gamma_{k-3} & \gamma_{k-2} \\ \gamma_2 & \gamma_1 & \gamma_0 & \cdots & \gamma_{k-4} & \gamma_{k-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \gamma_{k-2} & \gamma_{k-3} & \gamma_{k-4} & \cdots & \gamma_0 & \gamma_1 \\ \gamma_{k-1} & \gamma_{k-2} & \gamma_{k-3} & \cdots & \gamma_1 & \gamma_0 \end{bmatrix}_{k \times k} \quad (10.18)$$

$\Gamma_k$  is semi-positive definite.

$$\sum_{i=1}^k \sum_{j=1}^k \alpha_i \alpha_j \gamma_{|t_i - t_j|} \geq 0, \quad \forall k, \{t_1, \dots, t_k\}, \vec{\alpha} \quad (10.19)$$

- Partial Autocorrelation (PACF): correlation given information between two time points, original definition

$$\phi_{11} = \phi_1 \quad (10.20)$$

$$\phi_{kk} = \text{corr}(X_t - L(X_t | X_{t+1}, \dots, X_{t+k-1}), X_{t+k} - L(X_{t+k} | X_{t+1}, \dots, X_{t+k-1})), \quad k \geq 2 \quad (10.21)$$

where  $L(X_\tau | X_{t+1}, \dots, X_{t+k-1})$  is the **Best Linear Estimation** of linear model

$$X_\tau = \beta_0 + \beta_1 X_{t+1} + \dots + \beta_{k-1} X_{t+k-1} + \epsilon \quad (10.22)$$

deduction:

- Best MMSE linear estimation  $\hat{X}_\tau \equiv L(X_\tau | X_{t+1}, \dots, X_{t+k-1})$  satisfies<sup>93</sup>

$$\{\beta_0, \beta\} = \arg \min_{\beta_0, \beta} \mathbb{E} \left( \hat{X}_\tau - \beta_0 - \sum_{j=1}^{k-1} \beta_j X_{t+j} \right)^2 \quad (10.23)$$

solution: denote  $X = (X_{t+1}, \dots, X_{t+k-1})$ ,  $\beta = (\beta_1, \dots, \beta_{k-1})$

$$\hat{\beta} = \Sigma_X^{-1} \Sigma_{X, X_\tau} \quad (10.24)$$

$$\hat{\beta}_0 = \mathbb{E}(X_\tau) - \mathbb{E}(X)' \hat{\beta} \quad (10.25)$$

i.e.

$$L(X_\tau | X_{t+1}, \dots, X_{t+k-1}) = \mathbb{E}(X_\tau) + \Sigma_{X_\tau, X} \Sigma_X^{-1} (X - \mathbb{E}(X)) \quad (10.26)$$

Simplified case for zero-mean Weakly Stationary  $\mathbb{E}(X_t) = \mu$ ;  $\gamma_k, \Gamma_k$

$$L(X_{t+k} | X_{t+1}, \dots, X_{t+k-1}) = \mathbb{E}(X_{t+k}) + \Sigma_{X_{t+k}, X} \Sigma_X^{-1} (X - \mathbb{E}(X)) \quad (10.27)$$

$$= \gamma'_{k-1} \Gamma_{k-1}^{-1} X_{t+k-1:t+1} \quad (10.28)$$

Calculation formula for zero-mean Weakly Stationary:

<sup>93</sup>Detailed theory about MMSE and linear estimator see [section. 12.4.1, Linear MMSE Estimator](#).

– using determinant form

$$\phi_{11} = \rho_1 \quad (10.29)$$

$$\phi_{kk} = \frac{\begin{vmatrix} 1 & \rho_1 & \rho_2 & \cdots & \rho_{k-2} & \rho_1 \\ \rho_1 & 1 & \rho_1 & \cdots & \rho_{k-3} & \rho_2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \rho_{k-1} & \rho_{k-2} & \rho_{k-3} & \cdots & \rho_1 & \rho_k \end{vmatrix}_{k \times k}}{\begin{vmatrix} 1 & \rho_1 & \rho_2 & \cdots & \rho_{k-2} & \rho_{k-1} \\ \rho_1 & 1 & \rho_1 & \cdots & \rho_{k-3} & \rho_{k-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \rho_{k-1} & \rho_{k-2} & \rho_{k-3} & \cdots & \rho_1 & 1 \end{vmatrix}_{k \times k}} \quad (10.30)$$

– Levinson-Durbin's recursive formula

$$\phi_{11} = \rho_1 \quad (10.31)$$

$$\phi_{k+1,k+1} = \frac{\rho_{k+1} - \sum_{j=1}^k \phi_{k,j} \rho_{k+1-j}}{1 - \sum_{j=1}^k \phi_{k,j} \rho_j}, \quad k \geq 1 \quad (10.32)$$

$$\phi_{k+1,j} = \phi_{k,j} - \phi_{k+1,k+1} \phi_{k,k+1-j}, \quad j = 1, 2, \dots, k \quad (10.33)$$

where  $\phi_{k+1,j}$  here is a formal notation for recursion. But we will see its meaning in AR( $p$ ) model (equation. 10.90)

- Wold Decomposition: zero-mean weakly stationary time series can be decomposed as :

$$X_t = \sum_{j=-\infty}^{\infty} \phi_j \varepsilon_{t-j} + V_t \quad (10.34)$$

where

$$\phi_0 = 1 \quad (10.35)$$

$$\varepsilon_t \sim \text{WN}(0, \sigma^2) \quad (10.36)$$

- Spectrum of zero-mean weak stationary time series  $\{X_t\}$ :

$$X_t = \int_{\lambda} \xi(\lambda) e^{i\lambda t} d\lambda \quad (10.37)$$

We can use this form to construct ACF, ACVF, etc.

– Spectrum and ACVF: the fourier expansion of  $\gamma_k$  is denoted

$$\gamma_k = \text{cov}(X_t, X_{t+k}) \equiv \int_{\lambda} e^{i\lambda k} \nu(\lambda) d\lambda \quad (10.38)$$

and here a function  $F(\lambda) = \int \nu(\lambda) d\lambda$  is the **spectrum** of  $\gamma_k$ , and  $\nu(\lambda)$  is the **spectrum density**.

For  $k = 0, 1, 2, \dots$  (discrete time)

$$\gamma_k = \int_{-\pi}^{\pi} \nu(\lambda) e^{i\lambda k} d\lambda \quad (10.39)$$

and also use inverse fourier transform: for weak stationary TS  $X_t = \sum_{j=-\infty}^{\infty} \phi_j \varepsilon_{t-j}$ ,  $\varepsilon_t \sim \text{WN}(0, \sigma^2)$

$$\nu(\lambda) = \frac{1}{2\pi} \int_{\mathbb{R}} \gamma_k e^{-i\lambda k} dk = \frac{\sigma^2}{2\pi} \sum_{k=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} \phi_j \phi_{j-k} e^{-i\lambda k} = \frac{\sigma^2}{2\pi} \left| \sum_{j=-\infty}^{\infty} \phi_j e^{i\lambda j} \right|^2 \quad (10.40)$$

### 10.2.2 Statistics

To estimate the above  $\mu_t = \mu, \gamma_k, \rho_k, \phi_{kk}$  given a realization of  $\{X_t\}$ , say we have  $\{x_t\}_{t=1}^n$ , we can construct:

- Sample mean  $\mu$ :

$$\hat{\mu} = \hat{x}_n = \frac{1}{n} \sum_{t=1}^n x_t \quad (10.41)$$

$\hat{\mu}$  is the unbiased, consistent estimator, with

$$\sqrt{n}(\hat{\mu} - \mu) \xrightarrow{d} N(0, \sigma^2)$$

an estimator using spectrum:

$$\sqrt{n}(\hat{\mu} - \mu) \xrightarrow{d} N(0, 2\pi\nu(0)) \quad (10.42)$$

$$2\pi\nu(0) = \gamma_0 + 2 \sum_{j=1}^{\infty} \gamma_j = \sum_{j=-\infty}^{\infty} \gamma_j \quad (10.43)$$

- ACVF  $\gamma_k$ :

$$\hat{\gamma}_k = \frac{1}{n} \sum_{t=1}^{n-k} (x_t - \hat{\mu})(x_{t+k} - \hat{\mu}) \quad (10.44)$$

$$\hat{\hat{\gamma}}_k = \frac{1}{n-k} \sum_{t=1}^{n-k} (x_t - \hat{\mu})(x_{t+k} - \hat{\mu}) \quad (10.45)$$

Note for actual usage:

- We usually avoid estimation for  $k \sim n$  due to large error when  $n - k$  is small
- In most cases we use  $\hat{\gamma}_k$  rather than  $\hat{\hat{\gamma}}_k$ , for two reasons:
  - \* We often estimate  $\gamma_k$  for small  $k$ , which means  $\hat{\gamma}_k \approx \hat{\hat{\gamma}}_k$
  - \*  $\hat{\gamma}_k$  could guarantee the semi-positive-definition of  $\hat{\Gamma}_k$ :

$$\hat{\Gamma}_k = \{\hat{\gamma}_{i-j}\}_{i,j=1}^k \succeq 0 \quad (10.46)$$

asymptotic distribution: denote i.i.d. standard normal time series  $W_t \sim \text{i.i.d. } N(0, 1)$

$$\sqrt{n}(\hat{\gamma}_0 - \gamma_0, \hat{\gamma}_1 - \gamma_1, \dots, \hat{\gamma}_h - \gamma_h) \xrightarrow{d} (\xi_0, \xi_1, \dots, \xi_h) \quad (10.47)$$

$$\xi_j = \left( \frac{\sqrt{\mathbb{E}(\varepsilon^4)} - \sigma^4}{\sigma^2} \gamma_j \right) W_0 + \sum_{t=1}^{\infty} (\gamma_{t+j} + \gamma_{t-j}) W_t, \quad j \geq 0 \quad (10.48)$$

- ACF  $\rho_k$ :

$$\hat{\rho}_k = \frac{\hat{\gamma}_k}{\hat{\gamma}_0} = \frac{\sum_{t=1}^{n-k} (x_t - \hat{\mu})(x_{t+k} - \hat{\mu})}{\sum_{t=1}^{n-k} (x_t - \hat{\mu})^2} \quad (10.49)$$

asymptotic distribution: denote i.i.d. standard normal time series  $W_t \sim \text{i.i.d. } N(0, 1)$

$$\sqrt{n}(\hat{\gamma}_0 - \gamma_0, \hat{\gamma}_1 - \gamma_1, \dots, \hat{\gamma}_h - \gamma_h) \xrightarrow{d} (R_0, R_1, \dots, R_h) \quad (10.50)$$

$$R_j = \sum_{t=1}^{\infty} (\phi_{t+j}\rho_{t-j} - 2\rho_t\rho_j)W(t), \quad j \geq 1 \quad (10.51)$$

- PACF  $\phi_{kk}$ : take  $\hat{\rho}_k$  in the calculation equation of  $\phi_{kk}$ .



## Section 10.3 ARMA Model

Two of the basic modeling methods for time series: Auto-Regression (AR) and Moving-Average (MA)

### 10.3.1 Backshift Operator and Difference Equation

#### □ Backshift Operator $\mathcal{B}$

For clearer notation of ARMA and induce the solution, we first introduce backshift operator  $\mathcal{B}$  of time series: given time series  $\{X_t\}$ <sup>94</sup>

$$\mathcal{B}X_t = X_{t-1}, \quad \forall t \quad (10.56)$$

further it can be used as variable of function by Laurant function series expansion:

$$\psi(z) = \sum_{j=-\infty}^{\infty} \psi_j z^j \quad (10.57)$$

$$\psi(\mathcal{B}) = \sum_{j=-\infty}^{\infty} \psi_j \mathcal{B}^j \quad (10.58)$$

$$\psi(\mathcal{B})X_t = \sum_{j=-\infty}^{\infty} \psi_j \mathcal{B}^j X_t = \sum_{j=-\infty}^{\infty} \psi_j X_{t-j} \quad (10.59)$$

Linearity: for time series  $\{X_t\}, \{Y_t\}$ . r.v.  $U, V, W$ :

$$\phi(\mathcal{B})(UX_t + VY_t + W) = U\psi(\mathcal{B})X_t + V\psi(\mathcal{B})Y_t + W\psi(1) \quad (10.60)$$

#### □ Difference Equation

$p^{\text{th}}$  order ordinary difference equation:

$$X_t - [a_1X_{t-1} + a_2X_{t-2} + \dots + a_pX_{t-p}] = 0 \quad (10.61)$$

can be solved using backshift operator: define characteristic equation which would have  $p$  roots  $\zeta_j$

$$A(z) = 1 - [a_1z + a_2z^2 + \dots + a_pz^p] \quad (10.62)$$

$$= 1 - \sum_{j=1}^p a_j z^j \quad (10.63)$$

$$= \prod_{j=1}^p (1 - \zeta_j z) \quad (10.64)$$

<sup>94</sup>Backshift operator could be used to construct difference operator  $\Delta = (1 - \mathcal{B})$ , e.g.

$$\Delta X_t = (1 - \mathcal{B})X_t = X_t - X_{t-1} \quad (10.52)$$

$$\Delta^2 X_t = (1 - \mathcal{B})^2 X_t = X_t - 2X_{t-1} + X_{t-2} \quad (10.53)$$

$$\dots \quad (10.54)$$

or seasonal difference operator  $\Delta_k = (1 - \mathcal{B}^k)$ , e.g.

$$\Delta_4 X_t = (1 - \mathcal{B}^4)X_t = X_t - X_{t-4} \quad (10.55)$$

$$A(\mathcal{B}) = 1 - \sum_{j=1}^p a_j \mathcal{B}^j \quad (10.65)$$

$$= \prod_{j=1}^p (1 - \zeta_j \mathcal{B}) \quad (10.66)$$

similar to ODE, we can construct general solution from  $\zeta_j$ , and particular solution.<sup>95</sup>

### 10.3.2 AR Model

Auto-Regression model (of order  $p$ ) contains ( $p^{\text{th}}$  order) backshift on  $X_t$ :

$$X_t = \phi_1 X_{t-1} + \phi_2 X_{t-2} + \dots + \phi_p X_{t-p} + \varepsilon_t, \quad \varepsilon_t \sim \text{WN}(\mu_\varepsilon, \sigma^2) \quad (10.67)$$

or expressed in backshift operator with  $\phi(z) = 1 - \sum_{j=1}^p \phi_j z^j$ , where the root of  $\phi(z) = 0$  denoted  $\alpha_j$

$$\phi(\mathcal{B})X_t = \varepsilon_t, \quad \varepsilon_t \sim \text{WN}(\mu_\varepsilon, \sigma^2) \quad (10.68)$$

$$\phi(z) = 1 - \sum_{j=1}^p \phi_j z^j = \prod_{j=1}^p (1 - \alpha_j z) \quad (10.69)$$

□ **Properties and Solution:** (here we consider stationary case  $\mu_\varepsilon = 0$ )

- (Weak) Stationarity condition:

$$|\alpha_j| > 1, \quad \forall j \quad (10.70)$$

- Solution of  $X_t$ : using the expansion of  $\phi^{-1}$

$$\phi(z) = 1 - \sum_{j=1}^p \phi_j z^j \quad (10.71)$$

$$\phi^{-1}(z) = \sum_{j=0}^{\infty} \psi_j z^j, \quad \psi_0 = 1 \quad (10.72)$$

naturally expressed in the form of Wold Decomposition:

$$\phi(\mathcal{B})X_t = \varepsilon_t \Rightarrow X_t = \phi^{-1}(\mathcal{B})\varepsilon_t = \sum_{j=0}^{\infty} \psi_j \varepsilon_{t-j}, \quad \psi_0 = 1 \quad (10.73)$$

- ACF and ACVF:

$$\gamma_k = \sigma^2 \sum_{j=0}^{\infty} \psi_j \psi_{j+k} \quad (10.74)$$

$$\rho_k = \frac{\sum_{j=0}^{\infty} \psi_j \psi_{j+k}}{\sum_{j=0}^{\infty} \psi_j^2} \quad (10.75)$$

- Spectrum density  $\nu(\lambda)$ :

$$\nu(\lambda) = \frac{\sigma^2}{2\pi} \left| \sum_{j=0}^{\infty} \psi_j e^{i\lambda j} \right|^2 \quad (10.76)$$

$$= \frac{\sigma^2}{2\pi} \left| \phi^{-1}(e^{i\lambda}) \right|^2 \quad (10.77)$$

<sup>95</sup>Cases for multiple root see [https://www.math.pku.edu.cn/teachers/lidf/course/atsa/atsanotes/html/\\_atsanotes/atsa-lagdiff.html](https://www.math.pku.edu.cn/teachers/lidf/course/atsa/atsanotes/html/_atsanotes/atsa-lagdiff.html)

- Yule-Walker Equation: we have

$$\mathbb{E}(X_t X_{t-k}) = \phi_1 \mathbb{E}(X_{t-1} X_{t-k}) + \dots + \phi_p \mathbb{E}(X_{t-p} X_{t-k}) + \mathbb{E}(\varepsilon_t X_{t-k}), \quad \forall k = 1, 2, \dots, p \quad (10.78)$$

$$\Rightarrow \gamma_k = \phi_1 \gamma_{k-1} + \dots + \phi_p \gamma_{k-p}, \quad \forall k = 1, 2, \dots, p \quad (10.79)$$

and for  $k = 0$ :

$$\gamma_0 = \phi_1 \gamma_1 + \dots + \phi_p \gamma_p + \sigma^2 \quad (10.80)$$

write in matrix form to get Yule-Walker Equation:

$$\begin{bmatrix} \gamma_1 \\ \gamma_2 \\ \vdots \\ \gamma_p \end{bmatrix} = \begin{bmatrix} \gamma_0 & \gamma_1 & \cdots & \gamma_{p-1} \\ \gamma_1 & \gamma_0 & \cdots & \gamma_{p-2} \\ \vdots & \vdots & \ddots & \vdots \\ \gamma_{p-1} & \gamma_{p-2} & \cdots & \gamma_0 \end{bmatrix} \begin{bmatrix} \phi_1 \\ \phi_2 \\ \vdots \\ \phi_p \end{bmatrix} \quad (10.81)$$

$$\sigma^2 = \gamma_0 - \phi_1 \gamma_1 - \dots - \phi_p \gamma_p \quad (10.82)$$

or in dense matrix form (1):

$$\gamma = \Gamma \phi \quad (10.83)$$

$$\sigma^2 = \gamma_0 - \phi' \gamma \quad (10.84)$$

dense form (2):

$$\begin{bmatrix} -\sigma^2 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} \gamma_0 & \gamma_1 & \gamma_2 & \cdots & \gamma_p \\ \gamma_1 & \gamma_0 & \gamma_1 & \cdots & \gamma_{p-1} \\ \gamma_2 & \gamma_1 & \gamma_0 & \cdots & \gamma_{p-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \gamma_p & \gamma_{p-1} & \gamma_{p-2} & \cdots & \gamma_0 \end{bmatrix} \begin{bmatrix} -1 \\ \phi_1 \\ \phi_2 \\ \vdots \\ \phi_p \end{bmatrix} \quad (10.85)$$

- PACF: the coefficient of  $\text{AR}(p)$  has straight relation with  $\phi_{k,j}$ : for all given  $k \geq p$

$$(\phi_1, \dots, \phi_p, 0, \dots, 0) = (\phi_{k,1}, \dots, \phi_{k,p}, \phi_{k,p+1}, \dots, \phi_{k,k}) \quad (10.86)$$

(Note that  $\phi_{p,j} = \phi_{p+1,j} = \phi_{p+2,j} = \dots$  using Levinson-Durbin' recursion [equation. 10.31](#)).

□ **Estimation: Key focus is the estimation of  $\phi_i$ ,  $i = 1, 2, \dots, p$  and  $\sigma^2$  (assume a TS of  $\mu_\varepsilon = 0$ )**

Y-W Estimation and OLS Estimation are moment methods, asymptotically the same. MLE Estimation is usually more precise, but hard to calculate.

- Yule-Walker Estimation: use  $\gamma = \Gamma \phi$ . First estimate  $\hat{\gamma}$ , as well as  $\hat{\Gamma}$ , and get estimation for  $\phi, \sigma^2$

$$\hat{\phi} = \hat{\Gamma}^{-1} \hat{\gamma} \quad (10.87)$$

$$\hat{\sigma}^2 = \hat{\gamma}_0 - \hat{\gamma}' \hat{\Gamma}^{-1} \hat{\gamma} \quad (10.88)$$

Asymptotic distribution:

$$\sqrt{n}(\hat{\phi} - \phi) \xrightarrow{d} N_p(0, \sigma^2 \Gamma^{-1}) \quad (10.89)$$

- Levinson-Durbin's recursion for Yule-Walker Estimation: since PACF are the same as coefficients  $\phi_{k,j} = \phi_j$ , we can use Durbin's recursion to avoid calculation of  $\hat{\Gamma}^{-1}$

$$\hat{\phi}_{11} = \hat{\rho}_1 \quad (10.90)$$

$$\hat{\phi}_{k+1,k+1} = \frac{\hat{\rho}_{k+1} - \sum_{j=1}^k \hat{\phi}_{k,j} \hat{\rho}_{k+1-j}}{1 - \sum_{j=1}^k \hat{\phi}_{k,j} \hat{\rho}_j}, \quad k \geq 1 \quad (10.91)$$

$$\hat{\phi}_{k+1,j} = \hat{\phi}_{k,j} - \hat{\phi}_{k+1,k+1} \hat{\phi}_{k,k+1-j}, \quad j = 1, 2, \dots, k \quad (10.92)$$

$$\hat{\sigma}_0^2 = \hat{\gamma}_0 \quad (10.93)$$

$$\hat{\sigma}_k^2 = \hat{\sigma}_{k-1}^2 (1 - \hat{\phi}_{k,k}^2) \quad (10.94)$$

estimator:

$$\hat{\phi}_j = \hat{\phi}_{p,j} \quad (10.95)$$

- OLS Estimation: using the linear combination form of AR model:

$$\hat{\phi} = \arg \min_{\phi} \sum_{t=p+1}^n \left[ x_t - \sum_{j=1}^p \phi_j x_{t-j} \right]^2 \quad (10.96)$$

the solution is in the form of OLS estimator  $(X'X)^{-1}XY$ , with  $X, Y$  properly defined

- MLE Estimation: under normal assumption

$$\phi(\mathcal{B})X_t = \varepsilon_t, \quad \varepsilon_t \sim N(0, \sigma^2) \quad (10.97)$$

Likelihood: define  $\theta = \{\phi_1, \dots, \phi_p, \sigma^2\}$

$$L(\theta; x_1, \dots, x_n) = f(x_1, \dots, x_p | \theta) \prod_{t=p+1}^n f(x_t | x_{t-1}, \dots, x_1; \theta) \quad (10.98)$$

$$\approx \propto \prod_{t=p+1}^n f(x_t | x_{t-1}, \dots, x_1; \theta) \quad (10.99)$$

$$= \prod_{t=p+1}^n \frac{1}{\sqrt{2\pi\sigma^2}} \exp \left\{ -\frac{1}{2\sigma^2} (x_t - \sum_{j=1}^p \phi_j x_{t-j})^2 \right\} \quad (10.100)$$

$$= (2\pi\sigma^2)^{-(n-p)/2} \exp \left\{ -\frac{1}{2\sigma^2} \sum_{t=p+1}^n (x_t - \sum_{j=1}^p \phi_j x_{t-j})^2 \right\} \quad (10.101)$$

- Estimation to spectrum density:

$$\hat{\nu}(\lambda) = \frac{\hat{\sigma}^2}{2\pi} \left| 1 - \sum_{j=1}^{\hat{p}} \hat{\phi}_j e^{i\lambda j} \right|^{-2} \quad (10.102)$$

### 10.3.3 MA Model

Moving-Average model (of order  $q$ ) contains ( $q^{\text{th}}$  order) backshift on  $\varepsilon_t$ :

$$X_t = \varepsilon_t + \theta_1 \varepsilon_{t-1} + \dots + \theta_q \varepsilon_{t-q}, \quad \varepsilon_t \sim \text{WN}(\mu_\varepsilon, \sigma^2) \quad (10.103)$$

or expressed in backshift operator with  $\theta(z) = 1 + \sum_{j=1}^q \theta_j z^j$ , where the root of  $\theta(z) = 0$  denoted  $\kappa_j$

$$X_t = \theta(\mathcal{B})\varepsilon_t, \quad \varepsilon_t \sim \text{WN}(\mu_\varepsilon, \sigma^2) \quad (10.104)$$

$$\theta(z) = 1 + \sum_{j=1}^q \theta_j z^j = \prod_{j=1}^q (1 - \kappa_j z) \quad (10.105)$$

$$= \sum_{j=0}^q \theta_j z^j, \quad \theta_j = 1 \quad (10.106)$$

here we could note that AR( $p$ ) model has solution in the form of MA( $\infty$ ):

$$\text{AR}(p) : X_t = \sum_{j=0}^{\infty} \psi_j \varepsilon_{t-j}, \quad \psi_0 = 1 \quad (10.107)$$

□ **Properties and Solution: (here we consider stationary case  $\mu_\varepsilon = 0$ )**

- Invertibility: if and only if

$$|\kappa_j| > 1, \quad \forall j \quad (10.108)$$

- ACF and ACVF:

$$\gamma_k = \begin{cases} \sigma^2 \sum_{j=0}^{q-k} \theta_j \theta_{j+k}, & 0 \leq k \leq q \\ 0, & k > q \end{cases} \quad (10.109)$$

$$\rho_k = \begin{cases} \frac{\sum_{j=0}^{q-k} \theta_j \theta_{j+k}}{\sum_{j=0}^q \theta_j^2}, & 0 \leq k \leq q \\ 0, & k > q \end{cases} \quad (10.110)$$

- Solution:  $\hat{\theta}_j$  could solved from  $\{\gamma_k\}$

### 10.3.4 ARMA Model

Auto-Regression-Moving-Average model ARMA( $p, q$ ) in the form of

$$\phi(\mathcal{B})X_t = \theta(\mathcal{B})\varepsilon_t \quad (10.111)$$

$$\phi(z) = 1 - \sum_{j=1}^p \phi_j z^j = \prod_{j=1}^p (1 - \alpha_j z) \quad (10.112)$$

$$\theta(z) = 1 + \sum_{j=1}^q \theta_j z^j = \prod_{j=1}^q (1 - \kappa_j z) \quad (10.113)$$

□ **Properties and Solution: (here we consider stationary case  $\mu_\varepsilon = 0$ )**

- Solution:

$$X_t = \phi^{-1}(\mathcal{B})\theta(\mathcal{B})\varepsilon_t \equiv \Psi(\mathcal{B})\varepsilon_t \quad (10.114)$$

- Weak Stationarity: if and only if AR part is WS, i.e.

$$|\alpha_j| > 1, \quad \forall j \quad (10.115)$$

- Invertibility: if and only if MA part is invertible, i.e.

$$|\kappa_j| > 1, \quad \forall j \quad (10.116)$$

### 10.3.5 ARIMA Model

ARIMA( $p, d, q$ ) model adds an difference term  $\Delta^d = (1 - \mathcal{B})^d$  in ARMA( $p, q$ ):

$$\phi(\mathcal{B})(1 - \mathcal{B})^d X_t = \theta(\mathcal{B}) \quad (10.117)$$

## Section 10.4 Seasonal Model for Time Series

This part includes some ideas for modelling seasonal term (usually as well as trend term) in  $Y_t = \mathbf{T}_t + \mathbf{S}_t + X_t$ .

Usually we describe the trend term as the ‘mean’ of time series over time, and sensonal term with zero-mean and period  $P > 1$ .

表 9: Buys-Ballot Table of seasonal period  $s$

Period ( $i$ )	Season ( $j$ )				$\bar{y}_{i\cdot}$	$\hat{\sigma}_{i\cdot}^2$
	1	2	$\dots$	$s$		
1	$y_1$	$y_2$	$\dots$	$y_s$	$\bar{y}_{1\cdot}$	$\hat{\sigma}_{1\cdot}$
2	$y_{s+1}$	$y_{s+2}$	$\dots$	$y_{2s}$	$\bar{y}_{2\cdot}$	$\hat{\sigma}_{2\cdot}$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$
$m$	$y_{(m-1)s+1}$	$y_{(m-1)s+2}$	$\dots$	$y_{ms}$	$\bar{y}_{m\cdot}$	$\hat{\sigma}_{m\cdot}^2$
$\bar{y}_{\cdot j}$	$\bar{y}_{\cdot 1}$	$\bar{y}_{\cdot 2}$	$\dots$	$\bar{y}_{\cdot s}$	$\bar{y}_{\cdot\cdot}$	-
$\hat{\sigma}_{\cdot i}^2$	$\hat{\sigma}_{\cdot 1}^2$	$\hat{\sigma}_{\cdot 2}^2$	$\dots$	$\hat{\sigma}_{\cdot s}^2$	-	$\hat{\sigma}_{\cdot\cdot}^2$

### 10.4.1 Regression Model

A common functional description is polynomial trend + Fourier expansion senson, i.e.

$$Y_t = T_t + S_t + X_t \quad (10.118)$$

$$= \alpha_0 + \sum_{j=1}^m \alpha_j t^j + \sum_{j=1}^{\lfloor s/2 \rfloor} \left[ \beta_j \sin\left(\frac{2\pi}{s} jt\right) + \gamma_j \cos\left(\frac{2\pi}{s} jt\right) \right] \quad (10.119)$$

Note: for regression model,  $T_t$  and  $S_t$  are treated as invariant term.

Estimation of paramters  $\{\alpha_0, \alpha_j, \beta_j, \gamma_j\}$  use e.g. MSE estimator:

$$\{\hat{\alpha}_0, \hat{\alpha}_j, \hat{\beta}_j, \hat{\gamma}_j\} = \arg \min_{\{\alpha_0, \alpha_j, \beta_j, \gamma_j\}} \sum_{t \in T} [y_t - (T_t + S_t)]^2 \quad (10.120)$$

### 10.4.2 Moving Average Model

First estimate Trend term, then Seasonal term

Trend term is estimated by a symmetric moving average window  $\{\omega_j\}_{j=-w}^w$  with band width  $w$

$$\hat{T}_t = \sum_{j=-w}^w \omega_j y_{t-j} \quad (10.121)$$

$$\omega_j = \omega_{-j}, \quad j = -w, -w+1, \dots, w-1, w \quad (10.122)$$

$$\sum_{j=-w}^w \omega_j = 1 \quad (10.123)$$

then seasonal term is naturally estimated by

$$\hat{S}_t = y_t - \hat{T}_t$$

### 10.4.3 Seasonal ARIMA Model

Multiplicative seasonal ARIMA model with period  $s$  of  $Y_t$ :  $\text{ARIMA}(p, d, q) \times (P, D, Q)_s$

$$\Phi_P(\mathcal{B}^s)\phi_p(\mathcal{B})(1-\mathcal{B})^d(1-\mathcal{B}^s)^D Y_t = \Theta_Q(\mathcal{B}^s)\theta_q(\mathcal{B}^s)\varepsilon_t, \quad \varepsilon_t \sim \text{WN}(0, \sigma^2) \quad (10.124)$$

On the ACF plot of SARIMA, you should see peak at  $t_{\text{lag}} \propto s$

## Section 10.5 Model Selection and Diagnostics

### 10.5.1 Model Building of ARIMA

#### □ Box-Jenkins Approach for ARIMA Model:

1. Data Transformation: Note that in the general model  $Y_t = T_t + S_t + X_t$  we would expect a ‘stationary’ random term, thus a transform for stable variance is needed, see [section. 3.5.1](#) for detailed methods. Then we could preliminarily detect the Stationarity of sequence, e.g. by plotting.
2. Seasonal Term Detection: usually by plotting ACF plot & ACVF plot, further we could also use spectrum plot, seasonal subseries plot.
3. Stationarity Detection: Detect stationarity e.g. by unit-root test.
- 4.

### 10.5.2 Order Determination of ARIMA Model

#### □ Order Determination of $\text{AR}(p)$

- PACF test: use the proper of  $\phi_{k,k}$  for  $k \geq p$  where

$$\phi_{kk} = \begin{cases} \phi_p, & k \leq p \\ 0, & k > p \end{cases} \quad (10.125)$$

for all given  $k > p$ : Asymptotic distribution:

$$\sqrt{n}(\hat{\phi}_{k,1} - \phi_{k,1}, \dots, \hat{\phi}_{k,k} - \phi_{k,k}) \xrightarrow{d} N(0, \sigma^2 \Gamma_k^{-1}) \quad (10.126)$$

specially it could be proved that  $(\sigma^2 \Gamma_k^{-1})_{k,k} = (\sigma^2 \Gamma_k^{-1})_{1,1} = 1, \quad k > p.$

i.e. test statistics for AR( $p$ ):

$$\hat{\phi}_{k,k} \xrightarrow{d} N(0, 1), \quad w.r.t. H_0 : \phi_{k,k} = 0, \quad k > p \quad (10.127)$$

Plot  $\hat{\phi}_{k,k}$ - $k$  to determine the proper  $k$  as  $\hat{p}$ .

- AIC/BIC method: use  $\hat{p} = \arg \min AIC(k)$  or  $\arg \min BIC(k)$ :

$$AIC(k) = \ln \hat{\sigma}_k^2 + \frac{2k}{n} \quad (10.128)$$

$$BIC(k) = \ln \hat{\sigma}_k^2 + \frac{k \ln n}{n} \quad (10.129)$$

#### □ Order Determination of MA( $q$ )

- ACF test: use the cut off property of  $\rho_k$  of MA( $q$ ):

$$\rho_k = \begin{cases} \frac{\sum_{j=0}^{q-k} \theta_j \theta_{j+k}}{\sum_{j=0}^q \theta_j^2}, & 0 \leq k \leq q \\ 0, & k > q \end{cases} \quad (10.130)$$

use the asymptotic distribution of  $\hat{\rho}_m$  in [equation. 10.50](#), for  $m > q$ :

$$\sqrt{n} \hat{\rho}_m \xrightarrow{d} R_m \quad (10.131)$$

$$= \sum_{t=1}^{\infty} (\rho_{t+m} + \rho_{t-m} - \rho_t \rho_m) W_t \quad (10.132)$$

$$= \sum_{l=-q}^q \rho_l W_{l+m}, \quad m > q \quad (10.133)$$

$$\sim N(0, 1 + 2 \sum_{j=1}^q \rho_j^2) \quad (10.134)$$

i.e. test statistics for MA( $q$ ):

$$T_q(m) = \frac{\sqrt{n} \hat{\rho}_m}{\sqrt{1 + 2 \sum_{j=1}^q \hat{\rho}_j^2}} \xrightarrow{d} N(0, 1), \quad H_0 : \rho_m = 0, \quad m > q$$

- AIC/BIC method: use  $\hat{q} = \arg \min AIC(m)$  or  $\arg \min BIC(m)$ :

$$AIC(m) = \ln \hat{\sigma}_m^2 + \frac{2m}{n} \quad (10.135)$$

$$BIC(m) = \ln \hat{\sigma}_m^2 + \frac{m \ln n}{n} \quad (10.136)$$

#### □ Order Determination of ARMA( $p, q$ )

- AIC/BIC method:

$$\hat{p}, \hat{q} = \arg \min_{k,m} AIC(k, m) = \arg \min_{k,m} \ln \hat{\sigma}_{k,m}^2 + \frac{2(k+m)}{n} \quad (10.137)$$

$$\hat{p}, \hat{q} = \arg \min_{k,m} BIC(k, m) = \arg \min_{k,m} \ln \hat{\sigma}_{k,m}^2 + \frac{(k+m) \ln n}{n} \quad (10.138)$$

- EACF for ARIMA( $p, d, q$ ): Extended ACF forms a matrix for determining  $(p, d, q)$  using extended Yule-Walker Equation



### 10.5.3 Outlier Detection

Here we introduce two kinds of outlier in time series: Additive Outlier (AO) and Innovative Outlier (IO).

#### □ Notation for Outlier

- Step function in time series: a rise of value 1 at time  $\tau$ :

$$S_t^{(\tau)} = \begin{cases} 0, & t < \tau \\ 1, & t \geq \tau \end{cases}$$

- Pulse function in time series: a pulse of value 1 at time  $\tau$ :

$$P_t^{(\tau)} = (1 - \mathcal{B})S_t^{(\tau)} = \begin{cases} 0, & t \neq \tau \\ 1, & t = \tau \end{cases}$$

#### □ Additive Outlier

A pulse outlier of  $y$  at  $\tau$ :

$$\tilde{y}_t = y_t + \omega_A P_t^{(\tau)}$$

the outlier would not influence  $t \neq \tau$ , thus is additive.

#### □ Innovative Outlier

A pulse outlier of  $\varepsilon$  at  $\tau$ :

$$\varepsilon_\tau = \varepsilon_t + \omega_I$$

$t \neq \tau$  would also be influenced by this outlier.

## Section 10.6 Forecast of Time Series

### 10.6.1 MSE Forecast Criterion

The criterion for forecasting is to minimizing some loss function, usually taken as MSE loss:

$$\hat{X}_{\tau|t} = \arg \min_{X_\tau} \mathbb{E}[(X_\tau - X_{\tau|t})^2] = \mathbb{E}(X_{\tau|t}) \quad (10.139)$$

our mission is to construct a function  $g(\cdot)$  so that  $\hat{X}_{\tau|t} = g(\mathcal{F}_t)$  can act as the estimator.  $\mathcal{F}_t = \{X_t, X_{t-1}, X_{t-2}, \dots\}$  denotes the history until  $t$ .

### 10.6.2 Best Linear Estimator

A simple and straightforward method is a linear combination form of  $\mathcal{F}_t$ :

$$\hat{X}_{\tau|t} = \sum_{j=0}^{\infty} \beta_j X_{t-j} \quad (10.140)$$

$$\beta_j = \arg \min_{\{\beta_j\}} \mathbb{E} \left[ \left( X_\tau - \sum_{j=0}^{\infty} \beta_j X_{t-j} \right)^2 \right] \quad (10.141)$$

i.e.

$$\hat{X}_{\tau|t} = L(X_{\tau}|\mathcal{F}_t) \quad (10.142)$$

Solution was given in [equation. 10.27](#):

$$\beta = \Sigma_{X, \mathcal{F}_t}^{-1} \Sigma_{\mathcal{F}_t, X_{\tau}} \quad (10.143)$$

- e.g. ont-step forecast for zero-mean weakly stationary sequence:

$$\hat{X}_{t+1|t} = \sum_{j=0}^{\infty} \beta_j X_{t-j} \quad (10.144)$$

$$\vec{\beta} = \Gamma^{-1} \gamma \quad (10.145)$$

(For actual case, calculate for a proper truncation  $p$  for  $\hat{X}_{t+1|t} = \sum_{j=0}^p \beta_j X_{t-j}$  would be fine)

Best linear estimator is the best estimator for ARMA( $p, q$ ) with WN noise.

### 10.6.3 Forecast of AR( $p$ )

AR( $p$ ):

$$X_{t+1} = \sum_{j=1}^p \phi_j X_{t+1-j} + \varepsilon_{t+1} \quad (10.146)$$

1. First estimate coefficients, e.g. using Yule-Walker estimator  $\hat{\phi}_j, j = 1, 2, \dots, p$
2. Esitmate of  $X_{t+1|t}$

$$\hat{X}_{t+1|t} = \sum_{j=1}^p \hat{\phi}_j X_{t+1-j} \quad (10.147)$$

$$\hat{\sigma}_{t+1|t}^2 = \hat{\sigma}^2 = \hat{\gamma}_0 \quad (10.148)$$

3. Estimate of  $X_{t+h|t}$ : estimation conduct sequentially for  $h = 1, 2, \dots$ :

$$\hat{X}_{t+1|t} = \hat{\phi}_1 X_t + \hat{\phi}_2 X_{t-1} + \dots + \hat{\phi}_p X_{t+1-p} \quad (10.149)$$

$$\hat{X}_{t+2|t} = \hat{\phi}_1 \hat{X}_{t+1|t} + \hat{\phi}_2 X_t + \hat{\phi}_3 X_{t-1} + \dots + \hat{\phi}_p X_{t+2-p} \quad (10.150)$$

$$\hat{X}_{t+3|t} = \hat{\phi}_1 \hat{X}_{t+2|t} + \hat{\phi}_2 \hat{X}_{t+1|t} + \hat{\phi}_3 X_t + \hat{\phi}_4 X_{t-1} + \dots + \hat{\phi}_p X_{t+3-p} \quad (10.151)$$

$$\dots \quad (10.152)$$

### 10.6.4 Forecast of MA( $q$ )

MA( $q$ ):

$$X_{t+1} = \varepsilon_{t+1} + \sum_{j=1}^q \theta_j \varepsilon_{t+1-j} \quad (10.153)$$

1. First estimate coefficients  $\hat{\theta}_j, j = 1, 2, \dots, q$
2. Estimate of  $X_{t+h|t}$ : first for each  $k = 1, 2, \dots, t$ , calculate residual estimator:

$$\hat{\varepsilon}_k = X_k - L(X_k|\mathcal{F}_{k-1}) = X_k - L(X_k|X_{k-1}, \dots, X_1) \quad (10.154)$$

then calculate forecast:

$$\hat{X}_{t+h|t} = \begin{cases} \sum_{j=h}^q \hat{\theta}_j \hat{\varepsilon}_{t+1-j}, & h = 1, 2, \dots, q \\ 0, & h > q \end{cases} \quad (10.155)$$

### 10.6.5 Forecast of ARMA( $p, q$ )

ARMA( $p, q$ ):

$$\phi(\mathcal{B})X_t = \theta(\mathcal{B})\varepsilon_t \Rightarrow X_t = \phi^{-1}(\mathcal{B})\theta(\mathcal{B})\varepsilon_t \equiv \psi(\mathcal{B})\varepsilon_t \quad (10.156)$$

similarly estimate  $\psi_j$  and  $\varepsilon_j$  and forecast as MA( $\infty$ )

### 10.6.6 Forecast of ARIMA( $p, d, q$ )

## Chapter. XI 因果推断导论部分

Instructor: Wanlu Deng

### Section 11.1 Neyman-Rubin Potential Outcome Framework

Neyman-Rubin Framework (Donald B. Rubin, 1978), also called Potential Outcome Framework is based on **counterfactual outcome** inference to judge causal effect.

#### 11.1.1 Description of Causal Effect and Challenge

Causality concerns ‘what would happen when **an action** is applied to **a unit**’. Here the ‘unit’ is how causality is different from correlation.

- A unit is the physical object at that specific time, which is similar to the event in Einstein’s relativity.<sup>96</sup>
- An action is the treatment/intervention that could be **potentially** applied to the unit.

In this section we mainly focus on cases with binary intervention, i.e.<sup>97</sup>

$$\{\text{treatment, control}\} = \{1, 0\} \quad (11.1)$$

#### □ Potential Outcome

With this notation, the causal effect could be expressed by the **estimand** as follows by comparing the **potential outcomes**, here’s a commonly used form:

$$\tau := Y_{\text{treatment}} - Y_{\text{control}} := Y(1) - Y(0) \quad (11.2)$$

To estimate the causal effect (on a unit), we need to obtain both potential outcomes of  $Y(1)$  and  $Y(0)$ , but in the real world we can only observe one of them, say, the patient took the medicine, and we got  $Y(1)$ , while  $Y(0)$  is missing.

Relevant Notation:

- **Unit**: The atomic object in causal inference.  $i = 1, 2, \dots, N$
- **Treatment**  $W_i$ : (possible) assignment.
  - Treatment Group: Set of  $\{\text{Unit}_i | W_i = 1\}$ ;
  - Controlled Group: Set of  $\{\text{Unit}_i | W_i = 0\}$ .
- **Potential Outcome** (PO)  $Y_i$ : For each unit with action treatment(or control), the potential outcome  $Y(W = w)$ ,  $w = 0, 1$  is the ‘Eigen Outcome’ of the model, despite of what really happens. It can be seen as what would happen when the operation had not been done.
- **Observed Outcome**  $Y_i^{\text{obs}}$ : The actually happened outcome,  $Y_i^{\text{obs}} = Y_i(W = w_{\text{REAL\_CASE}}) := Y_i(W = w_i^{\text{obs}})$ .

<sup>96</sup>Which means that one object at different time  $((x, t) \& (x, t'))$  is not the same unit (event). However if the assumption of time independency is valid, then object in different time could be the same unit (usually less resonable for human subjects).

<sup>97</sup>Habitually we denote the more ‘active’ intervention as treatment, but in mathematical form they are symmetric.

- **Missing Outcome**  $Y_i^{\text{mis}}$ : The potential outcome when the  $w_i^{\text{mis}} := !w_i^{\text{obs}}$  would have been operated (it does exist but we cannot observe the ‘world-line’ where  $w_i^{\text{mis}}$  was operated, thus is unknown to us),  $Y_i^{\text{mis}} = Y_i(W_i = 1 - w_{\text{REAL\_CASE}}) := Y_i(W_i = w_i^{\text{mis}})$

$$Y_i^{\text{obs}} = Y_i(W_i^{\text{obs}}) = \begin{cases} Y_i(1) & W_i = 1 \\ Y_i(0) & W_i = 0 \end{cases} \quad (11.3)$$

$$Y_i^{\text{mis}} = Y_i(1 - W_i^{\text{obs}}) = \begin{cases} Y_i(0) & W_i = 1 \\ Y_i(1) & W_i = 0 \end{cases} \quad (11.4)$$

or in condensed notation

$$\begin{bmatrix} Y_i^{\text{obs}} \\ Y_i^{\text{mis}} \end{bmatrix} = \begin{bmatrix} W_i & 1 - W_i \\ 1 - W_i & W_i \end{bmatrix} \begin{bmatrix} Y_i(1) \\ Y_i(0) \end{bmatrix} \Leftrightarrow \begin{bmatrix} Y_i(1) \\ Y_i(0) \end{bmatrix} = \begin{bmatrix} W_i & 1 - W_i \\ 1 - W_i & W_i \end{bmatrix} \begin{bmatrix} Y_i^{\text{obs}} \\ Y_i^{\text{mis}} \end{bmatrix} \quad (11.5)$$

- **Causal Effect**  $\tau_i$  (defined by difference of PO): Difference between potential outcome,  $\tau_i = Y_i(W_i = 1) - Y_i(W_i = 0) = Y_i(1) - Y_i(0)$
- **Pre-Treatment Variables / Covariates**  $X_i$ : Some background elements that might attribute to treatment selection/potential outcome. Anyway they may cause confusion to causal inference. For example, the gender of patients  $X_i \in \{\text{female}, \text{male}\} := \{1, 0\}$ .
- **Subgroup**: Treatment/Control group could be further divided in subgroup according to covariates, e.g. categorical covariates  $X_i \in \mathcal{X}$

$$\{(X_i, Y_i, W_i)\} = \bigotimes_{\xi \in \mathcal{X}} \{(Y_i, W_i)\}_{i: X_i = \xi} \quad (11.6)$$

With the above basic notation, a dataset / sample can be expressed as

$$\mathcal{D} = \{(X_i, Y_i, W_i)\}_{i=1}^N \quad (11.7)$$

## □ Assignment Mechanism and Super Population

- Our observation sample is a **finite sample**  $\{X_i, Y_i\}_{i=1}^N$  in which  $Y_i$  is perceived fixed as potential outcome. And the above notation are studying the causal information within the finite sample. The randomness of the causal effect in the sample is the **assignment mechanism**  $W_i \sim f_{W|X,Y}$ . i.e. in finite sample, POs are fixed and actually different assignment mechanisms give randomized data (in a finite sample). So if we can control the assignment mechanism  $W|Y, X$ , which is the case for randomized experiment, then the assignment mechanism can help estimate the missing values. Some widely used mechanism includes Completely Randomized Experiment, Stratified Randomized Experiment, Pairwise Randomized Experiment, etc. Proper assignment can help avoid the influence of covariates (recall Simpson’s Paradox).
- Before that, the finite sample of  $\{X_i, Y_i\}_{i=1}^N$  is drawn from a **super population** with some distribution.

To summarize, The whole model has 2 stages of randomness: sampling from super population, and assign treatment to the finite sample.

$$\text{Super Population} \xrightarrow[\text{sample } N]{f_X, f_{Y|X}} \text{Finite Sample } \{X, Y\} \xrightarrow[\text{assignment}]{f_{W|X,Y}} \text{Observation } \mathcal{D} = \{X, Y, W\} \quad (11.8)$$

### 11.1.2 Assumptions

The null model is complicated, say, there could be multiple PO levels / interference between assignments / complex assignment mechanism, etc. There are some basic assumptions to help simplified the model.

**Note:** In actual usage of causal model, the assumptions should be checked.

- **SUTVA:** To solve the problem of omitted treatment (e.g.  $Y_i \in \{Y_i(0), Y_i(1), Y_i(2)\}$ ), and the intervention between units (e.g.  $Y_i(W_{j=1:N})$ ) to simplify the model, we usually put the assumption of SUTVA, which has two components:

- No Interference

$$Y_i(W_{j=1:N}) = Y_i(W_i) \quad (11.9)$$

- No Hidden Variation of Treatment:

$$Y_i(W_{j=1:N}) = Y_i(W_i) \in \{Y_i(1), Y_i(0)\}, \quad W_i \in \{1, 0\} := \mathbb{T}_i = \mathbb{T} \quad (11.10)$$

- **Regular Assignment Mechanisms (RAM)**

- **Individualistic Assignment:** Assignment probability of each unit does **not** depends on the covariants and PO of other units:

$$\mathbb{P}(W_i = w_i | X, Y(1), Y(0)) = \mathbb{P}(W_i = w_i | X_i, Y_i(1), Y_i(0)) \quad (11.11)$$

$$= \mathbb{P}(W_i | X_i, Y_i(1), Y_i(0))^{w_i} (1 - \mathbb{P}(W | X_i, Y_i(1), Y_i(0)))^{1-w_i}, \quad \forall i = 1, 2, \dots, N \quad (11.12)$$

Sometimes for simplification, denoted as

$$\mathbb{P}_i(W = 1 | X, Y(1), Y(0)) := q(X, Y(1), Y(0)) \quad (11.13)$$

- **Probabilistic Assignment:** Probility for both  $W_i = 1$  and  $W_i = 0$  are non-zero (to ensure a reasonable causal model)

$$0 < \mathbb{P}(W | X, Y(1), Y(0)) < 1, \quad \forall X, Y(1), Y(0) \quad (11.14)$$

- **Unconfounded Assignment:** Assignment mechanism is independent of PO

$$\mathbb{P}(W | X, Y(1), Y(0)) = \mathbb{P}(W | X) \quad (11.15)$$

when  $q(X, Y)$  mentioned above does *not* involve  $Y$ , i.e. with unconfoundedness, it is denoted as *propensity score*.

$$q(X, Y) := e(X), \quad \text{case } W \perp\!\!\!\perp Y | X \quad (11.16)$$

Note: Unconfoundedness is *not* testable (always invloves the missing value  $Y^{\text{mis}}$ ). We can only pre-design it (randomized experiment) or make it an appropriate assumption (RAM).

□ **With all the above assumptions, assignment mechanism can be simplified in the following form:**

$$\text{Assignment Mechanism: } \mathbb{P}(W|X, Y(1), Y(0)) = \frac{1}{Z} \prod_{i=1}^N e(X_i)^{W_i} (1 - e(X_i))^{1-W_i} \quad (11.17)$$

$$(11.18)$$

□ **Data Example**

表 10: Illustration of Causal Data

Unit $i$	Potential Outcomes		Assignment	Observation	Causal Estimand
	$Y_i(1)$	$Y_i(0)$	$W_i$	$Y_i^{\text{obs}}$	$Y_i(1) - Y_i(0)$
# 1	$Y_1(1)$	$Y_1(0)$	$W_1 = 1$	$Y_1^{\text{obs}} = Y_1(1)$	$Y_1(1) - Y_1(0)$
# 2	$Y_2(1)$	$Y_2(0)$	$W_2 = 0$	$Y_2^{\text{obs}} = Y_2(0)$	$Y_2(1) - Y_2(0)$
# 3	$Y_3(1)$	$Y_3(0)$	$W_3 = 0$	$Y_3^{\text{obs}} = Y_3(0)$	$Y_3(1) - Y_3(0)$
# 4	$Y_4(1)$	$Y_4(0)$	$W_4 = 1$	$Y_4^{\text{obs}} = Y_4(1)$	$Y_4(1) - Y_4(0)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

## Section 11.2 Inference to Causal Effect in Completely Randomized Experiment

First we focus on the randomness in finite sample, i.e. randomness of assignment mechanism. Specifically we usually consider the case of Completely Randomized Experiment (CRE):  $N_t$  in  $N$  items are given treatment and  $N_c = N - N_t$  in  $N$  are given control, and the assignment is given **randomly**.

$$\mathbb{P}(W|X, Y) = 1 / \binom{N}{N_t}, \quad W \in \mathbb{W}^{\text{CRE}} := \left\{ W : \sum_{i=1}^N W_i = N_t \right\} \quad (11.19)$$

The assumption of CRP is important in causal inference because it fixes the gap between  $Y^{\text{obs}}$  and  $Y^{\text{mis}}$  by randomly assign treatment/control.

### 11.2.1 Fisher's Exact $p$ -value

Test of Fisher's Sharp Null Hypothesis:

$$H_0 : \tau_i = 0, \forall i = 1, 2, \dots, N \rightsquigarrow H_a : \exists j \text{ s.t. } \tau_j \neq 0 \quad (11.20)$$

With the hypothesis, we could fill in all the  $Y^{\text{mis}}$  by  $Y_i^{\text{mis}} = Y_i^{\text{obs}} \forall i$ . And by traversing all possible  $\tilde{W}$  assignments and calculate corresponding  $\tau_{\tilde{W}}$ , we could calculate the Fisher's exact  $p$ -value

$$\hat{p} = \#(|\tau_{\tilde{W}}| \geq |\tau_W|) / \binom{N}{N_t} \quad (11.21)$$

Comments:

- Since the basic idea is traversing all  $W$ , so it could be applied to different designs of  $\tau$ , say

$$\hat{\tau}^{\text{diff}} = |\bar{Y}_t^{\text{obs}} - \bar{Y}_c^{\text{obs}}| \quad (11.22)$$

$$\hat{\tau}^{\text{median}} = |\text{med}_t(Y^{\text{obs}}) - \text{med}_c(Y^{\text{obs}})| \quad (11.23)$$

$$\hat{\tau}^{t\text{-stat}} = \left| \frac{\bar{Y}_t^{\text{obs}} - \bar{Y}_c^{\text{obs}}}{\sqrt{s_t^2/N_t + s_c^2/N_c}} \right| \quad (11.24)$$

$$\hat{\tau}^{\text{rank}} = |\bar{R}_t - \bar{R}_c|, \quad R_i = \sum_{j=1}^N \left( \mathbb{I}_{Y_j < Y_i} + \frac{1}{2} \mathbb{I}_{Y_j = Y_i} \right) - \frac{N}{2} \quad (11.25)$$

$$\hat{\tau}^{\text{reg}} = \arg \min_{\tau: (\beta_0, \beta_X, \tau)} \sum_{i=1}^N \left( Y_i^{\text{obs}} - \beta_0 - \tau W_i - X_i' \beta_X \right)^2 \quad (11.26)$$

Or even some other specially designed statistics on e.g. difference in variance

$$\hat{\tau}^{\text{var}} = v \hat{a}_t^{\text{obs}} / v \hat{a}_c^{\text{obs}} \quad (11.27)$$

- High computation complexity for large  $N$ . e.g. for  $N_t \approx \frac{N}{2}$

$$\text{flops} \sim \binom{N}{N_t} \sim 2^N \quad (11.28)$$

- Random simulation for large  $N$ : the  $p$ -value is actually

$$\hat{\mathbb{P}}(\text{more extreme } \hat{\tau}) = \hat{\mathbb{E}}[\mathbf{1}(\text{more extreme } \hat{\tau})] \quad (11.29)$$

which can be approached by random sampling

$$\hat{p} = \#(|\tau_{\tilde{W}}| \geq |\tau_W|) / \#(\text{sample}) \quad (11.30)$$

- A fiducial interval can be constructed. But generally speaking the hypothesis testing just help reject the sharp hypothesis, but cannot help determine the casual effect strength.

### 11.2.2 Neyman's Repeated Sampling Approach

Neyman's method uses the distribution of  $W$  for completely randomized experiment to obtain the property of the finite sample estimator

$$\hat{\tau}_{\text{fs}} = \bar{Y}_t^{\text{obs}} - \bar{Y}_c^{\text{obs}} = \frac{1}{N_t} \sum_{i=1}^N W_i Y_i(1) - \frac{1}{N_c} \sum_{i=1}^N (1 - W_i) Y_i(0) \quad (11.31)$$



- Property

$$\mathbb{E}_W[\hat{\tau}_{fs}] = \tau_{fs} = \frac{1}{N} \sum_{i=1}^N Y_i(1) - Y_i(0) \quad (11.32)$$

$$var_W(\hat{\tau}_{fs}) = \frac{S_t^2}{N_t} + \frac{S_c^2}{N_c} - \frac{S_{tc}^2}{N} \quad (11.33)$$

$$= \frac{N_c}{NN_t} S_t^2 + \frac{N_t}{NN_c} S_c^2 + \frac{2}{N} \rho_{tc} S_t S_c \quad (11.34)$$

$$\begin{cases} S_t^2 = \frac{1}{N-1} \sum_{i=1}^N (Y_i(1) - \bar{Y}(1))^2 \\ S_c^2 = \frac{1}{N-1} \sum_{i=1}^N (Y_i(0) - \bar{Y}(0))^2 \\ S_{tc}^2 = \frac{1}{N-1} \sum_{i=1}^N ([Y_i(1) - Y_i(0)] - [\bar{Y}(1) - \bar{Y}(0)])^2 = S_t^2 + S_c^2 - 2\rho_{tc} S_t S_c \\ \rho_{tc} = \frac{1}{(N-1)S_t S_c} \sum_{i=1}^N (Y_i(1) - \bar{Y}(1)) (Y_i(0) - \bar{Y}(0)) \end{cases} \quad (11.35)$$

- Estimator

$$\hat{\tau}_{fs} = \bar{Y}_t^{obs} - \bar{Y}_c^{obs} \quad (11.36)$$

$$\hat{var}(\hat{\tau}_{fs}) = \frac{s_t^2}{N_t} + \frac{s_c^2}{N_c} \quad (11.37)$$

$$\hat{var}_\rho(\hat{\tau}_{fs}) = \frac{N_c}{NN_t} s_t^2 + \frac{N_t}{NN_c} s_c^2 + \frac{2}{N} \rho s_t s_c, \quad -1 \leq \rho \leq 1 \quad (11.38)$$

$$\text{e.g. } \hat{var}_{\rho=1}(\hat{\tau}_{fs}) = \frac{s_t^2}{N_t} + \frac{s_c^2}{N_c} - \frac{(s_t - s_c)^2}{N} \leq \hat{var}(\hat{\tau}_{fs}) \quad (11.39)$$

$$\begin{cases} s_t^2 = \frac{1}{N_t-1} \sum_{i:W_i=1} (Y_i^{obs} - \bar{Y}_t^{obs})^2 \\ s_c^2 = \frac{1}{N_c-1} \sum_{i:W_i=0} (Y_i^{obs} - \bar{Y}_c^{obs})^2 \end{cases} \quad (11.40)$$

i.e.  $\hat{var}(\hat{\tau}_{fs})$  provides an upper bound of  $\hat{var}_\rho(\hat{\tau}_{fs})$  (equal when  $\rho = 1$ ). And  $\hat{var}(\hat{\tau}_{fs})$  also acts as the estimator at  $\tau_i = \text{const}, \forall i$ .<sup>98</sup>

- Confidence Interval

$$CI = \hat{\tau}_{fs} \pm N_{\alpha/2} \sqrt{\hat{var}(\hat{\tau}_{fs})} \quad (11.41)$$

$$CI_\rho = \hat{\tau}_{fs} \pm N_{\alpha/2} \sqrt{\hat{var}_\rho(\hat{\tau}_{fs})} \quad (11.42)$$

where the version with pre-specified  $\rho$  is applied to improve accuracy, if we have prior knowledge about  $\rho_{tc}$ .

- Hypothesis Testing

$$H_0 : \bar{Y}(1) - \bar{Y}(0) = 0 \iff H_a : \bar{Y}(1) - \bar{Y}(0) \neq 0 \quad (11.43)$$

and  $t$ -test

$$T = \frac{\hat{\tau}_{fs}}{\sqrt{\hat{var}(\hat{\tau}_{fs})}} \sim t_1 \quad (11.44)$$

<sup>98</sup>Actually in this case we should have  $s_t = s_c := s$  and the estimator reduces to  $\hat{var}(\hat{\tau}_{fs}) = s^2(1/N_t + 1/N_c)$

- Comment on three components  $S_t^2 / S_c^2 / S_{tc}^2$ : they each corresponds to the natural distribution of treatment / natural distribution of control / variation arises from assigning on finite sample.

So when dealing with the estimator under distribution of super population, in which we need to add the randomness of  $f_{X,Y}$  back, the  $S_{tc}^2$  term eliminates (which can be proven).

$$\mathbb{E}_{sp} [\hat{\tau}_{fs}] = \mathbb{E}_{sp} [\mathbb{E}_W [\hat{\tau}_{fs}]] = \tau_{sp} \quad (11.45)$$

$$var_{sp}(\hat{\tau}_{fs}) = \mathbb{E}_{sp} \left[ (\bar{Y}_t^{obs} - \bar{Y}_c^{obs} - \mathbb{E}_{sp} [\bar{Y}(1) - \bar{Y}(0)])^2 \right] = \frac{\sigma_t^2}{N_t} + \frac{\sigma_c^2}{N_c} \quad (11.46)$$

$$var_{sp}(\hat{\tau}_{fs}) = \frac{s_t^2}{N_t} + \frac{s_c^2}{N_c} \quad (11.47)$$

where  $\sigma^2$  is the variance under the distribution of super population  $Y|X, X$ .

$$\sigma_t^2(x) = var_{sp: Y|X} (Y(1)|X = x), \quad \sigma_t^2 = var_{sp} (Y(1)) \quad (11.48)$$

$$\sigma_c^2(x) = var_{sp: Y|X} (Y(0)|X = x), \quad \sigma_c^2 = var_{sp} (Y(0)) \quad (11.49)$$

$$\sigma_{ct}^2(x) = var_{sp: Y|X} (Y(1) - Y(0)|X = x), \quad \sigma_{ct}^2 = var_{sp} (Y(1) - Y(0)) \quad (11.50)$$

### 11.2.3 Regression Methods

Regression methods in Potential Outcome Framework is used to introduce covariates and lower the variance estimation, the idea is similar to variance decomposition in ANOVA.

#### □ Requisite Knowledge: M-Estimator

With data  $\mathcal{D}_n$  given, parameter estimation problem can usually be expressed in a **Maximization** problem with linear combination target function  $Q_n(\theta; \mathcal{D}_n)$

$$\hat{\theta}_n = \arg \max_{\theta} Q_n(\theta; \mathcal{D}_n) \quad (11.51)$$

e.g. for regression estimation  $Y = X\beta + \varepsilon$ ,  $\mathcal{D}_n = \{x_i, y_i\}_{i=1}^n = (X, Y)$

- OLS quadratic form  $\theta = \beta$

$$Q_n(\theta) := -\frac{1}{n} \sum_{i=1}^n (y_i - x_i' \beta)^2 = -\frac{1}{n} (Y - X\beta)' (Y - X\beta) \quad (11.52)$$

- MLE form with  $\varepsilon \sim f(\varepsilon; \phi)$ , and  $\theta = (\beta, \phi)$

$$Q_n(\theta) := \frac{1}{n} \sum_{i=1}^n f(y_i - x_i' \beta; \phi) \quad (11.53)$$

Denote the ground truth  $\theta^*$ , and the M-Estimator  $\hat{\theta}_n$  that maximizes  $Q_n$ . Then

$$\theta^* = \arg \max_{\theta} \mathbb{E}_{\mathcal{D}} [Q(\theta; \mathcal{D})] \quad (11.54)$$

$$\hat{\theta}_n = \arg \max_{\theta} Q_n \quad (11.55)$$

$$\text{with } Q_n \rightarrow \mathbb{E} [Q] \Rightarrow \hat{\theta}_n \rightarrow \theta^* \quad (11.56)$$

The solution  $\hat{\theta}_n$  is obtained at  $\frac{\partial Q_n(\theta)}{\partial \theta} = 0$ , so we first focus on first order derivative

$$\psi_n(\theta; \mathcal{D}_n) := \frac{\partial Q_n(\theta; \mathcal{D}_n)}{\partial \theta}, \quad \hat{\theta}_n = \arg_{\theta}(\psi_n(\theta; \mathcal{D}_n) = 0) \quad (11.57)$$

Note: a more important reason we study the property of  $\psi_n(\theta; \mathcal{D}_n)$  is that: we do **not** have an explicit expression of  $\hat{\theta}_n$  because it's just a maximizer.  $\psi_n(\theta; \mathcal{D}_n)$  together with Taylor expansion provide us with an approach to (asymptotically) express  $\hat{\theta}_n$  explicitly.

with LLN, we have

$$\psi_n(\theta^*; \mathcal{D}_n) \xrightarrow{d} \mathbb{E}[\psi(\theta^*)] = 0 \quad (11.58)$$

with CLT,  $\psi_n(\theta^*, \mathcal{D}_n)$  is a statistic asymptotically distributed normally:

$$\sqrt{n}(\psi_n(\theta^*; \mathcal{D}_n) - \mathbb{E}[\psi(\theta^*; \mathcal{D})]) = \sqrt{n}\psi_n(\theta^*; \mathcal{D}_n) \xrightarrow{d} N(0, \Sigma_{\psi}) \quad (11.59)$$

Taylor series of  $\psi_n(\cdot; \mathcal{D}_n)$  at  $\hat{\theta}_n$ :

$$\psi_n(\theta^*; \mathcal{D}_n) = 0 + \frac{\partial \psi_n(\theta = \hat{\theta}_n; \mathcal{D}_n)}{\partial \theta} (\theta^* - \hat{\theta}_n) + O((\theta^* - \hat{\theta}_n)^2) \quad (11.60)$$

$$\Rightarrow (\hat{\theta}_n - \theta^*) \approx \left( \frac{\partial \psi_n(\theta = \hat{\theta}_n; \mathcal{D}_n)}{\partial \theta} \right)^{-1} \psi_n(\theta^*; \mathcal{D}_n) \quad (11.61)$$

$$\Rightarrow \hat{\theta}_n \xrightarrow{d} N(\theta^*, \Gamma^{-1} \Sigma_{\psi} \Gamma^{-1} / n), \quad \Gamma := \frac{\partial \psi_n(\theta = \hat{\theta}_n; \mathcal{D}_n)}{\partial \theta} \quad (11.62)$$

and specifically if  $Q_n(\theta; \mathcal{D})$  is a log-likelihood, then  $\psi(\theta; \mathcal{D})$  here is Score function in [equation. 2.79](#). And  $\Sigma_{\psi} = I(\theta)$  is Fisher Information in [equation. 2.90](#). With the nice property of Fisher Information  $I(\theta) = \Sigma_{\psi} = -\mathbb{E}[\Gamma]$ , M-Estimator reduces to the asymptotic distribution of MLE in [equation. 2.68](#)

$$\hat{\theta}_n \xrightarrow{d} \left( \theta^*, \frac{I(\theta)^{-1}}{n} \right) \quad (11.63)$$

## □ Regression Model on Super Population

Motivation: regression model

$$Y_i^{\text{obs}} = \alpha + \tau \cdot W_i + f(X_i; \beta) + \varepsilon_i \quad (11.64)$$

concerns a quadratic loss function of the following form:

$$(\hat{\alpha}, \hat{\tau}, \hat{\beta})_{\text{ols}, X} = \arg \min_{(\alpha, \tau, \beta)} \frac{1}{N} \sum_{i=1}^N \left( Y_i^{\text{obs}} - \alpha - \tau \cdot W_i - f(X_i; \beta) \right)^2 := \arg \min_{(\alpha, \tau, \beta)} Q_N((\alpha, \tau, \beta); \{X_i, Y_i, W_i\}_{i=1}^N) \quad (11.65)$$

**Note :**

- Covariate dependency function  $f(\cdot; \beta)$  is a properly selected prior, e.g. linear regression  $X'\beta$
- In functional form it's the same as regression (reflects correlation), the causality comes from CRE of  $W_i$ .

**Solution:**

- Model without covariates

$$Y_i^{\text{obs}} = \alpha + \tau \cdot W_i + \varepsilon_i \quad (11.66)$$

OLS solution:

$$\hat{\tau}_{\text{ols}} = \frac{\sum_{i=1}^N (W_i - \bar{W})(Y_i^{\text{obs}} - \bar{Y}^{\text{obs}})}{\sum_{i=1}^N (W_i - \bar{W})^2} = \bar{Y}_t^{\text{obs}} - \bar{Y}_c^{\text{obs}} \quad (11.67)$$

$$\hat{\alpha}_{\text{ols}} = \bar{Y}^{\text{obs}} - \hat{\tau}_{\text{ols}} \cdot \bar{W} \quad (11.68)$$

$$\text{var}(\hat{\tau}) = \frac{\sigma_t^2}{N_t} + \frac{\sigma_c^2}{N_c} \quad (11.69)$$

$$s_t^2 = \hat{\sigma}_t^2 = \frac{1}{N-1} \sum_{i=1}^N W_i (Y_i^{\text{obs}} - \hat{Y}_i^{\text{obs}})^2 = \frac{1}{N-1} \sum_{i=1}^N W_i (Y_i^{\text{obs}} - \hat{\tau} - \hat{\alpha})^2 \quad (11.70)$$

$$s_c^2 = \hat{\sigma}_c^2 = \frac{1}{N-1} \sum_{i=1}^N (1 - W_i) (Y_i^{\text{obs}} - \hat{Y}_i^{\text{obs}})^2 = \frac{1}{N-1} \sum_{i=1}^N (1 - W_i) (Y_i^{\text{obs}} - \hat{\alpha})^2 \quad (11.71)$$

$$\hat{\text{var}}(\hat{\tau}_{\text{ols}}) = \frac{s_t^2}{N_t} + \frac{s_c^2}{N_c} = \hat{\text{var}}(\hat{\tau}_{\text{fs}}) \quad (11.72)$$

- Model with Covariates and Asymptotic Property:

$$Y_i^{\text{obs}} = \alpha + \tau \cdot W_i + f(X_i; \beta) + \varepsilon_i \quad (11.73)$$

The quadratic loss  $Q_N(\cdot)$  regression model gives a M-Estimator with ground truth as the parameters in super population

$$(\alpha, \tau, \beta)^* = \mathbb{E}_{\text{sp}}[(\alpha, \tau, \beta)] := (\alpha_{\text{sp}}, \tau_{\text{sp}}, \beta_{\text{sp}}) \quad (11.74)$$

OLS with covariates gives the same optimization solution to  $\tau$  as OLS without covariate:  $\hat{\tau}_{\text{ols},X} = \hat{\tau}_{\text{ols}} \rightarrow \tau^* = \tau_{\text{sp}}$ . And appending covariate dependency term can help **improve variance estimation**, with unbiasedness property kept.

e.g. for linear dependency  $f(X; \beta) = X' \beta$

$$\hat{\tau}_{\text{ols},X} = \hat{\tau}_{\text{ols}} \rightarrow \tau_{\text{sp}} \quad (11.75)$$

$$\sqrt{N}(\hat{\tau}_{\text{ols},X} - \tau_{\text{sp}}) \xrightarrow{d} N(0, (\Gamma^{-1} \Sigma_{\psi} \Gamma^{-1})_{22}) = N\left(0, \frac{\Sigma_{\psi,22}}{p^2(1-p)^2}\right) \quad (11.76)$$

$$\begin{cases} \Sigma_{\psi,22} = \mathbb{E}_{\text{sp}} \left[ \frac{\partial Q_N(\alpha, \tau, \beta)}{\partial(\alpha, \tau, \beta)} \frac{\partial Q_N(\alpha, \tau, \beta)}{\partial(\alpha, \tau, \beta)'} \right]_{22} \\ = \mathbb{E} \left[ (W_i - p)^2 (Y_i^{\text{obs}} - \alpha^* - \tau^* W_i - X_i' \beta^*)^2 \right] \\ p = \bar{W}_{N \rightarrow \infty} \end{cases} \quad (11.77)$$

using the asymptotic normality, we can construct variance estimation  $\hat{\text{var}}_{\text{hetero}}(\hat{\tau}_{\text{ols},X}) = \hat{\Sigma}_{\psi,22} / \hat{p}^2(1-\hat{p})^2$  to  $\hat{\tau}_{\text{ols},X}$  (with heteroskedasticity)

$$\hat{\text{var}}_{\text{hetero}}(\hat{\tau}_{\text{ols},X}) = \frac{1}{N(N - \dim(X_i) - 1)} \cdot \frac{\sum_{i=1}^N (W_i - \bar{W})^2 (Y_i^{\text{obs}} - \hat{\alpha}_{\text{ols},X} - \hat{\tau}_{\text{ols},X} W_i - X_i' \hat{\beta}_{\text{ols},X})^2}{\bar{W}^2 (1 - \bar{W})^2} \quad (11.78)$$

### 11.2.4 Model Based Inference using Bayesian Statistics

Motivation: how to use prior information about distribution? Basically with  $\mathbb{P}(W|Y, \theta)$ ,  $f(Y|\theta)$ ,  $\pi(\theta)$  we can construct any posterior distribution from

$$f(W, Y, X) = \mathbb{P}(W|Y, \theta) f(Y|\theta) \pi(\theta) \quad (11.79)$$

#### □ Bayesian Statistics Precap

Estimation target:  $f(Y^{\text{mis}}|Y^{\text{obs}}, W)$ , with assumptions

$$\text{CRE: } \mathbb{P}(W|Y, \theta) = \binom{N}{N_t}^{-1} \quad (11.80)$$

$$\text{Distribution: } \begin{pmatrix} Y(1) \\ Y(0) \end{pmatrix} \Big| \theta \sim f(Y|\theta), \text{ say } N \left( \begin{pmatrix} \mu_t \\ \mu_c \end{pmatrix}, \begin{pmatrix} \sigma_t^2 & \rho\sigma_t\sigma_c \\ \rho\sigma_t\sigma_c & \sigma_c^2 \end{pmatrix} \right), \quad \theta = [\mu_t, \mu_c, \sigma_t^2, \sigma_c^2, \rho] \quad (11.81)$$

$$\text{Prior: } \theta \sim \pi(\theta) \quad (11.82)$$

$$\text{Transformation: } (Y^{\text{obs}}, Y^{\text{mis}}) = g(Y(1), Y(0), W) \quad (11.83)$$

- Transformation between  $Y = [Y(1), Y(0)] \mapsto [Y^{\text{obs}}, Y^{\text{mis}}]$ :

$$f(Y^{\text{obs}}, Y^{\text{mis}}|W, \theta) = f(Y|W, \theta) \left| \frac{\partial Y(1), Y(0)}{\partial g(Y(1), Y(0), W)} \right| = \frac{f(Y, W|\theta)}{\int_y f(Y, W|\theta) dy} \left| \frac{\partial Y(1), Y(0)}{\partial g(Y(1), Y(0), W)} \right| \quad (11.84)$$

$$= \frac{f(W|Y, \theta) f(Y|\theta)}{\int_y f(W|Y, \theta) f(Y|\theta) dy} \left| \frac{\partial g(Y(1), Y(0), W)}{\partial Y(1), Y(0)} \right|^{-1} \quad (11.85)$$

$$\Rightarrow f(Y^{\text{mis}}|Y^{\text{obs}}, W, \theta) = \frac{f(Y^{\text{obs}}, Y^{\text{mis}}|W, \theta)}{\int_{y^{\text{mis}}} f(Y^{\text{obs}}, Y^{\text{mis}}|W, \theta) dy^{\text{mis}}} \quad (11.86)$$

- Calculating posterior of parameter

$$p(\theta|Y^{\text{obs}}, W) = \frac{\pi(\theta) \cdot f(Y^{\text{obs}}, W|\theta)}{f(Y^{\text{obs}}, W)} = \frac{\pi(\theta) \cdot \int_{y^{\text{mis}}} f(W|Y, \theta) f(Y^{\text{obs}}, Y^{\text{mis}}|\theta) dy^{\text{mis}}}{\int_{\theta} \pi(\theta) \cdot \int_{y^{\text{mis}}} f(W|Y, \theta) f(Y^{\text{obs}}, Y^{\text{mis}}|\theta) dy^{\text{mis}} d\theta} \quad (11.87)$$

- Marginal Integration

$$f(Y^{\text{mis}}|Y^{\text{obs}}, W) = \int_{\theta} f(Y^{\text{mis}}, \theta|Y^{\text{obs}}, W) d\theta \quad (11.88)$$

$$= \int_{\theta} f(Y^{\text{mis}}|Y^{\text{obs}}, W, \theta) p(\theta|Y^{\text{obs}}, W) d\theta \quad (11.89)$$

With the above (a little bit complex) steps we could estimate  $Y^{\text{mis}}$ , and also give the (bayesian posterior) distribution of  $\hat{\tau}$

$$f(\tau|Y^{\text{obs}}, W) = f(Y^{\text{obs}} - Y^{\text{mis}}|Y^{\text{obs}}, W) \quad (11.90)$$

Model with covariate involved need modification with assumptions as

$$f(Y(1), Y(0), X|\theta_{Y|X}, \theta_X) = f(Y(1), Y(0)|X, \theta_{Y|X}) \cdot f(X|\theta_X) \quad (11.91)$$

$$\pi(\theta_{Y|X}, \theta_X) = \pi(\theta_{Y|X}) \pi(\theta_X) \quad (11.92)$$

and corresponding intergrations need to consider intergral on  $X$ .

Usually computation of integral terms is complex, simulation methods like random integration can be used, see [section. 5.6](#) for a brief introduction.

(Detailed estimation methods would be complemented upon completion of Intro. to Bayesian Statistics next semester.)

## Section 11.3 More Assignment Mechanism and Observational Study

Some other classical randomized experiment are also used in causal experiments. This section includes Stratified Randomized Experiment (SRE) and Pairwise Randomized Experiment (PRE).

In more cases we can only deal with observational data, which means the assignment mechanism is beyond our control, thus some estimation is needed.

### 11.3.1 Other Classical Randomized Experiment

#### □ Stratified Randomized Experiment

Usually when we notice that some covariate  $X$  can have significant influence on  $\hat{\tau}$ , we consider a SRE by dividing data into stratum according to  $X$

$$\mathcal{S} : (N, N_t, N_c) \rightarrow \{(N(j), N_t(j), N_c(j))\}_{j=1}^J, \quad S_i := \mathcal{S}(X_i) = \text{Strata of } \mathcal{D}_i \in \{1, 2, \dots, J\} \quad (11.93)$$

with *proportion of strata*  $q(j)$  and *propensity score*  $e(j)$

$$q(j) := \frac{N(j)}{N} \quad e(j) = \frac{N_t(j)}{N(j)} \quad (11.94)$$

SRE Assignment Mechanism:

$$\mathbb{P}(W|S, Y) = \prod_{j=1}^J \left( \frac{N(j)}{N_t(j)} \right)^{-1}, \quad W \in \mathbb{W}^{\text{SRE}} := \left\{ W : \sum_{i=1}^N W_i \mathbb{I}_{S_i=j} = N(j), \forall j = 1, 2, \dots, J \right\} \quad (11.95)$$

With the notation above, the within-strata ACE  $\tau(j)$  estimation follows exactly the same estimation as in CRE, the key step is to *aggregate*  $\{\tau(j)\}_{j=1}^J \mapsto \tau$ .

- **Fisher's Exact  $p$ -Value:** with the same sharp null hypothesis

$$H_0 : \tau_i = 0, \forall i = 1, 2, \dots, N \iff H_a : \exists j \text{ s.t. } \tau_j \neq 0 \quad (11.96)$$

we can conduct similar testing by traversing  $W \in \mathbb{W}^{\text{SRE}}$ , with a slight modification on test statistics, e.g. using  $\hat{\tau}^{\text{diff}} \rightsquigarrow \hat{\tau}^{\text{diff}, \lambda}$  as example. Some other statistics used see [equation. 11.22](#)

$$\hat{\tau}^{\text{diff}} = \left| \bar{Y}_t^{\text{obs}} - \bar{Y}_c^{\text{obs}} \right| \quad (11.97)$$

$$\hat{\tau}^{\text{diff}, \lambda} = \left| \sum_{j=1}^J \lambda(j) \left( \bar{Y}_t^{\text{obs}}(j) - \bar{Y}_c^{\text{obs}}(j) \right) \right|, \quad w.r.t. \sum_{j=1}^J \lambda(j) = 1 \quad (11.98)$$

Note:  $\hat{\tau}^{\text{diff}, \mu}$  returns to  $\hat{\tau}^{\text{diff}}$  if  $\lambda$  is chosen as proportion of strata  $\lambda(j) = q(j)$ .

- Neyman's Repeated Sampling Approach: use similar aggregation method of weighting strata to form unbiased estimator

$$\hat{\tau}_{fs} = \sum_{j=1}^J q(j) \hat{\tau}(j), \quad \hat{\tau}(j) = \frac{1}{N_t(j)} \sum_{i:S_i=j} W_i Y_i^{\text{obs}} - \frac{1}{N_c(j)} \sum_{i:S_i=j} (1 - W_i) Y_i^{\text{obs}} \quad (11.99)$$

$$\text{var}(\hat{\tau}_{fs}) = \sum_{j=1}^J q(j)^2 \text{var}(\hat{\tau}(j)) = \sum_{j=1}^J q(j)^2 \left( \frac{S_t^2(j)}{N_t(j)} + \frac{S_c^2(j)}{N_c(j)} - \frac{S_{tc}^2(j)}{N(j)} \right) \quad (11.100)$$

$$\hat{\text{var}}(\hat{\tau}_{fs}) = \sum_{j=1}^J q(j)^2 \hat{\text{var}}(\hat{\tau}(j)) = \sum_{j=1}^J q(j)^2 \left( \frac{s_t^2(j)}{N_t(j)} + \frac{s_c^2(j)}{N_c(j)} \right) \quad (11.101)$$

$$\begin{cases} s_t^2(j) = \frac{1}{N_t(j) - 1} \sum_{i:S_i=j, W_i=1} (Y_i^{\text{obs}} - \bar{Y}_t^{\text{obs}}(j))^2 \\ s_c^2(j) = \frac{1}{N_c(j) - 1} \sum_{i:S_i=j, W_i=0} (Y_i^{\text{obs}} - \bar{Y}_c^{\text{obs}}(j))^2 \end{cases} \quad (11.102)$$

- Regression Method: basic stratified regression model:

$$Y_i^{\text{obs}} = \tau \cdot W_i + \sum_{j=1}^J \beta_j \mathbb{I}_{S_i=j} + \varepsilon_i \quad (11.103)$$

MMSE limit:

$$\hat{\tau}_{ols} \rightarrow \tau^* = \frac{1}{\sum_{k=1}^J q(k) e(k) (1 - e(k))} \sum_{j=1}^J q(j) e(j) (1 - e(j)) \tau_{sp}(j) \quad (11.104)$$

- Model Based Inference: Similar process as in CRE. We could further assess population average by setting *hyperparameter*  $\phi$

$$Y(j)|\theta(j) \sim f(Y|\theta(j)), \quad \theta_j|\phi \sim \pi_{\theta}(\theta_j|\phi), \quad \phi \sim \pi_{\phi}(\phi) \quad (11.105)$$

## □ Pairwise Randomized Experiment

Pairwise Randomized Experiment (PRE) can (in some sense) be vies as a special case that  $J = \frac{N}{2}$ , which can deal with continuous covariate cases. But a main difficult arises in variance estimation in Neyman's method.

To estimate the variance, we put assumption of constant causal effect within group, which gives

$$S_t^2(j) = S_c^2(j) \equiv S^2, \quad S_{tc}^2(j) = 0 \quad (11.106)$$

and we can access  $\hat{\text{var}}(\tau_{fs})$  as

$$\text{var}(\tau_{fs}) = \frac{4}{N} S^2 \quad (11.107)$$

$$\hat{\text{var}}(\tau_{fs}) = \frac{4}{N(N-2)} \sum_{i=1}^{N/2} (\hat{\tau}(j) - \bar{\tau})^2 \quad (11.108)$$

### 11.3.2 Observational Study with Regular Assignment Mechanisms

Recap RAM:

$$\begin{cases} \text{Individualistic: } \mathbb{P}(W_i|X, Y) = \mathbb{P}(W_i|X_i, Y_i) := q(X, Y) \\ \text{Probabilistic: } 0 < \mathbb{P}(W_i|X, Y) < 1 \\ \text{Unconfounded: } \mathbb{P}(W|X, Y) = \mathbb{P}(W|X) \end{cases} \Rightarrow \mathbb{P}(W|X, Y) = \frac{1}{Z} \prod_{i=1}^N e(X_i)^{W_i} (1 - e(X_i))^{1-W_i} \quad (11.109)$$

With the above assumptions and notations, propensity score  $e(x)$  can help fix the problem of Simpson's Paradox by

**Covariate Balance**

$$W_i \perp\!\!\!\perp X_i | e(X_i) \quad (11.110)$$

Note: there could be some other selection of balancing variable  $\epsilon(x)$ , in which  $e_i$  is the coarsest, i.e.  $e(x) = e(\epsilon(x))$

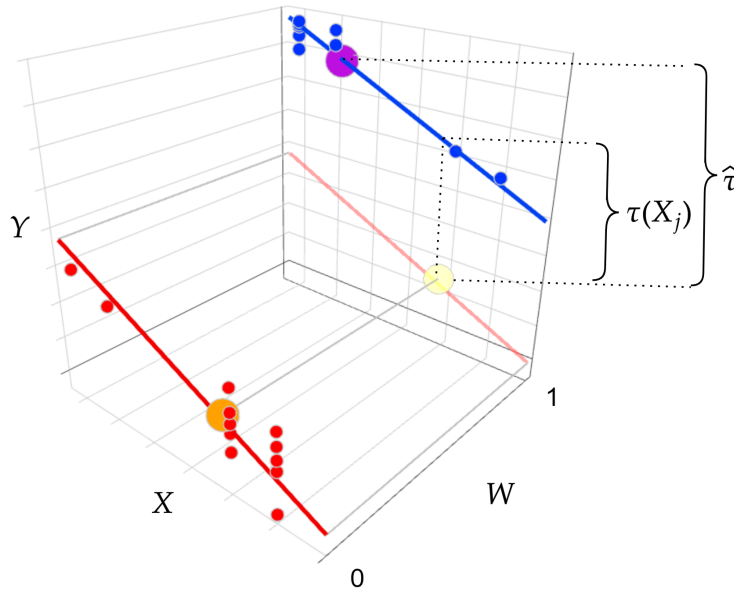


图 11: Illustration of covariate balance of propensity score (An example with linear dependence)

#### □ Statistical Inference to Propensity Score

Property of propensity score:

$$\Delta_{tc} := \mathbb{E}[e(X)|W=1] - \mathbb{E}[e(X)|W=0] = \frac{\text{var}(e(X))}{p(1-p)} \quad (11.111)$$

$$\text{var}(e(X)) = \mathbb{E} \left[ \left( \frac{f_t(X) - f_c(X)}{p f_t(X) + (1-p) f_c(X)} \right)^2 \right] \cdot p^2 (1-p)^2 \quad (11.112)$$

- Propensity Score test can be accessed by

$$\hat{\Delta}_{tc}^\ell = \frac{\bar{\ell}_t - \bar{\ell}_c}{\sqrt{(s_{\ell,t}^2 + s_{\ell,c}^2)/2}} \sim t_{N-2}, \quad \ell(x) = \ln \left( \frac{e(x)}{1-e(x)} \right) = \text{logistic}(x) \quad (11.113)$$

$$\hat{\Delta}_{tc}^\ell = 0 \iff \Delta_{tc} = 0 \iff \text{var}(e(X)) = 0 \iff f_t(x) = f_c(x) \quad (11.114)$$

- Estimate  $\hat{e}(X_i)$



- For categorical  $X$  with small  $|\mathcal{X}|$ , estimation

$$\hat{e}(x) = \frac{N(X_j)}{N} \quad (11.115)$$

- (Kernel) logistic regression is sometimes useful<sup>99</sup>

$$\hat{e}(x) = \hat{\mathbb{P}}(W_i = 1 | X_i = x; \beta) = \frac{e^{x'\beta}}{1 + e^{x'\beta}} \quad (11.116)$$

#### □ Useful Methods to Induce Propensity Score in Estimation

- Weighting: using the modulation of  $e(x)$  on  $\mathbb{P}(W|X)$

$$\begin{cases} \mathbb{E} \left[ \frac{Y_i^{\text{obs}} \cdot W_i}{e(X_i)} \right] = \mathbb{E} \left[ \frac{\mathbb{E}[Y_i(1)|X_i] \mathbb{E}[W_i|X_i]}{e(X_i)} \right] = \mathbb{E}[Y_i(1)] \\ \mathbb{E} \left[ \frac{Y_i^{\text{obs}} \cdot (1 - W_i)}{1 - e(X_i)} \right] = \mathbb{E} \left[ \frac{\mathbb{E}[Y_i(0)|X_i] \mathbb{E}[1 - W_i|X_i]}{1 - e(X_i)} \right] = \mathbb{E}[Y_i(0)] \end{cases} \quad (11.117)$$

to *weight* estimators through  $X$ : Horvitz-Thompson Estimator

$$\hat{\tau}^{\text{HT}} = \frac{1}{N} \sum_{i=1}^N \frac{W_i Y_i^{\text{obs}}}{\hat{e}(X_i)} - \frac{1}{N} \sum_{i=1}^N \frac{(1 - W_i) Y_i^{\text{obs}}}{1 - \hat{e}(X_i)} = \frac{1}{N} \sum_{i=1}^N \frac{(W_i - e(X_i)) \cdot Y_i^{\text{obs}}}{e(X_i) \cdot (1 - e(X_i))} \quad (11.118)$$

$$\hat{\tau}^{\text{HT,mod}} = \sum_{i=1}^N \lambda_i W_i Y_i^{\text{obs}} - \sum_{i=1}^N \lambda_i (1 - W_i) Y_i^{\text{obs}}, \quad \lambda_i = \begin{cases} \frac{1/\hat{e}(X_i)}{\sum_{k=1}^N W_k / \hat{e}(X_k)}, & W_i = 1 \\ \frac{1/(1 - \hat{e}(X_i))}{\sum_{k=1}^N (1 - W_k) / (1 - \hat{e}(X_k))}, & W_i = 0 \end{cases} \quad (11.119)$$

where the modification version is used to avoid extreme  $\hat{e}$  value.

The Horvitz-Thompson estimator is linked to stratified Neyman estimator [equation. 11.99](#) as

$$\hat{\tau}^{\text{strata}} = \sum_{j=1}^J q(j) \hat{\tau}(j) = \frac{1}{N} \sum_{i=1}^N \tilde{e}_i W_i Y_i^{\text{obs}} - \sum_{i=1}^N \tilde{e}_i (1 - W_i) Y_i^{\text{obs}}, \quad \tilde{e}_i = \begin{cases} \mathbb{I}_{S_i=j} \frac{1}{N_t(j)/N(j)}, & W_i = 1 \\ \mathbb{I}_{S_i=j} \frac{1}{N_c(j)/N(j)}, & W_i = 0 \end{cases} \quad (11.120)$$

where  $\tilde{e}_i$  is the propensity score for each strata.

- Blocking / Stratifying according to  $X$ , and then follows similar idea as SRE. (Because  $S(X_i)$  is still a covariate.)
- Matching ‘similar’ data points. e.g. for each data point  $(W_i = 1, Y_i, X_i)$ , select in  $\mathcal{D}_{W=1-W_i=0}$  for units with small distance  $d(X_i, X)$  as  $\mathcal{M}_i$ , and have a matching data

$$\{(W_i = 1, Y_i^{\text{obs}}, X_i, \mathcal{M}_i)\}, \quad \mathcal{M}_i = \{(W_j = 0, Y_j, X_j)\}_{d(X_i, X_j) \text{ small}} \quad (11.121)$$

and then

$$\hat{\tau} = \frac{1}{N_t} \sum_{i: W_i=1} (Y_i^{\text{obs}} - \bar{Y}_{\mathcal{M}_i}) \quad (11.122)$$

## Section 11.4 Pearl Causal Bayesian Framework

Pearl Bayesian Framework<sup>100</sup> (Judea Pearl, 1995) uses causal information on a graph to construct inference.

<sup>99</sup> Instruction of Kernel logistic regression see [section. 9.4.5](#).

<sup>100</sup> Also called Bayesian Network / Belief Network / Directed Acyclic Graphical (DAG) Model.

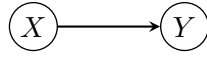
### 11.4.1 Causal Bayesian Network

The language of *Graph* is used to describe the causal relations.

#### □ Directed Acyclic Graph

In Pearl's causal network we focus on **Directed Acyclic Graphs** (DAGs) . Here are some key notions:

- ▷ DAG is a graph in which all edges are directed, and no path is a loop (acyclic).
- ▷ **Graph**  $\mathcal{G}$  is composed of a set of **Vertices** / Nodes  $\mathcal{V}$  and the **Edges**  $\mathcal{E}$  connecting them;  $\mathcal{G} = \{\mathcal{E}, \mathcal{V}\}$ .
  - **Adjacency**: Two vertices  $v_i, v_j$  are adjacent if they are linked by an edge  $e_{ij}$ .
  - **Path** : A (non-intersecting) routine tracing through edges to connect two vertices.
- ▷ **Direction** of edges: two vertices are connected by directed edge, pointing from the first to the second, say the following meta  $X \rightarrow Y$ .



in which  $X$  is a **parent** of  $Y$  and  $Y$  is a children of  $X$ . Parent of node  $v_i$  is denoted  $pa_i$

- **Skeleton** : The graph with all direction removed (looks like a graph with only nodes and line, without arrow).
- ▷ **Acyclic**: a graph without loop is acyclic. The structure is naturally required to make the causal structure healthy by clearly distinguish cause from effect.

#### □ Bayesian Network

A probability distribution  $\mathbb{P}(X_1, X_2, \dots, X_n)$  on vertices of graph has factorization given by conditional probability:

$$\mathbb{P}(X_1, \dots, X_n) = \mathbb{P}(X_{i_1} | X_{i_2}, \dots) \mathbb{P}(X_{i_2} | X_{i_3}, \dots) \dots \mathbb{P}(X_{i_{n-1}} | X_{i_n}) \mathbb{P}(X_n) \quad (11.123)$$

in which indices  $\{i_1, i_2, \dots, i_n\}$  can be any reshuffle of  $\{1, 2, \dots, n\}$ . But if we attach a graph on the probability to guide the factorization, the shuffle has to following some order, and form of conditional probability follows the Markov parents on DAG:

$$\mathbb{P}(X_1, \dots, X_n) = \prod_i \mathbb{P}(X_i | pa_i) \quad (11.124)$$

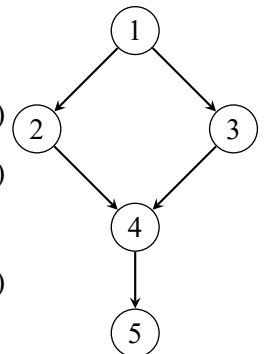
the r.v. sequence is **causal ordering** if  $X_i$  only dependent on  $X_{j:j < i}$ , i.e.  $pa_i \subset \{X_1, \dots, X_{i-1}\}$ .

Here's an example of Markov factorization on a DAG graph:

$$\mathbb{P}(X_1, X_2, X_3, X_4, X_5) = \mathbb{P}(X_5 | X_4) \mathbb{P}(X_1, X_2, X_3, X_4) \quad (11.125)$$

$$= \mathbb{P}(X_5 | X_4) \mathbb{P}(X_4 | X_2, X_3) \mathbb{P}(X_1, X_2, X_3) \quad (11.126)$$

$$= \mathbb{P}(X_5 | X_4) \mathbb{P}(X_4 | X_2, X_3) \mathbb{P}(X_3 | X_1) \mathbb{P}(X_2 | X_1) \mathbb{P}(X_1) \quad (11.127)$$



### □ Basic structures in a DAG

Starting from triplets in DAG as the key elements in a graph.

- Chain  $X \rightarrow Y \rightarrow Z$ , in which  $Y$  is the *mediator*. We have

$$\mathbb{P}(X, Z|Y) = \frac{\mathbb{P}(X) \mathbb{P}(Y|X) \mathbb{P}(Z|Y)}{\mathbb{P}(Y)} = \mathbb{P}(X|Y) \mathbb{P}(Z|Y) \quad (11.128)$$

i.e. we have a conditional independency in chain  $X \perp\!\!\!\perp Z|Y$

- Fork  $X \leftarrow Y \rightarrow Z$ . We have

$$\mathbb{P}(X, Z|Y) = \frac{\mathbb{P}(Y) \mathbb{P}(X|Y) \mathbb{P}(Z|Y)}{\mathbb{P}(Y)} = \mathbb{P}(X|Y) \mathbb{P}(Z|Y) \quad (11.129)$$

i.e. we have a conditional independency in chain  $X \perp\!\!\!\perp Z|Y$

- Collider  $X \rightarrow Y \leftarrow Z$ . In the collider,  $X \perp\!\!\!\perp Z$  marginally, but given  $Y$  are conditionally dependent. If there is no edge between  $X$  and  $Z$ , it's also called *v-structure*.

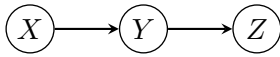


图 12: Chain

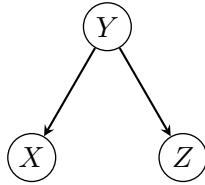


图 13: Fork

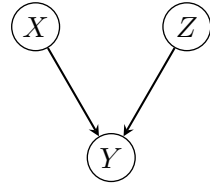


图 14: Collider

### □ d-Separation

d-separation in a graph is directional-separation of two vertices.

- Blocked path: a path  $p$  from  $X$  to  $Y$  is blocked by  $\{Z\}$  if for all triplets along the path:
  - **In**  $Z$ : the middle point of all chains and forks; and
  - **Not In**  $Z$ : the middle point itself of collider, and its descendants
- d-separation:  $X$  is d-separated from  $Y$  given  $Z$  if all paths  $p_{X \rightsquigarrow Y}$  are blocked by  $Z$ .

### □ Markov Compatibility

Markov compatibility is a match between DAG and probability distribution, a description of how  $\mathcal{G}$  represents  $\mathbb{P}(\cdot)$ .

A textbook definition is given as:

If a probability distribution  $\mathbb{P}(\cdot)$  admits a factorization  $\mathbb{P}(X) = \prod_i \mathbb{P}(X_i | pa_i)$  relative to DAG  $\mathcal{G}$ , then  $\mathbb{P}$  is *Markov Compatible* relative to  $\mathcal{G}$ .

Here ‘admits’ means that d-separation on graph finds its corresponding conditional probability.

$$X \perp\!\!\!\perp_{\mathcal{G}} Y|Z \Rightarrow X \perp\!\!\!\perp_{\mathbb{P}} Y|Z \quad (11.130)$$

Markov compatibility means that we can generate data following  $\mathbb{P}$  using  $\mathcal{G}$  as ‘Blueprint’.

Related notions and comments:

- I-Map: is a set of conditional independence statements read out from  $\mathcal{G}$ . If  $X$  and  $Y$  are d-separated by  $Z$  in  $\mathcal{G}$  (denoted  $X \perp\!\!\!\perp_{\mathcal{G}} Y|Z$ ), then we should have  $X \perp\!\!\!\perp_{\mathbb{P}} Y|Z$  in **every**  $\mathbb{P}$  distribution compatible with  $\mathcal{G}$ .

$$I(\mathcal{G}) = \{(X \perp\!\!\!\perp_{\mathcal{G}} Y|Z) : (X \perp\!\!\!\perp_{\mathbb{P}} Y|Z) \forall \mathbb{P} \text{ compatible with } \mathcal{G}\} \quad (11.131)$$

- From I-Map we can have definition of I-equivalence: i.e. if  $\mathcal{G}_1$  and  $\mathcal{G}_2$  yield the same I-map  $I(\mathcal{G}_1) = I(\mathcal{G}_2)$ .
- Note that Markov compatible states that  $X \perp\!\!\!\perp_{\mathcal{G}} Y|Z \Rightarrow X \perp\!\!\!\perp_{\mathbb{P}} Y|Z$  but not reversely, which means that  $I(\mathcal{G}) \subset I(\mathbb{P})$
- An concrete example: two r.v. are independently generated  $\mathbb{P}(X, Y) = \mathbb{P}(X) \mathbb{P}(Y)$ , i.e.  $I(\mathbb{P}) = X \perp\!\!\!\perp_{\mathbb{P}} Y$ . All the following graphs are markov compatible:

- $\mathcal{G}_0 : X \perp\!\!\!\perp Y$ , in which  $I(\mathcal{G}_0) = X \perp\!\!\!\perp_{\mathcal{G}} Y$
- $\mathcal{G}_1 : X \rightarrow Y$ , in which  $I(\mathcal{G}_1) = \emptyset$
- $\mathcal{G}_2 : X \leftarrow Y$ , in which  $I(\mathcal{G}_2) = \emptyset$

(because they all have  $I(\mathcal{G}_i) \subset I(\mathbb{P})$ )

- Perfect I-map: if  $I(\mathcal{G}) = I(\mathbb{P})$ .
- **Observational Equivalence:** A set of graphs are observational equivalent / belong to the same equivalent class if they encode the same conditional independencies.

Note that the key causal structures in DAGs are chain, fork, and collider; in which chain and fork imply the same conditional independence while collider is different. i.e.

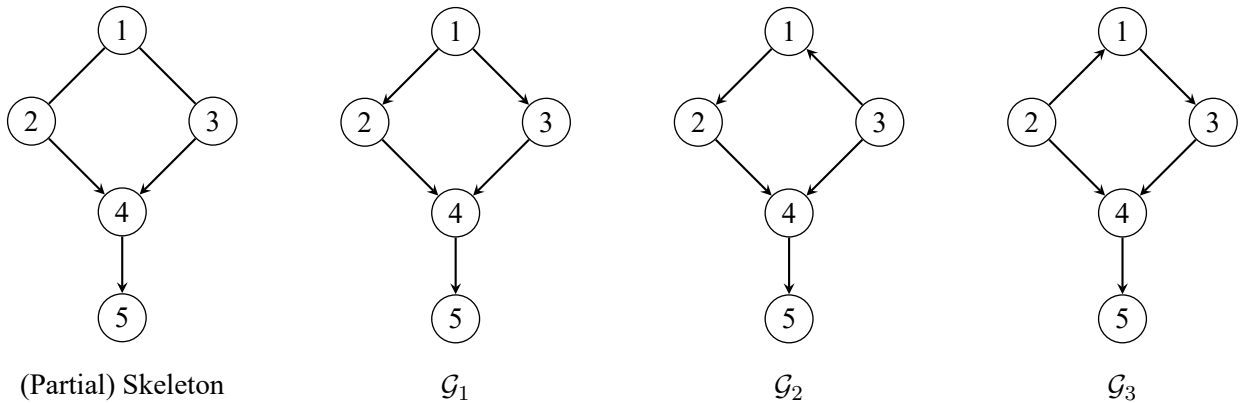
$$X \rightarrow Y \rightarrow Z, \quad X \leftarrow Y \leftarrow Z, \quad X \leftarrow Y \rightarrow Z \quad (11.132)$$

are observational equivalent by encoding  $X \perp\!\!\!\perp Z|Y$ .

The above argument gives the hint for identifying observational equivalent graphs:

- Having the same skeleton
- Having the same set of colliders.

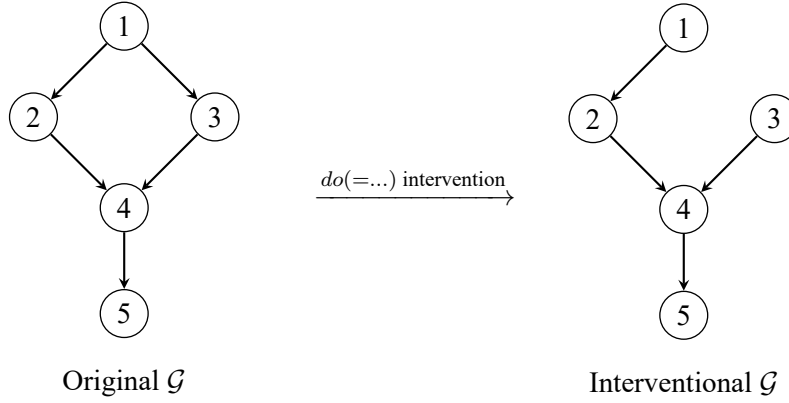
Here's an example of observational equivalent graphs:



Recall that in Rubin's Potential Outcome Framework, causality was induced by counterfactual side of potential outcomes  $Y^{\text{mis}} = Y(1 - W)$ . In Bayesian Network framework, causality is induced by **intervention**, in formulas expressed by  $do(\cdot)$  operator, e.g.

$$\mathbb{P}(X|do(Z = z)) \quad (11.133)$$

where  $Z \subset X$  is the set to conduct intervention on. Intervention would remove all 'incoming' edges to  $Z$ , as illustrated below an example of  $do(3 = \dots)$ :



Since intervention produces different subgraphs, we can obtain causality by comparing the probability distribution. e.g. in the simplest instance  $X \rightarrow Y$  v.s.  $X \leftarrow Y$ , intervention  $do(X = x)$  yields  $x \rightarrow Y$  and  $X \leftarrow Y$ , respectively, which have different observational outcome.

**Causal Bayesian Network (CBN)** is a DAG  $\mathcal{G}$  compatible with  $\mathcal{P}$ , if

- Notation: here  $\mathcal{P} = \{\mathbb{P}(X|do(Z = z)) : \forall Z \subset X\}$  is the set of all interventional probability distribution.
- $\forall \mathcal{P} \ni \mathbb{P}(X|do(Z = z))$  is compatible with  $\mathcal{G}$
- $\mathbb{P}(x_i|do(Z = z)) = 1$  if  $X_i \in Z$  and  $x_i = z_{\text{corresponding value}}$  ( $X_i = x_i$  is consistent with  $Z = z$ )
- $\mathbb{P}(X_i|pa_i)$  is invariant to interventions not involving  $X_i$  itself.

Comments:

- Note that intervention  $do(Z = z)$  cancels some edges, so it would only add new independencies, which holds  $I(\mathcal{G}) \subset I(\mathbb{P})$  (still compatible).
- With some intervention  $do(Z = z)$ , the *truncated* factorization of  $\mathbb{P}(\cdot)$  is

$$\mathbb{P}(X|do(Z = z)) = \prod_{i: X_i \notin Z} \mathbb{P}(X_i|pa_i) \quad (11.134)$$

## 11.4.2 Network Structure Learning

### □ IC/PC Algorithm

IC/PC Algorithm (Inductive Causation Algorithm with Peter & Clark Algorithm Refinement) is a constraint-based method. DAG is constructed through identifying conditional independencies.

Here illustrated with the following example with conditional independencies. Ground truth is shown on the right

$$X \not\perp\!\!\!\perp R|\mathcal{S}, \forall \mathcal{S} \subset \{Y, Z, W\}$$

$$X \not\perp\!\!\!\perp Z|\mathcal{S}, \forall \mathcal{S} \subset \{Y, R, W\}$$

$$X \not\perp\!\!\!\perp Y|\mathcal{S}, \forall \mathcal{S} \subset \{Z, R, W\}$$

$$Y \not\perp\!\!\!\perp Z|\mathcal{S}, \forall \mathcal{S} \subset \{X, R, W\}$$

$$Y \not\perp\!\!\!\perp W|\mathcal{S}, \forall \mathcal{S} \subset \{X, Z, R\}$$

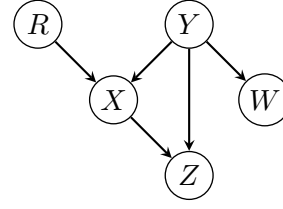
$$X \perp\!\!\!\perp W|Y$$

$$Y \perp\!\!\!\perp R$$

$$Z \perp\!\!\!\perp W|Y$$

$$Z \perp\!\!\!\perp R|\{X, Y\}$$

$$W \perp\!\!\!\perp R$$



1. Learning Skeleton: For all paris  $(a, b) \in \mathcal{V} \times \mathcal{V}$ :

- Connect  $a, b$  iff no  $\mathcal{S}_{ab}$  such that  $a \perp\!\!\!\perp b|\mathcal{S}_{ab}$  can be found. i.e.  $a, b$  have an edge if  $a \not\perp\!\!\!\perp b$  any set of other nodes.

$$X \not\perp\!\!\!\perp R|\mathcal{S}, \forall \mathcal{S} \subset \{Y, Z, W\}$$

$$X \not\perp\!\!\!\perp Z|\mathcal{S}, \forall \mathcal{S} \subset \{Y, R, W\}$$

$$X \not\perp\!\!\!\perp Y|\mathcal{S}, \forall \mathcal{S} \subset \{Z, R, W\}$$

$$Y \not\perp\!\!\!\perp Z|\mathcal{S}, \forall \mathcal{S} \subset \{X, R, W\}$$

$$Y \not\perp\!\!\!\perp W|\mathcal{S}, \forall \mathcal{S} \subset \{X, Z, R\}$$

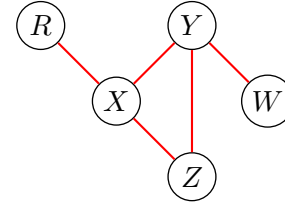
$$X \perp\!\!\!\perp W|Y$$

$$Y \perp\!\!\!\perp R$$

$$Z \perp\!\!\!\perp W|Y$$

$$Z \perp\!\!\!\perp R|\{X, Y\}$$

$$W \perp\!\!\!\perp R$$



2.  $v$ -structure Orientation: For all  $(a, b)$  with common neighbour  $c$  but not adjacent, i.e. have  $a-c-b$

- If  $c \notin \mathcal{S}_{ab}$ , then there is a  $v$ -structure  $a \rightarrow c \leftarrow b$

$$X \not\perp\!\!\!\perp R|\mathcal{S}, \forall \mathcal{S} \subset \{Y, Z, W\}$$

$$X \not\perp\!\!\!\perp Z|\mathcal{S}, \forall \mathcal{S} \subset \{Y, R, W\}$$

$$X \not\perp\!\!\!\perp Y|\mathcal{S}, \forall \mathcal{S} \subset \{Z, R, W\}$$

$$Y \not\perp\!\!\!\perp Z|\mathcal{S}, \forall \mathcal{S} \subset \{X, R, W\}$$

$$Y \not\perp\!\!\!\perp W|\mathcal{S}, \forall \mathcal{S} \subset \{X, Z, R\}$$

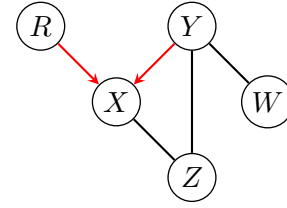
$$X \perp\!\!\!\perp W|Y$$

$$Y \perp\!\!\!\perp R$$

$$Z \perp\!\!\!\perp W|Y$$

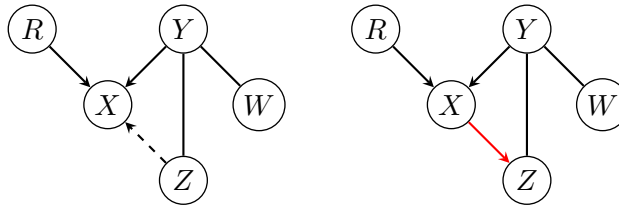
$$Z \perp\!\!\!\perp R|\{X, Y\}$$

$$W \perp\!\!\!\perp R$$

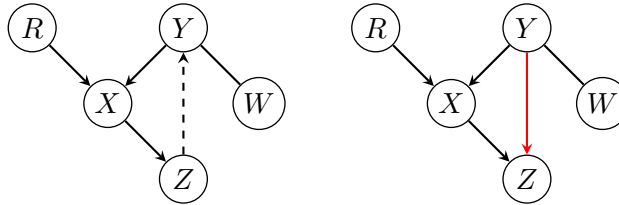


3. Meek's rule Orientation: orient as many edges as possible subject to:

- Alternative direction yields new  $v$ -structure



- Alternative direction yields cycle (acyclic rule is of more priority)



The above procedure can produce identify the DAG up to its observational equivalent class.

#### □ Search and Score Methods

We could also simply search in the space of all possible networks and select the one scoring the highest. Some frequently used metrics include AIC and BIC

$$\text{AIC} = -2 \log \mathbb{P}(\mathcal{G}|\hat{\theta}) + 2\text{dof}_{\mathcal{G}} \quad (11.135)$$

$$\text{BIC} = -2 \log \mathbb{P}(\mathcal{G}|\hat{\theta}) + \log n \cdot \text{dof}_{\mathcal{G}} \quad (11.136)$$

### 11.4.3 Network Parameter Learning

Basically, parameter learning (given BN structure) is simply estimating edge weights, denoted  $\Theta$ . Two basic methods are

- Bayesian approach

$$\arg \max_{\Theta} \mathbb{P}(\Theta|\mathcal{D}) \quad (11.137)$$

- Frequentist approach, e.g. with MLE loss+penalty form

$$\arg \min_{\Theta} -\log \mathbb{P}(\mathcal{D}|\Theta) + \lambda P(\Theta) \quad (11.138)$$

A trivial solution for categorical variables is

$$\hat{\theta}_{ijk} = \hat{\mathbb{P}}(X_i = j | pa_i = \vec{k}) = \frac{N_{ijk}}{N_{i \cdot k}}, \forall j = 1, \dots, J_i, \forall i \in \mathcal{V} \quad (11.139)$$

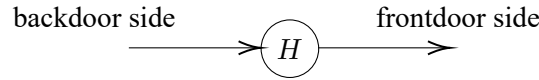
#### 11.4.4 Average Causal Effect Estimation

Average Causal Effects on BN are defined in terms of  $do(\cdot)$  operator,

$$\text{ACE}(Y|H) := \mathbb{E}[Y|do(H = h_1)] - \mathbb{E}[Y|do(H = h_2)] \quad (11.140)$$

Calculation of ACE given known BN relies on  $do$ -calculus. A  $do(\cdot)$  operator would cancel all edges pointing to the vertex, i.e. produce a modified graph  $\mathcal{G}_{do(\cdot)}$ , what we need to estimate is the probability in the modified graph.

$$\mathbb{E}_{\mathcal{G}}[Y|do(H)] \leftarrow \mathbb{P}_{\mathcal{G}}(Y|do(H)) \leftarrow \mathbb{P}_{\mathcal{G}_{do(H)}}(Y) \xleftarrow{\text{do-calculus}} \mathbb{P}_{\mathcal{G}}(\mathbf{X}) \leftarrow \text{Data} \quad (11.141)$$



#### □ $do$ -Calculus

- Module invariant

$$\mathbb{P}(X_i = x_i | do(PA_i = pa_i)) = \mathbb{P}(X_i = x_i | PA_i = pa_i) \quad (11.142)$$

#### ▷ The Adjustment Formula

$$\mathbb{P}(Y = y | do(X = x)) = \sum_{z \in \{pa_y\}} \mathbb{P}(Y = y | X = x, PA_y = z) \mathbb{P}(PA_y = z) \quad (11.143)$$

$$= \sum_{z \in \{pa_y\}} \frac{\mathbb{P}(X = x, Y = y, PA_y = z)}{\mathbb{P}(X = x | PA_y = z)} \quad (11.144)$$

in which we use the Markovian factorization on  $\mathcal{G}$

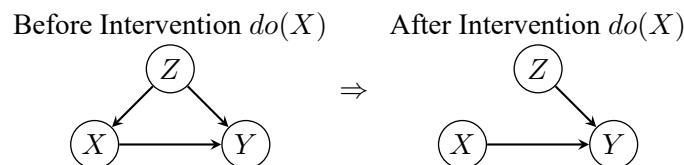
$$\mathbb{P}(X, Y, PA_y) = \mathbb{P}(Y|X, PA_y) \mathbb{P}(X|PA_y) \mathbb{P}(PA_y) \quad (11.145)$$

with  $X$  considered as assignment mechanism,  $Z$  considered as covariates, the formula shares the same idea as [equation. 11.117](#).

Through adjustment formula, we could obtain ACE from observed data (without intervention  $\rightsquigarrow$  with intervention).

**Example:** assessing  $Y|do(X = x)$ , in which  $PA_y = \{X, Z\}$  with  $X$  being fixed by  $do(X = x)$ .

$$\mathbb{P}(Y = y | do(X = x)) = \sum_{z \in \{z\}} \mathbb{P}(Y|X = x, Z = z) \mathbb{P}(Z = z) \quad (11.146)$$





### ▷ Backdoor criterion

Given  $(X, Y)$  in BN, a ‘backdoor set’  $Z$  is one such that  $Z$ :

- Blocks **all** paths with arrow onto  $X$  (i.e. backdoor side of  $X$  is blocked by  $Z$ )
- $Z$  contains **no** descendants of  $X$

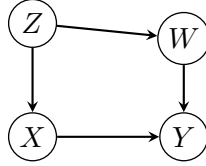
then we could use the backdoor variable set  $Z$  to have the **backdoor adjustment** of  $Y|do(X)$  as

$$\mathbb{P}(Y = y|do(X = x)) = \sum_z \mathbb{P}(Y = y|X = x, Z = z) \mathbb{P}(Z = z) \quad (11.147)$$

The selection of backdoor set  $Z$  is not unique. e.g. sometimes due to observability problem we could only obtain Partial DAG / have multiple methods to block the path, then we could pick proper nodes to form the backdoor set.

**Example:** assessing  $Y|do(X = x)$ , where  $Z$  is an observable while  $W$  is a hidden unobservable.

$$\mathbb{P}(Y = y|do(X = x)) = \sum_{z \in \{z\}} \mathbb{P}(Y|X = x, Z = z) \mathbb{P}(Z = z) \quad (11.148)$$



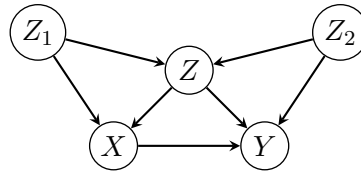
**Example:** assessing  $Y|do(X = x)$ .

$$\mathbb{P}(Y = y|do(X = x)) = \sum_{(z, z_1) \in \{(z, z_1)\}} \mathbb{P}(Y|X = x, Z = z, Z_1 = z_1) \mathbb{P}(Z = z, Z_1 = z_1) \quad (11.149)$$

$$= \sum_{(z, z_2) \in \{(z, z_2)\}} \mathbb{P}(Y|X = x, Z = z, Z_2 = z_2) \mathbb{P}(Z = z, Z_2 = z_2) \quad (11.150)$$

$$= \sum_{(z, z_1, z_2) \in \{(z, z_1, z_2)\}} \mathbb{P}(Y|X = x, Z = z, Z_1 = z_1, Z_2 = z_2) \quad (11.151)$$

$$\cdot \mathbb{P}(Z = z, Z_1 = z_1, Z_2 = z_2) \quad (11.152)$$



i.e. we could adjust for either  $(Z, Z_1)$  or  $(Z, Z_2)$  or  $(Z, Z_1, Z_2)$  as the backdoor set.

### ▷ Frontdoor criterion

Given  $(X, Y)$  in BN, a ‘frontdoor set’  $Z$  is one such that  $Z$ :

- Intercepts **all** paths from  $X$  to  $Y$  (i.e. frontdoor side of  $X$  is intercepted  $Z$ )
- **No** unblocked backdoor path from  $X$  to  $Z$ <sup>101</sup>
- **All** backdoor paths from  $Z$  to  $Y$  blocked by  $X$

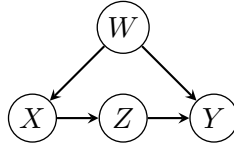
then we could use the frontdoor variable set  $Z$  to have the frontdoor adjustment of  $Y|do(X)$  as

$$\mathbb{P}(Y = y|do(X = x)) = \sum_z \mathbb{P}(Z = z|X = x) \sum_{x'} \mathbb{P}(Y = y|X' = x', Z = z) \mathbb{P}(X' = X') \quad (11.153)$$

<sup>101</sup>backdoor path from  $X$  to  $Z$  means containing a backdoor arrow of  $X$ , e.g. in the example,  $X \leftarrow W \rightarrow Y \leftarrow Z$ , is blocked.

**Example:** assessing  $Y|do(X = x)$ .

$$\mathbb{P}(Y = y|do(X = x)) = \sum_z \mathbb{P}(Z = z|X = x) \sum_{x'} \mathbb{P}(Y = y|X' = x', Z = z) \mathbb{P}(X' = X') \quad (11.154)$$



The derivation is using backdoor adjustment twice

$$\begin{cases} \mathbb{P}(Y = y|do(X = x)) = \sum_z \mathbb{P}(Y = y|do(Z = z)) \mathbb{P}(Z = z|do(X = x)) \\ \mathbb{P}(Y = y|do(Z = z)) = \sum_x \mathbb{P}(Y = y|Z = z, X = x) \mathbb{P}(X = x) \\ \mathbb{P}(Z = z|do(X = x)) = \mathbb{P}(Z = z|X = x) \end{cases} \quad (11.155)$$

▷ General Rules of *do*-calculus

#### 11.4.5 Instrumental Variable Method

## Chapter. XII 应用随机过程部分

Instructor: Pengkun Yang

### Section 12.1 Properties of Stochastic Process

#### 12.1.1 Basic Concepts

Some basic concepts about stochastic process / random process are introduced in [section. 10.2](#). Here's a brief recap.

A stochastic process is a mapping

$$\{X_t : t \in \mathcal{T}\} : \Omega \mapsto \mathcal{T} \times \mathbb{R} \quad (12.1)$$

- For given  $t \in \mathcal{T}$ ,  $X_t(\cdot)$  is a r.v. defined on  $\Omega$ .
- For given  $\omega \in \Omega$ ,  $X(\omega)$  is a function on  $\mathcal{T}$ , which is called sample path.

According to the continuity of index Fourier Transform set  $\mathcal{T}$  and sample path values, Stochastic process can be categorized in discrete / continuous Time + discrete / continuous State processes.

Some functions of stochastic processes include

- Mean function:

$$\mu_X(t) = \mathbb{E}[X_t] \quad (12.2)$$

- AutoCovariance function (ACVF):

$$\gamma_{s,t} := \text{cov}(X_s, X_t) \quad (12.3)$$

- AutoCorrelation function (ACF):

$$\rho_{s,t} := \text{corr}(X_s, X_t) = \frac{\gamma_{s,t}}{\sqrt{\gamma_{s,s}\gamma_{t,t}}} \quad (12.4)$$

- $n^{\text{th}}$  order CDF:

$$F_{X,n}(x_1, t_1; x_2, t_2; \dots; x_n, t_n) = \mathbb{P}(X_{t_1} \leq x_1, X_{t_2} \leq x_2, \dots, X_{t_n} \leq x_n) \quad (12.5)$$

#### 12.1.2 Properties of Discrete Time Markov Chain

A basic case for Markov Chain is Discrete Time Markov Chain (DTMC)

##### □ Notations and Properties of DTMC

- State: denote the state space / phase space of DTMC as

$$X_n \in \mathcal{S} \quad (12.6)$$

- Conditional Independency:

$$\mathbb{P}(X_{n+1} | X_0, X_1, \dots, X_n) = \mathbb{P}(X_{n+1} | X_n) \quad (12.7)$$

- State transition and transition probability matrix:

$$P^{(k)} = \{P_{ij}^{(k)}\} = \{\mathbb{P}(X_{k+1} = j | X_k = i)\}, \quad i, j \in \mathcal{S} \quad (12.8)$$

transition pr matrix  $P$  is called a (row) stochastic matrix, with

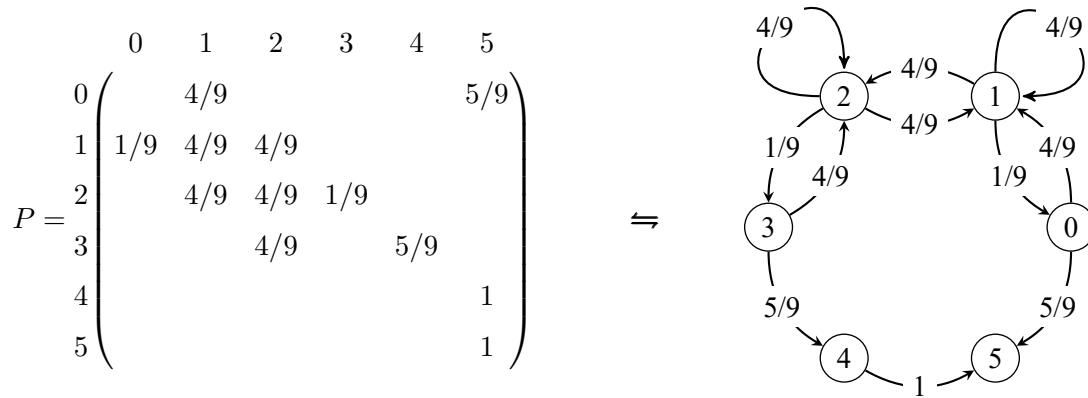
$$0 \leq P_{ij}^{(k)} \leq 1, \quad \sum_j P_{ij}^{(k)} = 1 \quad (12.9)$$

- Time homogeneity: transition probability is independent of step / time

$$P^{(k)} = P, \quad \forall k \quad (12.10)$$

we usually focus on time-homogeneous DTMC.

- State diagram: a useful way to visualize DTMC, in which vertices / nodes for states and edges / arrows for transition. Here's an example of 'Mickey Mouse' diagram with six states:



### □ Stationary Distribution

State transition between steps are like jumping in state diagram. Denote  $\pi(k)$  the probability distribution at step  $k$ , then a transition is

$$\pi(k+1) = \pi(k)P^{(k)} = \pi(k)P \quad (12.11)$$

A stationary distribution / equilibrium of DTMC is the eigen distribution of transition matrix

$$\pi^* = \pi^*P = \pi^*P^i, \quad \forall i \quad (12.12)$$

A sufficient condition for stationary state is the detailed balance condition

$$\pi_i^* = \sum_j \pi_j^* P_{ji} \quad (12.13)$$

$$\Leftrightarrow \pi^* \sum_{j \neq i} P_{ij} = \pi_i^* (1 - P_{ii}) = \sum_{j \neq i} \pi_j^* P_{ji} \quad (12.14)$$

$$\Leftrightarrow \pi_i P_{ij} = \pi_j P_{ji} \quad (12.15)$$

Some concepts related to stationary distribution

- Reachable: we can arrive at  $j$  starting from  $i$ , denoted  $i \rightsquigarrow j$

$$\exists n < \infty \text{ s.t. } \mathbb{P}(X_n = j | X_0 = i) > 0 \quad (12.16)$$

Sometimes I use the notation  $i \xrightarrow{k} j$  for ‘reaching  $j$  in  $k$  steps from  $i$ ’

- Irreducible: every state is reachable from any other states

$$i \rightsquigarrow j, \forall i, j \in \mathcal{S} \quad (12.17)$$

- Periodic: the period  $d_i$  for state  $i$  is the greatest common divisor (GCD) of step-to-come-back.

$$d_i := \gcd \{n : \mathbb{P}(X_n = i | X_0 = i) > 0\} \quad (12.18)$$

Irreducible DTMC has the same period for all states.

For any two states  $i, j$ , with periods  $d_i, d_j$ . Then  $d_i$  contains the following process:

$$\{i \xrightarrow{k_1} j \xrightarrow{m \times d_j} j \xrightarrow{k_n} i\}, \quad m \in \mathbb{N} \quad (12.19)$$

there are infinite elements. then

$$d_i = \gcd \{k_1 + k_2 + m d_j; m = 0, 1, 2, \dots\} \Rightarrow d_j = \text{multiple of } d_i \quad (12.20)$$

With the argument applied to all state pairs  $(i, j) \in \mathcal{S} \times \mathcal{S}$ , obviously  $d_i = d, \forall i \in \mathcal{S}$

- Aperiodic: is the case that  $d_i = 1$ , i.e. possible to come back anytime. For irreducible DTMC, if one state is aperiodic, then all are.

Naturally if a node is self looped  $P_{ii} > 0$  (e.g. node 1 or 2 in ‘Mickey Mouse’ loops back with pr 4/9), then all the states are aperiodic.

- Sojourn Time  $T_i$ : is the time to stay at the state

$$T_i \sim \text{Geo}(1 - P_{ii}) \quad (12.21)$$

- Classification of States.

Denote Hitting Time (without itself include)  $\tau_i^+$  and its mean

$$\tau_i^+ := \min\{k \geq 1 : X_k = i\} \quad (12.22)$$

$$\mu_i := \mathbb{E}[\tau_i^+ | X_0 = i] \quad (12.23)$$

– Recurrent State

$$\mathbb{P}(\tau_i^+ < \infty | X_0 = i) = 1 \quad (12.24)$$

in which

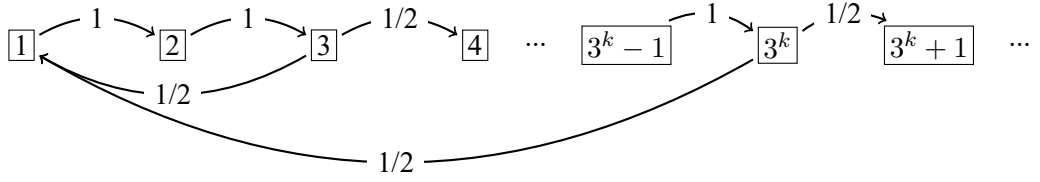
\* Positive Recurrent

$$\mu_i < \infty \quad (12.25)$$

## \* Null Recurrent

$$\mu_i = \infty \quad (12.26)$$

An example:



where

$$\mu_1 = \mathbb{E} [\tau_1^+ | X_0 = 1] = \sum_{i=1}^{\infty} \left(\frac{3}{2}\right)^i \rightarrow \infty \quad (12.27)$$

## – Transient State

$$\mathbb{P} (\tau_i^+ < \infty | X_0 = i) < 1 \quad (12.28)$$

□ DTMC: Irreducible & Aperiodic & Positive Recurrent  $\Rightarrow$  Unique Stationary Distribution  $\pi^*$  Exists

Given irreducible & aperiodic DTMC, we have

- All states have the same state classification: null recurrent / positive recurrent / transient.
- if all states are positive recurrent  $\mu_i < \infty$ , then stationary distribution exists.

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{l=1}^n (P^l)_{ij} = \frac{1}{\mu_i} \Rightarrow \pi_i(\infty) = (\pi(0)P^\infty)_i = \frac{1}{\mu_i}, \forall \pi(0) \quad (12.29)$$

- Further if states are positive recurrent  $\mu_i < \infty$ , then stationary distribution.

$$\pi^* = \frac{1}{\mu_i} \quad (12.30)$$

The proof is a little bit complicated, but an intuition is direct. For a realization of Markov Process  $\{X_t\}_{t=1}^\infty$ , in which  $\{X_{i_1}, X_{i_2}, \dots\}$  is the set that  $X_t = i$  for any given  $i$ , and  $\{u_{i_1}, u_{i_2}, \dots\} = \{i_1 - i_2, i_3 - i_2, \dots\}$  is the time-btw-event, i.e.  $u_{i_j} \sim_{i.i.d.} \tau_i^+ | X_0 = i, \forall j$ . Then

$$\lim_{n \rightarrow \infty} \sum_{t=1}^n \mathbb{I}_{X_t=i} = \lim_{k \rightarrow \infty} \frac{k}{u_{i_1} + u_{i_2} + \dots + u_{i_k}} = \frac{1}{\mathbb{E} [\tau_i^+ | X_0 = i]} = \frac{1}{\mu_i} \quad (12.31)$$

Comment: Ergodicity = irreducible & aperiodic condition. It *creates link between phase structure and time structure*, which makes  $\bar{u}$  (time-average) converge in an appropriate sense to  $\mu_i$  (phase-average).

Some algorithm about Markov Chain see [section. 5.6.2](#).

## □ Concrete examples of DTMC

- Random Walk
- Gambler's Model
- Branching Process

### 12.1.3 Properties of Continuous Time Markov Chain

Another case of Markov Chain is Continuous Time Markov Chain (CTMC)

#### □ Notations and Properties of CTMC

- Concepts of state and conditional independency are similar to DTMC

$$\mathbb{P}(X_{t_{n+1}} | X_{t_0}, X_{t_1}, \dots, X_{t_n}) = \mathbb{P}(X_{t_{n+1}} | X_{t_n}) \quad (12.32)$$

- Transition probability matrix

$$H(s, t) := \{H_{ij}(s, t)\} = \{\mathbb{P}(X_t = j | X_s = i)\}, \quad s < t \quad (12.33)$$

with a trivial case that  $H(t, t) = I$ . State transition could be expressed by matrix  $H(s, t)$  as

$$p(t) = p(s)H(s, t) \quad (12.34)$$

- Chapman-Kolmogorov Equation

$$H(r, t) = H(r, s)H(s, t), \quad r < s < t \quad (12.35)$$

- Time homogeneity: transition probability is independent of time interval:

$$H(s, t) = H(0, t - s) \quad (12.36)$$

- Generator of time homogeneous CTMC: The **Transition Rate Matrix** is

$$Q := \lim_{\delta \rightarrow 0} \frac{H(\delta) - H(0)}{\delta}, \quad H(\delta) = I + \delta Q + o(\delta) \quad (12.37)$$

with Chapman-Kolmogorov Equation we could see that  $Q$  is the generator of the transition matrix (group)

$$H(t) = \lim_{t=n\delta} \lim_{n \rightarrow \infty} H(\delta)^n = \lim_{n \rightarrow \infty} \left(I + \frac{t}{n}Q\right)^n = e^{Qt} \quad (12.38)$$

And note that  $H(t)$  has 1 row-sum,  $\sum_j (e^{Qt})_{ij} = 1$ :

$$0 = \frac{d \sum_j (e^{Qt})_{ij}}{dt} = \sum_{j,k} Q_{ik} (e^{Qt})_{kj} = \sum_k Q_{ik} = 0 \quad (12.39)$$

$$\Rightarrow Q_{ii} = - \sum_{k \neq i} Q_{ik}, \quad \forall i \quad (12.40)$$

i.e. generator  $Q$  has 0 row-sum.

Comment: with Gershgorin Circle Theorem<sup>102</sup>,  $Q$  as a diagonal dominant matrix, is negative definite, which guarantee the convergence of  $H(t) = e^{Qt} < \infty$

- Kolmogorov Forward Equation:<sup>103</sup>

$$\dot{p}(t) = \frac{dp(0)e^{Qt}}{dt} = p(0)e^{Qt}Q = p(t)Q \quad (12.42)$$

Kolmogorov forward could also be deduced for some other specifically defined event / probability.

<sup>102</sup>Detail see <https://vincent19.github.io/texts/DiagonalDominant/>.

<sup>103</sup>Note that  $Q$  and  $e^{Qt}$  are commutable

$$Qe^{Qt} = Q \sum_{i=0}^{\infty} \frac{Q^i t^i}{i!} = e^{Qt}Q \quad (12.41)$$

- Stationary Distribution: with  $\dot{\pi}^* = 0$  in Kolmogorov forward, stationary distribution of CTMC:

$$\pi^* = \pi^* H(t), \forall t \Leftrightarrow \pi^* Q = 0 \quad (12.43)$$

thus yield the detailed balance in CTMC version:

$$\pi^* Q = 0 \Leftrightarrow \pi_i^* q_{ij} = 0, \forall i, j \quad (12.44)$$

- Dynamics of CTMC: Each step (say,  $0 \rightsquigarrow t \rightsquigarrow t + \delta$ ) in state transitions in CTMC could be decomposed in two sub-steps:

$$\begin{cases} \text{Sojourn : } T_i \sim \mathbb{P}(t : X_\tau = i \forall 0 \leq \tau \leq t | X_0 = i) \\ \text{Jump : } p_{ij}^J \sim \mathbb{P}(X_{t+\delta} = j | X_t = i, X_{t+\delta} \neq i) \end{cases} \quad (12.45)$$

which has the following dynamics

$$\begin{cases} T_i \sim \varepsilon(-q_{ii}) \\ p_{ij}^J = (\delta_{ij} - 1) \frac{q_{ij}}{q_{ii}} \end{cases} \quad (12.46)$$

Where sojourn time  $T_i$  is a continuous correspondance of 12.21. In both versions it is memoryless.

#### □ CTMC: Irreducible & Non-explosive & Positive Recurrent $\Rightarrow$ Unique Stationary Distribution $\pi^*$

Given irreducible & non-explosive CTMC, we have

- All states have the same state classification: null recurrent / positive recurrent / transient
- Stationary distribution exists  $\Leftrightarrow$  all states are positive recurrent

$$\lim_{t \rightarrow \infty} p_i(t) = \frac{1}{-q_{ii}\mu_i} = \pi_i^* \quad (12.47)$$

#### □ Concrete examples of CTMC

- **Brownian Process**: CTMC with continuous states;
- **Poisson Process**: CTMC with discrete states;
- 

### 12.1.4 Independent Increment Process and Martingale

Motivation: Sometimes a process is a ‘summation of all past events’.

- Independent Increment: Def.  $\{X_t\}$  a **independent increment process** if  $\forall t_0 < t_1 < \dots < t_n, \forall n$

$$X_{t_n} - X_{t_{n-1}} \perp\!\!\!\perp X_{t_{n-1}} - X_{t_{n-2}} \perp\!\!\!\perp \dots \perp\!\!\!\perp X_{t_1} - X_{t_0} \quad (12.48)$$

- Martingale: Def.  $\{X_t\}$  a **Martingale** if  $\forall t_0 < t_1 < \dots < t_n, \forall n$

$$\mathbb{E}[X_{t_n} | X_{t_{n-1}}, \dots, X_{t_0}] = X_{t_{n-1}} \quad (12.49)$$

with a technical condition of bounded expectation  $\mathbb{E}[|X_t|] < \infty$ .



- Martingale: Def.  $\{X_t\}$  being a Martingale w.r.t.  $\{Y_t\}$  if

$$\mathbb{E}[X_{t_n} | Y_{t_{n-1}}, \dots, Y_{t_0}] = X_{t_{n-1}} \quad (12.50)$$

with bounded expectation  $\mathbb{E}[|X_t|] < \infty$ .

#### □ Concrete examples of independent increment processes

- **Brownian Process**: homogeneous events, probabilistic increment.
- **Poisson Process**: probabilistic events, homogeneous increment.

### 12.1.5 Ergodicity

## Section 12.2 Useful Instances of Stochastic Processes

### 12.2.1 Random Walk

Random walk is a renewal process  $X_n$  with each step  $W_i$  takes value  $\pm 1$

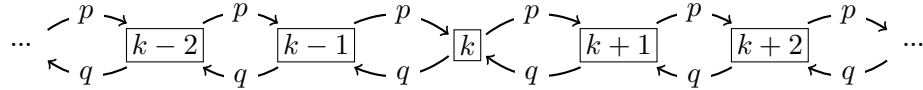
$$X_n := X_0 + \sum_{i=1}^n W_i \quad W_i = \begin{cases} +1 & \text{w.p. } p \\ -1 & \text{w.p. } q := 1 - p \end{cases} \quad (12.51)$$

where  $X_0 = k$  is the initial position.

#### □ Simple Random Walk

Simple random walk is the case with no ends, i.e.  $X_n \in \mathbb{Z}$

- State Diagram for Simple Random Walk



- Parameters

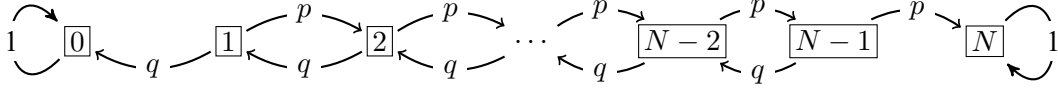
$$\begin{cases} \text{Mean Function : } \mu_n = k + n(2p - 1) \\ \text{Covariance : } \gamma_{m,n} = 4pq \min\{m, n\} \\ \text{CLT : } \frac{X_n - k - n(2p - 1)}{\sqrt{4npq}} \xrightarrow{d} N(0, 1) \end{cases} \quad (12.52)$$

### 12.2.2 Gambler's Model

Gambler's model is the case with one/two ends, usually one of the ends is denoted 0, as Gambler's ruin, and the other denoted  $N$  as Gambler's success.

Reaching 0 or  $N$  stops the chain, so are called 'absorbing state'.

- State Diagram of Gambler's model with two ends



- Gambler's Ruin / Success: Denote Hitting Time (allowing itself included)  $\tau_i = \min \{n \geq 0 : X_n = i\}$ , and probability of ruin  $r_i$  and probability of success  $s_i$  respectively

$$r_i := \mathbb{P}(X_{\tau_0} = 0 | X_0 = i) \quad (12.53)$$

$$s_i := \mathbb{P}(X_{\tau_N} = N | X_0 = i) \quad (12.54)$$

with iteration relation

$$s_i = p \cdot s_{i+1} + q \cdot s_{i-1}, \quad s_0 = 0, s_N = 1 \quad (12.55)$$

$$r_i = q \cdot r_{i+1} + p \cdot r_{i-1}, \quad r_0 = 1, r_N = 0 \quad (12.56)$$

we could get<sup>104</sup>

$$s_i = \frac{1 - (q/p)^i}{1 - (q/p)^N} \quad (12.59)$$

$$r_i = \frac{(q/p)^i - (q/p)^N}{1 - (q/p)^N} = 1 - s_i \quad (12.60)$$

- Mean Hitting Time  $T_{i \rightsquigarrow \{0, N\}}$  for  $i \rightsquigarrow \{0, N\}$ :  $T_{i \rightsquigarrow \{0, N\}} = \mathbb{E}[\min\{\tau_0, \tau_N\} | X_0 = i]$ :

$$T_{i \rightsquigarrow \{0, N\}} = p(1 + T_{i+1 \rightsquigarrow \{0, N\}}) + q(1 + T_{i-1 \rightsquigarrow \{0, N\}}), \quad T_{N \rightsquigarrow \{0, N\}} = T_{0 \rightsquigarrow \{0, N\}} = 0 \quad (12.61)$$

solution

$$T_{i \rightsquigarrow \{0, N\}} = \frac{(1 - (q/p)^i)(N - i)}{(1 - (q/p)^N)(p - q)} \quad (12.62)$$

- One-end case (greedy gambler) is just having  $N \rightarrow \infty$

$$r_i = \begin{cases} 1, & p \leq \frac{1}{2} \\ \left(\frac{q}{p}\right)^i, & p > \frac{1}{2} \end{cases} \quad (12.63)$$

Note: i.e. there is a phase transition at  $p = \frac{1}{2}$ .

<sup>104</sup>For the case  $q = p = 1/2$ , take the natural limit to get corresponding solution

$$s_i = \frac{k}{N} \quad (12.57)$$

$$r_i = 1 - \frac{k}{N} \quad (12.58)$$

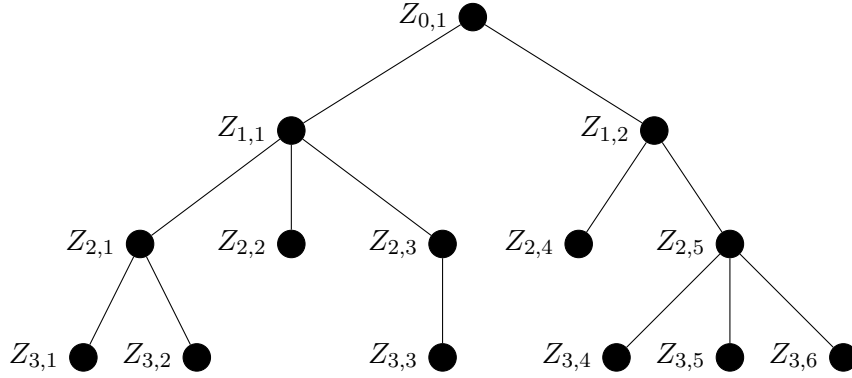
### 12.2.3 Branching Process

Branching process / Galton-Watson Tree focuses on the case of population growth / epidemic infection / nuclear fission chain reaction, etc. Each steps the state  $X_n$  denotes the number of individuals, update of state is given as

$$X_{t+1} = \sum_{j=1}^{X_t} Z_{t,j}, \quad Z_{t,j} \text{ i.i.d. } \sim Z_t, \quad X_0 = 1 \quad (12.64)$$

and we usually assume the simple case of  $Z_t$  i.i.d.  $\sim Z$ .

- State Diagram



- $z$ -transform for distribution of  $X_t$ :

$$\Pi_t(s) = \mathbb{E}[s^{X_t}] = \sum_{j=0}^{\infty} s^j \mathbb{P}(X_t = j) \quad L(s) = \mathbb{E}[s^Z] = \sum_{j=0}^{\infty} s^j \mathbb{P}(Z = j) \quad (12.65)$$

and

$$\Pi_t(s) = \sum_{j=0}^{\infty} \mathbb{E}[s^{X_t} | X_{t-1} = j] \mathbb{P}(X_{t-1} = j) \quad (12.66)$$

$$= \sum_{j=0}^{\infty} (L(s))^j \mathbb{P}(X_{t-1} = j) \quad (12.67)$$

$$= \Pi_{t-1}(L(s)) \quad (12.68)$$

$$(\Pi_1(s) = L(s)) = L^{(t)}(s) \quad (12.69)$$

- Mean and Variance:

$$\text{Mean : } \mu(t) = \Pi'_t(1) = \mu(0)^t \quad (12.70)$$

$$\text{Variance : } \text{var}(t) = \Pi''_t(1) + \Pi'_t(1) - [\Pi'_t(1)]^2 \quad (12.71)$$

- Extinction Probability

$$\theta_t = \mathbb{P}(X_t = 0) = \sum_{j=0}^{\infty} \theta_{t-1}^j \mathbb{P}(Z = j) \quad (12.72)$$

$$= L(\theta_{t-1}) \quad (12.73)$$

The eventual extinction is  $\theta^* = L(\theta^*)$ , the fixed point of  $L(\cdot)$ . There is a phase transformation at  $\mu = 1$

$$\mathbb{P}(\theta^* = 1) = \begin{cases} 1, & \mu \leq 1 \\ \text{the first root of } L(\theta) = \theta, & \mu > 1 \end{cases} \quad (12.74)$$

Convergence order at phase transition point:

$$\mathbb{P}(X_T > n) \sim \begin{cases} c_1 \mu^n, & \mu < 1 \\ \frac{c_2}{n}, & \mu = 1 \end{cases} \quad (12.75)$$

### 12.2.4 Brownian Motion

Motivation: Brownian motion / Wiener Process  $W_t$ <sup>105</sup> is similar to a random walk model with  $p = q = 1/2$ , but with initial state  $X_0 = 0$ , and ‘steps’ defined as ‘a short enough time segmentation’.

$$W_{t=\frac{k}{N}} := \frac{1}{\sqrt{N}} \sum_{i=1}^k \varpi_i, \quad \varpi_i \sim \text{i.i.d. Unif}\{+1, -1\} \quad (12.76)$$

and have  $N \rightarrow \infty$  as a Brownian Motion (Donsker Thm.)

Rigorous definition of **Brownian** / **Wiener Process**:  $\{W_t : T \geq 0\}$  with  $0 < \sigma^2 < \infty$  is Brownian if

1. Starts from 0:  $\mathbb{P}(W_0) = 1$
2. Independent increment:  $W_{t_1} - W_{s_1} \perp\!\!\!\perp W_{t_2} - W_{s_2}, \forall [t_1, s_1] \cap [t_2, s_2] = \emptyset$
3. Zero mean Normal:  $W_t - W_s \sim N(0, \sigma^2|t - s|)$
4. continuity:  $\mathbb{P}(W_t \text{ continuous}) = 1$

Properties:

- Parameters

$$\begin{cases} \text{Mean Function : } \mu(t) = 0 \\ \text{Covariance : } \gamma(t, s) = \sigma^2 \min\{s, t\} \end{cases} \quad (12.77)$$

- m.s. indifferntiable

$$\mathbb{E} \left[ \left( \frac{\partial W_t}{\partial t} \right)^2 \right] \rightarrow \infty \quad (12.78)$$

which is the reason why the plots for Brownian Motion always looks rugged.

- Conditional distribution / Brownian Bridge  $B_t$ :

$$B_t := W_t | W_T = 0 \sim N(0, \sigma^2 \frac{t(T-t)}{T}) \quad (12.79)$$

- Dependent increment: non-zero covariance

$$\gamma_{\text{Bridge}}(t, s) = \sigma^2 \left( \min\{t, s\} - \frac{ts}{T} \right) \quad (12.80)$$

<sup>105</sup>Symbol  $W_t$  for ‘Wiener’, sometimes uses  $B_t$  for ‘Brown’.

– Cross definition between Wiener Process and Brownian Bridge:

$$\begin{cases} B_t := W_t - \frac{t}{T}W_T \\ W_t := B_t + t\sigma^2 N(0, 1) \end{cases} \quad (12.81)$$

i.e. Brownian Bridge is independent of the terminal of its corresponding Wiener Process  $B_t \perp\!\!\!\perp W_T$ .

## 12.2.5 Poisson Process

Motivation: The accumulate events happens at random, with ‘happening rate’ of events as  $\lambda$

$$N_{t=\frac{k}{N}} := \sum_{i=1}^k \nu_i, \quad \nu_i \sim \text{i.i.d. Bern}\left(\frac{\lambda}{n}\right) \quad (12.82)$$

Rigorous Definition of **Poisson Process**:  $\{N_t : t \geq 0\}$  with rate  $\lambda > 0$  is Poisson if

- Counting Process  $N_t$ :  $N_0 = 0, N_t \in \mathbb{N}$
- Independent Increment:  $N_{t_1} - N_{s_1} \perp\!\!\!\perp N_{t_2} - N_{s_2}, \forall [t_1, s_1] \cap [t_2, s_2] = \emptyset$
- Poisson increment:  $N_t - N_s \sim P(\lambda(t-s)), t \geq s$ <sup>106</sup>

Properties:

- Parameters

$$\begin{cases} \text{Mean Function : } \mu(t) = \lambda t \\ \text{Covariance : } \gamma(t, s) = \lambda \min\{s, t\} \end{cases} \quad (12.83)$$

- Arrival time:  $N_{t_n} = n$  means there are  $n$  events before (and including)  $t_n$ , denoted  $\{t_1, t_2, \dots, t_n\}$ . PDF

$$f_{T_1, T_2, \dots, T_n}(t_1, t_2, \dots, t_n) = \lambda^n e^{-\lambda t_n} \mathbb{I}_{0 < t_1 < t_2 < \dots < t_n} \quad (12.84)$$

- Inter-event time: PDF of time-between-events  $\{u_1, u_2, \dots, u_n\} := \{t_1, t_2 - t_1, \dots, t_n - t_{n-1}\}$

$$f_{U_1, U_2, \dots, U_n}(u_1, u_2, \dots, u_n) = \prod_{i=1}^n \lambda e^{-\lambda u_i} \mathbb{I}_{u_i \geq 0} = \sim \otimes_i 1^n \varepsilon_i(\lambda) \quad (12.85)$$

i.e. time-between-events satisfies exponential distribution

$$U_i \sim \text{i.i.d. } \varepsilon(\lambda) \quad (12.86)$$

- Conditional distribution

$$f_{T_1, T_2, \dots, T_n | N_t = n}(t_1, t_2, \dots, t_n) = \frac{n!}{t^n} \mathbb{I}_{0 < t_1 < t_2 < \dots < t_n} \sim \text{Unif}(\mathbb{I}_{0 < t_1 < t_2 < \dots < t_n \leq t}) \quad (12.87)$$

is the PDF of order statistics<sup>107</sup> of i.i.d.  $\text{Unif}(0, t)$ .

- Poisson Process and Martingale:

$$\tilde{N}_t := N_t - \lambda t \sim \text{Martingale} \quad (12.88)$$

<sup>106</sup> A proof & another kind of definition concerning the intuition of ‘rate  $\lambda$ ’ is here: <https://vincent19.github.io//texts/Poisson/>.

<sup>107</sup> See equation. 1.47.

### 12.2.6 Birth-Death Process

Birth-death process looks like a one-end random-walk with ‘step’ as poisson r.v.(i.e. exponential time-interval) The transition rate & diagram are:

$$Q = \begin{matrix} & \begin{matrix} 0 & 1 & 2 & 3 & \dots \end{matrix} \\ \begin{matrix} 0 \\ 1 \\ 2 \\ 3 \\ \vdots \end{matrix} & \begin{pmatrix} -\lambda_0 & \lambda_0 & & & \\ \mu_1 & -\mu_1 - \lambda_1 & \lambda_1 & & \\ & \mu_2 & -\mu_2 - \lambda_2 & \lambda_2 & \\ & & \mu_3 & \ddots & \ddots \\ & & & \ddots & \ddots \end{pmatrix} \end{matrix} \quad \Leftrightarrow \quad \begin{matrix} & \lambda_0 & & \lambda_1 & & \lambda_2 & \dots \\ \boxed{0} & \leftarrow & \boxed{1} & \leftarrow & \boxed{2} & \leftarrow & \dots \\ & \mu_1 & & \mu_2 & & \mu_3 & \end{matrix}$$

- Kolmogorov forward: with a trivial notation that  $\lambda_{-1} = \mu_0 = 0$ , we have

$$\dot{p}_i(t) = \lambda_{i-1}p_{i-1}(t) + \mu_{i+1}p_{i+1}(t) - (\lambda_i + \mu_i)p_i(t) \quad (12.89)$$

- Stationary Distribution:  $\dot{\pi}^* = 0$  yields

$$(\lambda_i + \mu_i)\pi_i^* = \lambda_{i-1}\pi_{i-1}^* + \mu_{i+1}\pi_{i+1}^* \quad (12.90)$$

Solution:

$$\pi_i^* = \begin{cases} \frac{1}{Z} \frac{\lambda_0 \lambda_1 \dots \lambda_{i-1}}{\mu_1 \mu_2 \dots \mu_i}, & i \neq 0 \\ \frac{1}{Z}, & i = 0 \end{cases}, \quad Z = 1 + \sum_{j=1}^{\infty} \frac{\lambda_0 \lambda_1 \dots \lambda_{j-1}}{\mu_1 \mu_2 \dots \mu_j} \quad (12.91)$$

## Section 12.3 Applications

### 12.3.1 Innovation Sequence

Motivation of Innovation Sequence (新息序列): construction of linear MMSE  $L(X|Y_1, Y_2, \dots, Y_n) = L(X|Y)$ . Assume that  $\mathbb{E}[\vec{Y}] = 0$ , the prediction is

$$L(X|\vec{Y}) = \mathbb{E}[X] + \text{cov}(X, \vec{Y})\text{var}(\vec{Y})^{-1}\vec{Y} \quad (12.92)$$

which causes the problem of computation complexity when dimension  $n$  is large.

Innovation sequence fixed this problem by: instead of projecting on the whole linear combination  $\vec{Y}$  space of size  $(n+1)$ , we project on space of each  $Y_i$  sequentially. i.e. define an **innovation sequence**

$$\tilde{Y}_1 = Y_1 - \mathbb{E}[Y_1] = Y_1 - \mathbb{E}[Y_1] \quad (12.93)$$

$$\tilde{Y}_2 = Y_2 - L(Y_2|Y_1) = Y_2 - L(Y_2|\tilde{Y}_1) \quad (12.94)$$

$$\tilde{Y}_3 = Y_3 - L(Y_3|Y_2Y_1) = Y_3 - L(Y_3|\tilde{Y}_2\tilde{Y}_1) \quad (12.95)$$

$$\dots \quad (12.96)$$

$$\tilde{Y}_n = Y_n - L(Y_n|Y_{n-1} \dots Y_2Y_1) = Y_n - L(Y_n|\tilde{Y}_{n-1} \dots \tilde{Y}_2\tilde{Y}_1) \quad (12.97)$$

where ‘innovation’ means each  $\tilde{Y}_i$  contains the ‘new information without correlation with previous sequence’:  $\mathbb{E}[\tilde{Y}_i \tilde{Y}_j] = 0 \forall i \neq j$ . Computation of innovation sequence:

$$\tilde{Y}_k = Y_k - L(Y_k | \tilde{Y}_{k-1} \dots \tilde{Y}_1) = Y_k - \mathbb{E}[Y_k] - \sum_{j=1}^{k-1} \frac{\text{cov}(Y_k, \tilde{Y}_j)}{\text{var}(\tilde{Y}_j)} \tilde{Y}_j, \quad k = 1, 2, \dots, n \quad (12.98)$$

with a trivial notation that  $Y_0 = 1$

In this way a linear MMSE  $L(X | \vec{Y})$  could be written as

$$L(X | \vec{Y}) = L(X | \vec{Y}) = \mathbb{E}[X] + \sum_{i=1}^n \frac{\text{cov}(X, \tilde{Y}_i)}{\text{var}(\tilde{Y}_i)} \tilde{Y}_i = \mathbb{E}[X] + \sum_{i=1}^n L(X - \mathbb{E}[X] | \tilde{Y}_i) \quad (12.99)$$

I think the idea here is similar to Gram-Schmidt orthogonalization (section. 5.2.4), in which we also construct new components by eliminating projection on previous parts. As a result we have a set of orthogonal elements (here orthogonal means  $\mathbb{E}[\tilde{Y}_i \tilde{Y}_j] = 0$  and in Gram-Schmidt means  $q'_i q_j = 0, i \neq j$ ). And the result is a ‘change of basis’ of space.

### 12.3.2 Markov Decision Processes

In decision process/episode, say  $\{(s_t, a_t)\}_{t=0}^T$ , we need to determine a **policy**  $\pi_t$  to take **action**  $a_t$  given **state**  $s_t$  as

$$a_t \sim \pi_t(\cdot | s_t) \text{ or simply } a_t = \pi_t(s_t) \quad (12.100)$$

then (conditional) **transition** probability is a model pre-assumed, say

$$s_{t+1} \sim p_t(\cdot | s_t, a_t) \quad (12.101)$$

#### □ Optimization Target

The optimization target (in each step) is **reward function**

$$r_t(s_t, s_{t+1} | a_t) \quad (12.102)$$

The ‘cumulative reward’ from step  $t$  is denoted  $\mathcal{V}_{t \rightsquigarrow T}^{108}$

$$\mathcal{V}_{t \rightsquigarrow T}^{\pi_{t:T}}(s_t) = \mathbb{E}_{s_{t+1} \sim p(\cdot | s_t, a_t = \pi_t(s_t))} \left[ r_t(s_t, s_{t+1} | a_t = \pi_t(s_t)) + \gamma \mathcal{V}_{(t+1) \rightsquigarrow T}^{\pi_{(t+1):T}}(s_{t+1}) | s_t \right] \quad (12.103)$$

where **discount factor**  $\gamma < 1$  is induced to focus on recent rewards. By expanding all iteration terms we have

$$\mathcal{V}_{t \rightsquigarrow T}^{\pi_{t:T}}(s_t) = \mathbb{E}_{s_{(t+1):(T+1)}} \left[ \sum_{\tau=t}^T \gamma^{\tau-t} r_\tau(s_\tau, s_{\tau+1} | a_\tau = \pi_\tau(s_\tau)) | s_t \right] \quad (12.104)$$

and the final optimize goal is maximize total reward  $\mathcal{V}$

$$\pi_{0:T}^* = \arg \max_{\pi_{0:T}} \mathbb{E}_{s_0 \sim p_0(\cdot)} [\mathcal{V}_{0 \rightsquigarrow T}^{\pi_{0:T}}(s_0)] \quad (12.105)$$

$$= \arg \max_{\pi_{0:T}} \mathbb{E}_{s_{0:(T+1)}} \left[ \sum_{\tau=0}^T \gamma^\tau r_\tau(s_\tau, s_{\tau+1} | a_\tau = \pi_\tau(s_\tau)) \right] \quad (12.106)$$

Comments:

<sup>108</sup>In this subsection I usually use the superscript  $\cdot^{\pi_{t:T}}$  to specify the optimize target.

- The joint distribution of  $s_{t+1, T+1}$  has a complicated dependence on  $p_\tau(\cdot | s_\tau, a_\tau)$ , making the optimization hard to solve directly.
- Actually when making decision we should consider a complete process, i.e.  $T \rightarrow \infty$ , but note that with  $\gamma < 1$ , reward at far future is dispensable if rewards are upper-bounded  $r_\tau(s_\tau, s_{\tau+1} | a_\tau) \leq \tilde{r}$ , then

$$\sum_{\tau=T}^{\infty} \gamma^\tau r_\tau(s_\tau, s_{\tau+1} | a_\tau = \pi_\tau(s_\tau)) \leq \tilde{r} \frac{\gamma^T}{1-\gamma} \quad (12.107)$$

which can be bounded below  $\varepsilon \tilde{r}$  for a large enough **Effective Length**  $T_\varepsilon$

$$\tilde{r} \frac{\gamma^T}{1-\gamma} < \varepsilon \tilde{r} \Rightarrow T_\varepsilon \approx \frac{\log[(1-\gamma)\varepsilon]}{\log \gamma} \sim \mathcal{O}\left(\frac{1}{1-\gamma} \log \frac{1}{\varepsilon(1-\gamma)}\right) \sim \mathcal{O}\left(\frac{1}{1-\gamma}\right) \quad (12.108)$$

#### □ Algorithm

Solving all  $\pi_{0:T}$  jointly in [equation. 12.105](#) is complex. It would be wiser to use the iteration form [equation. 12.103](#) and *separate decision making*  $a_t$  and *processing*  $p(\cdot | s_t, a_t)$ . With expected rewards denoted

$$R_t(s_t, a_t) = \mathbb{E}_{s_{t+1} \sim p(\cdot | s_t, a_t)} [r_t(s_t, s_{t+1} | a_t) | s_t, a_t] \quad (12.109)$$

total reward  $\mathcal{V}_{t \rightsquigarrow T}$  could be written as

$$\mathcal{V}_{t \rightsquigarrow T}^{\pi_{t:T}}(s_t) = \mathbb{E}_{s_{t+1} \sim p(\cdot | s_t, a_t \sim \pi_t(s_t))} \left[ r_t(s_t, s_{t+1} | a_t \sim \pi_t(s_t)) + \gamma \mathcal{V}_{(t+1) \rightsquigarrow T}^{\pi_{(t+1):T}}(s_{t+1}) \middle| s_t \right] \quad (12.110)$$

$$= \mathbb{E}_{a_t \sim \pi(\cdot | s_t)} \left[ R_t(s_t, a_t) + \gamma \mathbb{E}_{s_{t+1} \sim p(\cdot | s_t, a_t)} \left[ \mathcal{V}_{(t+1) \rightsquigarrow T}^{\pi_{(t+1):T}}(s_{t+1}) \middle| s_t, a_t \right] \middle| s_t \right] \quad (12.111)$$

with the red part as **State-Value Function**, or **V-value**; the blue part as **Action-Value Function**, or **Q-value**

$$V_{t \rightsquigarrow T}^{\pi_{t:T}}(s_t) = \mathbb{E}_{a_t \sim \pi(\cdot | s_t)} \left[ Q_{t \rightsquigarrow T}^{\pi_{t:T}}(s_t, a_t) \middle| s_t \right] \quad (12.112)$$

$$Q_{t \rightsquigarrow T}^{\pi_{(t+1):T}}(s_t, a_t) = R_t(s_t, a_t) + \gamma \mathbb{E}_{s_{t+1} \sim p(\cdot | s_t, a_t)} \left[ V_{(t+1) \rightsquigarrow T}^{\pi_{(t+1):T}}(s_{t+1}) \middle| s_t, a_t \right] \quad (12.113)$$

Comments:

- The decision process  $(s_0, a_0) \rightsquigarrow (s_1, a_1) \rightsquigarrow \dots \rightsquigarrow (s_T, a_T)$  is Markovian in  $t = 0 \rightarrow T$  sense, while the reward propagation  $V_{T \rightsquigarrow T} \rightsquigarrow Q_{(T-1) \rightsquigarrow T} \rightsquigarrow V_{(T-1) \rightsquigarrow T} \rightsquigarrow \dots \rightsquigarrow Q_{0 \rightsquigarrow T} \rightsquigarrow V_{0 \rightsquigarrow T}$  is ‘Markovian’ in  $t = T \rightarrow 0$  sense. i.e. solution to optimal  $\pi^*$  obtained by maximizing total reward should go backward.
- Duality of optimal  $\{V_{t \rightsquigarrow T}^{\pi_{t:T}}\}_{t=0}^T$  (V-learning) and optimal  $\{Q_{t \rightsquigarrow T}^{\pi_{t:T}}\}_{t=0}^T$  (Q-learning): With  $R_t(s_t, a_t)$  actually a given function (for given model  $p(s_{\tau+1} | s_\tau, a_\tau)$ ),

$$\begin{cases} V_{t \rightsquigarrow T}^{\pi_{t:T}}(s_t) = \mathbb{E}_{a_t \sim \pi(\cdot | s_t)} \left[ R_t(s_t, a_t) + \gamma \mathbb{E}_{s_{t+1} \sim p(\cdot | s_t, a_t)} \left[ V_{(t+1) \rightsquigarrow T}^{\pi_{(t+1):T}}(s_{t+1}) \middle| s_t, a_t \right] \middle| s_t \right] \\ Q_{t \rightsquigarrow T}^{\pi_{(t+1):T}}(s_t, a_t) = R_t(s_t, a_t) + \gamma \mathbb{E}_{s_{t+1} \sim p(\cdot | s_t, a_t)} \left[ \mathbb{E}_{a_{t+1} \sim \pi(\cdot | s_{t+1})} \left[ Q_{(t+1) \rightsquigarrow T}^{\pi_{(t+1):T}}(s_{t+1}, a_{t+1}) \middle| s_{t+1} \right] \middle| s_{t+1}, a_{t+1} \right] \end{cases} \quad (12.114)$$

are equivalent, with the same optimization core  $\mathbb{E}_{a_t \sim \pi(\cdot | s_t)} [\cdot | s_t]$ .

△ Value function iteration for optimal policy  $\pi^*$ :

$$\pi_t^*(s) = \arg \max_a Q_t^*(s, a), \quad t = T, T-1, \dots, 0 \quad (12.115)$$



**Algorithm** Value Iteration

1.  $V_{T+1}^* \equiv 0$
2. for  $t = T, T-1, \dots, 1$ 
  - (a)  $Q$ -expectation step:

$$Q_t^*(s, a) = R_t(s, a) + \gamma \mathbb{E}_{\tilde{s} \sim p(\cdot | s, a)} [V_{t+1}^*(\tilde{s}) | s, a] \quad (12.116)$$

- (b)  $V$ -Optimal step:

$$\begin{cases} \pi_t^*(s) = \arg \max_a Q_t^*(s, a) \\ V_t^*(s) = \max_a Q_t^*(s, a) = Q_t^*(s, \pi_t^*(s)) \end{cases} \quad (12.117)$$

i.e. a  $(Q_t, V_t)$  ‘backward propagation’.

□ **Q-Learning**

Motivation: for some more complex cases, e.g.

- The functional form of reward  $r_t(s_t, s_{t+1} | a_t)$  or  $R_t(s_t, a_t)$  is unknown
- The transition probability  $s_{t+1} \sim p(\cdot | s_t, a_t)$  is unknown
- The phase space is too large to compute point wise

Note that the above optimize process **equation. 12.117** is an optimization w.r.t.  $Q_t(\cdot, \cdot)$ , we can first learn the functional form of  $Q(\cdot, \cdot)$  (or its function approximation), and thus get the policy  $\pi^*$ . The  $Q$ -learning process can have the following form:

$$\hat{Q}^{(\tau+1)}(s_t, a_t) \leftarrow \underbrace{\hat{Q}^{(\tau)}(s_t, a_t)}_{\text{current value}} + \alpha \cdot \underbrace{\left( R_t(s_t, a_t) + \gamma \cdot \underbrace{\max_a \hat{Q}^{(\tau)}(s_{t+1}, a)}_{\text{estimate of optimal future value}} - \underbrace{\hat{Q}^{(\tau)}(s_t, a_t)}_{\text{current value}} \right)}_{\text{new value (temporal difference target)}} \quad (12.118)$$

with some known *final/terminal* state  $\{s_{\text{final}}\}$ , where  $Q(s_{\text{final}}, a) \equiv 0, \forall a$

**Algorithm** Q-Learning

1. Initialize a tentative  $Q_t^{(0)}(\cdot, \cdot)$ , say  $Q \equiv 0$
2. for  $\tau = 0, 1, 2, \dots$  until  $Q(\cdot, \cdot)$  converge:
  - (a) Initialize some  $s_1$
  - (b) for  $t = 1, 2, \dots$  until  $s_t \in \{s_{\text{final}}\}$ : optimize the function form (approximation)  $Q(\cdot, \cdot)$

$$\hat{Q}^{(\tau+1)}(s_t, a_t) \leftarrow \hat{Q}^{(\tau)}(s_t, a_t) + \alpha \left( R_t(s_t, a_t) + \gamma \max_a \hat{Q}_{t+1}^{(\tau)}(s_{t+1}, a) - \hat{Q}_t^{(\tau)}(s_t, a_t) \right) \quad (12.119)$$

$$s_{t+1} \leftarrow p_t(s_t, a_t) \quad (12.120)$$

### 12.3.3 Karhunen-Loève Expansion

Karhunen-Loève Expansion (KL Expansion) is a continuous version of PCA [section. 4.3](#). The idea is a decomposition

$$X(t) = \sum_i X_i \phi_i(t) \quad (12.121)$$

i.e. we add an extra step in mapping

$$X(\cdot) : \Omega \mapsto \{X_i\} \mapsto \mathcal{T} \times \mathbb{R} \quad (12.122)$$

a special set of  $\{X_i, \phi_i\}$  is given by KL expansion.

#### □ Derivations

First note that  $R(s, t) := \mathbb{E}[X(s)X(t)]$  is a Kernel (see [equation. 9.77](#)), with positive semi-definition and symmetry. Then by Mercer's Thm., it has eigen-function decomposition

$$R(s, t) = \sum_i \lambda_i \phi_i(s) \phi_i(t) \Leftrightarrow \langle R(s, \cdot), \phi_i \rangle = \lambda_i \phi_i(s) \quad (12.123)$$

where eigen functions are orthonormal

$$\langle \phi_i, \phi_j \rangle := \int_{\mathcal{T}} \phi_i(\tau) \phi_j(\tau) d\tau = \delta_{ij} \quad (12.124)$$

using  $\{\phi_i\}$  as function basis, KL coefficients are r.v.

$$X_i = \langle X_t, \phi_i \rangle \quad (12.125)$$

with

$$\mathbb{E}[X_i X_j] = \langle \phi_i | X_t \rangle \langle X_t | \phi_j \rangle = \langle \phi_i | R(s, t) | \phi_j \rangle = \delta_{ij} \lambda_i \quad (12.126)$$

#### □ Other Concepts

- Total energy:

$$E = \mathbb{E}[\langle X_t, X_t \rangle] = \sum_i \lambda_i \quad (12.127)$$

- Rank:  $\text{rank}(\{\mathbb{E}[X_i X_j]\}) = \#(\lambda_i \neq 0)$  is also the rank of the process.

### 12.3.4 Kalman Filter

#### □ Model

**Kalman Filter** is an auto-regressive / iterative filter for estimating the **state**  $x_t$  from **observable**<sup>109</sup>  $z_t$ . The model structure, as in [figure. 15](#), is a Hidden Markov Model (HMM) with linear operator.

$$\text{State: } x_k = F_k x_{k-1} + w_k \quad (12.128)$$

$$\text{Observable: } z_k = H_k x_k + v_k \quad (12.129)$$

<sup>109</sup>Here I prefer the name as in Quantum mechanics 'Observable'.

where  $w_k, v_k$  is noise / random error, usually with (multivariate) Normal distribution

$$w_k \sim N(0, Q_k), \quad v_k \sim N(0, R_k) \quad (12.130)$$

the initial state denoted

$$x_0 \sim N(\hat{x}_{0|0}, P_{0|0}) \quad (12.131)$$

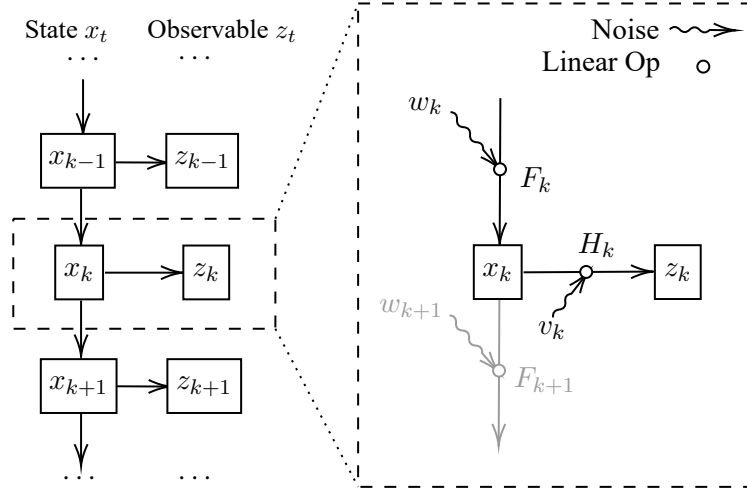


图 15: HMM structure of Kalman Filter

#### □ Algorithm

Motivation: what we could observe is  $\{z_k\}$  sequence, with pre-specified  $\{F_k, H_k, Q_k, R_k\}$ , which are part of the model. We hope to (linearly) estimate the value and variance of the hidden state  $x_k$

$$\text{value: } \hat{x}_{k|k-1} := L(x_k | z_1 \dots z_{k-1}) \quad (12.132)$$

$$\text{variance: } P_{k|k-1} := \text{var}(x_k - \hat{x}_{k|k-1}) \quad (12.133)$$

---

#### Algorithm Kalman Filter

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1. Initial State:  $x_0 \sim N(\hat{x}_{0|0}, P_{0|0})$ ; Model given  $\{F_k, H_k, Q_k, R_k\}$ ;

2. for  $k = 1, 2, \dots$

(a) State Predict:  $\cdot_{k-1|k-1} \mapsto \cdot_{k|k-1}$

$$\text{prior state: } \hat{x}_{k|k-1} = F_k \hat{x}_{k-1|k-1} \quad (12.134)$$

$$\text{prior cov: } P_{k|k-1} = F_k P_{k-1|k-1} F_k' + Q_k \quad (12.135)$$

(b) Information Update: weighting btw.  $\cdot_{k|k-1}$  and  $\cdot_k$

$$\text{innovation seq: } \tilde{z}_k = z_k - H_k \hat{x}_{k|k-1} \quad (12.136)$$

$$\text{innovation cov: } S_k = H_k P_{k|k-1} H_k' + R_k \quad (12.137)$$

$$\text{(Optimal) Kalman gain: } K_k = P_{k|k-1} H_k' S_k^{-1} \quad (12.138)$$

$$= P_{k|k-1} H_k' (H_k P_{k|k-1} H_k' + R_k)^{-1} \quad (12.139)$$


---

(c) State Update:  $\cdot_{k|k-1} \mapsto \cdot_{k|k}$

$$\text{posterior state: } \hat{x}_{k|k} = \hat{x}_{k|k-1} + K_k \tilde{z}_k \quad (12.140)$$

$$= \hat{x}_{k|k-1} + K_k (z_k - H_k \hat{x}_{k|k-1}) \quad (12.141)$$

$$= (I - K_k H_k) \hat{x}_{k|k-1} + K_k z_k \quad (12.142)$$

$$\text{posterior cov: } P_{k|k} = (I - K_k H_k) P_{k|k-1} (I - K_k H_k)' + K_k R_k K_k' \quad (12.143)$$

$$= (I - K_k H_k) P_{k|k-1} \quad (12.144)$$

## □ Derivation Details

- Key concepts in Kalman Filter:

$$\text{prior state: } \hat{x}_{k|k-1} = L(x_k | z_1 \dots z_{k-1}) \quad (12.145)$$

$$\text{prior covariance: } P_{k|k-1} = \text{var}(x_k - \hat{x}_{k|k-1}) \quad (12.146)$$

$$\text{posterior state: } \hat{x}_{k|k} = L(x_k | z_1 \dots z_k) \quad (12.147)$$

$$\text{posterior covariance: } P_{k|k} = \text{var}(x_k - \hat{x}_{k|k}) \quad (12.148)$$

$$\text{Kalman gain: } K_k \quad (12.149)$$

(a1) prior state prediction

$$\hat{x}_{k|k-1} = L(x_k | z_1 \dots z_{k-1}) = L(F_k x_{k-1} + w_k | z_1 \dots z_{k-1}) = F_k \hat{x}_{k-1|k-1} \quad (12.150)$$

(a2) prior covariance prediction

$$P_{k|k-1} = \text{var}(x_k - \hat{x}_{k|k-1}) = \text{var}(F_k(x_{k-1} - \hat{x}_{k-1|k-1}) + w_k) = F_k P_{k-1|k-1} F_k' + Q_k \quad (12.151)$$

(b1) innovation sequence of  $z_k$

$$\tilde{z}_k = z_k - L(z_k | z_1 \dots z_{k-1}) = z_k - L(H_k x_k + v_k | z_1 \dots z_{k-1}) = z_k - H_k \hat{x}_{k|k-1} \quad (12.152)$$

(b2) innovation sequence variance

$$S_k := \text{var}(\tilde{z}_k) = \text{var}(z_k - H_k \hat{x}_{k|k-1}) = \text{var}(H_k(x_k - \hat{x}_{k|k-1}) + v_k) = H_k P_{k|k-1} H_k' + R_k \quad (12.153)$$

(b3) Optimal Kalman gain is obtained by

$$\hat{x}_{k|k} = \hat{x}_{k|k-1} + L(x_k - \mathbb{E}[x] | \tilde{z}_k) = \hat{x}_{k|k-1} + \text{cov}(x_k, \tilde{z}_k) \text{var}(\tilde{z}_k)^{-1} \tilde{z}_k := \hat{x}_{k|k-1} + K_k \tilde{z}_k \quad (12.154)$$

i.e. Optimal Kalman gain in the combination coefficient in MMSE.

$$K_k = \text{cov}(x_k, \tilde{z}_k) \text{var}(\tilde{z}_k)^{-1} = \text{cov}(x_k, H_k(x_k - \hat{x}_{k|k-1}) + v_k) S_k^{-1} \quad (12.155)$$

$$= \text{cov}(x_k - \hat{x}_{k|k-1}, x_k - \hat{x}_{k|k-1}) H_k' S_k^{-1} \quad (12.156)$$

$$= P_{k|k-1} H_k' S_k^{-1} \quad (12.157)$$

here we use the property of MMSE

$$\text{cov}(\hat{x}_{k|k-1}, x_k - \hat{x}_{k|k-1}) = 0 \quad (12.158)$$

(c1) posterior state update

$$\hat{x}_{k|k} = \hat{x}_{k|k-1} + K_k \tilde{z}_k = (I - K_k H_k) \hat{x}_{k|k-1} + K_k z_k \quad (12.159)$$

(c2) posterior variance update

$$P_{k|k} = \text{var}(x_k - \hat{x}_{k|k}) = \text{var}(x_k - \hat{x}_{k|k-1} - K_k(z_k - H_k \hat{x}_{k|k-1})) \quad (12.160)$$

$$= \text{var}(x_k - \hat{x}_{k|k-1} - K_k(H_k x_k + v_k - H_k \hat{x}_{k|k-1})) \quad (12.161)$$

$$= \text{var}((I - K_k H_k)(x_k - \hat{x}_{k|k-1}) - K_k v_k) \quad (12.162)$$

$$= (I - K_k H_k) P_{k|k-1} (I - K_k H_k)' + K_k R_k K_k' \quad (12.163)$$

further if  $K_k$  takes optimal Kalman gain,

$$K_k S_k K_k' = P_{k|k-1} H_k' K_k' \quad (12.164)$$

we have a simplification

$$P_{k|k} = (I - K_k H_k) P_{k|k-1} (I - K_k H_k)' + K_k R_k K_k' \quad (12.165)$$

$$= P_{k|k-1} - K_k H_k P_{k|k-1} - P_{k|k-1} H_k' K_k' + K_k (H_k P_{k|k-1} H_k' + R_k) K_k' \quad (12.166)$$

$$= P_{k|k-1} - K_k H_k P_{k|k-1} - P_{k|k-1} H_k' K_k' + K_k S_k K_k' \quad (12.167)$$

$$= (I - K_k H_k) P_{k|k-1} \quad (12.168)$$

## □ Comments

- Optimality of Kalman Filter as a MMSE: in [equation. 12.160](#), posterior variance does **not** depend on a concrete form of Kalman gain, thus in which Kalman filter can be selected as some other ones  $\tilde{K}_k$  (e.g. to avoid numerical instability). The optimal Kalman gain is the one that minimizes  $\text{tr}(P_{k|k})$

$$K_k = \arg \min_K \text{tr}((I - K H_k) P_{k|k-1} (I - K H_k)' + K R_k K') \quad (12.169)$$

obtained by<sup>110</sup>

$$\frac{\partial \text{tr}(P_{k|k})}{\partial K} = -2(H_k P_{k|k-1})' + 2K_k S_k = 0 \Rightarrow K_k = P_{k|k-1} H_k' S_k^{-1} \quad (12.170)$$

- Role of Kalman gain  $K_k$ : in posterior update [equation. 12.159](#) we can see that  $K_k$  looks like a weighting factor btw. history information  $\hat{x}_{k|k-1}$  and new observation  $z_k$ .

$$\hat{x}_{k|k} = (I - K_k H_k) \hat{x}_{k|k-1} + K_k z_k \quad (12.171)$$

and note that the Kalman gain update

$$P_{k|k-1} = F_k P_{k-1|k-1} F_k' + Q_k \quad (12.172)$$

$$S_k = H_k P_{k|k-1} H_k' + R_k \quad (12.173)$$

$$K_k = P_{k|k-1} H_k' S_k^{-1} \quad (12.174)$$

$$P_{k|k} = (I - K_k H_k) P_{k|k-1} \quad (12.175)$$

<sup>110</sup>Matrix differentiation see [section. 4.1.2](#)

only involve  $\{F_k, H_k, Q_k, R_k\}$  and initial  $P_{0|0}$ . It means Kalman gain  $K_k$  could be computed offline. In actual application scenario we can just compute state iteratively

$$\hat{x}_{k|k-1} = F_k \hat{x}_{k-1|k-1} \quad (12.176)$$

$$\hat{x}_{k|k} = (I - K_k H_k) \hat{x}_{k|k-1} + K_k z_k \quad (12.177)$$

$$P_{k|k-1} = F_k P_{k-1|k-1} F_k' + Q_k \quad (12.178)$$

$$P_{k|k} = (I - K_k H_k) P_{k|k-1} \quad (12.179)$$

- Asymptotic form: when step  $k \rightarrow \infty$ , we may have limit

$$F_k \rightarrow F, \quad H_k \rightarrow H, \quad Q_k \rightarrow Q, \quad R_k \rightarrow R \quad (12.180)$$

then Kalman filter and variance estimation have asymptotic form by solving

$$P_\infty = F \left( P_\infty - P_\infty H' (H P_\infty H + R)^{-1} H P_\infty \right) F' + Q \quad (12.181)$$

$$K_\infty = P_\infty H' (H P_\infty H + R)^{-1} \quad (12.182)$$

and the asymptotic update

$$\hat{x}_{k+1} = F (I - K_\infty H) \hat{x}_k + F K_\infty z_k \quad (12.183)$$

- Extended Kalman Filter (EKF): Kalman filter assumes a linear model with noise. Usually it's a good-enough approximator to the real case. For non-linear case, i.e. Extended Kalman filter, has model

$$\text{State: } x_k = f_k(x_{k-1}) + w_k \quad (12.184)$$

$$\text{Observable: } z_k = h_k(x_k) + v_k \quad (12.185)$$

the update could be obtained by replacement

$$F_k = \frac{\partial}{\partial x} f_k(\hat{x}_{k-1|k-1}), \quad H_k = \frac{\partial}{\partial x} h_k(\hat{x}_{k|k-1}) \quad (12.186)$$

- Kalman-Bucy Filter is the continuous time version of Kalman filter, with model

$$\text{State: } \frac{dx(t)}{dt} = F(t)x(t) + w(t) \quad (12.187)$$

$$\text{Observable: } z(t) = H(t)x(t) + v(t) \quad (12.188)$$

where  $w(t)$ ,  $v(t)$  are white noise.

Kalman update:

$$\frac{d\hat{x}(t)}{dt} = (F(t) - K(t)H(t))\hat{x}(t) + K(t)z(t) \quad (12.189)$$

$$\frac{dP(t)}{dt} = F(t)P(t) + P(t)F(t)' + Q(t) - K(t)R(t)K(t)' \quad (12.190)$$

with Kalman gain

$$K(t) = P(t)H(t)'R(t)^{-1} \quad (12.191)$$

### 12.3.5 Linear Time Invariant Systems

Linear Time Invariant Systems (LTI Systems) models data generation process as a convolution

$$x(t) = \int_{\mathbb{R}} z(\tau)h(t - \tau) d\tau = (z * g)(t) \quad (12.192)$$

where  $\int$  for linear, and  $h(t - \tau)$  for time-invariant.

LTI systems could be conveniently parsed with Fourier Transform, introduced in [section. 12.4.3](#).

#### □ Cross Correlation Structure

Usually we consider weak stationary case, with notation:

$$\mu_X, \quad \mu_Z, \quad R_Z(t) = \mathbb{E}[z(s)z(s+t)], \quad \forall s, \quad R_{XZ}(t) = \mathbb{E}[x(s)z(s+t)], \quad \forall s \quad (12.193)$$

corresponding Fourier transform:

$$R_Z(t) \doteq S_Z(\omega), \quad R_{XZ}(t) \doteq S_{XZ}(\omega), \quad h(t) \doteq H(\omega) \quad (12.194)$$

Relations:

$$\mu_X = \sqrt{2\pi}\mu_Z H(0) \quad (12.195)$$

$$R_{XZ}(t) = (R_Z * h)(t) \quad (12.196)$$

$$R_X(t) = (h * R_Z * \tilde{h})(t), \quad \tilde{h}(\tau) = h(-\tau) \quad (12.197)$$

$$S_{XZ}(\omega) = \sqrt{2\pi}S_Z(\omega)H(\omega) \quad (12.198)$$

$$S_X(\omega) = 2\pi S_Z(\omega)|H(\omega)|^2 \quad (12.199)$$

### 12.3.6 Wiener Filter

Goal of Wiener Filter is to estimate some  $x_u$  from  $z_t : t \in [a, b]$  with a linear function in MMSE sense  $\hat{x}_u = L(x_u|z_t : t \in [a, b])$ :

$$\hat{x}_u = \int_a^b z_\tau h(\tau, u) d\tau, \quad w.r.t. h(\cdot) = \arg \min_h \mathbb{E}[(x_u - \hat{x}_u)^2] \quad (12.200)$$

the solution, as explained in [section. 12.4.1](#), satisfies  $\mathbb{E}[(x_u - \hat{x}_u)z_t] = 0, \quad \forall t \in [a, b]$ , which yields

$$R_{XZ}(u, t) = \int_a^b R_Z(t, \tau)h(u - \tau) d\tau, \quad \forall t \in [a, b] \quad (12.201)$$

usually we also consider weak stationary case, with  $[a, b] = \mathbb{R}$

$$R_{XZ}(u - t) = \int_{\mathbb{R}} R_Z(\tau - t)h(u - \tau) d\tau, \quad \forall t \in [a, b] \quad (12.202)$$

□ **Non-Causal Solution** A general solution  $L(x_u|z_t : t \in \mathbb{R})$  is easily obtained by Fourier transform, with the convolution expression of estimator

$$S_{XZ}(\omega) = \sqrt{2\pi}S_Z(\omega)H(\omega) \Rightarrow H(\omega) = \frac{S_{XZ}(\omega)}{\sqrt{2\pi}S_Z(\omega)} \quad (12.203)$$

with MSE<sup>111</sup>

$$\text{MSE} = \int_{\mathbb{R}} S_X(\omega) - \frac{|S_{XZ}(\omega)|^2}{S_Z(\omega)} d\omega \quad (12.204)$$

### □ Causal Solution

Causal solution demands that estimation cannot use future information, modelled as

$$\hat{x}_T = L(x_T | z_t : t \in (-\infty, 0]), \quad T > 0$$

i.e.

$$\begin{aligned} \hat{x}_T &= \int_{\mathbb{R}} z_\tau h(-\tau) d\tau, \quad w.r.t. h(\varsigma) = h(\varsigma)\eta(\varsigma) \\ R_{XZ}(T+t) &= \int_{\mathbb{R}} R_Z(\tau+t)h(-\tau) d\tau, \quad \forall t \geq 0 \end{aligned}$$

MMSE condition

$$[e^{i\omega T} S_{XZ}]_+ = [S_Z(\omega)H(\omega)]_+$$

where  $[\cdot]_+$  corresponds to the causal component of FT

$$\begin{aligned} f(t) &= \eta(t)f(t) + (1 - \eta(t))f(t) \triangleq [F(\omega)]_+ + [F(\omega)]_- \\ [F(\omega)]_+ &= \frac{1}{\sqrt{2\pi}} \int_0^\infty f(t)e^{-i\omega t} dt \end{aligned}$$

with factor decomposition  $S_Z(\omega) = S_Z^+(\omega)S_Z^-(\omega)$ , where  $S_Z^+$  is a causal function<sup>112</sup>, we have solution

$$H(\omega) = \frac{1}{S_Z^+} \left[ \frac{e^{i\omega T} S_{XZ}}{S_Z^-} \right]_+$$

Notes on causal function:

- Convolution is causal invariant:

$$(\eta f * \eta g)(t) = \int_0^\infty f(\tau)g(t-\tau) d\tau = 0 \text{ if } t < 0$$

## Section 12.4 Miscellanea

### 12.4.1 Minimum Mean Squared Estimator

Motivation: Here's a signal transmission process in which source is  $X \sim f_X$  and observation is  $\vec{Z} \sim f_Z$ , we need to find a (theoretically best) information process function  $g(\cdot)$  such that we can reproduce  $X$  with  $g(\vec{Z}) \in \mathcal{F}$  with

<sup>111</sup>Derivation uses Parseval's Thm. [equation. 12.233](#).

<sup>112</sup>An illustration: since convolution function is causal invariant, then

$$e^H = \sum_{i=0}^{\infty} \frac{H^i}{i!} \triangleq \sum_{i=0}^{\infty} \frac{(*h)^i}{i!}$$

is also causal invariant, i.e.  $H = [H]_+ \Rightarrow e^H = [e^H]_+$ , then we could have

$$S = S^+ S^- = e^{s^+ + s^-} = e^{[s]_+ + [s]_-}$$



minimum ‘error’ (Note that  $X$  and  $\vec{Z}$  can be dependent)., i.e.

$$\hat{g} = \arg \min_{g(\cdot) \in \mathcal{F}} \mathbb{E} \left[ (X - g(\vec{Z}))^2 \right] \quad (12.205)$$

which is the **Minimum Mean Squared Estimator (MMSE)**. <sup>113</sup>

#### □ General Solution to MMSE

The solution to MMSE is that

$$\hat{g}(\cdot) \text{ s.t. } \begin{cases} \hat{g}(\vec{Z}) \in \mathcal{F}(Z) \\ e := X - \hat{g}(\vec{Z}) \perp h(\vec{Z}), \quad \forall h(\vec{Z}) \in \mathcal{F}(Z) \end{cases} \quad (12.207)$$

here  $\perp$  in the sense that  $i \perp j \Leftrightarrow \mathbb{E}[ij] = 0$

Denote  $\mathcal{F}(Z) \ni g(Z) = \hat{g}(Z) + ch(Z)$ ,  $h(Z) \in \mathcal{F}(Z)$ , then

$$\mathbb{E}[(X - g(Z))^2] = \mathbb{E}[(X - \hat{g}(Z) - ch(Z))^2] \quad (12.208)$$

$$= \mathbb{E}[(X - \hat{g}(Z))^2] - 2c\mathbb{E}[(X - \hat{g}(Z))h(Z)] + c^2\mathbb{E}[h(Z)^2] \quad (12.209)$$

- If  $X - \hat{g}(\vec{Z}) \perp h(\vec{Z})$ :  $\mathbb{E}[(X - g(Z))^2] = \mathbb{E}[(X - \hat{g}(Z))^2] + c^2\mathbb{E}[h(Z)^2] \geq \mathbb{E}[(X - \hat{g}(Z))^2]$
- If  $X - \hat{g}(\vec{Z}) \not\perp h(\vec{Z})$ , then for  $|c|$  small enough we could have  $\mathbb{E}[(X - g(Z))^2] < \mathbb{E}[(X - \hat{g}(Z))^2]$ .

which gives that the above condition is necessary and sufficient.

The above expression is similar to the projection operator onto space  $\mathcal{F}$ , i.e.

$$\hat{g}(\cdot) = \Pi_{\mathcal{F}(\cdot)}(X), \quad \begin{cases} \Pi_{\mathcal{F}(\cdot)}(X) \in \mathcal{F} \\ X - \Pi_{\mathcal{F}(\cdot)}(X) \perp \mathcal{F} \end{cases} \quad (12.210)$$

#### □ Properties of Projection Operator $\Pi_{\mathcal{V}}$ (where function space $\mathcal{F}$ is a kind of linear space $\mathcal{V}$ )

- Linearity

$$\Pi_{\mathcal{V}}(aX + bY) = a\Pi_{\mathcal{V}}(X) + b\Pi_{\mathcal{V}}(Y) \quad (12.211)$$

- Project within subspace: for  $\mathcal{V}_2 \subset \mathcal{V}_1$

$$\Pi_{\mathcal{V}_2}(X) = \Pi_{\mathcal{V}_2}(\Pi_{\mathcal{V}_1}(X)) \quad (12.212)$$

- Projection onto orthogonal space: for  $\mathcal{V}_1 \perp \mathcal{V}_2$

$$\Pi_{\mathcal{V}_1 \oplus \mathcal{V}_2}(X) = \Pi_{\mathcal{V}_1}(X) + \Pi_{\mathcal{V}_2}(X) \quad (12.213)$$

#### □ Important Cases

<sup>113</sup>**Note:** the function space  $\mathcal{F}(\vec{Z})$  (by default) is the arbitrary measurable function space  $:= \mathcal{V}(\vec{Z})$ , but you can specifically select a proper one, e.g. linear combination of some power function  $\mathbb{V}(1, \vec{Z}, \vec{Z}^2) := \{a + bZ + cZ^2\}_{a,b,c \in \mathbb{R}} \subset \mathcal{F}(\vec{Z})$ .

I am not quite sure (actually I believe it's wrong lol) but maybe for some commonly used function form, we could view that

$$\mathcal{V}(\vec{Z}) \approx \mathbb{V}(\{\vec{Z}^p\}_{p=-\infty}^{\infty}) \quad (12.206)$$

- $\mathcal{F}(Z) = \mathcal{V}(Z)$ : Solution is

$$\mathbb{E}[X|Z] \quad (12.214)$$

in which

$$\begin{cases} \mathbb{E}[X|Z] \in \mathcal{F}(Z) \\ \mathbb{E}[(X - \mathbb{E}[X|Z])g(Z)] = \mathbb{E}[Xg(Z)] - \mathbb{E}[\mathbb{E}[g(Z)X|Z]] = 0 \end{cases} \quad (12.215)$$

- $\mathcal{F}(Z) = \text{const}$ : Solution is

$$\mathbb{E}[X] \quad (12.216)$$

in which

$$\begin{cases} \mathbb{E}[X] \in \mathcal{R} \\ \mathbb{E}[(X - \mathbb{E}[X])|\text{const}] = 0 \end{cases} \quad (12.217)$$

which is also a kind of variance definition:

$$\text{var}(X) := \min_{c \in \mathbb{R}} \mathbb{E}[(X - c)^2] \quad (12.218)$$

- $\mathcal{F}(Z) = \mathbb{V}(1, \vec{Z})$  i.e. linear combination of  $\vec{Z}$  as  $a + \vec{Z}'b$ . Solution is

$$L(X|\vec{Z}) := \mathbb{E}[X] + \text{cov}(X, \vec{Z})\text{var}(\vec{Z})^{-1}(\vec{Z} - \mathbb{E}[\vec{Z}]) \quad (12.219)$$

in which

$$\begin{cases} \mathbb{E}[X] + \text{cov}(X, \vec{Z})\text{var}(\vec{Z})^{-1}(\vec{Z} - \mathbb{E}[\vec{Z}]) \in \mathbb{V}(1, \vec{Z}) \\ \mathbb{E}[(X - L(X|\vec{Z}))(a + \vec{Z}'b)] = 0 \end{cases} \quad (12.220)$$

## 12.4.2 Conditional Independence

Conditional independence : say  $X$  and  $Z$  are conditionally independent given  $Y$ , i.e.  $X-Y-Z$

$$f_{X|YZ} = f_{X|Y} \Leftrightarrow f_{XZ|Y} = f_{X|Y}f_{Z|Y} \quad (12.221)$$

Further if  $(X, Y, Z) \sim N(\mu, \Sigma)$  (a joint Gaussian Dist.). Then

$$\text{cov}(X, Z) = \text{cov}(X, Y)\text{var}(Y)^{-1}\text{cov}(Y, Z) \quad (12.222)$$

it could be deduced using linear MMSE + innovation sequence of jointly Gaussian

$$\text{cov}(Z, X - L(X|Y)) = 0 \Rightarrow \text{cov}(X, Z) = \text{cov}(X, Y)\text{var}(Y)^{-1}\text{cov}(Y, Z) \quad (12.223)$$

or use [equation. 4.66](#), in which  $X_1 = (X, Z)$ ,  $X_2 = Y$

$$\Sigma_{X,Z|Y} = \begin{bmatrix} \Sigma_X - \Sigma_{XY} - \Sigma_Y^{-1}\Sigma_{YX} & \Sigma_{XZ} - \Sigma_{XY}\Sigma_Y^{-1}\Sigma_{YZ} \\ \Sigma_{ZX} - \Sigma_{ZY}\Sigma_Y^{-1}\Sigma_{YX} & \Sigma_Z - \Sigma_{ZY}\Sigma_Y^{-1}\Sigma_{YZ} \end{bmatrix} \Rightarrow \Sigma_{XZ} = \Sigma_{XY}\Sigma_Y^{-1}\Sigma_{YZ} \quad (12.224)$$

### 12.4.3 Fourier Transform and Convolution

#### □ Fourier Transform

Fourier Transform (FT)  $g(t) \rightleftharpoons G(\omega)$  is a link between time domain and frequency domain<sup>114</sup>

$$g(t) \rightleftharpoons G(\omega) : \begin{cases} g(t) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} G(\omega) e^{i\omega t} d\omega \\ G(\omega) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} g(t) e^{-i\omega t} dt \end{cases} \quad (12.225)$$

Fourier operator is denoted  $\mathcal{F}[\cdot]$

$$G = \mathcal{F}[g] \rightleftharpoons g = \mathcal{F}^{-1}[G] \quad (12.226)$$

Properties

- Linearity

$$\mathcal{F}[\alpha f + \beta g] = \alpha \mathcal{F}[f] + \beta \mathcal{F}[g] \quad (12.227)$$

- Time shifting / Frequency shifting

$$g(t - \tilde{t}) \rightleftharpoons G(\omega) e^{-i\omega \tilde{t}} \quad G(\omega - \tilde{\omega}) \rightleftharpoons g(t) e^{i\tilde{\omega} t} \quad (12.228)$$

- Convolution Thm.

$$\mathcal{F}[f * g] = \sqrt{2\pi} FG \quad (12.229)$$

where convolution operator is

$$(f * g)(t) = \int_{\mathbb{R}} f(\tau) g(t - \tau) d\tau \quad (12.230)$$

- Differentiation

$$\frac{d^k}{dt^k} g(t) \rightleftharpoons (i\omega)^k G(\omega) \quad (12.231)$$

- Duality

$$\mathcal{F}[\mathcal{F}[g(t)]] = \frac{1}{2\pi} g(-t) \quad (12.232)$$

- Parseval's Thm.:

$$\int_{\mathbb{R}} f(t) g^\dagger(t) dt = \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}} F(\omega_1) e^{i\omega_1 t} d\omega_1 \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}} G^\dagger(\omega_2) e^{-i\omega_2 t} d\omega_2 dt \quad (12.233)$$

$$= \int_{\omega_1} \int_{\omega_2} F(\omega_1) G^\dagger(\omega_2) \int_t \frac{1}{2\pi} e^{i(\omega_1 - \omega_2)t} dt d\omega_1 d\omega_2 \quad (12.234)$$

$$= \int_{\omega} F(\omega) G^\dagger(\omega) d\omega \quad (12.235)$$

(if the integration above can be properly defined.)

A physical intuition is the energy conservation in both time domain and spetrum domain (which is also a reason I prefer the  $\frac{1}{\sqrt{2\pi}}$  transform — no extra coefficient in this energy conservation)

$$\int_{\mathbb{R}} |f(t)|^2 dt = \int_{\omega} |F(\omega)|^2 d\omega \quad (12.236)$$

<sup>114</sup>For symmetry consideration, I usually use  $\frac{1}{\sqrt{2\pi}}$  in both transform and inversed.

Instances

- Dirac  $\delta$  function for unit impulse at  $t_0$

$$\int_{-\infty}^s \delta(t - t_0) dt = \eta(s - t_0) = \begin{cases} 0, & s < t_0 \\ 1, & s > t_0 \end{cases} \quad (12.237)$$

some commonly used definition of  $\delta$  function:

$$\delta(t) = \lim_{\Delta \rightarrow 0} \frac{1}{\Delta} \mathbb{I}_{-\Delta/2 < t < \Delta/2} \quad (12.238)$$

$$\delta(t) = \lim_{\Delta \rightarrow 0} \frac{1}{\pi \Delta} \text{sinc}(\Delta t) \quad (12.239)$$

Integration of Dirac  $\delta$  yields

$$\int_{\mathbb{R}} \delta(t - t_0) f(t) dt = f(t_0) \quad (12.240)$$

FT of Dirac  $\delta$  is harmonic wave

$$\delta(t - t_0) \triangleq \frac{1}{\sqrt{2\pi}} e^{-i\omega t_0}, \quad e^{i\omega_0 t} \triangleq \sqrt{2\pi} \delta(\omega - \omega_0) \quad (12.241)$$

- FT for periodic function  $g(t) = g(t + T)$  is Fourier series

$$\begin{cases} g(t) = \sum_{n=-\infty}^{\infty} c_n \cdot e^{i\frac{2\pi n}{T}t} \\ c_n = \frac{1}{T} \int_{\text{one period}} f(t) e^{-i\frac{2\pi n}{T}t} dt \end{cases} \quad (12.242)$$

where  $c_0$  is the DC component of the function.

- Discrete Time FT: discrete time case can be viewed as a sample of frequency  $T$  from continuous case

$$\begin{cases} g_T(t) = \sum_{n=-\infty}^{\infty} g(t) \delta(t - nT) \\ \mathcal{F}[g_T](\omega) = \frac{1}{\sqrt{2\pi}} \sum_{n=-\infty}^{\infty} g(nT) e^{-i\omega nT} \end{cases} \quad (12.243)$$

which dual with FT for periodic function.

## Chapter. XIII 贝叶斯统计导论部分

Instructor: Wanlu Deng

### Section 13.1 Calculation Preparation

Some useful calculation results / tricks are listed in this part, including r.v. distribution / integration, etc.

#### 13.1.1 Calculation

- Gamma Integral

$$\Gamma(z) \equiv \int_0^\infty t^{z-1} e^{-t} dt, \quad \operatorname{Re} t > 0$$

- scaling  $\lambda$  form

$$\int_0^\infty t^{z-1} e^{-\lambda t} dt = \frac{\Gamma(z)}{\lambda^z}$$

- Gaussian integral

$$\int_0^\infty t^p e^{-\alpha t^2} dt = \frac{\alpha^{-\frac{p+1}{2}}}{2} \Gamma\left(\frac{p+1}{2}\right)$$

with  $\alpha = \frac{1}{2\sigma^2}$  gives the normalization const of Gaussian distribution

$$\begin{cases} \int_{\mathbb{R}} e^{-\frac{t^2}{2\sigma^2}} dt = 2 \times \frac{\sqrt{2}\sigma}{2} \Gamma\left(\frac{1}{2}\right) = \sqrt{2\pi}\sigma \\ \int_{\mathbb{R}} t^2 \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{t^2}{2\sigma^2}} dt = \frac{2}{\sqrt{2\pi}\sigma} \times \frac{(2\sigma^2)^{3/2}}{2} \Gamma\left(\frac{3}{2}\right) = \sigma^2 \end{cases}$$

- complementary formula

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}$$

- special values

$$\Gamma(1) = 1, \quad \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

- recursion

$$\Gamma(z+1) = z\Gamma(z) \Rightarrow \begin{cases} \Gamma(n) = (n-1)!, & n \in \mathbb{N}^+ \\ \Gamma\left(n + \frac{1}{2}\right) = \sqrt{\pi} \frac{(2n-1)!!}{2^n}, & n \in \mathbb{N}^+ \end{cases}$$

(factorial expression)

- Beta Integral

$$B(p, q) \equiv \int_0^1 t^{p-1} (1-t)^{q-1} dt, \quad \operatorname{Re} p, \operatorname{Re} q > 0$$

- symmetry

$$B(p, q) = B(q, p)$$

- Gamma function expression

$$B(p, q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}, \text{ for integer } p, q = \frac{(p-1)!(q-1)!}{(p+q-1)!}$$

### 13.1.2 Useful Distribution Recap

- ( $n$ -dimensional) Normal distribution  $Z \sim N(\mu, \Sigma)$

$$f_Z(z) = (2\pi/\Sigma)^{-n/2} \exp \left[ -\frac{1}{2}(z - \mu)' \Sigma^{-1}(z - \mu) \right], \quad \text{with} \begin{cases} \mathbb{E}[Z] = \mu \\ \text{var}(z) = \Sigma \end{cases}$$

- Gamma distribution  $X \sim \Gamma(\alpha, \lambda)$

$$f_X(x) = \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x}, \quad \text{with} \begin{cases} \mathbb{E}[X] = \frac{\alpha}{\lambda} \\ \text{var}(X) = \frac{\alpha}{\lambda^2} \end{cases}$$

– summation

$$\Gamma\left(\sum_i \alpha_i, \lambda\right) = \sum_i \Gamma(\alpha_i, \lambda), \quad \Gamma(1, \lambda) = \varepsilon(\lambda)$$

- $\chi^2$ -distribution  $X \sim \chi_n^2$

$$f_X(x) = \frac{1}{2^{\frac{n}{2}} \Gamma(\frac{n}{2})} x^{\frac{n}{2}-1} e^{-\frac{x}{2}}, \quad \text{with} \begin{cases} \mathbb{E}[\chi_n^2] = n \\ \text{var}(\chi_n^2) = 2n \end{cases}$$

– relation to  $\Gamma$

$$\Gamma\left(\frac{n}{2}, \frac{1}{2}\right) = \chi_n^2$$

- Beta distribution  $X \sim \text{Beta}(\alpha, \beta)$

$$f_X(x) = \frac{x^{\alpha-1}(1-x)^{\beta-1}}{\text{Beta}(\alpha, \beta)} = \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1}, \quad \text{with} \begin{cases} \mathbb{E}[X] = \frac{\alpha}{\alpha + \beta} \\ \text{var}(X) = \frac{\alpha\beta}{(\alpha + \beta)^2(\alpha + \beta + 1)} \end{cases}$$

- $t$ -distribution  $T \sim t_\nu$

$$f_T(t) = \frac{\Gamma(\frac{\nu+1}{2})}{\sqrt{\nu\pi}\Gamma(\frac{\nu}{2})} \left(1 + \frac{t^2}{\nu}\right)^{-\frac{\nu+1}{2}} = \frac{1}{\sqrt{\nu}\text{Beta}(\frac{\nu}{2}, \frac{1}{2})} \left(1 + \frac{t^2}{\nu}\right)^{-\frac{\nu+1}{2}}, \quad \text{with} \begin{cases} \mathbb{E}[X] = 0 \\ \text{var}(X) = \frac{\nu}{\nu-2} \end{cases}$$

- Wishart distribution: a multi-dim version of  $\chi^2$ . If  $Z_1, \dots, Z_m$  i.i.d.  $\sim N_p(0, \Lambda)$ , then

$$W_p = \sum_{i=1}^m Z_i Z_i' \sim \text{Wishart}_m(\Lambda)$$

expression see [section. 4.2.3](#). Kernel term

$$f_W(w; p, m, \Lambda) \propto |w|^{\frac{m-p-1}{2}} \exp \left[ -\frac{1}{2} \text{tr}(\Lambda^{-1}w) \right], \quad w \in \mathbb{R}^{p \times p}$$

- Dirichlet distribution : A multi-parameter version of Beta distribution  $(x_1, x_2, \dots, x_J) \sim \text{Dirichlet}(\alpha_1, \alpha_2, \dots, \alpha_J)$ ,  
w.r.t.  $\sum_{j=1}^J x_j = 1$

$$f_X(x_1, x_2, \dots, x_J) = \frac{\Gamma\left(\sum_{j=1}^J \alpha_j\right)}{\prod_{j=1}^J \Gamma(\alpha_j)} \prod_{j=1}^J x_j^{\alpha_j-1}, \quad \sum_{j=1}^J x_j = 1$$

Beta distribution is the case of  $J = 2$ .

△ Inverse distribution. General formula for Inv- $f_X$

$$X \sim f_X(x), \quad Z = \frac{1}{X}, \quad f_Z(z) = \frac{1}{z^2} f_X\left(\frac{1}{z}\right)$$

Instances:

$$- \text{Inv-}\Gamma(\alpha, \lambda) = \frac{1}{\Gamma(\alpha, \lambda)}$$

$$f_Z(z) = \frac{\lambda^\alpha}{\Gamma(\alpha)} z^{-\alpha-1} e^{-\frac{\lambda}{z}}, \quad \text{with} \quad \begin{cases} \mathbb{E}[Z] = \frac{\lambda}{\alpha-1} \\ \text{var}(Z) = \frac{\lambda^2}{(\alpha-1)^2(\alpha-2)} \end{cases}$$

$$- (\text{scaled}) \text{Inv-}\chi^2(n, s^2) = \frac{ns^2}{\chi_n^2} = \text{Inv-}\Gamma\left(\frac{n}{2}, \frac{ns^2}{2}\right)$$

$$f_Z(z) = \frac{n^{\frac{n}{2}}}{2^{\frac{n}{2}} \Gamma(\frac{n}{2})} z^{-\frac{n}{2}-1} e^{-\frac{ns^2}{2z}}, \quad \text{with} \quad \begin{cases} \mathbb{E}[Z] = \frac{n}{n-2} s^2 \\ \text{var}(Z) = \frac{2n^2 s^4}{(n-2)^2(n-4)} \end{cases}$$

$$- Z \sim \text{Inv-Wishart}(\Lambda) \Leftrightarrow Z^{-1} \sim \text{Wishart}(\Lambda) \text{ }^{115}$$

$$f_Z(z) = f_W(z^{-1}; p, m, \Lambda) |z|^{-(p+1)} \propto |z|^{-\frac{m+p+1}{2}} \exp \left[ -\frac{1}{2} \text{tr}(\Lambda^{-1} z^{-1}) \right]$$

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<sup>115</sup>In R. and Python., functional input form is  $\text{Inv-Wishart}(\Lambda^{-1}) = (\text{Wishart}(\Lambda))^{-1}$

<sup>116</sup>Proof note for Jacobian  $\left| \frac{\partial A^{-1}}{\partial A} \right|$ : For an *arbitrary* matrix  $\left| \frac{\partial A^{-1}}{\partial A} \right| = |A|^{-2 \dim_A}$

1. First construct mapping  $\mathbb{R}^{p \times p} \mapsto \mathbb{R}^{p^2}$  e.g. by  $\vec{a}_I \equiv i p + j = A_{ij}$
2. Differentiation: where  $e_i$  is the unit vector on the  $i^{\text{th}}$  coord.

$$\begin{aligned} \frac{\partial A_{ij}^{-1}}{\partial A} &= \frac{\partial q'_i A^{-1} q_j}{\partial A} = \frac{\partial \text{tr}(A^{-1} q_j q'_i)}{\partial A} \\ &= -A^{-1} q_i q'_j A^{-1} = A_{:i}^{-1} A_{j:}^{-1} \\ \Rightarrow \frac{\partial A_{ij}^{-1}}{\partial A_{kl}} &= -A_{ki}^{-1} A_{jl}^{-1} \\ \Rightarrow \frac{\partial \vec{a}_I^{-1}}{\partial \vec{a}_J} &= -((A')^{-1} \otimes A^{-1})_{IJ} \end{aligned}$$

where  $\otimes$  is Kronecker product  $\mathbb{R}^{u \times u} \times \mathbb{R}^{v \times v} \mapsto \mathbb{R}^{uv \times uv}$  for

$$\left( \begin{matrix} U & \otimes & V \\ u \times u & & v \times v \end{matrix} \right)_{iu+j, kv+l} = U_{ik} V_{jl}$$

which has property

$$|U \otimes V| = |U|^v |V|^u$$

3. Determinant for Kronecker product

$$\left\| \frac{\partial A^{-1}}{\partial A} \right\| \equiv \left\| \frac{\partial \vec{a}^{-1}}{\partial \vec{a}} \right\| = \| -(A')^{-1} \otimes A^{-1} \| = |A|^{-2p}$$

Further here for Wishart distribution, we have a constraint for *positive definition*. The constraint causes a ‘degree of freedom reduction’ so  $\left| \frac{\partial A^{-1}}{\partial A} \right| = |A|^{-(\dim_A + 1)}$ .

## Section 13.2 Elements in Bayesian Model

Key idea: Bayesian rule

$$\mathbb{P}(X|Y) = \frac{\mathbb{P}(Y|X)\mathbb{P}(X)}{\mathbb{P}(Y)} = \frac{\mathbb{P}(Y|X)\mathbb{P}(X)}{\int_{\Omega_X} \mathbb{P}(Y|X)\mathbb{P}(X) dX}$$

In both Bayesian & Frequentist statistics, we care about updating our ‘belief’ on **parameter**.

$$\underbrace{\mathbb{P}(\theta|y)}_{\text{posterior}} = \frac{\mathbb{P}(\theta)\mathbb{P}(y|\theta)}{\mathbb{P}(y)} \propto \underbrace{\mathbb{P}(\theta)}_{\text{prior}} \underbrace{\mathbb{P}(y|\theta)}_{\text{data likelihood}}$$

### 13.2.1 Prior Selection

Selection of prior distribution  $p(\theta)$  could greatly influence posterior because it provides prior information about the parameter. The selection could be flexible, here are some frequently-used approaches

- Conjugate Prior: defined for the case that (conjugate) prior and posterior belong to the same distribution family.

<sup>117</sup>

$$p(\theta|y) \propto p(y|\theta)p(\theta) \in \mathcal{F}(\Theta), \forall p(y|\theta) \in \mathcal{F}(Y|\theta) \text{ \& } p(\theta) \in \mathcal{F}(\Theta)$$

Instances see

- Binomial Model
- Poisson Model
- Exponential Model
- UniNormal with known variance Model
- UniNormal with known mean Model
- Multinomial Model
- UniNormal Model
- MultiNormal Model

- Non-informative Prior: Jeffrey’s Prior. Idea is to choose a distribution ‘covariant’ with parameterization <sup>118</sup>. i.e. under different parameterization, say  $\theta \rightleftharpoons \phi$ , we should follow the same deduction method to get corresponding prior  $p_\theta \rightleftharpoons p_\phi$  that could covariant with parameter transform

$$p_\theta(\theta) = p_\phi(\phi(\theta)) \left| \frac{\partial \phi(\theta)}{\partial \theta} \right|$$

<sup>117</sup>Concept of distribution family see [section. 2.1](#). In this section we use notation

$$f(x; \theta) \in \mathcal{F}(\Theta)$$

to express the distribution family generated on parameter space  $(\alpha, \lambda) \in A \times \Lambda$ , e.g. family of  $\Gamma$  distribution

$$\mathcal{F}_\Gamma(A, \Lambda)$$

<sup>118</sup>此处 covariant 一词类似于广相中的“协变”义。



We notice that  $|I(\theta)|^{1/2}$  meet such requirement, which gives Jeffrey's Prior.

$$\begin{aligned} L(y|\theta) = L(y|\phi) &\Rightarrow |I(\theta)| = \left| \mathbb{E}_y \left[ \frac{\partial \log L(y|\theta)}{\partial \theta} \frac{\partial \log L(y|\theta)}{\partial \theta'} \right] \right| \\ &= \left| \mathbb{E}_y \left[ \frac{\partial L(y|\phi)}{\partial \phi} \frac{\partial L(y|\phi)}{\partial \phi'} \right] \right| \left| \frac{\partial \phi}{\partial \theta} \right|^2 \\ &= |I(\phi)| \left| \frac{\partial \phi}{\partial \theta} \right|^2 \end{aligned}$$

Note: Usually Jeffrey's is an improper prior (diverge).

- Other suitable prior that reflects our knowledge.

### 13.2.2 Posterior Distribution

After wisely select the prior, we can have it combined with data and get formula for posterior

$$p(\theta|y) \propto p(\theta)p(y|\theta)$$

which is a function of  $\theta$ , so we just need to take care of  $\theta$ -terms and normalization condition would help fixed the constant.

#### □ Calculation Trick

- Identify the distribution with the variable related term.

Example: Obtain the predictive distribution of Poisson model with conjugate Gamma distribution

$$\begin{aligned} p(y|\theta) &= \prod_{i=1}^N \frac{\theta^{y_i}}{y_i!} e^{-\theta} \\ p(\theta) &\sim \Gamma(\alpha, \beta) \\ p(\tilde{y}|y) &\propto \int \frac{\theta^{\tilde{y}}}{\tilde{y}!} e^{-\theta} \theta^{\alpha-1} e^{-\beta\theta} \theta^{N\bar{y}} e^{-N\theta} d\theta \\ &= \frac{1}{\tilde{y}!} \int \theta^{\alpha+N\bar{y}+\tilde{y}-1} e^{-(\beta+N+1)\theta} d\theta \\ &= \frac{1}{\tilde{y}!} \frac{\Gamma(\alpha+N\bar{y}+\tilde{y})}{(\beta+N+1)^{\alpha+N\bar{y}+\tilde{y}}} \\ &\propto \binom{\alpha+N\bar{y}+\tilde{y}-1}{\tilde{y}} \left( \frac{\beta+N}{\beta+N+1} \right)^{\alpha+N\bar{y}} \left( \frac{1}{\beta+N+1} \right)^{\tilde{y}} \\ &\sim \text{Neg-Binom}(\alpha+N\bar{y}, \beta+N) \end{aligned}$$

- Get marginal posterior with Conditional probability formula

$$p(\beta|y) = \frac{p(\alpha, \beta|y)}{p(\alpha|\beta, y)}$$

in which L.H.S. is free from  $\alpha$ , so R.H.S. should be invariant of  $\alpha$ , i.e. take the same value for any  $\alpha$  value. We can simplify the calculation by taking some convenient values.

Example: Marginal posterior distribution in Normal model.

$$\begin{aligned}
 p(\mu, \sigma^2 | y, \mu_0, \sigma_0^2 / \kappa_0; \nu_0, \sigma_0^2) &\sim N\text{-Inv-}\chi^2(\mu_n, \sigma_n^2 / \kappa_n; \nu_n, \sigma_n^2) \\
 p(\sigma^2 | \mu; y, \mu_0, \sigma_0^2 / \kappa_0; \nu_0, \sigma_0^2) &\sim \text{Inv-}\chi^2(\nu_n + 1, \frac{\nu_0^2 \sigma_0^2 + \kappa_0(\mu - \mu_0)^2 + NMSE}{\nu_n + 1}) \\
 p(\mu | y, \mu_0, \sigma_0^2 / \kappa_0; \nu_0, \sigma_0^2) &= \frac{p(\mu, \sigma^2 | y, \mu_0, \sigma_0^2 / \kappa_0; \nu_0, \sigma_0^2)}{p(\sigma^2 | \mu; y, \mu_0, \sigma_0^2 / \kappa_0; \nu_0, \sigma_0^2)} \\
 (\text{take } \sigma^2 = 1) \quad &\propto (\nu_0^2 \sigma_0^2 + \kappa_0(\mu - \mu_0)^2 + NMSE)^{-(\nu_n + 1)/2} \\
 &= \left( \nu_0^2 \sigma_0^2 + (N - 1)s^2 + \frac{\kappa_0 N}{\kappa_0 + N} (\bar{y} - \mu_0)^2 + \kappa_n (\mu - \mu_n)^2 \right)^{-(\nu_n + 1)/2} \\
 &\propto \left( 1 + \frac{\kappa_n (\mu - \mu_n)^2}{\nu_n \sigma_n^2} \right)^{-(\nu_n + 1)/2} \sim t_{\nu_n}(\mu_n, \sigma_n^2 / \kappa_n)
 \end{aligned}$$

### 13.2.3 Asymptotics

The Maximum A Posteriori (MAP) describes a point estimation by maximize posterior distribution of parameter, say

$$\hat{\theta} = \arg \max_{\theta} p(\theta | y) = \arg \max_{\theta} p(\theta) p(y | \theta)$$

The maximizer has consistency at large sample

$$p(\theta | y) \rightarrow \delta(\theta - \theta^*), \quad \text{as } n \rightarrow \infty$$

The maximizer would further give the asymptotic normal distribution centered around it. Use the Taylor series at  $\hat{\theta}$ :

$$\begin{aligned}
 \log p(\theta | y) &= \log p(\hat{\theta} | y) + \frac{1}{2}(\theta - \hat{\theta})' \left[ \frac{\partial \log p(\theta | y)}{\partial \theta \partial \theta'} \right]_{\theta = \hat{\theta}} (\theta - \hat{\theta}) + o(\theta^2) \\
 \Rightarrow p(\theta | y) &\rightarrow N(\hat{\theta}, \mathcal{I}(\hat{\theta})^{-1} / n)
 \end{aligned}$$

Note: Here  $\hat{\theta}$ , as is calculated from the data, is considered fixed, while  $\theta$  is the random one. It's just in contrast to frequentist's version where  $\hat{\theta}$  is random while  $\theta$  is fixed. (The estimator is also different, here maximizes posterior  $p(\theta | y)$ , frequentists maximize likelihood  $p(y | \theta)$ )

### 13.2.4 Predictive Distribution

Generally speaking we are studying the posterior predictive distribution

$$p_{\text{post}}(\tilde{y}) = \mathbb{E}_{\theta | y} [p(\tilde{y} | \theta)] = \int p(\tilde{y} | \theta) p(\theta | y) d\theta$$

Related concept:

- Expected log Prediction Distribution for New Data (elpd). Where the ground truth distribution of  $\tilde{y}$  is denoted  $\tilde{f}(\tilde{y})$ .

$$\text{elpd} \equiv \mathbb{E}_{\tilde{f}} \left[ \log \int p(\tilde{y} | \theta) p(\theta | y) d\theta \right] = \int_{\tilde{y}} \log \int_{\theta} p(\tilde{y} | \theta) p(\theta | y) d\theta d\tilde{y}$$

At large sample, the  $d\theta$  integration is dominated by  $\hat{\theta} = \arg \max_{\theta} p(\theta|y)$ , yielding<sup>119</sup>

$$\text{elpd} = \int_{\tilde{y}} \log \int_{\theta} p(\tilde{y}|\theta)p(\theta|y) d\theta \tilde{f}(\tilde{y}) d\tilde{y} \underset{N \rightarrow \infty}{\approx} \int_{\tilde{y}} \tilde{f}(\tilde{y}) \log p(\tilde{y}|\hat{\theta}) d\tilde{y} \equiv \text{elpd}_{\hat{\theta}}$$

## Section 13.3 Simulation

In Bayesian inference, the key target is posterior distribution

$$p(\theta|y) \propto p(\theta)p(y|\theta)$$

which can be

- Intractable: ugly kernel term.
  - High-dimensional: multi-dim parameter space  $\Theta$ .
  - Unnormalized: with an unknown normalize constant  $\frac{1}{p(y)}$
- so usually a closed form is not accessible. Simulation is needed to carry out further inference.

### 13.3.1 Random Number Generation and Simulation

Basic knowledge about simulation methods were covered in [section. 5.6](#). Here are some topic contents:

- **Linear Congruential Method for  $U(0, 1)$** : Other complicated distribution starts from uniform distributed r.v.
- **Quantile Method/Inverse Transform Method**: Use inverse CDF to obtain r.v.

$$X_i = F_X^{-1}(U_i) \sim f_X$$

- **Acceptance-Rejection Sampling**: Suitable for intractable CDF or high-dimensional cases.
- **MCMC**: Deal with high-dimensional case or unnormalized distribution.
- **Importance Sampling Estimator**: Quick way to obtain some observable.

### 13.3.2 Mean Field Approximation and Variation Bayesian Inference

## Section 13.4 Exactly Solvable Models

**Note** : In this section for a known/given parameter (i.e. we do **not** consider it an r.v., just a given param), we attach an fixed to label it, e.g.  $N(\mu, \sigma^2)$  for the case  $\sigma^2$  is given, and we only study the distribution of  $\mu$ .

<sup>119</sup>The integration method is called ‘steepest descent’.

### 13.4.1 Binomial Model

Generating process  $y_1, \dots, y_N$  i.i.d.  $\sim \text{Binom}(n, p)$

$$\text{Distribution: } f(y|p) = \binom{n}{y_i} p^y (1-p)^{n-y} \propto p^y (1-p)^{n-y}$$

$$\text{Likelihood: } L(y|p) \propto p^{\sum y_i} (1-p)^{Nn - \sum y_i} = p^{N\bar{y}} (1-p)^{N(n-\bar{y})}$$

$$\text{Score: } S(y|p) = \frac{N\bar{y}}{p} - \frac{N(n-\bar{y})}{1-p}$$

$$\text{Observed Info: } J(y|p) = \frac{N\bar{y}}{p^2} + \frac{N(n-\bar{y})}{(1-p)^2}$$

$$\text{Fisher Info: } I(p) = \frac{N}{p(1-p)}$$

- Conjugate prior: Beta distribution  $B(\alpha, \beta)$

$$p(p|\alpha, \beta) = \frac{1}{B(\alpha, \beta)} p^{\alpha-1} (1-p)^{\beta-1} \sim B(\alpha, \beta)$$

$$p(p|y, \alpha, \beta) \propto p^{\alpha-1} (1-p)^{\beta-1} p^{N\bar{y}} (1-p)^{N(n-\bar{y})} \sim B(\alpha + N\bar{y}, \beta + N(n-\bar{y}))$$

which suggests that prior distribution  $B(\alpha, \beta)$  looks like some ‘pre-drawn’ data.

- Jeffrey Prior:  $p(p) \sim B(\frac{1}{2}, \frac{1}{2})$ .

### 13.4.2 Poisson Model

Generating process  $y_1, \dots, y_N$  i.i.d.  $\sim P(\lambda)$

$$\text{Distribution: } f(y|\lambda) = \frac{\lambda^y}{y!} e^{-\lambda} \propto \lambda^y e^{-\lambda}$$

$$\text{Likelihood: } L(y|\lambda) \propto \lambda^{N\bar{y}} e^{-N\lambda}$$

$$\text{Score: } S(y|\lambda) = N\left(\frac{\bar{y}}{\lambda} - 1\right)$$

$$\text{Observed Info: } J(y|\lambda) = \frac{N\bar{y}}{\lambda^2}$$

$$\text{Fisher Info: } I(\lambda) = \frac{N}{\lambda}$$

- Conjugate Prior: Gamma Distribution  $\Gamma(\alpha, \beta)$

$$p(\lambda|\alpha, \beta) = \frac{\beta^\alpha}{\Gamma(\alpha)} \lambda^{\alpha-1} e^{-\beta\lambda} \sim \Gamma(\alpha, \beta)$$

$$p(\lambda|y, \alpha, \beta) \propto \lambda^{\alpha-1} e^{-\beta\lambda} \lambda^{N\bar{y}} e^{-N\lambda} \sim \Gamma(\alpha + N\bar{y}, \beta + N)$$

- Jeffrey Prior:  $p(\lambda) \sim \Gamma(\frac{1}{2}, 0)$ . (actually  $\beta \rightarrow 0^+$ , similar for followings)

### 13.4.3 Exponential Model

Generating process  $y_1, \dots, y_N$  i.i.d.  $\sim \varepsilon(\lambda)$

$$\text{Distribution: } f(y|\lambda) = \lambda e^{-\lambda y} \propto \lambda^y e^{-\lambda y}$$

$$\text{Likelihood: } L(y|\lambda) \propto \lambda^N e^{-\lambda N \bar{y}}$$

$$\text{Score: } S(y|\lambda) = \frac{N}{\lambda} - N \bar{y}$$

$$\text{Observed Info: } J(y|\lambda) = \frac{N}{\lambda^2}$$

$$\text{Fisher Info: } I(\lambda) = \frac{N}{\lambda^2}$$

- Conjugate Prior: Gamma Distribution  $\Gamma(\alpha, \beta)$

$$p(\lambda|\alpha, \beta) = \frac{\beta^\alpha}{\Gamma(\alpha)} \lambda^{\alpha-1} e^{-\beta \lambda} \sim \Gamma(\alpha, \beta)$$

$$p(\lambda|y, \alpha, \beta) \propto \lambda^{\alpha-1} e^{-\beta \lambda} \lambda^N e^{-N \bar{y} \lambda} \sim \Gamma(\alpha + N, \beta + N \bar{y})$$

- Jeffrey Prior:  $p(\lambda) \sim \Gamma(0, 0)$ .

### 13.4.4 Normal Model

#### □ Model with known variance $\sigma^2$

Generating process  $y_1, \dots, y_N$  i.i.d.  $\sim N(\mu, \sigma^2)$

$$\text{Distribution: } f(y|\mu) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{(y-\mu)^2}{2\sigma^2}\right] \propto \exp\left[-\frac{\mu^2 - 2y\mu}{2\sigma^2}\right]$$

$$\text{Likelihood: } L(y|\mu) \propto \exp\left[-\frac{N(\mu^2 - 2\bar{y}\mu)}{2\sigma^2}\right]$$

$$\text{Score: } S(y|\mu) = -\frac{N(\mu - \bar{y})}{\sigma^2}$$

$$\text{Observed Info: } J(y|\mu) = \frac{N}{\sigma^2}$$

$$\text{Fisher Info: } I(\mu) = \frac{N}{\sigma^2}$$

- Conjugate Prior: Normal Distribution  $N(\mu_0, \tau_0^2)$

$$p(\mu|\mu_0, \tau_0^2) = \frac{1}{\sqrt{2\pi\tau_0^2}} \exp\left[-\frac{(\mu - \mu_0)^2}{2\tau_0^2}\right] \sim N(\mu_0, \tau_0^2)$$

$$p(\mu|y, \mu_0, \tau_0^2) \propto \exp\left[-\frac{\mu^2 - 2\mu_0\mu}{2\tau_0^2} - \frac{N(\mu^2 - 2\bar{y}\mu)}{2\sigma^2}\right] \sim N\left(\frac{\frac{\mu_0}{\tau_0^2} + \frac{\bar{y}}{\sigma^2/N}}{\frac{1}{\tau_0^2} + \frac{1}{\sigma^2/N}}, \frac{1}{\frac{1}{\tau_0^2} + \frac{1}{\sigma^2/N}}\right)$$

- Jeffrey Prior:  $p(\mu) \propto 1 \sim N(\wedge, \infty)$ .

#### □ Model with known mean $\mu$

Generating process  $y_1, \dots, y_N$  i.i.d.  $\sim N(\mu, \sigma^2)$

$$\text{Distribution: } f(y|\sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{(y-\mu)^2}{2\sigma^2}\right]$$

$$\text{Likelihood: } L(y|\sigma^2) \propto \sigma^{-N} \exp\left[-\frac{1}{2\sigma^2} \sum_{i=1}^N (y_i - \mu)^2\right] \equiv \sigma^{-N} \exp\left[-\frac{NMSE}{2\sigma^2}\right]$$

$$\text{Score: } S(y|\sigma^2) = -\frac{N}{\sigma} + \frac{NMSE}{\sigma^3}$$

$$\text{Observed Info: } J(y|\sigma^2) = -\frac{N}{\sigma^2} + \frac{3NMSE}{\sigma^4}$$

$$\text{Fisher Info: } I(\sigma^2) = \frac{2N}{\sigma^2}$$

- Conjugate Prior:  $\text{Inv-}\chi^2(\nu_0, \sigma_0^2)$

$$p(\sigma^2|\nu_0, \sigma_0^2) = \frac{\nu_0^{\frac{\nu_0}{2}}}{2^{\frac{\nu_0}{2}} \Gamma(\frac{\nu_0}{2})} (\sigma^2)^{-\frac{\nu_0}{2}-1} e^{-\frac{\nu_0 \sigma_0^2}{2\sigma^2}} \sim \text{Inv-}\chi^2(\nu_0, \sigma_0^2)$$

$$p(\sigma^2|y, \nu_0, \sigma_0^2) \propto (\sigma^2)^{-\frac{\nu_0}{2}-1} e^{-\frac{\nu_0 \sigma_0^2}{2\sigma^2}} (\sigma^2)^{-N/2} \exp\left[-\frac{NMSE}{2\sigma^2}\right] \sim \text{Inv-}\chi^2\left(\nu_0 + N, \frac{\nu_0 \sigma_0^2 + NMSE}{\nu_0 + N}\right)$$

- Jeffrey Prior:  $p(\sigma^2) \propto \frac{1}{\sigma} \sim \text{Inv-}\chi^2(1, 0)$ .

#### □ Full model

Generating process  $y_1, \dots, y_N$  i.i.d.  $\sim N(\mu, \sigma^2)$

$$\text{Distribution: } f(y|\mu, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{(y-\mu)^2}{2\sigma^2}\right]$$

$$\text{Likelihood: } L(y|\mu, \sigma^2) \propto \sigma^{-N} \exp\left[-\frac{1}{2\sigma^2} \sum_{i=1}^N (y_i - \mu)^2\right] \equiv \sigma^{-N} \exp\left[-\frac{N(\bar{y} - \mu)^2 + (N-1)s^2}{2\sigma^2}\right]$$

$$\text{Score: } S(y|\mu, \sigma^2) = \begin{pmatrix} \frac{N}{\sigma^2}(\bar{y} - \mu) \\ -\frac{N}{2\sigma^2} + \frac{\sum (y_i - \mu)^2}{2(\sigma^2)^2} \end{pmatrix}$$

$$\text{Observed Info: } J(y|\mu, \sigma^2) = \begin{pmatrix} \frac{N}{\sigma^2} & \frac{N(\bar{y} - \mu)}{(\sigma^2)^2} \\ \frac{N(\bar{y} - \mu)}{(\sigma^2)^2} & -\frac{N}{2(\sigma^2)^2} + \frac{\sum (y_i - \mu)^2}{(\sigma^2)^3} \end{pmatrix}$$

$$\text{Fisher Info: } I(\mu, \sigma^2) = \begin{pmatrix} \frac{N}{\sigma^2} & 0 \\ 0 & \frac{N}{2(\sigma^2)^2} \end{pmatrix}$$

Another parameterization  $(\mu, \sigma^2) \mapsto (\mu, \log \sigma)$ :

$$\text{Score: } S(y|\mu, \log \sigma) = \begin{pmatrix} \frac{N}{\sigma^2}(\bar{y} - \mu) \\ -N + \frac{\sum (y_i - \mu)^2}{\sigma^2} \end{pmatrix}$$

$$\text{Observed Info: } J(y|\mu, \log \sigma) = \begin{pmatrix} \frac{N}{\sigma^2} & \frac{N(\bar{y} - \mu)}{(\sigma^2)^2} \\ \frac{N(\bar{y} - \mu)}{(\sigma^2)^2} & \frac{2\sum (y_i - \mu)^2}{\sigma^2} \end{pmatrix}$$

$$\text{Fisher Info: } I(\mu, \log \sigma) = \begin{pmatrix} \frac{N}{\sigma^2} & 0 \\ 0 & 2N \end{pmatrix}$$

- Conjugate Prior for  $(\mu, \sigma^2)$  parameterization: Normal-Inv- $\chi^2$  Distribution  $N\text{-Inv-}\chi^2(\mu_0, \sigma_0^2/\kappa_0; \nu_0, \sigma_0^2)$ , defined as  $p(\sigma^2) \times p(\mu|\sigma^2) = \text{Inv-}\chi^2(\nu_0, \sigma_0^2) \times N(\mu_0, \sigma_0^2/\kappa_0)$

$$p(\mu, \sigma^2 | \mu_0, \sigma_0^2/\kappa_0; \nu_0, \sigma_0^2) \propto (\sigma^2)^{-(\nu_0/2+1)} (\sigma^2)^{-1/2} \exp \left[ -\frac{1}{2\sigma^2} (\nu_0 \sigma_0^2 + \kappa_0 (\mu - \mu_0)^2) \right]$$

$$\sim N\text{-Inv-}\chi^2(\mu_0, \sigma_0^2/\kappa_0; \nu_0, \sigma_0^2)$$

$$p(\mu, \sigma^2 | y, \mu_0, \sigma_0^2/\kappa_0; \nu_0, \sigma_0^2) \propto (\sigma^2)^{-(\nu_0/2+1)} (\sigma^2)^{-1/2} \exp \left[ -\frac{1}{2\sigma^2} (\nu_0 \sigma_0^2 + \kappa_0 (\mu - \mu_0)^2) \right]$$

$$\times (\sigma^2)^{-N/2} \exp \left[ -\frac{N(\bar{y} - \mu)^2 + (N-1)s^2}{2\sigma^2} \right]$$

$$\sim N\text{-Inv-}\chi^2(\mu_n, \sigma_n^2/\kappa_n; \nu_n, \sigma_n^2) \quad \begin{cases} \mu_n &= \frac{\kappa_0}{\kappa_0 + N} \mu_0 + \frac{N}{\kappa_0 + N} \bar{y} \\ \kappa_n &= \kappa_0 + N \\ \nu_n &= \nu_0 + N \\ \nu_n \sigma_n^2 &= \nu_0 \sigma_0^2 + (N-1)s^2 + \frac{\kappa_0 N}{\kappa_0 + N} (\bar{y} - \mu_0)^2 \end{cases}$$

$$p(\mu | y, \sigma^2) \sim N \left( \frac{\frac{\kappa_0 \mu_0}{\sigma^2} + \frac{\bar{y}}{\sigma^2/N}}{\frac{\kappa_0}{\sigma^2} + \frac{1}{\sigma^2/N}}, \frac{1}{\frac{\kappa_0}{\sigma^2} + \frac{1}{\sigma^2/N}} \right) = N \left( \mu_n, \frac{\sigma^2}{\kappa_n} \right)$$

$$p(\mu | y, \mu_0, \sigma_0^2/\kappa_0; \nu_0, \sigma_0^2) \propto \left( 1 + \frac{\kappa_n (\mu - \mu_n)^2}{\nu_n \sigma_n^2} \right)^{-(\nu_n+1)/2} \sim t_{\nu_n}(\mu_n, \sigma_n^2/\kappa_n)$$

- Jeffrey Prior

- for  $(\mu, \sigma^2)$  parameterization *with independency assumption of  $(\mu, \sigma)$* :

$$p(\mu, \sigma^2) \propto 1 \times (\sigma^2)^{-1} = \sigma^{-2}$$

- for  $(\mu, \sigma^2)$  parameterization *without independency assumption of  $(\mu, \sigma)$* :

$$p(\mu, \sigma^2) \propto \sigma^{-3}$$

- for  $(\mu, \log \sigma)$  parameterization *with independency assumption of  $(\mu, \sigma)$* :

$$p(\mu, \log \sigma) \propto 1 \times 1 = 1$$

- for  $(\mu, \log \sigma)$  parameterization *without independency assumption of  $(\mu, \sigma)$* :

$$p(\mu, \log \sigma) \propto \sigma^{-1}$$

### 13.4.5 Multinomial Model

(One sample item here) Generating process  $(y_1, y_2, \dots, y_J) \sim \text{Multino}(n; \theta_1, \theta_2, \dots, \theta_J)$ , *w.r.t.*  $\sum_{j=1}^J \theta_j = 1$ ,  $\sum_{j=1}^J y_j = n$

$$\text{Distribution: } f(y|\theta) = \binom{n}{y_1 \dots y_J} \prod_{j=1}^J \theta_j^{y_j}, \quad \sum_{j=1}^J \theta_j = 1, \quad \sum_{j=1}^J y_j = n$$

$$\text{Likelihood: } L(y|\theta) \propto \prod_{j=1}^J \theta_j^{y_j}, \quad \sum_{j=1}^J \theta_j = 1$$

the score function and Fisher information are slightly different because of the constraint  $\sum_j \theta_j = 1$ , i.e.  $\vec{\theta} \in \mathbb{R}^{J-1} \subset \mathbb{R}^J$ . Fortunately **for multinomial** the transformation function **happens to** reserve the  $\det(I(\theta))$ , i.e. we could simply ‘pretend’ their independence to get

$$\text{Fisher Info: } \det[I(\theta)] = \frac{1}{\theta_1 \theta_2 \dots \theta_J}, \quad \sum_{j=1}^J \theta_j = 1$$

- Conjugate Prior: Dirichlet Distribution  $\text{Dirichlet}(\alpha_1, \dots, \alpha_J)$

$$p(\theta|\alpha) = \frac{\Gamma(\alpha_1 + \dots + \alpha_J)}{\Gamma(\alpha_1) \dots \Gamma(\alpha_J)} \prod_{j=1}^J \theta_j^{\alpha_j-1} \sim \text{Dirichlet}(\alpha_1, \dots, \alpha_J)$$

$$p(\theta|y, \alpha) \propto \prod_{j=1}^J \theta_j^{\alpha_j-1} \prod_{j=1}^J \theta_j^{y_j} \sim \text{Dirichlet}(\alpha_1 + y_1, \dots, \alpha_J + y_J)$$

- Jeffrey Prior:  $p(\theta) \propto \prod_{j=1}^J \theta_j^{-1/2} \sim \text{Dirichlet}(\frac{1}{2}, \dots, \frac{1}{2})$ .

### 13.4.6 Multi-Normal Model

Generating process  $y_1, \dots, y_N$  i.i.d.  $\sim N_d(\mu, \Sigma)$

$$\text{Distribution: } f(y|\mu, \Sigma) = \frac{1}{(2\pi)^{-d/2} |\Sigma|^{1/2}} \exp \left[ -\frac{1}{2} (y - \mu)' \Sigma^{-1} (y - \mu) \right]$$

$$\begin{aligned} \text{Likelihood: } L(y|\mu, \Sigma) &\propto |\Sigma|^{-N/2} \exp \left[ -\frac{1}{2} \sum_{i=1}^N (y_i - \mu)' \Sigma^{-1} (y_i - \mu) \right] \\ &= |\Sigma|^{-N/2} \exp \left[ -\frac{1}{2} \text{tr}(\Sigma^{-1} S_0) \right] \end{aligned}$$

$$\begin{aligned} \text{where } S_0 &\equiv \sum_{i=1}^N (y_i - \mu)(y_i - \mu)' = N(\bar{y} - \mu)(\bar{y} - \mu)' + \sum_{i=1}^N (y_i - \bar{y})(y_i - \bar{y})' \\ &\equiv N(\bar{y} - \mu)(\bar{y} - \mu)' + S \end{aligned}$$

- Conjugate Prior: Normal-Inv-Wishart Distribution  $N\text{-Inv-Wishart}(\mu_0, \Lambda_0^2/\kappa_0; \nu_0, \Lambda_0)$ , defined as  $p(\Sigma) \times p(\mu|\Sigma) = \text{Inv-Wishart}(\nu_0, \Lambda_0) \times N(\mu_0, \Sigma/\kappa_0)$

$$\begin{aligned} p(\mu, \Sigma|\mu_0, \Lambda_0^2/\kappa_0; \nu_0, \Lambda_0) &\propto (\Sigma)^{-(\nu_0/2+1)} (\Sigma)^{-p/2} \exp \left[ -\frac{1}{2} \text{tr}(\Lambda_0 \Sigma^{-1}) - \frac{\kappa_0}{2} (\mu - \mu_0)' \Sigma^{-1} (\mu - \mu_0) \right] \\ &\sim N\text{-Inv-Wishart}(\mu_0, \Lambda_0/\kappa_0; \nu_0, \Lambda_0) \end{aligned}$$

$$\begin{aligned} p(\mu, \Sigma|y, \mu_0, \Lambda_0/\kappa_0; \nu_0, \Lambda_0) &\propto (\Sigma)^{-(\nu_0/2+1)} (\Sigma)^{-p/2} \exp \left[ -\frac{1}{2} \text{tr}(\Lambda_0 \Sigma^{-1}) - \frac{\kappa_0}{2} (\mu - \mu_0)' \Sigma^{-1} (\mu - \mu_0) \right] \\ &\quad \times (\Sigma)^{-N/2} \exp \left[ -\frac{1}{2} \text{tr}([N(\bar{y} - \mu)(\bar{y} - \mu)' + S] \Sigma^{-1}) \right] \end{aligned}$$

$$\sim N\text{-Inv-Wishart}(\mu_n, \Lambda_n/\kappa_n; \nu_n, \Lambda_n) \quad \left\{ \begin{array}{l} \mu_n = \frac{\kappa_0}{\kappa_0 + N} \mu_0 + \frac{N}{\kappa_0 + N} \bar{y} \\ \kappa_n = \kappa_0 + N \\ \nu_n = \nu_0 + N \\ \sigma_n^2 = \Lambda_0 + S + \frac{\kappa_0 N}{\kappa_0 + N} (\bar{y} - \mu_0)(\bar{y} - \mu_0)' \end{array} \right.$$



$$p(\mu|y, \Sigma) \sim N(\mu_n, \frac{\Sigma}{\kappa_n})$$

$$p(\mu|y) \sim t_{\nu_n-d+1}(\mu_n, \frac{\Lambda_n}{\kappa_n(\nu_n-d+1)})$$

Note: When generalizing from  $\text{Inv-}\chi^2$  to  $\text{Inv-Wishart}$ , there's a slight change  $\nu_0\sigma_0^2 \mapsto \Lambda_0$ .

- Jeffrey Prior:  $p(\mu, \Sigma) \propto |\Sigma|^{-(d+1)/2} \sim N\text{-Inv-Wishart}(\wedge, \infty; -1, 0)$

### 13.4.7 Hierarchical Binomial Model

Generating process:  $y_j \sim \text{Binom}(n_j, \theta_j)$ ,  $\theta_j \sim B(\alpha, \beta)$ ,  $j = 1, 2, \dots, J$

$$p(\theta, \alpha, \beta|y) \propto p(\alpha, \beta) \prod_{j=1}^J \frac{1}{B(\alpha, \beta)} \theta_j^{\alpha-1} (1-\theta_j)^{\beta-1} \prod_{j=1}^J \theta_j^{y_j} (1-\theta_j)^{n_j-y_j}$$

$$p(\theta|\alpha, \beta, y) \propto \prod_{j=1}^J \theta_j^{\alpha+y_j-1} (1-\theta_j)^{\beta+n_j-y_j-1} \sim B(\alpha+y_j, \beta+n_j-y_j)$$

$$p(\alpha, \beta|y) \propto p(\alpha, \beta) \prod_{j=1}^J \frac{B(\alpha+y_j, \beta+n_j-y_j)}{B(\alpha, \beta)}$$

Note:  $p(\alpha, \beta)$  should be thin-tailed to avoid divergence at  $\alpha, \beta \rightarrow \infty$ .

### 13.4.8 Hierarchical Normal Model

Generating process:  $y_{ij} \sim N(\theta_j, \sigma^2)$ ,  $\theta_j \sim N(\mu, \tau^2)$ ,  $i = 1, 2, \dots, n_j$ ,  $j = 1, 2, \dots, J$

$$p(\theta, \mu, \tau|y) \propto p(\mu, \tau) \prod_{j=1}^J N(\theta_j|\mu, \tau^2) \prod_{j=1}^J N(\bar{y}_j|\theta_j, \sigma^2/n_j)$$

$$p(\theta|\mu, \tau, y) \sim \prod_{j=1}^J N\left(\frac{\frac{\bar{y}_j}{\sigma^2/n_j} + \frac{\mu}{\tau^2}}{\frac{1}{\sigma^2/n_j} + \frac{1}{\tau^2}}, \frac{1}{\frac{1}{\sigma^2/n_j} + \frac{1}{\tau^2}}\right)$$

$$p(\mu, \tau|y) \propto p(\mu, \tau) \prod_{j=1}^J N(\bar{y}_j|\mu, \frac{\sigma^2}{n_j} + \tau^2)$$

$$p(\mu|\tau, y) \sim N(\tilde{\mu}, \tilde{V}), \quad \text{with } p(\mu|\tau) \propto 1$$

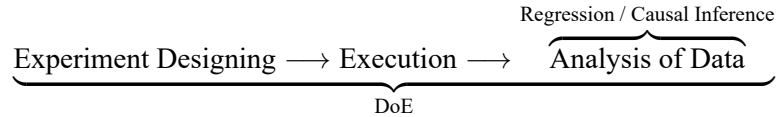
$$\left\{ \begin{array}{l} \tilde{\mu} \equiv \frac{\sum_{j=1}^J \frac{\bar{y}_j}{\sigma^2/n_j + \tau^2}}{\sum_{j=1}^J \frac{1}{\sigma^2/n_j + \tau^2}} \\ \tilde{V} \equiv \frac{1}{\sum_{j=1}^J \frac{1}{\sigma^2/n_j + \tau^2}} \end{array} \right.$$

$$p(\tau|y) \propto p(\tau) \left(\tilde{V}\right)^{1/2} \prod_{j=1}^J (\sigma^2/n_j + \tau^2)^{-1/2} \exp\left[-\frac{(\bar{y}_j - \tilde{\mu})^2}{2(\sigma^2/n_j + \tau^2)}\right]$$

## Chapter. XIV 实验设计与分析部分

Instructor: Zaiying Zhou

Design of Experiment (DoE) aims at understanding the cause-and-effect relation in systems (thus shares lots of similar language as Causal Inference). DoE is one step beyond Linear Regression where  $X$ s are passively drawn while in DoE we are *deliberately* designing them to be more precise / more efficient in studying  $Y$ - $X$  relation.



### □ Philosophy of DoE

- **Randomize** :
- **Replicate** :
- **Blocking** :

## Section 14.1 Statistical Inference Methods for Models

Basic inference methods are introduced in [section. 2.3.3](#) (interval estimation) and [section. 2.4.2](#) (hypothesis testing). ANOVA in Regression is introduced in [section. 3.3.4](#). Preliminary introductions to factor model include [section. 3.1.2](#) and [Chapter. 8](#). Listed here for review.

### 14.1.1 One Sample Inference

With  $X_1, X_2, \dots, X_n$  i.i.d.  $\sim N(\mu, \sigma^2)$ :

- at null hypothesis  $H_0 : \mu = \mu_0$ , with known variance:

$$Z_0 = \frac{\sqrt{n}(\bar{X} - \mu_0)}{\sigma} \sim N(0, 1)$$

- at null hypothesis  $H_0 : \mu = \mu_0$ , with unknown variance:

$$t_0 = \frac{\sqrt{n}(\bar{X} - \mu_0)}{s} \sim t_{n-1}$$

- at null hypothesis  $H_0 : \sigma = \sigma_0$ , with unknown mean

$$\chi_0^2 = \frac{(n-1)s^2}{\sigma_0^2} \sim \chi_{n-1}^2$$

### 14.1.2 Two Sample Comparison

Two sample comparison with Normal assumption is just similar to one-sample mean comparison. Usually the key problem is to find a  $t$ -statistics and get the *dof* for denominator.

Two sample:  $X_{11}, X_{12}, \dots, X_{1n_1}$  i.i.d.  $\sim N(\mu_1, \sigma_1^2)$ ;  $X_{21}, X_{22}, \dots, X_{2n_2}$  i.i.d.  $\sim N(\mu_2, \sigma_2^2)$ .

- at null hypothesis  $H_0 : \mu_1 - \mu_2 = \Delta_0$ , with known variance

$$z_0 = \frac{(\bar{X}_1 - \bar{X}_2) - \Delta_0}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}} \sim N(0, 1)$$

- at null hypothesis  $H_0 : \mu_1 - \mu_2 = \Delta_0$ , with unknown but same variance

$$t_0 = \frac{(\bar{X}_1 - \bar{X}_2) - \Delta_0}{s_{\text{pooled}} \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}} \sim t_{n_1+n_2-2}, \quad s_{\text{pooled}}^2 = \frac{(n_1-1)s_1^2 + (n_2-1)s_2^2}{n_1+n_2-2}$$

- at null hypothesis  $H_0 : \mu_1 - \mu_2 = \Delta_0$ , with unknown variance (Welch-Satterthwaite approximation for the Behrens-Fisher problem<sup>120</sup>).

$$t_0^{\text{Welch}} = \frac{(\bar{X}_1 - \bar{X}_2) - \Delta_0}{\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}} \approx t_\nu, \quad \nu = \frac{\left(\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}\right)^2}{\frac{(s_1^2/n_1)^2}{n_1+1} + \frac{(s_2^2/n_2)^2}{n_2+1}} - 2$$

- at null hypothesis  $H_0 : \sigma_1 = \sigma_2$ , with unknown mean

$$F_0 = \frac{s_1^2}{s_2^2} \sim F_{n_1-1, n_2-2}$$

### 14.1.3 One Way ANOVA

Generalization from two-sample  $t$ -test to Factor ANOVA: Use the trick that  $F \sim t^2$ , e.g.

$$t_0^2 = \frac{((\bar{X}_1 - \bar{X}_2) - \Delta_0)^2}{s_{\text{pooled}}^2 \left(\frac{1}{n_1} + \frac{1}{n_2}\right)} \sim F_{1, n_1+n_2-2}$$

in which the nominator is ‘difference in mean’, denominator is ‘fluctuation’, i.e. in ANOVA language, variation caused by group difference MSR and variation caused by random effect MSE.

Model: (Factor model with balance design here. Cell mean model / unbalanced design see [section. 8.1.1](#))

$$Y_{ij} = \mu + \alpha_i + \varepsilon_{ij} \quad \varepsilon_{ij} \sim N(0, \sigma^2), \quad i = 1, 2, \dots, a, \quad j = 1, 2, \dots, n, \quad w.r.t. \sum_{i=1}^a \alpha_i = 0$$

Solution could be obtained by traditional way  $[\mu, \alpha_1, \dots, \alpha_{a-1}] = (X'X)^{-1}XY$  with notation [equation. 3.10](#).

$$\hat{\mu} = \bar{Y}_{..} = \frac{1}{na} \sum_{i=1}^a \sum_{j=1}^n Y_{ij} \quad (14.1)$$

$$\hat{\alpha}_i = \frac{1}{n} \sum_{j=1}^n Y_{ij} - \hat{\mu} \quad (14.2)$$

$$s_i^2 = \frac{1}{n-1} \sum_{j=1}^n (Y_{ij} - \bar{Y}_{i.})^2 \quad (14.3)$$

$$s^2 = \frac{(n-1) \sum_{i=1}^a s_i^2}{n_T - a} \quad (14.4)$$

<sup>120</sup>This is the output in `t.test(x1,x2, paired = FALSE, var.equal = FALSE)`

And ANOVA Table

Source of Var	SS	dof	MS	$\mathbb{E}(\text{MS})$
$\alpha_i$	$\text{SS}\alpha = \sum_{i=1}^a (\bar{Y}_{i\cdot} - \bar{Y}_{\cdot\cdot})^2$	$a - 1$	$\frac{\text{SS}\alpha}{a - 1}$	$\sigma^2 + \frac{n \sum_{i=1}^a \alpha_i^2}{a - 1}$
$\sigma^2$	$\text{SSE} = \sum_{i=1}^a \sum_{j=1}^n (Y_{ij} - \bar{Y}_{i\cdot})^2$	$n_T - a$	$\frac{\text{SSE}}{n(a - 1)}$	$\sigma^2$

Test for group difference

$$F_0 = \frac{\text{MS}\alpha}{\text{MSE}} \sim F_{a-1, n_T-a}$$

#### □ General Linear Test Point of View

General Linear Test in linear regression see [section. 3.4.6](#). The idea is to compare a full model and a reduced model

$$\begin{cases} \text{Full Model :} & Y_{ij} = \mu + \alpha_i + \varepsilon_{ij} \\ \text{Reduced Model :} & Y_{ij} = \mu + \varepsilon_{ij} \end{cases}$$

in this case the General Linear Test  $F$  is

$$F^{\text{GLT}} = \frac{(\text{SSE}_R - \text{SSE}_F) / (\text{dof}_R - \text{dof}_F)}{\text{SSE}_F / \text{dof}_F} = \frac{\text{MS}\alpha}{\text{MSE}} = F_0 \sim F_{a-1, n_T-a}$$

#### □ Likelihood Ratio Test Point of View

Detail theory of LRT see [section. 2.4.3](#). the test statistics is

$$\Lambda = \frac{\sup_{\mu; \alpha=0} L(Y; \mu, \alpha)}{\sup_{\mu; \alpha} L(Y; \mu, \alpha)} = \left( \frac{\text{SSTotal}}{\text{SSE}} \right)^{n_T/2}$$

and we have a bijection between  $\Lambda$  and  $F_0$ .

#### □ Homoscedasticity Assumption Diagnostics

- Bartlett's Test for  $H_0 : \sigma_1 = \dots = \sigma_a$

$$T = \frac{(n_T - a) \log \frac{\sum_{i=1}^a (n-1) \text{MS}_i}{n_T - a} - \sum_{i=1}^a (n-1) \log \text{MS}_i}{1 + \frac{1}{3(a-1)} \left( \sum_{i=1}^a \frac{1}{n-1} - \frac{1}{n_T - a} \right)} \approx \chi_{a-1}^2$$

the idea is GeomMean = ArithMean when all are equal.

- Levene's Test

$$T = (\text{ANOVA of } |y_{ij} - \bar{y}_{i\cdot}|) \approx F_{a-1, n_T-a}$$

- Welch's ANOVA

A generalized version of Welch's Test in two-sample  $t$ -test.

#### □ Multiple Comparison

Target: When compare level pairs, say some  $(\alpha_i, \alpha_j)$  pairs, i.e. there are multiple tests, we need to adjust the testing procedure to avoid multiple comparison hazard.<sup>121</sup>

- Fisher's Least Significant Difference (LSD) without correction

$$t_{ij} = \frac{\bar{Y}_{i\cdot} - \bar{Y}_{j\cdot}}{\sqrt{\text{MSE} \left( \frac{1}{n_i} + \frac{1}{n_j} \right)}} \sim t_{N-a}$$

rejection region construction use  $t_{N-a, 1-\alpha/2}$

- Fisher's Least Significant Difference (LSD) with Bonferroni correction: rejection region use  $t_{N-a, 1-\alpha/2m}$
- Tukey's Honestly Significant Difference (HSD): Under  $H_0 : \alpha_1 = \dots = \alpha_a$ , treat  $Y_{ij}$  as sample of  $\mu_i$ . We could study the range of  $\{\bar{Y}_{i\cdot}\}_{i=1}^a$

$$q = \frac{\max\{\bar{Y}_{i\cdot}\}_{i=1}^a - \min\{\bar{Y}_{i\cdot}\}_{i=1}^a}{s\sqrt{n}} \sim q_{a, n_T-a}$$

where  $q_{\cdot, \cdot}$  is **Tukey's studentized range distribution**

- Scheffé's Method by testing contrasts. A  $\phi \equiv \sum_{i=1}^a \xi_i \alpha_i$  w.r.t.  $\sum_{i=1}^a \xi_i = 0$  is called a contrast.

$$F_0 = \left( \frac{\hat{\phi}}{\sqrt{\widehat{\text{var}}(\phi)}} \right)^2 / (a-1) = \frac{(\sum \xi_i \bar{Y}_{i\cdot})^2}{(a-1)\text{MSE} \sum \xi_i^2 / n_i} \sim F_{a-1, N-a}$$

- Benjamini-Hochberg Method for False Discovery Rate control:

Interval Construction follows similar method, see **section. 3.6**.

#### 14.1.4 Multi Factor ANOVA

<sup>121</sup> An intuition: If we simply test each of  $m$  tests at  $\alpha_i = 0.05$ , then the overall type-I error is

$$\alpha = 1 - \prod_{i=1}^m (1 - \alpha_i) > 0.05$$

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