

Time Series Analysis

Fourier analysis

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Informatik



Outline

- Fourier analysis
- Sinusoids
- Fitting sinusoids with known frequencies
- Harmonic analysis with Fourier frequencies
- Orthogonality of complex exponentials
- Discrete Fourier transform
- Fast Fourier transform
- Aliasing and the Nyquist frequency

Fourier analysis

Decomposition of time series into a **sum of sinusoidal components**.

- Linear combination of sine and cosine waves, each with specific **frequency** and **amplitude**.

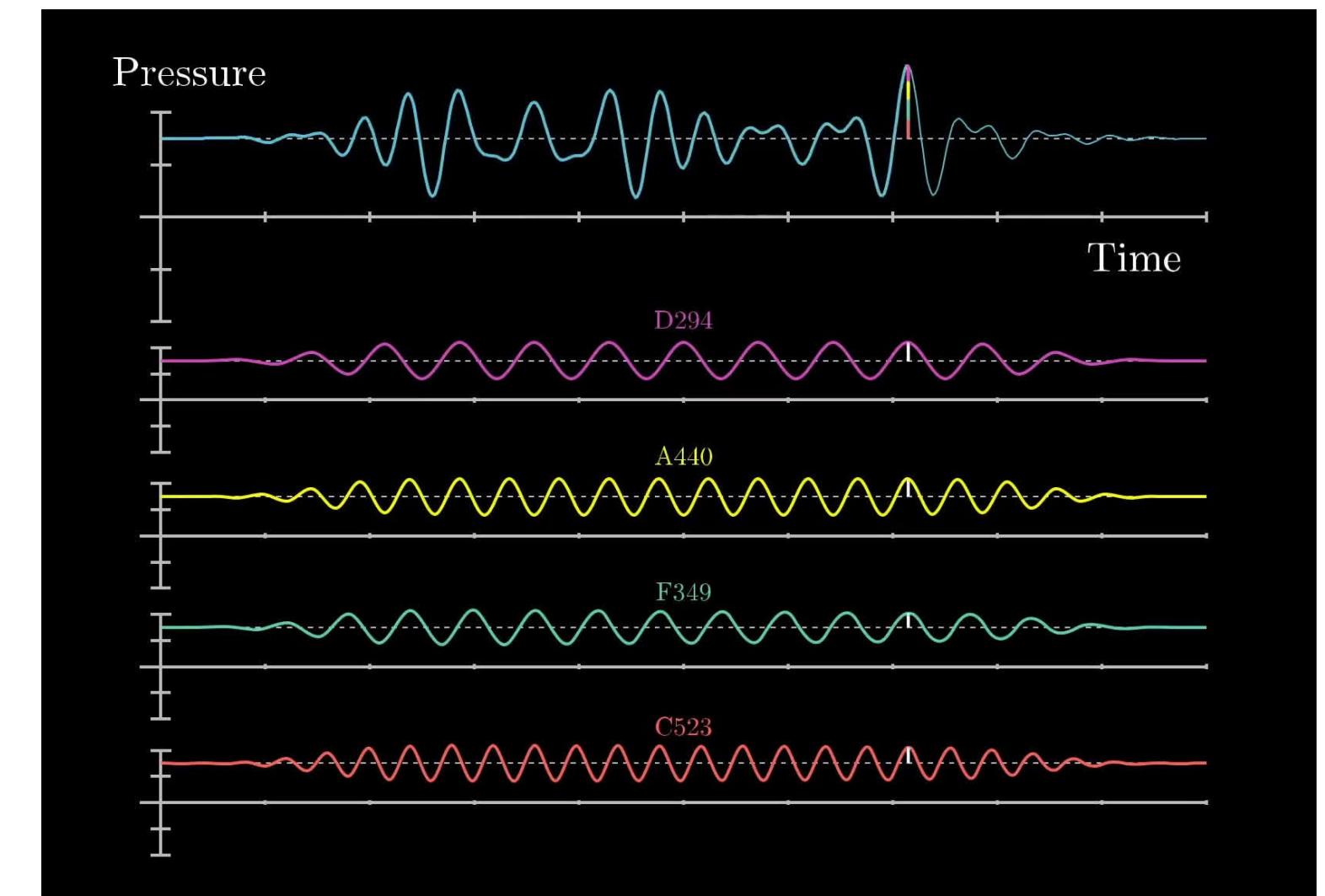
Describe fluctuations in time series by **comparing** them with **sinusoids**.

Periodic fluctuations

- Detect seasonality (simple or multiple).
- Identify dominant frequencies.

Aperiodic fluctuations

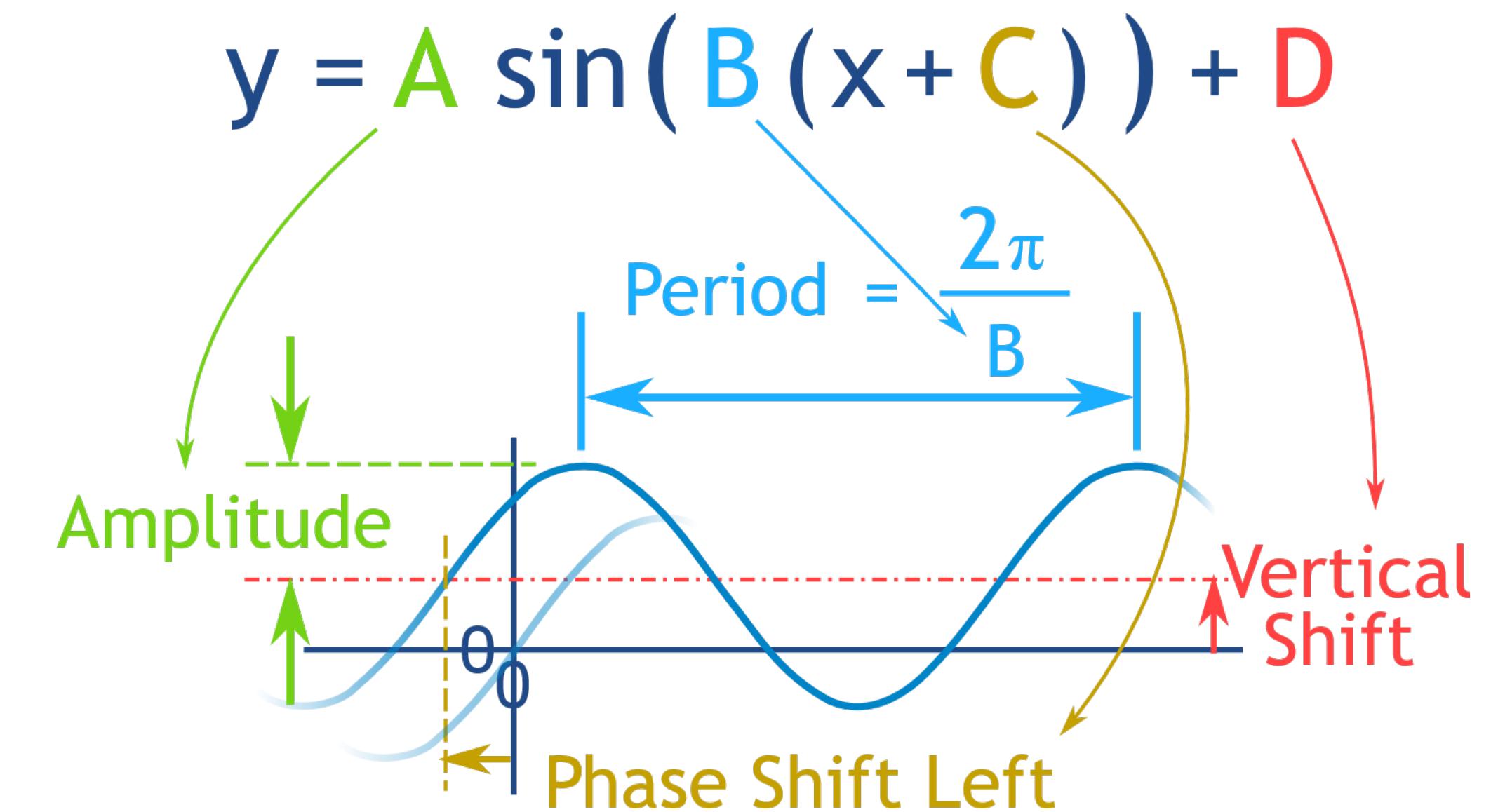
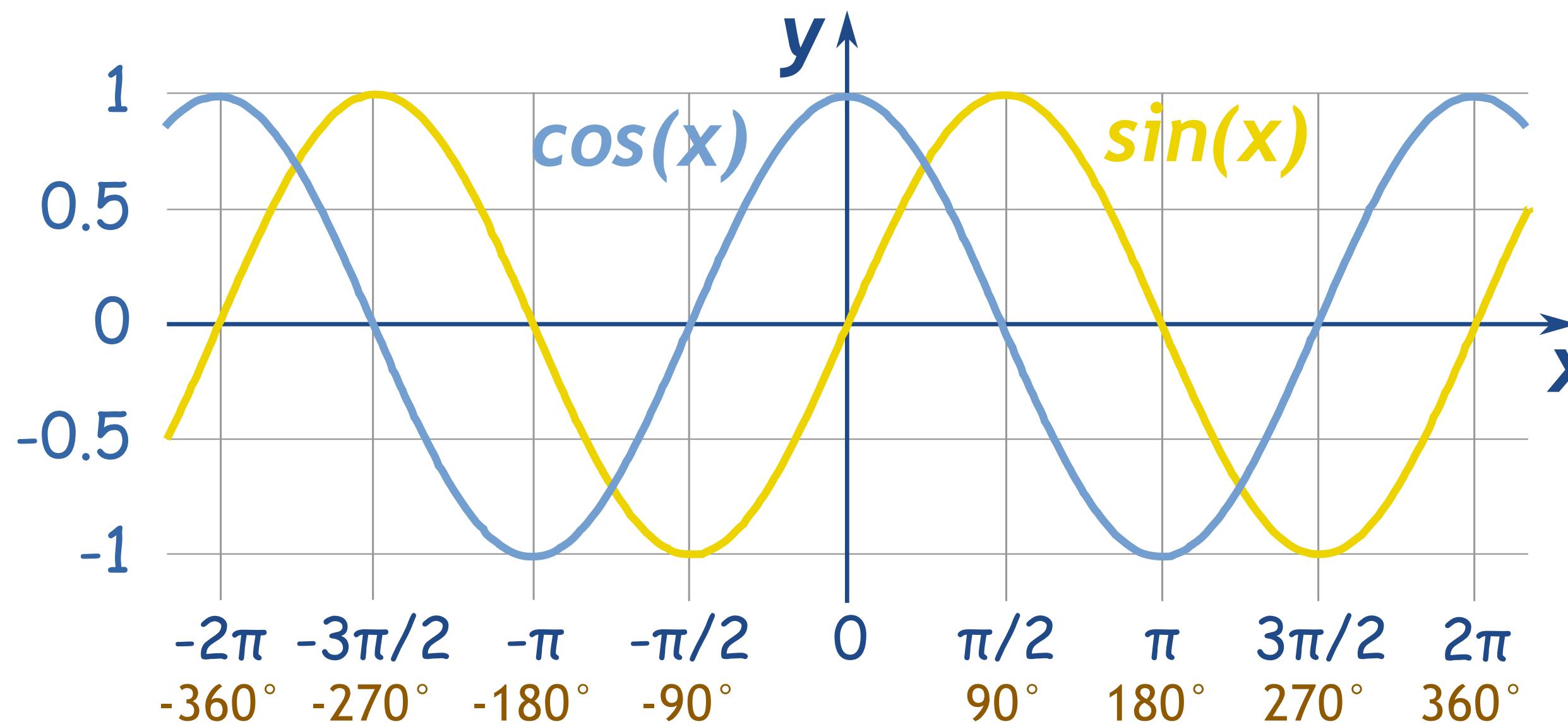
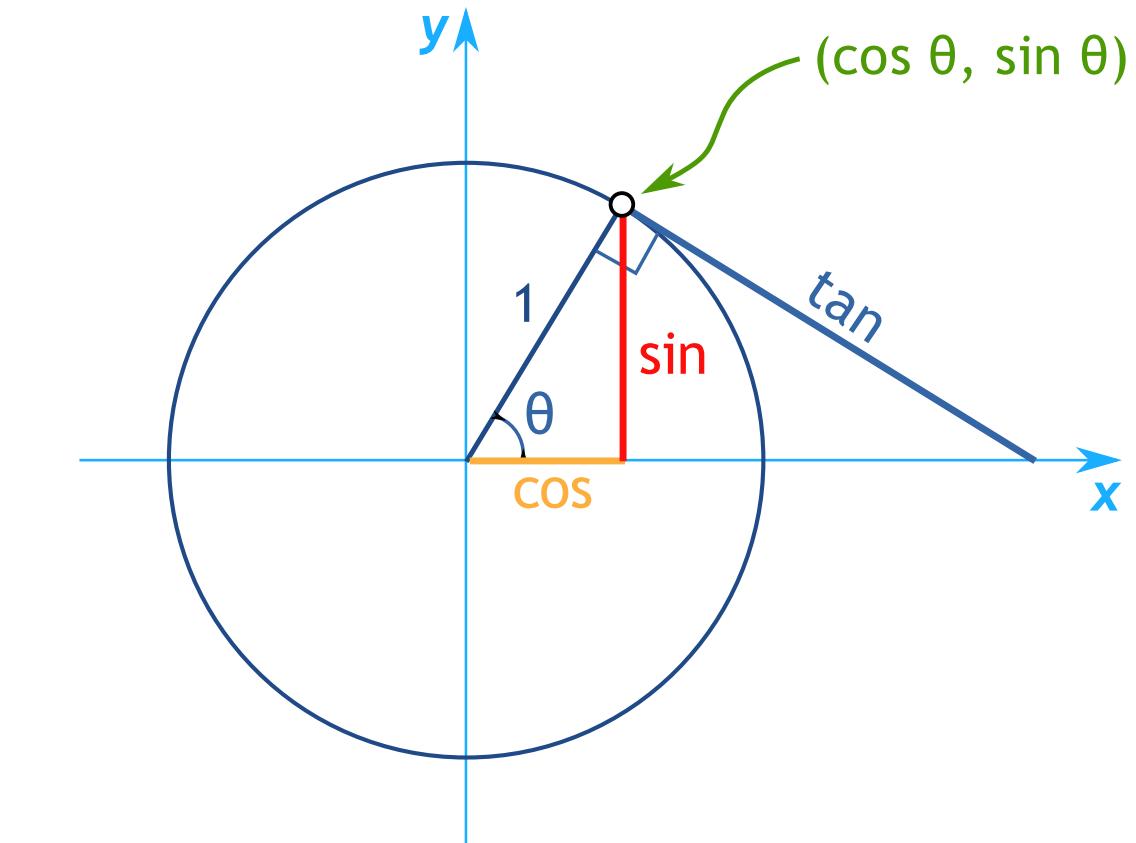
- Describe the frequency distribution of oscillations.
- Distinguish noise from signal.



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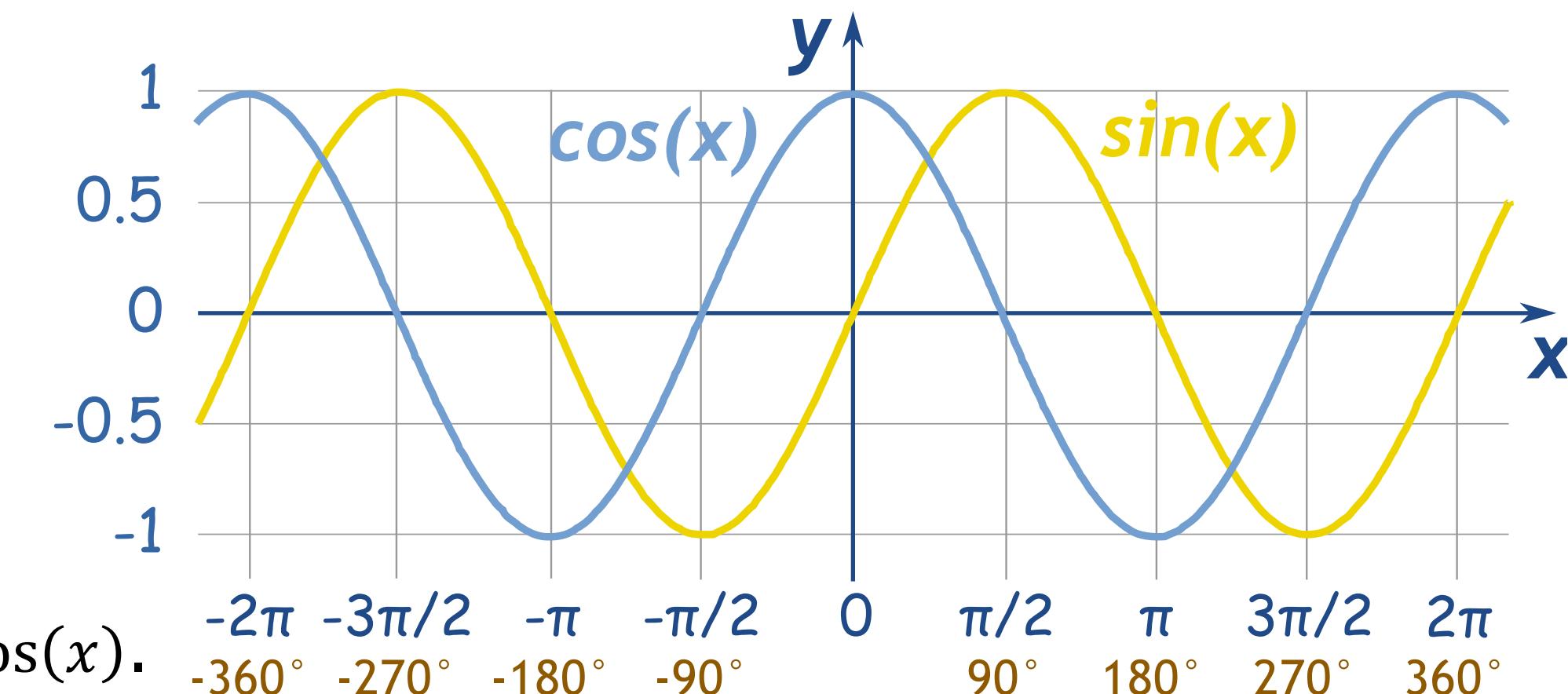
Recap – Trigonometry concepts (1)

Sinusoids are periodic waves whose waveform is the **cosine/sine** function.



Recap – Trigonometry concepts (2)

Sine and cosine are the **same waveform**, just shifted: $\sin\left(x \pm \frac{\pi}{2}\right) = \pm \cos(x)$.



They are periodic $\sin(x + 2\pi) = \sin(x)$ and $\cos(x + 2\pi) = \cos(x)$

- The **period** $P = 2\pi$ means the function completes **one full cycle** every 2π units along the x-axis.
- The **frequency** $f = 1/P = 1/2\pi$ tells **how many cycles occur per unit of x**.

Sine is an **odd** function $\sin(-x) = -\sin(x)$, while cosine is an **even** function $\cos(-x) = \cos(x)$.

Pythagorean identity: $\sin^2 x + \cos^2 x = 1$.

Angle sum: $\sin(\alpha \pm \beta) = \sin(\alpha)\cos(\beta) \pm \cos(\alpha)\sin(\beta)$ and $\cos(\alpha \pm \beta) = \cos(\alpha)\cos(\beta) \mp \sin(\alpha)\sin(\beta)$

In the continuous case, two sinusoids are **orthogonal** (the integral of their product is zero) over any interval spanning a **complete period** of the functions (provided they are **not vertically shifted**).

Sinusoids

A sinusoid of **frequency** f (cycles per unit time) or period $P = 1/f$ (in units of time) can be written as

$$x_t = R \cos(2\pi(f \cdot t + \phi))$$

where R is the **amplitude** and ϕ is the **phase**.

Orthogonality of sinusoids: length n sampled sinusoidal signal segments are **orthogonal** \Leftrightarrow their frequencies have an **integer multiple** k of periods in n samples (i.e., we can write $f = k/n$ with $k \in [0, n - 1]$) .

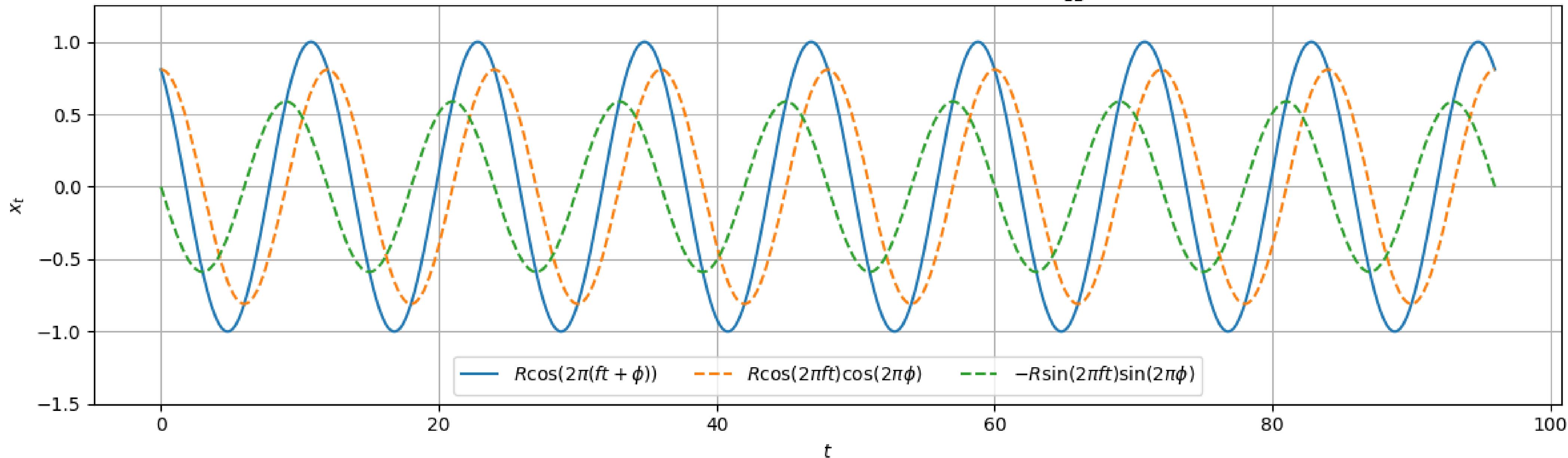
A **sum of sinusoids** with common frequency f is another sinusoid with the same frequency

$$\begin{aligned} x_t &= R \cos(2\pi(ft + \phi)) \\ &= R(\cos(2\pi ft) \cos(2\pi\phi) - \sin(2\pi ft) \sin(2\pi\phi)) \\ &= A \cos(2\pi ft) + B \sin(2\pi ft) \\ &= A c_t + B s_t \end{aligned}$$

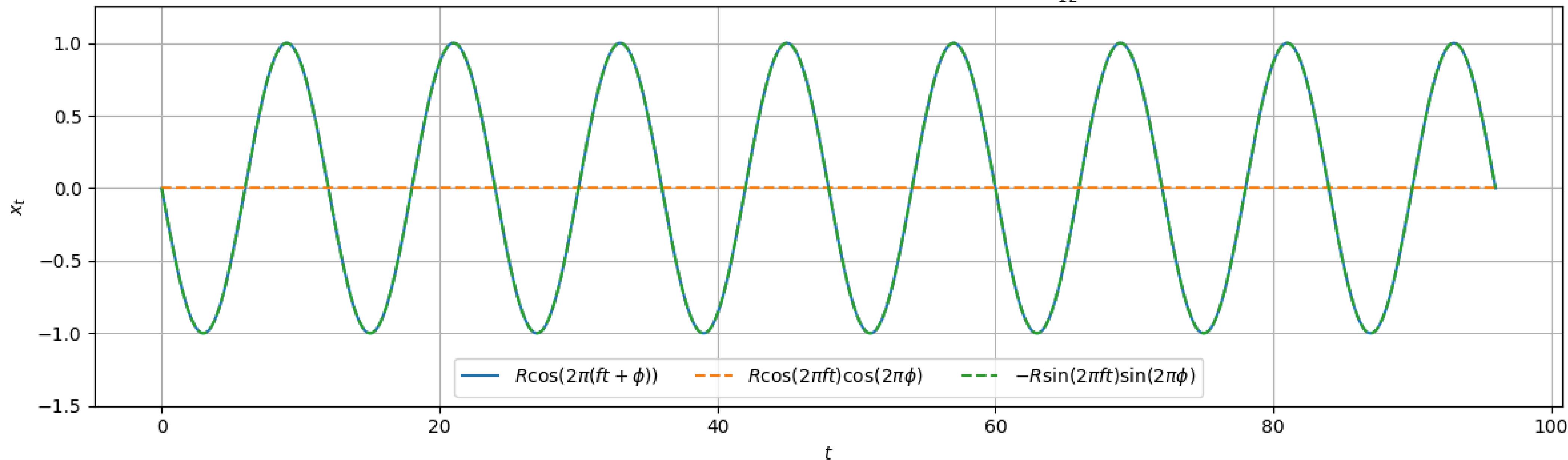
with $A = R \cos(2\pi\phi)$, $B = -R \sin(2\pi\phi)$, $c_t = \cos(2\pi ft)$ and $s_t = \sin(2\pi ft)$.

Then $R = \sqrt{A^2 + B^2}$ and $\phi = \arctan\left(-\frac{B}{A}\right)/2\pi$.

Sum of sinusoids with the same frequency ($R = 1, f = \frac{1}{12}, \phi = 0.1$)



Sum of sinusoids with the same frequency ($R = 1, f = \frac{1}{12}, \phi = 0.25$)



Fitting sinusoids (1)

Given a time series realization $\{x_0, \dots, x_{n-1}\}$ with known period $P = 1/f$,

The data can be modeled as

$$x_t = \mu + Ac_t + Bs_t + e_t$$

with e_t the residual at time t , $c_t = \cos(2\pi ft)$ and $s_t = \sin(2\pi ft)$.

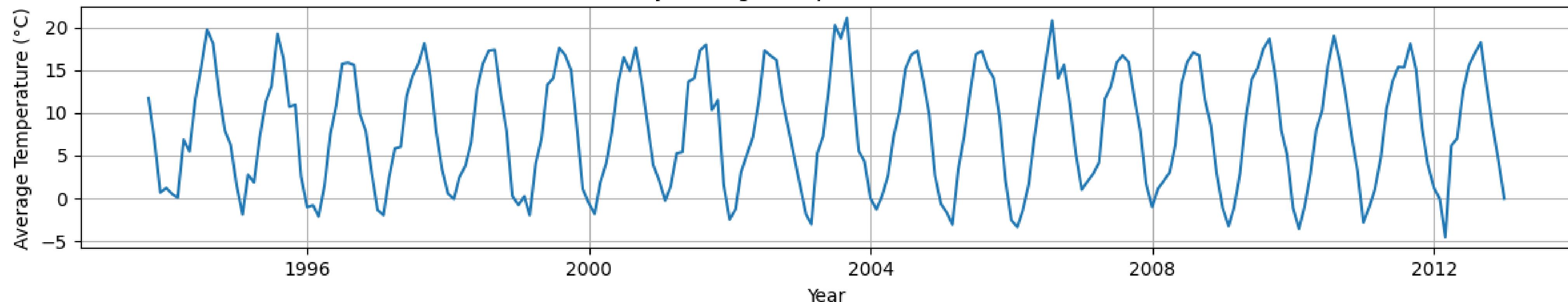
Apply OLS: $\begin{pmatrix} \hat{\mu} \\ \hat{A} \\ \hat{B} \end{pmatrix} = \min_{\mu, A, B} \sum_{i=0}^{n-1} e_i^2 = (X^T X)^{-1} X^T Y$ with $Y = \begin{pmatrix} x_0 \\ \vdots \\ x_{n-1} \end{pmatrix}$ and $X = \begin{pmatrix} 1 & c_0 & s_0 \\ \vdots & \vdots & \vdots \\ 1 & c_{n-1} & s_{n-1} \end{pmatrix}$

X columns are **orthogonal** provided n is an **integer multiple** of P .

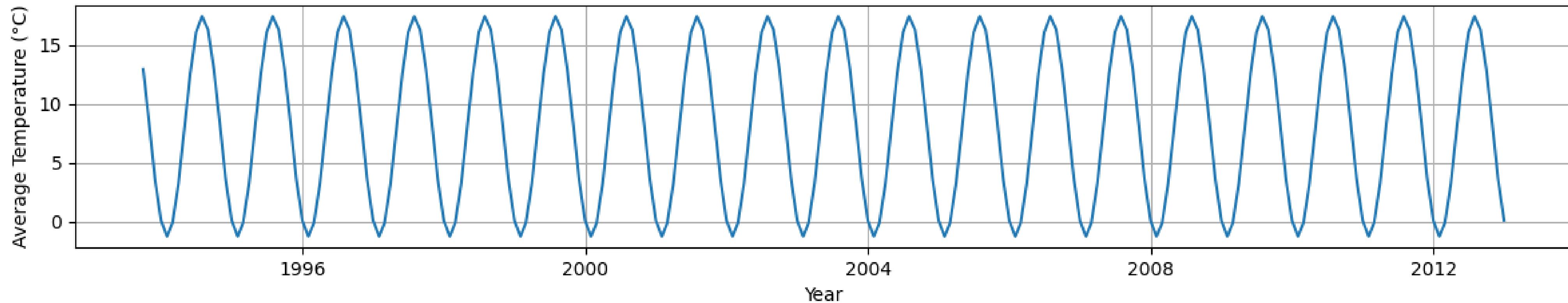
In this case, this optimization problem can be solved with

$$\hat{\mu} = \frac{1}{n} \sum_{t=0}^{n-1} x_t, \quad \hat{A} = \frac{2}{n} \sum_{t=0}^{n-1} x_t c_t, \quad \hat{B} = \frac{2}{n} \sum_{t=0}^{n-1} x_t s_t$$

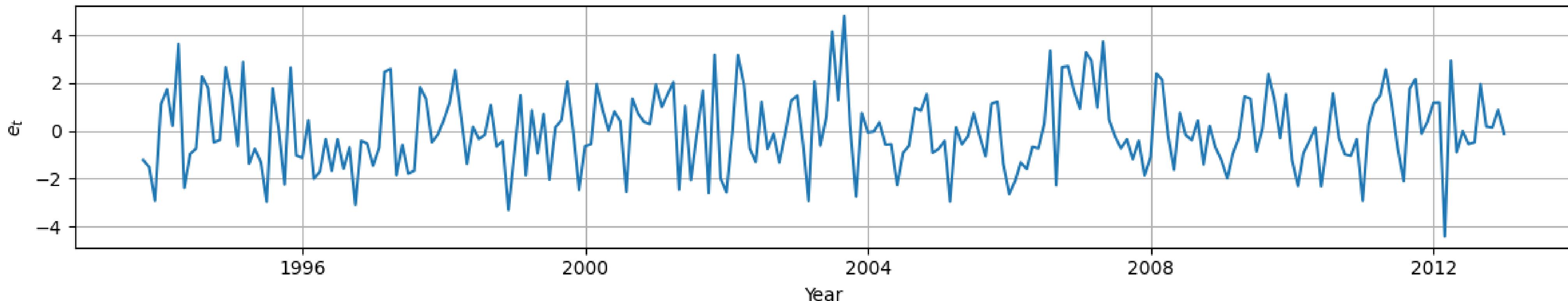
Monthly average temperature in Switzerland



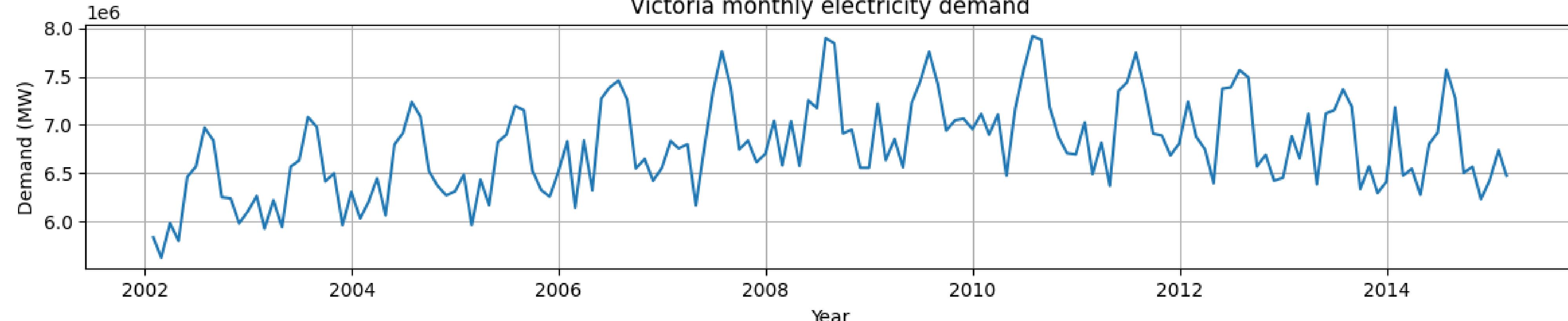
Fitted sinusoid + mean



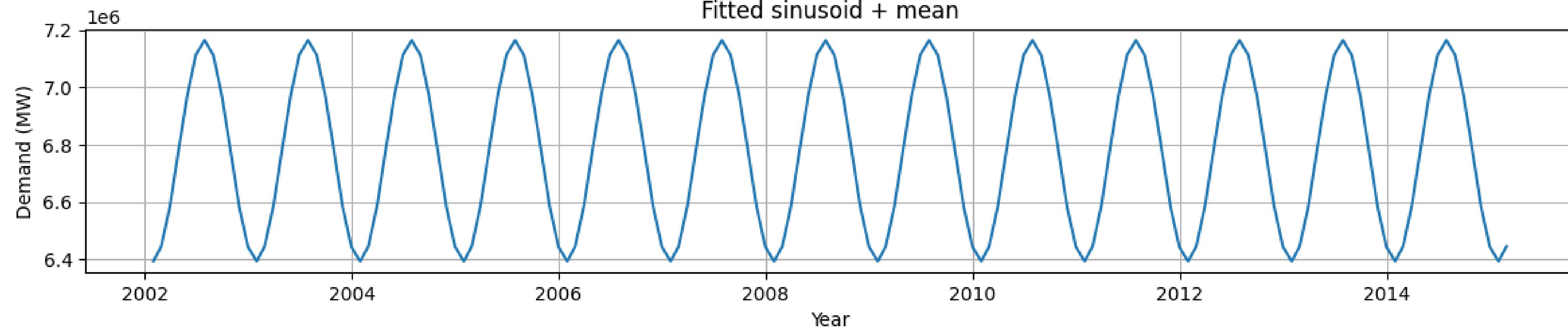
Residuals



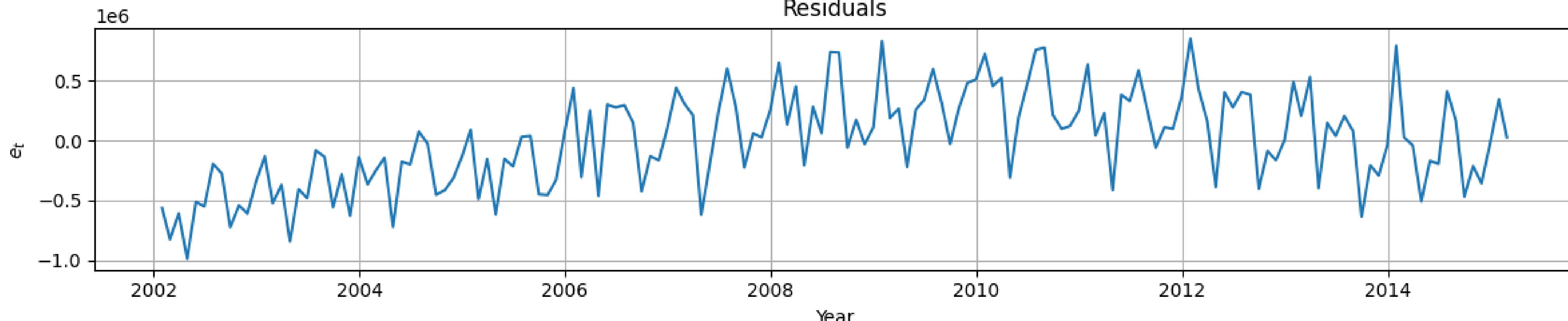
Victoria monthly electricity demand



Fitted sinusoid + mean



Residuals



Fitting sinusoids (2)

When the TS involves k seasonalities of known periods P_1, \dots, P_k with $P_j = 1/f_j$ for $j = 1, \dots, k$,

The data can be modeled as

$$x_t = \mu + \sum_{j=1}^k (A_j c_{t,j} + B_j s_{t,j}) + e_t$$

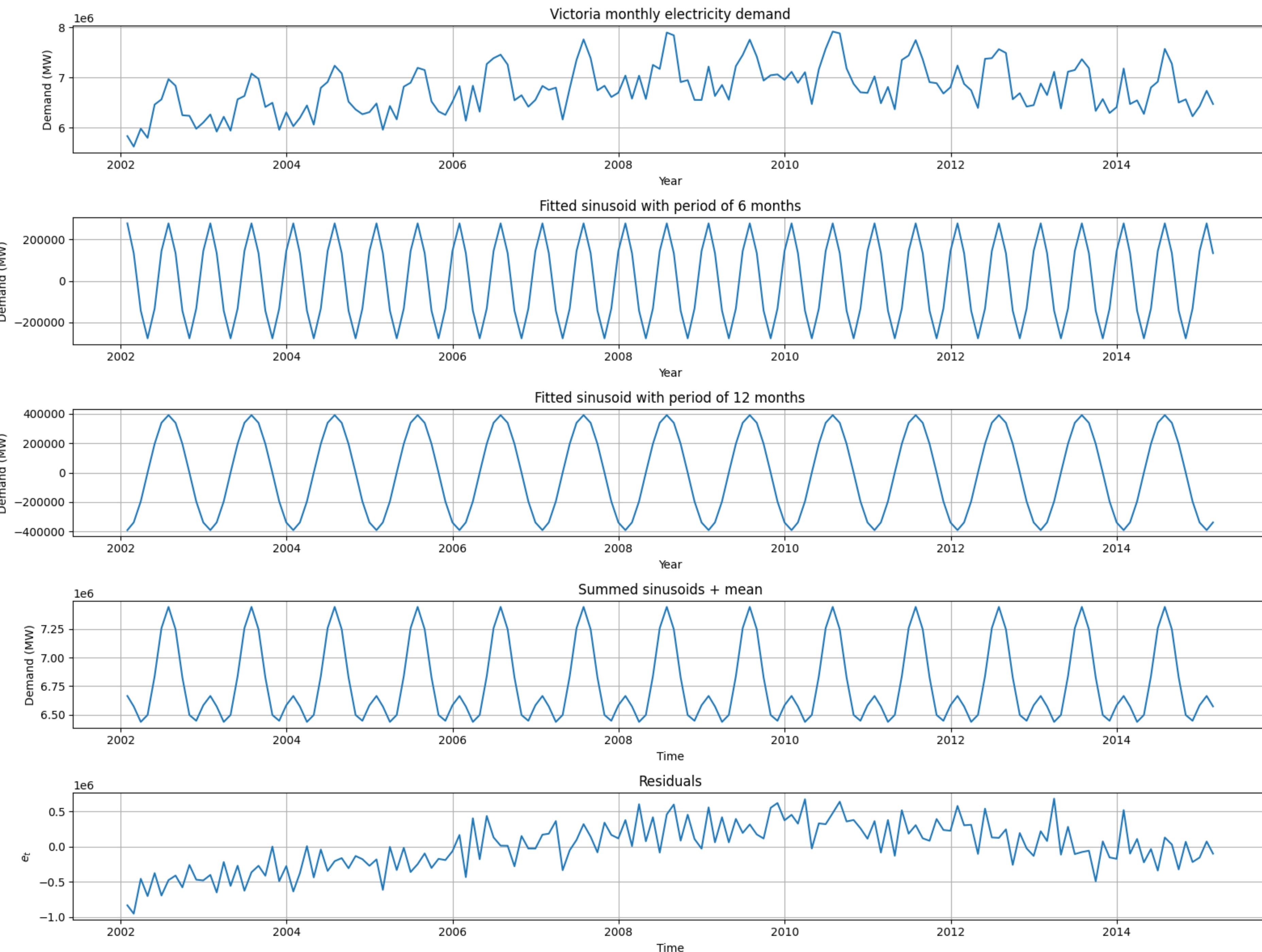
with $c_{t,j} = \cos(2\pi f_j t)$, $s_{t,j} = \sin(2\pi f_j t)$ and e_t the residual at time t .

The design matrix becomes $X = \begin{pmatrix} 1 & c_{0,1} & s_{0,1} & \dots & c_{0,k} & s_{0,k} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & c_{n-1,1} & s_{n-1,1} & \dots & c_{n-1,k} & s_{n-1,k} \end{pmatrix}$

X columns are **orthogonal** provided n is an **integer multiple of all periods**.

In this case, this optimization problem can be solved with

$$\hat{\mu} = \frac{1}{n} \sum_{t=0}^{n-1} x_t, \quad \hat{A}_j = \frac{2}{n} \sum_{t=0}^{n-1} x_t c_{t,j}, \quad \hat{B}_j = \frac{2}{n} \sum_{t=0}^{n-1} x_t s_{t,j}$$



Harmonic analysis with Fourier frequencies

Given a TS realization $\{x_0, \dots, x_{n-1}\}$, the **Fourier frequencies** are frequencies of the form $f_j = j/n$ with $j = 0, 1, \dots, k$.

- Due to aliasing, we consider only frequencies satisfying $0 \leq f_j \leq \frac{1}{2}$ (since $\Delta = 1$) i.e., $k = \begin{cases} n/2, & n \text{ is even} \\ (n-1)/2, & n \text{ is odd} \end{cases}$.

These frequencies are **harmonics** (integer multiples) with respect to the span of the data i.e., each of the corresponding periods $P_j = n/j$ is **repeated j times**.

- Sinusoids at the Fourier frequencies (X columns) are **orthogonal** over the span of the data n .

The TS realization can be **exactly** represented as a **sum of k sinusoids at the Fourier frequencies**:

$$x_t = \mu + \sum_{j=1}^k (A_j c_{t,j} + B_j s_{t,j})$$

$$\mu = \frac{1}{n} \sum_{t=0}^{n-1} x_t = \frac{A_0}{2}, \quad A_j = \frac{2}{n} \sum_{t=0}^{n-1} x_t c_{t,j}, \quad B_j = \frac{2}{n} \sum_{t=0}^{n-1} x_t s_{t,j}$$

Recap – Complex numbers

Complex numbers \mathbb{C} **extend** real numbers \mathbb{R} with the **imaginary unit** i that satisfies $i^2 = 1$.

- $z = Re(z) + i Im(z)$ with $Re(z)$ the **real part** and $Im(z)$ the **imaginary part** of z .

Complex numbers can also be defined in **polar coordinates**: $z = |z|(\cos(\varphi) + i \sin(\varphi))$.

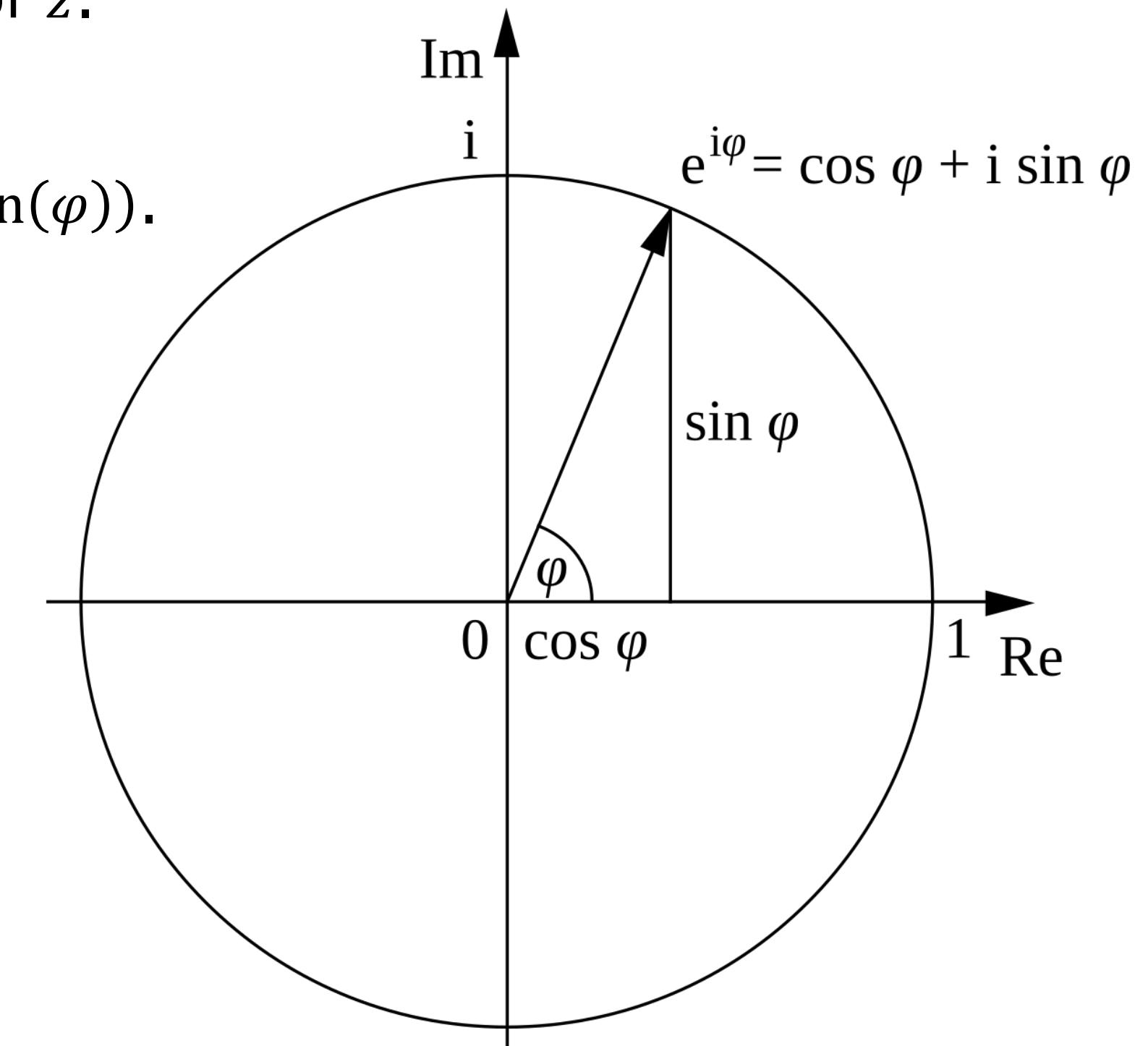
- $Re(z) = |z| \cos(\varphi)$ and $Im(z) = |z| \sin(\varphi)$.
- **Magnitude** $|z| = \sqrt{Re(z)^2 + Im(z)^2}$ and **phase** $\varphi = \arctan(Im(z)/Re(z))$.

Euler formula: $e^{i\varphi} = \cos(\varphi) + i \sin(\varphi)$. Hence, $e^{i\varphi} = -e^{i(\varphi+\pi)}$.

- Any $z \in \mathbb{C}$ can be defined as $z = |z|e^{i\varphi}$.
- $\cos(\varphi) = \frac{1}{2}(e^{i\varphi} + e^{-i\varphi})$ and $\sin(\varphi) = \frac{1}{2i}(e^{i\varphi} - e^{-i\varphi})$.

The **complex conjugate** of z is defined as $\bar{z} = Re(z) - i Im(z) = |z|e^{-i\varphi}$.

- $\bar{\bar{z}} = z$ and the complex conjugate of $x \in \mathbb{R}$ is $\bar{x} = x$.
- $z\bar{z} = Re(z)^2 + Im(z)^2 = |z|^2$ where $|z|$ is the modulus of z .



Gunther, wikimedia

Orthogonality of complex exponentials

The sum of complex numbers **equally spaced over the unit circle** $z_{t,j} = e^{2\pi i \frac{j}{n} t}$ (called the **n-th roots of unity** as $z_{t,j}$ satisfies $z_{t,j}^n = 1$) **over one period** n is zero for any non-zero integer $0 < t < n$,

$$\sum_{j=0}^{n-1} e^{2\pi i \frac{j}{n} t} = \delta_{t0} n = \begin{cases} n & \text{if } t = 0 \\ 0 & \text{otherwise}, \end{cases} \quad \text{with } \delta_{kl} = \begin{cases} 1 & \text{if } k = l \\ 0 & \text{otherwise} \end{cases} \text{ the Kronecker delta}$$

Proof: When $t = 0$, we have $\sum_{j=0}^{n-1} e^0 = n$. When $t \neq 0$, the sum is a geometric series:

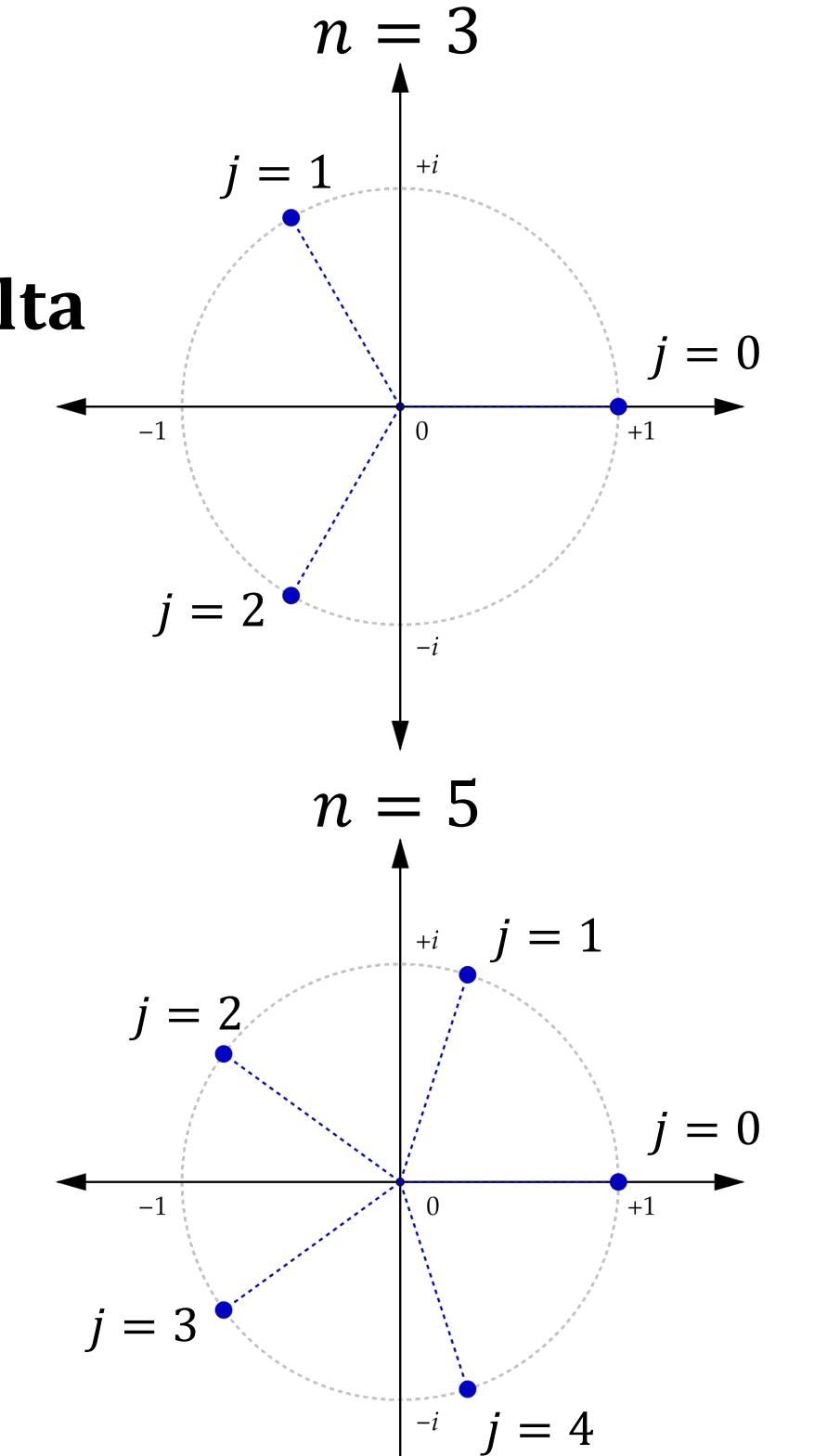
$$\sum_{j=0}^{n-1} e^{2\pi i \frac{j}{n} t} = \sum_{j=0}^{n-1} \left(e^{2\pi i \frac{1}{n} t} \right)^j = 1 + \left(e^{2\pi i \frac{1}{n} t} \right)^1 + \left(e^{2\pi i \frac{1}{n} t} \right)^2 + \dots + \left(e^{2\pi i \frac{1}{n} t} \right)^{n-1}$$

(recall that $\sum_{j=0}^{n-1} r^j = \frac{1-r^n}{1-r}$ and that $e^{2\pi i t} = 1$ for integer t)

$$= \frac{1 - \left(e^{2\pi i \frac{1}{n} t} \right)^n}{1 - e^{2\pi i \frac{1}{n} t}} = \frac{1 - e^{2\pi i t}}{1 - e^{2\pi i \frac{1}{n} t}} = 0$$

Such complex exponentials are **orthogonal** over a complete cycle: taking $\mathbf{z}_k = \{z_{0,k}, \dots z_{n-1,k}\}$,

$$\langle \mathbf{z}_k, \mathbf{z}_l \rangle = \sum_{t=0}^{n-1} z_k \bar{z}_l = \sum_{t=0}^{n-1} e^{2\pi i \frac{k}{n} t} e^{-2\pi i \frac{l}{n} t} = \sum_{t=0}^{n-1} e^{2\pi i \frac{t}{n} (k-l)} = \delta_{kl} n$$



Discrete Fourier transform (DFT)

Given a TS realization $x = \{x_0, \dots, x_{n-1}\}$, the **Discrete Fourier Transform** $\mathcal{F}(x) = X = \{X_0, \dots, X_{n-1}\}$ is defined as

$$\mathcal{F}(x)_j = X_j = \frac{1}{n} \sum_{t=0}^{n-1} x_t e^{-2\pi i f_j t}, \quad j = 0, 1, \dots, k, \dots, n-1$$

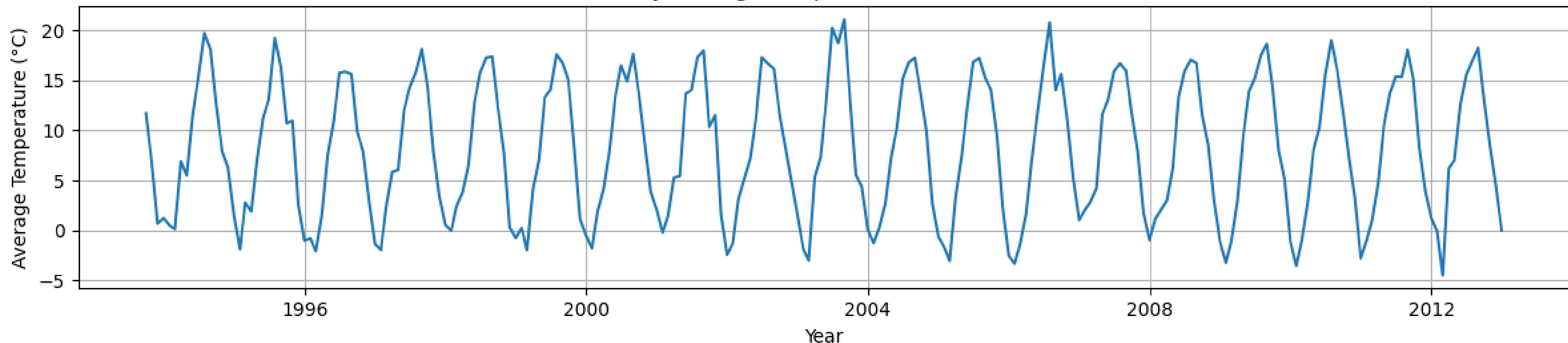
with $f_j = j/n$ the **Fourier frequencies** and $k = \begin{cases} n/2, & n \text{ is even} \\ (n-1)/2, & n \text{ is odd} \end{cases}$.

For **real** x_t we observe that $X_{n-j} = \bar{X}_j$ (**conjugate symmetry**) i.e., the frequencies f_j for $j > k$ are **redundant**:

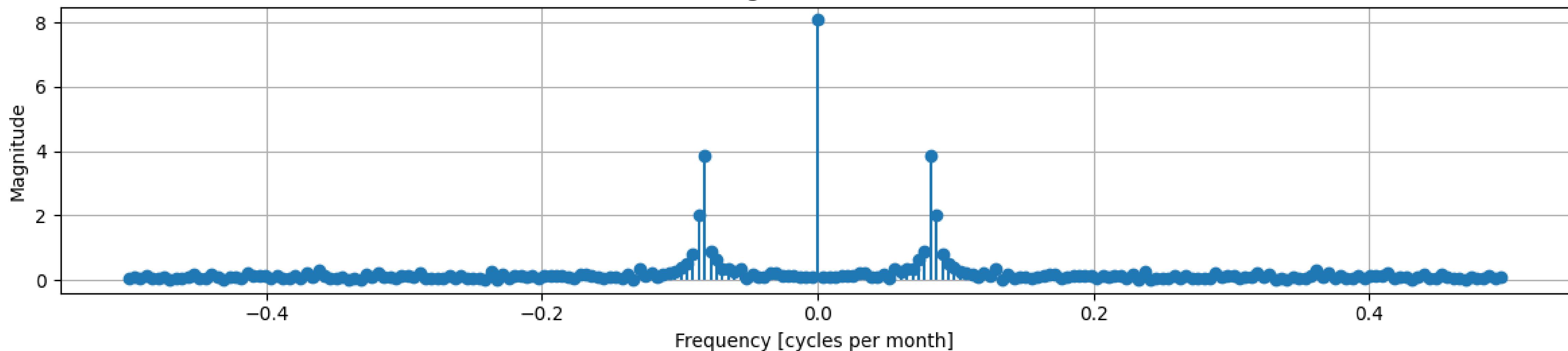
$$\begin{aligned} X_{n-j} &= \frac{1}{n} \sum_{t=0}^{n-1} x_t e^{-2\pi i \left(\frac{n-j}{n}\right)t} \\ &= \frac{1}{n} \sum_{t=0}^{n-1} x_t e^{-2\pi i t} e^{2\pi i \frac{j}{n} t} \\ &= \frac{1}{n} \sum_{t=0}^{n-1} x_t \cdot 1 \cdot e^{2\pi i f_j t} = \bar{X}_j \end{aligned}$$

$(e^{2\pi i k} = 1 \text{ for integer } k)$

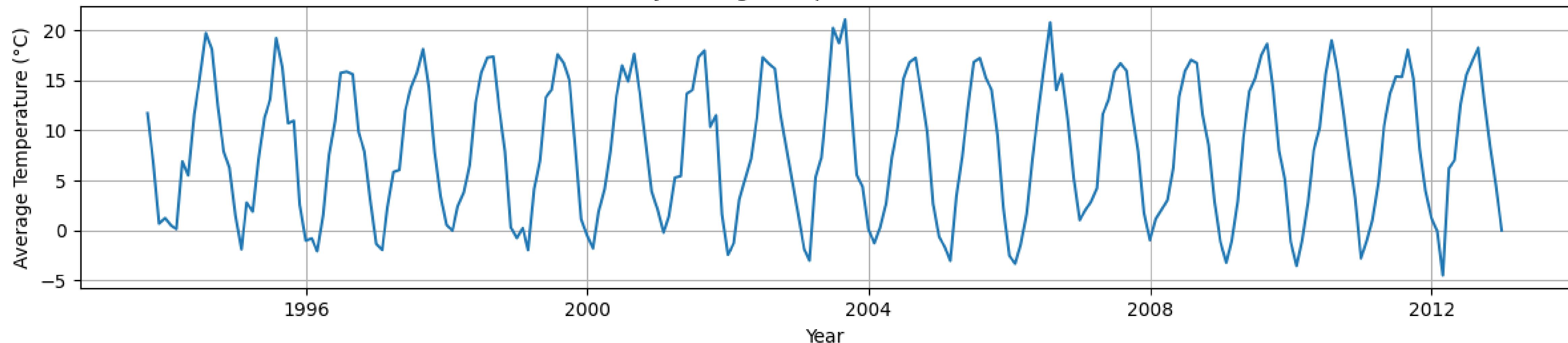
Monthly average temperature in Switzerland



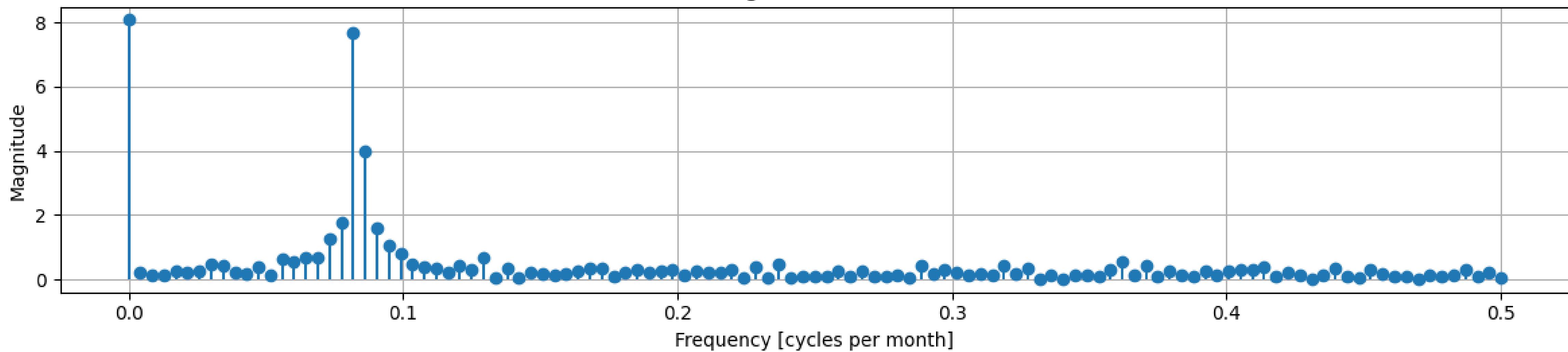
DFT Magnitude (Two-sided)



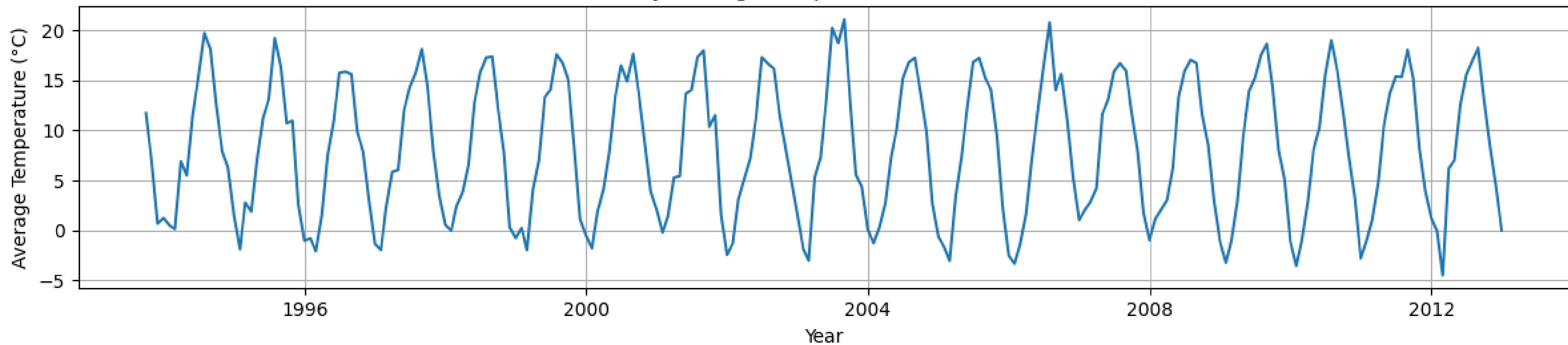
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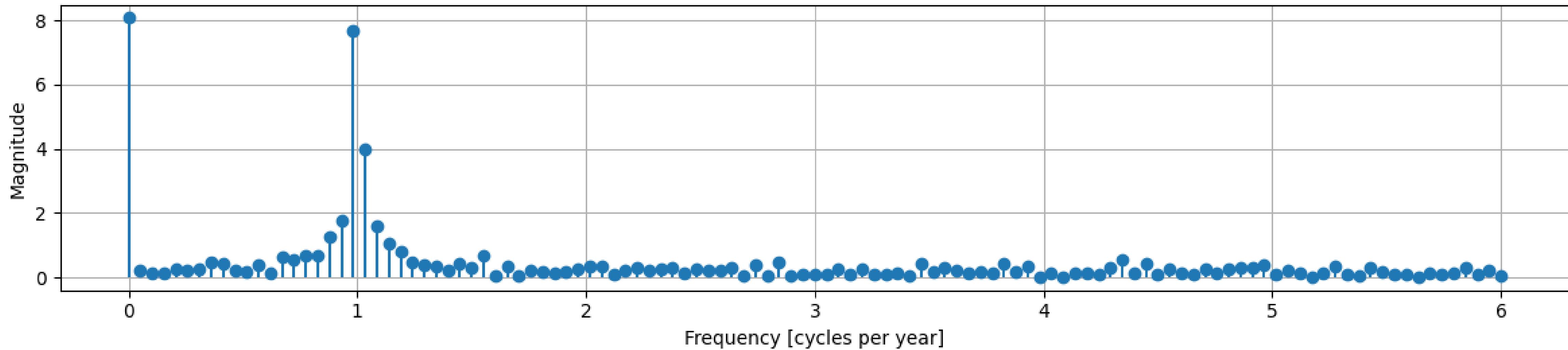
DFT Magnitude (One-sided)



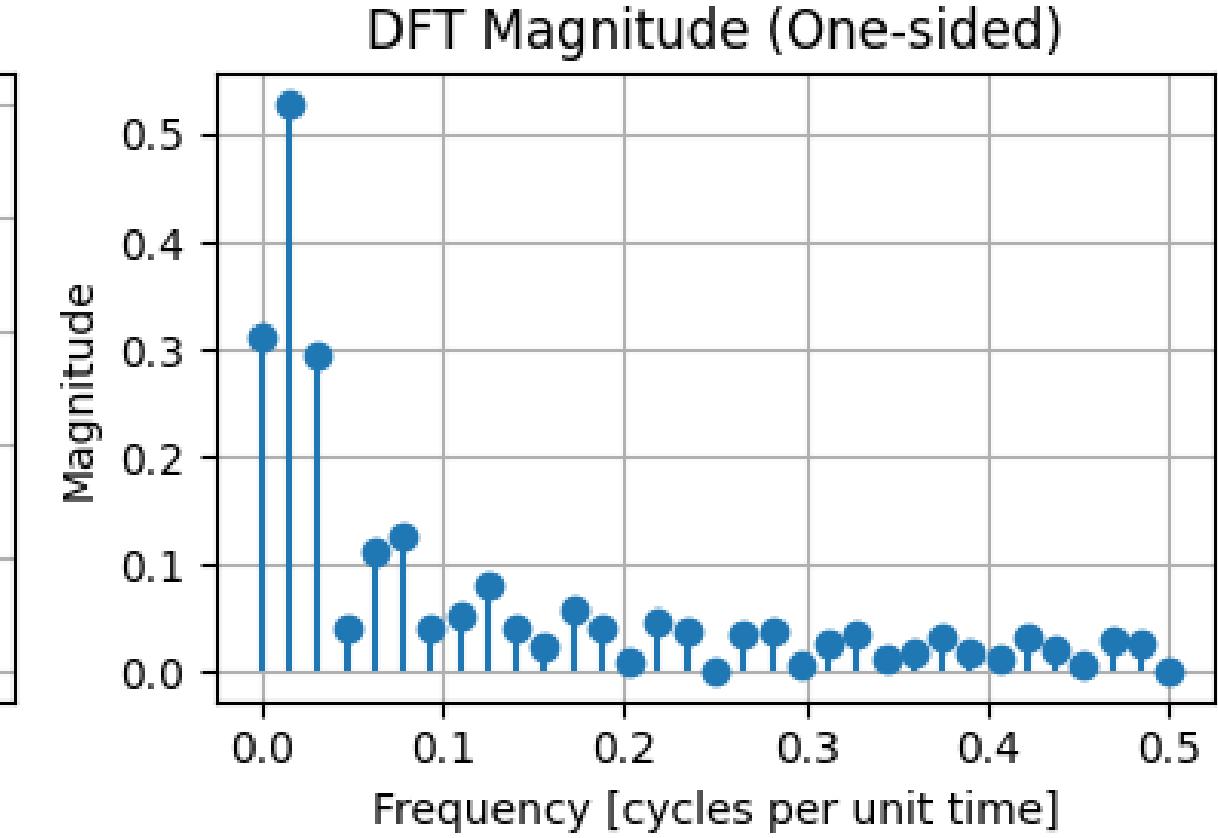
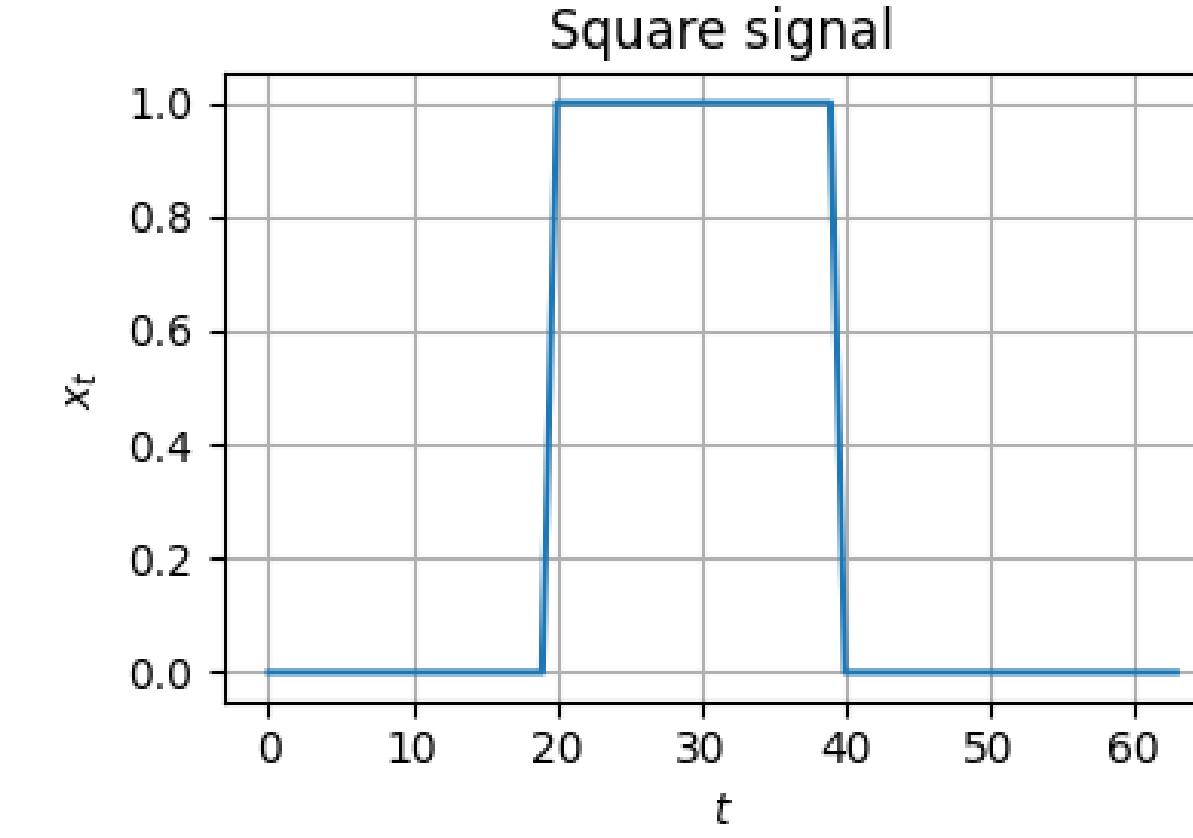
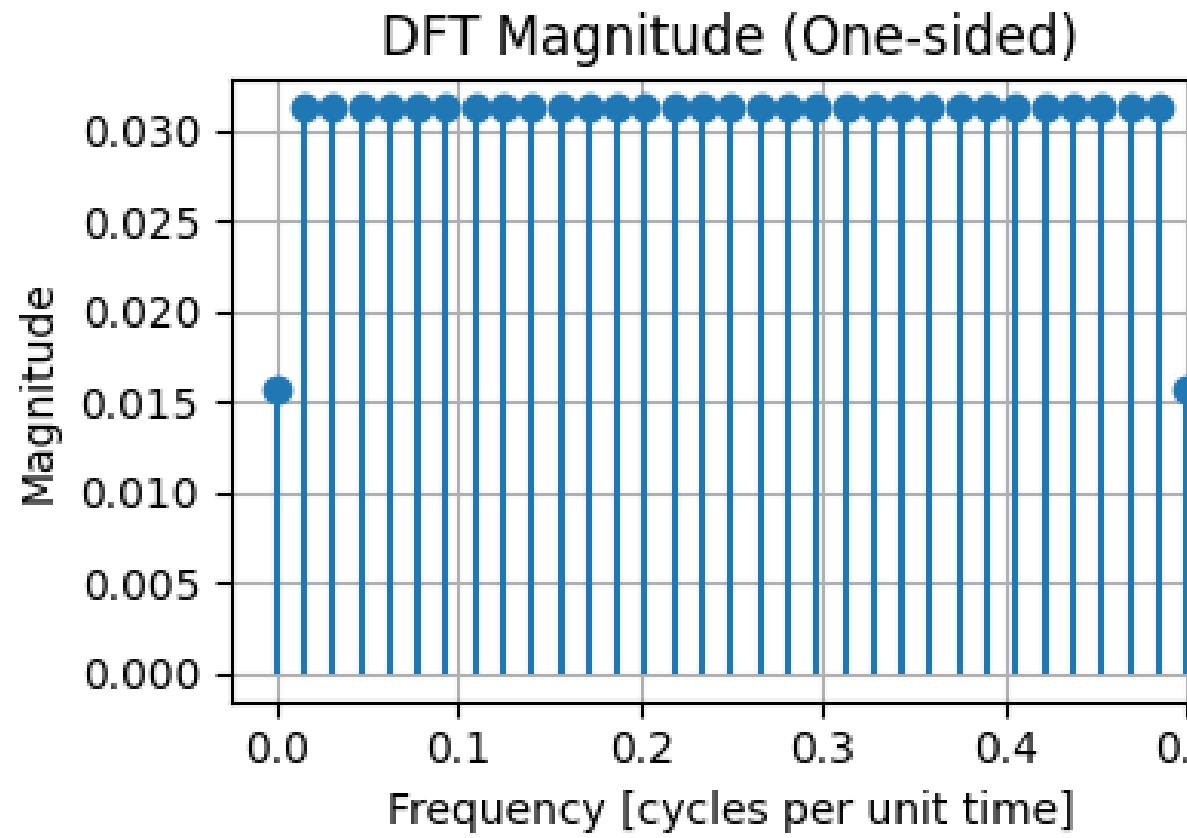
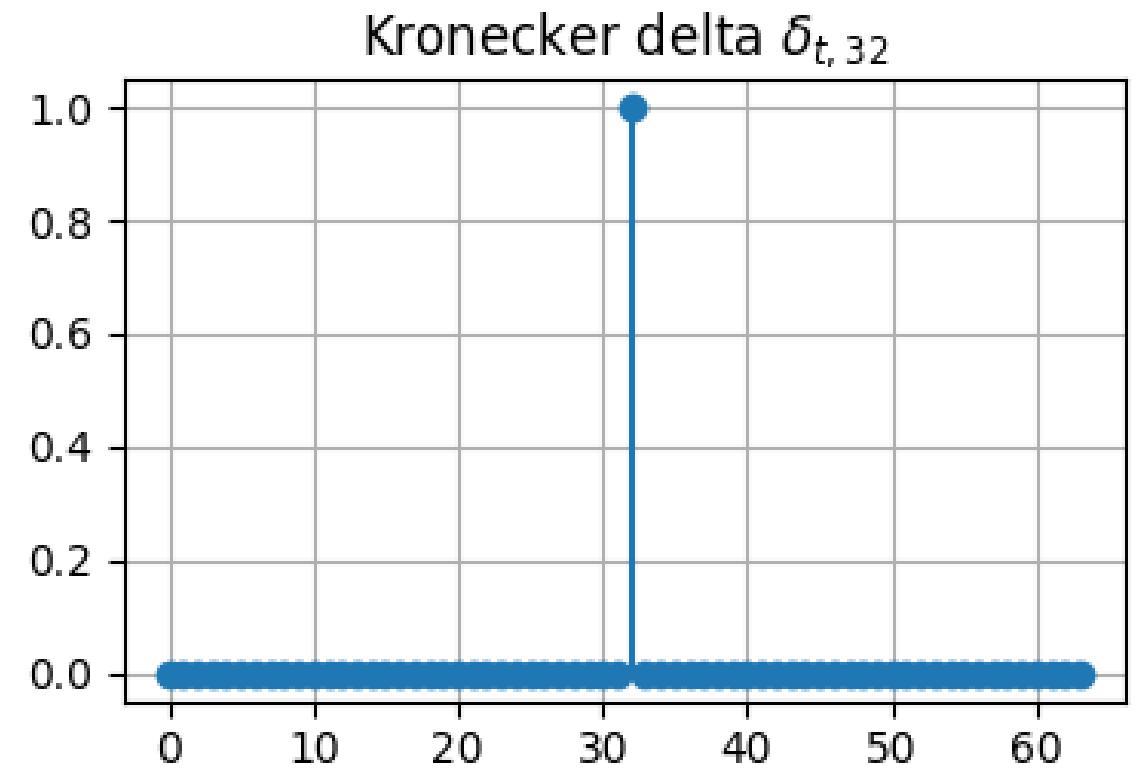
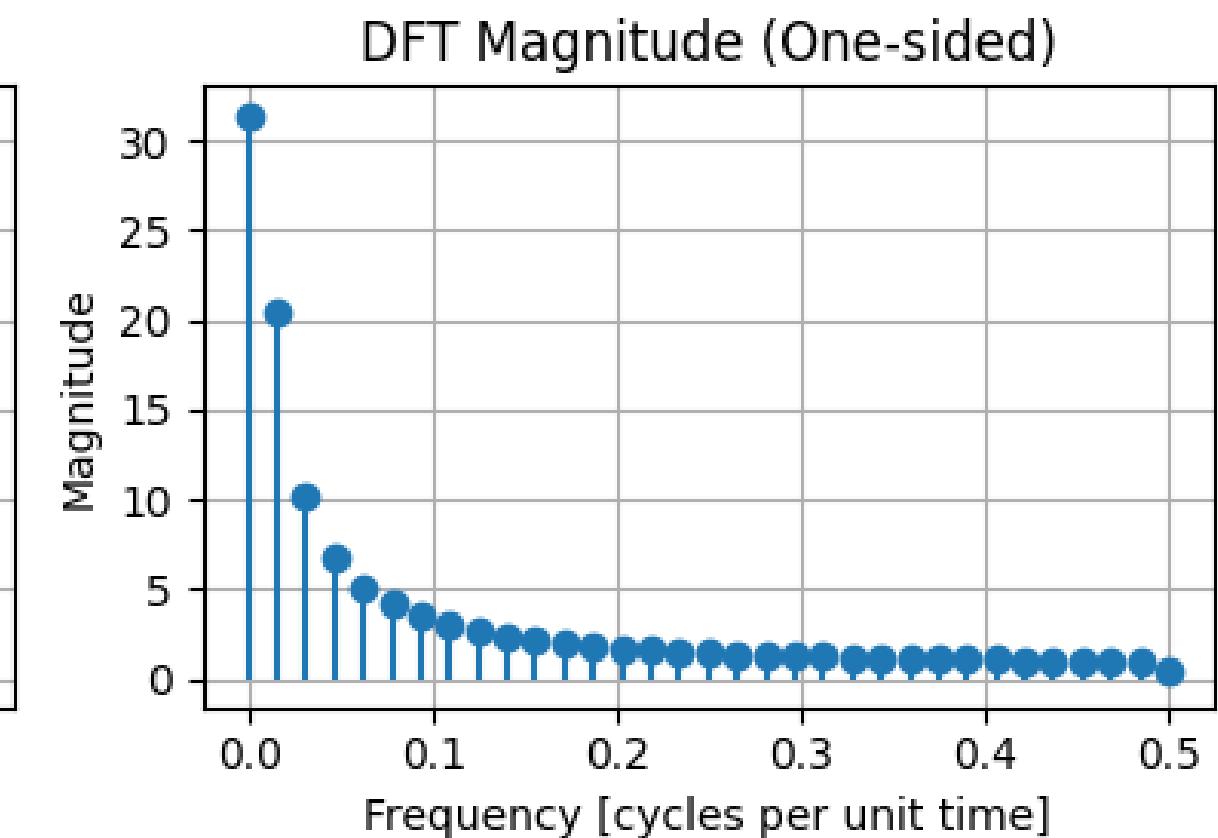
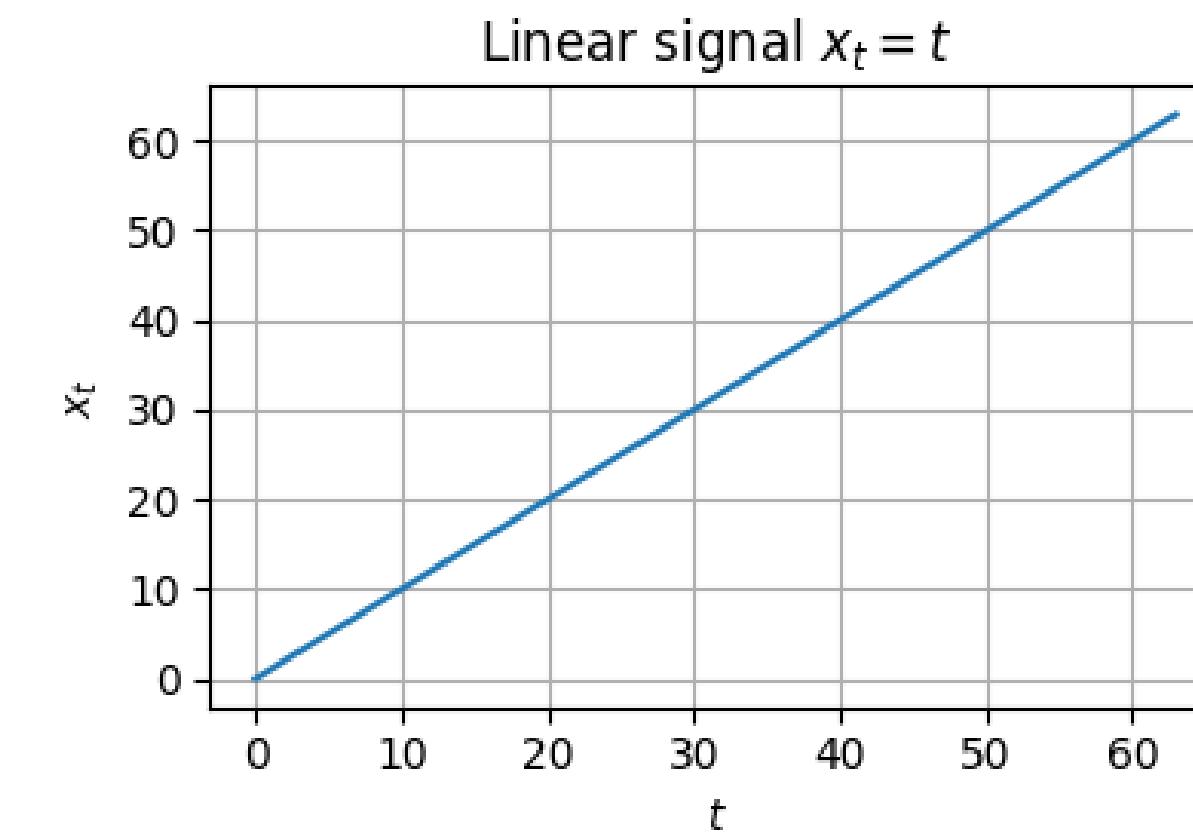
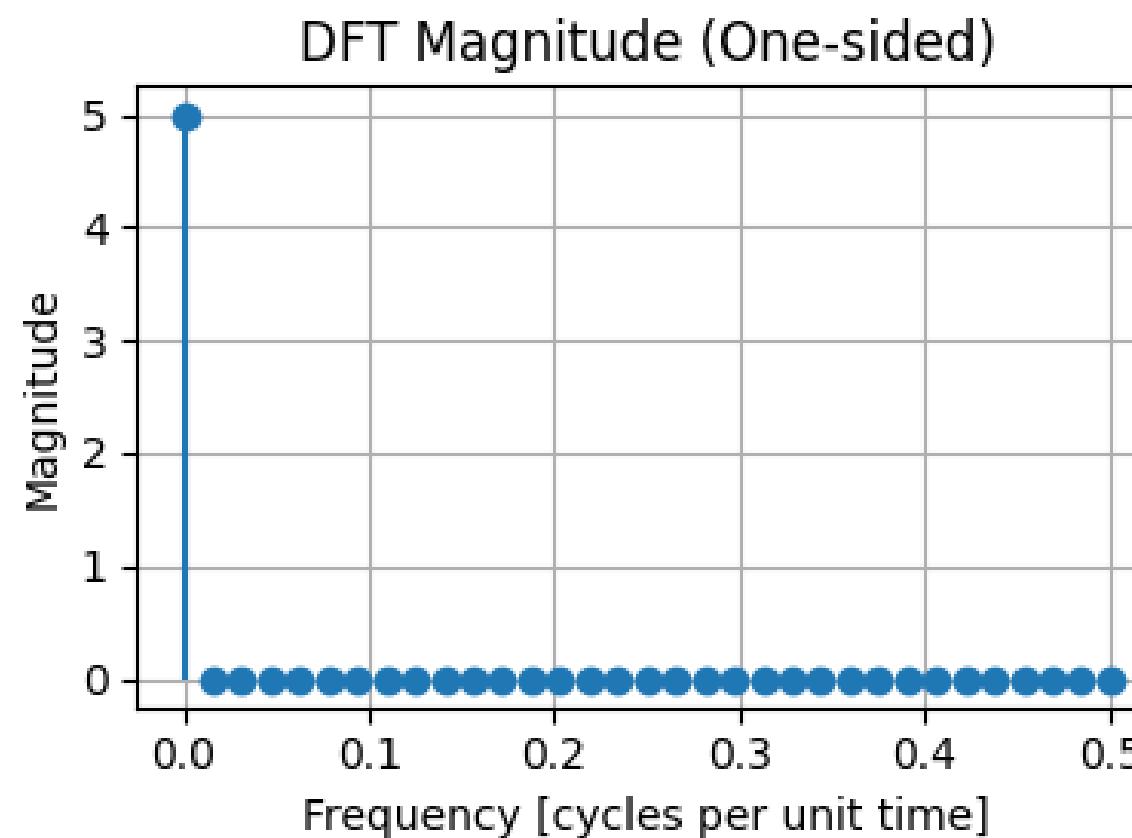
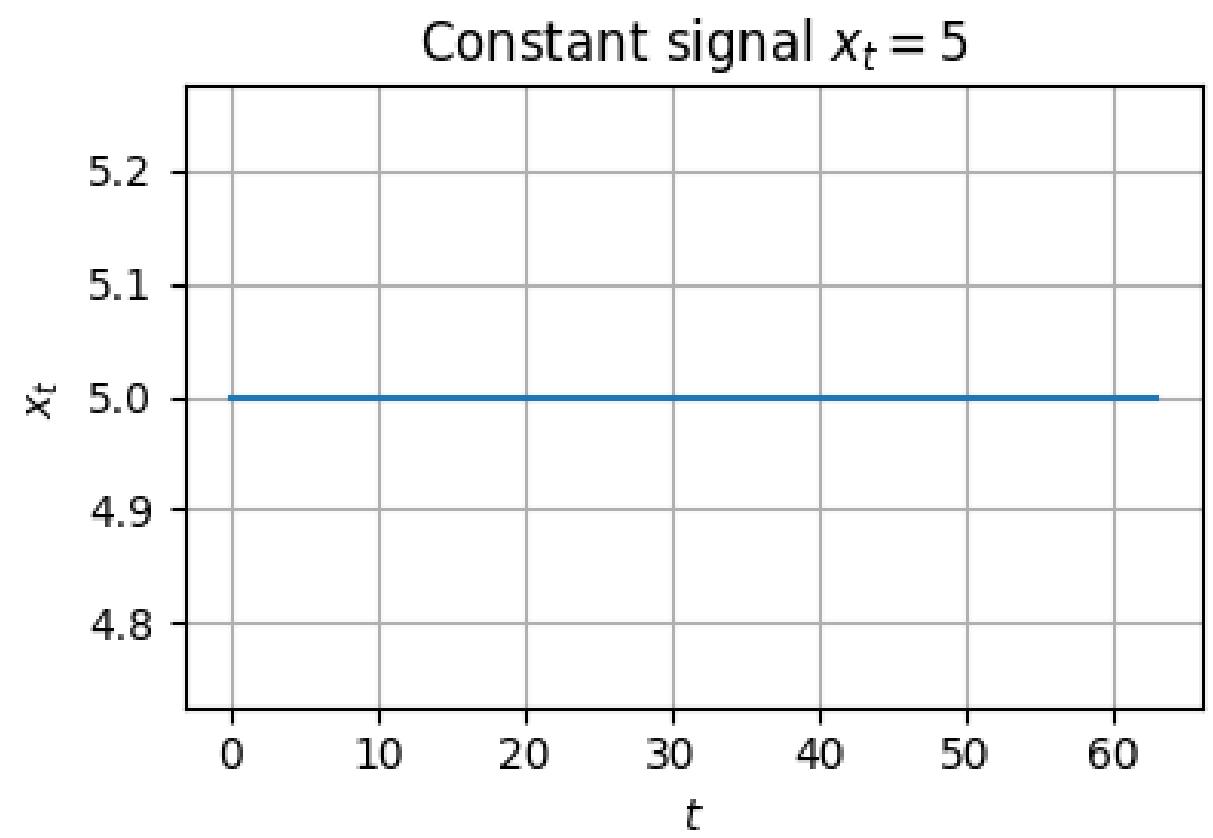
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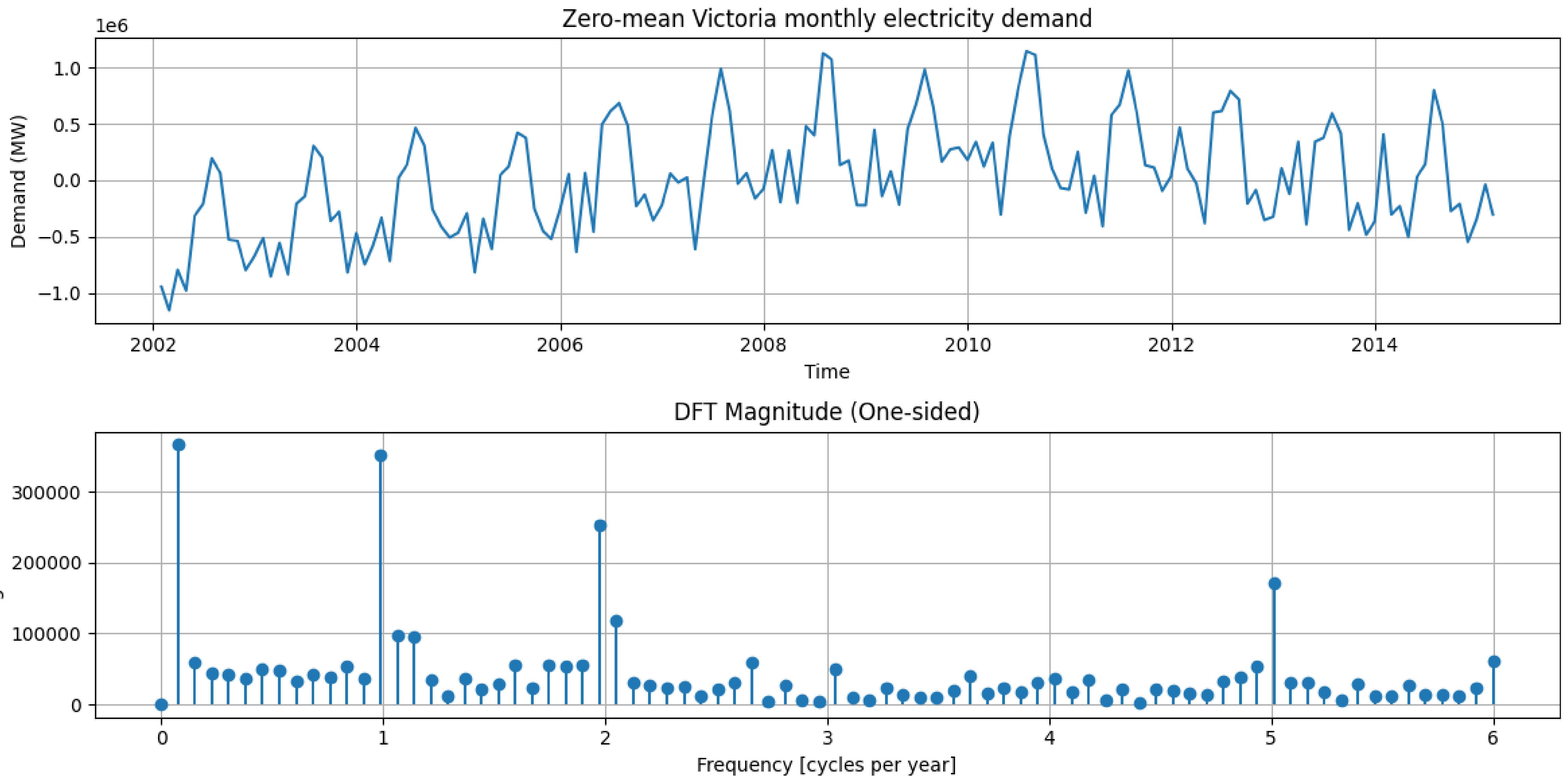


DFT Magnitude (One-sided)



Discrete Fourier transform of simple signals





Discrete Fourier transform – Connection with sinusoids

$(\sin(-x) = -\sin(x) \text{ and } \cos(-x) = \cos(x))$

(recall that $A_j = \frac{2}{n} \sum_{t=0}^{n-1} x_t c_{t,j}$, $B_j = \frac{2}{n} \sum_{t=0}^{n-1} x_t s_{t,j}$)

(observe that for real input, the sinusoid magnitude is split in half $|X_j| = \frac{1}{2} \sqrt{A_j^2 + B_j^2} = |\overline{X_{n-j}}|$)

$$\begin{aligned}
 \mathcal{F}(x)_j &= X_j = \frac{1}{n} \sum_{t=0}^{n-1} x_t e^{-2\pi i f_j t} \\
 &= \frac{1}{n} \sum_{t=0}^{n-1} x_t (\cos(-2\pi f_j t) + i \sin(-2\pi f_j t)) \\
 &= \frac{1}{n} \sum_{t=0}^{n-1} x_t (\cos(2\pi f_j t) - i \sin(2\pi f_j t)) \\
 &= \frac{1}{n} \sum_{t=0}^{n-1} x_t (c_{t,j} - i s_{t,j}) \\
 &= \underbrace{\frac{1}{n} \sum_{t=0}^{n-1} x_t c_{t,j}}_{Re(X_j) = \frac{A_j}{2}} - i \underbrace{\frac{1}{n} \sum_{t=0}^{n-1} x_t s_{t,j}}_{Im(X_j) = \frac{B_j}{2}} \\
 &= \frac{A_j}{2} - i \frac{B_j}{2}
 \end{aligned}$$

Inverse discrete Fourier transform

Given $\mathcal{F}(x) = X = \{X_0, \dots, X_{n-1}\}$, the **inverse DFT** $\mathcal{F}^{-1}(X) = x = \{x_0, \dots, x_{n-1}\}$ is defined as

$$x_t = \mathcal{F}^{-1}(X)_t = \sum_{j=0}^{n-1} X_j e^{2\pi i f_j t}, \quad t = 0, 1, \dots, n-1$$

Proof:

$$\begin{aligned} \mathcal{F}^{-1}(X)_t &= \sum_{j=0}^{n-1} \left(\frac{1}{n} \sum_{k=0}^{n-1} x_k e^{-2\pi i f_j k} \right) e^{2\pi i f_j t} \\ &= \frac{1}{n} \sum_{j=0}^{n-1} \sum_{k=0}^{n-1} x_k e^{2\pi i f_j (t-k)} = \frac{1}{n} \sum_{j=0}^{n-1} \sum_{k=0}^{n-1} x_k e^{2\pi i \frac{j}{n} (t-k)} \\ &\quad (\text{orthogonality of complex exponentials}) \\ &= \frac{1}{n} \sum_{k=0}^{n-1} x_k \underbrace{\sum_{j=0}^{n-1} e^{2\pi i \frac{j}{n} (t-k)}}_{= \delta_{kt} n} = \frac{1}{n} x_t n = x_t \end{aligned}$$

Inverse discrete Fourier transform – Connection with sinusoids

Given a **real** TS realization $x = \{x_0, \dots x_{n-1}\}$,

$$\begin{aligned}
x_t &= \mathcal{F}^{-1}(X)_t = \sum_{j=0}^{n-1} X_j e^{2\pi i f_j t} \\
&= X_0 + \sum_{j=1}^k (X_j e^{2\pi i f_j t} + X_{n-j} e^{2\pi i f_{n-j} t}) & k = \begin{cases} n/2, & n \text{ is even} \\ (n-1)/2, & n \text{ is odd} \end{cases} \\
&= X_0 + \sum_{j=1}^k (X_j e^{2\pi i f_j t} + \bar{X}_j e^{-2\pi i f_j t}) & (\text{conjugate symmetry for real input}) \\
&= \left(\frac{A_0}{2} - i \frac{B_0}{2} \right) + \sum_{j=1}^k \left(\left(\frac{A_j}{2} - i \frac{B_j}{2} \right) (c_{t,j} + i s_{t,j}) + \left(\frac{A_j}{2} + i \frac{B_j}{2} \right) (c_{t,j} - i s_{t,j}) \right) \\
&= \frac{A_0}{2} + \sum_{j=1}^k \left(\left(\frac{A_j}{2} c_{t,j} + \frac{A_j}{2} i s_{t,j} - i \frac{B_j}{2} c_{t,j} + \frac{B_j}{2} s_{t,j} \right) + \left(\frac{A_j}{2} c_{t,j} - \frac{A_j}{2} i s_{t,j} + i \frac{B_j}{2} c_{t,j} + \frac{B_j}{2} s_{t,j} \right) \right) \\
&= \mu + \sum_{j=1}^k (A_j c_{t,j} + B_j s_{t,j}) & \mu = \frac{A_0}{2}
\end{aligned}$$

Some properties of the discrete Fourier transform

Given two **real** TS realizations $x = \{x_0, \dots, x_{n-1}\}$ and $y = \{y_0, \dots, y_{n-1}\}$, their DFT $X = \{X_0, \dots, X_{n-1}\}$ and $Y = \{Y_0, \dots, Y_{n-1}\}$,

Linearity: let $z_t = ax_t + by_t$ then $Z_j = aX_j + bY_j$ for any scalars a, b .

Shifting

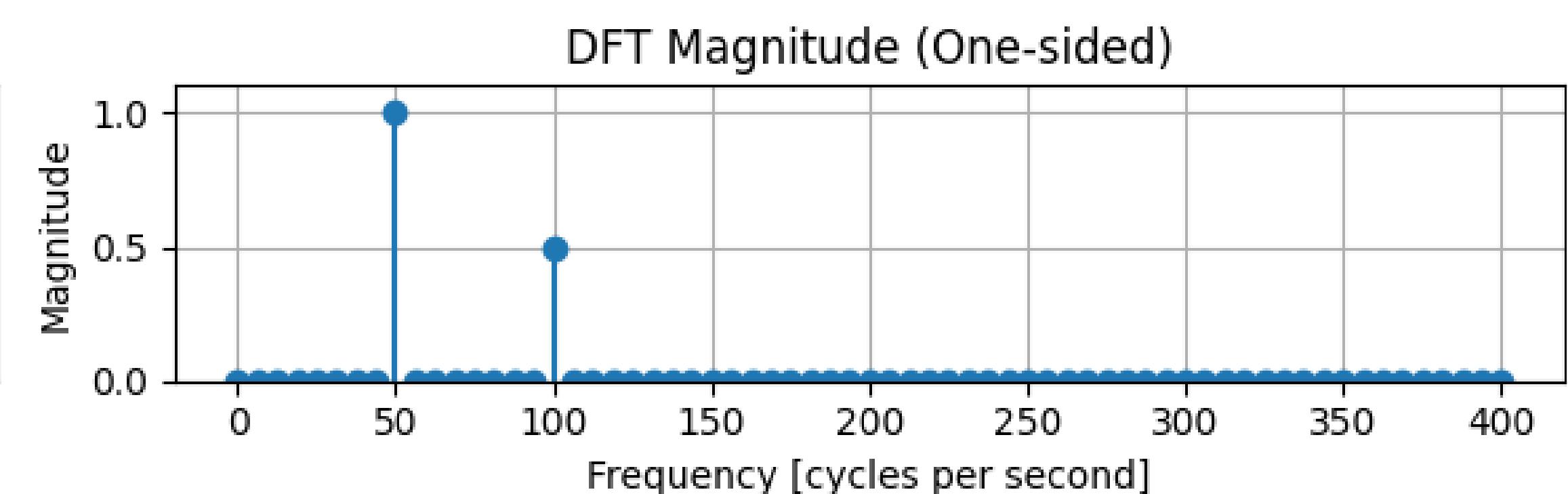
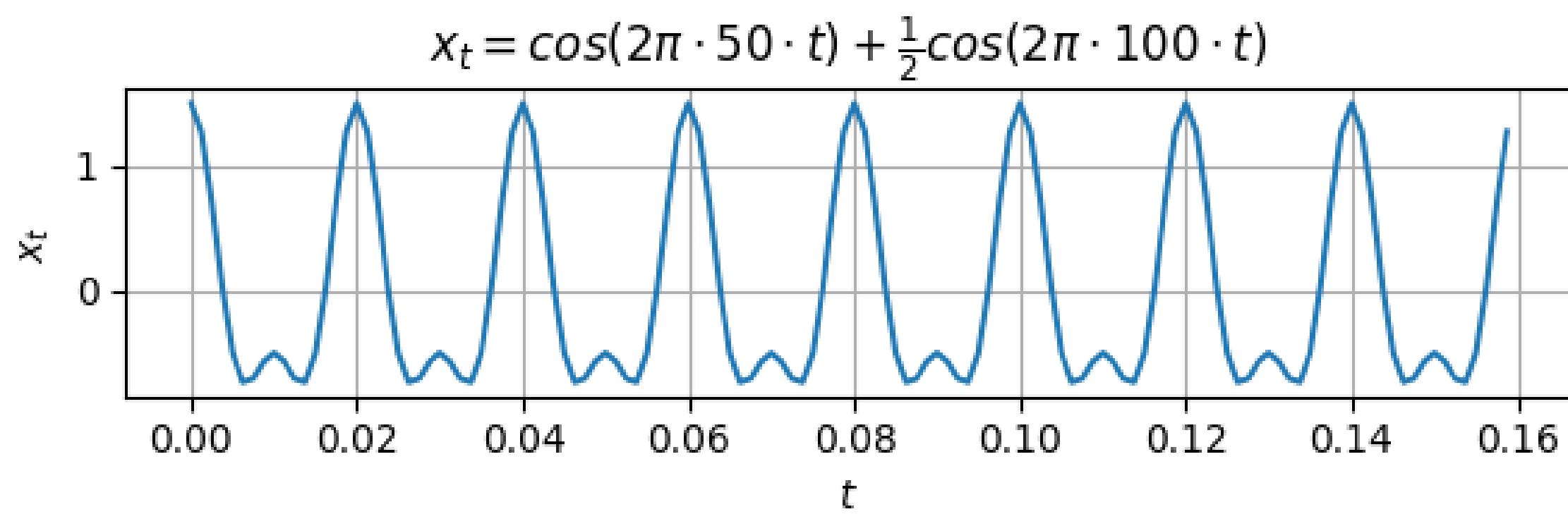
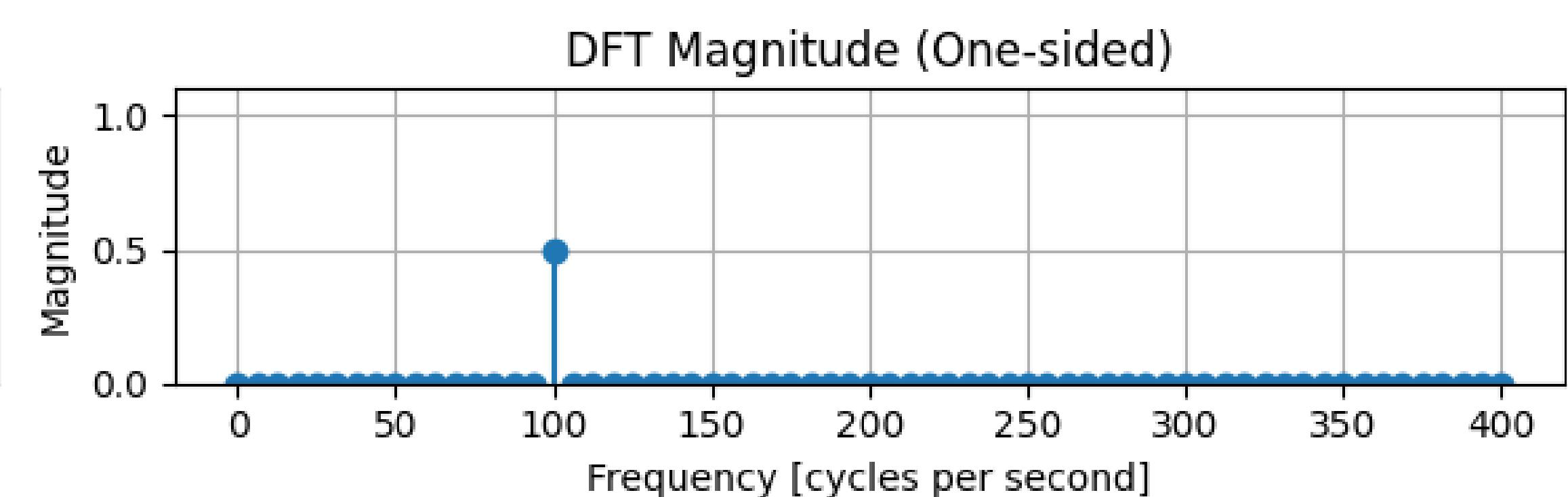
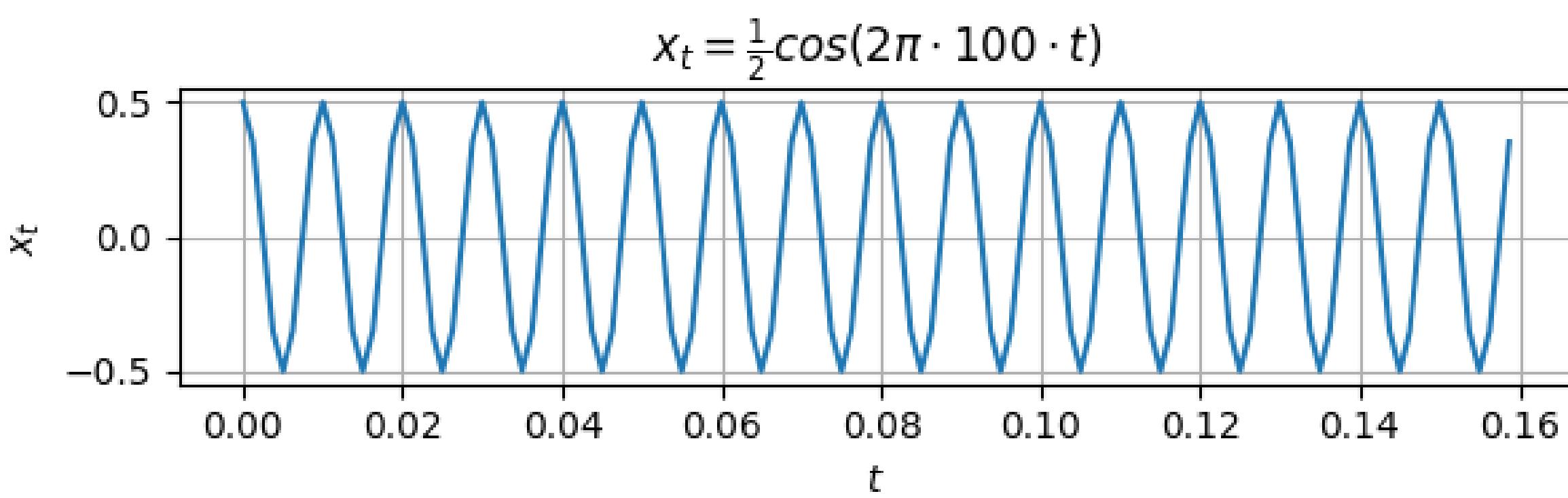
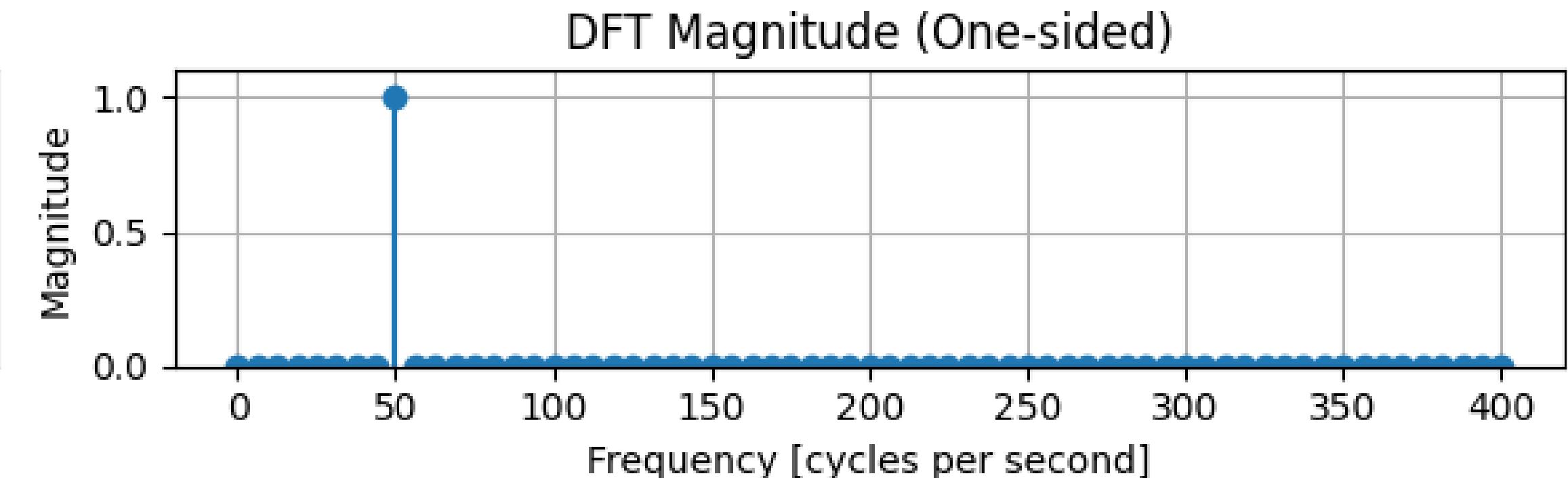
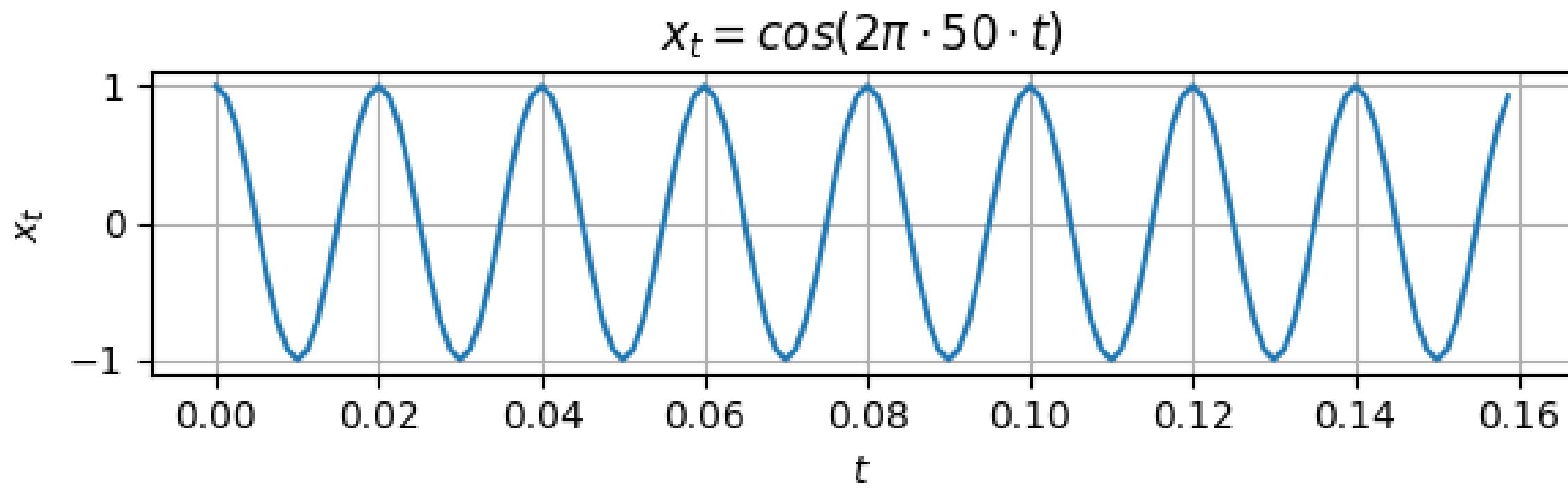
- Time domain (**lagging/leading TS**): let $z_t = x_{t-m}$ then $Z_j = X_j e^{-2\pi i f_j m}$
- Frequency domain (**phase shift**): let $Z_j = X_{j-m}$ then $Z_j = \mathcal{F}(x e^{2\pi i f_m t})$

Convolution and multiplication: $\mathcal{F}(x \circledast y) = X \cdot Y$ and $\mathcal{F}(x \cdot y) = X \odot Y$

Parseval's theorem: $\sum_{t=0}^{n-1} x_t \bar{y}_t = n \sum_{j=0}^{n-1} X_j \bar{Y}_j$

- Special case, the Plancherel theorem: $\sum_{t=0}^{n-1} |x_t|^2 = n \sum_{j=0}^{n-1} |X_j|^2$ *(recall that $x_t \bar{x}_t = |x_t|^2$)*

Discrete derivative: let $z_t = x_t - x_{t-1}$ then $Z_t = X_j (1 - e^{-2\pi i f_j})$

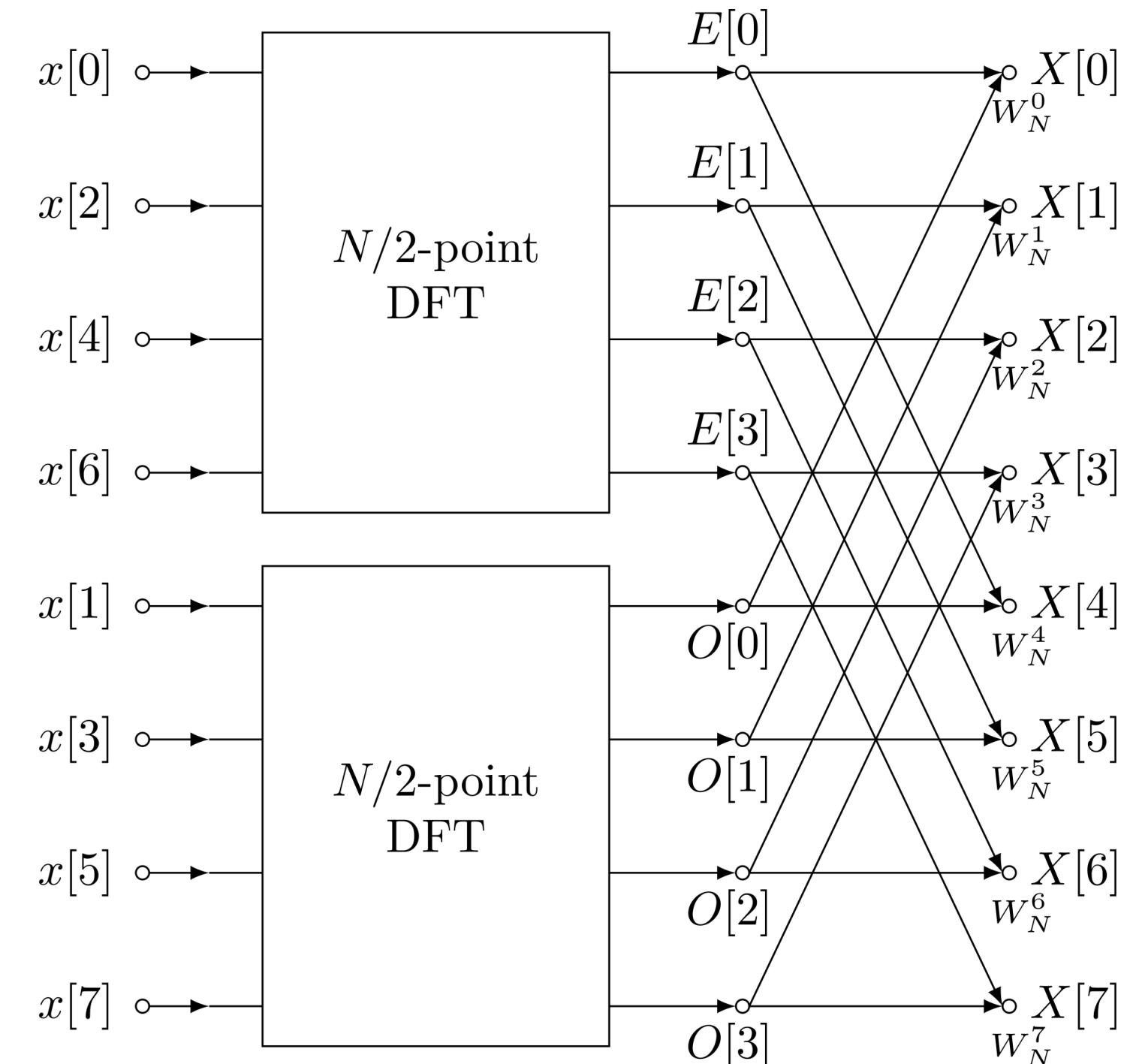


Fast Fourier transform (FFT)

Computing the DFT requires $\mathcal{O}(n^2)$ operations.

- Each of the n coefficients X_j (except for $j = 0$) requires $n - 1$ multiplications and $n - 1$ additions.

Fast Fourier transform algorithms reduce this cost to $\mathcal{O}(n \log n)$ operations.



Yangwenbo99, wikipedia

They adopt a **divide-and-conquer** strategy, and **recursively split** the DFT of size $n = n_1 n_2$ into n_1 smaller DFTs of size n_2 along with $\mathcal{O}(n)$ multiplications by complex roots of unity (called twiddle factors).

- In practice, implementations avoid the recursion by rearranging the algorithm.

Typical implementations split into two DFTs of size $n/2$ at each step, **requiring n to be a power of 2** (radix-2 FFT), but generally any factorization of n is possible.

Aliasing and the Nyquist frequency

Consider the signal $x_t = \cos(2\pi f \Delta t)$ sampled at interval Δ i.e., x_t is observed at time Δt .

Increasing f from 0, the **fastest observable oscillation** is reached when $f = \frac{1}{2\Delta}$ (called the **Nyquist frequency**)

- In this example, $x_t = \cos(\pi t) = (-1)^t$.

Higher frequencies are folded down into the interval $[0, \frac{1}{2\Delta}]$: taking f such that $\frac{1}{2\Delta} < f < \frac{1}{\Delta}$, and $f' = \frac{1}{\Delta} - f$, we get

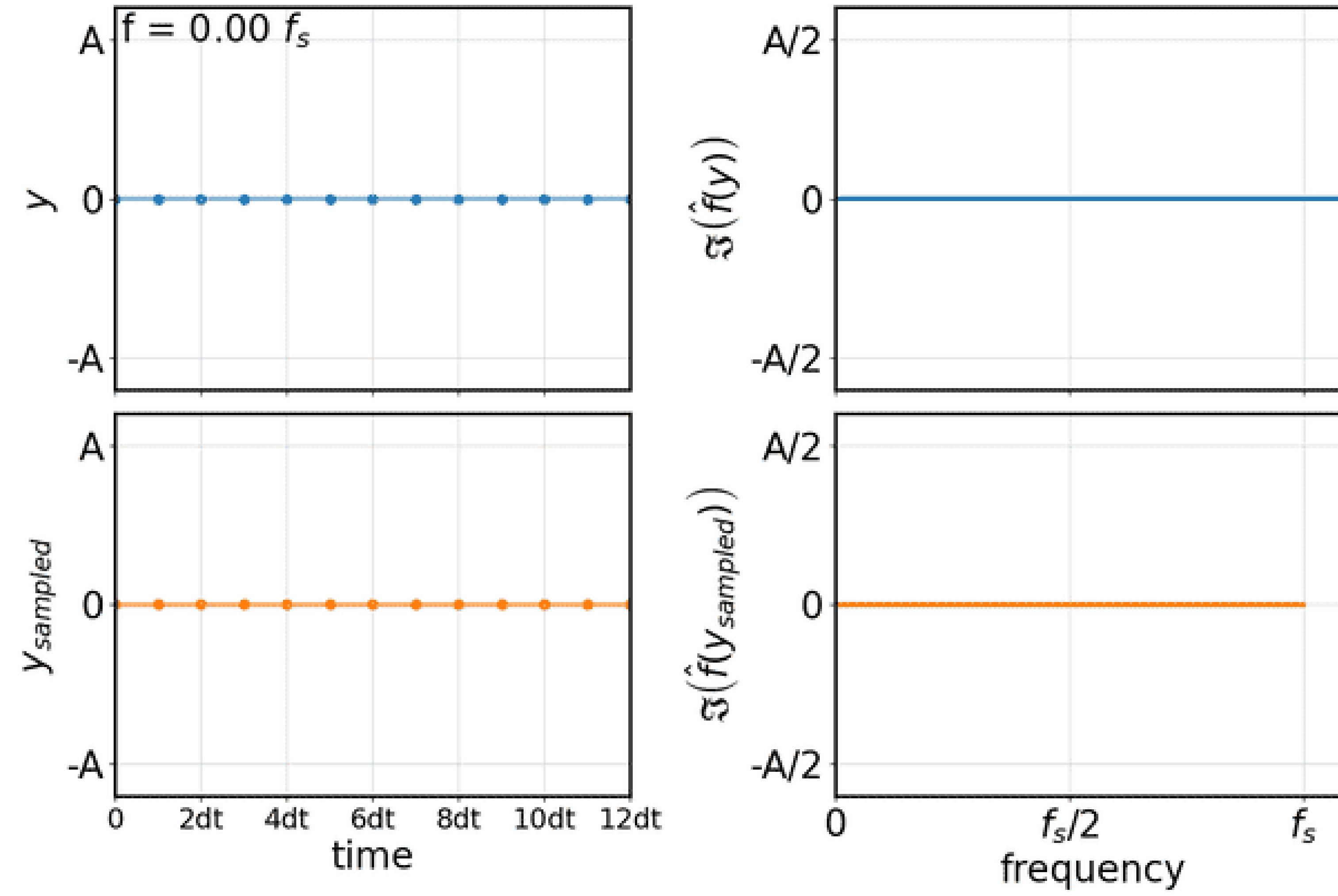
$$x_t = \cos(2\pi f \Delta t) = \cos\left(2\pi\left(\frac{1}{\Delta} - f'\right)\Delta t\right) = \cos(2\pi(t - f'\Delta t)) = \cos(2\pi f' \Delta t)$$

Then f and f' are indistinguishable, hence **aliases** of each other.

- For arbitrary f the observed frequency is $f' = |f - k\Delta|$ with k chosen such that $f' \leq \frac{1}{2\Delta}$

To capture a signal's frequency without aliasing, the sampling rate must be at least **twice** the signal's frequency (**Nyquist-Shannon sampling theorem**).

Aliasing



Davidjessop, wikipedia

Exercise

Review 3Blue1Brown article on the Fourier transform.

Simulate different synthetic signals (sinusoids, square waves, impulses, ...).

- Compute DFT and analyze frequencies.
- Test the different properties of the DFT (linearity, time-shifting, ...).
- Review the effect of different sampling rates on the DFT.

Model real-world time series.

- Fit sinusoids with known frequencies.
- Compute DFT and analyze dominant frequencies.
- Compare empirical results with your initial expectations.
- Review how the different DFT components contribute to the signal.