1. Suppose you have a sinusoidal signal with an actual frequency of $f = 15$ Hz. If the signal is sampled at a rate of $\Delta = 20$ Hz, what frequency would be observed in the sampled data. Explain the phenomena.

Solution: The Nyquist frequency is given by $f_N = \frac{\Delta}{2} = 10$ Hz. According to the Nyquist-Shannon sampling theorem, the sampling frequency Δ must be at least twice the frequency of the signal to avoid aliasing. Since the actual frequency $f = 15$ Hz is greater than the Nyquist frequency, aliasing occurs.

The observed (aliased) frequency can be calculated as:

$$f_{\text{observed}} = |f - n\Delta|, \quad \text{where } n \text{ is chosen so that } f_{\text{observed}} \leq f_N.$$

In this case:

$$f_{\text{observed}} = |15 - 20| = 5 \text{ Hz}.$$

Thus, the observed frequency in the sampled data will be 5 Hz. This occurs because sampling at 20 Hz is insufficient to resolve the 15 Hz signal, resulting in a lower frequency being misrepresented in the data.

2. Compute the discrete Fourier transform of the realization $\boldsymbol{x} = \{2, 1, 0, 1\}$ and then compute the inverse discrete Fourier transform to retrieve the original data.

Solution: The DFT is computed as follows:

$$X_0 = \frac{1}{4}\sum_{t=0}^{3} x_t e^{-2\pi i\cdot 0\cdot t} = \frac{1}{4}(2+1+0+1) = 1$$

$$\begin{aligned} X_1 &= \frac{1}{4}\sum_{t=0}^{3} x_t e^{-2\pi i\cdot\frac{1}{4}\cdot t} = \frac{1}{4}\sum_{t=0}^{3} x_t e^{-i\frac{\pi}{2}t} \\ &= \frac{1}{4}\sum_{t=0}^{3} x_t \left(\cos\left(\frac{\pi}{2}t\right) - i\sin\left(\frac{\pi}{2}t\right)\right) \\ &= \frac{1}{4}(2\cdot 1 + 1\cdot -i + 0\cdot -1 + 1\cdot i) = \frac{1}{2} \end{aligned}$$

$$\begin{aligned} X_2 &= \frac{1}{4}\sum_{t=0}^{3} x_t e^{-2\pi i\cdot\frac{2}{4}\cdot t} = \frac{1}{4}\sum_{t=0}^{3} x_t e^{-i\pi t} \\ &= \frac{1}{4}\sum_{t=0}^{3} x_t \left(\cos(\pi t) - i\sin(\pi t)\right) \\ &= \frac{1}{4}\sum_{t=0}^{3} x_t(-1)^t = \frac{1}{4}(2-1+0-1) = 0 \end{aligned}$$

$$
\begin{aligned}
X_3 &= \frac{1}{4}\sum_{t=0}^{3} x_t e^{-2\pi i\cdot\frac{3}{4}\cdot t} = \frac{1}{4}\sum_{t=0}^{3} x_t e^{-i\frac{3\pi}{2}t} \\
&= \frac{1}{4}\sum_{t=0}^{3} x_t \left(\cos\left(\frac{3\pi}{2}t\right) - i\sin\left(\frac{3\pi}{2}t\right)\right) \\
&= \frac{1}{4}(2\cdot 1 + 1\cdot i + 0\cdot -1 + 1\cdot -i) = \frac{1}{2}
\end{aligned}
$$

Thus, the DFT is $\boldsymbol{X} = \{1, \frac{1}{2}, 0, \frac{1}{2}\}$.

The inverse DFT is computed as follows:

$$
\begin{aligned}
x_0 &= \sum_{j=0}^{3} X_j e^{2\pi i\cdot\frac{j}{4}\cdot 0} = \sum_{j=0}^{3} X_j \\
&= 1 + \frac{1}{2} + 0 + \frac{1}{2} = 2
\end{aligned}
$$

$$
\begin{aligned}
x_1 &= \sum_{j=0}^{3} X_j e^{2\pi i\cdot\frac{j}{4}\cdot 1} = \sum_{j=0}^{3} X_j e^{i\frac{\pi}{2}j} \\
&= 1e^0 + \frac{1}{2}e^{i\frac{\pi}{2}} + 0e^{i\pi} + \frac{1}{2}e^{i\frac{3\pi}{2}} \\
&= 1 + \frac{1}{2}(i) + 0 + \frac{1}{2}(-i) = 1,
\end{aligned}
$$

$$
\begin{aligned}
x_2 &= \sum_{j=0}^{3} X_j e^{2\pi i\cdot\frac{j}{4}\cdot 2} = \sum_{j=0}^{3} X_j e^{i\pi j} \\
&= 1e^0 + \frac{1}{2}e^{i\pi} + 0e^{i2\pi} + \frac{1}{2}e^{i3\pi} \\
&= 1 + \frac{1}{2}(-1) + 0 + \frac{1}{2}(-1) = 0
\end{aligned}
$$

$$
\begin{aligned}
x_3 &= \sum_{j=0}^{3} X_j e^{2\pi i\cdot\frac{j}{4}\cdot 3} = \sum_{j=0}^{3} X_j e^{i(\frac{3\pi}{2})j} \\
&= 1e^0 + \frac{1}{2}e^{i\frac{3\pi}{2}} + 0e^{i3\pi} + \frac{1}{2}e^{i\frac{9\pi}{2}} \\
&= 1 + \frac{1}{2}(-i) + 0 + \frac{1}{2}(i) = 1.
\end{aligned}
$$

Thus, the inverse DFT recovers the original time series $\boldsymbol{x} = \{2, 1, 0, 1\}$, verifying that our DFT and IDFT computations are correct.

3. Evaluate and represent graphically

$$S = \sum_{j=0}^{8} e^{2\pi i \frac{j}{10}}$$

Solution: This expression can be rewritten as

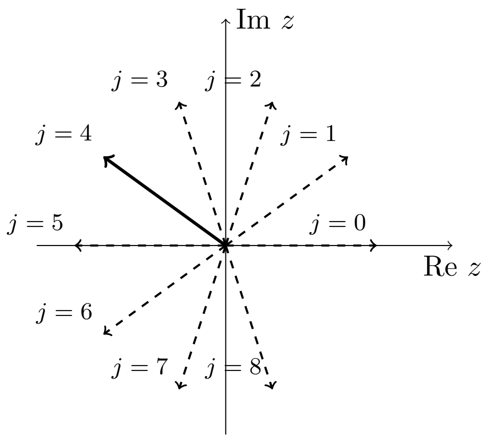
$$S = \sum_{j=0}^{9} e^{2\pi i \frac{j}{10}} - e^{2\pi i \frac{9}{10}}$$

This is a sum of complex numbers equally spaced over the unit circle of the form $z = e^{2\pi i \frac{j}{n} t}$ with $n = 10$ and $t = 1$. Since we are summing over a complete period n, the sum $\sum_{j=0}^{9} e^{2\pi i \frac{j}{10}}$ evaluates to 0, therefore

$$\begin{aligned} S &= -e^{2\pi i \frac{9}{10}} = e^{i(\frac{9}{10}2\pi + \pi)} = e^{i\frac{14}{10}2\pi} \\ &= e^{i\frac{7}{52\pi}} = e^{i\frac{2}{5}2\pi} \end{aligned}$$

Thus S is a complex number lying on the unit circle, with an angle of $\frac{4\pi}{5}$ radians (144°) from the positive real axis.

For the graphical representation, we can show the individual terms in the sum as vectors equally spaced on the unit circle in the complex plane (shown as dashed arrows), each corresponding to $e^{2\pi i \frac{j}{10}}$. Since the sum covers most of a full circle but stops at $j = 8$, we end up with a resultant vector pointing in the direction of $e^{i\frac{2}{5}2\pi}$ (shown as solid arrow), with the final result as shown below.



4. Consider a sinusoid $x_t = R\cos(2\pi(ft + \phi))$. Determine the effect of changing the time origin and scale $u = \frac{t-a}{b}$ on the amplitude, phase, and frequency.

Solution: The original time variable is given by $t = a + bu$. The signal becomes

$$\begin{aligned} y_u &= x_{a+bu} \\ &= R\cos(2\pi(fa + fbu + \phi)) \\ &= R\cos(2\pi((fb)u + (fa + \phi)) \\ &= R\cos(2\pi(f'u + \phi') \end{aligned}$$

Thus, the amplitude R remains unchanged, the frequency is multiplied by the inverse of the change in time scale b and the phase is shifted by the product of the change of time origin a and the original frequency f.

This implies that the amplitude of a sinusoid depends neither on the time origin nor the scale of the time variable.

5. In the lecture, we saw that fitting a sinusoid $x_t = R\cos(2\pi(ft + \phi))$ with known frequency f to a time series $\{x_0, \ldots, x_{n-1}\}$ could be achieved by minimizing the sum of squared residuals.

   Derive the solution of this optimization problem, assuming that the number of observations n is an integer multiple k of the period $P = 1/f$, i.e., $n = kP$.

Solution: We aim to minimize the sum of squared residuals:

$$\min_{\mu,A,B} \sum_{t=0}^{n-1} e_t^2 = \sum_{t=0}^{n-1} (x_t - \mu - Ac_t - Bs_t)^2 \, ,$$

where $c_t = \cos(2\pi ft)$ and $s_t = \sin(2\pi ft)$.

To find the optimal values $\hat{\mu}$, $\hat{A}$, and $\hat{B}$, we set the partial derivatives of the sum of squares with respect to each parameter to zero and solve for the roots.

The partial derivative of the sum of squares with respect to μ is:

$$\frac{\partial}{\partial\mu} \sum_{t=0}^{n-1} (x_t - \mu - Ac_t - Bs_t)^2 = \sum_{t=0}^{n-1} 2\,(x_t - \mu - Ac_t - Bs_t)\,(-1).$$

Setting this equal to zero, we get:

$$\sum_{t=0}^{n-1} (x_t - \mu - Ac_t - Bs_t) = 0.$$

Bringing μ to the left side:

$$
\begin{aligned}
\sum_{t=0}^{n-1} \mu = n\mu &= \sum_{t=0}^{n-1} x_t - A \sum_{t=0}^{n-1} c_t - B \sum_{t=0}^{n-1} s_t \\
&= \sum_{t=0}^{n-1} x_t - A \sum_{t=1}^{kP} c_t - B \sum_{t=1}^{kP} s_t \\
&= \sum_{t=0}^{n-1} x_t - Ak \sum_{t=1}^{P} c_t - Bk \sum_{t=1}^{P} s_t \\
&= \sum_{t=0}^{n-1} x_t
\end{aligned}
$$

The sums $\sum_{t=1}^{P} c_t$ and $\sum_{t=1}^{P} s_t$ are zero due to the periodicity and symmetry of the sine and cosine functions, so we have:

$$\hat{\mu} = \frac{1}{n} \sum_{t=0}^{n-1} x_t.$$

The partial derivative with respect to A is:

$$\frac{\partial}{\partial A} \sum_{t=0}^{n-1} (x_t - \mu - Ac_t - Bs_t)^2 = \sum_{t=0}^{n-1} 2\,(x_t - \mu - Ac_t - Bs_t)\,(-c_t).$$

Setting this equal to zero, we get:

$$\sum_{t=0}^{n-1} c_t\,(x_t - \mu - Ac_t - Bs_t) = 0.$$

Expanding and rearranging terms to solve for A, we have:

$$\sum_{t=0}^{n-1} x_t c_t - \mu \sum_{t=0}^{n-1} c_t - A \sum_{t=0}^{n-1} c_t^2 - B \sum_{t=0}^{n-1} c_t s_t = 0.$$

Since $\sum_{t=0}^{n-1} c_t = 0$ (described in the derivation of $\hat{\mu}$), this simplifies to:

$$A \sum_{t=0}^{n-1} c_t^2 + B \sum_{t=0}^{n-1} c_t s_t = \sum_{t=0}^{n-1} x_t c_t.$$

Using the angle sum identity $\cos(2\pi f t)\sin(2\pi f t) = \frac{1}{2}\sin(4\pi f t)$,

$$\sum_{t=0}^{n-1} c_t s_t = \sum_{t=0}^{n-1} \frac{1}{2}\sin(4\pi f t) = \frac{1}{2} k \sum_{t=1}^{P} \sin(2\pi(2f)t) = 0$$

Since the period of $\sin(2\pi(2f)t)$ is $P/2$, the sum is evaluated over an integer multiples of the period and results in 0.

Now using the angle sum and the Pythagorean identities together, we get

$$\begin{aligned}\sum_{t=0}^{n-1} c_t^2 &= \sum_{t=0}^{n-1} \frac{1+\cos(4\pi f t)}{2} \\ &= \frac{1}{2}\left(\sum_{t=0}^{n-1} 1 + \sum_{t=0}^{n-1} \cos(4\pi f t)\right) \\ &= \frac{1}{2}\left(n + k\sum_{t=1}^{P} \cos(2\pi(2f)t)\right) = \frac{n}{2}\end{aligned}$$

Since the period of $\cos(2\pi(2f)t)$ is $P/2$, the sum is evaluated over an integer multiples of the period and results in 0.

We can now simplify the expression for $\hat{A}$ as:

$$\hat{A} = \frac{2}{n}\sum_{t=0}^{n-1} x_t c_t.$$

The partial derivative with respect to B is:

$$\frac{\partial}{\partial B}\sum_{t=0}^{n-1}(x_t - \mu - Ac_t - Bs_t)^2 = \sum_{t=0}^{n-1} 2\,(x_t - \mu - Ac_t - Bs_t)\,(-s_t).$$

Setting this equal to zero, we get:

$$\sum_{t=0}^{n-1} s_t\,(x_t - \mu - Ac_t - Bs_t) = 0.$$

Expanding and rearranging terms to solve for B, we have:

$$\sum_{t=0}^{n-1} x_t s_t - \mu\sum_{t=0}^{n-1} s_t - A\sum_{t=0}^{n-1} c_t s_t - B\sum_{t=0}^{n-1} s_t^2 = 0.$$

Since $\sum_{t=0}^{n-1} s_t = 0$ (described in the derivation of $\hat{\mu}$), and $\sum_{t=0}^{n-1} c_t s_t = 0$ (described in the derivation of $\hat{A}$), this simplifies to:

$$B\sum_{t=0}^{n-1} s_t^2 = \sum_{t=0}^{n-1} x_t s_t$$

Using the angle sum and Pythagorean identities together, we get

$$\begin{aligned}\sum_{t=0}^{n-1} s_t^2 &= \sum_{t=0}^{n-1} \frac{1-\cos(4\pi f t)}{2} \\ &= \frac{1}{2}\left(\sum_{t=0}^{n-1} 1 - \sum_{t=0}^{n-1} \cos(4\pi f t)\right) \\ &= \frac{1}{2}\left(n - k\sum_{t=1}^{P} \cos(2\pi(2f)t)\right) = \frac{n}{2}.\end{aligned}$$

Following the same arguments described in the derivation of $\hat{A}$.

We can now simplify the expression for $\hat{B}$ as:

$$\hat{B} = \frac{2}{n} \sum_{t=0}^{n-1} x_t s_t.$$