1. (a) Why are confidence intervals important? How can they be computed and under what conditions? Provide an example to illustrate their interpretation.

   (b) When is it more appropriate to use scaled performance metrics versus scale-dependent performance metrics? Explain the differences and provide examples to illustrate scenarios where each is most suitable.

   (c) How does the approach to splitting time series data differ between classification tasks and forecasting tasks? Explain the reasoning behind these differences and provide an example to illustrate the appropriate splitting method for each task.

   (d) The ETS model for Holt-Winters method is a generalization of the ETS model for simple exponential smoothing. Should it then be always preferred? Discuss the trade-offs and the scenarios where each model is more appropriate.

2. Consider the stochastic process $X_t = -2t + W_t + \frac{1}{2}W_{t-1}$ where $W_t \sim \mathcal{N}(0, \sigma^2)$.

   (a) Compute the mean and auto-covariance and determine whether X_t is stationary.

   (b) Transform the process to make it stationary without using decomposition methods, and compute the mean and auto-covariance of the resulting process.

   (c) Plot the theoretical auto-correlation function of both process and discuss any differences.

3. Consider the ARMA model $X_t = X_{t-1} - \frac{1}{4}X_{t-2} + W_t - \frac{1}{4}W_{t-1}$ where $W_t \sim \mathcal{N}(0, \sigma^2)$.

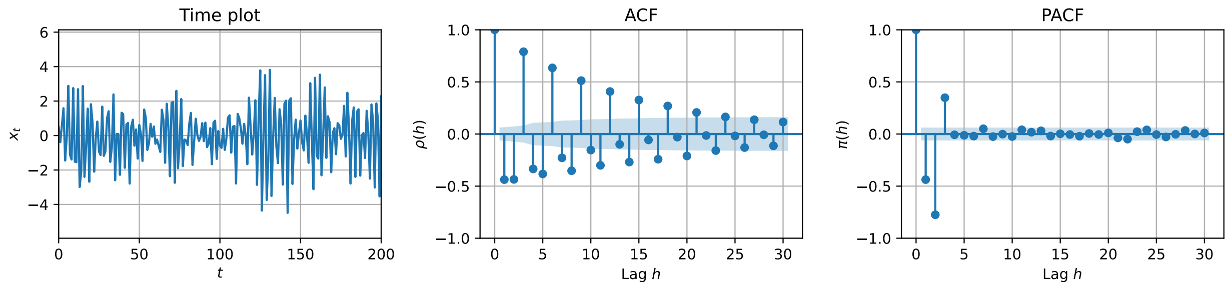
   (a) Determine the order (p,q) of the model and derive the characteristic polynomials for the autoregressive (AR) and moving average (MA) components.

   (b) Assess whether the model is stationary.

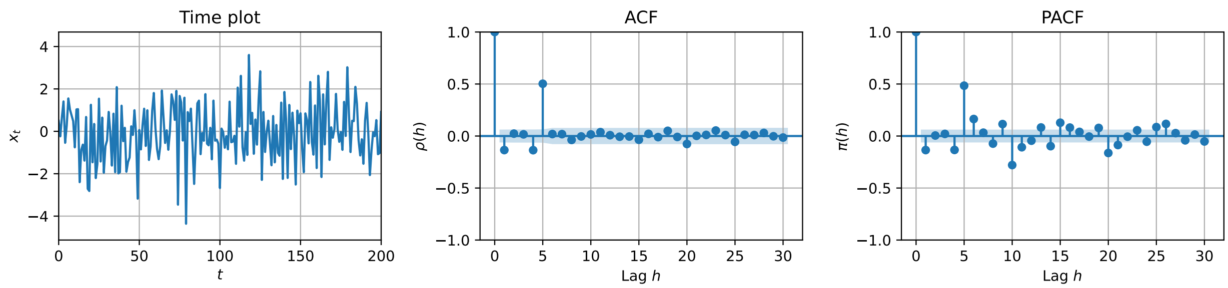
   (c) Evaluate whether the model is invertible and provide an explanation of the concept.

   (d) Considering the realization $\{2, \frac{3}{2}, \frac{9}{8}\}$, compute the one, two, three step-ahead forecasts assuming the process starts at $t = 1$ with initial conditions $x_0 = w_0 = 0$.

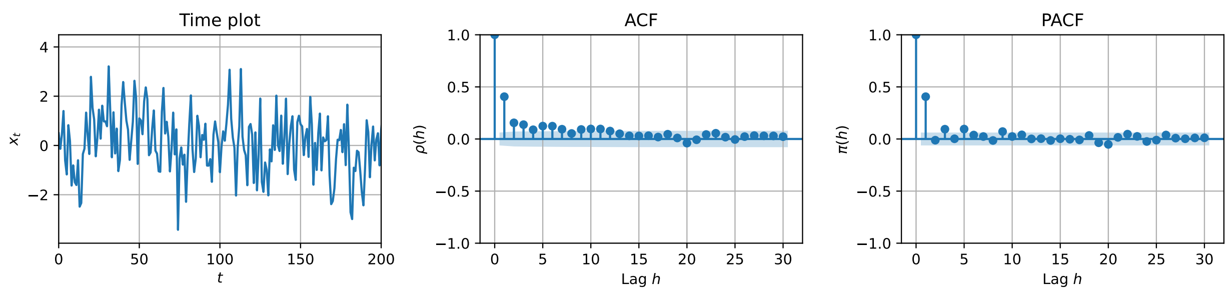
   (e) Determine the appropriate order (p,q) for an ARMA model to fit each time series shown in fig. 1, using the provided sample autocorrelation function (ACF) and partial autocorrelation function (PACF). Justify your choices.



(a)



(b)



(c)

Figure 1: Time series realizations.

4. Consider the ETS(A,N,N) model defined as

$$x_t = \ell_{t-1} + \epsilon_t,$$

$$\ell_t = \ell_{t-1} + \alpha\epsilon_t,$$

where $\epsilon_t \sim \mathcal{N}(0, \sigma^2)$. The observed values are

$$x_1 = 10, \quad x_2 = 12, \quad x_3 = 13.$$

Assume the initial level is $\ell_0 = 9$.

(a) Compute the level updates ℓ_1, ℓ_2, and ℓ_3 using the smoothing parameter $\alpha = 0.2$.

(b) Repeat the computations for ℓ_1, ℓ_2, and ℓ_3 using $\alpha = 0.8$.

(c) Compare the previous results and discuss how the choice of the smoothing parameter affects the model behavior.

(d) Taking $\alpha = 0.2$, compute the forecasts $\hat{x}_{4|3}$ and $\hat{x}_{10|3}$.

(e) The forecast variance of the model is given by:

$$\hat{\sigma}_h^2 = \hat{\sigma}_e^2 \left(1 + \alpha^2(h-1)\right),$$

with $\hat{\sigma}_e^2$ the residuals variance. Calculate the 95% confidence intervals for both forecasts assuming normally distributed residuals. Explain the difference.

(f) An ETS(A,N,N) model was fitted to monthly data with a fixed smoothing parameter α. Interpret each of the residual analysis plots shown in fig. 2. Suggest specific ways to improve performance based on your observations.

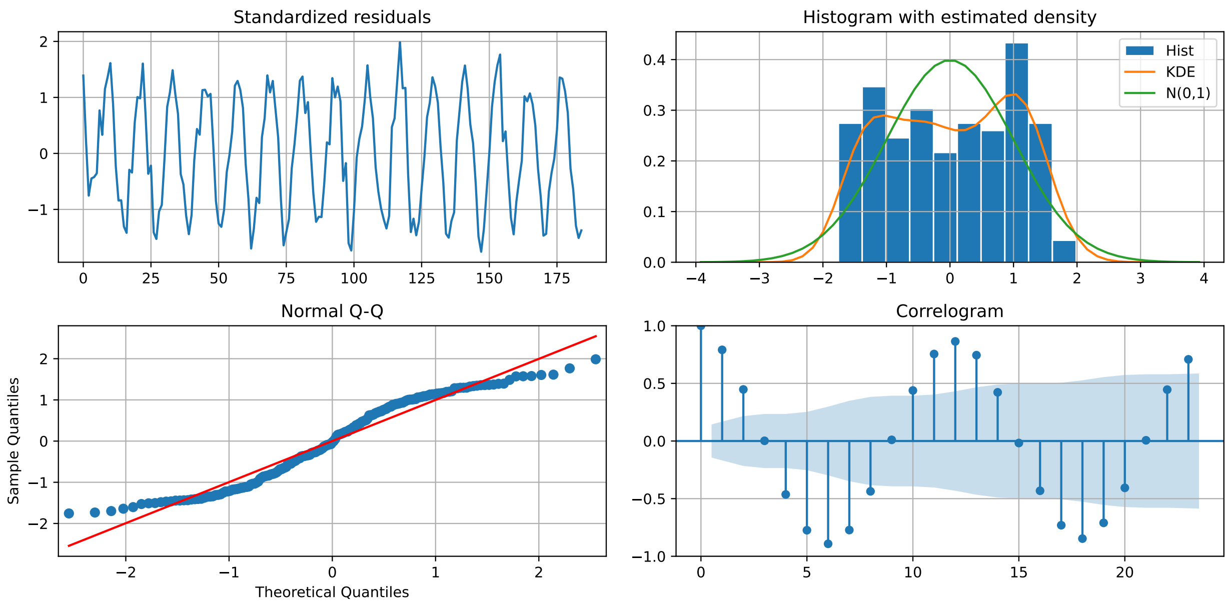


Figure 2: Residual analysis plots.