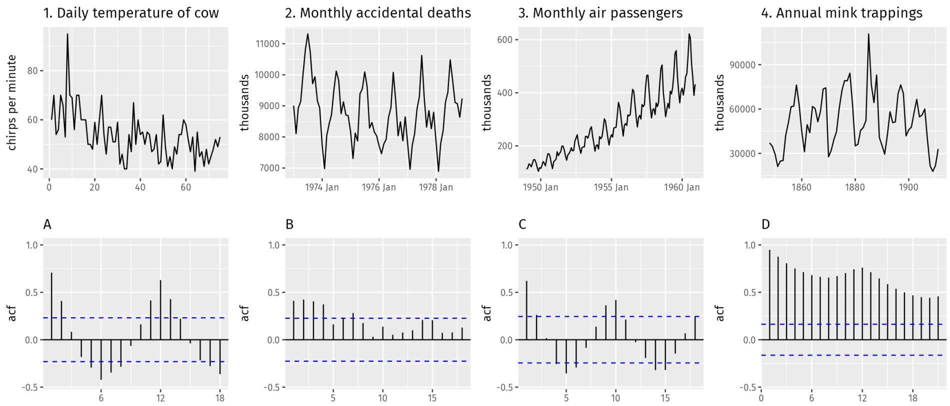
1. The time plots and ACF plots in Figure 1 correspond to four different time series. Your task is to match each time plot in the first row with one of the ACF plots in the second row.



Source: Section 2.10 Exercise 9 from *Forecasting: Principles and Practice* (3rd ed).

Solution: In the ACF plot, different time series patterns manifest through distinct auto-correlation behaviors:

- Trend: slow, gradual decay of auto-correlations across increasing lags.
- Seasonality: sharp, significant peaks in the ACF at lags that correspond to the length of the seasonal period. These peaks occur at multiples of the seasonal period.
- Cyclic: oscillating auto-correlations that alternate between positive and negative values. The lags where the ACF crosses the zero line typically mark the length of the cycle, giving a more sinusoidal pattern compared to seasonality.
- White noise: no significant correlations at any lag, with values randomly hovering around zero.

While both seasonality and cycles involve repeating patterns, seasonality shows sharp, regular peaks at multiples of the seasonal period, indicating a strict periodicity, whereas cycles have a more gradual and smooth alternation of positive and negative autocorrelations, reflecting a less rigid, more sinusoidal pattern.

With these considerations in mind, the exercise time plots can be matched to the following ACF plots:

- Daily temperature of cow (trend): B
- Monthly accidental deaths (seasonality): A
- Monthly air passengers (trend + seasonality): D
- Annual mink trappings (cyclic): C

2. For each of the provided stochastic processes, compute its mean, variance, and auto-covariance $cov(X_s, X_t)$, and determine whether it qualifies as white noise. In the following,

consider that W_t is an iid white noise process with $Var(W_t) = \sigma^2$.

(a) $X_t = 5 + W_t$.

Solution:

- $E[X_t] = E[5 + W_t] = 5 + E[W_t] = 5$
- $Var(X_t) = Var(5 + W_t) = Var(W_t) = \sigma^2$
- $cov(X_s, X_t) = cov(5 + W_s, 5 + W_t) = cov(W_s, W_t) = \begin{cases} \sigma^2 & \text{if } s = t \\ 0 & \text{otherwise} \end{cases}$

X_t is *not* white noise since its mean is not zero.

(b) $X_t = t \cdot W_t$.

Solution:

- $E[X_t] = E[t \cdot W_t] = t \cdot E[W_t] = 0$
- $Var(X_t) = t^2 \cdot Var(W_t) = t^2\sigma^2$
- $cov(X_s, X_t) = s \cdot t \cdot cov(W_s, W_t) = \begin{cases} t^2\sigma^2 & \text{if } s = t \\ 0 & \text{otherwise} \end{cases}$

X_t is *not* white noise since its variance and auto-covariance are not constant.

(c) $X_t = 2W_t + W_{t-1}$.

Solution:

- $E[X_t] = E[2W_t + W_{t-1}] = 2E[W_t] + E[W_{t-1}] = 0$
- $Var(X_t) = Var(2W_t + W_{t-1}) = 4Var(W_t) + Var(W_{t-1}) = 5\sigma^2$ since W_t is iid.
- Considering $s = t + 1$, the auto-covariance can be expanded as follows (since $cov(W_s, W_t) = 0 \ \forall s \neq t$):

$$\begin{aligned} cov(X_{t+1}, X_t) &= cov(2W_{t+1} + W_t, 2W_t + W_{t-1}) \\ &= 4cov(W_{t+1}, W_t) + 2cov(W_{t+1}, W_{t-1}) \\ &\quad + 2cov(W_t, W_t) + cov(W_t, W_{t-1}) \\ &= 2cov(W_t, W_t) = 2\sigma^2 \neq 0 \end{aligned}$$

X_t is *not* white noise since its auto-covariance is not zero for all lags.

(d) $X_t = (-1)^t W_t$

Solution:

- $E[X_t] = E[(-1)^t W_t] = (-1)^t E[W_t] = 0$.
- $Var(X_t) = Var((-1)^t W_t) = (-1)^{2t} Var(W_t) = Var(W_t) = \sigma^2$.
- Considering $s \neq t$, the auto-covariance can be expanded as follows:

$$\begin{aligned} cov(X_s, X_t) &= cov((-1)^s W_s, (-1)^t W_t) \\ &= (-1)^s (-1)^t cov(W_s, W_t) \\ &= (-1)^{s+t} cov(W_s, W_t) = 0 \end{aligned}$$

X_t is white noise since it is zero-mean, has constant variance and its RVs are uncorrelated.

3. Show that a 3×5 MA is equivalent to a 7-term weighted moving average with weights $\left[\frac{1}{15}, \frac{2}{15}, \frac{3}{15}, \frac{3}{15}, \frac{3}{15}, \frac{2}{15}, \frac{1}{15}\right]$.
Source: Section 3.7 Exercise 6 from *Forecasting: Principles and Practice* (3rd ed).

Solution: The 5-term moving average is defined as:

$$y_j = \frac{1}{5}(x_{j-2} + x_{j-1} + x_j + x_{j+1} + x_{j+2})$$

For the 3-term moving average, we use: $z_t = \frac{1}{3}(y_{t-1} + y_t + y_{t+1})$.
Substituting the expression, we get:

$$\begin{aligned} z_t = \frac{1}{3}\Bigg(&\frac{1}{5}(x_{t-3} + x_{t-2} + x_{t-1} + x_t + x_{t+1}) \\ &+ \frac{1}{5}(x_{t-2} + x_{t-1} + x_t + x_{t+1} + x_{t+2}) \\ &+ \frac{1}{5}(x_{t-1} + x_t + x_{t+1} + x_{t+2} + x_{t+3})\Bigg) \end{aligned}$$

$$= \frac{1}{15}(x_{t-3} + 2x_{t-2} + 3x_{t-1} + 3x_t + 3x_{t+1} + 2x_{t+2} + x_{t+3}),$$

which simplifies to a 7-term weighted moving average with weights:

$$\left[\frac{1}{15}, \frac{2}{15}, \frac{3}{15}, \frac{3}{15}, \frac{3}{15}, \frac{2}{15}, \frac{1}{15}\right]$$

4. Figures 1 and 2 show the result of decomposing the number of persons in the civilian labour force in Australia each month from February 1978 to August 1995.
Source: Section 3.7 Exercise 9 from *Forecasting: Principles and Practice* (3rd ed).
   (a) Write about 3–5 sentences describing the results of the decomposition. Pay particular attention to the scales of the graphs in making your interpretation.

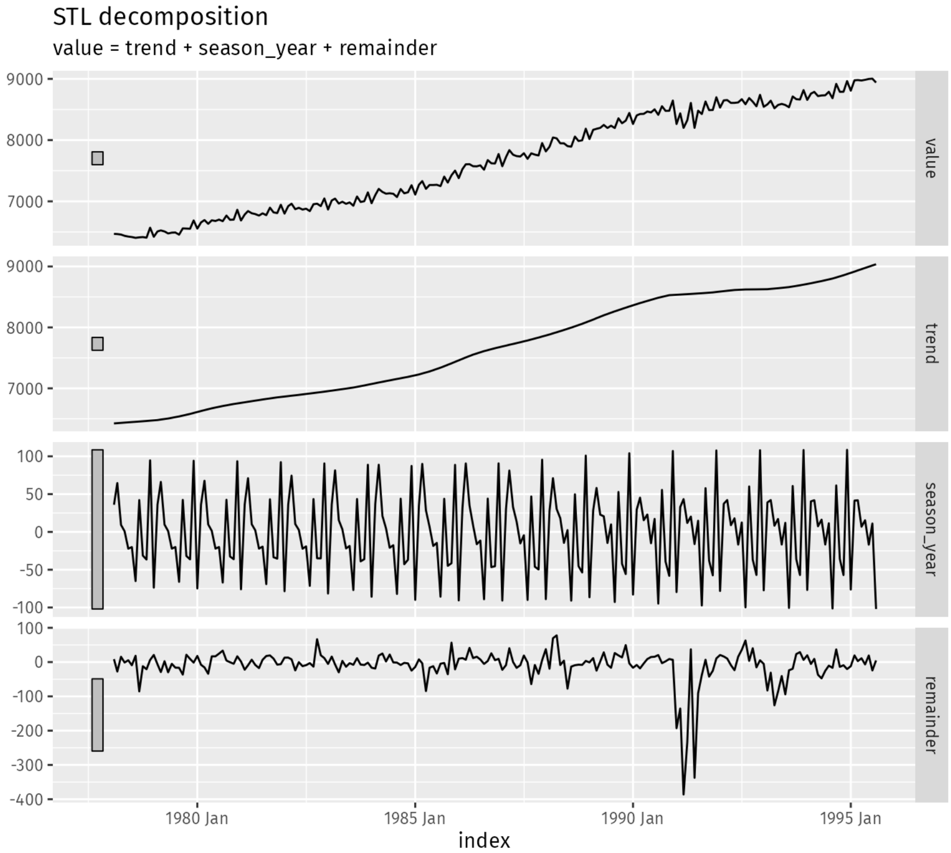


Figure 1: Decomposition of the number of persons in the civilian labour force in Australia each month from February 1978 to August 1995.

Solution:

- The Australian labour force has been decomposed into 3 components (trend, seasonality, and remainder) using an STL decomposition.
- The trend element has been captured well by the decomposition, as it smoothly increases with a similar pattern to the data. The trend is of the same scale as the data (indicated by similarly sized grey bars), and contributes most to the decomposition (having the smallest scale bar).
- The seasonal component changes slowly throughout the series, with the second seasonal peak diminishing as time goes on – this component is the smallest contribution original data (having the largest scale bar).
- The remainder is well-behaved until 1991/1992 when there is a sharp drop. There also appears to be a smaller drop in 1993/1994. There is sometimes some leakage of the trend into the remainder component when the trend window is too large. This appears to have happened here. It would be better if the recession of 1991-1992, and the smaller dip in 1993, were both included in the trend estimate rather than the remainder estimate. This would require a smaller trend window than what was used.

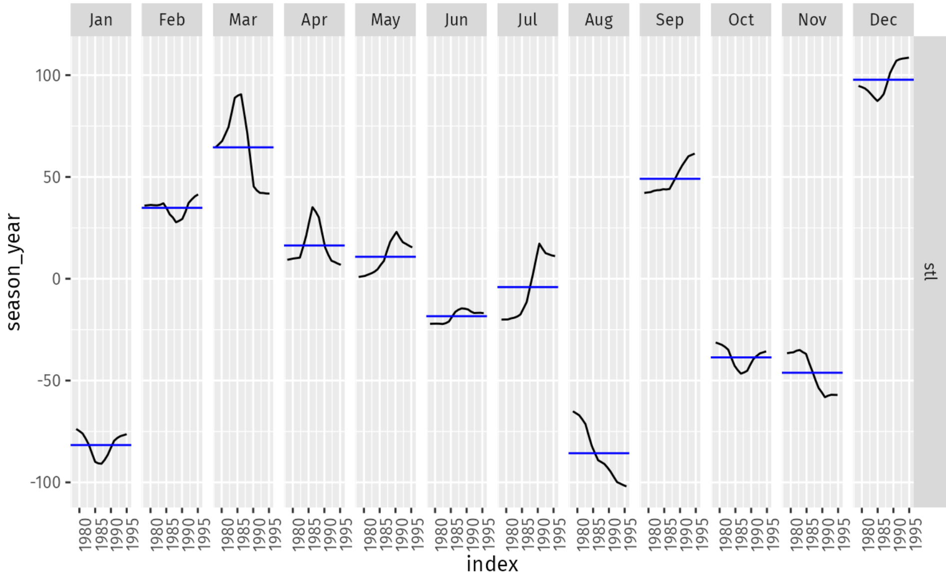


Figure 2: Seasonal component from the decomposition shown in Figure 1.

- In the bottom graph, the seasonal component is shown using a sub-series plot. December is the highest employment month, followed by March and September. The seasonal component changes mostly in March (with a decrease in the most recent years). July and August are the months with the next largest changes. The least changing is June with the rest are somewhere between these. December and September show increases in the most recent years.

(b) Is the recession of 1991/1992 visible in the estimated components?

Solution: Yes. The remainder shows a substantial drop during 1991 and 1992 coinciding with the recession.