1. The following table provides the quarterly sales of a company. Use simple exponential smoothing with $\alpha = 0.1$ and $l_0 = 100$ to forecast the sales for the year 2025.

Year	Q4 2023	Q1 2024	Q2 2024	Q3 2024
Sales	200	160	150	160

Solution: Using simple exponential smoothing (SES), the level is updated each quarter using the formula:

$$l_t = \alpha x_t + (1-\alpha) l_{t-1}$$

where l_t is the level for the current period, α is the smoothing constant, x_t is the actual sales, and l_{t-1} is the level of the previous period.

Starting with $l_0 = 100$, we compute:

$$\begin{aligned}
l_1 &= 0.1 \times 200 + 0.9 \times 100 = 20 + 90 = 110 \\
l_2 &= 0.1 \times 160 + 0.9 \times 110 = 16 + 99 = 115 \\
l_3 &= 0.1 \times 150 + 0.9 \times 115 = 15 + 103.5 = 118.5 \\
l_4 &= 0.1 \times 160 + 0.9 \times 118.5 = 16 + 106.65 = 122.65
\end{aligned}$$

Since the SES forecast equation is $\hat{x}_{t+1|t} = l_t$, the forecast for the sales in the fourth quarter of 2024, as well as for each quarter of 2025, is 122.65.

2. Derive the observation and state equations of ETS(M,A,A).

Solution: Recall the forecast equation: $\hat{x}_{i+h|i} = (l_i + hb_i)s_{i+h-P\lceil h/P \rceil}$

$$\begin{aligned}
&\Rightarrow \hat{x}_{i+1|i} = (l_i + b_i)s_{i+1-P} \\
&\Rightarrow \hat{x}_{t|t-1} = (l_{t-1} + b_{t-1})s_{t-P}
\end{aligned}$$

With multiplicative errors: $\epsilon_t = \frac{x_t - \hat{x}_{t|t-1}}{\hat{x}_{t|t-1}} = \frac{x_t - (l_{t-1}+b_{t-1})s_{t-P}}{(l_{t-1}+b_{t-1})s_{t-P}}$

Observation equation: $x_t = (l_{t-1} + b_{t-1})s_{t-P}(1+\epsilon_t)$

Level state equation:

$$\begin{aligned}
l_t &= \alpha \frac{x_t}{s_{t-P}} + (1-\alpha)(l_{t-1} + b_{t-1}) \\
&= l_{t-1} + b_{t-1} + \alpha \left(\frac{x_t}{s_{t-P}} - (l_{t-1} + b_{t-1}) \right) \\
&= l_{t-1} + b_{t-1} + \alpha \frac{x_t - (l_{t-1} + b_{t-1})s_{t-P}}{s_{t-P}} \\
&= l_{t-1} + b_{t-1} + \alpha \epsilon_t (l_{t-1} + b_{t-1}) \\
&= (l_{t-1} + b_{t-1})(1 + \alpha \epsilon_t)
\end{aligned}$$

Trend state equation:

$$\begin{aligned}
b_t &= \beta^*(l_t - l_{t-1}) + (1-\beta^*)b_{t-1} \\
&= \beta^*((l_{t-1} + b_{t-1})(1+\alpha\epsilon_t) - l_{t-1}) + b_{t-1} - \beta^* b_{t-1} \\
&= \alpha\beta^*(l_{t-1} + b_{t-1})\epsilon_t + \beta^* b_{t-1} + b_{t-1} - \beta^* b_{t-1} \\
&= b_{t-1} + \beta(l_{t-1} + b_{t-1})\epsilon_t
\end{aligned}$$

with $\beta = \alpha\beta^*$.

Seasonal state equation:

$$\begin{aligned} s_t &= \gamma \frac{x_t}{l_{t-1} + b_{t-1}} + (1-\gamma)s_{t-P} \\ &= \gamma s_{t-P}(1+\epsilon_t) + (1-\gamma)s_{t-P} \\ &= s_{t-P}(1+\gamma\epsilon_t) \end{aligned}$$

3. In this exercise, we will study the uncertainty of the ETS(A,N,N) model.

   Source: Section 8.8 Exercise 17-18 from *Forecasting: Principles and Practice* (3rd ed).

   (a) Show that the forecast variance is given by $\sigma_h^2 = \sigma^2(1 + \alpha^2(h-1))$

   Solution: An ETS(A,N,N) model is defined as

$$\begin{aligned} x_t &= l_{t-1} + \epsilon_t \\ l_t &= l_{t-1} + \alpha\epsilon_t \end{aligned}$$

   where $\epsilon \sim$ iid $\mathcal{N}(0, \sigma^2)$.

   The h-step forecast is constant $\hat{x}_{t+h|t} = l_t$, however this represents only the expected value. The forecast variance grows over time because the observed value x_{t+h} is influenced by the accumulation of all intermediate error terms:

$$\begin{aligned} x_{t+h} &= l_{t+h-1} + \epsilon_{t+h} \\ &= l_{t+h-2} + \alpha\epsilon_{t+h-1} + \epsilon_{t+h} \\ &= l_{t+h-3} + \alpha\epsilon_{t+h-2} + \alpha\epsilon_{t+h-1} + \epsilon_{t+h} \\ &= l_t + \epsilon_{t+h} + \alpha\sum_{i=1}^{h-1}\epsilon_{t+h-i} \end{aligned}$$

   Thus, the h-step forecast variance is

$$\begin{aligned} Var(x_{t+h}|x_1, ...x_t) &= Var(l_t + \epsilon_{t+h} + \alpha\sum_{i=1}^{h-1}\epsilon_{t+h-i}) \\ &= Var(\epsilon_{t+h}) + \alpha^2 Var(\sum_{i=1}^{h-1}\epsilon_{t+h-i}) \\ &= \sigma^2 + \alpha^2(h-1)\sigma^2 \\ &= \sigma^2(1 + \alpha^2(h-1)) \end{aligned}$$

   (b) Write down the 95% prediction intervals as a function of l_t, α, h, σ, assuming normally distributed errors.

   Solution: The confidence interval is $l_t \pm 1.96\sigma\sqrt{1 + \alpha^2(h-1)}$.

4. Analyze the following time plots. Which ETS models would be appropriate?
   Source: Section 8.8 from *Forecasting: Principles and Practice* (3rd ed).

   (a) Figure 1 showing the Australian gas production.

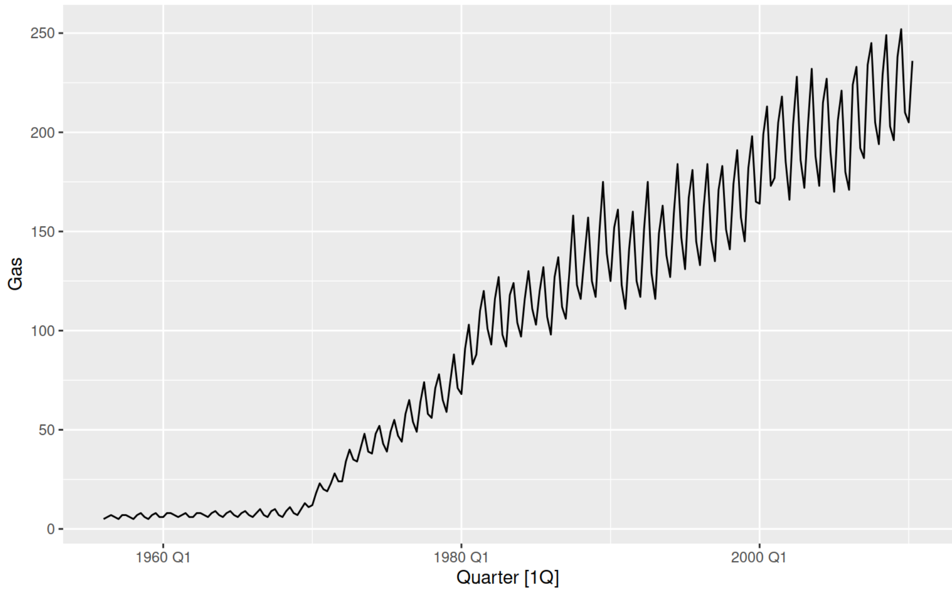

Figure 1: Time plot of the Australian gas production.

> **Solution:** The time series presents increasing seasonality fluctuations as well as an upward trend. Clearly the variance increases with the level of the series so the models ETS({A,M},{A,Ad},M) should be suitable.

   (b) Figure 2 showing the quantity of Canadian lynx trapping.

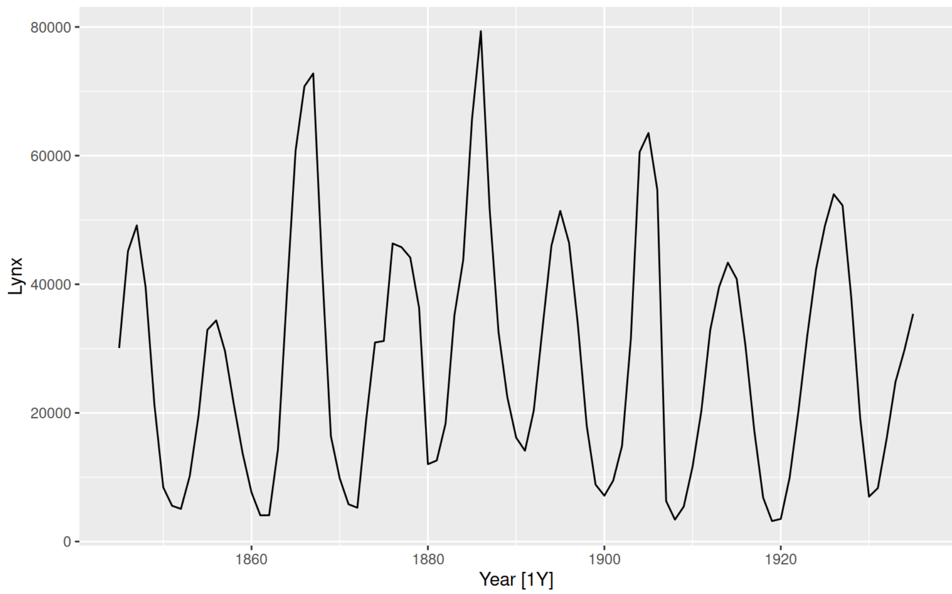

Figure 2: Time plot of the Canadian lynx trapping.

> **Solution:** The time series presents cycles and no clear trend. The variance does not seem to increase with the level of the series. The ETS(A,N,N) model should be suitable. Note that this will smooth out the cyclic behavior of the lynx data. In general, ETS models are not designed to handle cyclic data.

(c) Figure 3 showing the total domestic overnight trips across Australia.

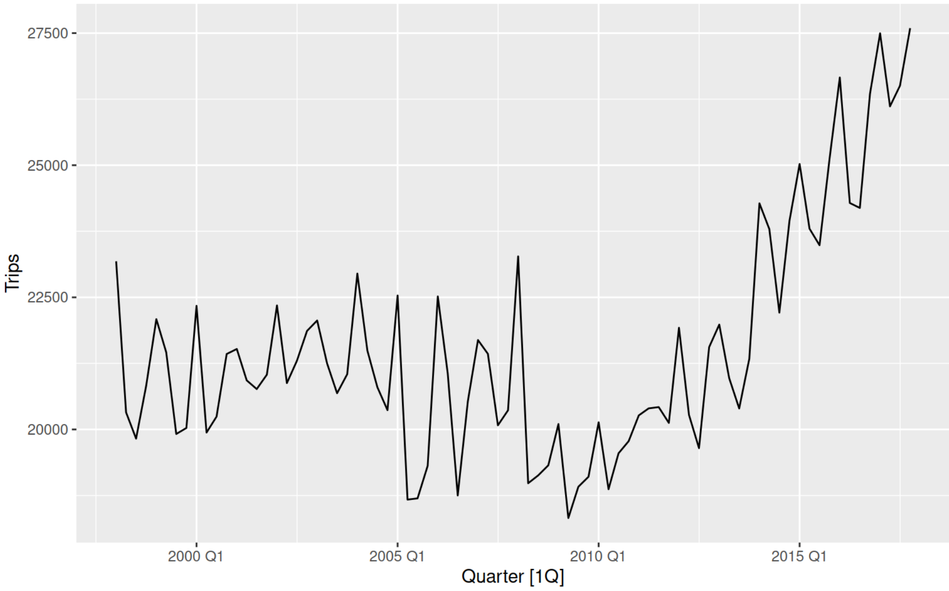

Figure 3: Time plot of the total domestic overnight trips across Australia.

> **Solution:** The data is seasonal and shows an increasing trend after 2010. The variance does not seem to increase with the level of the series. Therefore, the ETS(A,{A,Ad},A) models should be suitable.