

1. Derive the spectral density of the following stationary stochastic processes. $X_t = t + W_t$, where W_t is an iid white noise process with $Var(W_t) = \sigma^2$. For each process, consider that W_t is an iid white noise process with $Var(W_t) = \sigma^2$.

- (a) $X_t = 5 + W_t$.
- (b) $X_t = 2W_t + W_{t-1}$.
- (c) $X_t = (-1)^t W_t$
- (d) $X_t = W_t - \frac{1}{2}W_{t-2}$.

2. Derive the auto-covariance from the spectral density $f(\omega) = \cos(2\pi\omega)$.

For this derivation, it is useful to note the following result for the integral of a complex exponential:

$$\int_{-1/2}^{1/2} e^{2\pi i \omega k} d\omega = \begin{cases} 1 & \text{if } k = 0, \\ \frac{\sin(\pi k)}{\pi k} & \text{if } k \neq 0. \end{cases}$$

3. Derive the mean, variance, auto-covariance, and spectral density of a mixture of periodic series:

$$X_t = \sum_{j=1}^k (A_j c_{t,j} + B_j s_{t,j}),$$

where $A_j, B_j \sim \mathcal{N}(0, \sigma_j^2)$ are uncorrelated $\forall j$, $c_{t,j} = \cos(2\pi\omega_j t)$, and $s_{t,j} = \sin(2\pi\omega_j t)$.

4. Derive the periodogram of the following time series realization with n observations:

$$x_t = a c_{t,j} + b s_{t,j},$$

with a, b constants, ω_j the j -th Fourier frequency, $c_{t,j} = \cos(2\pi\omega_j t)$, and $s_{t,j} = \sin(2\pi\omega_j t)$.