Time Series Analysis

Fourier analysis

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Informatik

Outline

- Fourier analysis
- Sinusoids
- Fitting sinusoids with known frequencies
- Harmonic analysis with Fourier frequencies
- Orthogonality of complex exponentials
- Discrete Fourier transform
- Fast Fourier transform
- Aliasing and the Nyquist frequency

Fourier analysis

Decomposition of time series into a **sum of sinusoidal components**.

- Linear combination of sine and cosine waves, each with specific **frequency** and **amplitude**.

Describe fluctuations in time series by **comparing** them with **sinusoids**.

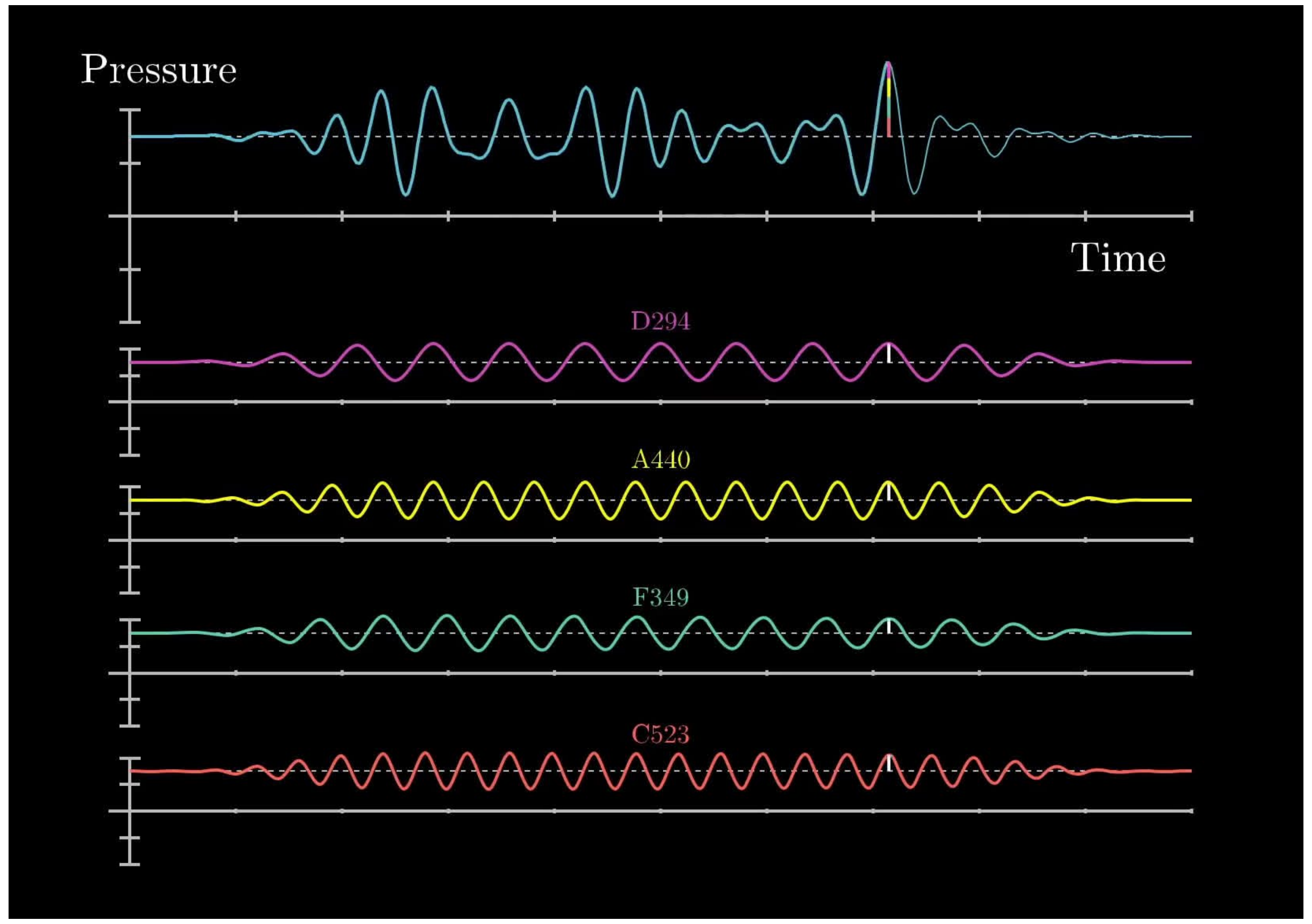
Periodic fluctuations

- Detect seasonality (simple or multiple).
- Identify dominant frequencies.

Aperiodic fluctuations

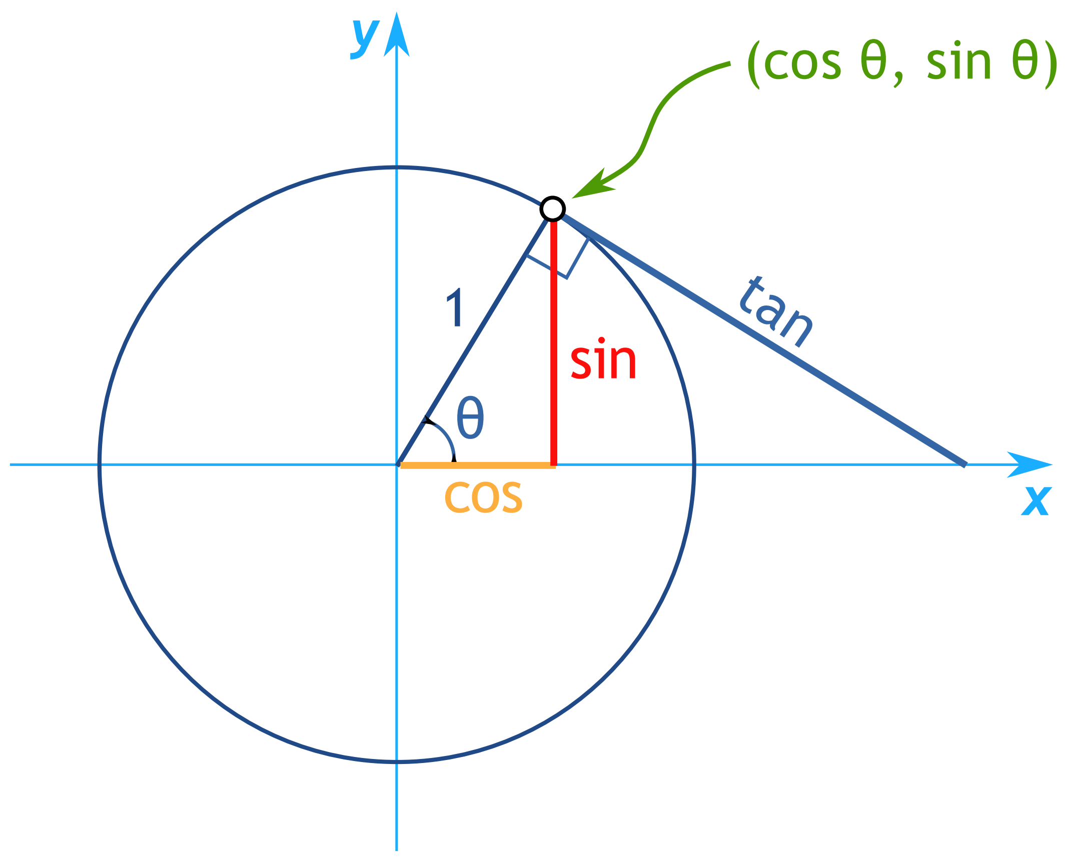
- Describe the frequency distribution of oscillations.
- Distinguish noise from signal.

(and clean up)

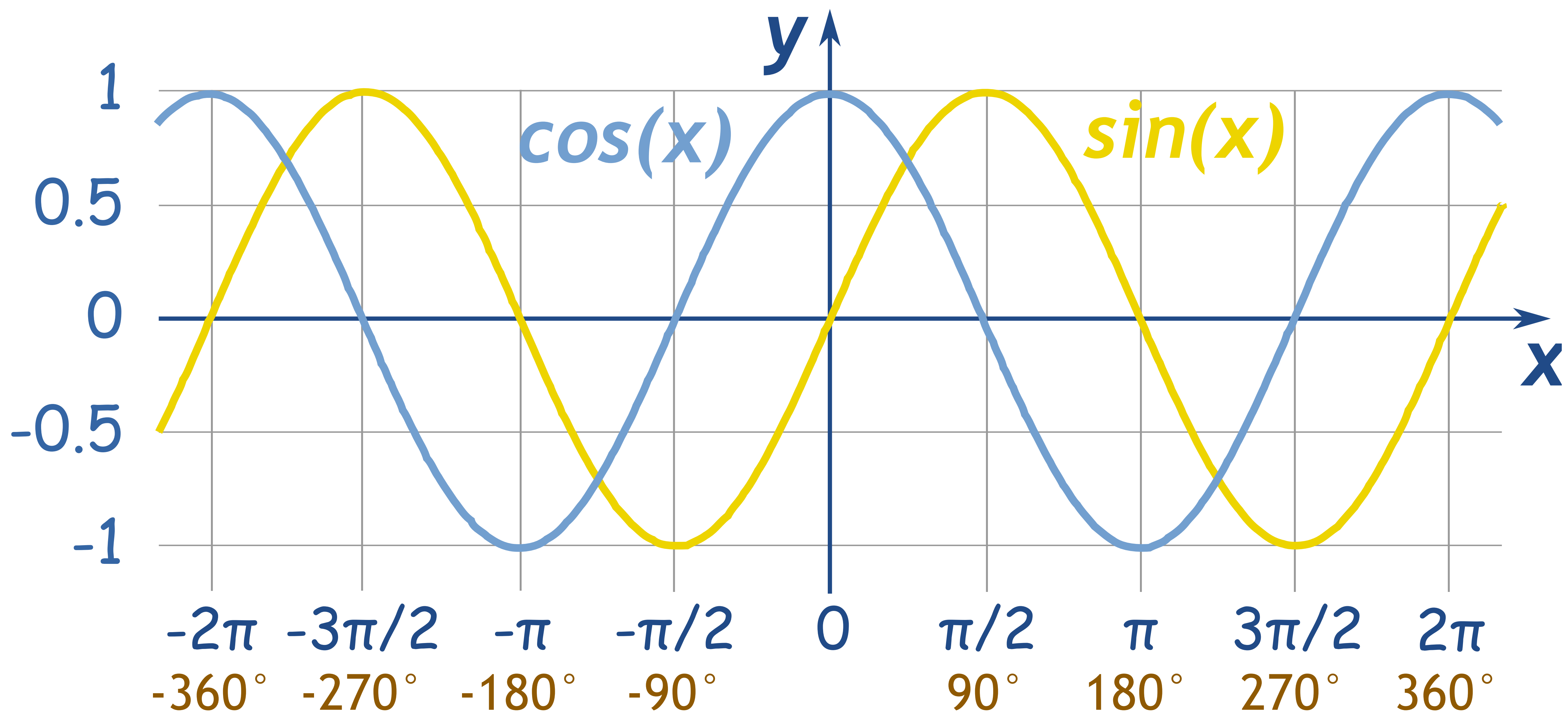


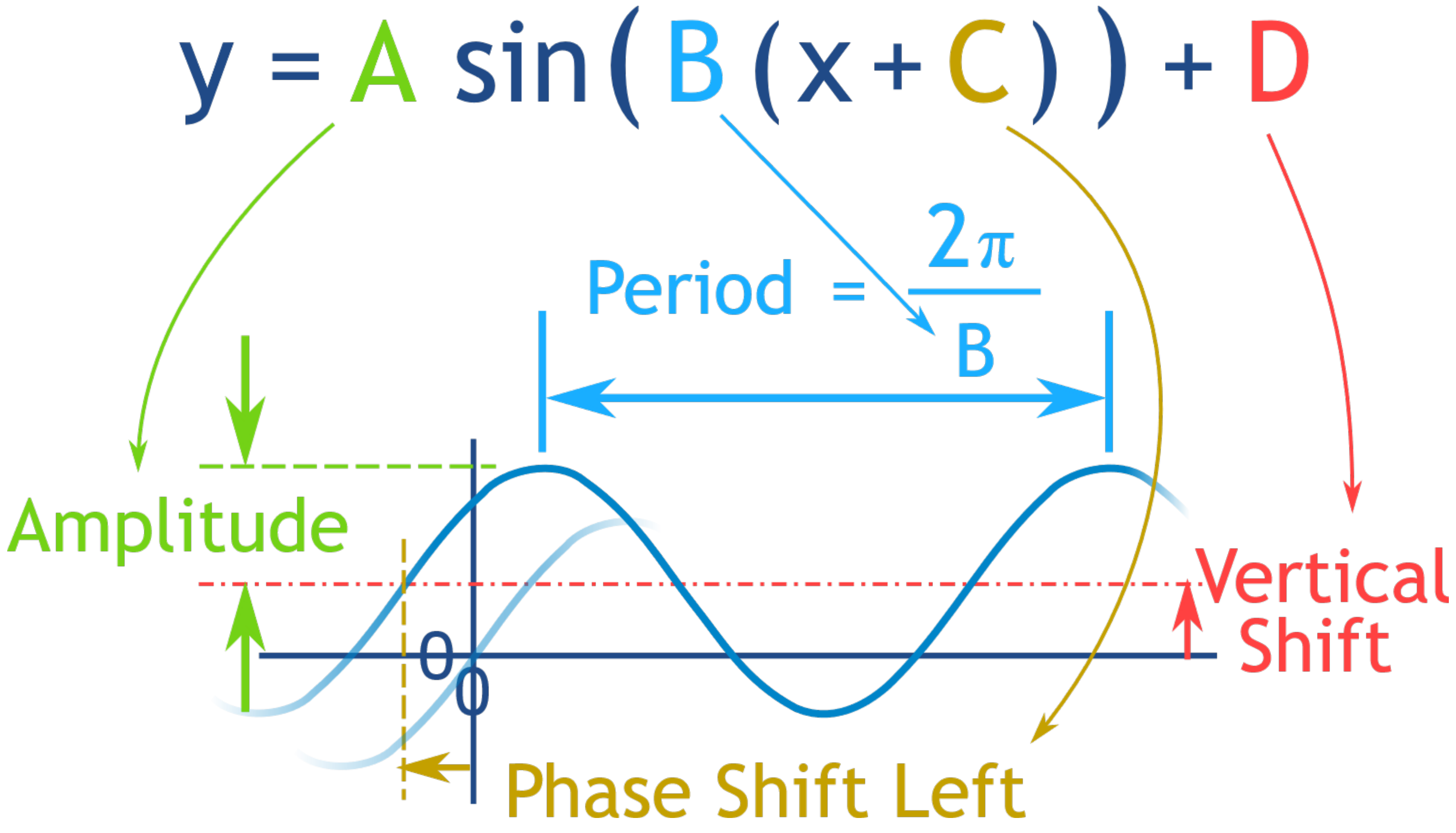
3Blue1Brown

Recap – Trigonometry concepts (1)

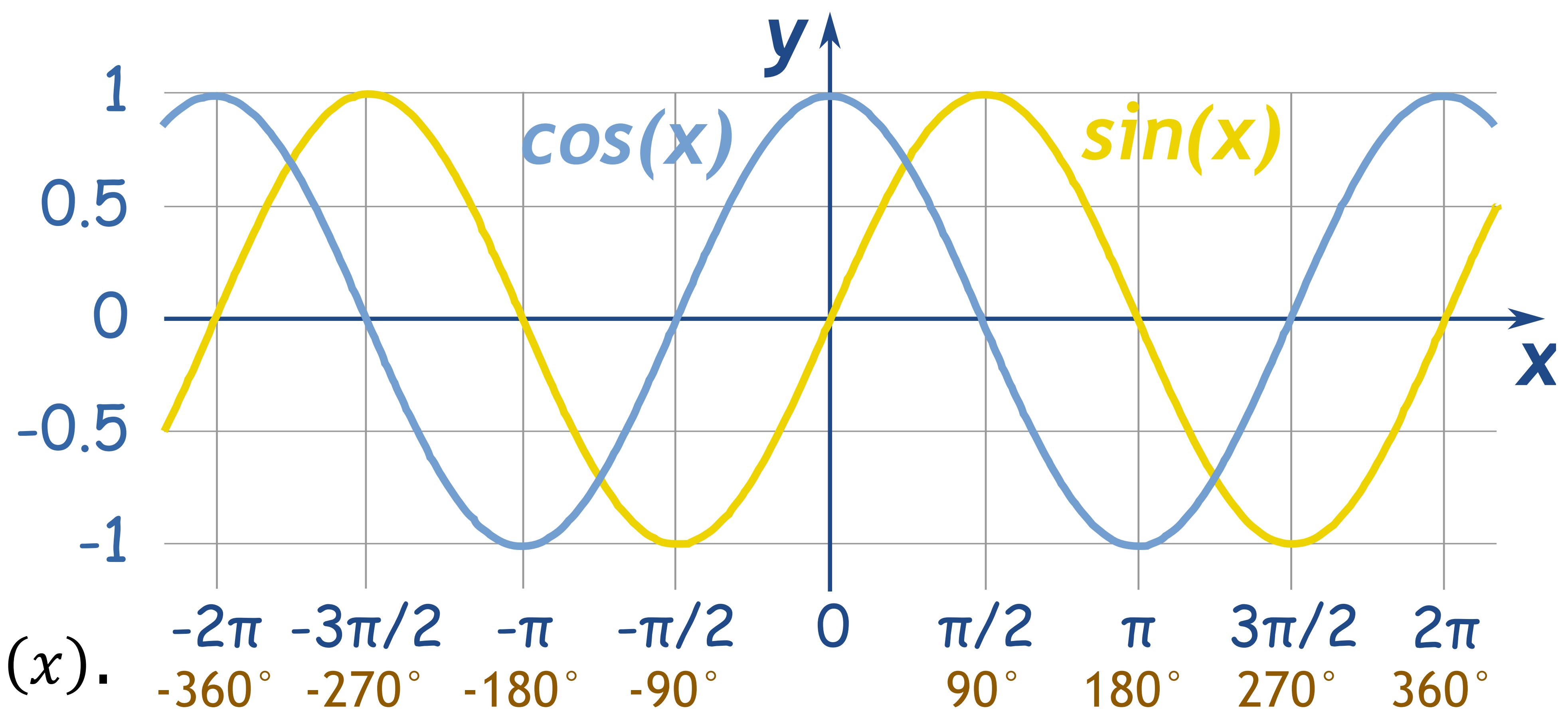


Sinusoids are periodic waves whose waveform is the **cosine**/**sine** function.





Recap – Trigonometry concepts (2)



Sine and cosine are the **same waveform**, just shifted: $\sin\left(x \pm \frac{\pi}{2}\right) = \pm\cos(x)$.

They are periodic $\sin(x+2\pi) = \sin(x)$ and $\cos(x+2\pi) = \cos(x)$

- The **period** $\mathrm{P} = 2\pi$ means the function completes **one full cycle** every 2π units along the x-axis.
- The **frequency** $\mathrm{f} = 1/\mathrm{P} = 1/2\pi$ tells **how many cycles occur per unit of x**.

Sine is an **odd** function $\sin(-x) = -\sin(x)$, while cosine is an **even** function $\cos(-x) = \cos(x)$.

Pythagorean identity: $\sin^2 x + \cos^2 x = 1$.

Angle sum: $\sin(\alpha \pm \beta) = \sin(\alpha)\cos(\beta) \pm \cos(\alpha)\sin(\beta)$ and $\cos(\alpha \pm \beta) = \cos(\alpha)\cos(\beta) \mp \sin(\alpha)\sin(\beta)$

In the continuous case, two sinusoids are **orthogonal** (the integral of their product is zero) over any interval spanning a **complete period** of the functions (provided they are **not vertically shifted**).

Sinusoids

A sinusoid of **frequency** f (cycles per unit time) or period $\mathrm{P} = 1/f$ (in units of time) can be written as

$$x_t = Rcos\big(2\pi(f \cdot t + \phi)\big)$$

where R is the **amplitude** and ϕ is the **phase**.

Orthogonality of sinusoids: length n sampled sinusoidal signal segments are **orthogonal** ⇔ their frequencies have an **integer multiple** k of periods in n samples (i.e., we can write $f = k/n$ with $k \in [0, n-1]$) .

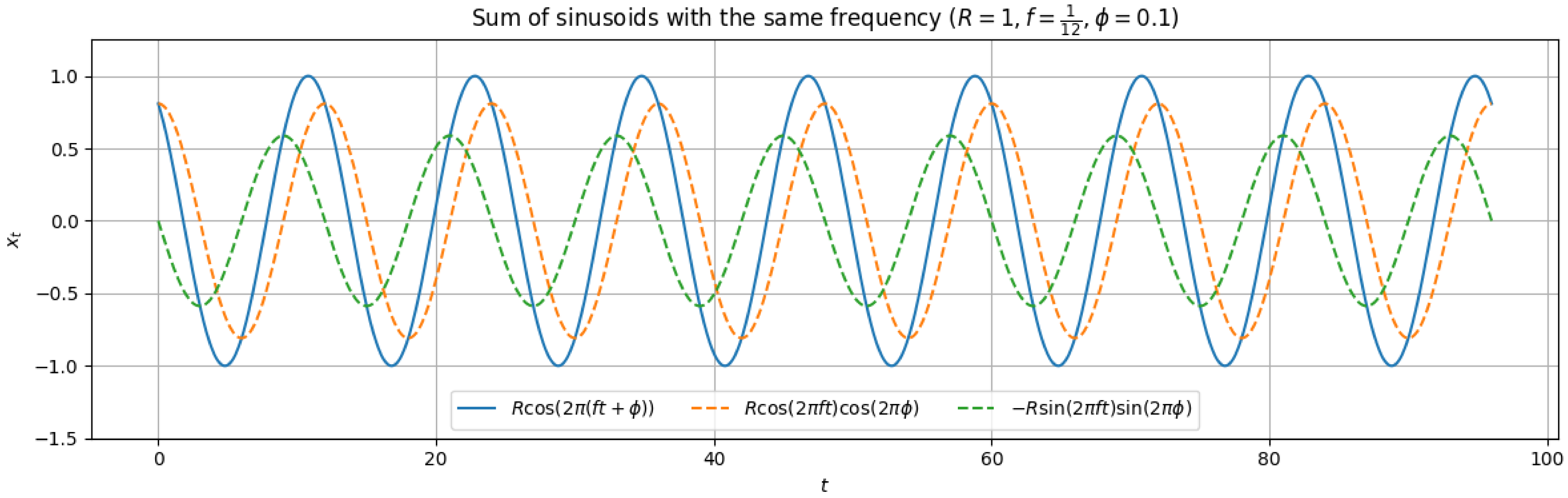
number of sample needs to be a multiple of the periods, so that the orthogonality hols

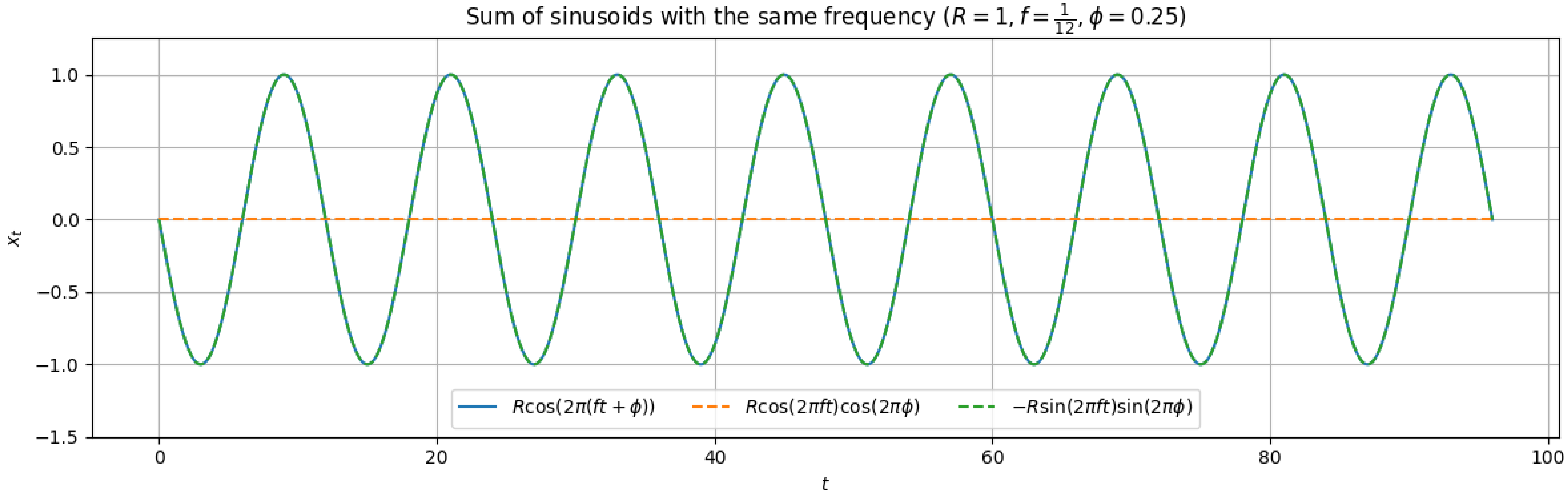
A **sum of sinusoids** with common frequency f is another sinusoid with the same frequency

$$\begin{aligned} x_t &= Rcos\big(2\pi(ft + \phi)\big) \\ &= R(cos(2\pi ft)\cos(2\pi\phi) - sin(2\pi ft)\sin(2\pi\phi)) \\ &= Acos(2\pi ft) + \mathrm{B}sin(2\pi ft) \\ &= Ac_t + Bs_t \end{aligned}$$

with $A = R\cos(2\pi\phi)$, $\mathrm{B} = -\mathrm{R}\sin(2\pi\phi)$, $c_t = cos(2\pi ft)$ and $s_t = sin(2\pi ft)$.

Then $R = \sqrt{A^2 + B^2}$ and $\phi = \arctan\left(-\frac{B}{A}\right)/2\pi$.


Sum of sinusoids with the same frequency ($R = 1, f = \frac{1}{12}, \phi = 0.1$)
x_t
t
$R\cos(2\pi(ft + \phi))$
$R\cos(2\pi ft)\cos(2\pi\phi)$
$-R\sin(2\pi ft)\sin(2\pi\phi)$


Sum of sinusoids with the same frequency ($R = 1, f = \frac{1}{12}, \phi = 0.25$)
x_t
t
$R\cos(2\pi(ft + \phi))$
$R\cos(2\pi ft)\cos(2\pi\phi)$
$-R\sin(2\pi ft)\sin(2\pi\phi)$

Fitting sinusoids (1)

Given a time series realization $\{x_{\mathbf{0}}, \dots x_{\boldsymbol{n-1}}\}$ with known period $\mathrm{P} = 1/f$,

The data can be modeled as

$$x_t = \mu + Ac_t + \mathrm{B}s_t + e_t$$

When frequency unknown: estimate by counting peaks and taking mean

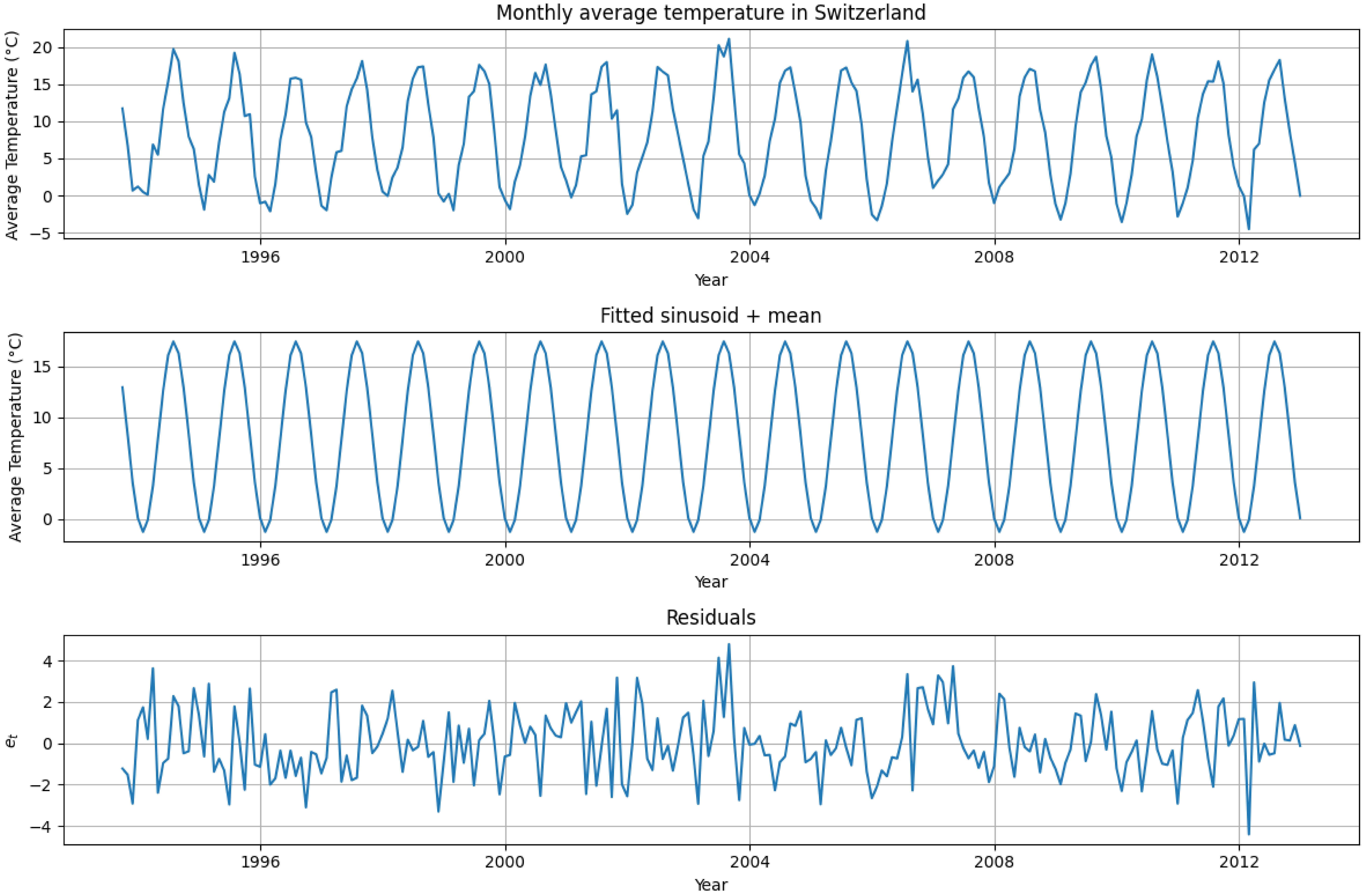
with e_t the residual at time t, $c_t = cos(2\pi f t)$ and $s_t = sin(2\pi f t)$.

Apply OLS: $\begin{pmatrix} \hat{\mu} \\ \hat{A} \\ \hat{B} \end{pmatrix} = \min\limits_{\mu,A,B} \sum_{i=0}^{n-1} e_i^2 = \left(X^T X\right)^{-1} X^T Y$ with $Y = \begin{pmatrix} x_0 \\ \vdots \\ x_{n-1} \end{pmatrix}$ and $X = \begin{pmatrix} 1 & c_0 & s_0 \\ \vdots & \vdots & \vdots \\ 1 & c_{n-1} & s_{n-1} \end{pmatrix}$

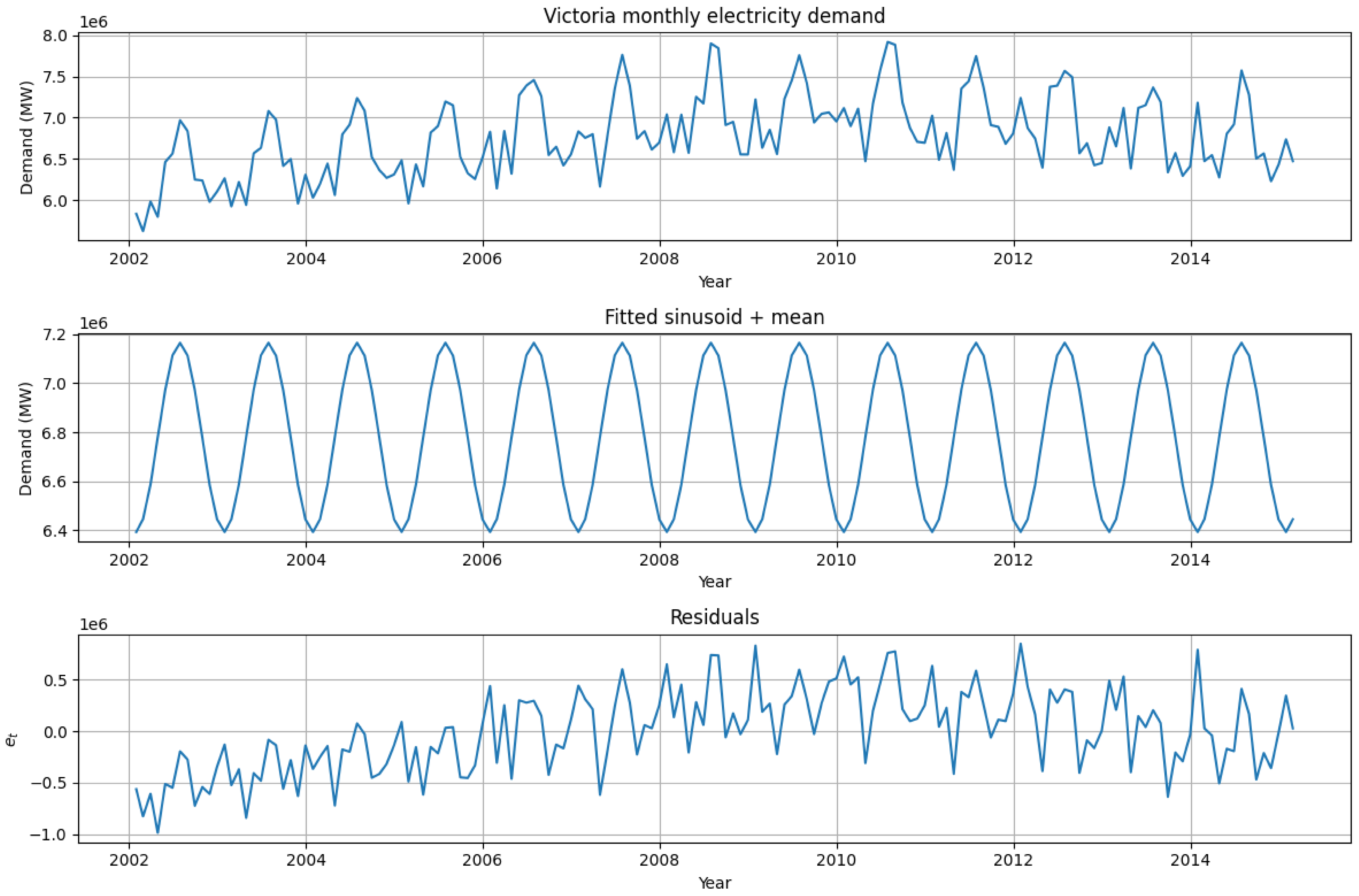
X columns are **orthogonal** provided n is an **integer multiple** of P.

In this case, this optimization problem can be solved with

$$\hat{\mu} = \frac{1}{n} \sum_{t=0}^{n-1} x_t\,, \qquad \hat{A} = \frac{2}{n} \sum_{t=0}^{n-1} x_t c_t\,, \qquad \hat{B} = \frac{2}{n} \sum_{t=0}^{n-1} x_t s_t$$



Monthly average temperature in Switzerland
Average Temperature (°C)
Year
Fitted sinusoid + mean
Average Temperature (°C)
Year
Residuals
e_t
Year


Victoria monthly electricity demand
1e6
Demand (MW)
Year
Fitted sinusoid + mean
Residuals
e_t
2002
2004
2006
2008
2010
2012
2014

Fitting sinusoids (2)

When the TS involves k seasonalities of known periods $\mathrm{P_1, \dots P_k}$ with $\mathrm{P_j} = 1/f_j$ for $j = 1, \dots k$,

The data can be modeled as

$$x_t = \mu + \sum_{j=1}^{k} \left(A_j c_{t,j} + \mathrm{B}_j s_{t,j} \right) + e_t$$

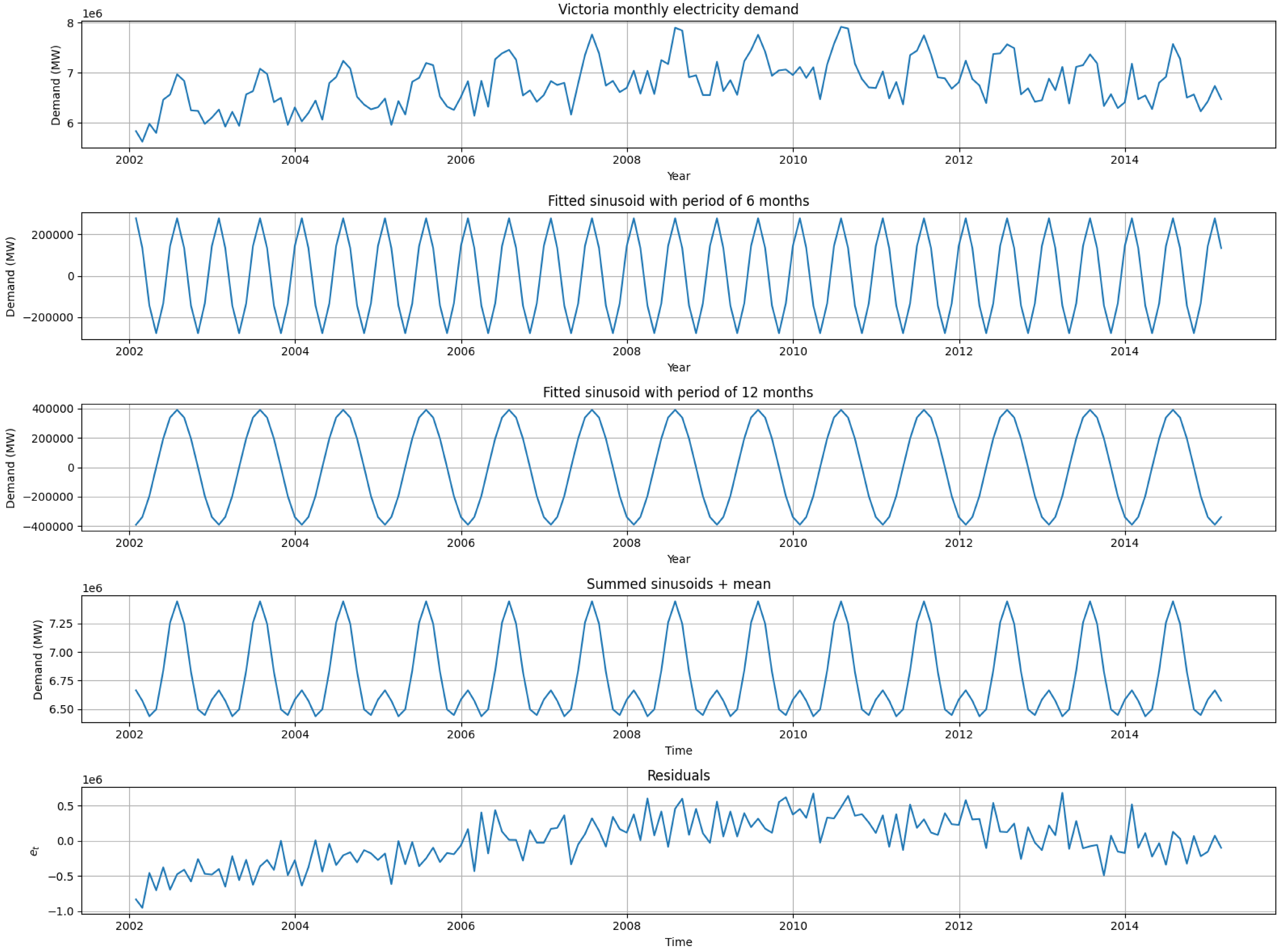
with $c_{t,j} = cos\left(2\pi f_j t\right)$, $s_{t,j} = sin\left(2\pi f_j t\right)$ and e_t the residual at time t.

The design matrix becomes $X = \begin{pmatrix} 1 & c_{0,1} & s_{0,1} & \cdots & c_{0,k} & s_{0,k} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & c_{n-1,1} & s_{n-1,1} & \cdots & c_{n-1,k} & s_{n-1,k} \end{pmatrix}$

X columns are **orthogonal** provided n is an **integer multiple of all periods**.

In this case, this optimization problem can be solved with

$$\hat{\mu} = \frac{1}{n} \sum_{t=0}^{n-1} x_t \,, \qquad \hat{A}_j = \frac{2}{n} \sum_{t=0}^{n-1} x_t c_{t,j} \,, \qquad \hat{B}_{\mathrm{j}} = \frac{2}{n} \sum_{t=0}^{n-1} x_t s_{t,j}$$


Victoria monthly electricity demand
Fitted sinusoid with period of 6 months
Fitted sinusoid with period of 12 months
Summed sinusoids + mean
Residuals
Demand (MW)
Year
Time
1e6

Harmonic analysis with Fourier frequencies

Given a TS realization $\{x_0, \dots x_{n-1}\}$, the **Fourier frequencies** are frequencies of the form $\mathrm{f_j} = j/n$ with $j = \mathbf{0}, 1, \dots, k$.

- Due to aliasing, we consider only frequencies satisfying $0 \leq \mathrm{f_j} \leq \frac{1}{2}$ (since $\Delta = 1$) i.e., $k = \begin{cases} n/2\,, & n\ is\ even \\ (n-1)/2\,, & n\ is\ odd \end{cases}$.

These frequencies are **harmonics** (integer multiples) with respect to the span of the data i.e., each of the corresponding periods $\mathrm{P_j} = n/j$ is **repeated *j* times**.

- Sinusoids at the Fourier frequencies (X columns) are **orthogonal** over the span of the data n.

The TS realization can be **exactly** represented as a **sum of k sinusoids at the Fourier frequencies**:

$$x_t = \mu + \sum_{j=\mathbf{1}}^{k} \left(A_j c_{t,j} + \mathrm{B}_j s_{t,j} \right)$$

$$\mu = \frac{1}{n} \sum_{t=0}^{n-1} x_t = \frac{A_0}{2}, \qquad A_j = \frac{2}{n} \sum_{t=0}^{n-1} x_t c_{t,j}\,, \qquad B_j = \frac{2}{n} \sum_{t=0}^{n-1} x_t s_{t,j}$$

Recap – Complex numbers

Complex numbers $\mathbb{C}$ **extend** real numbers $\mathbb{R}$ with the **imaginary unit** i that satisfies $i^2 = -1$.

- $z = Re(z) + i\, Im(z)$ with $Re(z)$ the **real part** and $Im(z)$ the **imaginary part** of z.

Complex numbers can also be defined in **polar coordinates**: $z = |z|(\cos(\varphi) + i \sin(\varphi))$.

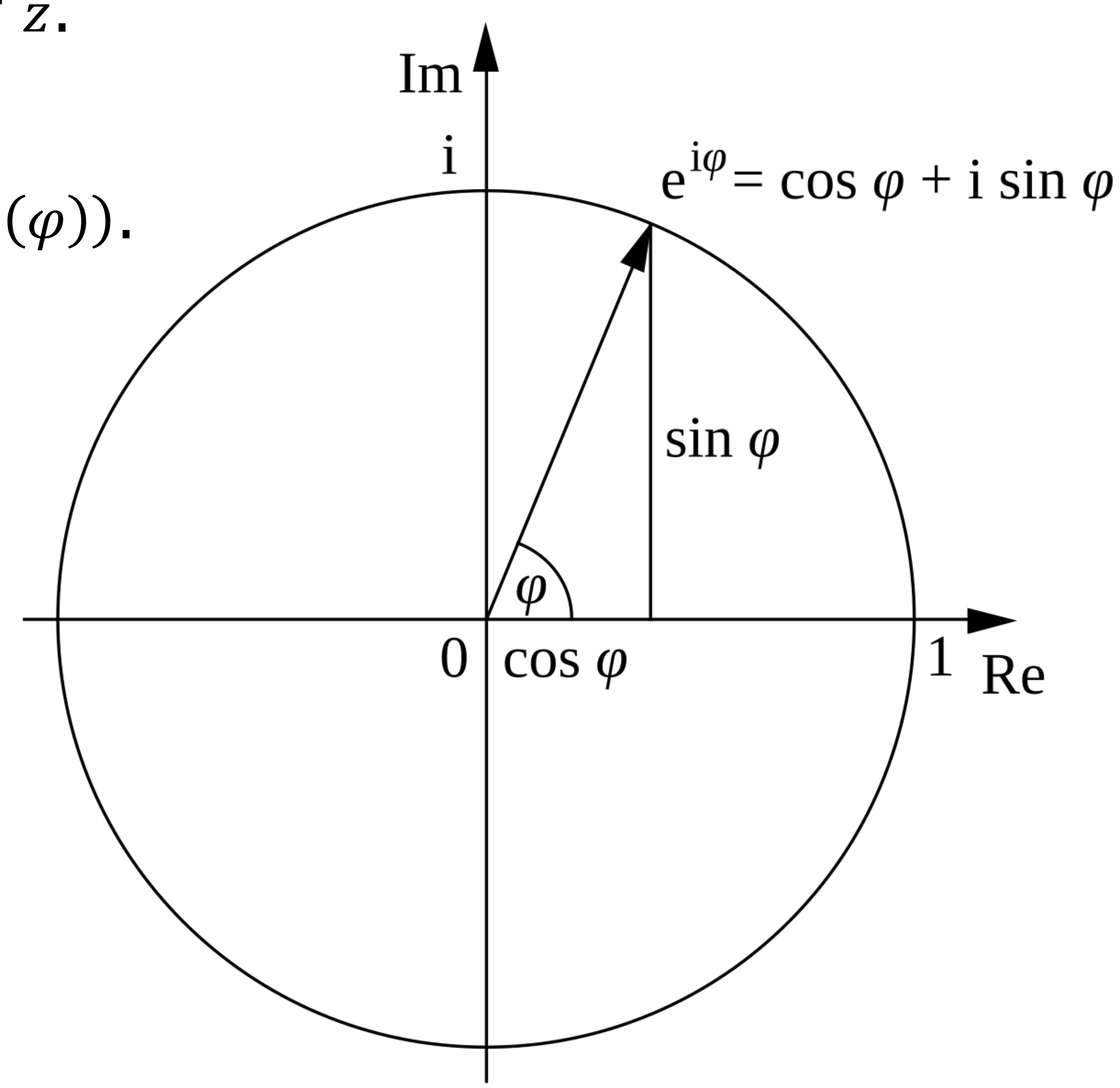
- $Re(z) = |z| \cos(\varphi)$ and $Im(z) = |z| \sin(\varphi)$.
- **Magnitude** $|z| = \sqrt{Re(z)^2 + Im(z)^2}$ and **phase** $\varphi = \arctan(Im(z)/Re(z))$.

Euler formula: $\mathrm{e}^{\mathrm{i}\varphi} = \cos(\varphi) + i \sin(\varphi)$. Hence, $\mathrm{e}^{\mathrm{i}\varphi} = -\mathrm{e}^{\mathrm{i}(\varphi+\pi)}$.

- Any $z \in \mathbb{C}$ can be defined as $z = |z|\mathrm{e}^{\mathrm{i}\varphi}$.
- $\cos(\varphi) = \frac{1}{2}\left(\mathrm{e}^{\mathrm{i}\varphi} + \mathrm{e}^{-\mathrm{i}\varphi}\right)$ and $\sin(\varphi) = \frac{1}{2i}\left(\mathrm{e}^{\mathrm{i}\varphi} - \mathrm{e}^{-\mathrm{i}\varphi}\right)$.

The **complex conjugate** of z is defined as $\bar{z} = Re(z) - i\, Im(z) = |z|\mathrm{e}^{-\mathrm{i}\varphi}$.

- $\bar{\bar{z}} = z$ and the complex conjugate of $\mathrm{x} \in \mathbb{R}$ is $\bar{x} = x$.
- $z\bar{z} = Re(z)^2 + Im(z)^2 = |z|^2$ where $|z|$ is the modulus of z.

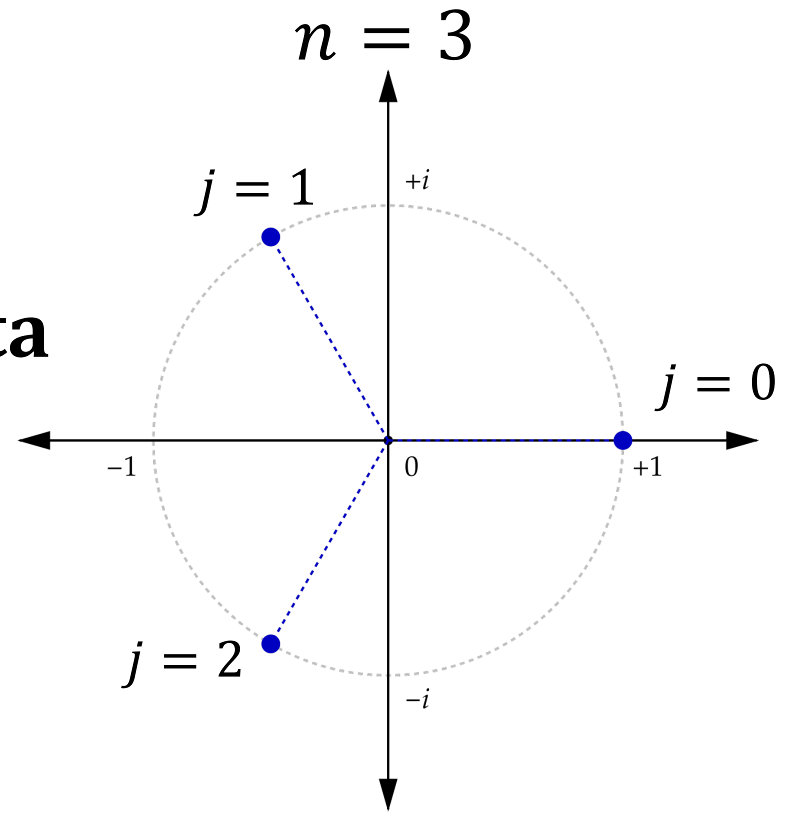


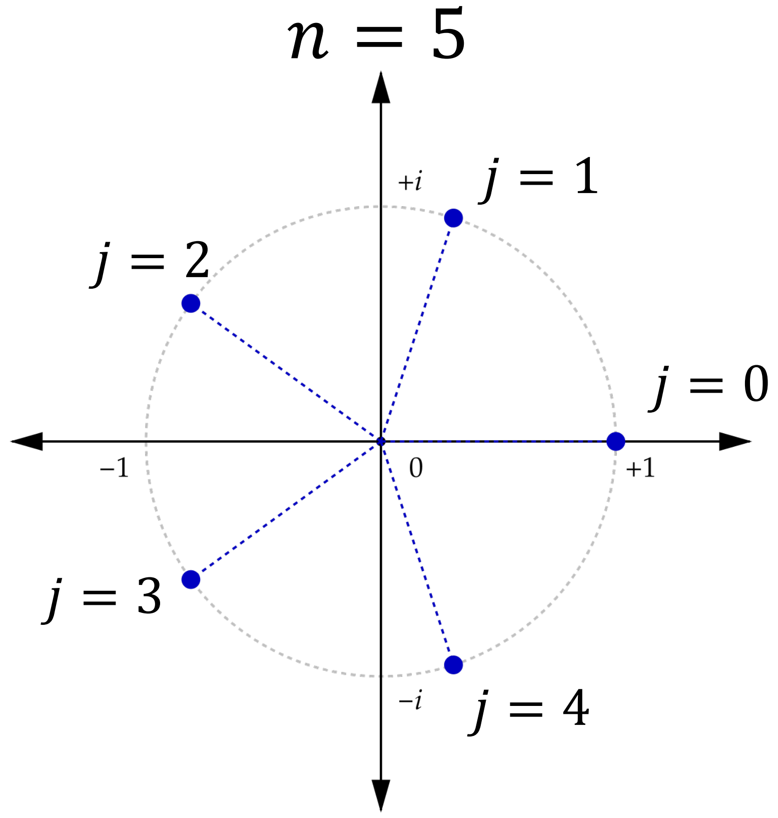
Gunther, wikimedia

Orthogonality of complex exponentials

The sum of complex numbers **equally spaced over the unit circle** $z_{t,j} = e^{2\pi i \frac{j}{n} t}$ (called the **n-th roots of unity** as $z_{t,j}$ satisfies ${z_{t,j}}^n = 1$) **over one period** n is zero for any non-zero integer $0 < t < n$,

$$\sum_{j=0}^{n-1} e^{2\pi i \frac{j}{n} t} = \delta_{t0} n = \begin{cases} n & \text{if } t = 0 \\ 0 & \text{otherwise} \end{cases}, \qquad \text{with } \delta_{kl} = \begin{cases} 1 & \text{if } k = l \\ 0 & \text{otherwise} \end{cases} \text{ the } \textbf{Kronecker delta}$$





Loadmaster, wikimedia

Proof: When $t = 0$, we have $\sum_{j=0}^{n-1} e^0 = n$. When $t \neq 0$, the sum is a geometric series:

$$\sum_{j=0}^{n-1} e^{2\pi i \frac{j}{n} t} = \sum_{j=0}^{n-1} \left(e^{2\pi i \frac{1}{n} t}\right)^j = 1 + \left(e^{2\pi i \frac{1}{n} t}\right)^1 + \left(e^{2\pi i \frac{1}{n} t}\right)^2 + \cdots \left(e^{2\pi i \frac{1}{n} t}\right)^{n-1}$$

(recall that $\sum_{j=0}^{n-1} r^j = \frac{1-r^n}{1-r}$ and that $e^{2\pi i t} = 1$ for integer t)

$$= \frac{1 - \left(e^{2\pi i \frac{1}{n} t}\right)^n}{1 - e^{2\pi i \frac{1}{n} t}} = \frac{1 - e^{2\pi i t}}{1 - e^{2\pi i \frac{1}{n} t}} = 0$$

Such complex exponentials are **orthogonal** over a complete cycle: taking $\boldsymbol{z}_k = \{z_{0,k}, \dots z_{n-1,k}\}$,

$$\langle \boldsymbol{z}_k, \boldsymbol{z}_l \rangle = \sum_{t=0}^{n-1} z_k \overline{z_l} = \sum_{t=0}^{n-1} e^{2\pi i \frac{k}{n} t} e^{-2\pi i \frac{l}{n} t} = \sum_{t=0}^{n-1} e^{2\pi i \frac{t}{n}(k-l)} = \delta_{kl} n$$

Discrete Fourier transform (DFT)

Given a TS realization $\boldsymbol{x} = \{x_0, \dots x_{n-1}\}$, the **Discrete Fourier Transform** $\mathcal{F}(\boldsymbol{x}) = \boldsymbol{X} = \{X_0, \dots X_{n-1}\}$ is defined as

$$\mathcal{F}(\boldsymbol{x})_j = X_j = \frac{1}{n}\sum_{t=0}^{n-1} x_t e^{-2\pi i f_j t}, \qquad j = 0,1,\dots,k,\dots n-1$$

with $f_j = j/n$ the **Fourier frequencies** and $\mathrm{k} = \begin{cases} n/2, & n \; is \; even \\ (n-1)/2, & n \; is \; odd \end{cases}$.

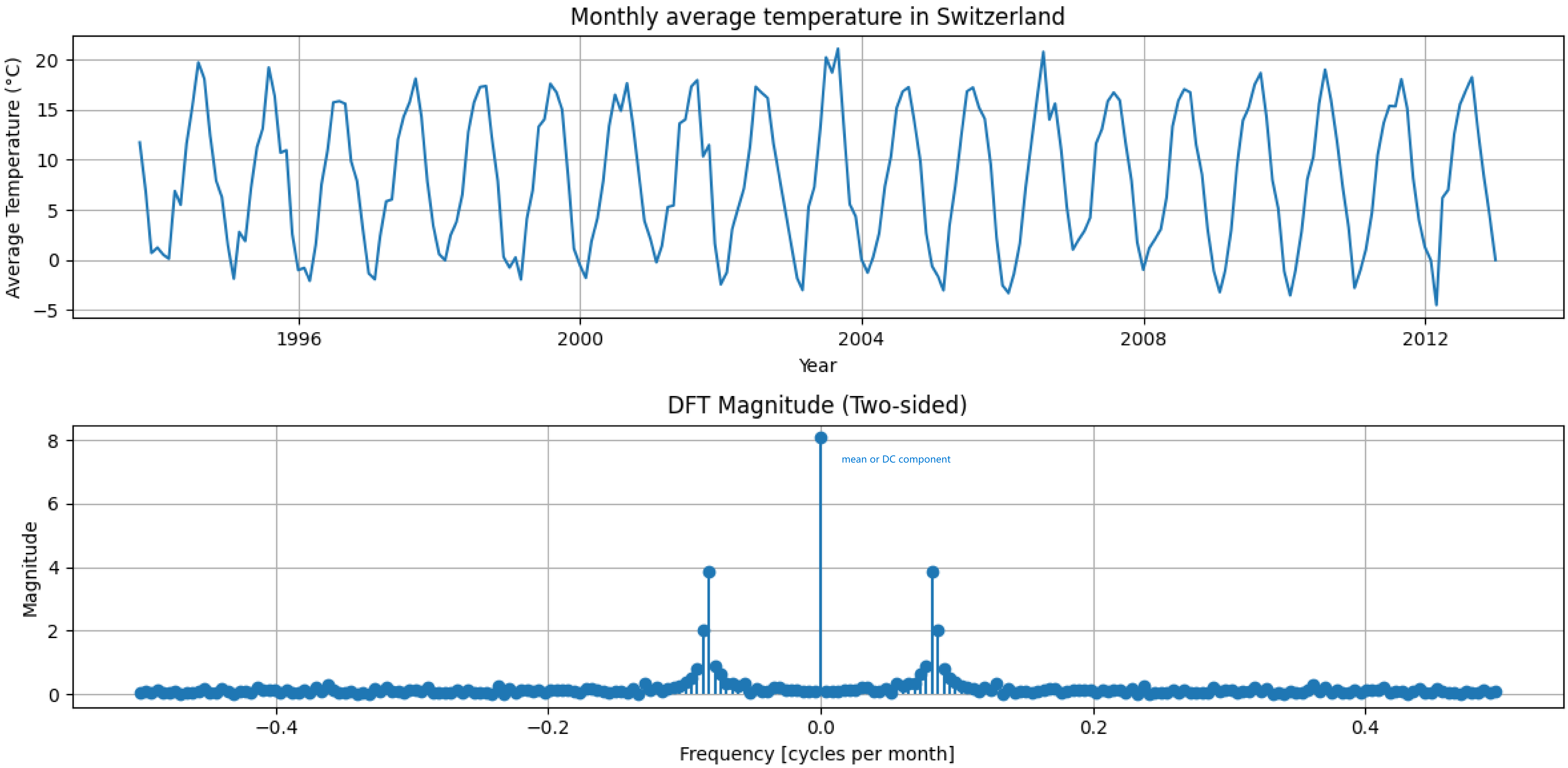
For **real** x_t we observe that $X_{n-j} = \bar{X}_j$ (**conjugate symmetry**) i.e., the frequencies f_j for $j > k$ are **redundant**:

$$X_{n-j} = \frac{1}{n}\sum_{t=0}^{n-1} x_t e^{-2\pi i \left(\frac{n-j}{n}\right) t}$$

$$= \frac{1}{n}\sum_{t=0}^{n-1} x_t e^{-2\pi i t} e^{2\pi i \frac{j}{n} t}$$

($e^{2\pi i k} = 1$ for integer k)

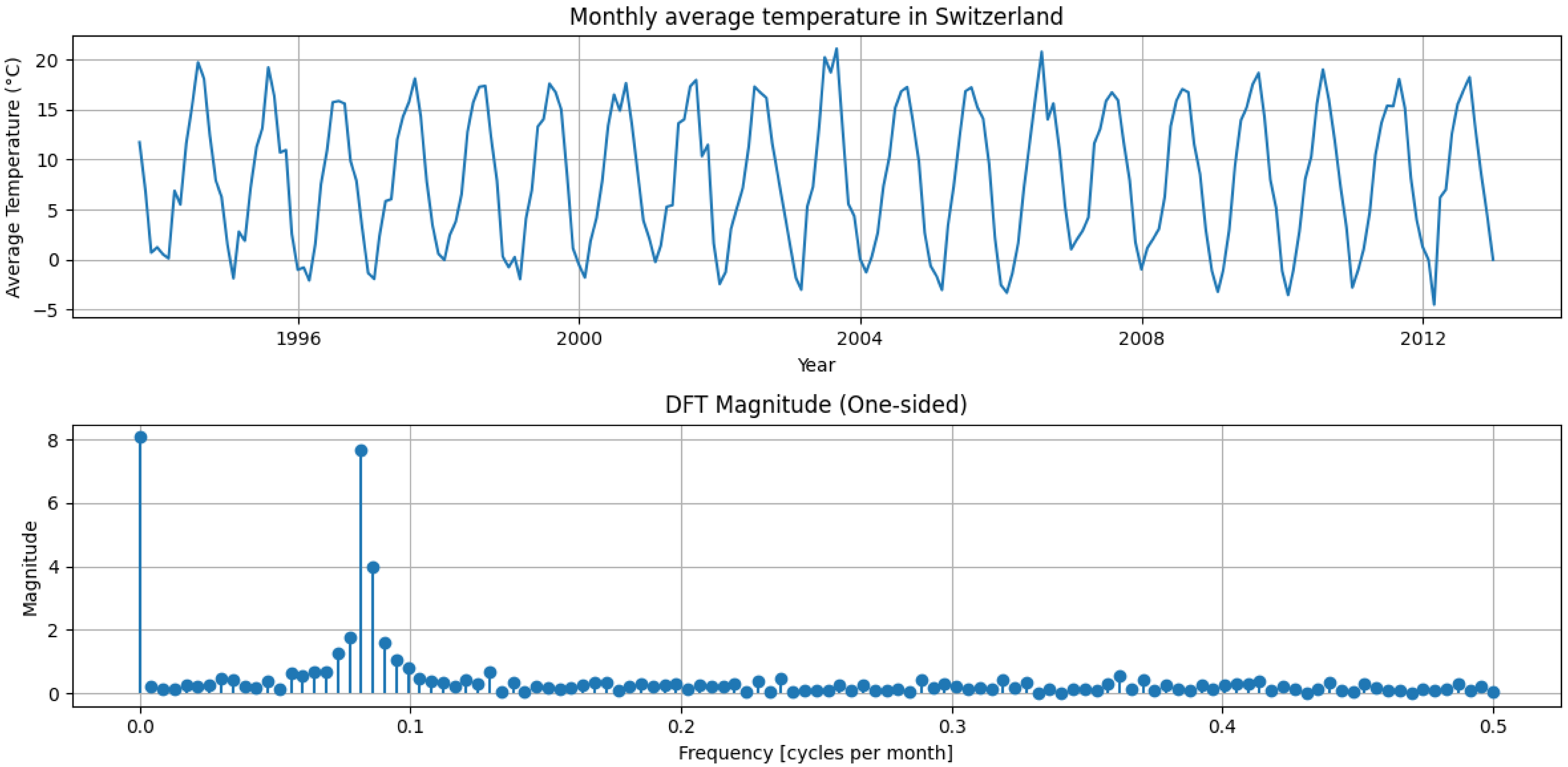
$$= \frac{1}{n}\sum_{t=0}^{n-1} x_t \cdot 1 \cdot e^{2\pi i f_j t} = \bar{X}_j$$



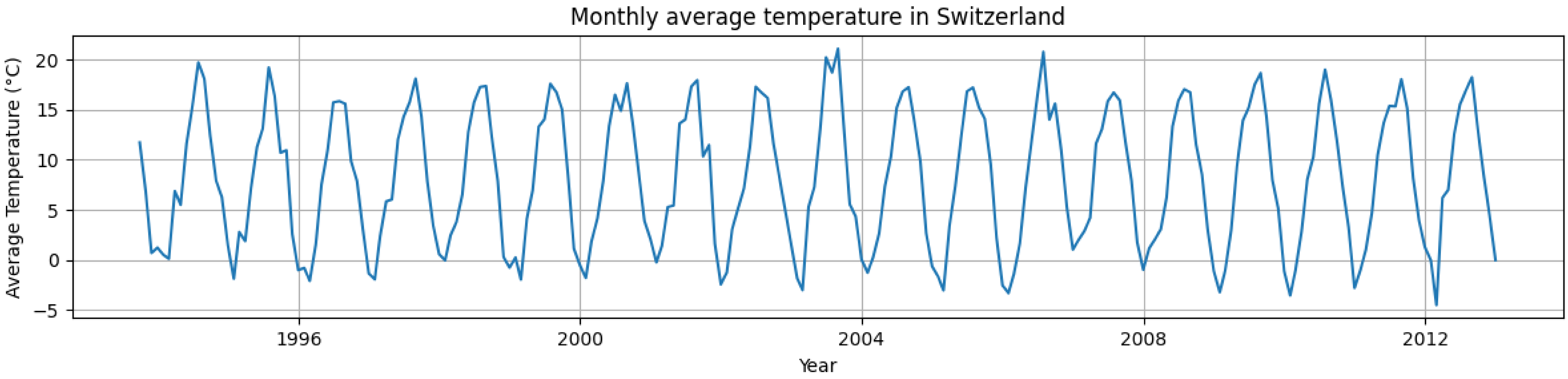
data points are per month, hence frequency per time unit --> frequency by month

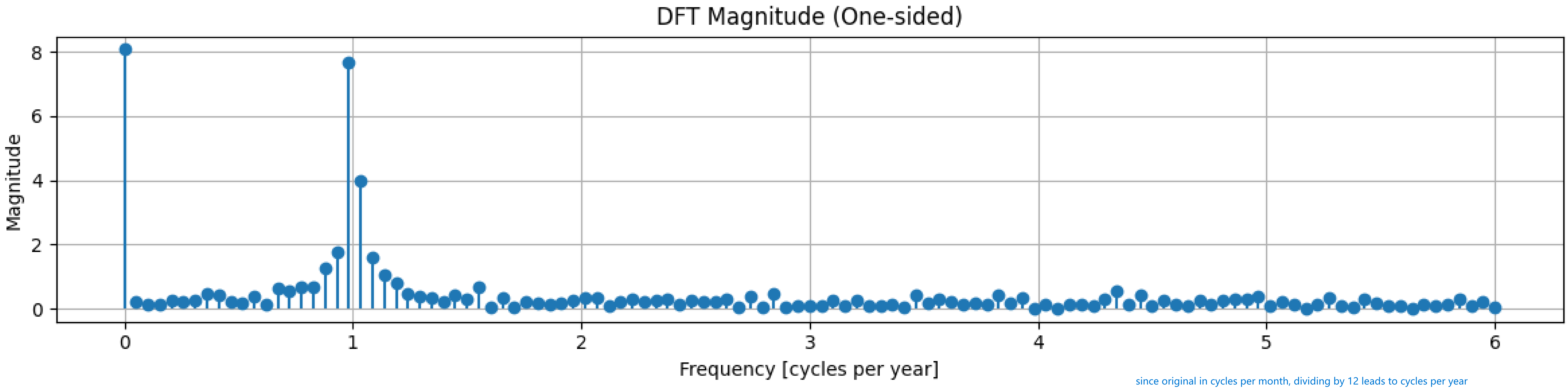
X_{n-j} for $j = 0, \dots n/2 - 1$ correspond to **negative frequencies** (counterclockwise rotations)

X_j for $j = 0, \dots n/2$ correspond to **positive frequencies** (counterclockwise rotations)



Dominant frequency at 0.078 cycles per month i.e., 0.931 cycles per year.



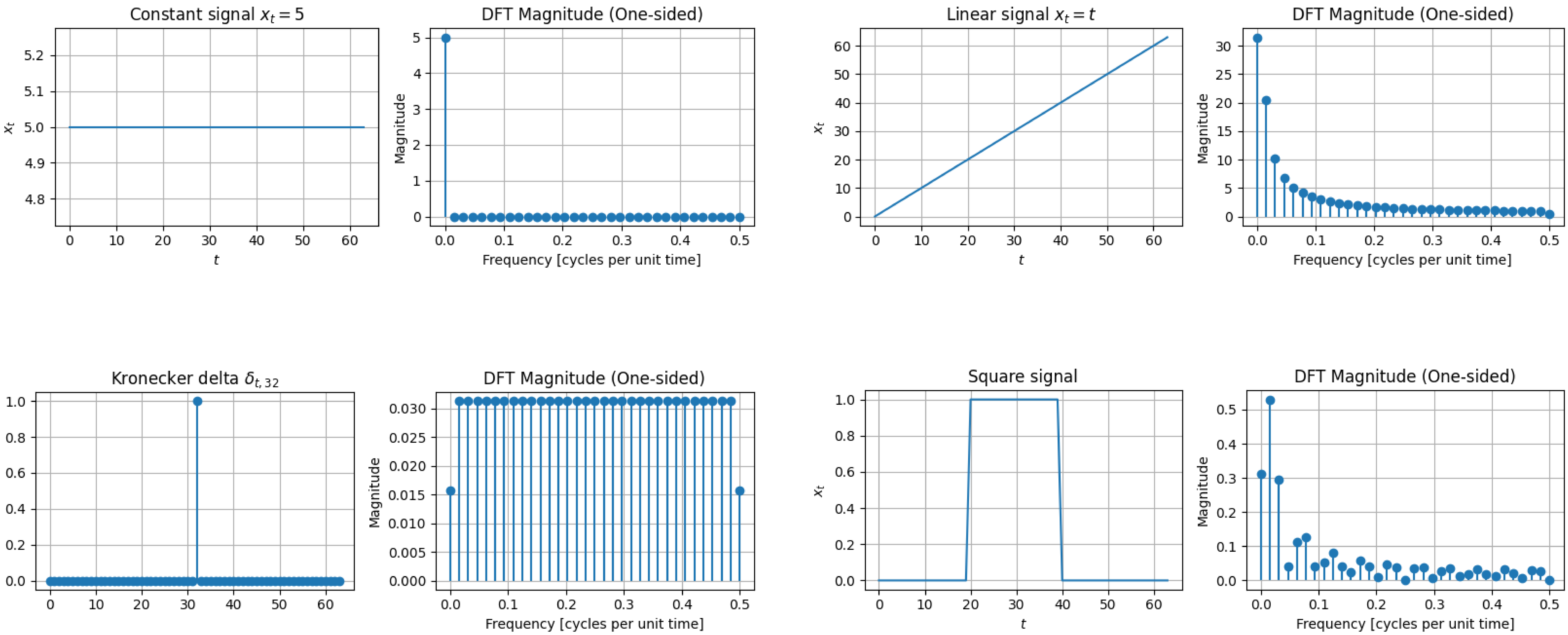


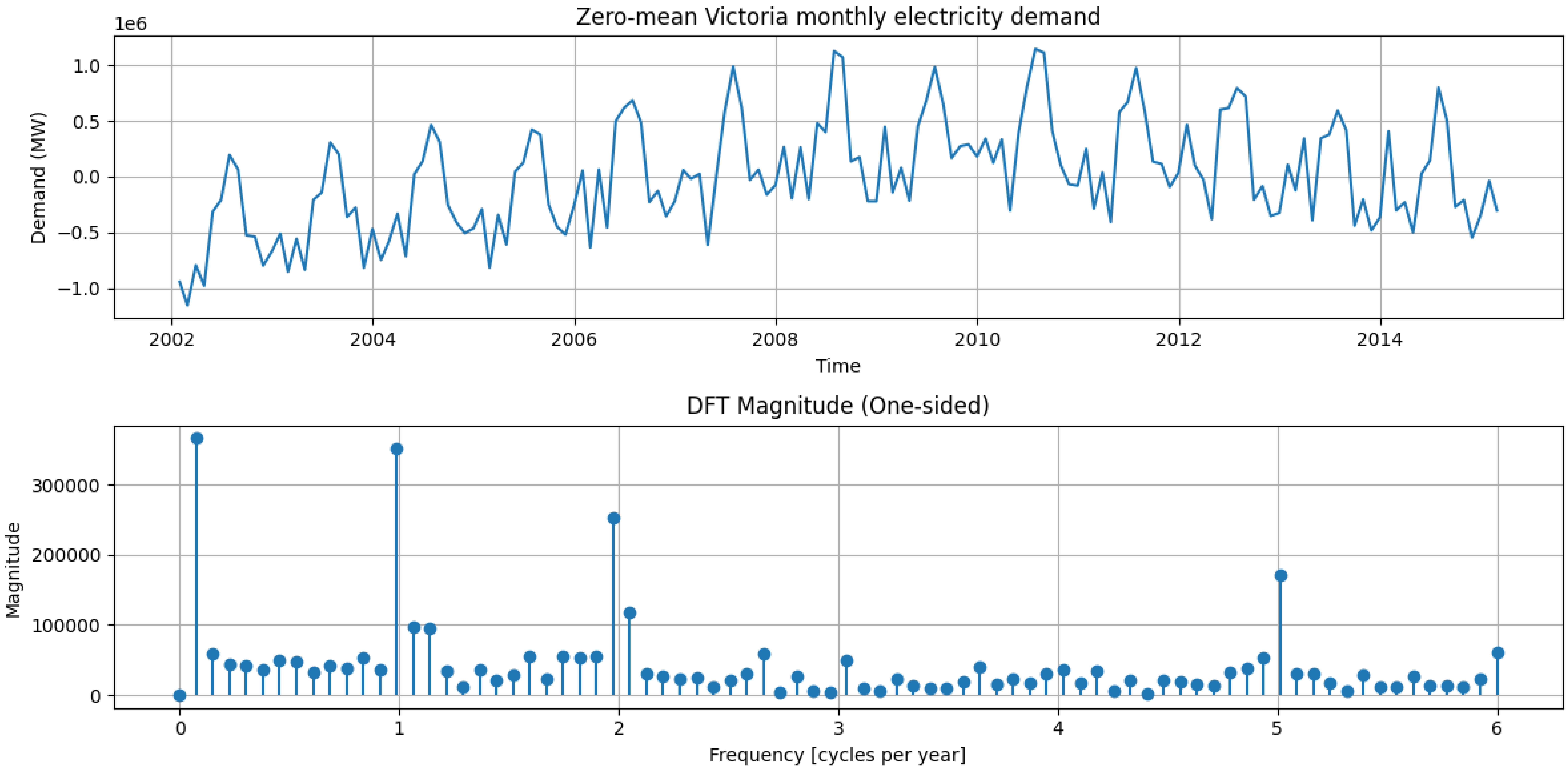
since original in cycles per month, dividing by 12 leads to cycles per year

magnitude tells how important a frequency is in the signal

cycles per year?

Discrete Fourier transform of simple signals




Zero-mean Victoria monthly electricity demand
1e6
1.0
0.5
0.0
−0.5
−1.0
Demand (MW)
2002
2004
2006
2008
2010
2012
2014
Time
DFT Magnitude (One-sided)
300000
200000
100000
0
Magnitude
0
1
2
3
4
5
6
Frequency [cycles per year]

Discrete Fourier transform – Connection with sinusoids

$$\mathcal{F}(\boldsymbol{x})_j = X_j = \frac{1}{n}\sum_{t=0}^{n-1} x_t e^{-2\pi i f_j t}$$

$$= \frac{1}{n}\sum_{t=0}^{n-1} x_t\big(\cos(-2\pi f_j t) + i\sin(-2\pi f_j t)\big)$$

($sin(-x) = -sin(x)$ and $cos(-x) = cos(x)$)

$$= \frac{1}{n}\sum_{t=0}^{n-1} x_t\big(\cos(2\pi f_j t) - i\sin(2\pi f_j t)\big)$$

$$= \frac{1}{n}\sum_{t=0}^{n-1} x_t\big(c_{t,j} - i\, s_{t,j}\big)$$

(recall that $A_j = \frac{2}{n}\sum_{t=0}^{n-1} x_t c_{t,j}$, $B_j = \frac{2}{n}\sum_{t=0}^{n-1} x_t s_{t,j}$)

$$= \underbrace{\frac{1}{n}\sum_{t=0}^{n-1} x_t c_{t,j}}_{Re(X_j) = \frac{A_j}{2}} - i\underbrace{\frac{1}{n}\sum_{t=0}^{n-1} x_t s_{t,j}}_{Im(X_j) = \frac{B_j}{2}}$$

(observe that for real input, the sinusoid magnitude is split in half $|X_j| = \frac{1}{2}\sqrt{A_j^2 + B_j^2} = |\overline{X_{n-j}}|$)

$$= \frac{A_j}{2} - i\frac{B_j}{2}$$

Inverse discrete Fourier transform

Given $\mathcal{F}(\boldsymbol{x}) = \boldsymbol{X} = \{X_0, \dots X_{n-1}\}$, the **inverse DFT** $\mathcal{F}^{-1}(\boldsymbol{X}) = \boldsymbol{x} = \{x_0, \dots x_{n-1}\}$ is defined as

$$x_t = \mathcal{F}^{-1}(\boldsymbol{X})_t = \sum_{j=0}^{n-1} X_j e^{2\pi i f_j t}, \qquad t = 0,1, \dots n-1$$

Proof:

$$\mathcal{F}^{-1}(\boldsymbol{X})_t = \sum_{j=0}^{n-1} \left(\frac{1}{n} \sum_{k=0}^{n-1} x_k e^{-2\pi i f_j k} \right) e^{2\pi i f_j t}$$

$$= \frac{1}{n} \sum_{j=0}^{n-1} \sum_{k=0}^{n-1} x_k e^{2\pi i f_j (t-k)} = \frac{1}{n} \sum_{j=0}^{n-1} \sum_{k=0}^{n-1} x_k e^{2\pi i \frac{j}{n}(t-k)}$$

(orthogonality of complex exponentials)

$$= \frac{1}{n} \sum_{k=0}^{n-1} x_k \underbrace{\sum_{j=0}^{n-1} e^{2\pi i \frac{j}{n}(t-k)}}_{= \delta_{kt} n = \begin{cases} n & \text{if } k = t \\ 0 & \text{otherwise} \end{cases}} = \frac{1}{n} x_t \, n = x_t$$

Inverse discrete Fourier transform – Connection with sinusoids

Given a **real** TS realization $\boldsymbol{x} = \{x_0, \dots x_{n-1}\}$,

$$x_t = \mathcal{F}^{-1}(\boldsymbol{X})_t = \sum_{j=0}^{n-1} X_j e^{2\pi i f_j t}$$

$$= X_0 + \sum_{j=1}^{k} \left(X_j e^{2\pi i f_j t} + X_{n-j} e^{2\pi i f_{n-j} t} \right)$$

$$k = \begin{cases} n/2\,, & n \text{ is even} \\ (n-1)/2\,, & n \text{ is odd} \end{cases}$$

(conjugate symmetry for real input)

$$= X_0 + \sum_{j=1}^{k} \left(X_j e^{2\pi i f_j t} + \bar{X}_j e^{-2\pi i f_j t} \right)$$

(recall that $X_j = \frac{A_j}{2} - i\frac{B_j}{2}$)

$$= \left(\frac{A_0}{2} - i\frac{B_0}{2}\right) + \sum_{j=1}^{k} \left(\left(\frac{A_j}{2} - i\frac{B_j}{2}\right)(c_{t,j} + i\, s_{t,j}) + \left(\frac{A_j}{2} + i\frac{B_j}{2}\right)(c_{t,j} - i\, s_{t,j}) \right)$$

($B_0 = 0$)

$$= \frac{A_0}{2} + \sum_{j=1}^{k} \left(\left(\frac{A_j}{2} c_{t,j} + \frac{A_j}{2} i\, s_{t,j} - i\frac{B_j}{2} c_{t,j} + \frac{B_j}{2} s_{t,j} \right) + \left(\frac{A_j}{2} c_{t,j} - \frac{A_j}{2} i\, s_{t,j} + i\frac{B_j}{2} c_{t,j} + \frac{B_j}{2} s_{t,j} \right) \right)$$

($\mu = \frac{A_0}{2}$)

$$= \mu + \sum_{j=1}^{k} \left(A_j c_{t,j} + B_j s_{t,j} \right)$$

Some properties of the discrete Fourier transform

Given two **real** TS realizations $\boldsymbol{x} = \{x_0, \dots x_{n-1}\}$ and $\boldsymbol{y} = \{y_0, \dots y_{n-1}\}$, their DFT $\boldsymbol{X} = \{X_0, \dots X_{n-1}\}$ and $\boldsymbol{Y} = \{Y_0, \dots Y_{n-1}\}$,

Linearity: let $z_t = ax_t + by_t$ then $Z_j = aX_j + bY_j$ for any scalars a, b.

Shifting

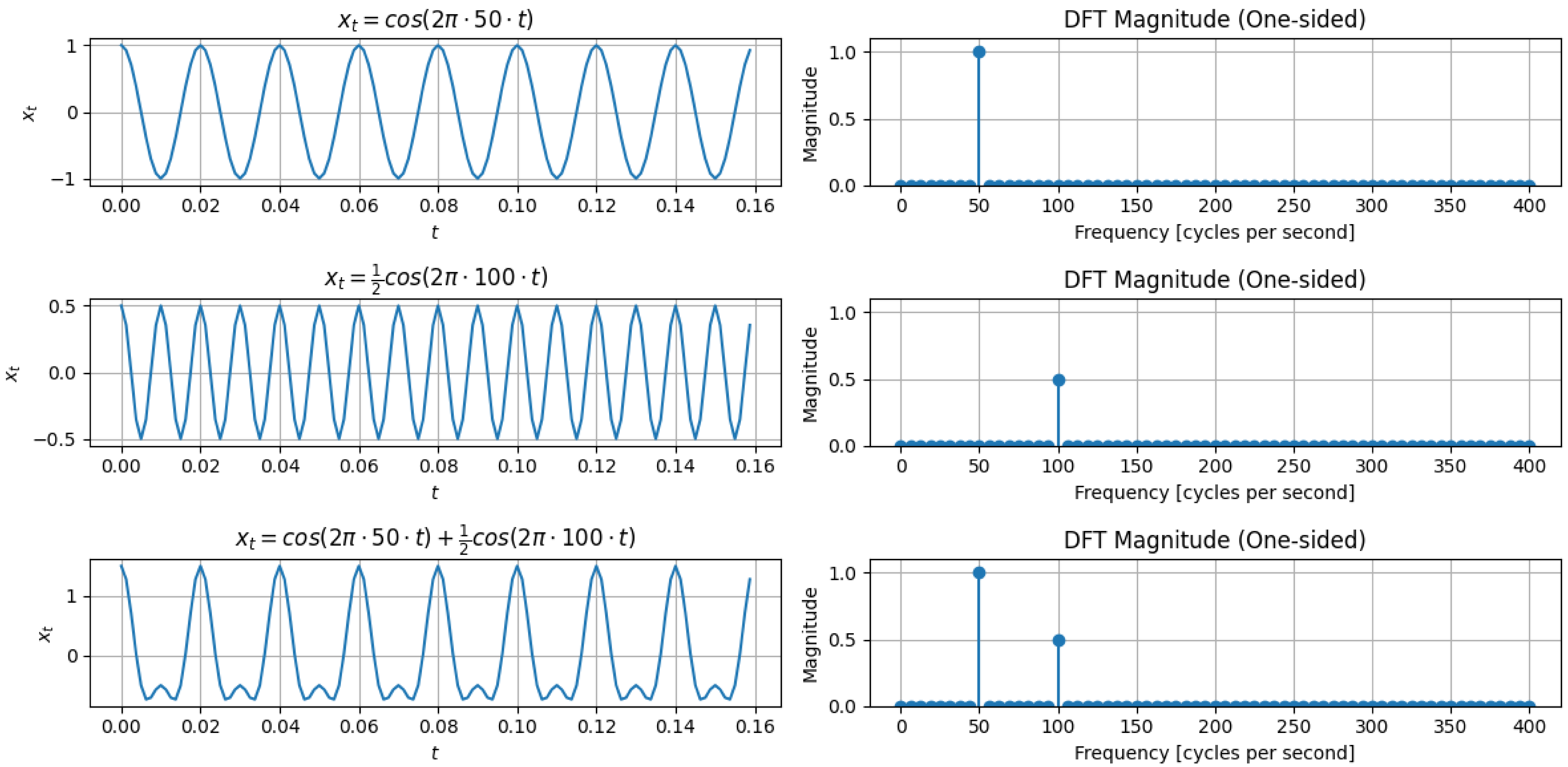
- Time domain (**lagging/leading TS**): let $z_t = x_{t-m}$ then $Z_j = X_j e^{-2\pi i f_j m}$
- Frequency domain (**phase shift**): let $Z_j = X_{j-m}$ then $Z_j = \mathcal{F}\left(\boldsymbol{x} e^{2\pi i f_m t}\right)$

Convolution and multiplication: $\mathcal{F}(\boldsymbol{x} \circledast \boldsymbol{y}) = \boldsymbol{X} \cdot \boldsymbol{Y}$ and $\mathcal{F}(\boldsymbol{x} \cdot \boldsymbol{y}) = \boldsymbol{X} \circledast \boldsymbol{Y}$

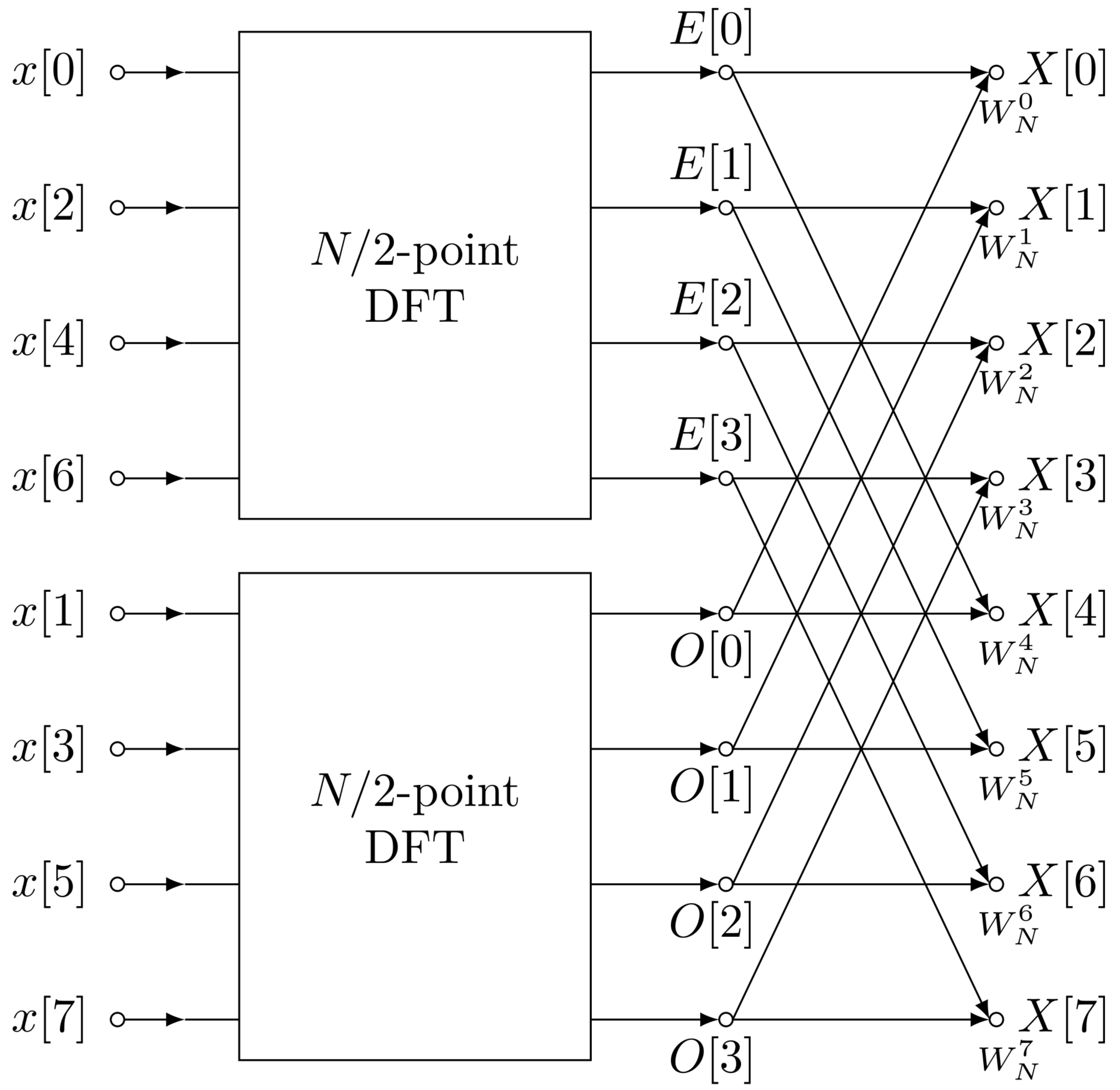
Parseval's theorem: $\sum_{t=0}^{n-1} x_t \bar{y}_t = n \sum_{j=0}^{n-1} X_j \bar{Y}_j$

- Special case, the Plancherel theorem: $\sum_{t=0}^{n-1} |x_t|^2 = n \sum_{j=0}^{n-1} \left|X_j\right|^2$ *(recall that $x_t \bar{x}_t = |x_t|^2$)*

Discrete derivative: let $z_t = x_t - x_{t-1}$ then $Z_t = X_j\left(1 - e^{-2\pi i f_j}\right)$


$x_t = cos(2\pi \cdot 50 \cdot t)$
DFT Magnitude (One-sided)
$x_t = \frac{1}{2}cos(2\pi \cdot 100 \cdot t)$
DFT Magnitude (One-sided)
$x_t = cos(2\pi \cdot 50 \cdot t) + \frac{1}{2}cos(2\pi \cdot 100 \cdot t)$
DFT Magnitude (One-sided)
x_t
t
Magnitude
Frequency [cycles per second]

Fast Fourier transform (FFT)



Yangwenbo99, wikimedia

Computing the DFT requires $\boldsymbol{O(n^2)}$ operations.

- Each of the n coefficients X_j (except for $j = 0$) requires $n - 1$ multiplications and $n - 1$ additions.

Fast Fourier transform algorithm**s** reduce this cost to $\boldsymbol{O(n \log n)}$ operations.

They adopt a **divide-and-conquer** strategy, and **recursively split** the DFT of size $n = n_1 n_2$ into n_1 smaller DFTs of size n_2 along with $O(n)$ multiplications by complex roots of unity (called twiddle factors).

- In practice, implementations avoid the recursion by rearranging the algorithm.

Typical implementations split into two DFTs of size $n/2$ at each step, **requiring $\boldsymbol{n}$ to be a power of 2** (radix-2 FFT), but generally any factorization of n is possible.

Aliasing and the Nyquist frequency

only in discrete domain

Consider the signal $\mathrm{x_t} = cos(2\pi f \Delta t)$ sampled at interval Δ i.e., $\mathrm{x_t}$ is observed at time $\Delta\mathrm{t}$.

Increasing f from 0, the **fastest observable oscillation** is reached when $f = \frac{1}{2\Delta}$ (called the **Nyquist frequency**)

- In this example, $\mathrm{x_t} = cos(\pi t) = (-1)^t$.

Higher frequencies are folded down into the interval $\left[0, \frac{1}{2\Delta}\right]$: taking f such that $\frac{1}{2\Delta} < f < \frac{1}{\Delta}$, and $f' = \frac{1}{\Delta} - f$, we get

$$\mathrm{x_t} = cos(2\pi f \Delta t) = cos\left(2\pi\left(\frac{1}{\Delta} - f'\right)\Delta t\right) = cos\big(2\pi(\mathrm{t} - \mathrm{f}'\Delta t)\big) = cos(2\pi f' \Delta t)$$

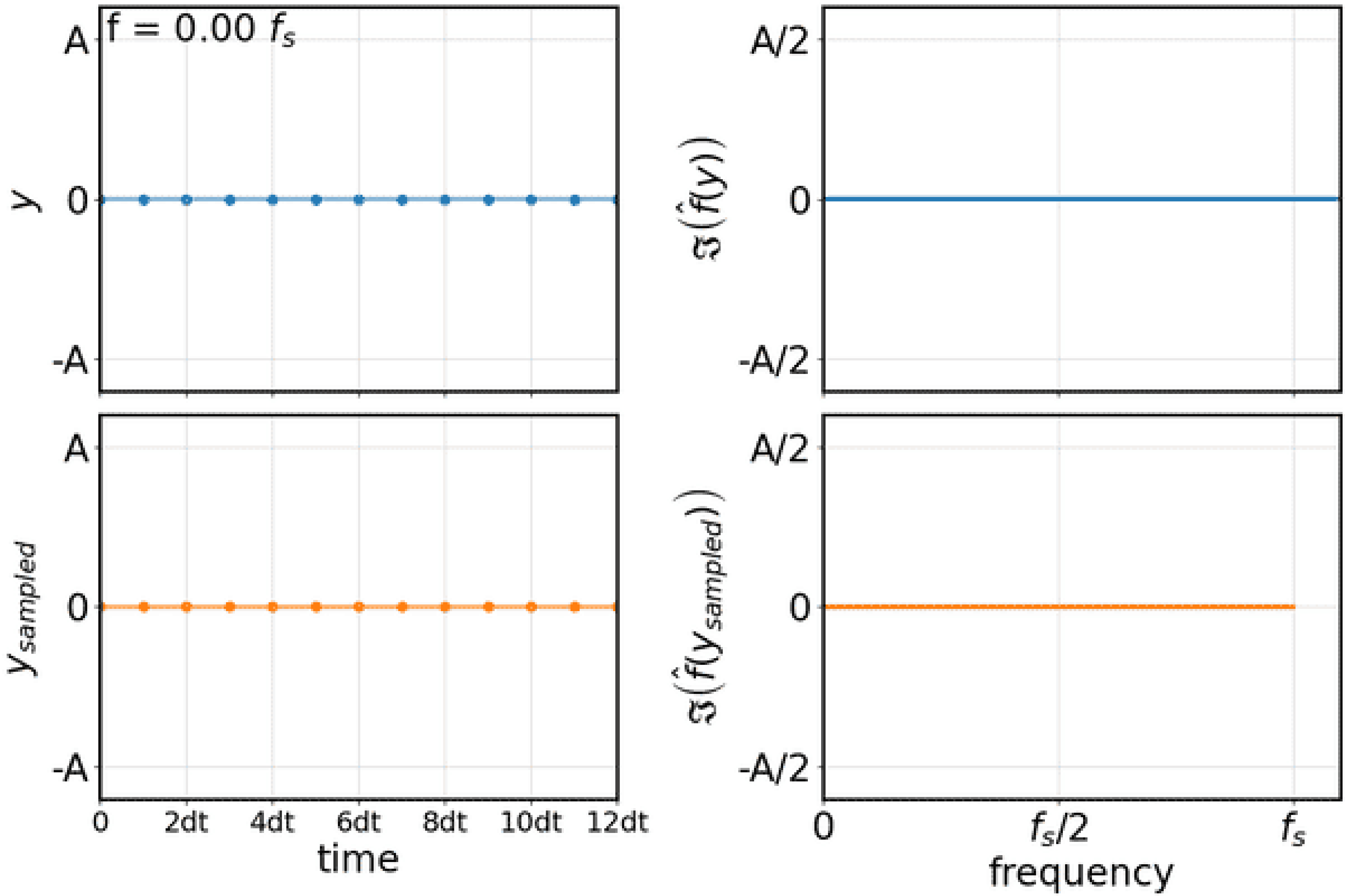
Then f and f' are indistinguishable, hence **aliases** of each other.

- For arbitrary f the observed frequency is $f' = |f - k\Delta|$ with k chosen such that $f' \leq \frac{1}{2\Delta}$

To capture a signal's frequency without aliasing, the sampling rate must be at least **twice** the signal's frequency (**Nyquist-Shannon sampling theorem**).

From here on, the sampling interval is adopted as the **unit of time** i.e., $\Delta = 1$.

Aliasing



Davidjessop, wikimedia

Exercise

Review 3Blue1Brown article on the Fourier transform.

Simulate different synthetic signals (sinusoids, square waves, impulses, …).

- Compute DFT and analyze frequencies.
- Test the different properties of the DFT (linearity, time-shifting, …).
- Review the effect of different sampling rates on the DFT.

Model real-world time series.

- Fit sinusoids with known frequencies.
- Compute DFT and analyze dominant frequencies.
- Compare empirical results with your initial expectations.
- Review how the different DFT components contribute to the signal.