HSLU Hochschule Luzern

Time Series Analysis

ETS

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Informatik

Outline

- Simple exponential smoothing
- (Damped) Holt's linear trend
- Holt-Winters methods
- Time series decompositions with Holt-Winters
- Innovation state space models – ETS
- Prediction confidence intervals
- Connection to ARIMA

Simple exponential smoothing (SES)

Given a TS realization $\{x_1, x_2, \dots x_t, \dots x_n\}$, the 1-step ahead forecast is modeled as a weighted average (avg_w) of the **current observation** x_t and the **previous 1-step ahead forecast**.

$$\hat{x}_{t+1|t} = \alpha x_t + (1-\alpha)\hat{x}_{t|t-1} \text{ with } \alpha \in [0,1].$$

Choosing a **large (small)** α value assigns **more weight to recent (older) observations**.

Substituting the intermediate forecasts: $\hat{x}_{t+1|t} = \alpha x_t + \alpha(1-\alpha)x_{t-1} + \alpha(1-\alpha)^2 x_{t-2} + \cdots$

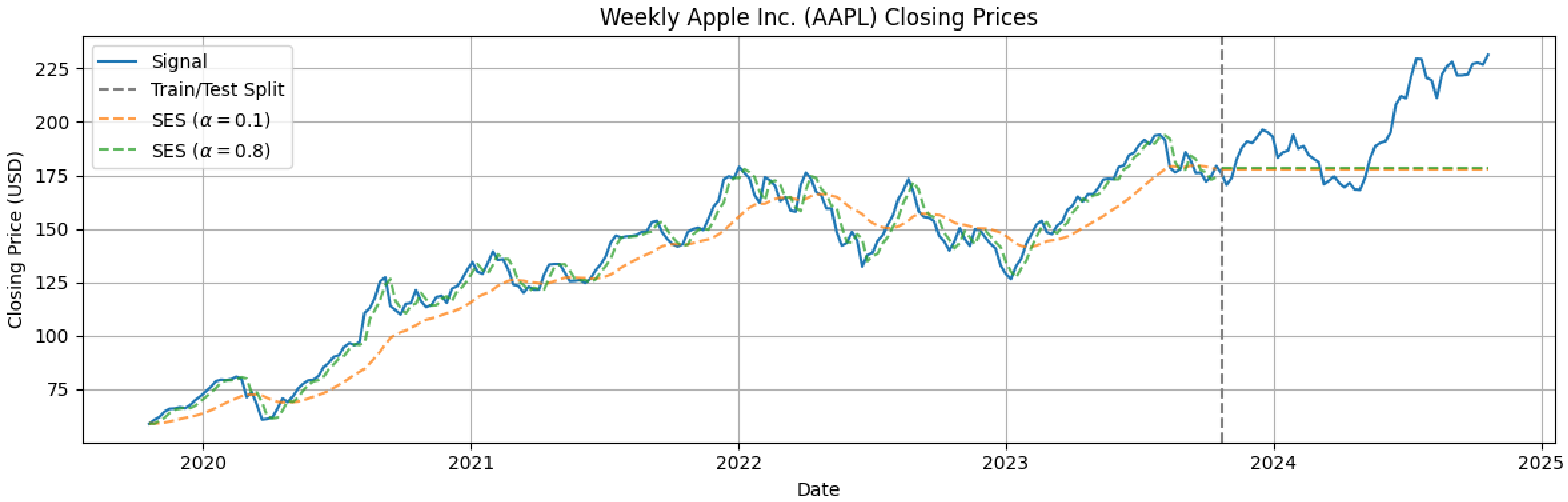
$$\hat{x}_{t+1|t} = \alpha \sum_{i=0}^{t-1}(1-\alpha)^i x_{t-i} + (1-\alpha)^t l_0 \text{ with } \hat{x}_{1|0} \text{ denoted as } l_0.$$

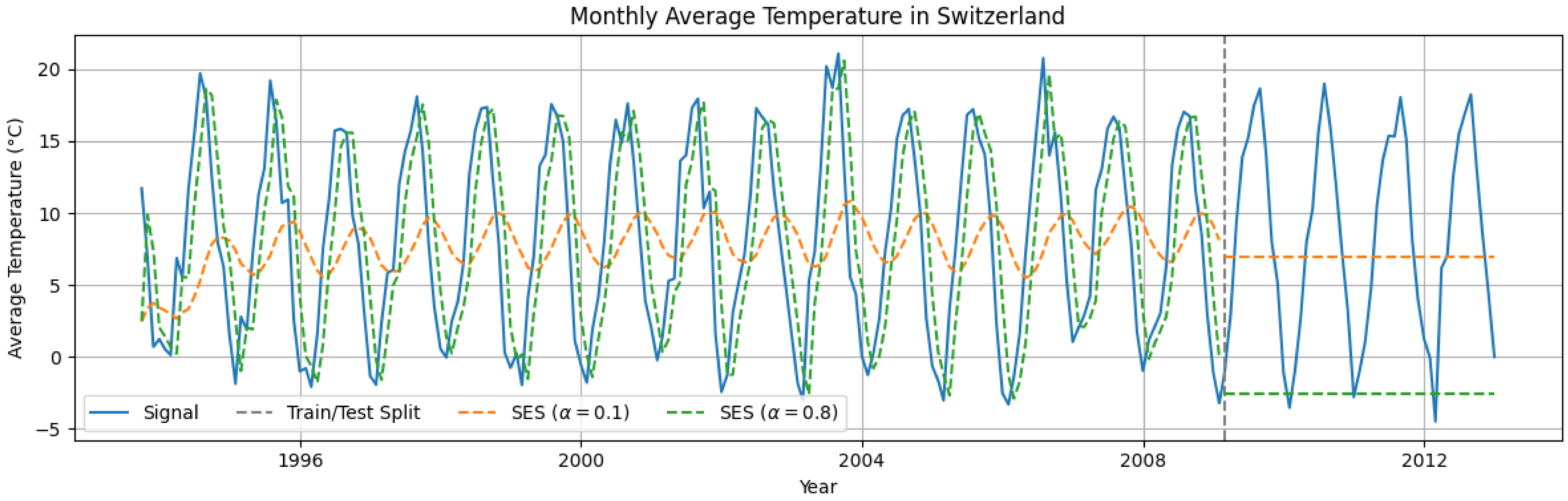
i.e., forecast as weighted averages of past observations with exponentially decaying weights.

SES generates **flat forecast trajectories**

$$\hat{x}_{t+h|t} = \alpha\hat{x}_{t+h-1|t} + (1-\alpha)\hat{x}_{t+h-1|t} = \hat{x}_{t+h-1|t} = \cdots = \hat{x}_{t+1|t}$$

SES assumes the data fluctuate randomly around a **constant mean level**: **data without trend or seasonality**.


Weekly Apple Inc. (AAPL) Closing Prices
Signal
Train/Test Split
SES (α = 0.1)
SES (α = 0.8)
Closing Price (USD)
Date


Monthly Average Temperature in Switzerland
Signal
Train/Test Split
SES (α = 0.1)
SES (α = 0.8)
Average Temperature (°C)
Year

SES component form

Represent model components **separately**.

SES only considers the **level** l_t of the time series.

Forecast equation: $\hat{x}_{t+h|t} = l_t$

Level equation: $l_t = \alpha x_t + (1-\alpha) l_{t-1}$ with $\alpha \in [0,1]$

l_t is the avg_w of observation x_t and the previous level i.e., the previous 1-step ahead forecast $\hat{x}_{t|t-1} = l_{t-1}$.

Estimate α and l_0 from $\{x_1, x_2, \dots x_T\}$ by minimizing the **sum of squared errors** (SSE)

$$SSE = \sum_{i=1}^{T} \left(x_t - \hat{x}_{t|t-1}\right)^2$$

Holt's linear trend method

Extend SES by adding an estimate of the level's **rate of change**, called the **trend component**.

Forecast equation: $\hat{x}_{t+h|t} = l_t + hb_t$
Level equation: $l_t = \alpha x_t + (1-\alpha)(l_{t-1} + b_{t-1})$
Trend equation: $b_t = \beta^*(l_t - l_{t-1}) + (1-\beta^*)b_{t-1}$ with $\alpha, \beta^* \in [0,1]$

l_t is the avg$_\mathrm{w}$ of observation x_t and the previous 1-step ahead forecast $\hat{x}_{t|t-1} = l_{t-1} + b_{t-1}$.
b_t is the avg$_\mathrm{w}$ of the current trend estimate $l_t - l_{t-1}$ and the previous trend b_{t-1}.

Generates **linear forecast trajectories** function of h.

Estimate $\alpha, l_0, \beta^*, b_0$ from $\{x_1, x_2, \dots x_T\}$ by minimizing the SSE.

Damped Holt’s linear trend method

Purely linear trend tend to **over-forecast** especially for longer forecast horizons → consider **damped trend**.

Forecast equation: $\hat{x}_{t+h|t} = l_t + b_t \sum_{i=1}^{h} \phi^i$

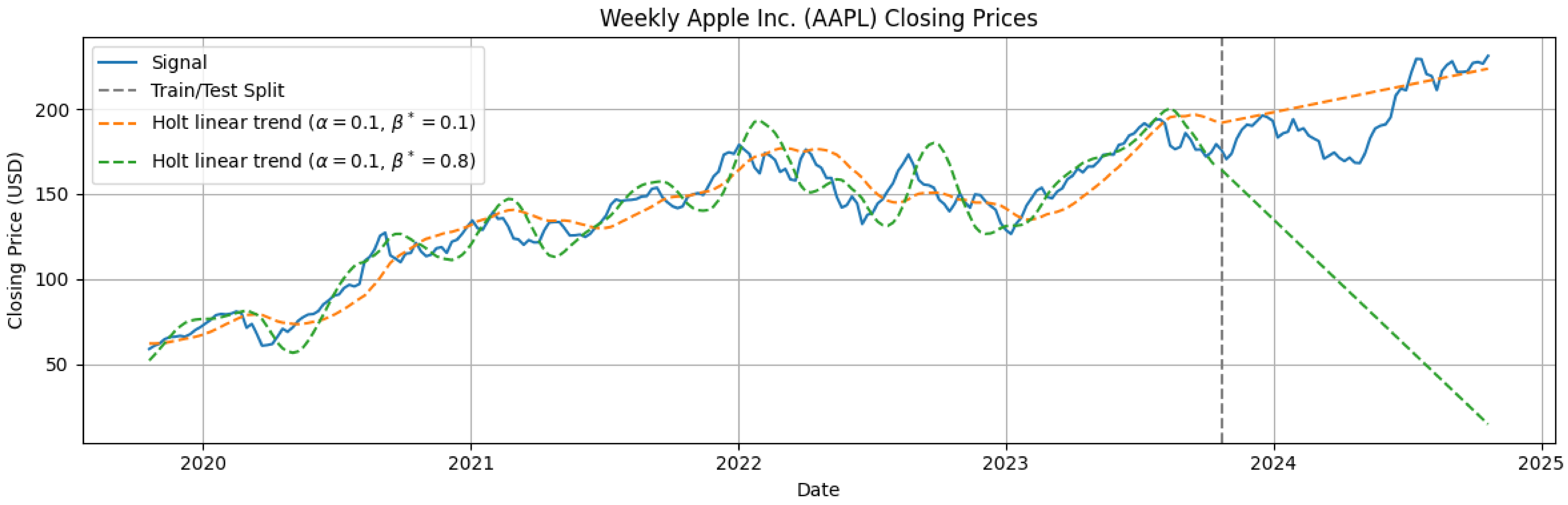
Level equation: $l_t = \alpha x_t + (1-\alpha)(l_{t-1} + \phi b_{t-1})$

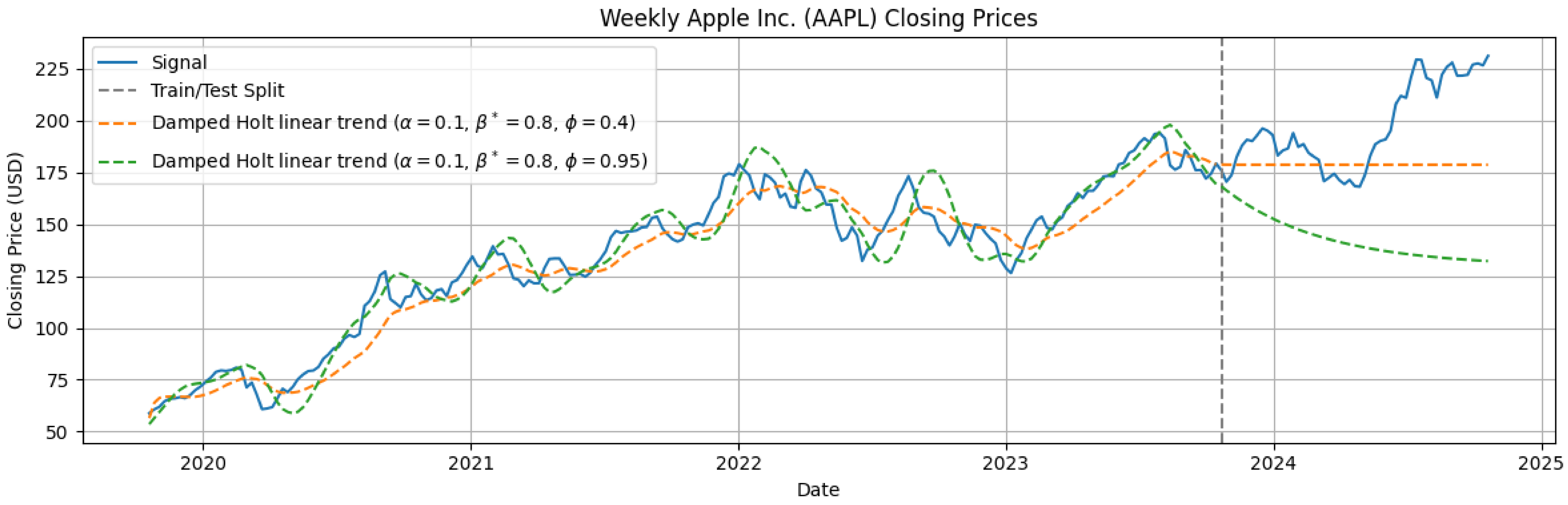
Trend equation: $b_t = \beta^*(l_t - l_{t-1}) + (1-\beta^*)\phi b_{t-1}$ with $\alpha, \beta^*, \phi \in [0,1]$

For $\phi = 1$, the method is equivalent to Holt’s linear trend method.
The damping effect is **stronger (weaker)** for **smaller (larger)** $\boldsymbol{\phi}$ values.

For long horizons (and provided $\phi < 1$), the trend **tends to a constant** since $\lim_{h\to\infty} \hat{x}_{t+h|t} = l_t + b_t \frac{\phi}{1-\phi}$.

Estimate $\alpha, l_0, \beta^*, b_0, \phi$ from $\{x_1, x_2, \dots x_T\}$ by minimizing the SSE. In practice ϕ typically lies within $[0.8, 0.98]$.


Weekly Apple Inc. (AAPL) Closing Prices
Signal
Train/Test Split
Holt linear trend ($\alpha = 0.1$, $\beta^* = 0.1$)
Holt linear trend ($\alpha = 0.1$, $\beta^* = 0.8$)
Closing Price (USD)
Date


Weekly Apple Inc. (AAPL) Closing Prices
Signal
Train/Test Split
Damped Holt linear trend ($\alpha = 0.1$, $\beta^* = 0.8$, $\phi = 0.4$)
Damped Holt linear trend ($\alpha = 0.1$, $\beta^* = 0.8$, $\phi = 0.95$)
Closing Price (USD)
Date

Holt-Winters additive method

Extend Holt's linear trend method by adding an **estimate of the seasonal component** of the time series.

Forecast equation: $\hat{x}_{t+h|t} = l_t + hb_t + s_{t+h-P\lceil h/P \rceil}$
Level equation: $l_t = \alpha(x_t - s_{t-P}) + (1-\alpha)(l_{t-1} + b_{t-1})$
Trend equation: $b_t = \beta^*(l_t - l_{t-1}) + (1-\beta^*)b_{t-1}$
Seasonal equation: $s_t = \gamma^*(x_t - l_t) + (1-\gamma^*)s_{t-P}$ with $\alpha, \beta^*, \gamma^* \in [0,1]$

l_t is the avg$_\mathrm{w}$ of the seasonality adjusted observation $x_t - s_{t-P}$ and the previous 1-step ahead non-seasonal forecast.
b_t is the avg$_\mathrm{w}$ of the current trend estimate $l_t - l_{t-1}$ and the previous trend b_{t-1}.
s_t is the avg$_\mathrm{w}$ of the level adjusted observation $x_t - l_t$ and the seasonal index from the previous period.

Replacing l_t in the seasonal equation yields $s_t = \gamma(x_t - l_{t-1} - b_{t-1}) + (1-\gamma)s_{t-P}$ with $\gamma = \gamma^*(1-\alpha)$ and $\gamma \in [0, 1-\alpha]$.
s_t is the avg$_\mathrm{w}$ of the current seasonal index estimate $x_t - l_{t-1} - b_{t-1}$ and the seasonal index from the previous period.

Estimate $\alpha, l_0, \beta^*, b_0, \gamma, s_{0,-1,-2,\ldots-(P-1)}$ from $\{x_1, x_2, \ldots x_T\}$ by minimizing the SSE.

Holt-Winters multiplicative method

Use Holt-Winters' **additive method** when the seasonal variations are roughly **constant**.

- $s_{\mathrm{t}+1,\ldots t+P}$ are expressed in absolute terms with the same unit as the TS and sum up to approximately zero.

When the seasonal variations are roughly **proportional** to the TS level, consider either non-linear transforms or the Holt-Winters' **multiplicative method**.

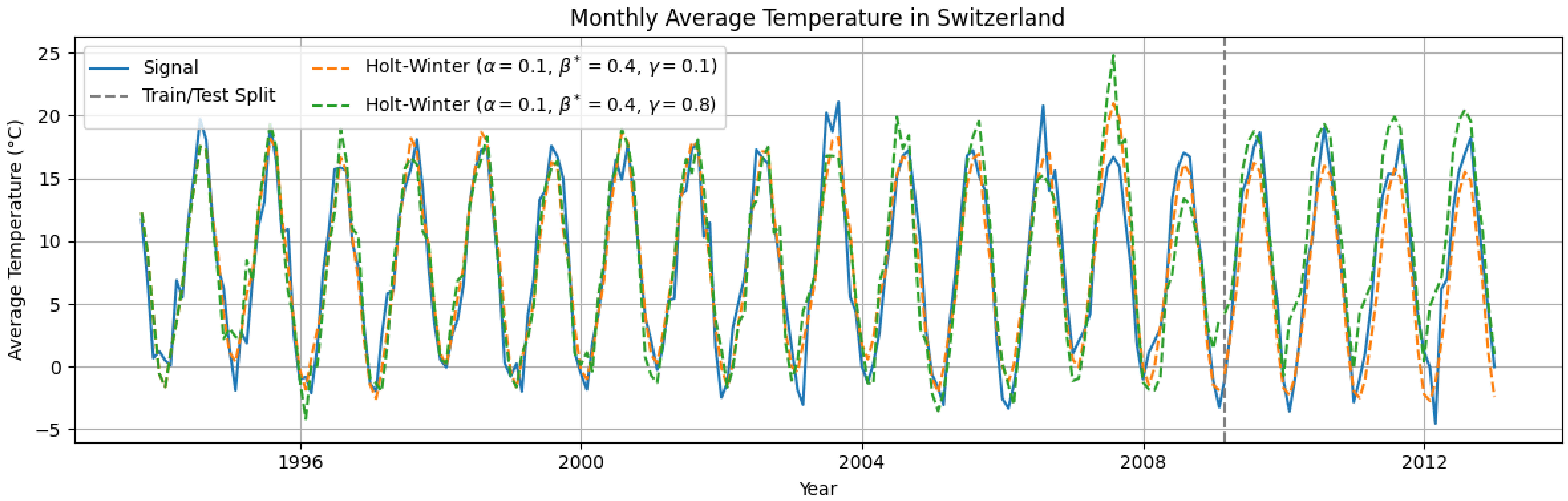
- $s_{\mathrm{t}+1,\ldots t+P}$ are expressed in relative terms and sum up to approximately P.

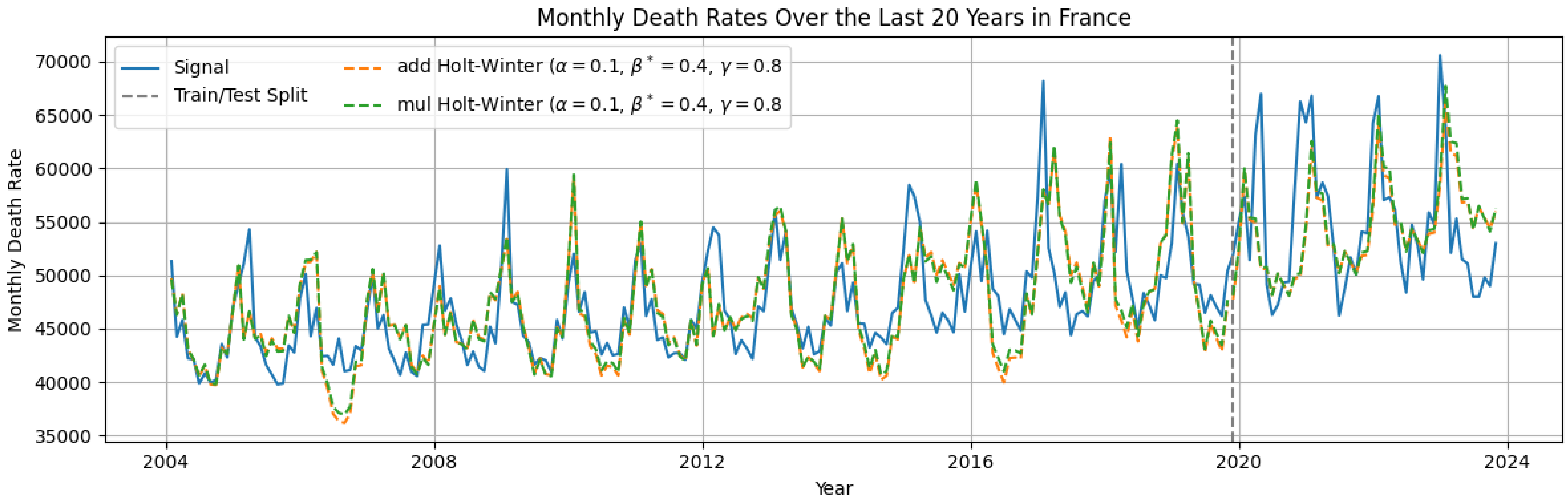
Forecast equation: $\hat{x}_{t+h|t} = (l_t + hb_t)s_{t+h-P\lceil h/P \rceil}$

Level equation: $l_t = \alpha \frac{x_t}{s_{t-P}} + (1-\alpha)(l_{t-1} + b_{t-1})$

Trend equation: $b_t = \beta^*(l_t - l_{t-1}) + (1-\beta^*)b_{t-1}$

Seasonal equation: $s_{\mathrm{t}} = \gamma \frac{x_t}{l_{t-1}+b_{t-1}} + (1-\gamma)s_{t-P}$ with $\alpha, \beta^* \in [0,1]$ and $\gamma \in [0, 1-\alpha]$


Monthly Average Temperature in Switzerland
Signal
Train/Test Split
Holt-Winter ($\alpha = 0.1$, $\beta^* = 0.4$, $\gamma = 0.1$)
Holt-Winter ($\alpha = 0.1$, $\beta^* = 0.4$, $\gamma = 0.8$)
Average Temperature (°C)
Year


Monthly Death Rates Over the Last 20 Years in France
Signal
Train/Test Split
add Holt-Winter ($\alpha = 0.1$, $\beta^* = 0.4$, $\gamma = 0.8$
mul Holt-Winter ($\alpha = 0.1$, $\beta^* = 0.4$, $\gamma = 0.8$
Monthly Death Rate
Year

Holt-Winters' damped method

Damped trend and additive seasonality

Forecast equation: $\hat{x}_{t+h|t} = l_t + b_t \sum_{i=1}^{h} \phi^i + s_{t+h-P\lceil h/P \rceil}$

Level equation: $l_t = \alpha(x_t - s_{t-P}) + (1-\alpha)(l_{t-1} + \phi b_{t-1})$

Trend equation: $b_t = \beta^*(l_t - l_{t-1}) + (1-\beta^*)\phi b_{t-1}$

Seasonal equation: $s_t = \gamma(x_t - l_{t-1} - \phi b_{t-1}) + (1-\gamma)s_{t-P}$ with $\alpha, \beta^*, \phi \in [0,1]$ and $\gamma \in [0, 1-\alpha]$

Damped trend and multiplicative seasonality

Forecast equation: $\hat{x}_{t+h|t} = \left(l_t + b_t \sum_{i=1}^{h} \phi^i\right) s_{t+h-P\lceil h/P \rceil}$

Level equation: $l_t = \alpha \frac{x_t}{s_{t-P}} + (1-\alpha)(l_{t-1} + \phi b_{t-1})$

Trend equation: $b_t = \beta^*(l_t - l_{t-1}) + (1-\beta^*)\phi b_{t-1}$

Seasonal equation: $\mathrm{s_t} = \gamma \frac{x_t}{l_{t-1} - \phi b_{t-1}} + (1-\gamma)s_{t-P}$ with $\alpha, \beta^*, \phi \in [0,1]$ and $\gamma \in [0, 1-\alpha]$

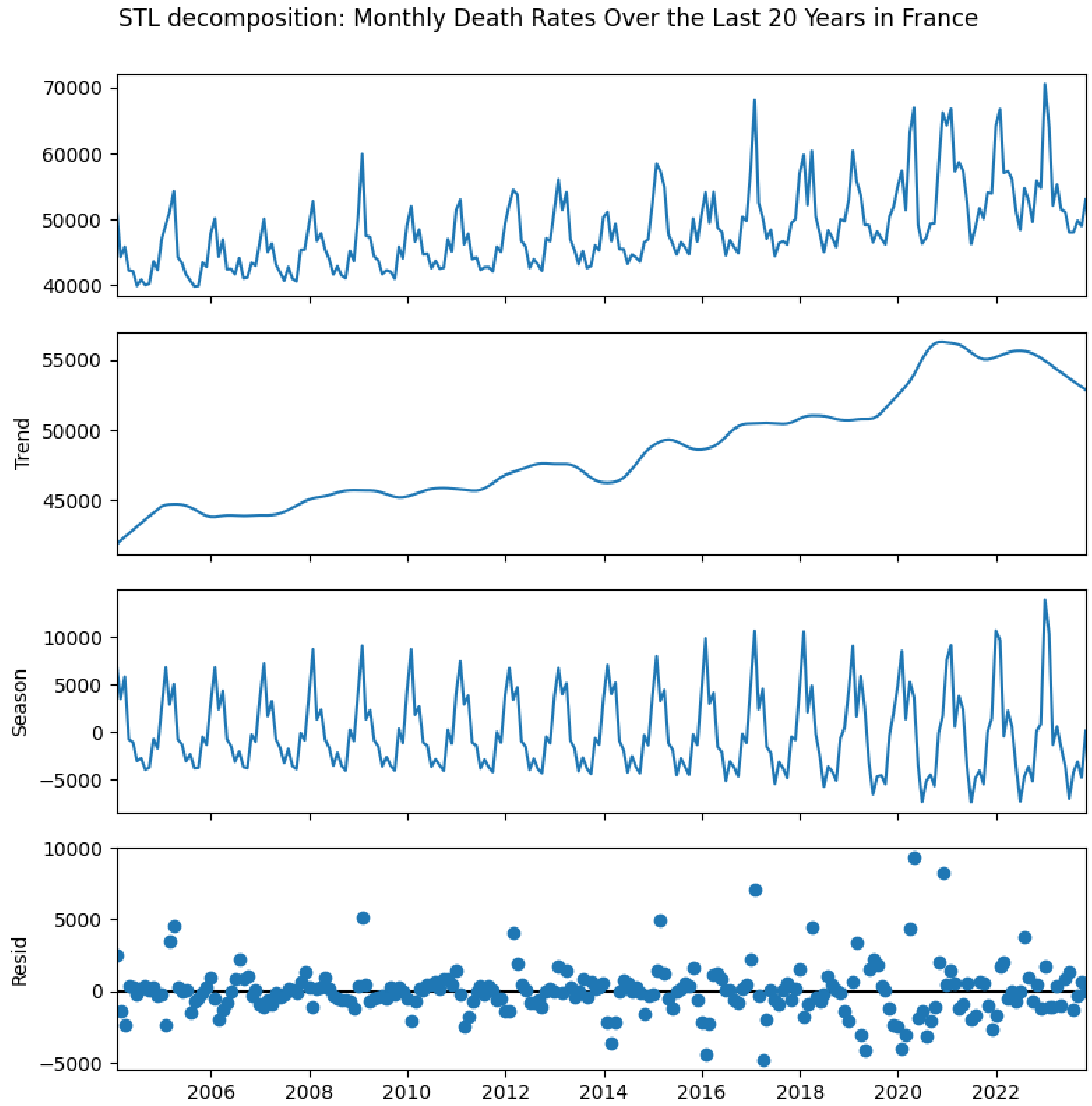
Time series decomposition with Holt-Winters

Holt-Winters’ components provides a decomposition comparable to decomposition methods like STL decomposition.

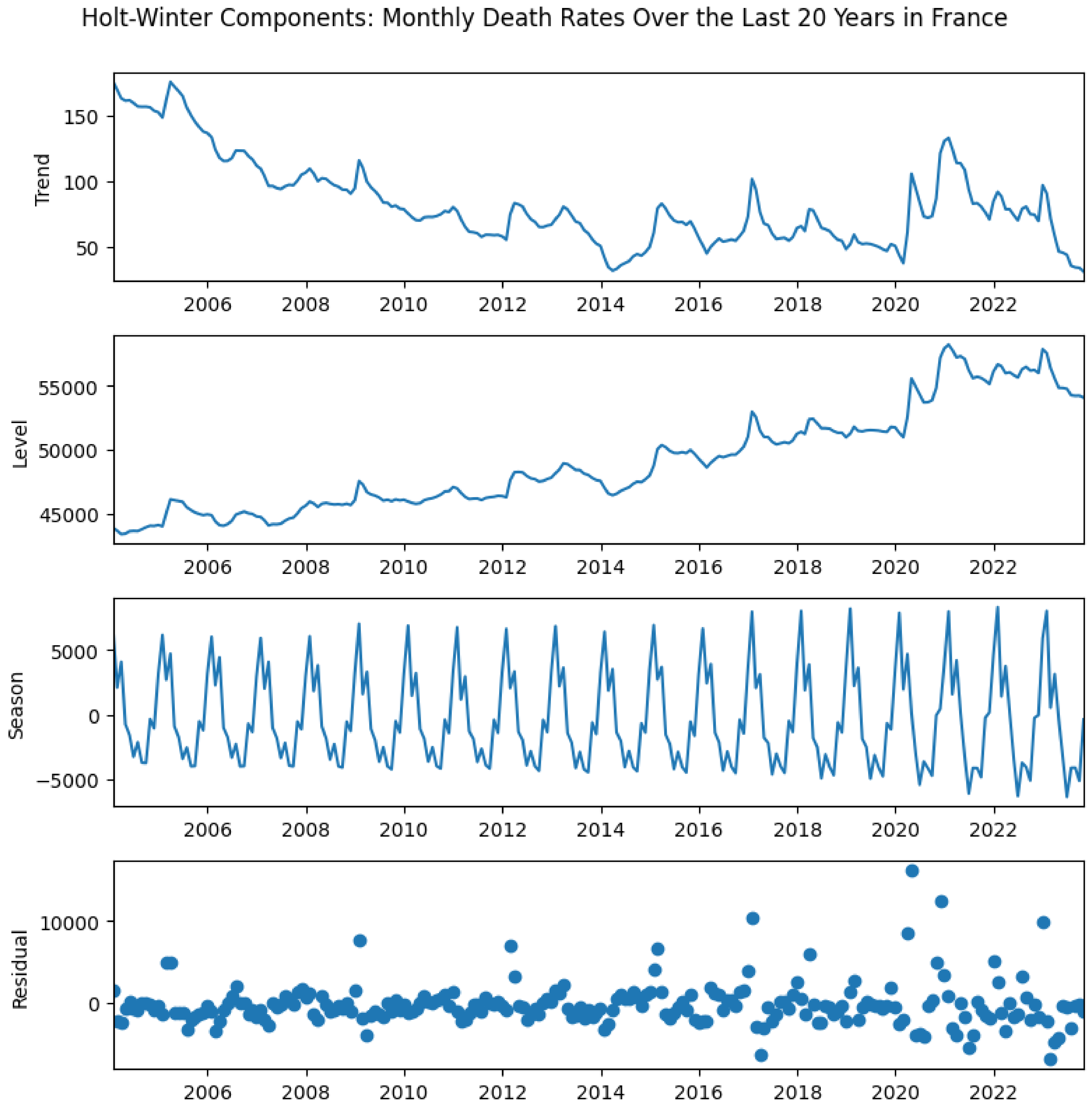
Note that Holt-Winters’ **level component** corresponds to the **trend component** of decomposition methods.

Holt-Winters’ **trend component** has **no counterpart** in decomposition methods.

Holt-Winters’ trend component can be understood as the projected **direction** or **trajectory** of the time series.



STL decomposition: Monthly Death Rates Over the Last 20 Years in France
Trend
Season
Resid



Holt-Winter Components: Monthly Death Rates Over the Last 20 Years in France
Trend
Level
Season
Residual

Innovation state space models – ETS

The methods described so far are **deterministic** and cannot provide **uncertainty estimates**.

Innovation state space models are stochastic processes that describe how the current state of the system depends on its **previous state** and a **random innovation term**.

- They can produce a **forecast distribution** that enables to compute **prediction intervals**.
- Their parameters are estimated with **maximum likelihood estimation**.
- The goodness of fit can be evaluated with **information criteria**.

We consider the Exponential smoothing with Trend and Seasonality (ETS) class of innovation state space models.

- The innovation term is a random error process $\mathrm{E_t} \sim iid\ \mathcal{N}(0, \sigma^2)$
- Realizations can be expressed in absolute terms (**additive errors**): $\varepsilon_t = x_t - \hat{x}_{t|t-1}$
- Realizations can be expressed in relative terms (**multiplicative errors**): $\varepsilon_t = \frac{x_t - \hat{x}_{t|t-1}}{\hat{x}_{t|t-1}}$
- Choose multiplicative errors when the (residuals) variance is proportional to the level of the TS.

Simple exponential smoothing with error process

Recall the **forecast equation** $\hat{x}_{i+h|i} = l_i \overset{h=1}{\Longleftrightarrow} \hat{x}_{i+1|i} = l_i \overset{i=t-1}{\Longleftrightarrow} \hat{x}_{t|t-1} = l_{t-1}$

With **additive errors** $\varepsilon_t = x_t - \hat{x}_{t|t-1} = x_t - l_{t-1}$

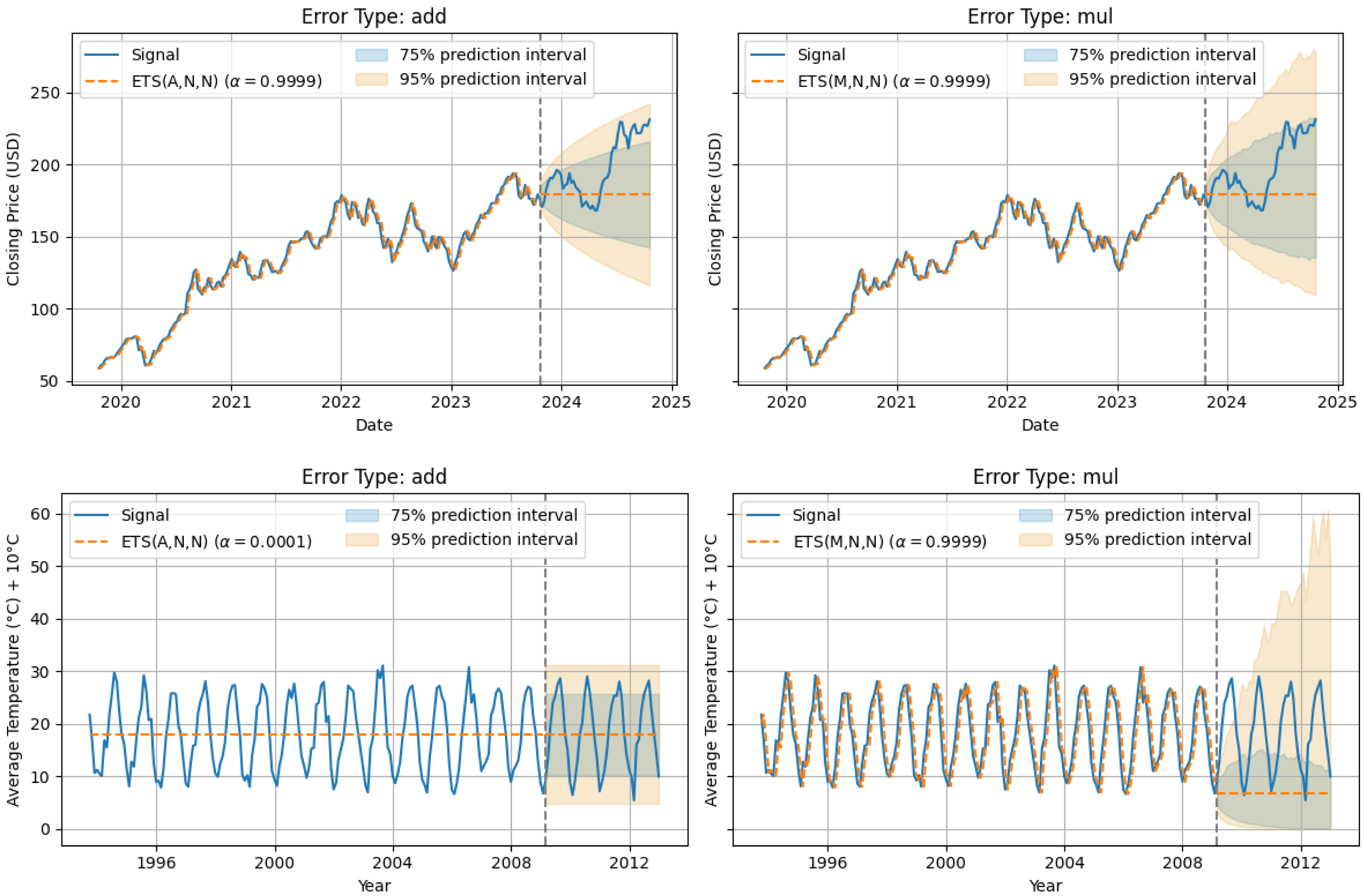
Observation equation: $x_t = l_{t-1} + \epsilon_t$

Level state equation: $l_t = \alpha x_t + (1-\alpha) l_{t-1} = l_{t-1} + \alpha \underbrace{(x_t - l_{t-1})}_{\epsilon_t} = l_{t-1} + \alpha\epsilon_t$ with $\alpha \in [0,1]$

With **multiplicative errors** $\varepsilon_t = \frac{x_t - \hat{x}_{t|t-1}}{\hat{x}_{t|t-1}} = \frac{x_t - l_{t-1}}{l_{t-1}}$

Observation equation: $x_t = l_{t-1}(1 + \epsilon_t)$

Level state equation: $l_t = l_{t-1} + \alpha \underbrace{(x_t - l_{t-1})}_{\epsilon_t l_{t-1}} = l_{t-1} + \alpha\epsilon_t l_{t-1} = l_{t-1}(1 + \alpha\epsilon_t)$ with $\alpha \in [0,1]$


Error Type: add
Signal
ETS(A,N,N) ($\alpha = 0.9999$)
75% prediction interval
95% prediction interval
Closing Price (USD)
Date
Error Type: mul
Signal
ETS(M,N,N) ($\alpha = 0.9999$)
75% prediction interval
95% prediction interval
Closing Price (USD)
Date
Error Type: add
Signal
ETS(A,N,N) ($\alpha = 0.0001$)
75% prediction interval
95% prediction interval
Average Temperature (°C) + 10°C
Year
Error Type: mul
Signal
ETS(M,N,N) ($\alpha = 0.9999$)
75% prediction interval
95% prediction interval
Average Temperature (°C) + 10°C
Year

Damped Holt’s linear trend with error process

Recall the **forecast equation** $\hat{x}_{i+h|i} = l_i + b_i \sum_{j=1}^{h} \phi^j \overset{h=1}{\Longleftrightarrow} \hat{x}_{i+1|i} = l_i + \phi b_i \overset{i=t-1}{\Longleftrightarrow} \hat{x}_{t|t-1} = l_{t-1} + \phi b_{t-1}$

With **additive errors** $\varepsilon_t = x_t - \hat{x}_{t|t-1} = x_t - (l_{t-1} + \phi b_{t-1})$

Observation equation: $x_t = l_{t-1} + \phi b_{t-1} + \epsilon_t$

Level state equation: $l_t = \alpha x_t + (1-\alpha)(l_{t-1} + \phi b_{t-1}) = l_{t-1} + \phi b_{t-1} + \alpha \epsilon_t$

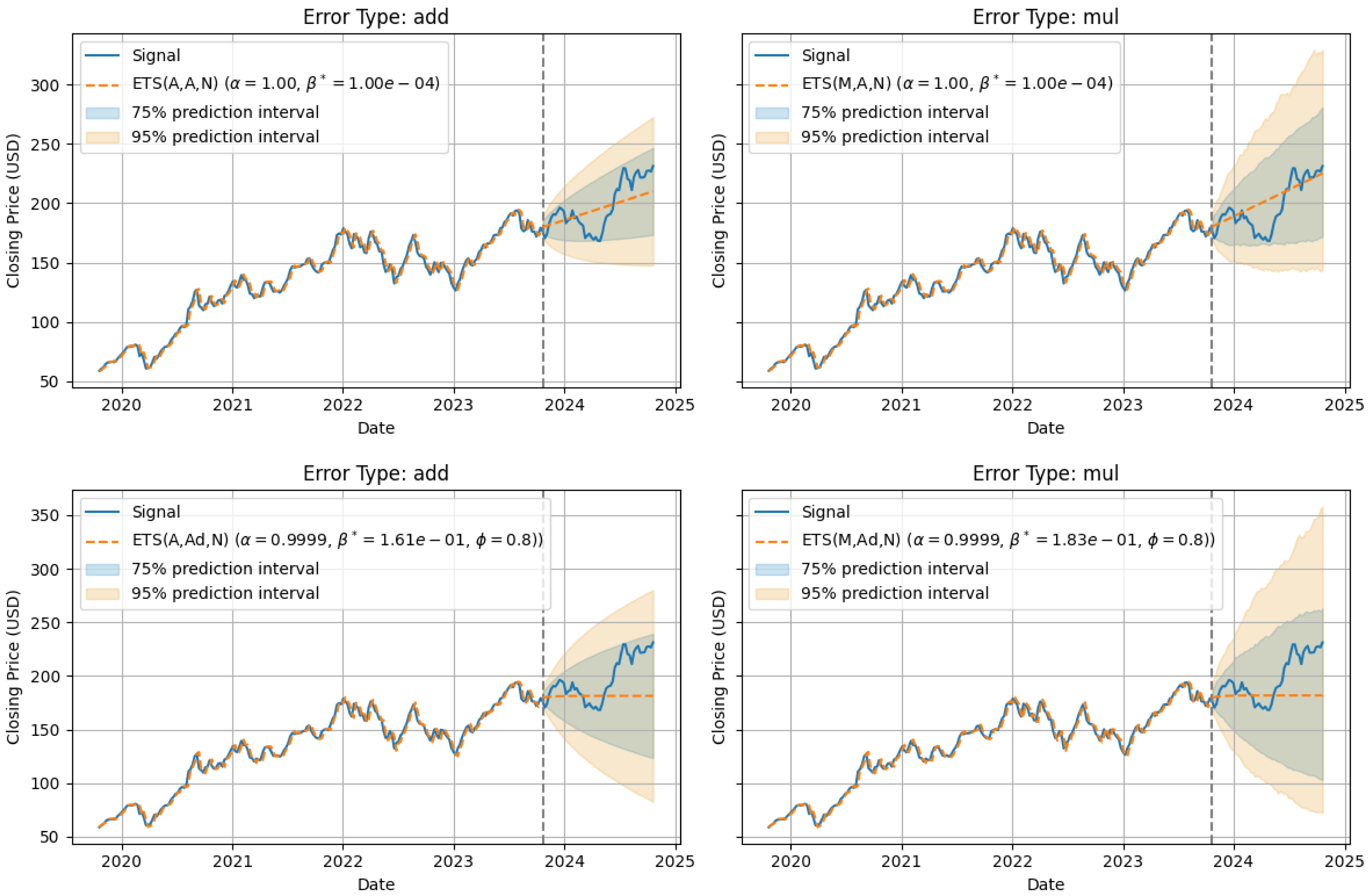
Trend state equation: $b_t = \beta^*(l_t - l_{t-1}) + (1-\beta^*)\phi b_{t-1} = \phi b_{t-1} + \beta \epsilon_t$ with $\alpha, \beta^*, \phi \in [0,1]$ and $\beta = \alpha\beta^*$

With **multiplicative errors** $\varepsilon_t = \frac{x_t - \hat{x}_{t|t-1}}{\hat{x}_{t|t-1}} = \frac{x_t - (l_{t-1} + \phi b_{t-1})}{l_{t-1} + \phi b_{t-1}}$

Observation equation: $x_t = (l_{t-1} + \phi b_{t-1})(1 + \epsilon_t)$

Level state equation: $l_t = (l_{t-1} + \phi b_{t-1})(1 + \alpha \epsilon_t)$

Trend state equation: $b_t = \phi b_{t-1} + \beta(l_{t-1} + \phi b_{t-1})\epsilon_t$ with $\alpha, \beta^*, \phi \in [0,1]$ and $\beta = \alpha\beta^*$



Error Type: add
Signal
ETS(A,A,N) ($\alpha = 1.00$, $\beta^* = 1.00e-04$)
75% prediction interval
95% prediction interval
Closing Price (USD)
Date
Error Type: mul
Signal
ETS(M,A,N) ($\alpha = 1.00$, $\beta^* = 1.00e-04$)
75% prediction interval
95% prediction interval
Closing Price (USD)
Date
Error Type: add
Signal
ETS(A,Ad,N) ($\alpha = 0.9999$, $\beta^* = 1.61e-01$, $\phi = 0.8$))
75% prediction interval
95% prediction interval
Closing Price (USD)
Date
Error Type: mul
Signal
ETS(M,Ad,N) ($\alpha = 0.9999$, $\beta^* = 1.83e-01$, $\phi = 0.8$))
75% prediction interval
95% prediction interval
Closing Price (USD)
Date

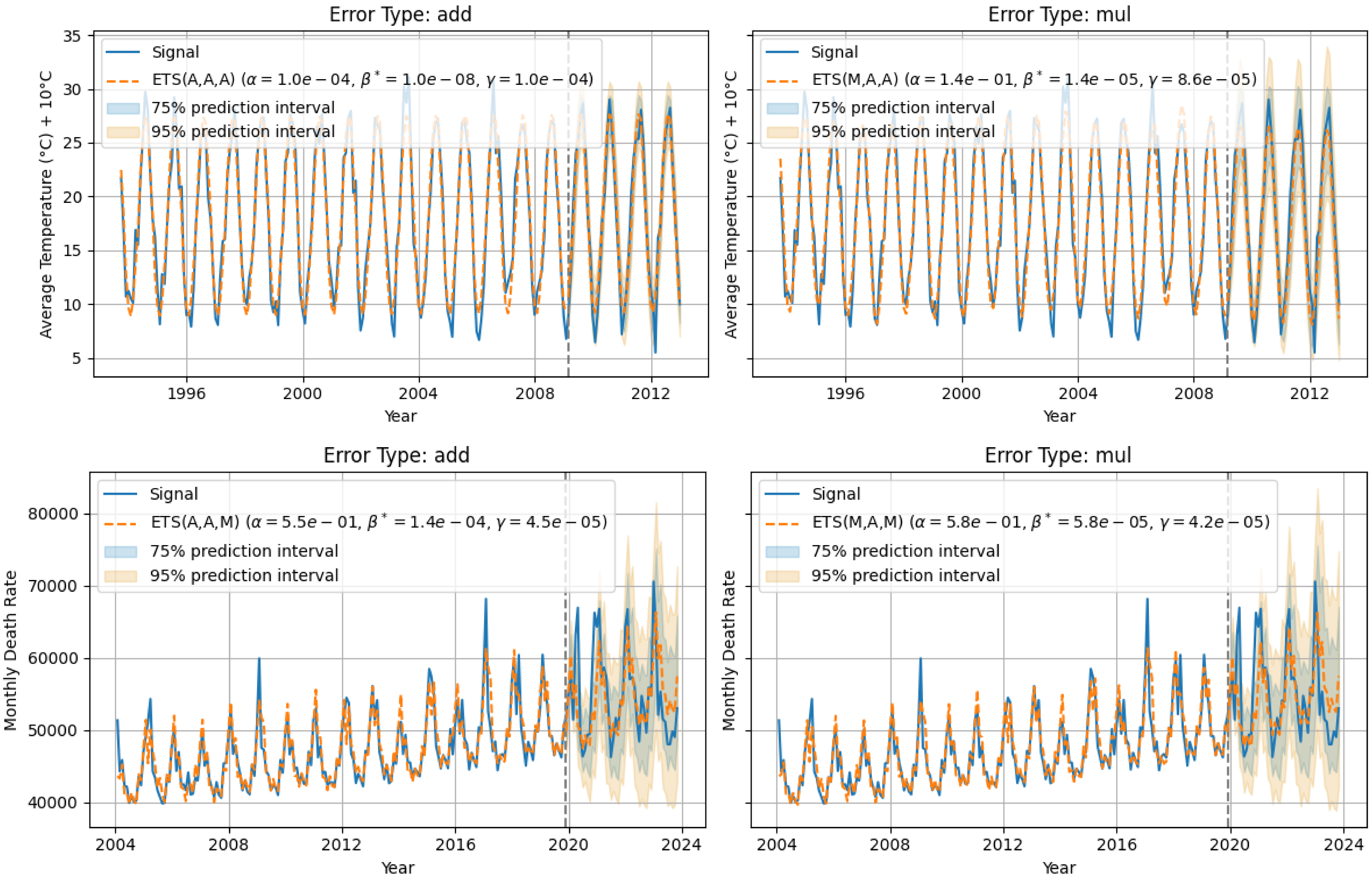
Holt-Winters additive method with additive error process

Recall the **forecast equation** $\hat{x}_{i+h|i} = l_i + hb_i + s_{i+h-P\lceil h/P \rceil} \overset{h=1}{\Longleftrightarrow} \hat{x}_{i+1|i} = l_i + b_i + s_{i+1-P} \overset{i=t-1}{\Longleftrightarrow} \hat{x}_{t|t-1} = l_{t-1} + b_{t-1} + s_{t-P}$

SES with **additive errors** $\varepsilon_t = x_t - \hat{x}_{t|t-1} = x_t - l_{t-1} - b_{t-1} - s_{t-P}$

Observation equation: $x_t = l_{t-1} + b_{t-1} + s_{t-P} + \epsilon_t$
Level state equation: $l_t = l_{t-1} + b_{t-1} + \alpha\epsilon_t$
Trend state equation: $b_t = b_{t-1} + \beta\epsilon_t$
Seasonal state equation: $s_t = \gamma(x_t - l_{t-1} - b_{t-1}) + (1-\gamma)s_{t-P} = s_{t-P} + \gamma\epsilon_t$

with $\alpha, \beta^* \in [0,1]$, $\beta = \alpha\beta^*$ and $\gamma \in [0,1-\alpha]$



Error Type: add
Signal
ETS(A,A,A) ($\alpha = 1.0e-04$, $\beta^* = 1.0e-08$, $\gamma = 1.0e-04$)
75% prediction interval
95% prediction interval
Average Temperature (°C) + 10°C
Year
Error Type: mul
Signal
ETS(M,A,A) ($\alpha = 1.4e-01$, $\beta^* = 1.4e-05$, $\gamma = 8.6e-05$)
75% prediction interval
95% prediction interval
Average Temperature (°C) + 10°C
Year
Error Type: add
Signal
ETS(A,A,M) ($\alpha = 5.5e-01$, $\beta^* = 1.4e-04$, $\gamma = 4.5e-05$)
75% prediction interval
95% prediction interval
Monthly Death Rate
Year
Error Type: mul
Signal
ETS(M,A,M) ($\alpha = 5.8e-01$, $\beta^* = 5.8e-05$, $\gamma = 4.2e-05$)
75% prediction interval
95% prediction interval
Monthly Death Rate
Year

ETS nomenclature

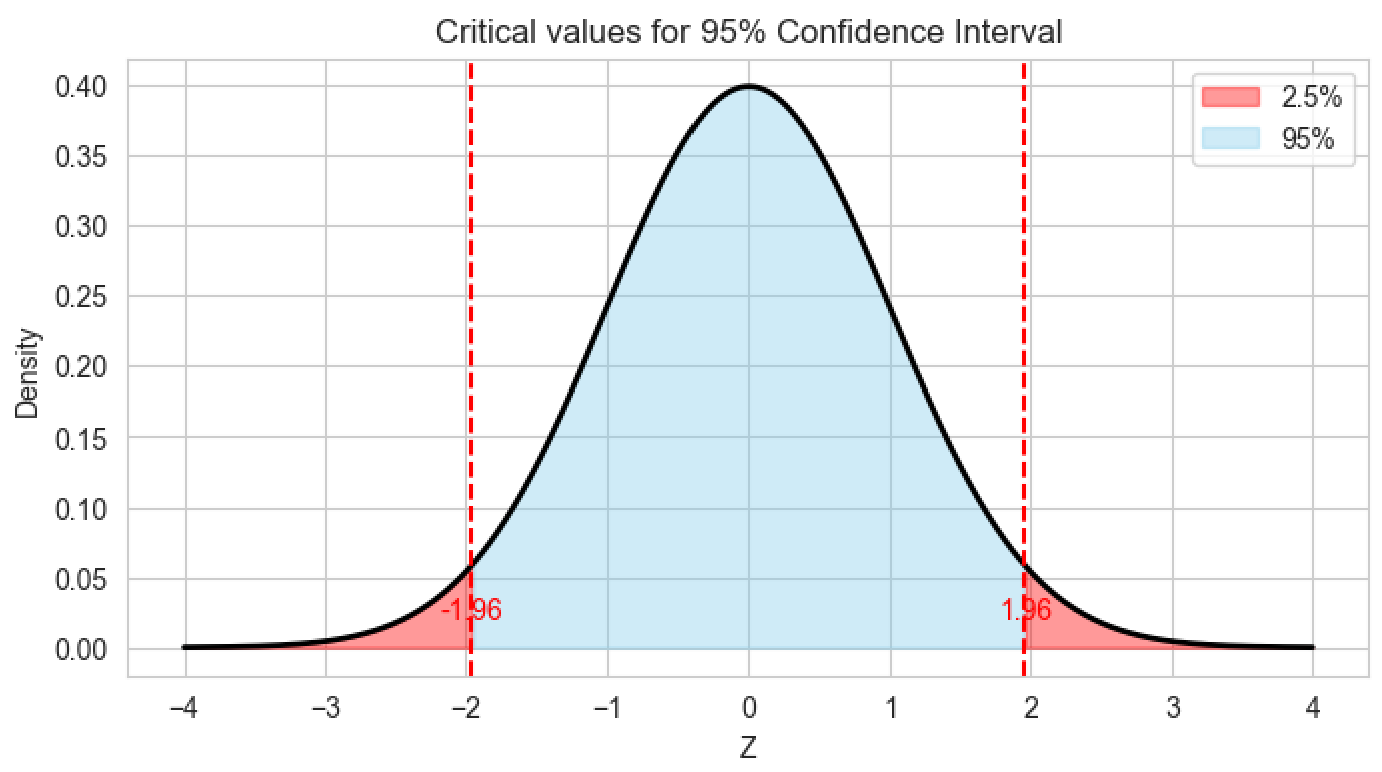
ETS models are labeled as ETS(·,·,·) for respectively the error, trend, and seasonal types.

- Error is either additive (A) or multiplicative (M)
- Trend is either none (N), (A), additive damped (A_d).
- Seasonal is either (N), (A) or (M)

Multiplicative (damped) ($M_{(d)}$) trend methods are not considered since they tend to produce poor forecast.

Name	**Method**
ETS(A,N,N)	SES with additive error
ETS(A,A,N)	Holt’s linear trend with additive error
ETS(M,A_d,N)	Damped Holt’s linear trend with multiplicative error
ETS(M,A,A)	Additive Holt-Winters method with multiplicative error
ETS(M,A_d,M)	Damped multiplicative Holt-Winters method with multiplicative error

Prediction confidence intervals



Confidence intervals provide a measure of **forecast uncertainty**.

When the residuals are **normally distributed**, the CI of $\hat{x}_{T+h|T}$ is $\hat{x}_{T+h|T} \pm z_{1-\alpha/2}\hat{\sigma}_{h|T}$

- $z_{1-\alpha/2}$ is the critical value of the normal distribution e.g., for 95% CI $z_{1-\alpha/2} = 1.96$
- $\hat{\sigma}_{h|T}$ is an **estimate of the standard deviation** of the h-step forecast with $\hat{\sigma}_1 = \hat{\sigma}_e$ (residuals std)

For $h > 1$, $\hat{\sigma}_{h|T}$ depends on the ETS model and how the **innovations accumulate**:

ETS(A,N,N)	ETS(A,A,N)	ETS(A,N,A)
$\hat{\sigma}^2_{h\|T} = \hat{\sigma}^2_e\left(1 + \alpha^2(h-1)\right)$	$\hat{\sigma}^2_{h\|T} = \hat{\sigma}^2_e\left(1 + (h-1)\left(\alpha^2 + \alpha\beta h + \frac{1}{6}\beta^2 h(2h-1)\right)\right)$	$\hat{\sigma}^2_{h\|T} = \hat{\sigma}^2_e\left(1 + \alpha^2(h-1) + \gamma\left\lfloor\frac{h}{P}\right\rfloor(2\alpha + \gamma)\right)$

Use **bootstrapping** for **non-normal** residuals that are **uncorrelated** and have **constant variance**.

Connection to ARIMA (1)

Consider a zero-mean ARIMA(0,1,1) where $\theta = \alpha - 1$:

$$\Phi(B)\nabla x_t = c + \Theta(B) w_t$$
$$\Leftrightarrow \nabla x_t = w_t + (\alpha - 1) w_{t-1}$$

Setting $y_t = \nabla x_t$:

$$y_t = w_t + (\alpha - 1) w_{t-1} \Leftrightarrow w_t = y_t + (1 - \alpha) w_{t-1}$$

Rewriting the model as an AR(∞):

$$\begin{aligned} w_t &= y_t + (1-\alpha) w_{t-1} \\ &= y_t + (1-\alpha)(y_{t-1} + (1-\alpha) w_{t-2}) \\ &= y_t + (1-\alpha) y_{t-1} + (1-\alpha)^2 (y_{t-2} + (1-\alpha) w_{t-3}) \\ &= \cdots \\ &= y_t + \sum_{i=1}^{\infty} (1-\alpha)^i y_{t-i} \end{aligned}$$

Connection to ARIMA (2)

Substituting back $y_t = \nabla x_t = x_t - x_{t-1}$:

$$x_t - x_{t-1} = w_t - \sum_{i=1}^{\infty} (1-\alpha)^i (x_{t-i} - x_{t-i-1})$$

Expanding the sum, choosing two different indexes for clarity:

$$\begin{aligned}
x_t &= x_{t-1} + w_t - \sum_{i=1}^{\infty} (1-\alpha)^i x_{t-i} + \sum_{j=1}^{\infty} (1-\alpha)^j x_{t-j-1} \\
&= x_{t-1} + w_t - \sum_{i=1}^{\infty} (1-\alpha)^i x_{t-i} + \sum_{k=2}^{\infty} (1-\alpha)^{k-1} x_{t-(k-1)-1} && (j = k-1) \\
&= w_t - \sum_{i=1}^{\infty} (1-\alpha)^i x_{t-i} + \sum_{k=1}^{\infty} (1-\alpha)^{k-1} x_{t-k} \\
&= w_t + \sum_{i=1}^{\infty} (1-\alpha)^{i-1} x_{t-i} \big(1 - (1-\alpha)\big) = w_t + \sum_{i=1}^{\infty} \alpha (1-\alpha)^{i-1} x_{t-i} && (k = i)
\end{aligned}$$

Connection to ARIMA (3)

Consider the 1-step ahead prediction:

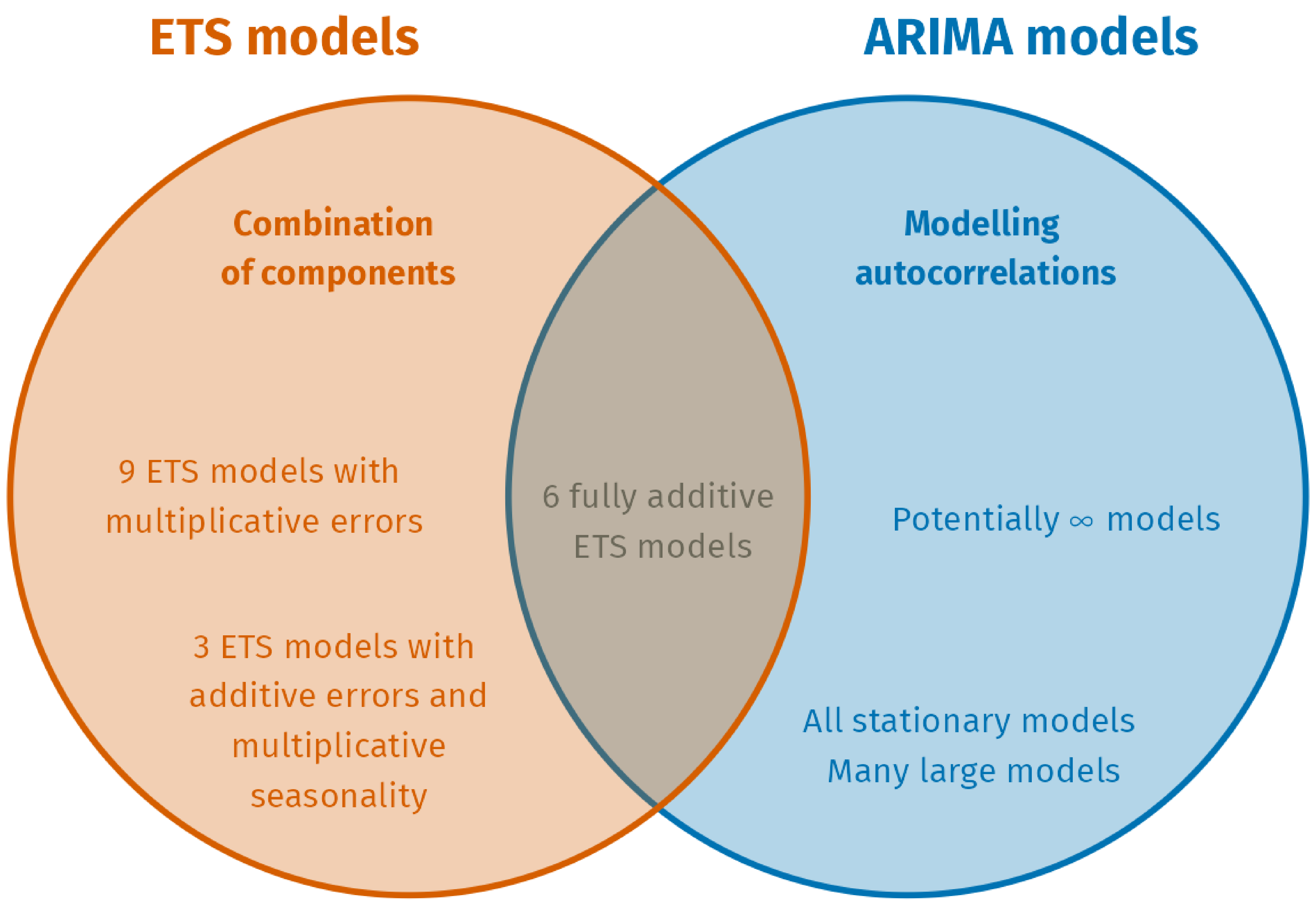
$$\begin{aligned}
\hat{x}_{t+1|t} &= \underbrace{w_{t+1}}_{0} + \sum_{i=1}^{\infty} \alpha(1-\alpha)^{i-1} x_{t+1-i} \\
&= \alpha x_t + \sum_{i=2}^{\infty} \alpha(1-\alpha)^{i-1} x_{t+1-i} \\
&= \alpha x_t + \sum_{k=1}^{\infty} \alpha(1-\alpha)^{k} x_{t-k} && (k = i-1) \\
&= \alpha x_t + (1-\alpha) \sum_{k=1}^{\infty} \alpha(1-\alpha)^{k-1} x_{t-k} \\
&= \alpha x_t + (1-\alpha)\hat{x}_{t|t-1}
\end{aligned}$$

$\hat{x}_{t+1|t}$ is the avg_w of the current observation x_t and the previous prediction $\hat{x}_{t|t-1}$.
Thus, the zero-mean ARIMA(0,1,1) is **equivalent** to simple exponential smoothing.

Connection to ARIMA (4)

ETS models are non-stationary while some ARIMA models are stationary.

- ETS models with seasonality or non-damped trend have two unit-roots i.e., require two levels of differencing.
- Other ETS models have one unit root i.e., require one level of differencing.



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Fully additive ETS	ARIMA equivalent
ETS(A,N,N)	ARIMA(0,1,1)
ETS(A,A,N)	ARIMA(0,2,2)
ETS(A,A_d,N)	ARIMA(1,1,2)
ETS(A,N,A)	ARIMA(0,1,P)$(0,1,0)_P$
ETS(A,A,A)	ARIMA(0,1,P+1)$(0,1,0)_P$
ETS(A,A_d,A)	ARIMA(1,0,P+1)$(0,1,0)_P$

Exercise

Review statsmodels notebook on exponential smoothing.

Simulate and analyze different synthetic signals
- Compare variations of ETS error, trend, seasonality
- Confirm equivalence with ARIMA models when applicable.
- Estimate the model parameters from the generated data.

Model real-world time series
- Compare ETS models with information criteria and residual analysis
- Compare with ARIMA performance using cross-validation
- Compare prediction intervals
- Compare ETS components with decomposition methods