1. Consider the stochastic process $X_t = t + W_t$, where W_t is an iid white noise process with $Var(W_t) = \sigma^2$.

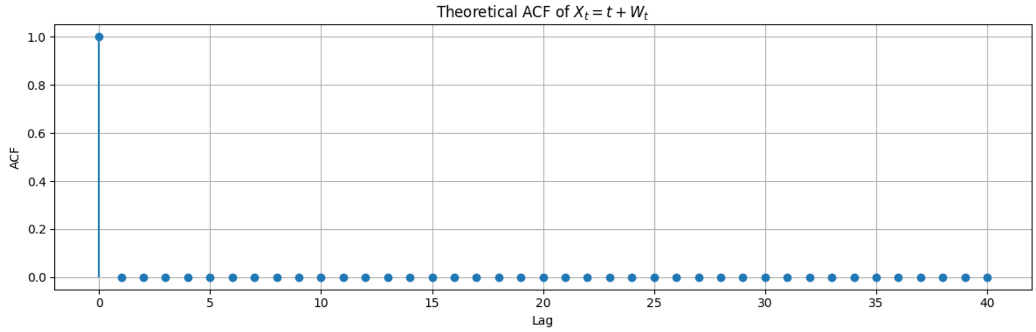
   (a) Compute the auto-covariance function of X_t. Based on this, what would the theoretical ACF be if we tried to apply the standard formula? Comment on whether this is valid.

Solution: Start by calculating the auto-covariance:

$$\begin{aligned} cov(X_{t+h}, X_t) &= cov(t + h + W_{t+h}, t + W_t) = cov(W_{t+h}, W_t) \\ &= \begin{cases} \sigma^2 & \text{if } h = 0 \\ 0 & \text{otherwise} \end{cases} = \gamma(h) \end{aligned}$$

If we now apply the usual ACF definition, we obtain

$$\rho(h) = \frac{\gamma(h)}{\gamma(0)} = \begin{cases} 1 & \text{if } h = 0 \\ 0 & \text{otherwise} \end{cases}$$



This calculation formally looks like the ACF of white noise, but in fact X_t is nonstationary (because of the linear trend). Strictly speaking the theoretical ACF is not well-defined for X_t — what we computed corresponds only to the stationary white noise component W_t.

   (b) Figure 1 represents the ACF generated from a realization of X_t with 1000 values. Explain why it looks different from your result above.

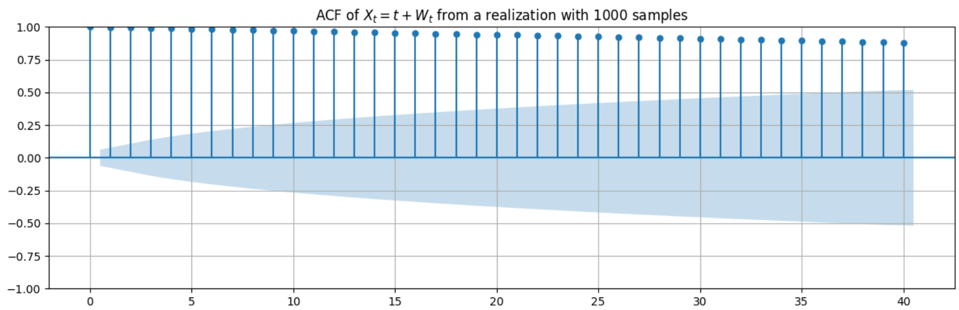


Figure 1: ACF from a realization of X_t with 1000 values

Solution: The sample ACF is calculated directly on the nonstationary series X_t, which contains a deterministic upward trend. This trend induces strong correlations between time points, making the ACF appear persistent at many lags.

In contrast, the theoretical calculation reflects only the stationary white noise component and ignores the trend. This explains the discrepancy.

2. For each of the provided stochastic processes, compute its mean, variance, and auto-covariance $cov(X_s, X_t)$, and determine whether it is stationary. In the following, consider that W_t is an iid white noise process with $Var(W_t) = \sigma^2$.

(a) $X_t = 5 + W_t$.

Solution:

- $E[X_t] = E[5 + W_t] = 5 + E[W_t] = 5$
- $Var(X_t) = Var(5 + W_t) = Var(W_t) = \sigma^2$.
- $cov(X_{t+h}, X_t) = cov(5+W_{t+h}, 5+W_t) = cov(W_{t+h}, W_t) = \begin{cases} \sigma^2 & \text{if } h = 0 \\ 0 & \text{otherwise} \end{cases}$

X_t is stationary since its mean and auto-covariance are independent of t.

(b) $X_t = t \cdot W_t$.

Solution:

- $E[X_t] = E[t \cdot W_t] = t \cdot E[W_t] = 0$.
- $Var(X_t) = t^2 \cdot Var(W_t) = t^2\sigma^2$.
- $cov(X_{t+h}, X_t) = t(t+h) \cdot cov(W_{t+h}, W_t) = \begin{cases} t^2\sigma^2 & \text{if } h = 0 \\ 0 & \text{otherwise} \end{cases}$

X_t is *not* stationary since auto-covariance is time-dependent.

(c) $X_t = 2W_t + W_{t-1}$.

Solution:

- $E[X_t] = E[2W_t + W_{t-1}] = 2E[W_t] + E[W_{t-1}] = 0$ since $E[W_t] = 0$.
- $Var(X_t) = Var(2W_t + W_{t-1}) = Var(2W_t) + Var(W_{t-1}) = 4\sigma^2 + \sigma^2 = 5\sigma^2$ since W_t is an iid white noise process.
- The auto-covariance can be expanded as follows:

$$\begin{aligned} cov(X_{t+h}, X_t) &= cov(2W_{t+h} + W_{t+h-1}, 2W_t + W_{t-1}) \\ &= 4cov(W_{t+h}, W_t) + 2cov(W_{t+h}, W_{t-1}) \\ &\quad + 2cov(W_{t+h-1}, W_t) + cov(W_{t+h-1}, W_{t-1}) \\ &= \begin{cases} 5\sigma^2 & \text{if } h = 0 \\ 2\sigma^2 & \text{if } h = \pm 1 \\ 0 & \text{otherwise} \end{cases} \\ &= \gamma(h) \end{aligned}$$

X_t is stationary since its mean and auto-covariance are independent of t.

(d) $X_t = (-1)^t W_t$

Solution:

- $E[X_t] = E[(-1)^t W_t] = (-1)^t E[W_t] = 0$.
- $Var(X_t) = Var((-1)^t W_t) = (-1)^{2t} Var(W_t) = Var(W_t) = \sigma^2$.
- The auto-covariance can be expanded as follows:

$$\begin{aligned} cov(X_{t+h}, X_t) &= cov((-1)^{t+h} W_{t+h}, (-1)^t W_t) \\ &= (-1)^{t+h}(-1)^t cov(W_s, W_t) \\ &= (-1)^{2t+h} cov(W_{t+h}, W_t) \\ &= \begin{cases} \sigma^2 & \text{if } h = 0 \\ 0 & \text{otherwise} \end{cases} \\ &= \gamma(h) \end{aligned}$$

X_t is stationary since its mean and auto-covariance are independent of t.

3. Make each of the following stochastic processes stationary without using decomposition, and review any induced artificial dependencies. In the following, consider that W_t is an iid white noise process with $Var(W_t) = \sigma^2$.

(a) $X_t = t^2 + W_t$

Solution: The time series has a quadratic trend. We apply second-order differencing.

$$\begin{aligned} \nabla^2 X_t &= (1-B)^2 X_t = (1 - 2B + B^2) X_t \\ &= (t^2 + W_t) - 2((t-1)^2 + W_{t-1}) + ((t-2)^2 + W_{t-2}) \\ &= (t^2 + W_t) - 2(t^2 - 2t + 1 + W_{t-1}) + (t^2 - 4t + 4 + W_{t-2}) \\ &= 2 + W_t - 2W_{t-1} + W_{t-2} \end{aligned}$$

The quadratic trend is fully removed by the second-order differencing. To confirm that $\nabla^2 X_t$ is stationary, we check its mean and auto-covariance structure.

- $E[\nabla^2 X_t] = 2$

- The auto-covariance can be expanded as follows:

$$
\begin{aligned}
cov(\nabla^2 X_{t+h}, \nabla^2 X_t) &= cov(W_{t+h} - 2W_{t+h-1} + W_{t+h-2}, W_t - 2W_{t-1} + W_{t-2}) \\
&= \mathrm{cov}(W_{t+h}, W_t) - 2\mathrm{cov}(W_{t+h}, W_{t-1}) + \mathrm{cov}(W_{t+h}, W_{t-2}) \\
&\quad - 2\mathrm{cov}(W_{t+h-1}, W_t) + 4\mathrm{cov}(W_{t+h-1}, W_{t-1}) - 2\mathrm{cov}(W_{t+h-1}, W_{t-2}) \\
&\quad + \mathrm{cov}(W_{t+h-2}, W_t) - 2\mathrm{cov}(W_{t+h-2}, W_{t-1}) + \mathrm{cov}(W_{t+h-2}, W_{t-2}) \\
&= \begin{cases} 6\sigma^2 & \text{if } h = 0 \\ -4\sigma^2 & \text{if } h = \pm 1 \\ \sigma^2 & \text{if } h = \pm 2 \\ 0 & \text{otherwise} \end{cases} \\
&= \gamma(h)
\end{aligned}
$$

$\nabla^2 X_t$ is stationary since its mean and auto-covariance are independent of t. To review the induced artificial dependencies, we check X_t auto-covariance.

$$
\begin{aligned}
cov(X_{t+h}, X_t) &= cov((t+h)^2 + W_{t+h}, t^2 + W_t) \\
&= cov(W_{t+h}, W_t) = \begin{cases} \sigma^2 & \text{if } h = 0 \\ 0 & \text{otherwise} \end{cases}
\end{aligned}
$$

The chosen differencing operation introduces synthetic auto-correlations of $-\frac{2}{3}$ at $h = \pm 1$ and $\frac{1}{6}$ at $h = \pm 2$.

(b) $X_t = \cos\left(\frac{2\pi t}{12}\right) + W_t$

Solution: The time series has a seasonal component with period 12. We apply seasonal differencing with lag 12.

$$
\begin{aligned}
\nabla_{12} X_t &= (1 - B^{12}) X_t \\
&= (\cos\left(\frac{2\pi t}{12}\right) + W_t) - (\cos\left(\frac{2\pi(t-12)}{12}\right) + W_{t-12}) \\
&= W_t - W_{t-12}
\end{aligned}
$$

The seasonal component is fully removed by the seasonal differencing. To confirm that $\nabla_{12} X_t$ is stationary, we check its mean and auto-covariance structure.

- $E[\nabla_{12} X_t] = 0$
- The auto-covariance can be expanded as follows:

$$
\begin{aligned}
cov(\nabla_{12} X_{t+h}, \nabla_{12} X_t) &= cov(W_{t+h} - W_{t+h-12}, W_t - W_{t-12}) \\
&= cov(W_{t+h}, W_t) - cov(W_{t+h}, W_{t-12}) \\
&\quad - cov(W_{t+h-12}, W_t) + cov(W_{t+h-12}, W_{t-12}) \\
&= \begin{cases} 2\sigma^2 & \text{if } h = 0 \\ -\sigma^2 & \text{if } h = \pm 12 \\ 0 & \text{otherwise} \end{cases}
\end{aligned}
$$

$\nabla_{12} X_t$ is stationary since its mean and auto-covariance are independent of t. To review the induced artificial dependencies, we check X_t auto-covariance.

$$\begin{aligned} cov(X_{t+h}, X_t) &= cov(\cos\left(\frac{2\pi(t+h)}{12}\right) + W_{t+h}, \cos\left(\frac{2\pi t}{12}\right) + W_t) \\ &= cov(W_{t+h}, W_t) = \begin{cases} \sigma^2 & \text{if } h = 0 \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

The chosen differencing operation introduces synthetic auto-correlations of $-\frac{1}{2}$ at $h = \pm 12$.

(c) $X_t = t + \cos\left(\frac{2\pi t}{6}\right) + W_t$

Solution: This time series has both a linear trend and a seasonal component with period 6. We apply seasonal differencing with lag 6.

$$\begin{aligned} \nabla_6 X_t &= (1 - B^6) X_t \\ &= (t + \cos\left(\frac{2\pi t}{6}\right) + W_t) - (t - 6 + \cos\left(\frac{2\pi(t-6)}{6}\right) + W_{t-6}) \\ &= 6 + W_t - W_{t-6} \end{aligned}$$

Both the trend and seasonal components are fully removed by the seasonal differencing. To confirm that $\nabla_6 X_t$ is stationary, we check its mean and auto-covariance structure.

- $E[\nabla_6 X_t] = 6$
- The auto-covariance can be expanded as follows:

$$\begin{aligned} cov(\nabla_6 X_{t+h}, \nabla_6 X_t) &= cov(6 + W_{t+h} - W_{t+h-6}, 6 + W_t - W_{t-6}) \\ &= cov(W_{t+h}, W_t) - cov(W_{t+h}, W_{t-6}) \\ &\quad - cov(W_{t+h-6}, W_t) + cov(W_{t+h-6}, W_{t-6}) \\ &= \begin{cases} 2\sigma^2 & \text{if } h = 0 \\ -\sigma^2 & \text{if } h = \pm 6 \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

$\nabla_6 X_t$ is stationary since its mean and auto-covariance are independent of t. To review the induced artificial dependencies, we check X_t auto-covariance.

$$\begin{aligned} cov(X_{t+h}, X_t) &= cov(t + h + \cos\left(\frac{2\pi(t+h)}{6}\right) + W_{t+h}, t + \cos\left(\frac{2\pi t}{6}\right) + W_t) \\ &= cov(W_{t+h}, W_t) = \begin{cases} \sigma^2 & \text{if } h = 0 \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

The chosen differencing operation introduces synthetic auto-correlations of $-\frac{1}{2}$ at $h = \pm 6$.

(d) $X_t = t^2 + \sin\left(\frac{2\pi t}{4}\right) + W_t$

Solution: The time series has both a quadratic trend and a seasonal component with period 4. We apply seasonal differencing with lag 4.

$$\begin{aligned}\nabla_4 X_t &= (1 - B^4)X_t \\ &= (t^2 + \sin\left(\frac{2\pi t}{4}\right) + W_t) - ((t-4)^2 + \sin\left(\frac{2\pi(t-4)}{4}\right) + W_{t-4}) \\ &= 8t - 16 + W_t - W_{t-4}\end{aligned}$$

The seasonal components is fully removed by the seasonal differencing but a linear trend remain. We apply first-order differencing.

$$\begin{aligned}\nabla\nabla_4 X_t &= (1 - B)\nabla_4 X_t = (1 - B)(1 - B^4)X_t \\ &= (8t - 16 + W_t - W_{t-4}) - (8(t-1) - 16 + W_{t-1} - W_{t-5}) \\ &= 8 + W_t - W_{t-1} - W_{t-4} + W_{t-5}\end{aligned}$$

To confirm that $\nabla\nabla_4 X_t$ is stationary, we check its mean and auto-covariance structure.

- $E[\nabla\nabla_4 X_t] = 8$
- $cov(\nabla\nabla_4 X_{t+h}, \nabla\nabla_4 X_t) = \begin{cases} 4\sigma^2 & \text{if } h = 0 \\ -2\sigma^2 & \text{if } h = \pm 1 \\ -2\sigma^2 & \text{if } h = \pm 4 \\ \sigma^2 & \text{if } h = \pm 5 \\ 0 & \text{otherwise} \end{cases}$

$\nabla\nabla_4 X_t$ is stationary since its mean and auto-covariance are independent of t. To review the induced artificial dependencies, we check X_t auto-covariance.

$$\begin{aligned}cov(X_{t+h}, X_t) &= cov((t+h)^2 + \sin\left(\frac{2\pi(t+h)}{4}\right) + W_{t+h}, t^2 + \sin\left(\frac{2\pi t}{4}\right) + W_t) \\ &= cov(W_{t+h}, W_t) = \begin{cases} \sigma^2 & \text{if } h = 0 \\ 0 & \text{otherwise} \end{cases}\end{aligned}$$

The chosen differencing operation introduces synthetic auto-correlations of $-\frac{1}{4}$ at $h = \pm 1, \pm 4$ and $\frac{1}{4}$ at $h = \pm 5$.