1. Given the AR(2) model $X_t = \frac{1}{2}X_{t-1} + \frac{1}{5}X_{t-2} + W_t$ where $W_t \sim \mathcal{N}(0, \sigma^2)$, determine whether the process is stationary.

> **Solution:** The characteristic polynomial of the AR(2) model is given by:
>
> $$1 - \frac{1}{2}B - \frac{1}{5}B^2 = 0$$
>
> Solving this quadratic equation:
>
> $$r_{1,2} = \frac{\frac{1}{2} \pm \sqrt{\frac{1}{4} + \frac{4}{5}}}{-\frac{2}{5}} = -\frac{5}{4} \pm \frac{5}{2}\sqrt{\frac{21}{20}} = \frac{-5 \pm \sqrt{105}}{4} \Rightarrow |r_{1,2}| > 1$$
>
> The roots magnitudes are greater than 1, meaning the process is stationary.

2. Consider the MA(1) process $X_t = W_t + \theta W_{t-1}$, where $W_t \sim \mathcal{N}(0, \sigma^2)$. Show that the autocorrelation function (ACF) is zero for all lags greater than 1.

> **Solution:** The autocovariance function of an MA(1) process is:
>
> $$\begin{aligned}\gamma(h) &= cov(X_{t+h}, X_t) = cov(W_{t+h} + \theta W_{t+h-1}, W_t + \theta W_{t-1}) \\ &= cov(W_{t+h}, W_t) + \theta cov(W_{t+h}, W_{t-1}) \\ &\quad + \theta cov(W_{t+h-1}, W_t) + \theta^2 cov(W_{t+h-1}, W_{t-1}) \\ &= \begin{cases} \sigma^2(\theta^2 + 1) & \text{if } h = 0 \\ \sigma^2\theta^2 & \text{if } h = \pm 1 \\ 0 & \text{otherwise} \end{cases}\end{aligned}$$
>
> Thus, the autocorrelation function is:
>
> $$\rho(0) = 1, \quad \rho(1) = \frac{\theta\sigma^2}{(1+\theta^2)\sigma^2} = \frac{\theta}{1+\theta^2}, \quad \rho(h) = 0 \quad \text{for} \quad h > 1.$$
>
> This confirms that the ACF is zero for all lags greater than 1.

3. Consider the stationary ARMA(1,1) process $X_t = \phi X_{t-1} + W_t + \theta W_{t-1}$ where $W_t \sim \mathcal{N}(0, \sigma^2)$. Derive the MA representation of X_t.

> **Solution:** Express the ARMA(1,1) process using the backshift operator:
>
> $$(1 - \phi B)X_t = (1 + \theta B)W_t$$
>
> The process can be rewritten as:
>
> $$X_t = \frac{1}{1 - \phi B}(1 + \theta B)W_t$$

Since X_t is stationary, we can consider $|\phi| < 1$, and expand the fraction as a geometric series:

$$\frac{1}{1-\phi B} = \sum_{i=0}^{\infty} (\phi B)^i = 1 + \phi B + \phi^2 B^2 + \phi^3 B^3 + \cdots$$

Substituting into the expression for X_t:

$$X_t = (1 + \phi B + \phi^2 B^2 + \phi^3 B^3 + \cdots)(1 + \theta B) W_t$$

Expanding the product:

$$\begin{aligned} X_t &= (1 + (\theta + \phi)B + \phi(\theta + \phi)B^2 + \cdots) W_t \\ &= W_t + \sum_{i=1}^{\infty} \theta'_i W_{t-i} \text{ with } \theta'_i = \phi^{i-1}(\theta + \phi) \end{aligned}$$

Thus, the ARMA(1,1) process can be written as an infinite-order MA process with these coefficients.

4. Consider the following two ARMA processes:

$$X_t = \frac{1}{2} X_{t-1} - \frac{1}{4} W_{t-1} + W_t$$

and

$$Y_t = \frac{5}{6} Y_{t-1} - \frac{1}{6} Y_{t-2} - \frac{7}{12} W_{t-1} + \frac{1}{12} W_{t-2} + W_t$$

where $W_t \sim \mathcal{N}(0, \sigma^2)$. Show that X_t and Y_t are equivalent.

Solution: We rewrite the processes in backshift operator notation. The first process X_t can be rearranged to:

$$(1 - \frac{1}{2}B) X_t = (1 - \frac{1}{4}B) W_t.$$

Now for the second process Y_t:

$$(1 - \frac{5}{6}B + \frac{1}{6}B^2) Y_t = (1 - \frac{7}{12}B + \frac{1}{12}B^2) W_t.$$

The characteristic polynomials can be factored as:

$$(1 - \frac{1}{3}B)(1 - \frac{1}{2}B) Y_t = (1 - \frac{1}{3}B)(1 - \frac{1}{4}B) W_t.$$

which can be simplified as:

$$(1 - \frac{1}{2}B) Y_t = (1 - \frac{1}{4}B) Y_t.$$

Both ARMA processes exhibit the same behavior and are thus equivalent.

5. Rewrite the following models as ARMA processes:

(a) ARIMA(1,1,1)

Solution: The ARIMA(1,1,1) process is given by:

$$\Phi(B)\nabla X_t = \Theta(B)W_t,$$

Expanding the characteristic polynomials and differencing operator:

$$(1 - \phi B)(X_t - X_{t-1}) = (1 + \theta B)W_t,$$

which simplifies to the following ARMA(2,1) process:

$$X_t = (1 + \phi)X_{t-1} - \phi X_{t-2} + W_t + \theta W_{t-1}.$$

(b) SARIMA(0,0,0)$(1,1,1)_{\mathrm{s}}$

Solution: The SARIMA(0,0,0)$(1,1,1)_{\mathrm{s}}$ is given by:

$$\Phi'(B_s)\nabla_s X_t = \Theta'(B_s)W_t,$$

Expanding the characteristic polynomials and differencing operator:

$$(1 - \phi' B^s)(X_t - X_{t-s}) = (1 + \theta' B^s)W_t,$$

which simplifies to the following ARMA(2,1) process:

$$X_t = (1 + \phi')X_{t-s} - \phi' X_{t-2s} + W_t + \theta' W_{t-s}.$$