

# Time Series Analysis

## Fourier analysis

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**Informatik**



# Outline

- Fourier analysis
- Sinusoids
- Fitting sinusoids with known frequencies
- Harmonic analysis with Fourier frequencies
- Orthogonality of complex exponentials
- Discrete Fourier transform
- Fast Fourier transform
- Aliasing and the Nyquist frequency

# Fourier analysis

Decomposition of time series into a **sum of sinusoidal components**.

- Linear combination of sine and cosine waves, each with specific **frequency** and **amplitude**.

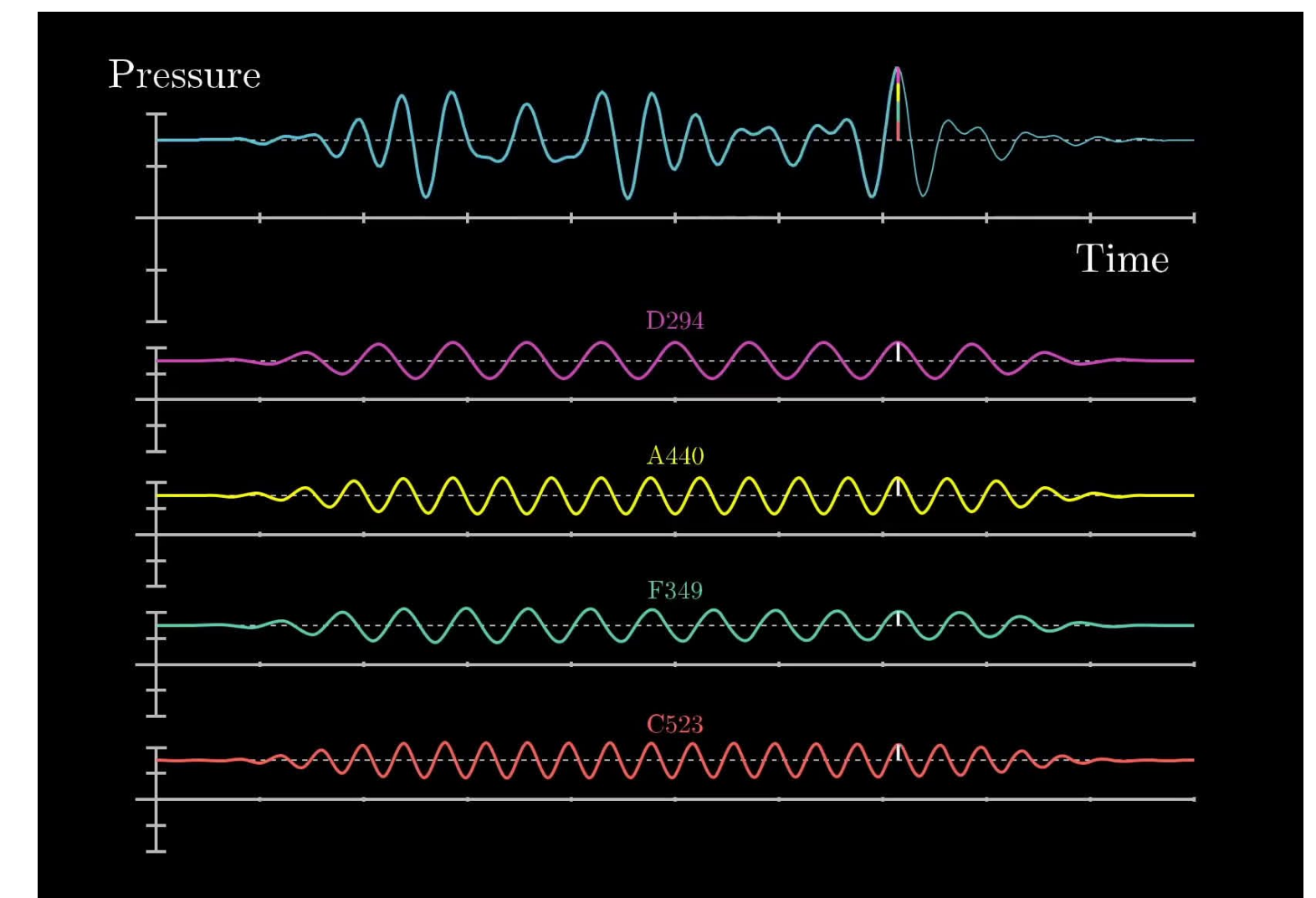
Describe fluctuations in time series by **comparing** them with **sinusoids**.

## **Periodic** fluctuations

- Detect seasonality (simple or multiple).
- Identify dominant frequencies.

## **Aperiodic** fluctuations

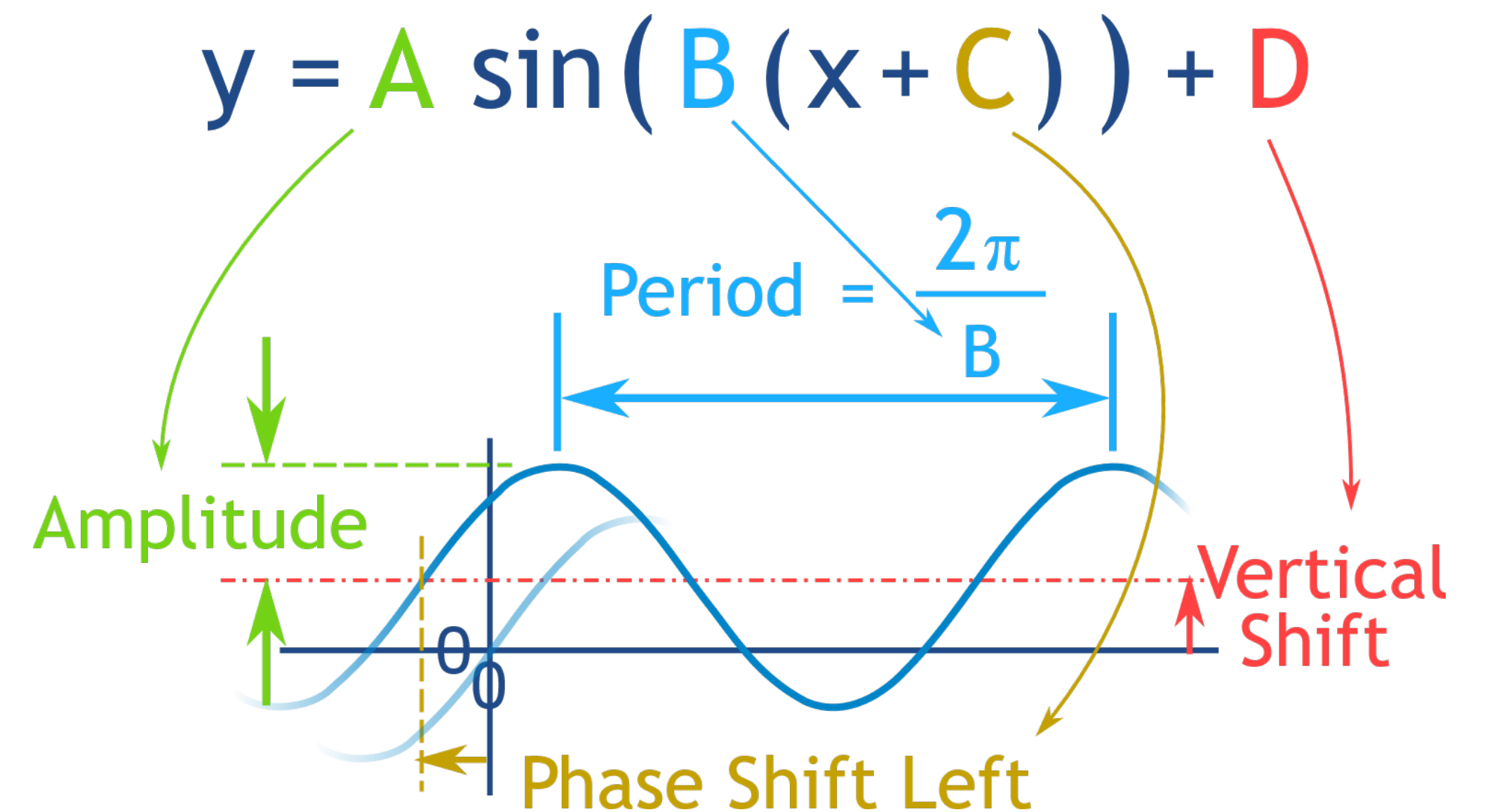
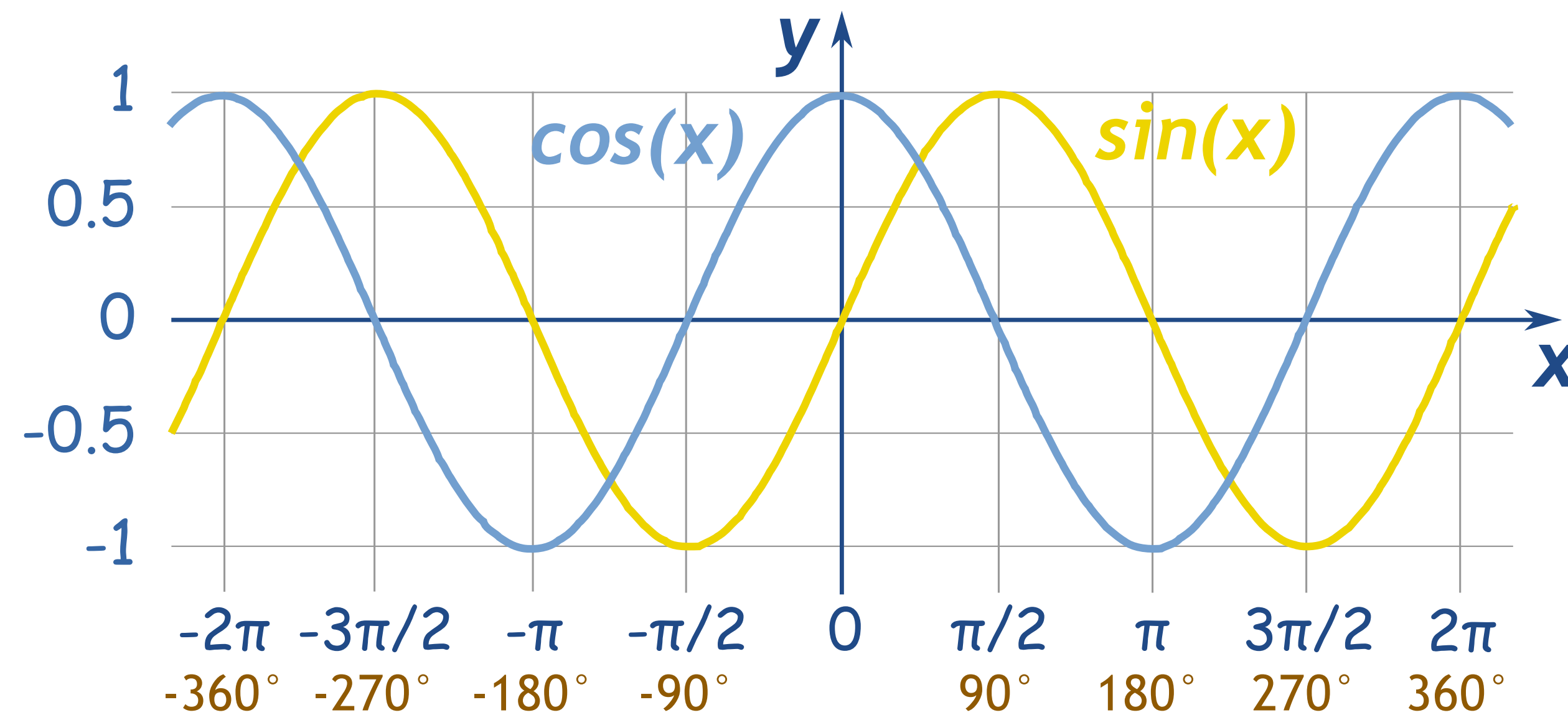
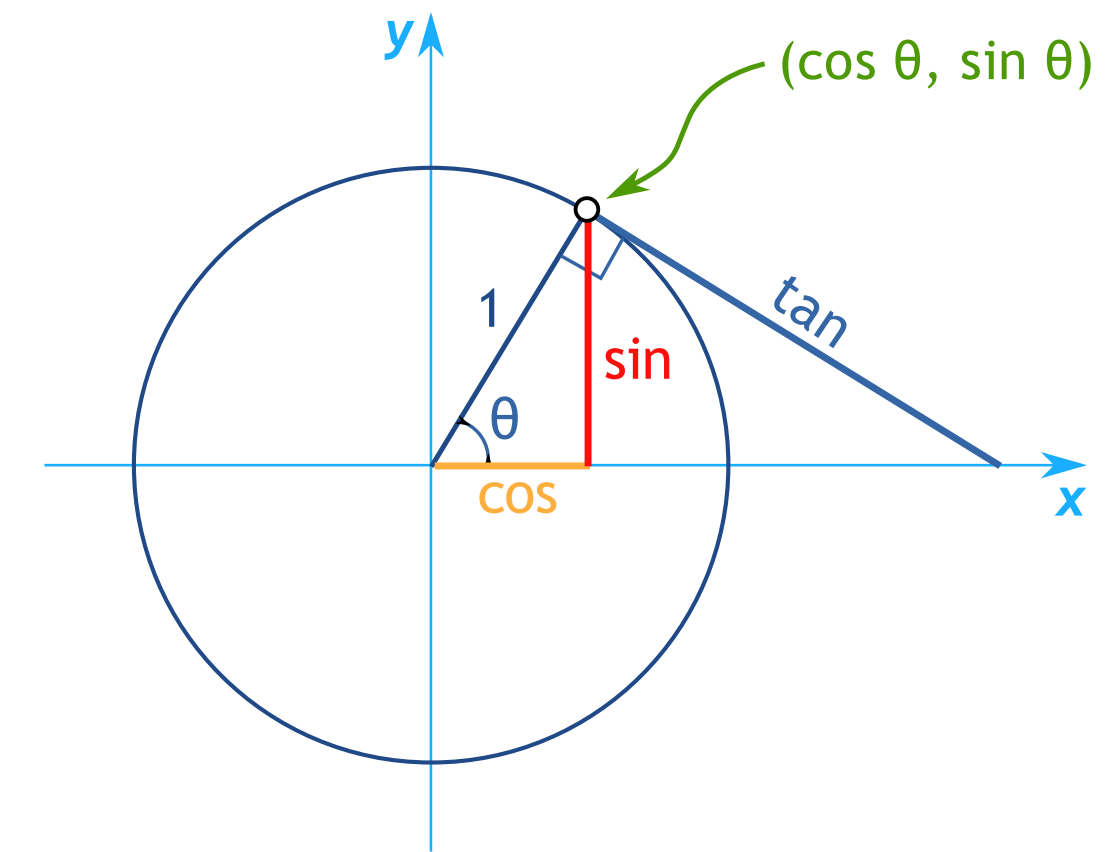
- Describe the frequency distribution of oscillations.
- Distinguish noise from signal.



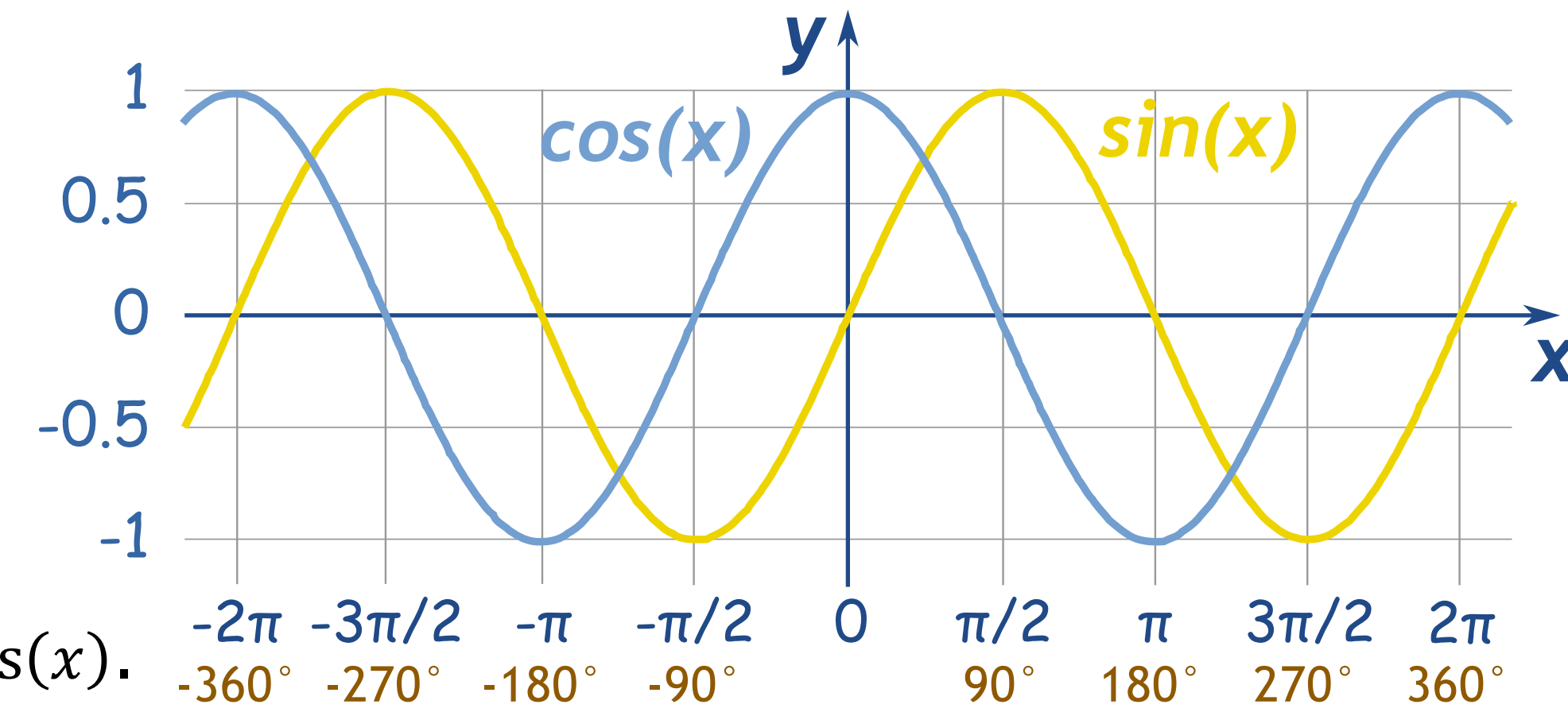
3Blue1Brown

## Recap – Trigonometry concepts (1)

**Sinusoids** are periodic waves whose waveform is the **cosine/sine** function.



## Recap – Trigonometry concepts (2)



Sine and cosine are the **same waveform**, just shifted:  $\sin\left(x \pm \frac{\pi}{2}\right) = \pm \cos(x)$ .

They are periodic  $\sin(x + 2\pi) = \sin(x)$  and  $\cos(x + 2\pi) = \cos(x)$

- The **period**  $P = 2\pi$  means the function completes **one full cycle** every  $2\pi$  units along the x-axis.
- The **frequency**  $f = 1/P = 1/2\pi$  tells **how many cycles occur per unit of x**.

Sine is an **odd** function  $\sin(-x) = -\sin(x)$ , while cosine is an **even** function  $\cos(-x) = \cos(x)$ .

**Pythagorean identity:**  $\sin^2 x + \cos^2 x = 1$ .

**Angle sum:**  $\sin(\alpha \pm \beta) = \sin(\alpha) \cos(\beta) \pm \cos(\alpha) \sin(\beta)$  and  $\cos(\alpha \pm \beta) = \cos(\alpha) \cos(\beta) \mp \sin(\alpha) \sin(\beta)$

In the continuous case, two sinusoids are **orthogonal** (the integral of their product is zero) over any interval spanning a **complete period** of the functions (provided they are **not vertically shifted**).

# Sinusoids

A sinusoid of **frequency**  $f$  (cycles per unit time) or period  $P = 1/f$  (in units of time) can be written as

$$x_t = R \cos(2\pi(f \cdot t + \phi))$$

where  $R$  is the **amplitude** and  $\phi$  is the **phase**.

**Orthogonality of sinusoids:** length  $n$  sampled sinusoidal signal segments are **orthogonal**  $\Leftrightarrow$  their frequencies have an **integer multiple**  $k$  of periods in  $n$  samples (i.e., we can write  $f = k/n$  with  $k \in [0, n - 1]$ ) .

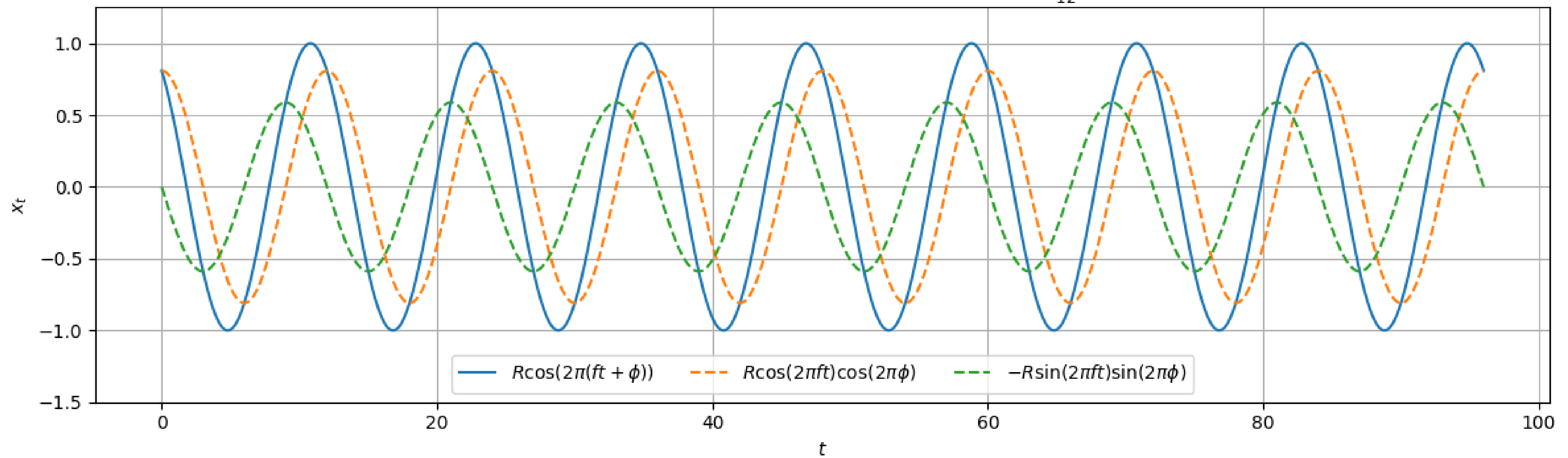
A **sum of sinusoids** with common frequency  $f$  is another sinusoid with the same frequency

$$\begin{aligned} x_t &= R \cos(2\pi(ft + \phi)) \\ &= R(\cos(2\pi ft) \cos(2\pi \phi) - \sin(2\pi ft) \sin(2\pi \phi)) \\ &= A \cos(2\pi ft) + B \sin(2\pi ft) \\ &= A c_t + B s_t \end{aligned}$$

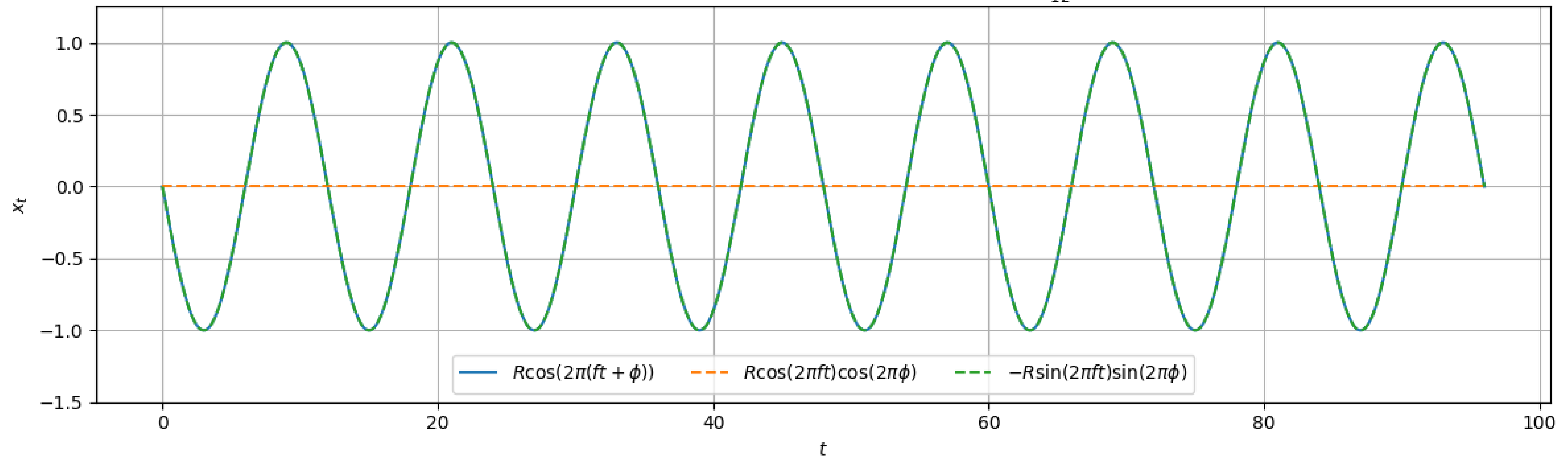
with  $A = R \cos(2\pi \phi)$ ,  $B = -R \sin(2\pi \phi)$ ,  $c_t = \cos(2\pi ft)$  and  $s_t = \sin(2\pi ft)$ .

Then  $R = \sqrt{A^2 + B^2}$  and  $\phi = \arctan\left(-\frac{B}{A}\right)/2\pi$ .

Sum of sinusoids with the same frequency ( $R = 1, f = \frac{1}{12}, \phi = 0.1$ )



Sum of sinusoids with the same frequency ( $R = 1, f = \frac{1}{12}, \phi = 0.25$ )



## Fitting sinusoids (1)

Given a time series realization  $\{x_0, \dots, x_{n-1}\}$  with known period  $P = 1/f$ ,

The data can be modeled as

$$x_t = \mu + Ac_t + Bs_t + e_t$$

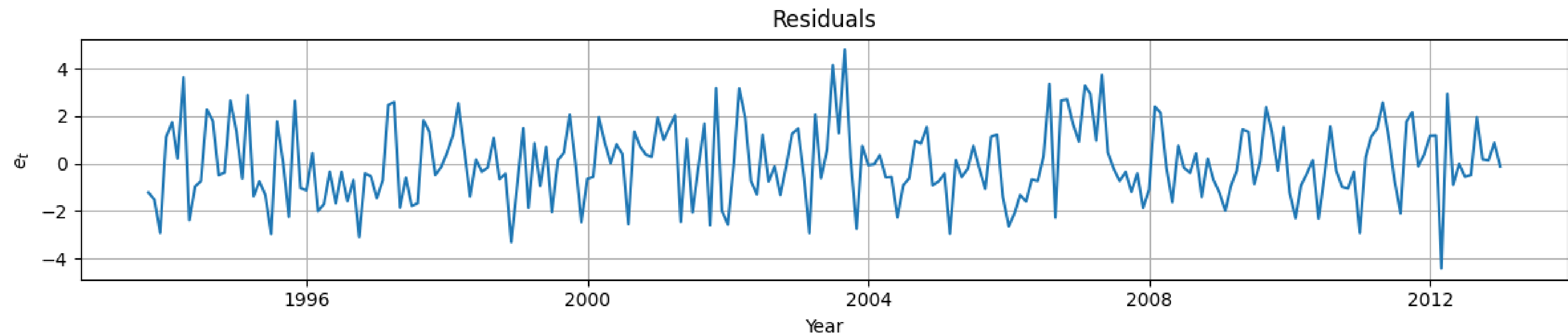
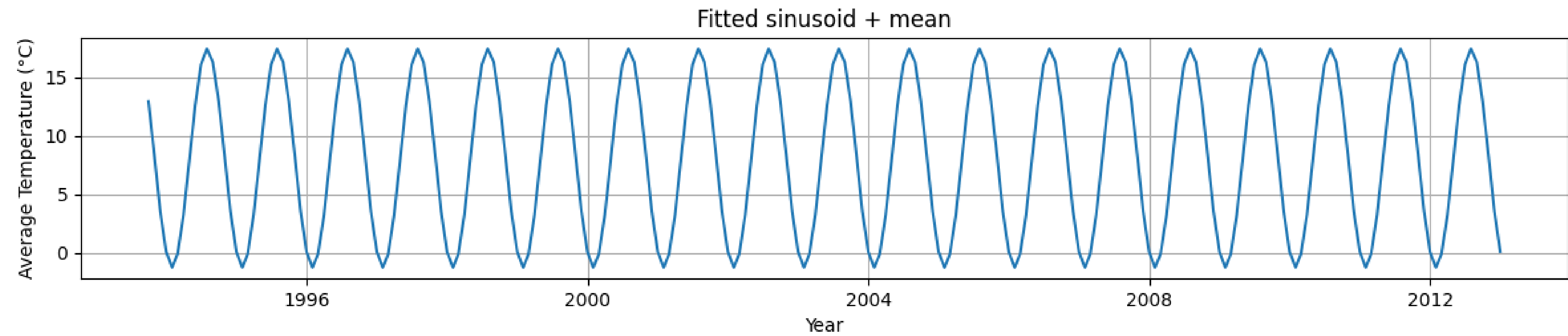
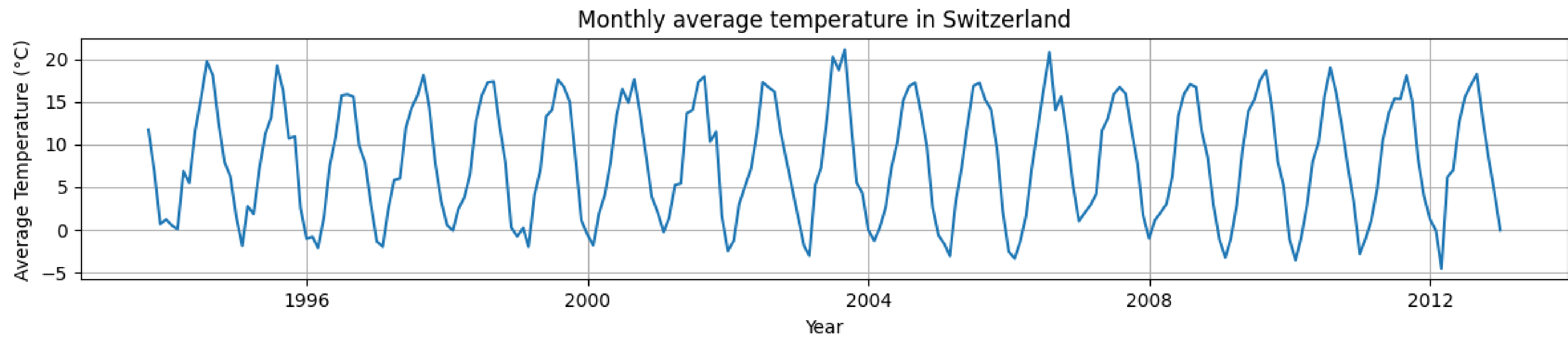
with  $e_t$  the residual at time  $t$ ,  $c_t = \cos(2\pi ft)$  and  $s_t = \sin(2\pi ft)$ .

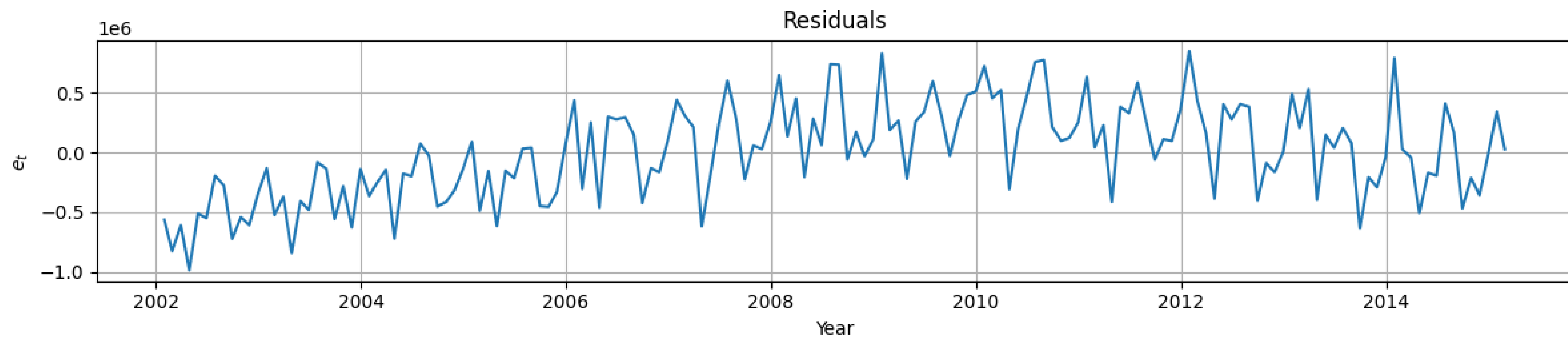
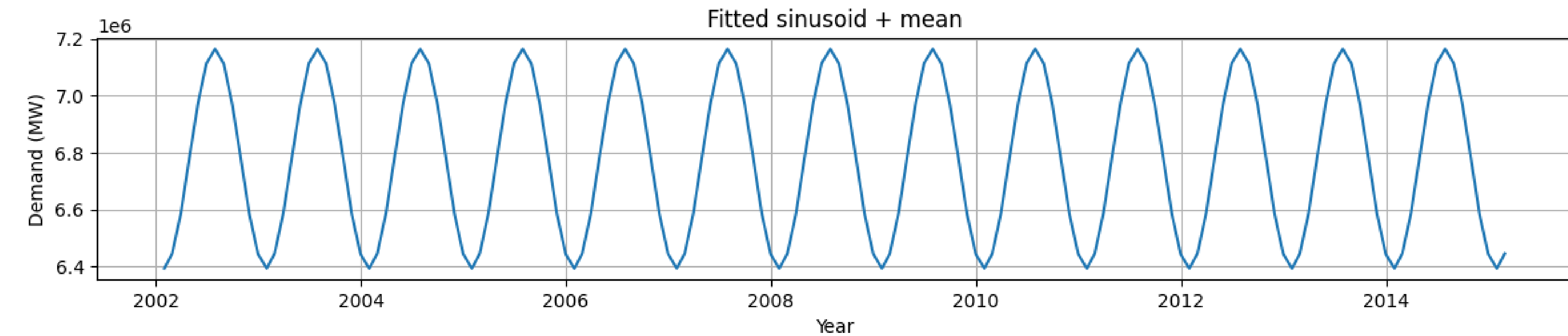
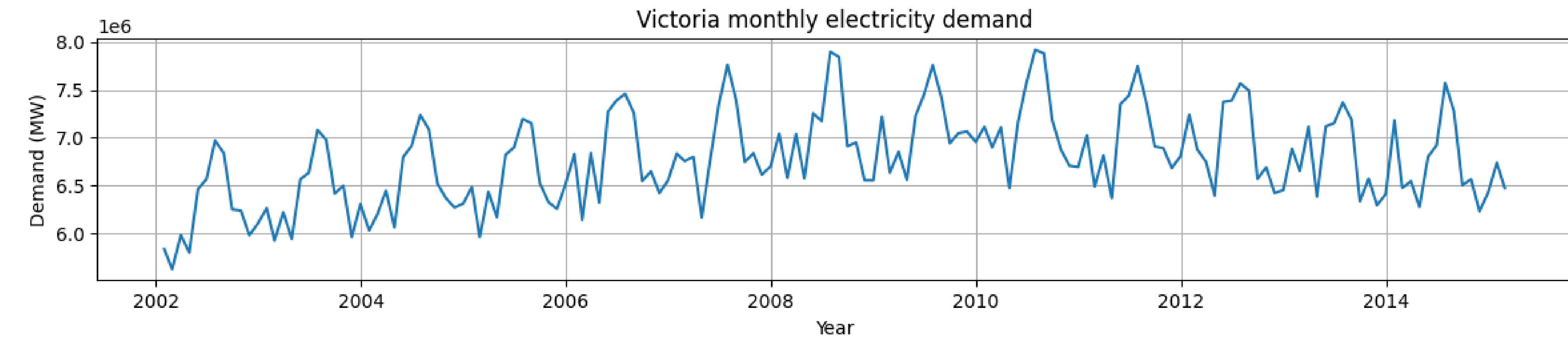
Apply OLS:  $\begin{pmatrix} \hat{\mu} \\ \hat{A} \\ \hat{B} \end{pmatrix} = \min_{\mu, A, B} \sum_{i=0}^{n-1} e_i^2 = (X^T X)^{-1} X^T Y$  with  $Y = \begin{pmatrix} x_0 \\ \vdots \\ x_{n-1} \end{pmatrix}$  and  $X = \begin{pmatrix} 1 & c_0 & s_0 \\ \vdots & \vdots & \vdots \\ 1 & c_{n-1} & s_{n-1} \end{pmatrix}$

$X$  columns are **orthogonal** provided  $n$  is an **integer multiple** of  $P$ .

In this case, this optimization problem can be solved with

$$\hat{\mu} = \frac{1}{n} \sum_{t=0}^{n-1} x_t, \quad \hat{A} = \frac{2}{n} \sum_{t=0}^{n-1} x_t c_t, \quad \hat{B} = \frac{2}{n} \sum_{t=0}^{n-1} x_t s_t$$





## Fitting sinusoids (2)

When the TS involves  $k$  seasonalities of known periods  $P_1, \dots, P_k$  with  $P_j = 1/f_j$  for  $j = 1, \dots, k$ ,

The data can be modeled as

$$x_t = \mu + \sum_{j=1}^k (A_j c_{t,j} + B_j s_{t,j}) + e_t$$

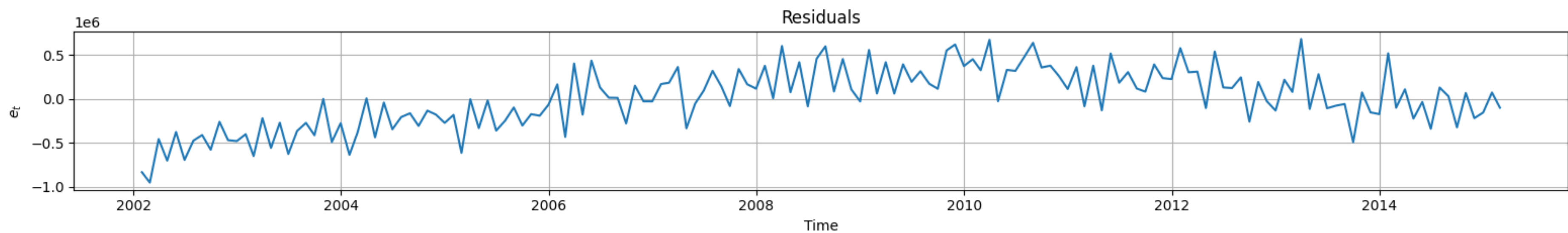
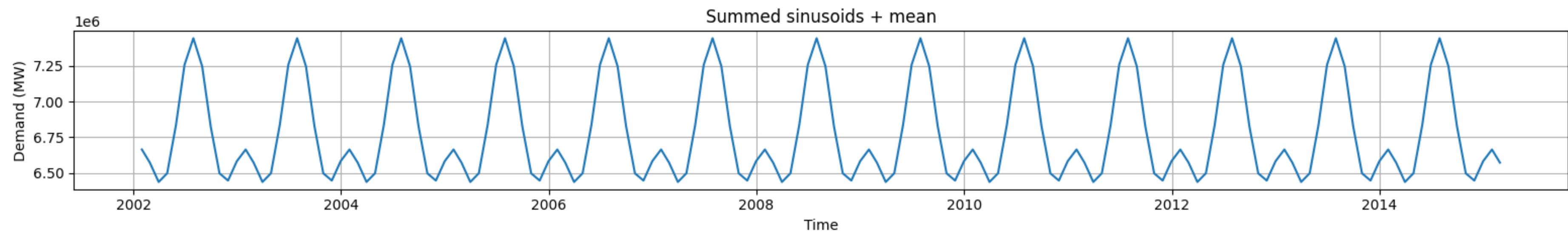
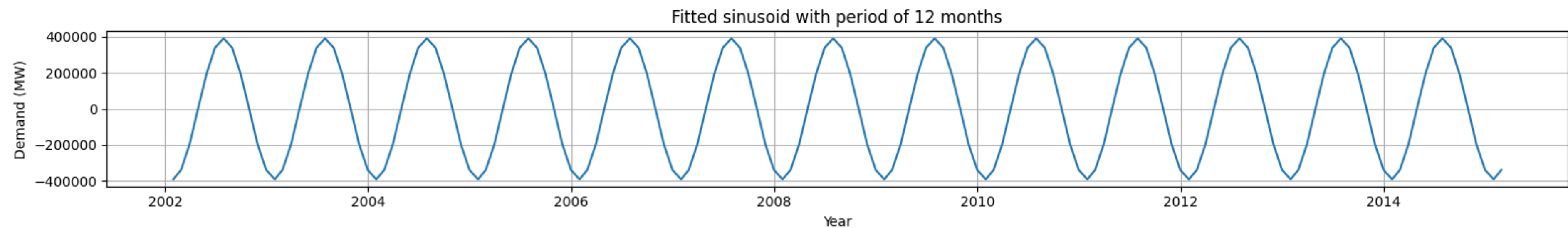
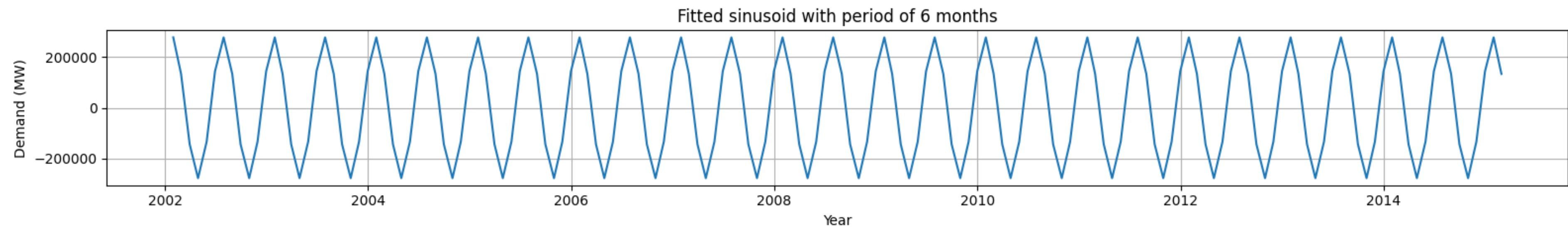
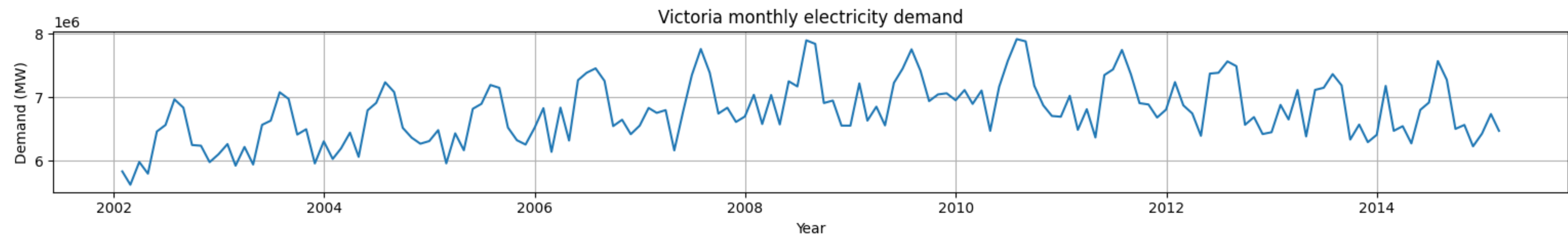
with  $c_{t,j} = \cos(2\pi f_j t)$ ,  $s_{t,j} = \sin(2\pi f_j t)$  and  $e_t$  the residual at time  $t$ .

The design matrix becomes  $X = \begin{pmatrix} 1 & c_{0,1} & s_{0,1} & \dots & c_{0,k} & s_{0,k} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & c_{n-1,1} & s_{n-1,1} & \dots & c_{n-1,k} & s_{n-1,k} \end{pmatrix}$

$X$  columns are **orthogonal** provided  $n$  is an **integer multiple of all periods**.

In this case, this optimization problem can be solved with

$$\hat{\mu} = \frac{1}{n} \sum_{t=0}^{n-1} x_t, \quad \hat{A}_j = \frac{2}{n} \sum_{t=0}^{n-1} x_t c_{t,j}, \quad \hat{B}_j = \frac{2}{n} \sum_{t=0}^{n-1} x_t s_{t,j}$$



# Harmonic analysis with Fourier frequencies

Given a TS realization  $\{x_0, \dots, x_{n-1}\}$ , the **Fourier frequencies** are frequencies of the form  $f_j = j/n$  with  $j = 0, 1, \dots, k$ .

- Due to aliasing, we consider only frequencies satisfying  $0 \leq f_j \leq \frac{1}{2}$  (since  $\Delta = 1$ ) i.e.,  $k = \begin{cases} n/2, & n \text{ is even} \\ (n-1)/2, & n \text{ is odd} \end{cases}$

These frequencies are **harmonics** (integer multiples) with respect to the span of the data i.e., each of the corresponding periods  $P_j = n/j$  is **repeated  $j$  times**.

- Sinusoids at the Fourier frequencies ( $X$  columns) are **orthogonal** over the span of the data  $n$ .

The TS realization can be **exactly** represented as a **sum of  $k$  sinusoids at the Fourier frequencies**:

$$x_t = \mu + \sum_{j=1}^k (A_j c_{t,j} + B_j s_{t,j})$$
$$\mu = \frac{1}{n} \sum_{t=0}^{n-1} x_t = \frac{A_0}{2}, \quad A_j = \frac{2}{n} \sum_{t=0}^{n-1} x_t c_{t,j}, \quad B_j = \frac{2}{n} \sum_{t=0}^{n-1} x_t s_{t,j}$$

# Recap – Complex numbers

Complex numbers  $\mathbb{C}$  **extend** real numbers  $\mathbb{R}$  with the **imaginary unit**  $i$  that satisfies  $i^2 = -1$ .

- $z = \operatorname{Re}(z) + i \operatorname{Im}(z)$  with  $\operatorname{Re}(z)$  the **real part** and  $\operatorname{Im}(z)$  the **imaginary part** of  $z$ .

Complex numbers can also be defined in **polar coordinates**:  $z = |z|(\cos(\varphi) + i \sin(\varphi))$ .

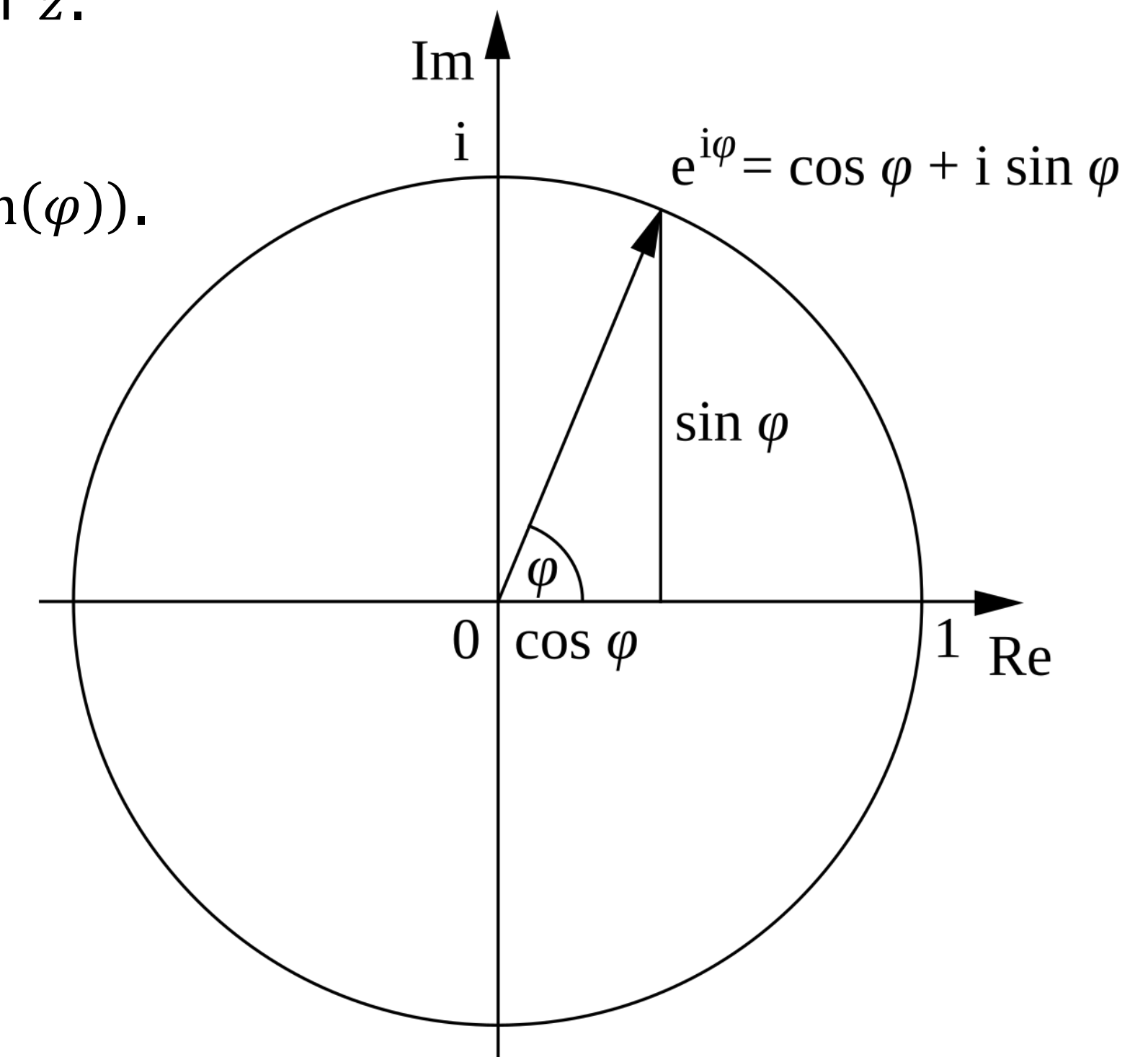
- $\operatorname{Re}(z) = |z| \cos(\varphi)$  and  $\operatorname{Im}(z) = |z| \sin(\varphi)$ .
- **Magnitude**  $|z| = \sqrt{\operatorname{Re}(z)^2 + \operatorname{Im}(z)^2}$  and **phase**  $\varphi = \arctan(\operatorname{Im}(z)/\operatorname{Re}(z))$ .

**Euler formula**:  $e^{i\varphi} = \cos(\varphi) + i \sin(\varphi)$ . Hence,  $e^{i\varphi} = -e^{i(\varphi+\pi)}$ .

- Any  $z \in \mathbb{C}$  can be defined as  $z = |z|e^{i\varphi}$ .
- $\cos(\varphi) = \frac{1}{2}(e^{i\varphi} + e^{-i\varphi})$  and  $\sin(\varphi) = \frac{1}{2i}(e^{i\varphi} - e^{-i\varphi})$ .

The **complex conjugate** of  $z$  is defined as  $\bar{z} = \operatorname{Re}(z) - i \operatorname{Im}(z) = |z|e^{-i\varphi}$ .

- $\bar{\bar{z}} = z$  and the complex conjugate of  $x \in \mathbb{R}$  is  $\bar{x} = x$ .
- $z\bar{z} = \operatorname{Re}(z)^2 + \operatorname{Im}(z)^2 = |z|^2$  where  $|z|$  is the modulus of  $z$ .



Gunther, wikimedia

# Orthogonality of complex exponentials

The sum of complex numbers **equally spaced over the unit circle**  $z_{t,j} = e^{2\pi i \frac{j}{n} t}$  (called the **n-th roots of unity** as  $z_{t,j}$  satisfies  $z_{t,j}^n = 1$ ) **over one period**  $n$  is zero for any non-zero integer  $0 < t < n$ ,

$$\sum_{j=0}^{n-1} e^{2\pi i \frac{j}{n} t} = \delta_{t0} n = \begin{cases} n & \text{if } t = 0 \\ 0 & \text{otherwise} \end{cases}, \quad \text{with } \delta_{kl} = \begin{cases} 1 & \text{if } k = l \\ 0 & \text{otherwise} \end{cases} \text{ the **Kronecker delta**}$$

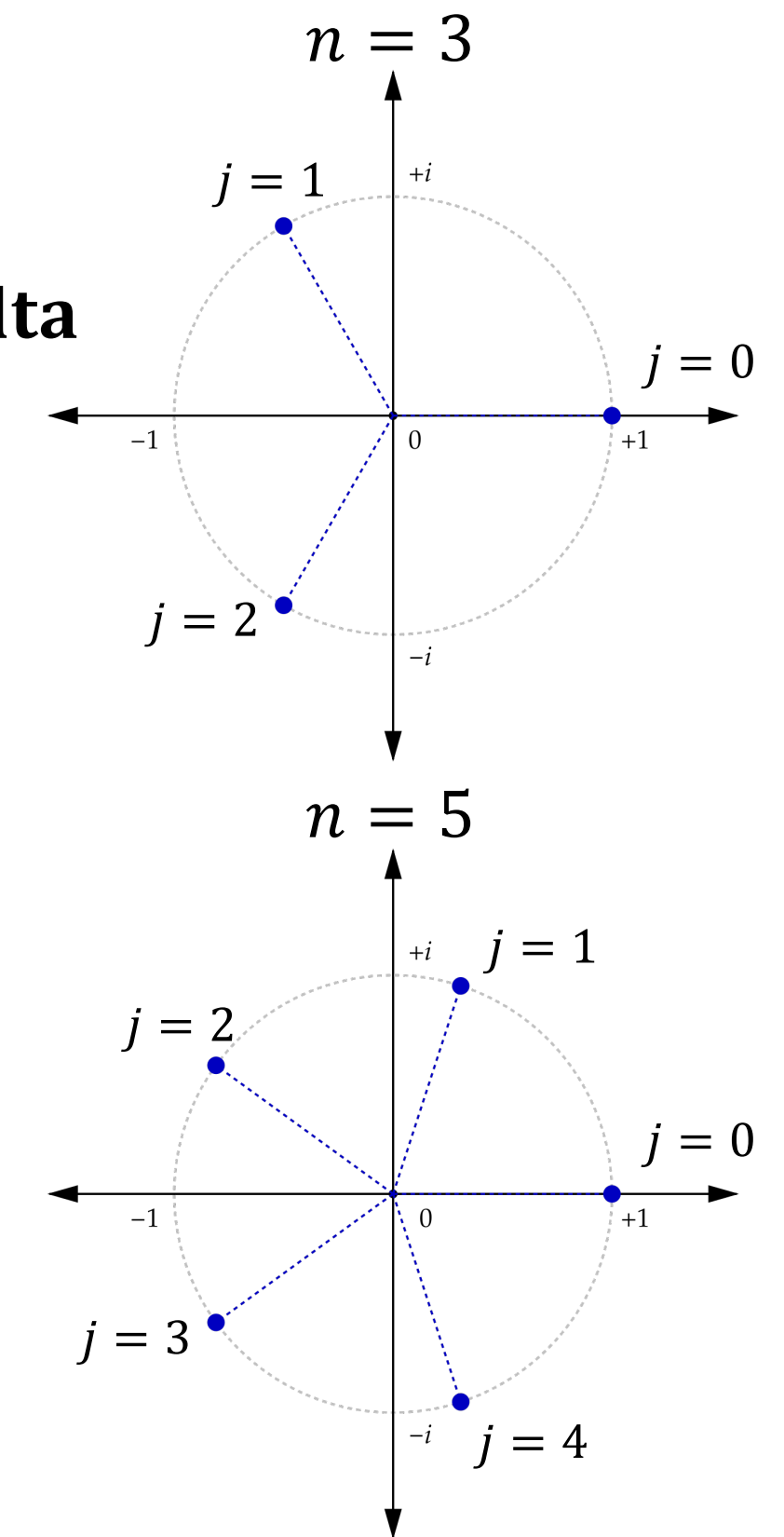
Proof: When  $t = 0$ , we have  $\sum_{j=0}^{n-1} e^0 = n$ . When  $t \neq 0$ , the sum is a geometric series:

$$\sum_{j=0}^{n-1} e^{2\pi i \frac{j}{n} t} = \sum_{j=0}^{n-1} \left( e^{2\pi i \frac{1}{n} t} \right)^j = 1 + \left( e^{2\pi i \frac{1}{n} t} \right)^1 + \left( e^{2\pi i \frac{1}{n} t} \right)^2 + \dots + \left( e^{2\pi i \frac{1}{n} t} \right)^{n-1}$$

$$\begin{aligned} & \text{(recall that } \sum_{j=0}^{n-1} r^j = \frac{1-r^n}{1-r} \text{ and} \\ & \text{that } e^{2\pi i t} = 1 \text{ for integer } t) \\ & = \frac{1 - \left( e^{2\pi i \frac{1}{n} t} \right)^n}{1 - e^{2\pi i \frac{1}{n} t}} = \frac{1 - e^{2\pi i t}}{1 - e^{2\pi i \frac{1}{n} t}} = 0 \end{aligned}$$

Such complex exponentials are **orthogonal** over a complete cycle: taking  $\mathbf{z}_k = \{z_{0,k}, \dots, z_{n-1,k}\}$ ,

$$\langle \mathbf{z}_k, \mathbf{z}_l \rangle = \sum_{t=0}^{n-1} z_k \bar{z}_l = \sum_{t=0}^{n-1} e^{2\pi i \frac{k}{n} t} e^{-2\pi i \frac{l}{n} t} = \sum_{t=0}^{n-1} e^{2\pi i \frac{t}{n} (k-l)} = \delta_{kl} n$$



Loadmaster, wikimedia

# Discrete Fourier transform (DFT)

Given a TS realization  $x = \{x_0, \dots, x_{n-1}\}$ , the **Discrete Fourier Transform**  $\mathcal{F}(x) = X = \{X_0, \dots, X_{n-1}\}$  is defined as

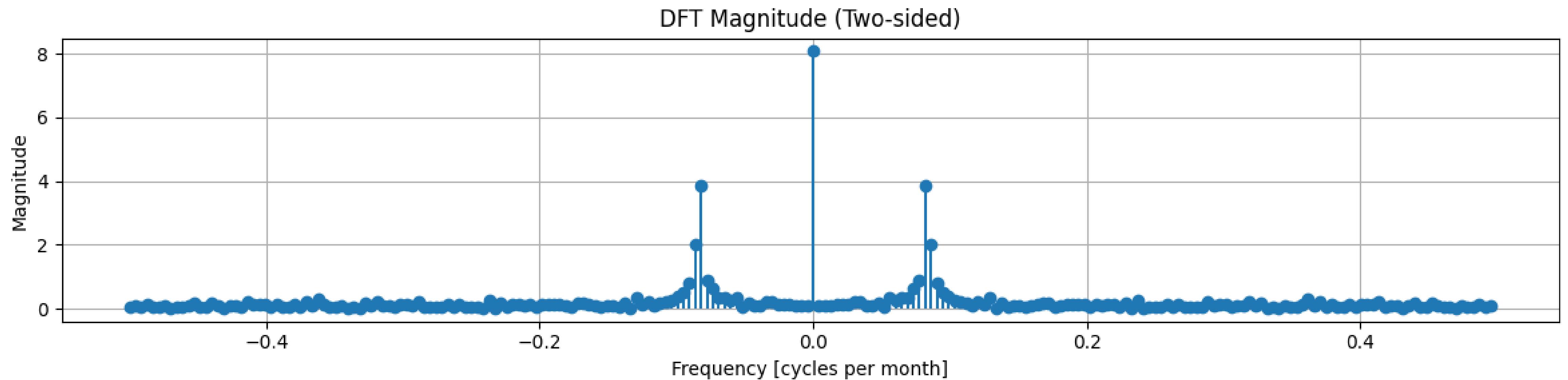
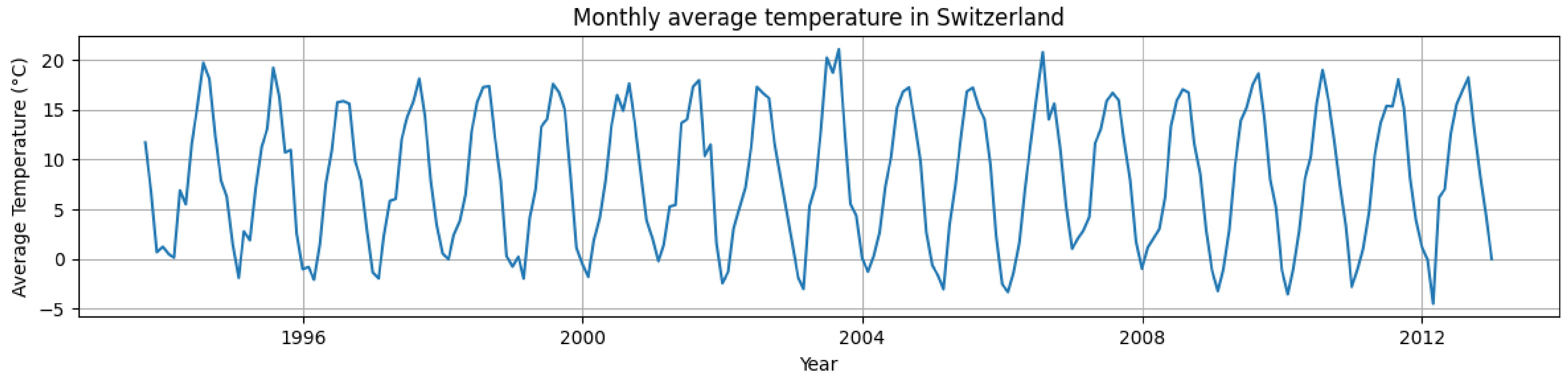
$$\mathcal{F}(x)_j = X_j = \frac{1}{n} \sum_{t=0}^{n-1} x_t e^{-2\pi i f_j t}, \quad j = 0, 1, \dots, k, \dots, n-1$$

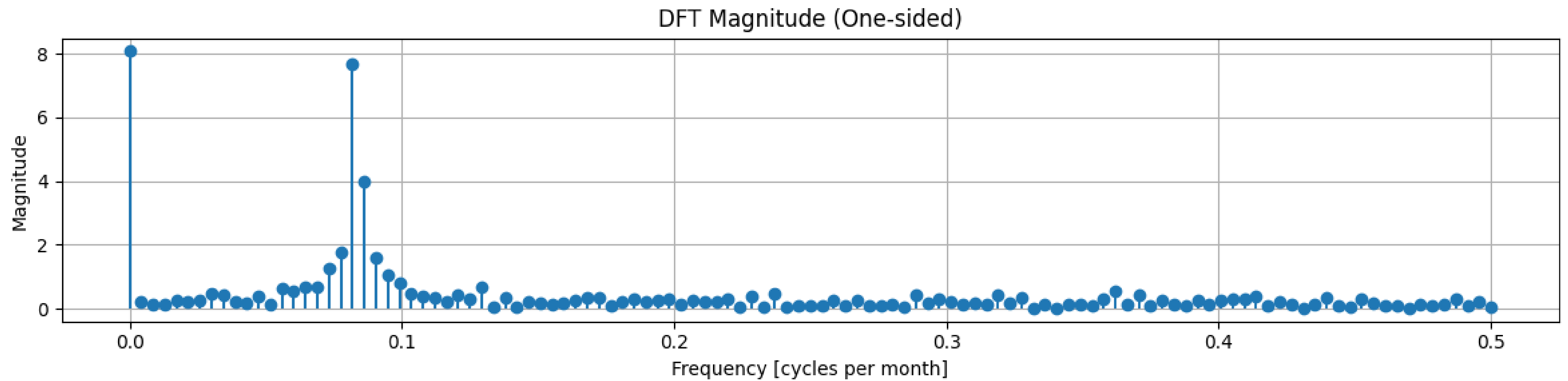
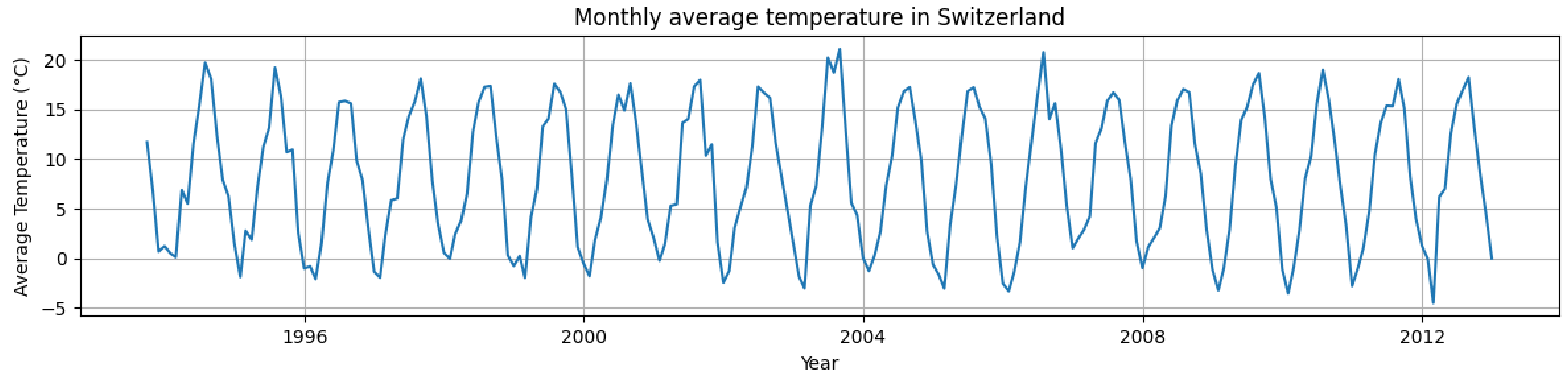
with  $f_j = j/n$  the **Fourier frequencies** and  $k = \begin{cases} n/2, & n \text{ is even} \\ (n-1)/2, & n \text{ is odd} \end{cases}$

For **real**  $x_t$  we observe that  $X_{n-j} = \bar{X}_j$  (**conjugate symmetry**) i.e., the frequencies  $f_j$  for  $j > k$  are **redundant**:

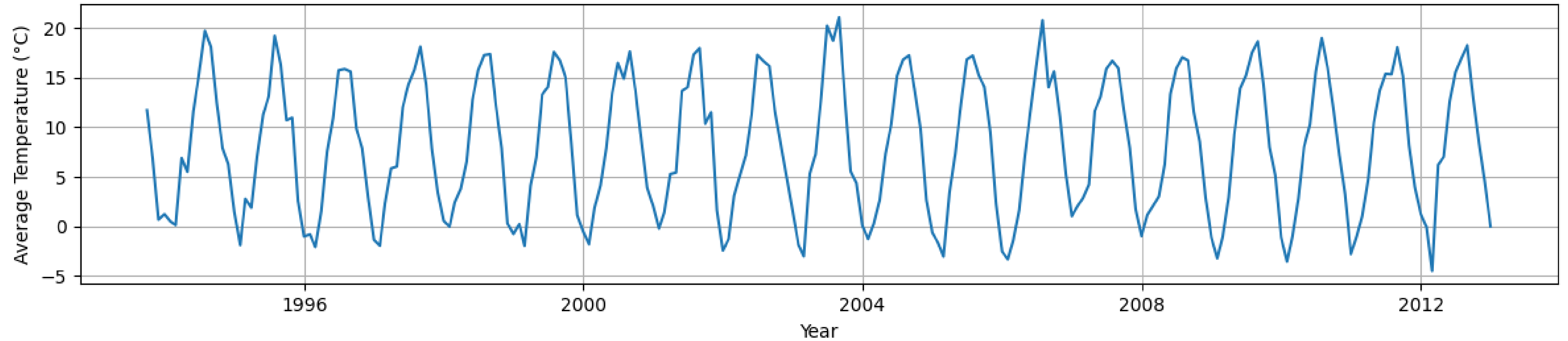
$$\begin{aligned} X_{n-j} &= \frac{1}{n} \sum_{t=0}^{n-1} x_t e^{-2\pi i \left(\frac{n-j}{n}\right)t} \\ &= \frac{1}{n} \sum_{t=0}^{n-1} x_t e^{-2\pi i t} e^{2\pi i \frac{j}{n}t} \\ &= \frac{1}{n} \sum_{t=0}^{n-1} x_t \cdot 1 \cdot e^{2\pi i f_j t} = \bar{X}_j \end{aligned}$$

( $e^{2\pi i k} = 1$  for integer  $k$ )

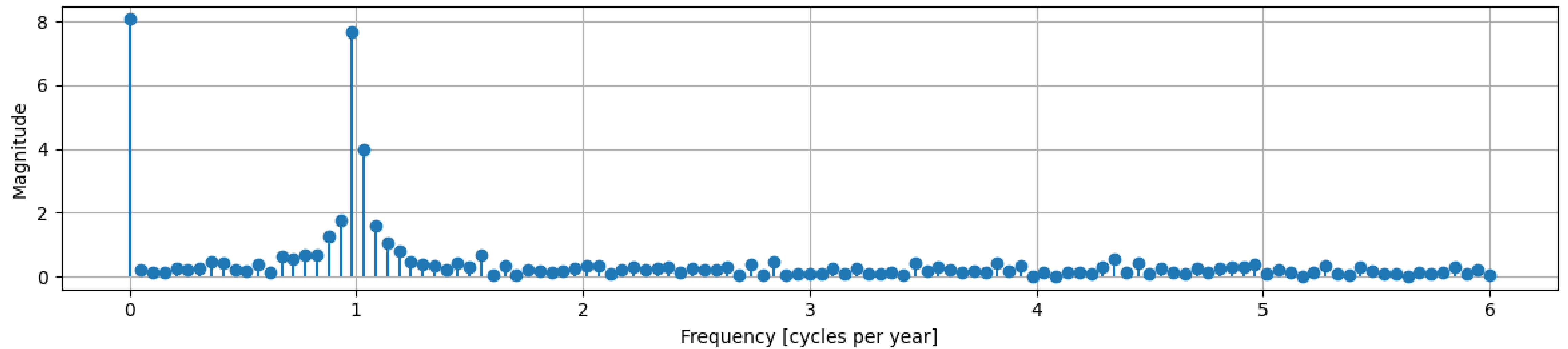




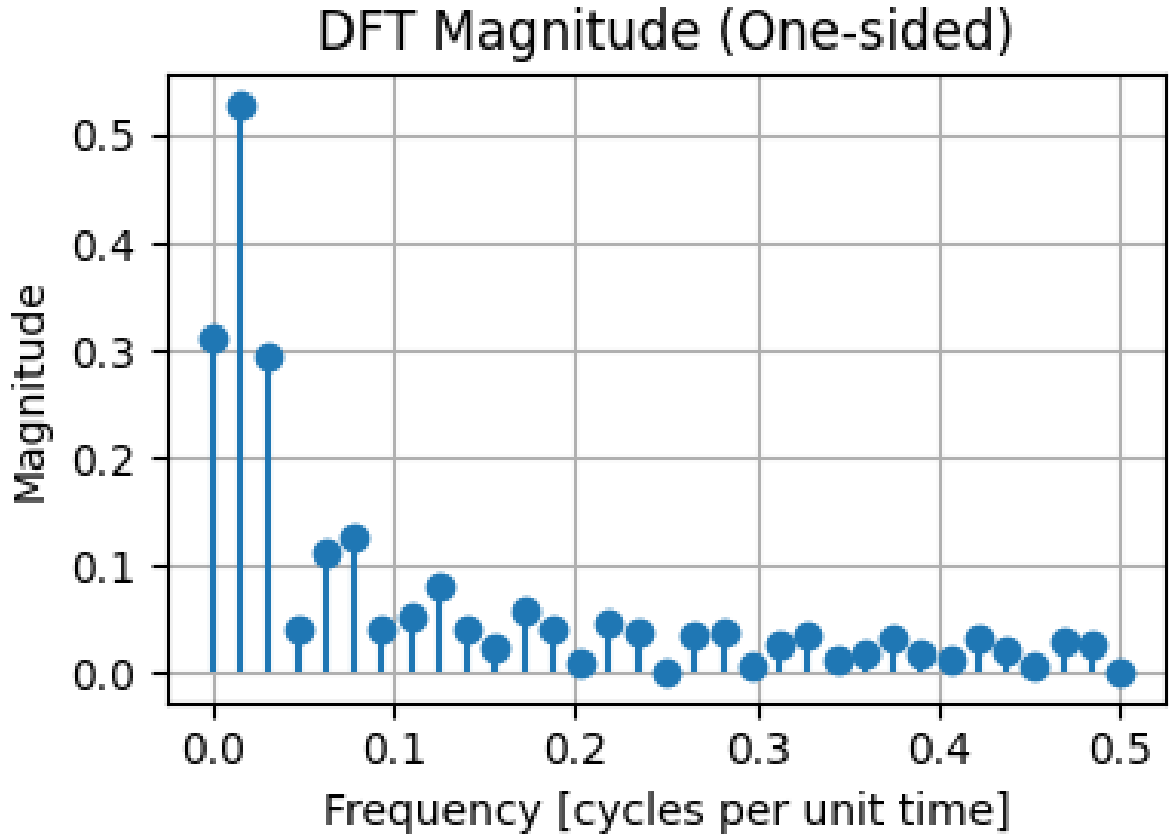
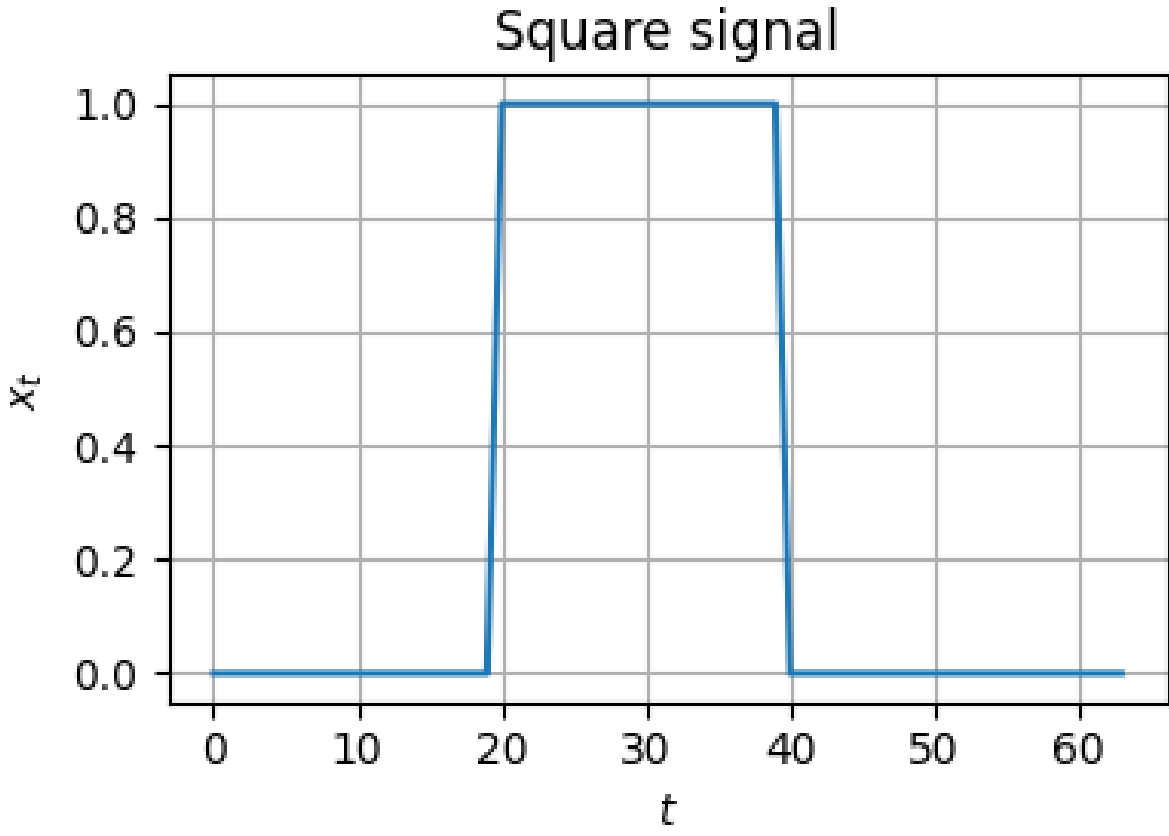
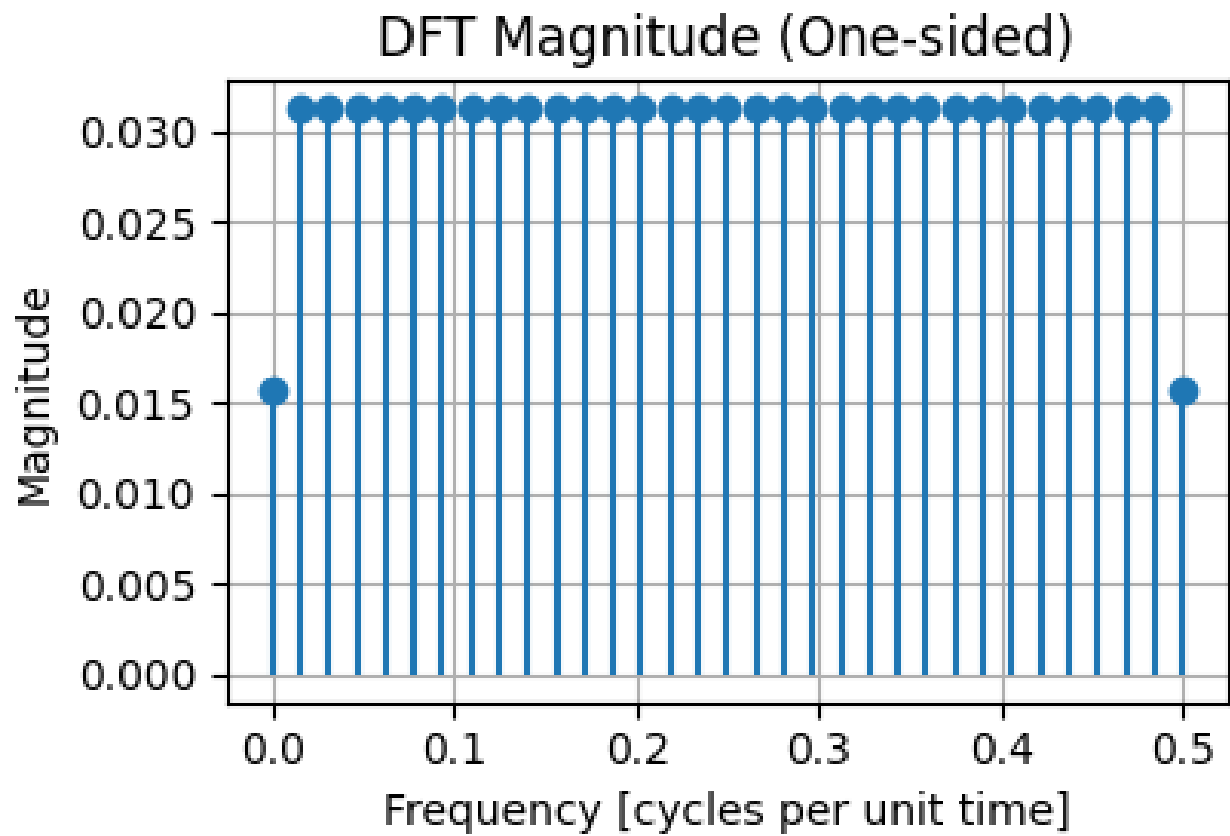
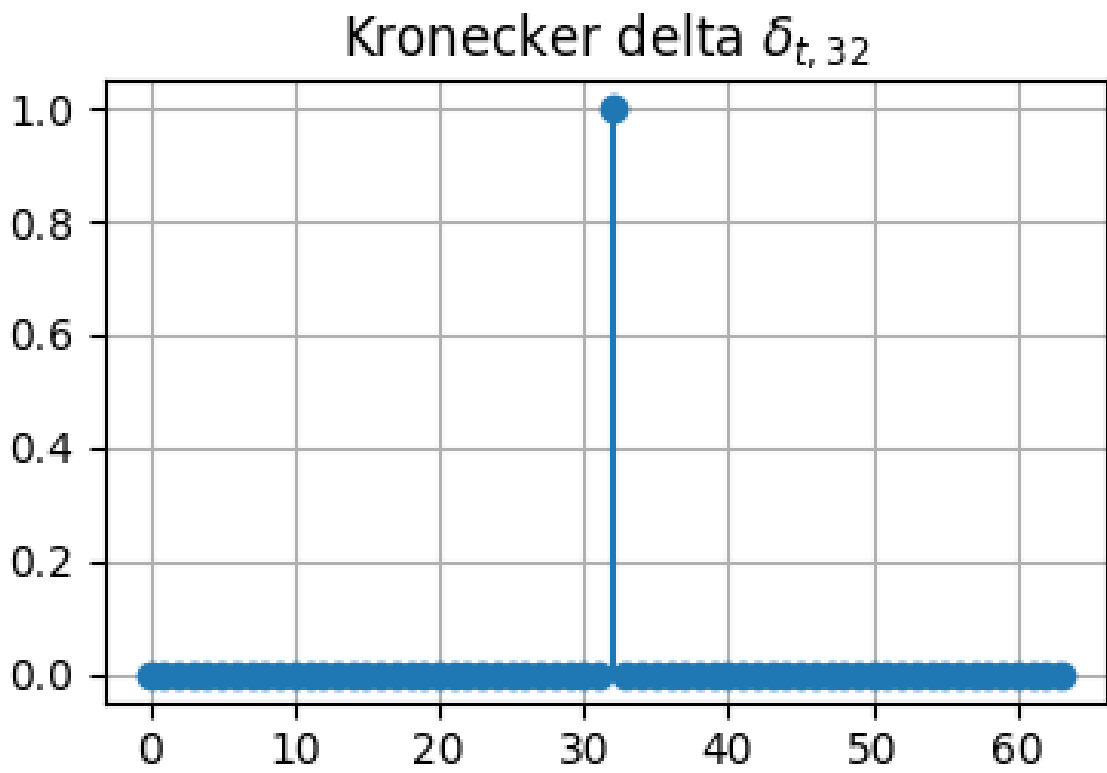
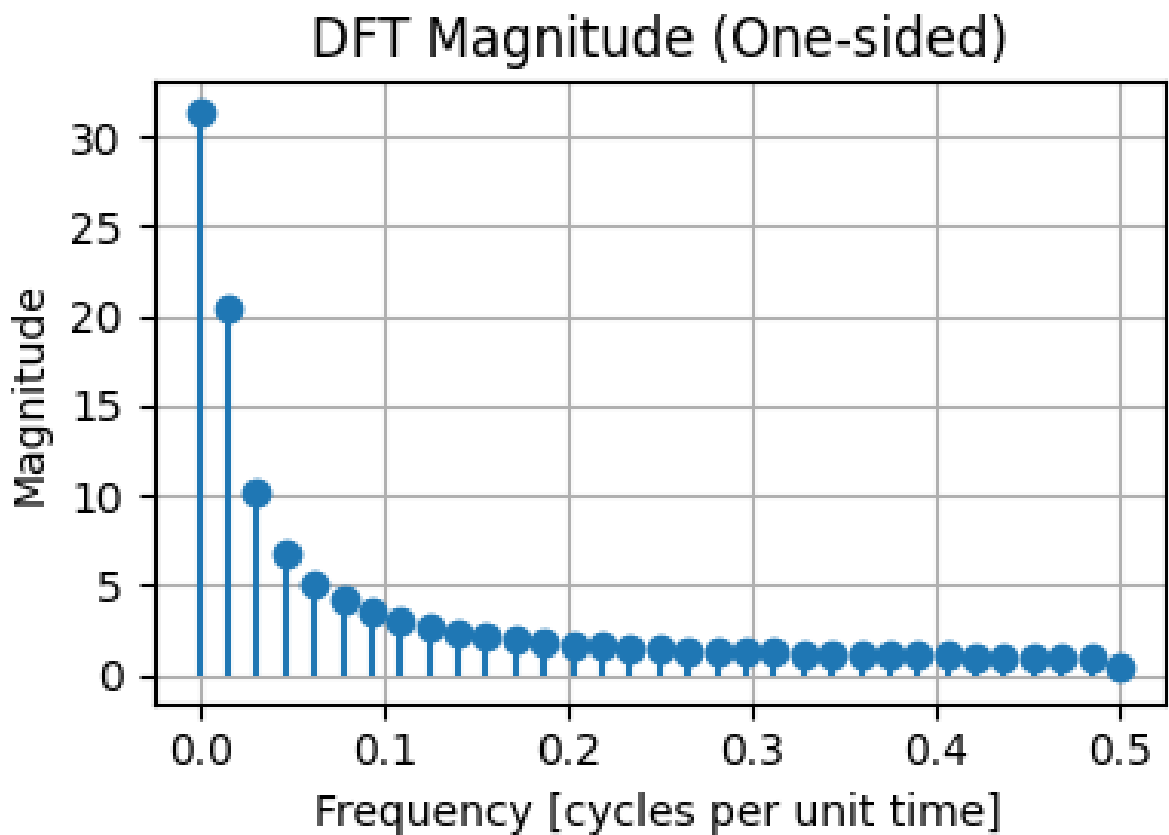
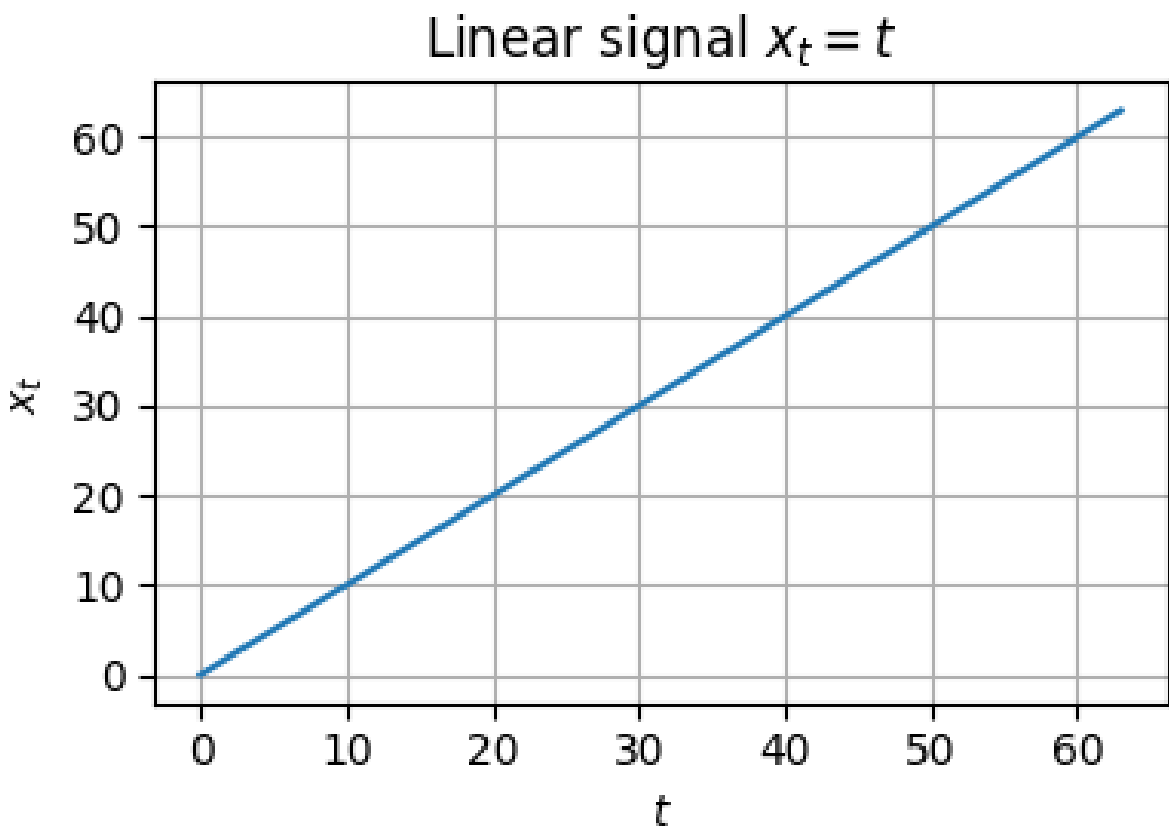
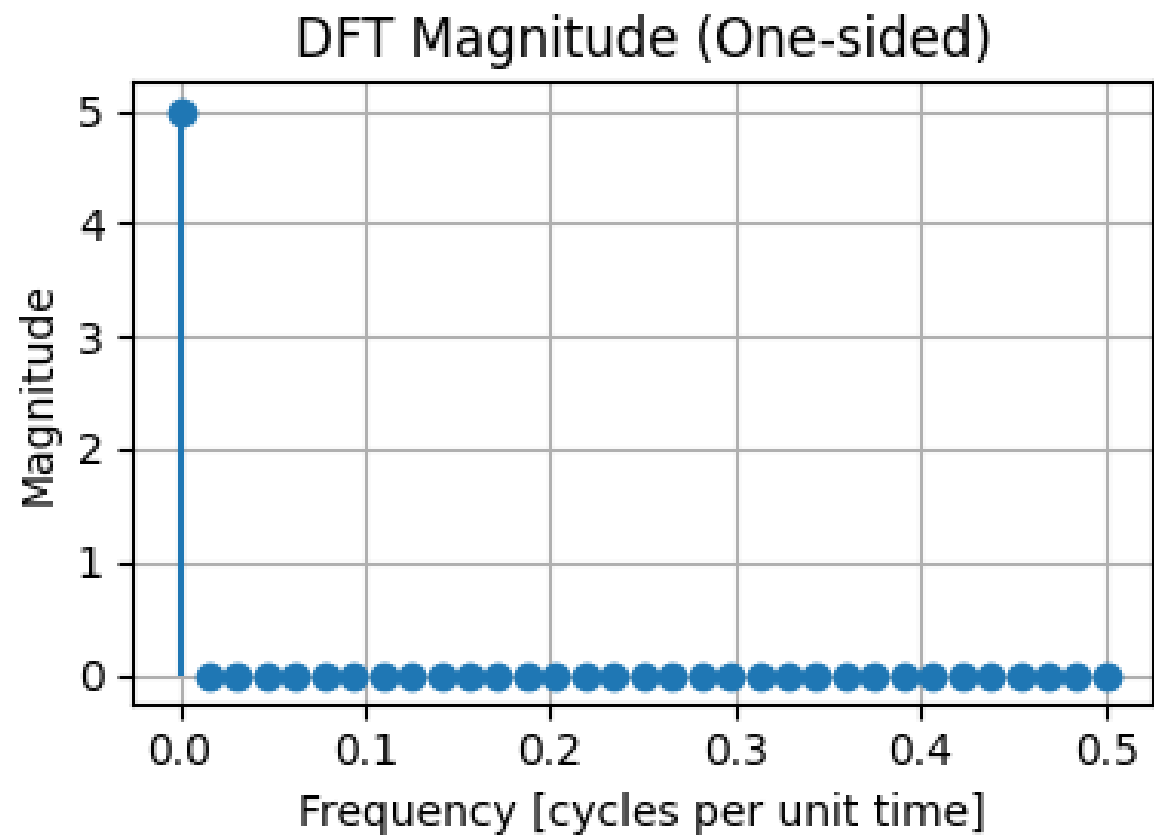
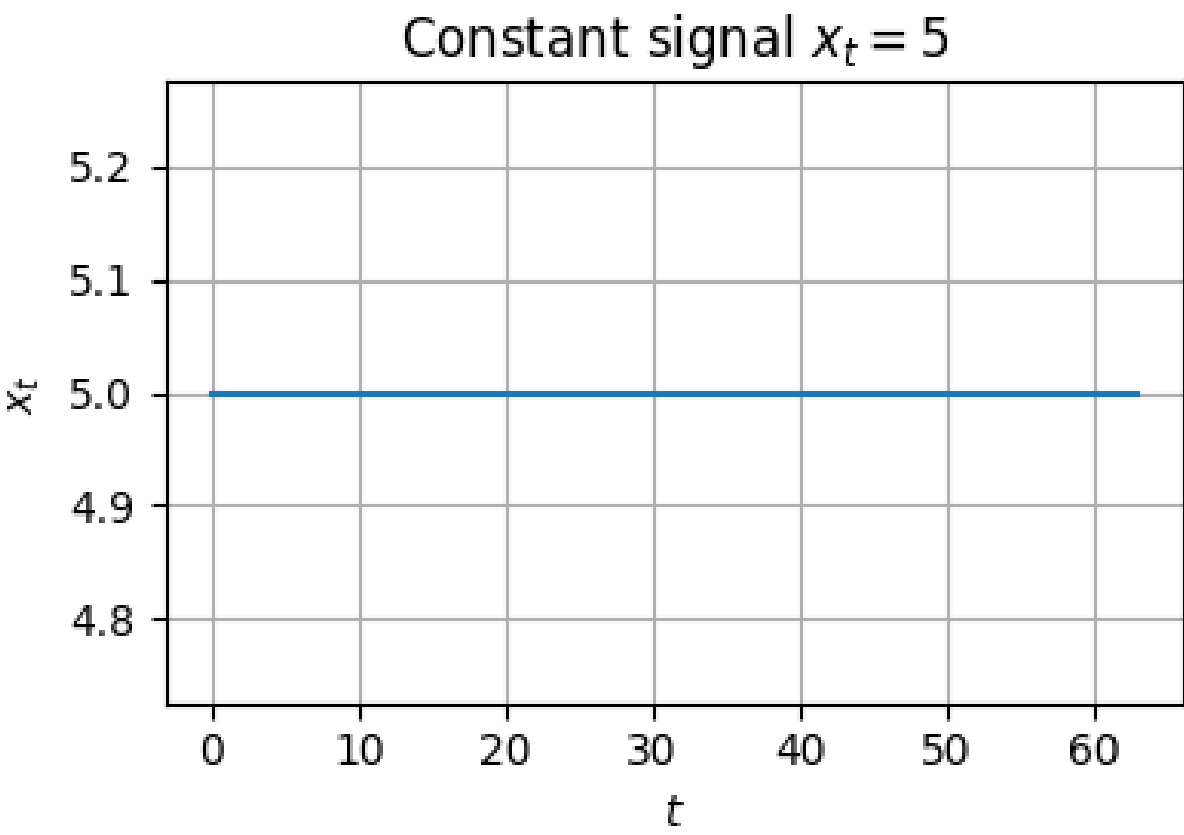
Monthly average temperature in Switzerland

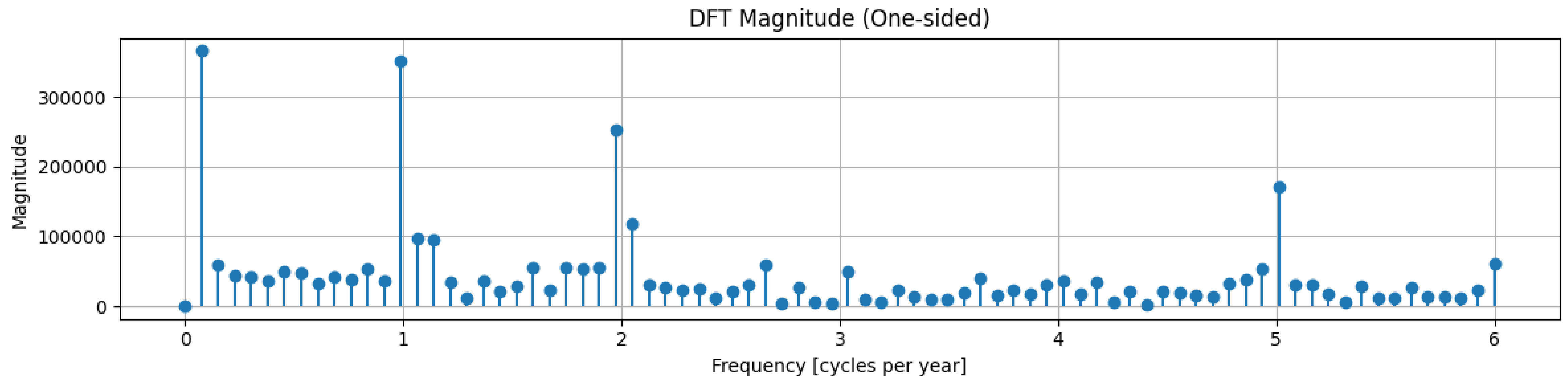
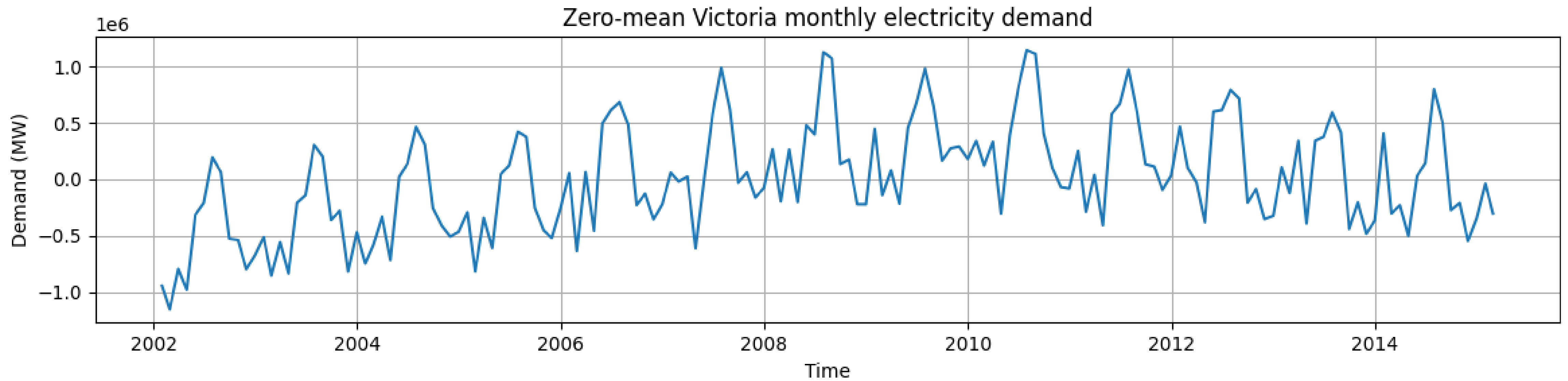


DFT Magnitude (One-sided)



# Discrete Fourier transform of simple signals





# Discrete Fourier transform – Connection with sinusoids

$$\begin{aligned}
 \mathcal{F}(\mathbf{x})_j = X_j &= \frac{1}{n} \sum_{t=0}^{n-1} x_t e^{-2\pi i f_j t} \\
 &= \frac{1}{n} \sum_{t=0}^{n-1} x_t (\cos(-2\pi f_j t) + i \sin(-2\pi f_j t)) \\
 &= \frac{1}{n} \sum_{t=0}^{n-1} x_t (\cos(2\pi f_j t) - i \sin(2\pi f_j t)) \\
 &= \frac{1}{n} \sum_{t=0}^{n-1} x_t (c_{t,j} - i s_{t,j}) \\
 &= \underbrace{\frac{1}{n} \sum_{t=0}^{n-1} x_t c_{t,j}}_{\text{Re}(X_j) = \frac{A_j}{2}} - i \underbrace{\frac{1}{n} \sum_{t=0}^{n-1} x_t s_{t,j}}_{\text{Im}(X_j) = \frac{B_j}{2}} \\
 &= \frac{A_j}{2} - i \frac{B_j}{2}
 \end{aligned}$$

*(recall that  $A_j = \frac{2}{n} \sum_{t=0}^{n-1} x_t c_{t,j}$ ,  $B_j = \frac{2}{n} \sum_{t=0}^{n-1} x_t s_{t,j}$ )*

*(observe that for real input, the sinusoid magnitude is split in half  $|X_j| = \frac{1}{2} \sqrt{A_j^2 + B_j^2} = |\overline{X_{n-j}}|$ )*

# Inverse discrete Fourier transform

Given  $\mathcal{F}(x) = \mathbf{X} = \{X_0, \dots, X_{n-1}\}$ , the **inverse DFT**  $\mathcal{F}^{-1}(\mathbf{X}) = \mathbf{x} = \{x_0, \dots, x_{n-1}\}$  is defined as

$$x_t = \mathcal{F}^{-1}(\mathbf{X})_t = \sum_{j=0}^{n-1} X_j e^{2\pi i f_j t}, \quad t = 0, 1, \dots, n-1$$

Proof:

$$\begin{aligned} \mathcal{F}^{-1}(\mathbf{X})_t &= \sum_{j=0}^{n-1} \left( \frac{1}{n} \sum_{k=0}^{n-1} x_k e^{-2\pi i f_j k} \right) e^{2\pi i f_j t} \\ &= \frac{1}{n} \sum_{j=0}^{n-1} \sum_{k=0}^{n-1} x_k e^{2\pi i f_j (t-k)} = \frac{1}{n} \sum_{j=0}^{n-1} \sum_{k=0}^{n-1} x_k e^{2\pi i \frac{j}{n} (t-k)} \end{aligned}$$

(orthogonality of complex exponentials)

$$\begin{aligned} &= \frac{1}{n} \sum_{k=0}^{n-1} x_k \underbrace{\sum_{j=0}^{n-1} e^{2\pi i \frac{j}{n} (t-k)}}_{= \delta_{kt} n} = \frac{1}{n} x_t n = x_t \end{aligned}$$

# Inverse discrete Fourier transform – Connection with sinusoids

Given a **real** TS realization  $x = \{x_0, \dots, x_{n-1}\}$ ,

$$\begin{aligned} x_t = \mathcal{F}^{-1}(\mathbf{X})_t &= \sum_{j=0}^{n-1} X_j e^{2\pi i f_j t} \\ &= X_0 + \sum_{j=1}^k (X_j e^{2\pi i f_j t} + X_{n-j} e^{2\pi i f_{n-j} t}) \\ &= X_0 + \sum_{j=1}^k (X_j e^{2\pi i f_j t} + \bar{X}_j e^{-2\pi i f_j t}) \end{aligned}$$

$$k = \begin{cases} n/2, & n \text{ is even} \\ (n-1)/2, & n \text{ is odd} \end{cases}$$

*(conjugate symmetry for real input)*

$$\begin{aligned} & \text{(recall that } X_j = \frac{A_j}{2} - i \frac{B_j}{2} \text{)} \\ & = \left( \frac{A_0}{2} - i \frac{B_0}{2} \right) + \sum_{j=1}^k \left( \left( \frac{A_j}{2} - i \frac{B_j}{2} \right) (c_{t,j} + i s_{t,j}) + \left( \frac{A_j}{2} + i \frac{B_j}{2} \right) (c_{t,j} - i s_{t,j}) \right) \\ & \text{(} B_0 = 0 \text{)} \\ & = \frac{A_0}{2} + \sum_{j=1}^k \left( \left( \frac{A_j}{2} c_{t,j} + \frac{A_j}{2} i s_{t,j} - i \frac{B_j}{2} c_{t,j} + \frac{B_j}{2} s_{t,j} \right) + \left( \frac{A_j}{2} c_{t,j} - \frac{A_j}{2} i s_{t,j} + i \frac{B_j}{2} c_{t,j} + \frac{B_j}{2} s_{t,j} \right) \right) \\ & \text{(} \mu = \frac{A_0}{2} \text{)} \\ & = \mu + \sum_{j=1}^k (A_j c_{t,j} + B_j s_{t,j}) \end{aligned}$$

# Some properties of the discrete Fourier transform

Given two **real** TS realizations  $\mathbf{x} = \{x_0, \dots, x_{n-1}\}$  and  $\mathbf{y} = \{y_0, \dots, y_{n-1}\}$ , their DFT  $\mathbf{X} = \{X_0, \dots, X_{n-1}\}$  and  $\mathbf{Y} = \{Y_0, \dots, Y_{n-1}\}$ ,

**Linearity:** let  $z_t = ax_t + by_t$  then  $Z_j = aX_j + bY_j$  for any scalars  $a, b$ .

## Shifting

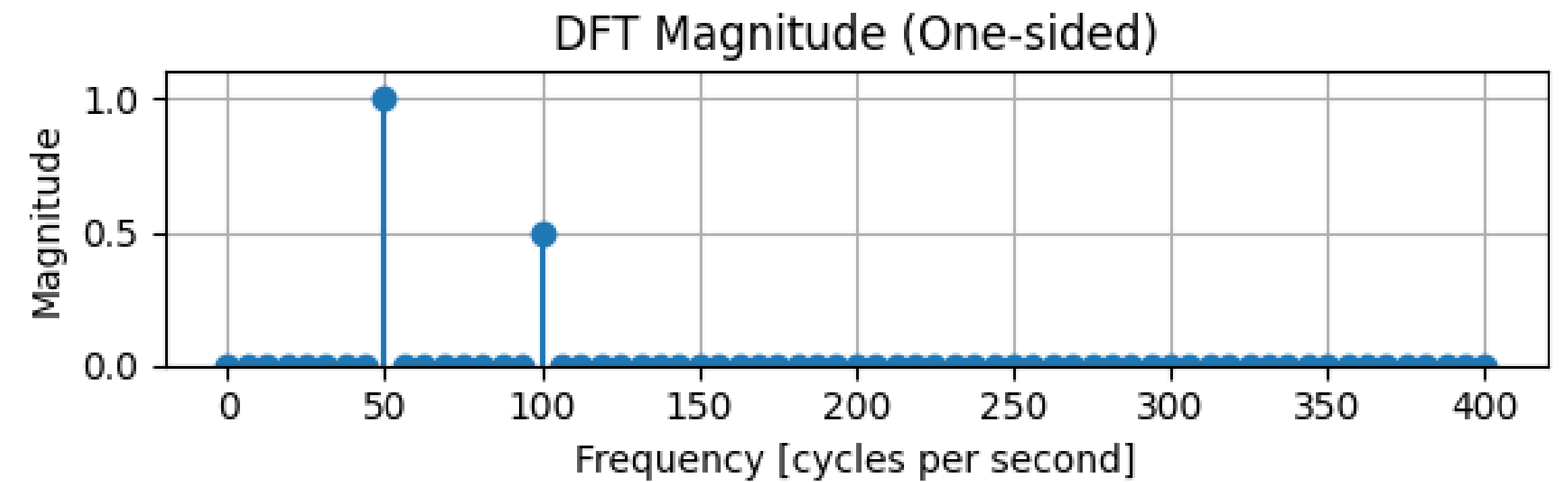
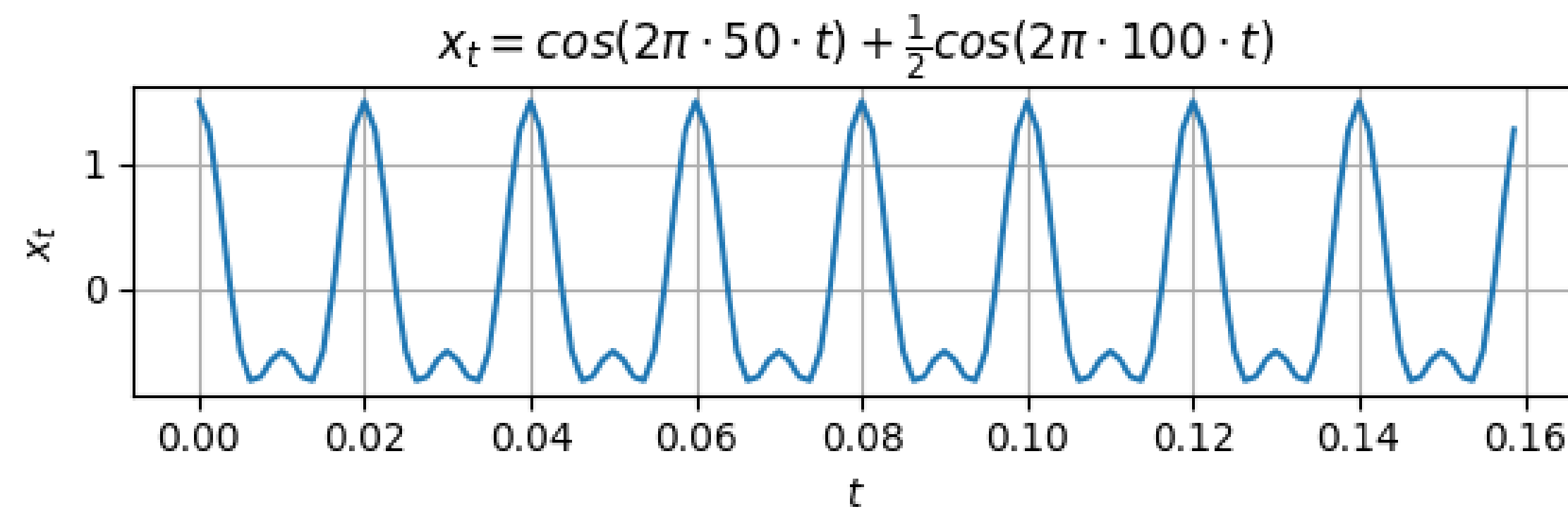
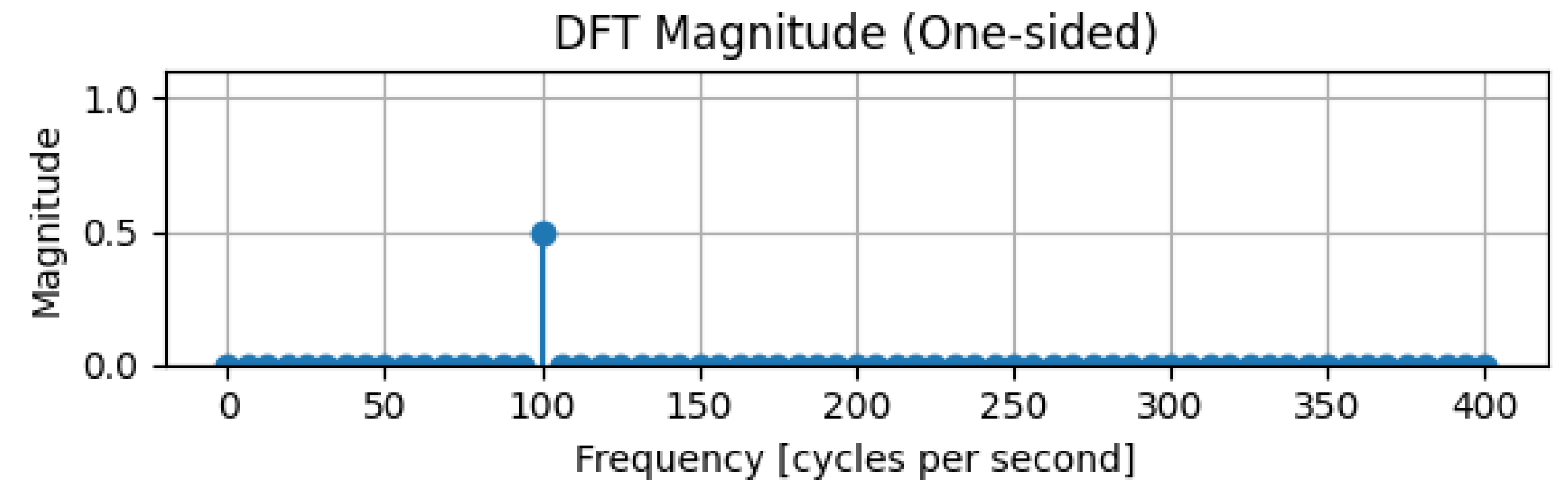
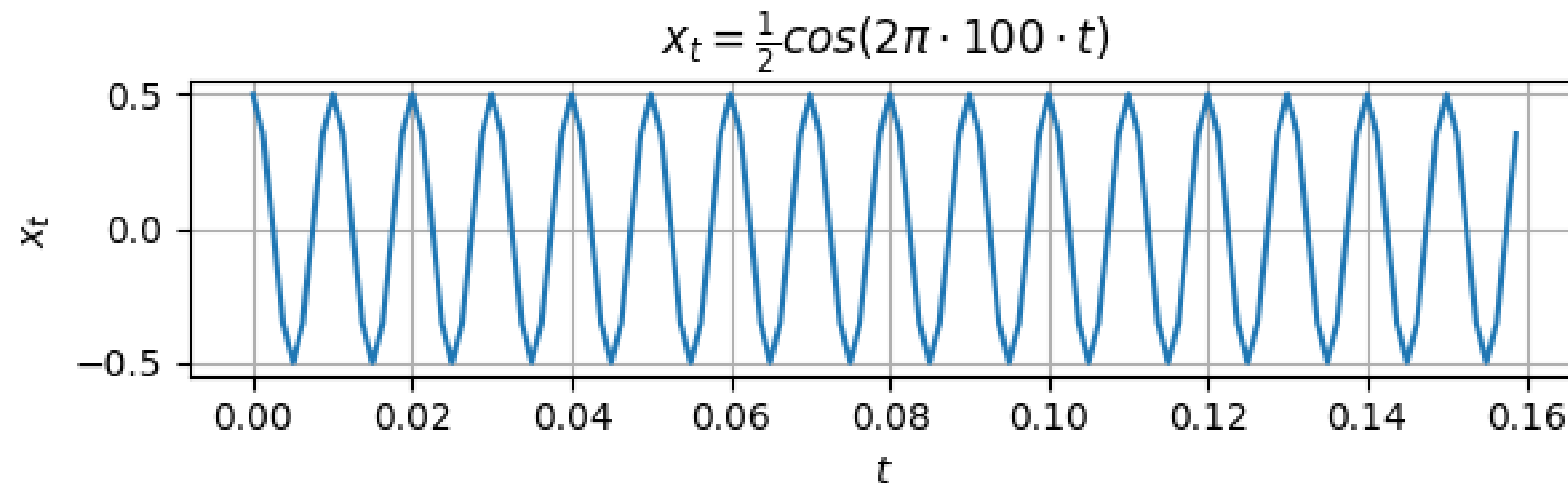
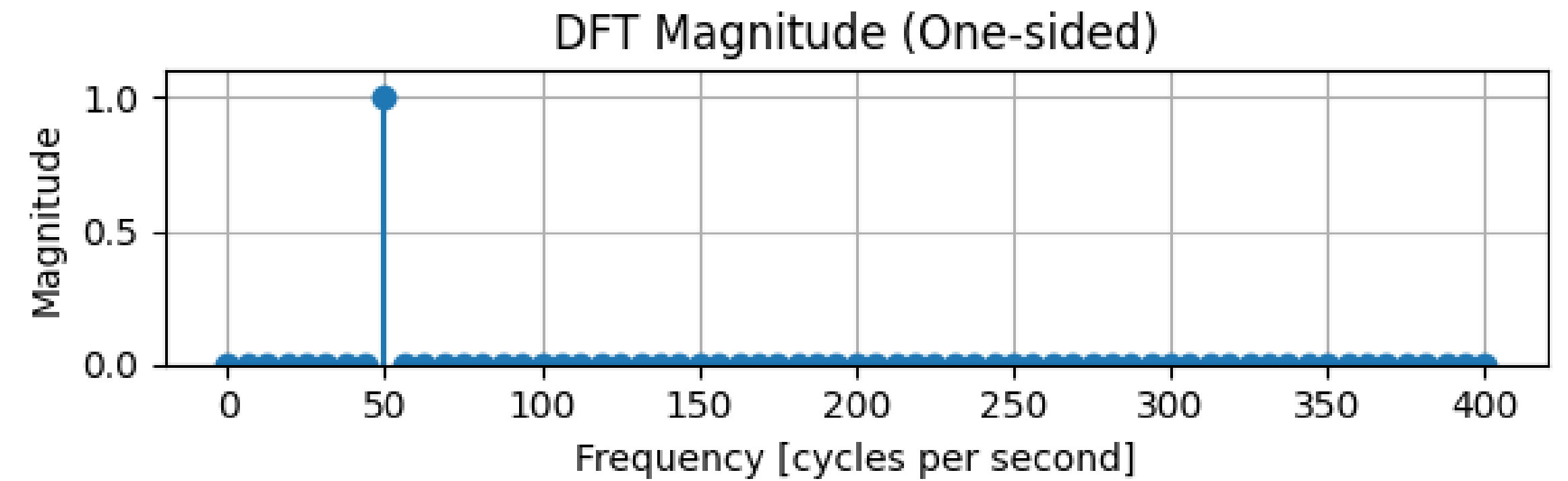
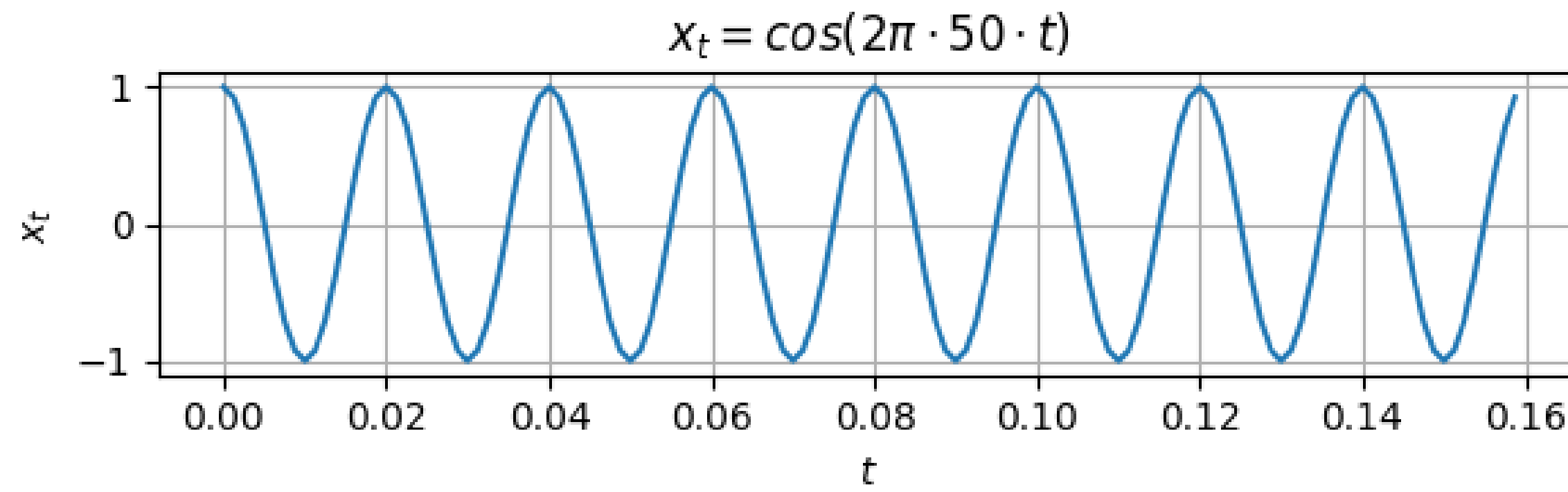
- Time domain (**lagging/leading TS**): let  $z_t = x_{t-m}$  then  $Z_j = X_j e^{-2\pi i f_j m}$
- Frequency domain (**phase shift**): let  $Z_j = X_{j-m}$  then  $Z_j = \mathcal{F}(x e^{2\pi i f_m t})$

**Convolution and multiplication:**  $\mathcal{F}(x \circledast y) = X \cdot Y$  and  $\mathcal{F}(x \cdot y) = X \circledast Y$

**Parseval's theorem:**  $\sum_{t=0}^{n-1} x_t \bar{y}_t = n \sum_{j=0}^{n-1} X_j \bar{Y}_j$

- Special case, the Plancherel theorem:  $\sum_{t=0}^{n-1} |x_t|^2 = n \sum_{j=0}^{n-1} |X_j|^2$  (recall that  $x_t \bar{x}_t = |x_t|^2$ )

**Discrete derivative:** let  $z_t = x_t - x_{t-1}$  then  $Z_j = X_j (1 - e^{-2\pi i f_j})$



# Fast Fourier transform (FFT)

Computing the DFT requires  $O(n^2)$  operations.

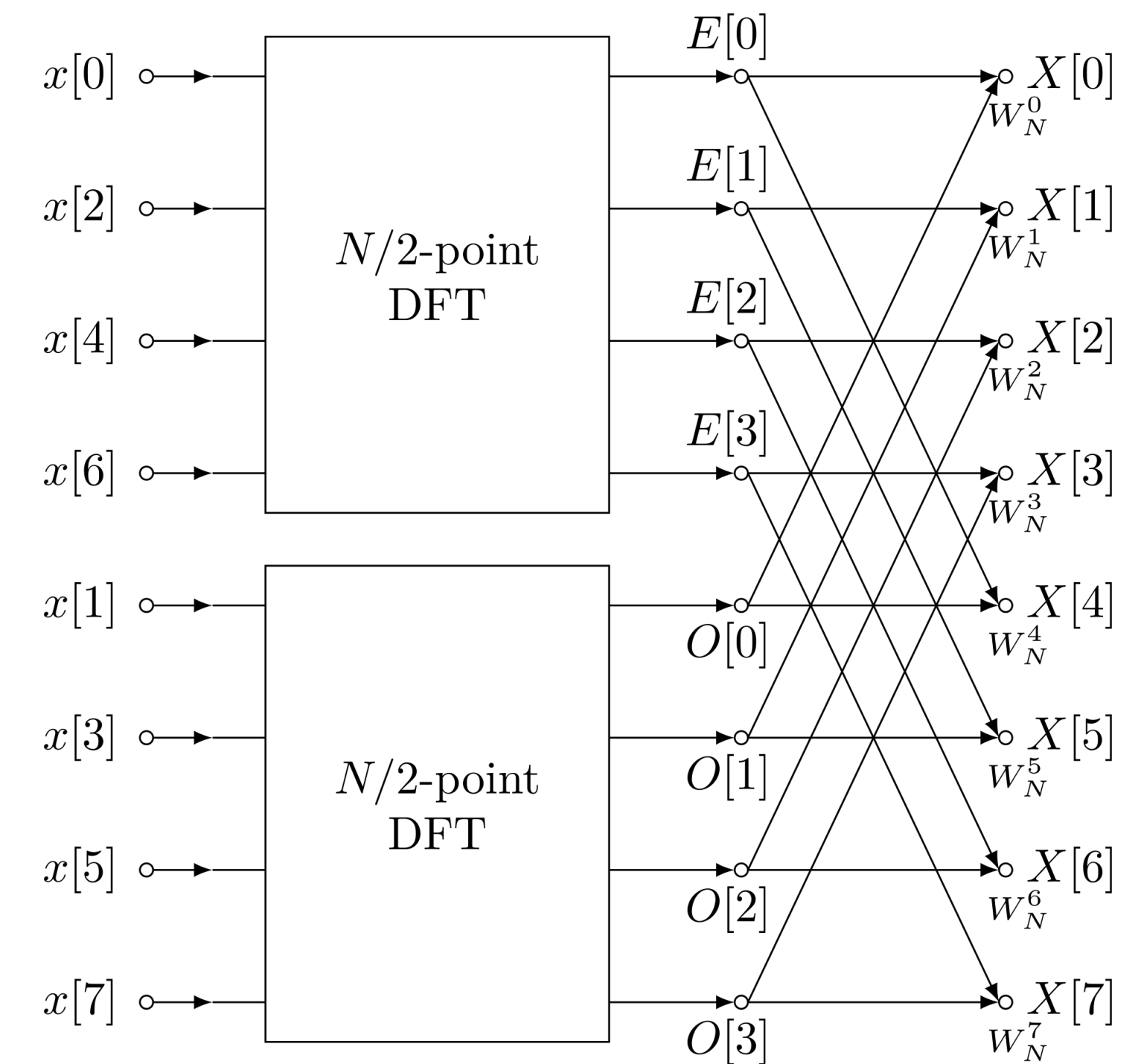
- Each of the  $n$  coefficients  $X_j$  (except for  $j = 0$ ) requires  $n - 1$  multiplications and  $n - 1$  additions.

Fast Fourier transform algorithms reduce this cost to  $O(n \log n)$  operations.

They adopt a **divide-and-conquer** strategy, and **recursively split** the DFT of size  $n = n_1 n_2$  into  $n_1$  smaller DFTs of size  $n_2$  along with  $O(n)$  multiplications by complex roots of unity (called twiddle factors).

- In practice, implementations avoid the recursion by rearranging the algorithm.

Typical implementations split into two DFTs of size  $n/2$  at each step, **requiring  $n$  to be a power of 2** (radix-2 FFT), but generally any factorization of  $n$  is possible.



Yangwenbo99, wikimedia

# Aliasing and the Nyquist frequency

Consider the signal  $x_t = \cos(2\pi f \Delta t)$  sampled at interval  $\Delta$  i.e.,  $x_t$  is observed at time  $\Delta t$ .

Increasing  $f$  from 0, the **fastest observable oscillation** is reached when  $f = \frac{1}{2\Delta}$  (called the **Nyquist frequency**)

- In this example,  $x_t = \cos(\pi t) = (-1)^t$ .

Higher frequencies are folded down into the interval  $[0, \frac{1}{2\Delta}]$ : taking  $f$  such that  $\frac{1}{2\Delta} < f < \frac{1}{\Delta}$ , and  $f' = \frac{1}{\Delta} - f$ , we get

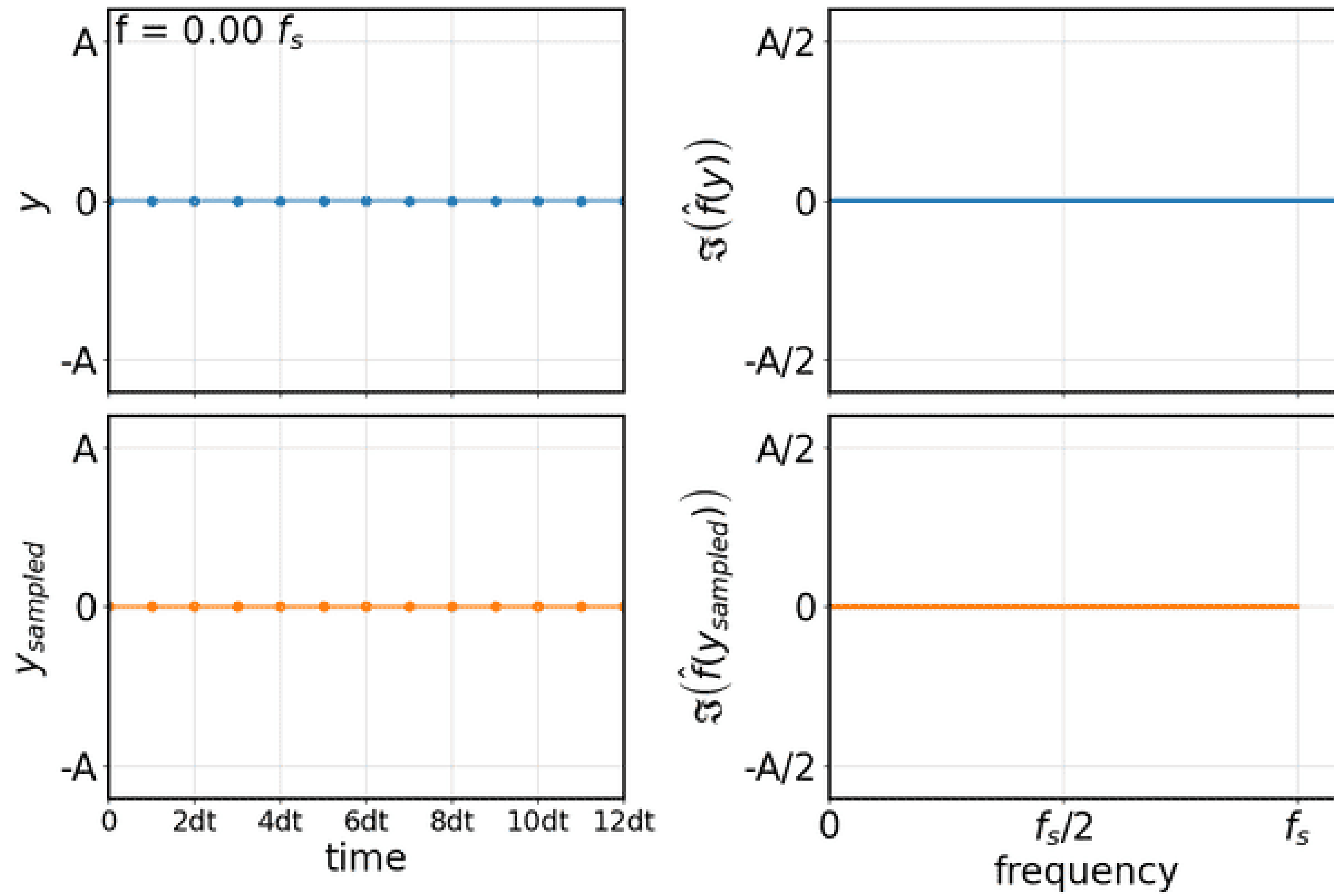
$$x_t = \cos(2\pi f \Delta t) = \cos\left(2\pi\left(\frac{1}{\Delta} - f'\right)\Delta t\right) = \cos(2\pi(t - f'\Delta t)) = \cos(2\pi f' \Delta t)$$

Then  $f$  and  $f'$  are indistinguishable, hence **aliases** of each other.

- For arbitrary  $f$  the observed frequency is  $f' = |f - k\Delta|$  with  $k$  chosen such that  $f' \leq \frac{1}{2\Delta}$

To capture a signal's frequency without aliasing, the sampling rate must be at least **twice** the signal's frequency (**Nyquist-Shannon sampling theorem**).

# Aliasing



Davidjessop, wikimedia

# Exercise

Review 3Blue1Brown article on the Fourier transform.

Simulate different synthetic signals (sinusoids, square waves, impulses, ...).

- Compute DFT and analyze frequencies.
- Test the different properties of the DFT (linearity, time-shifting, ...).
- Review the effect of different sampling rates on the DFT.

Model real-world time series.

- Fit sinusoids with known frequencies.
- Compute DFT and analyze dominant frequencies.
- Compare empirical results with your initial expectations.
- Review how the different DFT components contribute to the signal.