

# Inter-IIT Tech Meet Prep

## *Week 2: Graph Theory*

MathSoc IIT Delhi

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# 1 Graphs and Terminology

**Definition.** A (simple) graph  $G = (V, E)$  consists of a set of vertices  $V$  and a set of edges  $E$ , where each edge is an unordered pair  $\{u, v\}$  with  $u, v \in V$  and  $u \neq v$ .

Graphs can be:

- **Simple:** No loops (edges from a vertex to itself) and no multiple edges between the same pair of vertices.
- **Multigraphs:** Graphs that may include multiple edges between the same pair of vertices.
- **Loops:** Edges that connect a vertex to itself.
- **Directed Graphs (Digraphs):** Edges are ordered pairs  $(u, v)$ , indicating direction.
- **Weighted Graphs:** Edges carry a numerical weight or cost.
- **Degree:** The degree of a vertex is the number of edges incident to it.
- **Subgraph:** A graph formed from a subset of the vertices and edges of another graph.

Some common types of graphs:

- **Complete Graph  $K_n$ :** Every pair of  $n$  vertices is connected by an edge.
- **Cycle Graph  $C_n$ :** Vertices form a single cycle with  $n$  edges.

# 2 Graph Representations

Graphs can be represented as:

- **Adjacency Matrix:** An  $n \times n$  matrix where entry  $a_{ij}$  is 1 if there is an edge between  $i$  and  $j$ .
- **Adjacency List:** A list where each vertex stores a list of its adjacent vertices.

**Example.** For a graph with vertices  $\{1, 2, 3\}$  and edges  $\{(1, 2), (2, 1), (2, 3), (3, 2)\}$ , the adjacency list is:

$$1 \rightarrow [2], \quad 2 \rightarrow [1, 3], \quad 3 \rightarrow [2]$$

### 3 Trees and Forests

**Definition.** A **tree** is a connected graph with no cycles. A **forest** is a graph with no cycles; it may be disconnected and is simply a collection of trees.

**Theorem.** A graph is a tree  $\iff$  it is connected and has  $n - 1$  edges (where  $n$  is the number of vertices).

Trees appear frequently in algorithms (e.g., DFS, BFS, MSTs).

### 4 Paths and Walks

**Walk:** A sequence of vertices such that each pair of consecutive vertices is connected by an edge.

**Path:** A walk with no repeated vertices.

**Cycle:** A path that starts and ends at the same vertex.

**Theorem (Handshake Lemma).** The sum of degrees of all vertices in a graph is twice the number of edges:

$$\sum_{v \in V} \deg(v) = 2|E|$$

### 5 Connectedness and Components

**Definition.** A graph is connected if there exists a path between every pair of vertices.

A **connected component** is a maximal set of connected vertices.

### 6 Bipartite Graphs

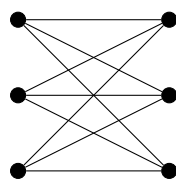
**Definition.** A graph is **bipartite** if its vertex set  $V$  can be divided into two disjoint sets  $A$  and  $B$  such that every edge connects a vertex from  $A$  to a vertex from  $B$ , and there are no edges between vertices within the same set.

An easy way to check if a graph is bipartite is to color it using 2 colors, such that no two nodes of the same color are adjacent (share a common edge).

**Example:** A tree is always bipartite.

**Theorem.** A graph is bipartite  $\iff$  it contains no odd-length cycles.

**Complete Bipartite Graph.** A complete bipartite graph  $K_{m,n}$  has  $m$  vertices in set  $A$  and  $n$  in set  $B$ , with every vertex in  $A$  connected to every vertex in  $B$ .



$K_{3,3}$

## 7 Eulerian Paths and Circuits

**Definition.** An *Eulerian Path* is a path that visits every edge exactly once.

An *Eulerian Circuit* is an Eulerian path that starts and ends at the same vertex.

**Theorem.** A connected graph has:

- An Eulerian circuit  $\iff$  all vertices have even degree.
- An Eulerian path but not a circuit  $\iff$  exactly two vertices have odd degree.

## 8 Hamiltonian Paths and Cycles

**Definition.** A *Hamiltonian Path* visits every vertex exactly once.

A *Hamiltonian Cycle* is a Hamiltonian path that starts and ends at the same vertex.

**Note.** Unlike Eulerian cycles, there is no simple necessary and sufficient condition for Hamiltonian cycles.

**Theorem (Dirac's Theorem).** If  $G$  is a simple graph with  $n \geq 3$  vertices and every vertex has degree  $\geq n/2$ , then  $G$  has a Hamiltonian cycle.

## 9 Planarity and Kuratowski's Theorem

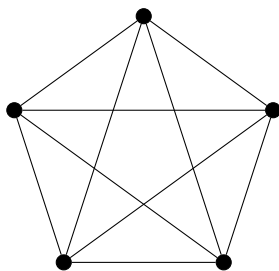
Some graphs can be drawn on paper such that no edges cross, these are called **planar graphs**.

**Definition.** A graph is **planar** if it can be drawn in the plane without any edges crossing.

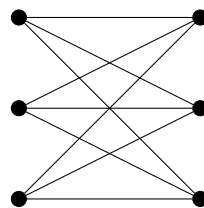
The complete graph on 5 vertices  $K_5$ , and the complete bipartite graph  $K_{3,3}$ , are **not** planar.

**Theorem (Kuratowski's Theorem).** A graph is non-planar if and only if it contains a subgraph that can be transformed into  $K_5$  or  $K_{3,3}$  by replacing edges with paths (called a subdivision).

These two graphs are the “forbidden minors” of planar graphs — spotting them inside a larger graph is a powerful way to show that planarity is impossible.



$K_5$



$K_{3,3}$

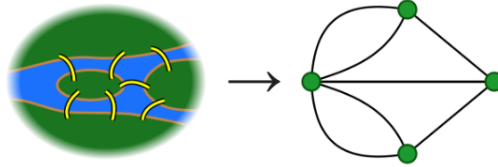
**Fun Fact:** The famous “three houses and three utilities” puzzle — where each house must connect to each utility without crossing lines — is actually just an attempt to draw  $K_{3,3}$  on the plane!

## 10 Königsberg Bridges

In the city of Königsberg, there were seven bridges connecting four land-masses. The challenge: can one walk through the city crossing each bridge exactly once?

**Hint:** Think of Eulerian Path

Euler showed the answer is **no**, and in doing so, laid the foundation of graph theory. Can you prove it?



## 11 This Week's Problem

### Problem 1

Prove that in any graph, the number of vertices with odd degree is even.

### Problem 2

Show that at any party, there are always at least two people with exactly the same number of friends at the party.

### Problem 3

There are  $2n$  people in a room where each person is enemies with at most  $n - 1$  people in the room. Prove that the  $2n$  people can sit at a circular table so that no two enemies are sitting next to each other.

### Problem 4

Prove or disprove: every connected graph with an Eulerian path has at most two vertices of odd degree.

### Problem 5

Let  $G$  be a graph with  $n$  vertices and more than  $\frac{(n-1)(n-2)}{2}$  edges. Prove that  $G$  is connected.

### Problem 6

Let  $G$  be a connected graph with all vertices of even degree. Prove that for any edge  $e \in G$ , there exists an Eulerian circuit containing  $e$ .

### **Problem 7 (Construction)**

Construct a simple graph with  $2k$  vertices where every vertex has degree  $k - 1$ , but the graph contains no Hamiltonian cycle.

### **Puzzle**

Given  $n$  vertices, how many edges must a graph have to guarantee it contains a triangle? (Hint: Explore Turán's theorem.)