INTER-IIT TECH MEET PREP

Week 3: Combinatorics

MathSoc, IIT Delhi

Contents

1 Four Basic Counting Principles

Definition 1.1 (Addition Principle). *If an event can occur in m ways and another event in n ways, and they cannot happen simultaneously, then the total number of ways is*

$$m + n.$$

Definition 1.2 (Multiplication Principle). *If an event can occur in m ways and, following that, another in n ways, then the total number of combined outcomes is*

$$m \times n.$$

Definition 1.3 (Subtraction Principle). *If the total number of outcomes is T and the number of undesired outcomes is U, then the number of desired outcomes is*

$$T - U.$$

Definition 1.4 (Division Principle). *If each outcome is counted k times due to symmetry or repetition, divide the total count by k to get the number of distinct outcomes.*

$$\frac{\textit{total count}}{k}.$$

2 Pigeonhole Principle

Theorem 2.1 (Basic Pigeonhole Principle). *If $n + 1$ or more objects are placed into n boxes, then at least one box contains two or more objects.*

Theorem 2.2 (Generalized Pigeonhole Principle). *If N objects are placed into k boxes, then at least one box contains at least*

$$\left\lceil \frac{N}{k} \right\rceil$$

objects.

Theorem 2.3. *Let $q_1, q_2, \ldots, q_n$ be positive integers. If*

$$q_1 + q_2 + \cdots + q_n - n + 1$$

objects are distributed into n boxes, then one box contains at least q_i objects.

3 Ramsey's Theorem

Theorem 3.1. *If $m \geq 2$ and $n \geq 2$ are integers, then there exists a minimum positive integer p such that*

$$K_p \to (K_m, K_n).$$

Definition 3.2 (Ramsey Number). *K_n is the complete graph on n vertices. The notation*

$$K_p \to (K_m, K_n)$$

*means that every red-blue coloring of the edges of the complete graph K_p contains either a red K_m or a blue K_n subgraph. The smallest such p is called the **Ramsey number** $R(m, n)$.*

4 Generating Permutations

Definition 4.1. *For n distinct objects, the total number of permutations is*

$$n!.$$

If objects are repeated, with multiplicities $n_1, n_2, \ldots, n_k$, the number of distinct permutations is:

$$\frac{n!}{n_1! \cdot n_2! \cdots n_k!}.$$

5 Pascal's Triangle and Binomial Theorem

Definition 5.1 (Pascal's Triangle). *A triangular array where each entry is the sum of the two directly above it:*

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}.$$

Theorem 5.2 (Binomial Theorem). *For a non-negative integer n,*

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}.$$

Also,

$$\binom{n}{k} = \binom{n}{n-k}.$$

Definition 5.3 (Unimodal Sequence). *The binomial coefficients form a unimodal sequence:*

$$\binom{n}{0} < \binom{n}{1} < \cdots < \binom{n}{\lfloor n/2 \rfloor} > \cdots > \binom{n}{n}.$$

The maximum value occurs at $k = \lfloor n/2 \rfloor$.

Definition 5.4 (Extended Binomial Theorem). *For a negative integer $-n$, where n is positive,*

$$(1+x)^{-n} = \sum_{k=0}^{\infty} (-1)^k \binom{n+k-1}{k} x^k, \quad |x| < 1.$$

Definition 5.5 (Multinomial Theorem).

$$(x_1 + x_2 + \cdots + x_t)^n = \sum \binom{n}{n_1, n_2, \ldots, n_t} x_1^{n_1} x_2^{n_2} \cdots x_t^{n_t},$$

where the sum is over all nonnegative integer solutions to

$$n_1 + n_2 + \cdots + n_t = n.$$

Definition 5.6 (Catalan Numbers).

$$C_n = \frac{1}{n+1}\binom{2n}{n} = \binom{2n}{n} - \binom{2n}{n+1}.$$

6 Inclusion-Exclusion Principle and Derangements

Theorem 6.1 (Inclusion-Exclusion Principle). *The number of elements in a set S having none of the properties $P_1, P_2, \ldots, P_m$ is:*

$$|A_1^c \cap A_2^c \cap \cdots \cap A_m^c| = |S| - \sum |A_i| + \sum |A_i \cap A_j| - \sum |A_i \cap A_j \cap A_k| \\ + \cdots + (-1)^m |A_1 \cap A_2 \cap \cdots \cap A_m|,$$

where the sums are taken over all 1-, 2-, ..., *m-subsets of* $\{1, 2, ..., m\}$.

Definition 6.2 (Derangement). *A derangement is a permutation with no fixed points.*

The number of derangements of n objects is

$$D_n = n! \left(1 - \frac{1}{1!} + \frac{1}{2!} - \cdots + \frac{(-1)^n}{n!}\right),$$

and satisfies the recurrence

$$D_n = nD_{n-1} + (-1)^n.$$

7 Generating Functions

Definition 7.1. *For a sequence $\{a_n\}$, the generating function $G(x)$ is defined as*

$$G(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots .$$

Example 7.2. ***Problem:*** *Find the generating function for the number of n-combinations of apples, bananas, oranges, and pears, where:*

- *Number of apples is even,*
- *Number of bananas is odd,*
- *Number of oranges is between 0 and 4,*
- *Number of pears is at least 1.*

Let e_1, e_2, e_3, e_4 represent the counts respectively, satisfying

$$e_1 + e_2 + e_3 + e_4 = n.$$

Generating function:

$$g(x) = (1 + x^2 + x^4 + \cdots)(x + x^3 + x^5 + \cdots)(1 + x + x^2 + x^3 + x^4)(x + x^2 + x^3 + \cdots).$$

Simplify each factor:

$$1 + x^2 + x^4 + \cdots = \frac{1}{1 - x^2}, \quad x + x^3 + x^5 + \cdots = \frac{x}{1 - x^2},$$

$$1 + x + x^2 + x^3 + x^4 = \frac{1 - x^5}{1 - x}, \quad x + x^2 + x^3 + \cdots = \frac{x}{1 - x}.$$

Thus,

$$g(x) = \frac{1}{1 - x^2} \cdot \frac{x}{1 - x^2} \cdot \frac{1 - x^5}{1 - x} \cdot \frac{x}{1 - x} = \frac{x^2(1 - x^5)}{(1 - x^2)^2 (1 - x)^2}.$$

Therefore, the coefficients in the Taylor series for this rational function count the number of combinations of the type considered.

8 Fibonacci Numbers and Recurrence Relations

Definition 8.1 (Fibonacci Numbers)**.** *Defined by the recurrence:*

$$f_n = f_{n-1} + f_{n-2}, \quad f_1 = 1, \quad f_2 = 1.$$

Theorem 8.2 (Closed Form for Fibonacci Numbers)**.**

$$f_n = \frac{\phi^n - \psi^n}{\sqrt{5}},$$

where

$$\phi = \frac{1+\sqrt{5}}{2}, \quad \psi = \frac{1-\sqrt{5}}{2}.$$

Definition 8.3 (General Linear Recurrence)**.**

$$a_n = r_1 a_{n-1} + r_2 a_{n-2} + \cdots + r_k a_{n-k}.$$

Theorem 8.4 (Solving Linear Recurrences)**.** *Solve the characteristic polynomial:*

$$x^k - r_1 x^{k-1} - r_2 x^{k-2} - \cdots - r_k = 0,$$

with roots $r_1, r_2, \ldots, r_k$. Then

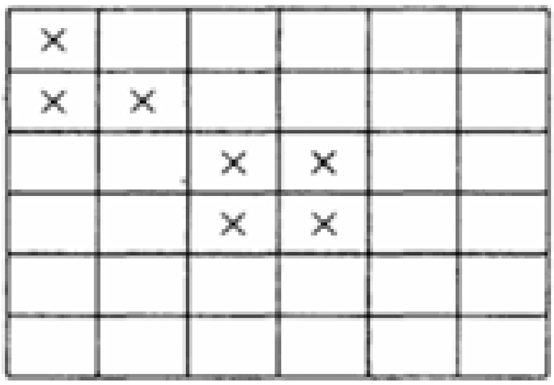
$$a_n = \sum_{i=1}^{k} c_i r_i^n,$$

where c_i are constants determined by initial conditions.

9 Problems

Problem 1: Nonattacking Rooks

Determine the number of ways to place six nonattacking rooks on the following 6-by-6 board, with forbidden positions as shown.



Problem 2: Lawyer's Route

A big city lawyer works n blocks north and n blocks east of her place of residence. Every day she walks $2n$ blocks to work.

How many routes are possible if she never crosses (but may touch) the diagonal line from home to office?

Problem 3: Towers of Hanoi

There are three pegs and n circular disks of increasing size on one peg, with the largest disk on the bottom. These disks are to be transferred, one at a time, onto another of the pegs, with the provision that at no time is one allowed to place a larger disk on top of a smaller one.

Determine the number of moves necessary for the transfer.

Problem 4: Divisibility Among Integers

Show that if 100 integers are chosen from $1, 2, \ldots, 200$, and one of the integers chosen is less than 16, then there are two chosen numbers such that one of them is divisible by the other.