Advanced Macroeconomics II: Assignment

by Sannah Tijani & Fynn Lohre April 25, 2023

We hereby declare that the answers to the given assignment are entirely our own, resulting from our own work effort only. Our team members contributed to the answers of the assignment in the following proportions:

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Contents

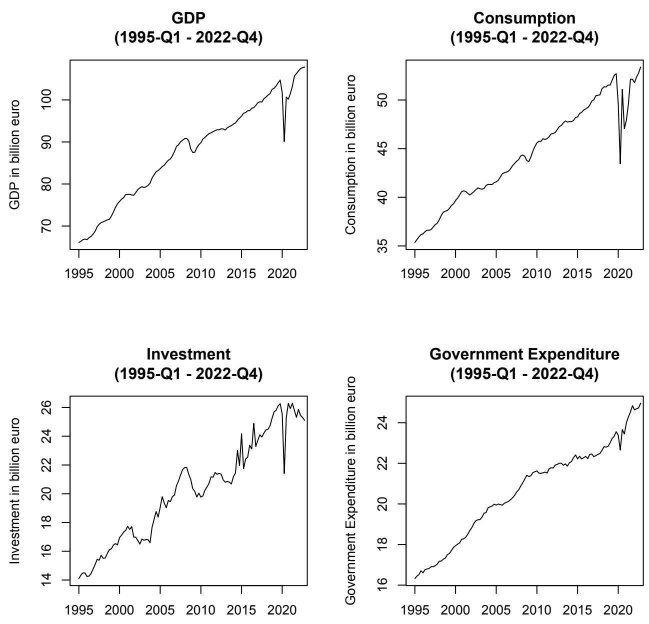
Problem 1

[...] [Based on EuroStat Data]

- Construct and report time series plots of the original raw (growing) data for Y, C, I, and G. Report sample means of C/Y, I/Y, G/Y.
- Obtain the stationary (cyclical) component of these macroeconomic time series through a method of your choice (e.g., by HP-filtering the logged time series for Y, C, I, G). Construct and report time series plots of these cyclical components.
- For the full length of your downloaded data, construct and report a summary table on business cycle stylized facts (standard deviations, relative standard deviations and contemporaneous output correlations) of the cyclical component of these macroeconomic time series. Now do the same for the following two subsample splits: start of your data until 2007Q4, 2008Q1 until end. Discuss and interpret your results!

The following results are based on: **Belgium**[1]
Respective additional files: R_Script.R, Data_Basis_2.xlsx

Time Series Plots



Note: GDP (Y) is "Gross domestic product at market prices"; Consumption (C) is "Final consumption expenditure of households"; Investment (I) is "Gross fixed capital formation" & Government Expenditure (G) is "Final consumption expenditure of general government" - The data is measured in chain linked volumes (2010), million euro as well as seasonally and calendar adjusted.

Figure 1: Key Timeseries of Beligum (1995 - 2022)

[1] Data from 2020+ is "provisional", due to accounting changes (see ESA 1995) reported data starts with 1995

Table 1: Sample Means Belgium

Variable	Mean
C/Y	0.5080
I/Y	0.2294
G/Y	0.2359

Cyclical Component

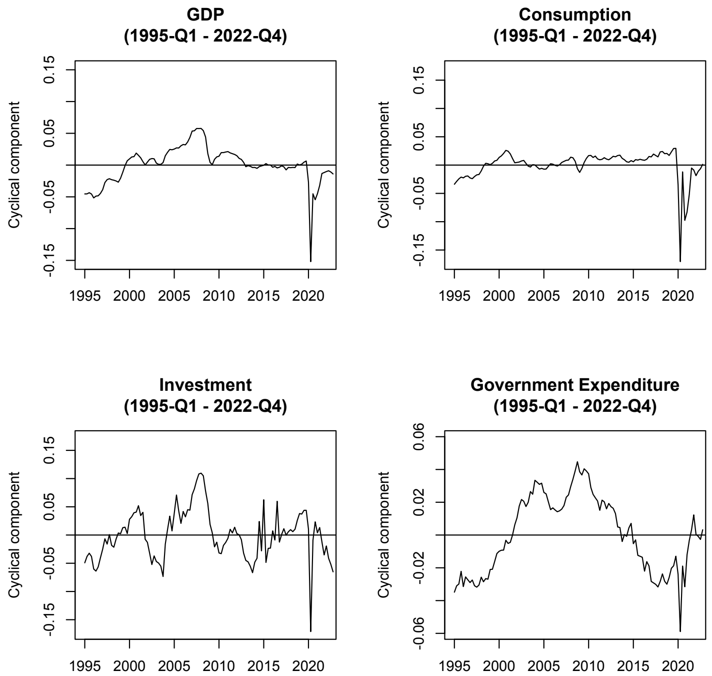


Figure 2: Cyclical Component obtained by Deterministic Linear Detrending of Logarithmized Variables

Summary Business Cycle Stylized Facts

Table 2: Business Cycle Statistics for Belgium - 1995-Q1 - 2022-Q4

Variable	Standard Deviation	Relative Standard Deviation	Contemporaneous Output Correlation
Y	0.0289	1.0000	1.0000
C	0.0252	0.8715	0.6992
I	0.0439	1.5175	0.6800
G	0.0234	0.8079	0.7204

Note: All variables are in logarithms and have been detrended by deterministic linear detrending.

Table 3: Business Cycle Statistics for Belgium - 1995-Q1 - 2007-Q4

Variable	Standard Deviation	Relative Standard Deviation	Contemporaneous Output Correlation
Y	0.0095	1.0000	1.0000
C	0.0118	1.2359	0.6372
I	0.0343	3.5995	0.7210
G	0.0095	1.0002	-0.3272

Note: All variables are in logarithms and have been detrended by deterministic linear detrending.

Table 4: Business Cycle Statistics for Belgium - 2008-Q1 - 2022-Q4

Variable	Standard Deviation	Relative Standard Deviation	Contemporaneous Output Correlation
Y	0.0214	1.0000	1.0000
C	0.0301	1.4021	0.8243
I	0.0426	1.9826	0.5858
G	0.0153	0.7133	0.3094

Note: All variables are in logarithms and have been detrended by deterministic linear detrending.

Discussion and Interpretation:

The first table shows that the GDP of Belgium is primarily composed of consumption, which accounts for 50 percent. This is followed by government expenditure, and finally by investment. They both account approximatively for 23 % of GDP. Moreover, we observed that the ratio does not add to one which is probably due to the missing part of net export.

The second table reports the business cycle statistics for Belgium over the full sample period from 1995-Q1 to 2022-Q4. We find that the standard deviation of output (Y) is 0.029, indicating moderate volatility of the business cycle. The relative standard deviations of consumption (C) and government spending (G) are lower than that of output, at 0.871 and 0.808, respectively, suggesting that these variables are less volatile than output. Investment (I) has the highest relative standard deviation at 1.518, indicating that it is more volatile than output. The contemporaneous output correlation of consumption and government spending with output is 0.699 and 0.720, respectively, while investment has a somewhat lower contemporaneous correlation of 0.680. We also observed that the three variables are positively correlated with output.

The third and fourth tables report the business cycle statistics for Belgium in two sub-periods: 1995-Q1 to 2007-Q4, and 2008-Q1 to 2022-Q4. We find that the standard deviation of output in the first sub-period is much lower at 0.0095 than in the full sample period, indicating a period of relative stability. Consumption has a higher relative standard deviation in the first sub-period at 1.236 than in the full sample period, while investment has a much higher relative standard deviation at 3.600, indicating a highly volatile period for investment. Government spending has a negative contemporaneous correlation with output in the first sub-period, indicating a countercyclical relationship. We notice that all variables became more volatile following the subprime crisis, this increase in volatility primarily impacted the output and consumption. This can be interpreted as an increased uncertainty in the economy following 2007 Q4.

The impact of the financial crisis can be seen in the second sub-period. In the period after the financial crisis (2008-2022), the standard deviation of output increased, reflecting the increased volatility and uncertainty in the economy. This increase in volatility was also reflected in the higher relative standard deviation of consumption, indicating it was more sensitive to changes in the business cycle. In contrast, the relative standard deviation of government spending declined in the post-crisis period. This could be due to government intervention in the economy, such as fiscal stimulus measures or changes in public spending priorities, which helped to stabilize the economy and reduce its sensitivity to changes in the business cycle. The contemporaneous output correlation of consumption and government spending with output also increased in the post-crisis period. This could be due to the fact that these variables are more closely tied to overall economic activity, as consumers and governments may be more cautious in their spending during times of economic uncertainty.

Overall, the business cycle statistics suggest that the Belgian economy has experienced moderate volatility in output and consumption, but higher volatility in investment, with some countercyclicality in government spending. It's interesting to see after this assignment if this countercyclicality was Belgian specific or the general "zeitgeist".

Problem 2

[...]

- Set up the social planner's intertemporal optimization problem and, step by step, solve for the first order optimality conditions. List the variables whose optimal time paths are described by the optimality conditions and be sure you end up with as many equations (label them) as you have variables in your system. To define investment, realize it is governed by the capital law of motion. If you substituted out for Lagrange multiplier λ_t, you should, this way, arrive at a system of 8 equations (forming a nonlinear expectational system of difference equations) in 8 variables: Y_t, C_t, I_t, K_t, N_t, EN_t, A_t, P_t.
- Provide an economic interpretation for each of your first order and equilibrium conditions (that is, explain their meaning in words).

Equations

Baseline

$$Max(U) = Max(E_0 \sum_{t=0}^{\infty} B^t \frac{C_t^{1-\sigma} - 1}{1-\sigma} + \theta log(1-N_t))^{2} \tag{1}$$

Constraints:

$$C_t + K_{t+1} + P_t EN_t \leq A_t K_t^{\alpha} N_t^{\gamma} EN_t^{1-\alpha-\gamma} + (1-\delta)K_t \tag{2}$$

$$C_t, K_{t+1} \leq 0 \tag{3}$$

Optimization problem:

$$L = E_0 \sum_{t=0}^{\infty} B^t [\frac{C_t^{1-\sigma} - 1}{1-\sigma} + \theta log(1-N_t) + \lambda_t (A_t K_t^{\alpha} N_t^{\gamma} EN_t^{1-\alpha-\gamma} + (1-\delta)K_t - C_t - K_{t+1} - P_t EN_t]) \tag{4}$$

[2] The handwritten solutions are in the appendix.

FOC

W.r.t. C_t: (Economic Interpretation)

$$C_t^{-\sigma} = \lambda_t \tag{5}$$

W.r.t. N_t: (Economic Interpretation)

$$\frac{\theta}{(1-N_t)(\gamma A_t K_t^{\alpha} N_t^{\gamma-1} EN_t^{1-\alpha-\gamma})} = \lambda_t \tag{6}$$

W.r.t. K_{t+1}: (Economic Interpretation)

$$BE_t[\lambda_{t+1}(\alpha A_{t+1} K_{t+1}^{\alpha-1} N_{t+1}^{\gamma} EN_{t+1}^{1-\alpha-\gamma} + (1-\delta)] = \lambda_t \tag{7}$$

W.r.t. EN_t: (Economic Interpretation)

$$(1-\alpha-\gamma)A_t K_t^{\alpha} N_t^{\gamma} EN_t^{-\alpha-\gamma} = P_t \tag{8}$$

W.r.t. λ_t: (Economic Interpretation)

$$A_t K_t^{\alpha} N_t^{\gamma} EN_t^{1-\alpha-\gamma} + (1-\delta)K_t = C_t + K_{t+1} + P_t EN_t \tag{9}$$

For the final 8 equations, we will first equate equations 5 and 7 to obtain the Euler equations. Then, we will repeat the process with equations 5 and 6. Next, we will use the law of motion to obtain an equation for I_t. Finally, we will use the production function to obtain an equation for income.

System of 8 Equations

$$C_t^{-\sigma} = BE_t[C_{t+1}^{-\sigma}(\alpha A_{t+1} K_{t+1}^{\alpha-1} N_{t+1}^{\gamma} EN_{t+1}^{1-\alpha-\gamma} + (1-\delta)] \tag{10}$$

$$C_t^{-\sigma} = \frac{\theta}{(1-N_t)(\gamma A_t K_t^{\alpha} N_t^{\gamma-1} EN_t^{1-\alpha-\gamma})} \tag{11}$$

$$P_t = (1-\alpha-\gamma)A_t K_t^{\alpha} N_t^{\gamma} EN_t^{-\alpha-\gamma} \tag{12}$$

$$A_t K_t^{\alpha} N_t^{\gamma} EN_t^{1-\alpha-\gamma} + (1-\delta)K_t = C_t + K_{t+1} + P_t EN_t \tag{13}$$

$$Y_t = A_t K^{\alpha} N_t^{\gamma} EN_t^{1-\alpha-\gamma} \tag{14}$$

$$I_t = K_{t+1} - (1-\delta) \tag{15}$$

$$logA_{t+1} = \rho_A logA_t + \epsilon_{A,t+1} \tag{16}$$

$$logP_{t+1} = \rho_P logP_t + \epsilon_{P,t+1} \tag{17}$$

Economic Interpretation

FOC I:

The l.h.s. of the equation is the marginal utility of consumption, where σ is the coefficient of relative risk aversion. The r.h.s. represents the Lagrange multiplier, which is the shadow price of consumption in the household's optimization problem. The FOC, thus, states that at the optimum, the marginal utility of consumption is equal to its shadow price. In other words, relaxing the constraint by one unit induces an increase in utility equal to the marginal utility of consuming that unit.

FOC II:

It states that the marginal utility of leisure (l.h.s.) must be equal to the Lagrange multiplier λ_t. The expression in the denominator is the marginal product of labor, which reflects how much output increases if an additional unit of labor is used, taking into account the productivity of capital (K_t^{α}), of energy ($EN_t^{1-\alpha-\gamma}$), and of labor ($N_t^{\gamma-1}$). The term $(1-N_t)$ reflects the fact that as more labor is used, the worker has less leisure time available, and hence, the marginal utility of leisure decreases. The parameter θ reflects the labor supply preference.

FOC III:

The l.h.s. of the equation can be seen as the expected value of the Lagrange multiplier λ_{t+1}, which represents the shadow price of capital. This shadow price captures the marginal benefit of an additional unit of capital in the following(!) period, taking into account the expected returns to that capital investment. On the r.h.s. of the equation, we have the current value of the Lagrange multiplier λ_t, which represents the shadow price of income or consumption. This shadow price captures the marginal benefit of an additional unit of income or consumption in the current period. The term in the square brackets represents the marginal cost of investing in capital. It includes two components: (i) the marginal product of capital, which is the additional output generated by an additional unit of capital investment and (ii) the depreciation of the existing capital stock.

Therefore, this equation implies that firms will invest in additional physical capital as long as the expected marginal benefit of that investment, captured by the shadow price of capital, exceeds the marginal cost of that investment, captured by the marginal product of capital net of depreciation.

FOC IV:

The l.h.s. of the equation is the expression for the quantity of energy used in production, which is a function of the capital stock (K_t), labor input (N_t), energy input (EN_t), and productivity (A_t). The parameters α, γ, and $1-\alpha-\gamma$ represent the shares of output that go to capital, labor, and energy, respectively.

Thus, the equation states that the price of energy is proportional to the quantity of energy used in production. This makes sense, as an increase in the level of production would lead to an increase in energy consumption, which would in turn increase the demand for energy and, hence, its price

FOC V:

This equation is the intertemporal budget constraint of the household. It states that the sum of the value of goods consumed in period t, the investment in new capital goods K_{t+1}, and the energy used $P_t EN_t$ (l.h.s.) must be equal to the sum of the value of capital goods held at the beginning of the period K_t and the value of new capital goods acquired during the period $(1-\delta)K_t$ (r.h.s.).

Problem 3

- Solve for the steady state. Take parameter values to be $\beta = 0.99$ $\sigma = 1$, $\theta = 3.48$, $\alpha = 0.3$, $\delta = 0.025$, $\overline{A} = 1$,$\overline{P} = 1$, $\rho_A = 0.95$, $\sigma_A = 0.007$, $\rho_P = 0.5$, $\sigma_P = 0.00001$. Given these, compute the steady state values of $\overline{C}, \overline{K}, \overline{N}, \overline{Y}, \overline{I}, \overline{EN}, \overline{A}, \overline{P}$ [...]
- Write a Dynare code that solves the RBC model you solved for in Question 2, so that you obtain a solution in terms of **percentage deviations** from steady state (i.e., variables in logs or in terms of hat variables). [...]
- Construct and plot impulse responses (for 20 periods) to a 1% productivity decrease (i.e. $\epsilon_{A;t}$ $= 0{:}01$ in period t $= 1$ and $\epsilon_{A;t} = 0$ for all other t $\neq 1$) [...]
- Repeat the same for a 10 percent energy price shock in period 1 [...]
- Log-linearize all 8 first order and equilibrium conditions.
- Solve the above model by mapping them into matrix format and using a linear rational expectations solution algorithm like the one used in klein.m. Having solved for the steady state, and having log- linearized, map your resulting log-linear system into format: $AE_t z_{t+1} = Bz_t$. Write a code where you define these matrices and solve for the policy functions via the klein.m algorithm. Report the policy functions you obtain. If you want you can also derive impulse responses as in exercises check if your results are identical to question c. and d.

Computing Steady State

We start by computing the ratio appearing in our system (step-by-step-derivations).

Table 5: Calculated Ratios of RBC Model

Variable	Computed Value
$\overline{K}/\overline{Y}$	8.547
$\overline{EN}/\overline{Y}$	0.0500
$\overline{C}/\overline{Y}$	0.7363

Afterwards we are able to compute the steady state of all variables.

Table 6: Steady State Values of RBC Model

Variable	Computed Value
$\overline{N}$	0.2023
$\overline{Y}$	0.4326
$\overline{C}$	0.3185
$\overline{K}$	3.6971
$\overline{EN}$	0.0216
$\overline{I}$	0.0924
$\overline{P}$	1.0000
$\overline{A}$	1.0000

Dynare: Solving RBC Model

Respective additional files: Computing_RBC.mod

First, we set up our RBC Model s.t. we can compute the steady state, like we did by hand in order to see if any mistakes happened:

Table 7: Policy Function Matrix computed by Dynare

Variable	Computed Value
$\overline{N}$	0.2023
$\overline{Y}$	0.4326
$\overline{C}$	0.3185
$\overline{K}$	3.6971
$\overline{EN}$	0.0216
$\overline{I}$	0.0924
$\overline{P}$	1.0000
$\overline{A}$	1.0000

We can extract it from the full policy and transition function generated by Dynare:

Table 8: Policy and Transformation Function computed by Dynare

	C	K	N	Y	I	EN	A	P
Const.	0.3185	3.6971	0.2023	0.4326	0.0924	0.0216	1.0000	1.0000
K_{-1}	0.0436	0.9410	-0.0183	0.0102	-0.0340	0.0005	0.0000	0.0000
P_{-1}	-0.0013	-0.0205	-0.0079	-0.0229	-0.0205	-0.0120	0.0000	0.5000
A_{-1}	0.1462	0.5319	0.1923	0.7138	0.5319	0.0357	0.9500	0.0000
ϵ_A	0.1539	0.5599	0.2024	0.7514	0.5599	0.0376	1.0000	0.0000
ϵ_P	-0.0026	-0.0409	-0.0158	-0.0458	-0.0409	-0.0239	0.0000	1.0000

As observable, model and calculations are similar, letting us conclude that no typos/miscalculations happened.

If we now logarithmize our model, we end up with the respective percentage changes, given by the following table:

Table 9: Policy and Transformation Function computed by Dynare (log-Model)

	Y	C	I	K	N	EN	A	P
Const.	-0.8380	-1.1440	-2.3813	1.3076	-1.5978	-3.8337	0.000	0.000
K_{-1}	0.0868	0.5064	-1.3589	0.9410	-0.3346	0.0868	0.0000	0.0000
P_{-1}	-0.0530	-0.0041	-0.2214	-0.0055	-0.0390	-0.5530	0.0000	0.5000
A_{-1}	1.6501	0.4589	5.7552	0.1439	0.9502	1.6501	0.9500	0.0000
ϵ_A	1.7370	0.4831	6.0581	0.1515	1.0002	1.7370	1.0000	0.0000
ϵ_P	-0.1060	-0.0082	-0.4428	-0.0111	-0.0780	-1.1060	0.0000	1.0000

Dynare: Shocks

Respective additional files: I_Shock_RBC.mod & II_Shock_RBC.mod

Productivity Shock

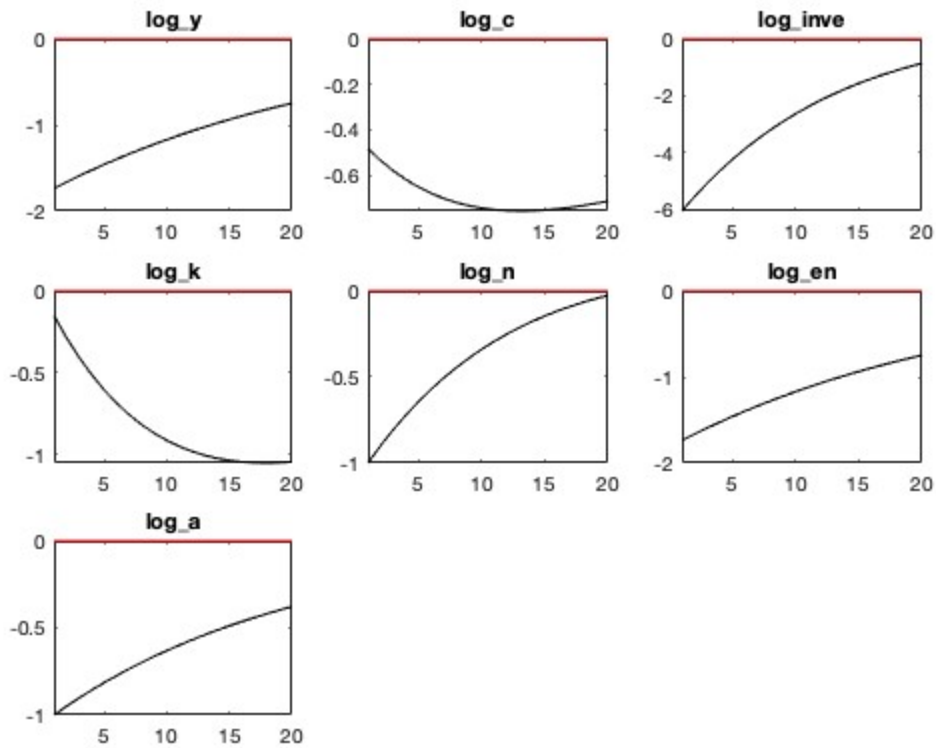


Figure 3: IRF (20 Periods) of a 1% Productivity Decrease in Period I; Values displayed in percent

We can observe in our estimated IRF (20 Periods) that the productivity shock (negative, -1%) leads, due to the lower productivity, to a lower level of output (Y). The effect is close to -2% in the first period estimated and converges back to -1% in period 20.

When productivity decreases, firms are less productive in using inputs, including labor, to produce output. As a result, the demand for labor decreases, and, thus, leads to less employment (N) and a lower real wage rate. In the estimated IRF, we see that the decrease in labor supply is initially 1% in period one but then converges back to zero by period 20. This is because in the long run, workers adjust their expectations about the wage rate, which leads to a return to the original equilibrium level of labor supply. Additionally, as the productivity shock persists, firms have to adjust and shift more towards labor and, thus, offset the initial decrease in labor supply.

In terms of consumption (C) the IRF is close to a U-shape, which means that consumption initially decreases in response to the shock and then gradually recovers over time, which could be due to the consumption smoothing mentioned in class. The initial decrease of -0.4% in consumption occurs because the negative productivity price shock reduces the economy's output and income, which reduces households' ability to consume. The gradual recovery in consumption occurs as households adjust their consumption behavior in response to the shock, such as by increasing their saving or reducing their debt levels, to maintain their desired consumption levels over time. The fact that the entire IRF of consumption is negative (between -0.4% ad -0.06%) indicates that the productivity shock has a negative effect on consumption for the whole period estimated.

The reason for the initial small decrease in capital (K) of 0.1% is likely due to the fact that the shock only affects productivity, and not directly capital. Afterwards our estimation shows two effects, (i) the substitution towards labor, already mentioned, and (ii) the decrease in investment (IN) of 6% due to the productivity shock and induced lower output.

Furthermore, the induced lower output leads to approx. -2% energy demand, as firms a producing less (according to the estiamted IRF). The converge back to steady state is in line with the converge of the output.

Energy Price Shock

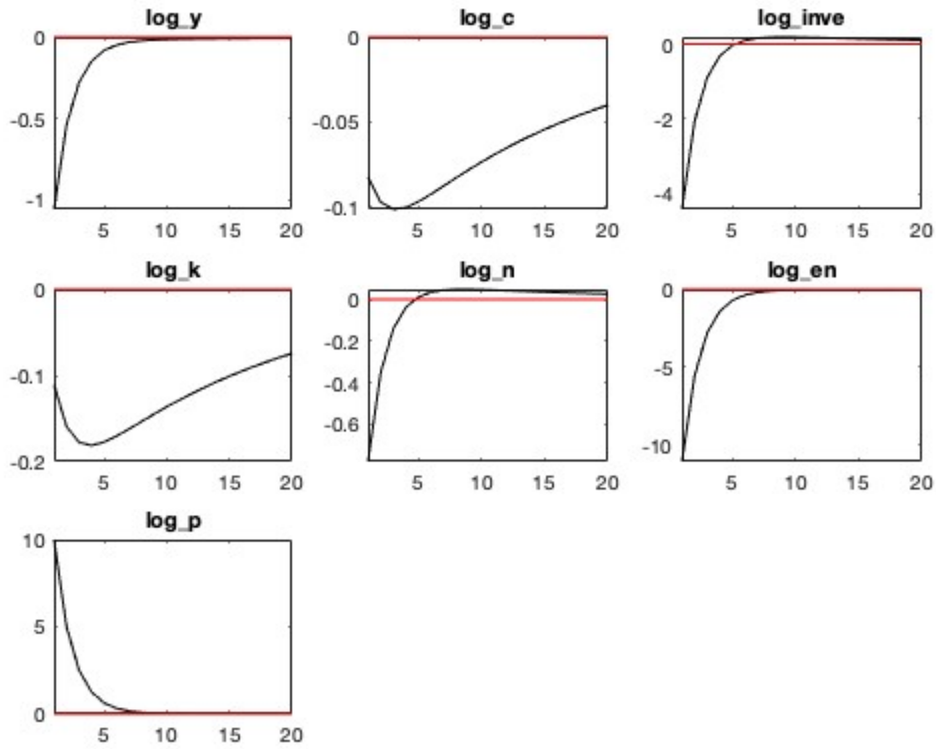


Figure 4: IRF (20 Periods) of a 10% Energy Price Increase in Period I; Values displayed in percent

We can observe in our estimated IRF (20 Periods) that the energy price shock (positive, 10%) decreases the productivity of the economy, as the same output can c.p. only be produced to higher costs. This is why firms are substituting away from energy, leading to 10% energy usage decrease. As a result, the level of output (Y) decreases in the short run by 0.1%, whereas the medium-run-effect (c.a. 10 years+) is close to zero. It's visible that the shock is less persistent, than the previous productivity shock.

However, in response to the decrease in output, the economy adjusts by reducing its real wage rate, which leads to an increase in employment (N) after a short decline. This adjustment is captured by the FOC w.r.t. N_t, which implies that the marginal disutility of work decreases as the level of employment increases. This adjustment helps to partially offset the decrease in output resulting from the productivity shock.

In terms of consumption (C) the IRF is close to a U-shape, which means that consumption initially decreases in response to the shock and then gradually recovers over time, which could be due to the consumption smoothing mentioned in class. The initial decrease of -0.075% in consumption occurs because the positive energy price shock reduces the economy's output and income, which reduces households' ability to consume. The gradual recovery in consumption occurs as households adjust their consumption behavior in response to the shock, such as by increasing their saving or reducing their debt levels, to maintain their desired consumption levels over time. The fact that the entire IRF of consumption is negative (between -0.01% ad -0.005%) indicates that the productivity shock has a negative effect on consumption for the whole period estimated.[3] Nevertheless, the effect is less in magnitude and persistence than the previously estimated productivity shock.

The initial decrease in the capital stock (K) occurs because the energy price shock reduces, as mentioned, the economy's output and income, which in turn reduces the amount of investment (I) that firms can undertake. With less investment, the capital stock then decreases due to depreciation. The gradual recovery in the capital stock occurs as firms adjust their investment behavior in response to the shock, such as by delaying or reducing investment in the short run but eventually increasing investment to restore the capital stock to its long-run equilibrium level.

[3] We tried a longer horizon and observe a converge back to zero.

Log-Linearization

All the steps of the derivation are in the appendix, we will just show the final results below.

$$-\sigma\hat{c}_t = E_t[-\sigma\hat{c}_{t+1} + (1-\beta(1-\delta)(\hat{A}_{t+1} + (\alpha-1)\hat{K}_{t+1} + \gamma\hat{N}_{t+1} + (1-\alpha-\gamma)\hat{EN}_{t+1}] \tag{18}$$

$$\overline{Y}\hat{Y}_t = \frac{\theta}{\gamma}\overline{CN}\hat{C}_t + (\frac{\theta}{\gamma}\overline{CN} + \overline{NY})\hat{N}_t + \overline{NY}\hat{Y}_t \tag{19}$$

$$\overline{ENP}(\hat{EN}_t + \hat{P}_t) = (1-\alpha-\gamma)\overline{Y}\hat{Y}_t \tag{20}$$

$$\overline{Y}(\hat{A} + \hat{\alpha}K_t + \gamma\hat{N}_t + (1-\alpha-\gamma)\hat{EN}_t) + (1-\delta)\overline{K}\hat{K} = \overline{C}\hat{C}_t + \overline{K}\hat{K}_{t+1} + \overline{PEN}(\hat{P}_t + \hat{EN}_t) \tag{21}$$

$$\hat{Y}_t = \hat{A}_t + \alpha\hat{K}_t + \gamma\hat{N}_t + (1-\alpha-\gamma)\hat{EN}_t \tag{22}$$

$$\hat{I} = \frac{\hat{K}_{t+1}}{\delta} + \frac{1-\delta}{\delta}\hat{K}_t \tag{23}$$

$$Log(A_{t+1}) = \rho_A logA_t + \epsilon_{A,t+1} \tag{24}$$

$$Log(P_{t+1}) = \rho_P LogP_t + \epsilon_{P,t+1} \tag{25}$$

Klein.M

Respective additional files: III_KleinM_RBC.mod

Using matrix A und B, as calculated in the appendix part for the log-linearization calculations:

$$A = \begin{pmatrix} (1-\beta(1-\delta))(\alpha-1) & (1-\beta(1-\delta)) & 0 & -\sigma & 0 & 0 & (1-\beta(1-\delta))\gamma & (1-\beta(1-\delta))(1-\alpha-\gamma) \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \bar{K}/\bar{Y} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1/\delta & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$B = \begin{pmatrix} 0 & 0 & 0 & -\sigma & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & (\theta/\gamma)\bar{C}\bar{N} & 0 & -\bar{Y}+\bar{N}\bar{Y} & \bar{N}((\theta/\gamma)\bar{C}+\bar{Y}) & 0 \\ 0 & 0 & \bar{P}\bar{E}\bar{N} & 0 & 0 & -\bar{Y}(1-\alpha-\gamma) & 0 & \bar{P}\bar{E}\bar{N} \\ \alpha+(1-\delta)\bar{K}/\bar{Y} & 1 & -\bar{P}\bar{E}\bar{N}/\bar{Y} & -\bar{C}/\bar{Y} & 0 & 0 & \gamma & -\bar{P}\bar{E}\bar{N}/\bar{Y}+1-\alpha-\gamma \\ -\alpha & (-1) & 0 & 0 & 0 & 1 & -\gamma & -(1-\alpha-\gamma) \\ (1-\delta)/\delta & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & \rho_A & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \rho_P & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Using the generalized Schur decomposition we end up with:

$$G = \begin{pmatrix} 0.4886 & 0.5244 & -0.0047 \\ -1.6450 & 7.0862 & -0.2598 \\ -0.0740 & 2.2531 & -0.0220 \\ -0.4489 & 1.3827 & -0.0138 \\ -0.0753 & 2.2911 & -1.0223 \end{pmatrix} \quad \wedge \quad H = \begin{pmatrix} 0.9339 & 0.1772 & -0.0065 \\ -0.0000 & 0.9500 & 0.0000 \\ 0.0000 & -0.0000 & 0.5000 \end{pmatrix}$$

Appendix

FOC Calculations

Question 2)

$$\mathrm{Max}(U) = \mathrm{Max}\left(E_0 \sum_{t=0}^{\infty} B^t \frac{C_t^{1-\sigma} - 1}{1-\sigma} + \theta \log(1-N_t)\right)$$

Constraints: 1) $C_t + K_{t+1} + P_t EN_t \leq A_t K_t^{\alpha} N_t^{\gamma} EN_t^{1-\alpha-\gamma} + (1-\delta)K_t$

2) $C_t,\ K_{t+1} \geq 0$

$$L = E_0 \sum_{t=0}^{\infty} B^t \left(\frac{C_t^{1-\sigma} - 1}{1-\sigma} + \theta \log(1-N_t) + \lambda_t \left[A_t K_t^{\alpha} N_t^{\gamma} EN_t^{1-\alpha-\gamma} + (1-\delta)K_t - C_t - K_{t+1} - P_t\ EN_t \right] \right)$$

FOC: $\frac{\partial L}{\partial C_t} \stackrel{!}{=} 0$

$\frac{\partial L}{\partial N_t} \stackrel{!}{=} 0$

$\frac{\partial L}{\partial K_{t+1}} \stackrel{!}{=} 0$

$\frac{\partial L}{\partial EN_t} \stackrel{!}{=} 0$

$\frac{\partial L}{\partial \lambda_t} \stackrel{!}{=} 0$

Note: Expanse of Lagrangian:

$$\begin{aligned} L = E_t \Big[& B^t \left(\frac{C_t^{1-\sigma} - 1}{1-\sigma} \right) + \theta \log(1-N_t) + \lambda_t \big[A_t K_t^{\alpha} N_t^{\gamma} EN_t^{1-\alpha-\gamma} + (1-\delta)K_t - C_t - \underline{K_{t+1}} - P_t\ EN_t \\ & + B^{t+1} \left(\frac{C_{t+1}^{1-\sigma} - 1}{1-\sigma} \right) + \theta \log(1-N_{t+1}) + \lambda_{t+1} \big[A_{t+1} \underline{K_{t+1}^{\alpha}} N_{t+1}^{\gamma} EN_{t+1}^{1-\alpha-\gamma} \\ & + (1-\delta) \underline{K_{t+1}} - C_{t+1} - K_{t+2} - P_{t+1}\ EN_{t+1} + \dots \Big] \end{aligned}$$

1

Annexe: Step-by-step derivation of all FOC:

$$\frac{\partial L}{\partial C_t} = 0 \Rightarrow E_t\left[B^t\left((1-\sigma)\cdot\frac{1}{1-\sigma}\cdot C_t^{1-\sigma-1} + \lambda_t\cdot(-1)\right)\right] = 0$$

$$\Leftrightarrow E_t\left[B^t\left(C_t^{-\sigma} - \lambda_t\right)\right] = 0$$

$$\Leftrightarrow \quad C_t^{-\sigma} - \lambda_t = 0$$

$$\Leftrightarrow \quad \boxed{C_t^{-\sigma} = \lambda_t} \qquad \text{or} \quad U'(C_t) = \lambda_t$$

$$\frac{\partial L}{\partial N_t} = 0 \Rightarrow E_t\left[B^t\left(\theta\cdot\frac{(-1)}{1-N_t} + \lambda_t\cdot\gamma\cdot A_t K_t^{\alpha} N_t^{\gamma-1} EN_t^{1-\alpha-\gamma}\right)\right] = 0$$

$$\Leftrightarrow B^t\,\frac{\theta}{1-N_t} = B^t \lambda_t\, \gamma\, A_t K_t^{\alpha} N_t^{\gamma-1} EN_t^{1-\alpha-\gamma}$$

$$\Leftrightarrow \frac{\theta}{(1-N_t)}\cdot\frac{1}{(\gamma A_t K_t^{\alpha} N_t^{\gamma-1} EN_t^{1-\alpha-\gamma})} = \lambda_t$$

$$\frac{\partial L}{\partial K_{t+1}} = 0 \Rightarrow E_t\left[B^t\cdot\lambda_t(-1) + B^{t+1}\cdot\lambda_{t+1}\cdot\alpha A_{t+1} K_{t+1}^{\alpha-1} N_{t+1}^{\gamma} EN_{t+1}^{1-\alpha-\gamma}\right.$$

$$\left. + \lambda_{t+1}\cdot(1-\delta)\right] = 0$$

$$\Leftrightarrow -B^t\lambda_t + E_t B^t\left[B\,\lambda_{t+1}\left(\alpha A_{t+1} K_{t+1}^{\alpha-1} N_{t+1}^{\gamma} EN_{t+1}^{1-\alpha-\gamma} + (1-\delta)\right)\right] = 0$$

$$\Leftrightarrow B\, E_t\left[\lambda_{t+1}\left(\alpha A_{t+1} K_{t+1}^{\alpha-1} N_{t+1}^{\gamma} EN_{t+1}^{1-\alpha-\gamma} + (1-\delta)\right)\right] = \lambda_t$$

2

$$\frac{\partial L}{\partial EN_t} = 0 \Rightarrow E_t \left[B^t \cdot \lambda_t \cdot \left[(1-\alpha-\gamma) A_t K_t^{\alpha} EN_t^{1-\alpha-\gamma-1} - P_t \right] = 0 \right.$$

$$\Leftrightarrow \quad B^t \lambda_t (1-\alpha-\gamma) A_t K_t^{\alpha} N_t^{\gamma} EN_t^{-\alpha-\gamma} = B^t \lambda_t P_t$$

$$\Leftrightarrow \quad (1-\alpha-\gamma) A_t K_t^{\alpha} N_t^{\gamma} EN^{-\alpha-\gamma} = P_t$$

$$\frac{\partial L}{\partial \lambda_t} = 0 \Rightarrow A_t K_t^{\gamma} N_t^{\gamma} EN_t^{1-\alpha-\gamma} + (1-\delta) K_t - C_t - K_{t+1} - P_t EN_t = 0$$

$$\Leftrightarrow \quad A_t K_t^{\alpha} N_t^{\gamma} EN_t^{1-\alpha-\gamma} + (1-\delta) K_t = C_t + K_{t+1} + P_t EN_t$$

Thus we obtain:

(1) $$C_t^{-\sigma} = \lambda_t$$

(2) $$\frac{\theta}{(1-N_t)(\gamma A_t K_t^{\alpha} N_t^{\gamma-1} EN_t^{1-\alpha-\gamma})} = \lambda_t$$

(3) $$B E_t \left[\lambda_{t+1} (\alpha A_{t+1} K_{t+1}^{\alpha-1} N_{t+1}^{\gamma} EN_{t+1}^{1-\alpha-\gamma} + (1-\delta)) \right] = \lambda_t$$

(4) $$(1-\alpha-\gamma) A_t K_t^{\alpha} N_t^{\gamma} EN_t^{1-\alpha-\gamma} = P_t$$

(5) $$A_t K_t^{\alpha} N_t^{\gamma} EN_t^{1-\alpha-\gamma} + (1-\delta) K_t = C_t + K_{t+1} + P_t EN_t$$

3

Now we can equate the 1st and 2nd FOC to get the Euler equation: $C_t^{-\sigma} = \beta E_t[\lambda_{t+1}(\alpha A_{t+1} K_{t+1}^{\alpha-1} N_{t+1}^{\gamma} EN_{t+1}^{1-\alpha-\gamma} + (1-\delta)]$

We can also equate: $C_t^{-\sigma} = \dfrac{\theta}{(1-N_t)(\gamma A_t K_t^{\alpha} N_t^{\gamma-1} EN_t^{1-\alpha-\gamma})}$

Given that the investment is governed by the law of motion we have: $I_t = K_{t+1} - (1-\delta)K_t$

And we know that $Y_t = A_t K_t^{\alpha} N_t^{\gamma} EN_t^{1-\alpha-\gamma}$

Thus we obtain a system of 8 equations:

① $C_t^{-\sigma} = \beta E_t[C_{t+1}^{-\sigma}(\alpha A_{t+1} K_{t+1}^{\alpha-1} N_{t+1}^{\gamma} EN_{t+1}^{1-\alpha-\gamma} + (1-\delta))]$

② $C_t^{-\sigma} = \dfrac{\theta}{(1-N_t)(\gamma A_t K_t^{\alpha} N_t^{\gamma-1} EN_t^{1-\alpha-\gamma}}$

③ $P_t = (1-\alpha-\gamma) A_t K_t^{\alpha} N_t^{\gamma} EN_t^{1-\alpha-\gamma-1}$

④ $A_t K_t^{\alpha} N_t^{\gamma} EN_t^{1-\alpha-\gamma} + (1-\delta)K_t = C_t + K_{t+1} + P_t EN_t$

⑤ $Y_t = A_t K_t^{\alpha} N_t^{\gamma} EN_t^{1-\alpha-\gamma}$

⑥ $I_t = K_{t+1} - (1-\delta)K_t$

⑦ $\log A_{t+1} = \rho_A \log A_t + \varepsilon_{A,t+1}$

⑧ $\log P_{t+1} = \rho_p \log P_t + \varepsilon_{p,t+1}$

4

Steady State Calculations

3a) To solve for the steady states we will start by the 2 exogenous variables:

$$\bar{A} = 1$$

$$\bar{P} = 1$$

* We replace the variables in the Euler equations by their steady-states:

$$\bar{C}^{-\sigma} = \beta \left[\bar{C}^{-\sigma} \left(\alpha \bar{A} \bar{K}^{\alpha-1} \bar{N}^{\gamma} \overline{EN}^{1-\alpha-\gamma} + (1-\delta) \right) \right]$$

$$\Leftrightarrow \quad 1 = \beta \left[\alpha \frac{\bar{Y}}{\bar{K}} + (1-\delta) \right]$$

$$\Leftrightarrow \quad \frac{1}{\beta} - (1-\delta) = \alpha \frac{\bar{Y}}{\bar{K}}$$

$$\Leftrightarrow \quad \frac{\alpha}{\frac{1}{\beta} - (1-\delta)} = \frac{\bar{K}}{\bar{Y}}$$

$$\Leftrightarrow \quad \frac{0.3}{\frac{1}{0.99} - 0.975} \approx 8.547 = \frac{\bar{K}}{\bar{Y}}$$

* We then use eq(12)

$$\bar{P} = (1-\alpha-\gamma) \bar{A} \bar{K}^{\alpha} \bar{N}^{\gamma} \overline{EN}^{1-\alpha-\gamma-1}$$

$$1 = 0.05 \frac{\bar{Y}}{\overline{EN}} \quad \Leftrightarrow \quad \frac{\overline{EN}}{\bar{Y}} = 0.05$$

* Now in eq(13)

$$\bar{A} \bar{K}^{\alpha} \bar{N}^{\gamma} \overline{EN}^{1-\alpha-\gamma} + (1-\delta)\bar{K} = \bar{C} + \bar{K} + \bar{P}\,\overline{EN}$$

$$\bar{Y} + (1-\delta)\bar{K} = \bar{C} + \bar{K} + \overline{EN}$$

$$1 + (1-\delta)\frac{\bar{K}}{\bar{Y}} - 1\frac{\bar{K}}{\bar{Y}} - \frac{\overline{EN}}{\bar{Y}} = \frac{\bar{C}}{\bar{Y}}$$

$$1 - 0.025\frac{\bar{K}}{\bar{Y}} - \frac{\overline{EN}}{\bar{Y}} = \frac{\bar{C}}{\bar{Y}} \simeq 0.7363$$

- We can now use the last 3 results to compute the steady states value of $N_t = \bar{N}$ using eq (11):

$$\bar{C}^{-\sigma} = \frac{\theta}{(1-\bar{N})\,(\gamma A \bar{K}^{\alpha} \bar{N}^{\gamma-1} \overline{EN}^{1-\alpha-\gamma})}$$

$$\Leftrightarrow \quad \frac{1}{\bar{C}} = \frac{\theta}{\gamma \bar{Y}} \cdot \frac{\bar{N}}{(1-\bar{N})}$$

$$\Leftrightarrow \quad \frac{1}{\bar{C}} = \frac{\theta}{\gamma \bar{Y}} \cdot \frac{1}{(1/\bar{N} - 1)}$$

$$\frac{1}{\bar{N}} - 1 = \frac{\bar{C}}{\bar{Y}} \cdot \frac{\theta}{\gamma}$$

$$\frac{1}{\bar{N}} = \left(\frac{\bar{C}}{\bar{Y}} \cdot \frac{\theta}{\gamma} + 1\right)$$

$$\bar{N} = \frac{1}{\left(\frac{\bar{C}}{\bar{Y}} \frac{\theta}{\gamma} + 1\right)} \approx 0.2023$$

- We can now compute $\bar{Y}$ with eq (14)

$$\bar{Y} = A \bar{K}^{\alpha} \bar{N}^{\gamma} \overline{EN}^{1-\alpha-\gamma}$$

$$\frac{\bar{Y}}{\bar{Y}^{\alpha} \, \bar{Y}^{1-\alpha-\gamma}} = \left(\frac{\bar{K}}{\bar{Y}}\right)^{\alpha} \cdot \bar{N}^{\gamma} \cdot \left(\frac{\overline{EN}}{\bar{Y}}\right)^{1-\alpha-\gamma}$$

$$\bar{Y}^{-\gamma} = \left(\frac{\bar{K}}{\bar{Y}}\right)^{\alpha} \bar{N}^{\gamma} \left(\frac{\overline{EN}}{\bar{Y}}\right)^{1-\alpha-\gamma}$$

$$\bar{Y} = \left(\frac{\bar{K}}{\bar{Y}}\right)^{\frac{\alpha}{\gamma}} \bar{N} \left(\frac{\overline{EN}}{\bar{Y}}\right)^{\frac{1-\alpha-\gamma}{\gamma}} \approx 0.4326$$

We can now use the following to calculate $\bar{K}$, $\bar{C}$, $\overline{EN}$:

$$\bar{K} = \frac{\bar{K}}{\bar{Y}} \cdot \bar{Y}, \quad \bar{C} = \frac{\bar{C}}{\bar{Y}} \cdot \bar{Y}, \quad \overline{EN} = \frac{\overline{EN}}{\bar{Y}} \cdot \bar{Y}$$

Log-Lin Calculations

e) Log-Linearization :

* Defining log-deviation from steady-states:

$$\hat{C}_t = \log\left(\frac{C_t}{\bar{C}}\right) \iff C_t = \bar{C}\, e^{\hat{C}_t}$$

$$\hat{N}_t = \log\left(\frac{N_t}{\bar{N}}\right) \iff N_t = \bar{N} \cdot e^{\hat{N}_t}$$

$$\hat{K}_t = \log\left(\frac{K_t}{\bar{K}}\right) \iff K_t = \bar{K} \cdot e^{\hat{K}_t}$$

$$\hat{EN}_t = \log\left(\frac{EN_t}{\bar{EN}}\right) \iff EN_t = \bar{EN} \cdot e^{\hat{EN}_t}$$

$$\hat{Y}_t = \log\left(\frac{Y_t}{\bar{Y}}\right) \iff Y_t = \bar{Y} \cdot e^{\hat{Y}_t}$$

$$\hat{I}_t = \log\left(\frac{I_t}{\bar{I}}\right) \iff I_t = \bar{I} \cdot e^{\hat{I}_t}$$

$$\hat{A}_t = \log\left(\frac{A_t}{\bar{A}}\right) = \log(A_t) \iff A_t = \bar{A} \cdot e^{\hat{A}_t}$$

$$\hat{P}_t = \log\left(\frac{P_t}{\bar{P}}\right) = \log(P_t) \iff P_t = \bar{P} \cdot e^{\hat{P}_t}$$

e) Euler equation:

$$C_t^{-\sigma} = \beta E_t \left[C_{t+1}^{-\sigma} \left(\alpha A_{t+1} K_{t+1}^{\alpha-1} N_{t+1}^{\gamma} EN_{t+1}^{1-\alpha-\gamma} + (1-\delta) \right) \right]$$

$$(\bar{C} e^{\hat{c}_t})^{-\sigma} = \beta E_t \Big[(\bar{C} e^{\hat{c}_{t+1}})^{-\sigma} \Big(\alpha (\bar{A} \cdot e^{\hat{A}_{t+1}}) (\bar{K} e^{\hat{k}_{t+1}})^{\alpha} (\bar{N} e^{\hat{N}_{t+1}})^{\gamma}$$

$$(\overline{EN}\, e^{\widehat{EN}_{t+1}})^{1-\alpha-\gamma} + (1-\delta) \Big]$$

$$LHS = \bar{C}^{-\sigma} e^{\hat{c}_t \cdot (-\sigma)}$$

Taylor approximation at $\hat{c}_t = 0$

$$LHS \approx \bar{C}^{-\sigma} \cdot e^{\bar{\hat{c}}_t (-\sigma)} + C^{-\sigma} \cdot (-\sigma) \cdot e^{-\bar{\hat{c}}_t \cdot (-\sigma)} (\hat{c}_t - \bar{\hat{c}}_t)$$

$$\approx \bar{C}^{-\sigma} + \bar{C}^{-\sigma} \cdot (-\sigma) (\hat{c}_t) \approx (1 - \sigma \cdot \hat{c}_t) \bar{C}^{-\sigma}$$

RHS Taylor approximation at (1) $\hat{C}_{t+1} = 0$, (2) $\hat{A}_{t+1} = 0$

(4) $\hat{K}_{t+1} = 0$, (3) $\hat{N}_{t+1} = 0$, (5) $\widehat{EN}_{t+1} = 0$

$$\approx \beta E ((\bar{C} e^{\bar{\hat{c}}})^{-\sigma} \Big[\alpha (\bar{A} \cdot e^{\bar{\hat{A}}_{t+1}}) (\bar{K} \cdot e^{\bar{\hat{K}}_{t+1}})^{1-\alpha} (\bar{N} \cdot e^{\bar{\hat{N}}_{t+1}})^{\gamma} (\overline{EN} \cdot e^{\bar{\widehat{EN}}_{t+1}})^{1-\alpha-\gamma}$$

$$+ (1-\delta) \Big]$$

(1)

$$+ \beta E_t (\bar{C}^{-\sigma} e^{\bar{\hat{c}}_{t+1} (-\sigma)} \cdot (-\sigma)) (\alpha \bar{A} e^{\hat{A}_{t+1}}) (\bar{K} e^{\hat{k}_{t+1}})^{\alpha} (\bar{N} e^{\hat{N}_{t+1}})^{\gamma}$$

$$(EN e^{\hat{N}_{t+1}})^{\gamma} (1-\alpha)) (\hat{C}_{t+1} - \bar{\hat{C}}_{t+1})$$

(2)

$$+ \beta E_t ((\bar{C} e^{\bar{\hat{c}}_{t+1}})^{-\sigma} (\alpha \bar{A} e^{\bar{\hat{A}}_{t+1}}) (\bar{K} e^{\bar{\hat{k}}_{t+1}}) (\bar{N} e^{\bar{\hat{N}}_{t+1}})^{\gamma} (\overline{EN}\, e^{\bar{\widehat{EN}}_{t+1}})^{1-\alpha-\gamma}$$

$$(\hat{A}_{t+1} - \bar{\hat{A}}_{t+1})$$

(3)

$$+ \beta E_t (\bar{C} e^{\bar{\hat{c}}_{t+1}})^{-\sigma} (\alpha \cdot \gamma (\bar{A} e^{\bar{\hat{A}}_{t+1}}) (\bar{K} e^{\bar{\hat{k}}_{t+1}})^{1-\alpha} (\bar{N} e^{\bar{\hat{N}}_{t+1}})^{\gamma}$$

$$(\overline{EN} e^{\overline{EN}_{t+1}})^{1-\alpha-\gamma} (\hat{N}_{t+1} - \hat{\bar{N}}_{t+1})$$

$$\vdots$$

$$\boxed{\bar{C}^{-\sigma}} + \bar{C}^{-\sigma} \cdot (-\sigma)\, \hat{c}_t = \boxed{B\left(\bar{C}^{-\sigma}\left(\alpha \bar{A} \bar{K}^{\alpha-1} N^{\gamma} EN^{1-\alpha-\gamma} + (1-\delta)\right)\right]}$$

$$+ B\left(\bar{C}^{-\sigma}(-\sigma)\right)\left(\alpha \bar{A} \bar{K}^{\alpha-1} \bar{N}^{\gamma} \overline{EN}^{1-\alpha-\gamma} + (1-\delta)\right) E(\hat{C}_{t+1})$$

$$+ B\left(\bar{C}^{-\sigma}\right)\left(\alpha \bar{A} \bar{K}^{1-\alpha} \bar{N}^{\gamma} \overline{EN}^{1-\alpha-\gamma}\right) E(\hat{A}_{t+1})$$

$$+ B\left(\bar{C}^{-\sigma}\right)\left(\alpha(\alpha-1) \bar{A} \bar{K}^{\alpha-1} \bar{N}^{\gamma} \overline{EN}^{1-\alpha-\gamma}\right) E(\hat{K}_{t+1})$$

$$+ B\left(\bar{C}^{-\sigma}\right)\left(\alpha\gamma \bar{A} \bar{K}^{1-\alpha} \bar{N}^{\gamma} \overline{EN}^{1-\alpha-\gamma}\right) E(\hat{N}_{t+1})$$

$$+ B\left(\bar{C}^{-\sigma}\right)\left(\alpha(1-\alpha-\gamma) \bar{A} \bar{K}^{1-\alpha} \bar{N}^{\gamma} \overline{EN}^{1-\alpha-\gamma}\right) E(\widehat{EN}_{t+1})$$

$$\Leftrightarrow -\sigma \hat{c}_t = -\sigma E(\hat{C}_{t+1}) + $$

$$+ \underline{B}\Big[\underline{\alpha \bar{A} \bar{K}^{1-\alpha} \bar{N}^{\gamma} \overline{EN}^{1-\alpha-\gamma}} E(\hat{A}_{t+1}) +$$

$$\underline{\alpha}(\alpha-1) \underline{\bar{A} \bar{K}^{\alpha-1} \bar{N}^{\gamma} EN^{1-\alpha-\gamma}} E(\hat{K}_{t+1}) +$$

$$\underline{\alpha} \cdot \gamma \underline{\bar{A} \bar{K}^{1-\alpha} \bar{N}^{\gamma} \overline{EN}^{1-\alpha-\gamma}} E(\hat{N}_{t+1}) +$$

$$\underline{\alpha}(1-\alpha-\gamma) \underline{\bar{A} \bar{K}^{1-\alpha} \bar{N}^{\gamma} \overline{EN}^{1-\alpha-\gamma}} E(\widehat{EN}_{t+1})\Big]$$

note: SS: $\bar{C}^{-\sigma} = B\left(\bar{C}^{-\sigma}\left(\alpha \bar{A} \bar{K}^{\alpha-1} \bar{N}^{\gamma} \overline{EN}^{1-\alpha-\gamma} + (1-\delta)\right)\right) \quad \Big\downarrow : \bar{C}^{-\sigma}$

$$1 = B\left[\alpha \bar{A} \bar{K}^{\alpha-1} \bar{N}^{\gamma} \overline{EN}^{1-\alpha-\gamma} + (1-\delta)\right]$$

$$\frac{1}{B} - (1-\delta) = \alpha \bar{A} \bar{K}^{\alpha-1} \bar{N}^{\gamma} \overline{EN}^{1-\alpha-\gamma}$$

$$1 - B(1-\delta) = \underline{B \alpha \bar{A} \bar{K}^{\alpha-1} \bar{N}^{\gamma} \overline{EN}^{1-\alpha-\gamma}}$$

$$-\sigma \hat{c}_t = E_t\left[-\sigma \hat{c}_{t+1} + \left(1 - B(1-\delta)\right)\left(\hat{A}_{t+1} + (\alpha-1)\hat{K}_{t+1} + \gamma \hat{N}_{t+1} + (1-\alpha-\gamma)\widehat{EN}_{t+1}\right)\right]$$

2) Before log-linearizing equations & we are going to make some simplification:

$$C_t^{-\sigma} = \frac{\theta}{(1-N_t)\ (\gamma A_t K^{\alpha} N_t^{\gamma-1} \varepsilon N_t^{1-\alpha-\gamma}}$$

$$C_t^{-\sigma} = \frac{\theta \quad N_t}{(1-N_t)\ \gamma\ Y_t}$$

$$(1-N_t)\ \gamma\ Y_t\ C_t^{-\sigma} = \theta\ N_t$$

$$\gamma Y_t C_t^{-\sigma} - N_t Y_t C_t^{-\sigma} \gamma = \theta N_t$$

$$Y_t C_t^{-\sigma} - N_t Y_t C_t^{-\sigma} = \frac{\theta}{\gamma} N_t$$

(we know $\sigma = 1$)

$$Y_t = \frac{\theta}{\gamma} N_t \cdot C_t + N_t Y_t$$

$\leadsto$ log linearize: $\bar{Y} e^{\hat{Y}_t} = \frac{\theta}{\gamma} \bar{N} e^{\hat{N}_t}\ \bar{C} e^{\hat{C}_t} + \bar{Y} e^{\hat{Y}_t}\ \bar{N} e^{\hat{N}_t}$

RHS: $\bar{Y} + \bar{Y}\hat{Y}$

LHS: $\frac{\theta}{\gamma}\bar{N}\bar{C} + \bar{Y}\bar{N} + \left(\frac{\theta}{\gamma}\bar{N}\bar{C} + \bar{Y}\bar{N}\right)\hat{N}_t + \left(\sigma\frac{\theta}{\gamma}\bar{N}\bar{C}\right)\hat{C}_t$
$+ \bar{Y}\hat{Y}_t \bar{N}$

$$\Rightarrow \overset{SS}{\bar{Y}} + \bar{Y}\hat{Y} = \overset{SS}{\left(\frac{\theta}{\gamma}\bar{N}\bar{C} + \bar{Y}\bar{N}\right)} + \left(\frac{\theta}{\gamma}\bar{N}\bar{C} + \bar{Y}\bar{N}\right)\hat{N}_t + \left(\sigma\frac{\theta}{\gamma}\bar{N}\bar{C}\right)\hat{C}_t + \bar{Y}\hat{Y}_t\bar{N}$$

$$\Leftrightarrow \bar{Y}\hat{Y}_t = \frac{\theta}{\gamma}\bar{C}\bar{N}\hat{C}_t + \left(\frac{\theta}{\gamma}\bar{C}\bar{N} + \bar{N}\bar{Y}\right)\hat{N}_t + \bar{N}\bar{Y}\hat{Y}_t$$

equation 3: $P_t = (1-\alpha-\gamma)\, A_t\, K_t^{\alpha}\, N_t^{\gamma}\, EN_t^{1-\alpha-\gamma-1}$

$\iff P_t = (1-\alpha-\gamma) \dfrac{Y_t}{EN_t}$

$\iff P_t = (1-\alpha-\gamma)\, Y_t \cdot EN_t^{-1}$

$\iff EN_t P_t = (1-\alpha-\gamma)\, Y_t$

log-linearization:

$$\overline{EN}\, e^{\hat{EN}_t}\, \overline{P}\, e^{\hat{P}_t} = (1-\alpha-\gamma)\, \overline{Y}\, e^{\hat{Y}_t}$$

RHS: $\underbrace{\overline{EN}\ \overline{P}}_{SS} + \overline{EN}\,(\hat{EN}_t)\overline{P} + \overline{P}\,(\hat{P}_t)\,\overline{EN}$

LHS: $\overbrace{(1-\alpha-\gamma)\,\overline{Y}}^{SS} + 1-\alpha-\gamma\ \overline{Y}\ \hat{Y}_t$

$\Rightarrow \overline{EN}\,\overline{P}\,(\hat{EN}_t + \hat{P}_t) = (1-\alpha-\gamma)\,\overline{Y}\,\hat{Y}_t$

(4)

$$\bar{A}e^{\hat{A}_t}\bar{K}^{\alpha}e^{\alpha\hat{K}_t}\bar{N}e^{\gamma\hat{N}_t}\overline{EN}^{(1-\alpha-\gamma)}e^{\hat{EN}_t(1-\alpha-\gamma)}+(1-\delta)\bar{K}e^{\hat{K}_t}=\bar{C}e^{\hat{C}_t}+\bar{K}e^{\hat{K}_{t+1}}+\bar{P}e^{\hat{P}_t}\overline{EN}e^{\hat{EN}_t}$$

$$\text{RHS}: \bar{A}\bar{K}^{\alpha}\bar{N}^{\gamma}\overline{EN}^{(1-\alpha-\gamma)}+(1-\delta)\bar{K}+\bar{A}\bar{K}^{\alpha}\bar{N}^{\gamma}\overline{EN}^{1-\alpha-\gamma}(\hat{A}_t)$$
$$+\alpha\bar{A}\bar{K}^{\alpha}\bar{N}^{\gamma}\overline{EN}^{(1-\alpha-\gamma)}(\hat{K}_t)+\gamma\bar{A}\bar{K}^{\alpha}\bar{N}^{\gamma}\overline{EN}^{(1-\alpha-\gamma)}(\hat{N}_t)$$
$$+(1-\alpha-\gamma)\bar{A}\bar{K}^{\alpha}\bar{N}^{\gamma}\overline{EN}^{1-\alpha-\gamma}(\hat{EN}_t)+(1-\delta)\bar{K}(\hat{K}_t)$$

$$\approx \bar{A}\bar{K}^{\alpha}\bar{N}^{\gamma}\overline{EN}^{(1-\alpha-\gamma)}(1+\hat{A}_t+\alpha\hat{K}_t+\gamma\hat{N}_t+(1-\alpha-\gamma)\hat{EN}_t)+(1-\delta)$$
$$(1+\bar{K}(\hat{K}_t))$$
$$=\bar{Y}(1+\hat{A}_t+\alpha\hat{K}_t+\gamma\hat{N}_t+(1-\alpha-\gamma)\hat{EN}_t)+[(1-\delta)\bar{K}](1+\hat{K}_t)$$

$$\text{LHS}: \bar{C}+\bar{K}+\bar{P}\,\overline{EN}+\bar{C}(\hat{C}_t)+\bar{K}(\hat{K}_{t+1})+\bar{P}\,\overline{EN}(\hat{P}_t)+\bar{P}\,\overline{EN}(\hat{EN}_t)$$

$$\text{RHS}=\text{LHS}: \bar{Y}(1+\hat{A}+\alpha\hat{K}+\gamma\hat{N}+(1-\alpha-\gamma)\hat{EN}_t)+[(1-\delta)\bar{K}](1+\hat{K}_t)$$
$$=\underline{\bar{C}+\bar{K}+\bar{P}\,\overline{EN}}+\bar{C}(\hat{C}_t)+\bar{K}(\hat{K}_{t+1})+\bar{P}\,\overline{EN}(\hat{P}_t)+\bar{P}\,\overline{EN}(\hat{EN}_t)$$

At ss: $\bar{A}\bar{K}^{\alpha}\bar{N}^{\gamma}\overline{EN}^{1-\alpha-\gamma}+(1-\delta)\bar{K}=\bar{C}+\bar{K}+\bar{P}\,\overline{EN}$

$\Leftrightarrow \quad \underline{\bar{Y}+(1-\delta)\bar{K}}=\underline{\bar{C}+\bar{K}+\bar{P}\,\overline{EN}}$

$$\Leftrightarrow \bar{Y}(\hat{A}_t+\alpha\hat{K}_t+\gamma\hat{N}_t+(1-\alpha-\gamma)\hat{EN}_t)+(1-\delta)\bar{K}\hat{K}_t=\bar{C}(\hat{C}_t)+\bar{K}(\hat{K}_{t+1})+\bar{P}\,\overline{EN}(\hat{P}_t+\hat{EN}_t)$$

$$\Leftrightarrow \hat{A}_t+\alpha\hat{K}_t+\gamma\hat{N}_t+(1-\alpha-\gamma)\hat{EN}_t+(1-\delta)\frac{\bar{K}}{\bar{Y}}\hat{K}_t=\frac{\bar{C}}{\bar{Y}}(\hat{C}_t)+\frac{\bar{K}}{\bar{Y}}(\hat{K}_{t+1})+\frac{\bar{P}\,\overline{EN}}{\bar{Y}}(\hat{P}_t+\hat{EN}_t)$$

5) $\bar{Y} e^{\hat{y}_t} = \bar{A} e^{\hat{A}_t} \bar{K}^{\alpha} e^{\alpha \hat{K}_t} \bar{N} e^{\gamma \hat{N}_t} \overline{EN} e^{\widehat{EN}_t (1-\alpha-\gamma)}$

RHS: $\bar{Y} + \bar{Y}(\hat{y}_t)$

LHS: $\bar{A} \bar{K}^{\alpha} \bar{N}^{\gamma} \overline{EN}^{1-\alpha-\gamma} + \bar{A} \bar{K}^{\alpha} \bar{N}^{\gamma} \overline{EN}^{(1-\alpha-\gamma)} (\hat{K}_t) + \gamma \bar{A} \bar{K}^{\alpha} \bar{N}^{\gamma} \overline{EN}^{(1-\alpha-\gamma)} (\hat{N}_t)$

$+ (1-\alpha-\gamma) \bar{A} \bar{K}^{\alpha} \bar{N}^{\gamma} \overline{EN}^{1-\alpha-\gamma} (\widehat{EN}_t) + \bar{A} \bar{K}^{\alpha} \bar{N}^{\gamma} \overline{EN}^{1-\alpha-\gamma} (\hat{A}_t)$

SS: $\bar{Y} = \bar{A} \bar{K}^{\alpha} \bar{N}^{\gamma} \overline{EN}^{1-\alpha-\gamma}$

$\Leftrightarrow \bar{Y}(\hat{y}_t) = \bar{Y}(\hat{A}_t + \alpha \hat{K}_t + \gamma \hat{N}_t + (1-\alpha-\gamma) \widehat{EN}_t)$

$\Leftrightarrow \hat{y}_t = \hat{A}_t + \alpha \hat{K}_t + \gamma \hat{N}_t + (1-\alpha-\gamma) \widehat{EN}_t$

⑥ $\bar{I} e^{\hat{I}_t} = \bar{K} e^{\hat{K}_{t+1}} - (1-\delta) \bar{K} e^{\hat{K}_t}$

RHS: $\bar{I} + \bar{I}(\hat{I}_t) = \bar{I}(1+\hat{I}_t)$

LHS: $\bar{K} - (1-\delta)\bar{K} + \bar{K}(\hat{K}_{t+1}) - (1-\delta)\bar{K}(\hat{K}_t)$

$= \bar{K} - \bar{K} + \delta\bar{K} + \bar{K}(\hat{K}_{t+1}) + \bar{K}(1-\delta)\hat{K}_t$

$= \bar{K}(\delta + \hat{K}_{t+1} + (1-\delta)\hat{K}_t)$

$\Leftrightarrow \bar{I}(1 + \hat{I}_t) = \bar{K}(\delta + \hat{K}_{t+1} + (1-\delta)\hat{K}_t)$

At SS: $\bar{I} = \bar{K} - (1-\delta)\bar{K}$

$\bar{I} = \bar{K} - \bar{K} + \delta\bar{K}$

$\bar{I} = \delta\bar{K}$

$\Leftrightarrow \bar{I}\hat{I}_t = \bar{K}(\hat{K}_{t+1} + (1-\delta)\hat{K}_t)$ $\downarrow$: SS

$\hat{I}_t = \dfrac{\hat{K}_{t+1}}{\delta} + \dfrac{(1-\delta)}{\delta}\hat{K}_t$

3 e)

(1) $-\sigma \hat{c}_t = E_t\left[-\sigma \hat{c}_{t+1} + (1-\beta(1-\delta))(\hat{A}_{t+1} + (\alpha-1)\hat{K}_{t+1} + \gamma\hat{N}_{t+1} + (1-\alpha-\gamma)E\hat{N}_{t+1})\right]$

(2) $\bar{Y}\hat{Y}_t = \frac{\theta}{\gamma}\bar{C}\bar{N}\hat{C}_t + \left(\frac{\theta}{\gamma}\bar{C}\bar{N} + \bar{N}\bar{Y}\right)\hat{N}_t + \bar{N}\bar{Y}\hat{Y}_t$

(3) $\overline{EN}\,\bar{P}\,(E\hat{N}_t + \hat{P}_t) = (1-\alpha-\gamma)\bar{Y}\hat{Y}_t$

(4) $\bar{Y}(\hat{A} + \alpha\hat{K}_t + \gamma\hat{N}_t + (1-\alpha-\gamma)E\hat{N}_t) + (1-\delta)\bar{K}\hat{K}_t = \bar{C}(\hat{C}_t) + \bar{K}(\hat{K}_{t+1}) + \bar{P}\overline{EN}(\hat{P} + \widehat{EN}_t)$

(5) $\hat{Y}_t = \hat{A}_t + \alpha\hat{K}_t + \gamma\hat{N}_t + (1-\alpha-\gamma)E\hat{N}_t$

(6) $\hat{I}_t = \frac{\hat{K}_{t+1}}{\delta} + \frac{(1-\delta)\hat{K}_t}{\delta}$

(7) $\log A_{t+1} = \rho_A \log A_t + \varepsilon_{A,t+1}$

(8) $\log P_{t+1} = \rho_P \log P_t + \varepsilon_{P,t+1}$

→ arise from $\hat{A}_{t+1} = \log\left(\frac{A_{t+1}}{\bar{A}}\right) = \log(A_{t+1})$

$\hat{P}_{t+1} = \log\left(\frac{P_{t+1}}{\bar{P}}\right) = \log(P_{t+1})$