PROBABILITY AND RANDOM VARIABLES Assignment 3

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Download latex-tikz codes from

https://github.com/VARSHITHAGANJI/AI1103_Probability_Assignment/blob/main/Assignment3.tex

Problem

GATE 2016 (MA) Question 48

Let $X_1, X_2, X_3, \cdots$ be a sequence of i.i.d uniform $(0, 1)$ random variables. Then the value of

$$\lim_{n\to\infty} \Pr\left(-\ln\left(1 - X_1\right) - \cdots - \ln\left(1 - X_n\right) > n\right)$$

is equal to

Solution

$$f_{X_i}(x) = \begin{cases} 1 & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

Let $Y_1, Y_2, \cdots,$ be another sequence of random variables where $Y_i = -\ln\left(1 - X_i\right), i = 1, 2, 3, \cdots$

$$f_{Y_i}(x) = \frac{f_{X_i}(x)}{\frac{dY_i}{dX_i}}$$

$$f_{Y_i}(x) = \begin{cases} e^{-x} & x > 0 \\ 0 & \text{otherwise} \end{cases}$$

From the above probability function, we have all Y_i's to be exponential random variables.

$$Y_i \sim \text{Exp}(1)$$

$$\Rightarrow \mu = 1, \sigma^2 = 1$$

The required probability is

$$\lim_{n\to\infty} \Pr\left(\sum_{i=1}^{n} Y_i > n\right) \tag{0.0.1}$$

$$= \lim_{n\to\infty} \Pr\left(\overline{Y_n} > 1\right) \tag{0.0.2}$$

Consider

$$Z = \lim_{n\to\infty} \sqrt{n}\left(\frac{\overline{Y_n} - \mu}{\sigma}\right) \tag{0.0.3}$$

Since $\overline{Y_n} > 1$, we have $Z > 0$.
By central limit theorem, we have Z to be a standard normal distribution.

$$Z \sim \mathcal{N}(0, 1) \tag{0.0.4}$$

$$\lim_{n\to\infty} \Pr\left(\overline{Y_n} > 1\right) = \Pr(Z > 0) \tag{0.0.5}$$

$$= \int_0^\infty f_Z(x)\, dx \tag{0.0.6}$$

$$= \int_0^\infty \frac{e^{\frac{-x^2}{2}}}{\sqrt{2\pi}}\, dx \tag{0.0.7}$$

$$\tag{0.0.8}$$

solving the guassian integral, we have

$$\lim_{n\to\infty} \Pr\left(\overline{Y_n} > 1\right) = \frac{1}{2} \tag{0.0.9}$$

$$\therefore \lim_{n\to\infty} \Pr\left(-\ln\left(1 - X_1\right) - \cdots - \ln\left(1 - X_n\right) > n\right) = 0.5 \tag{0.0.10}$$