Assignment 4 - Linear Forms Q2.23

Ganji Varshitha - AI20BTECH11009

September 10, 2021

Question

Linear Forms Q2.23

Find the shortest distance between the lines

$$L_1 : \mathbf{x} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \lambda_1 \begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix} \tag{1}$$

$$L_2 : \mathbf{x} = \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} + \lambda_2 \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} \tag{2}$$

Solution

Prerequisites

The general equation of a line in 3D plane can be written as :

$$\mathbf{x} = \mathbf{a} + \lambda \mathbf{b} \tag{3}$$

where **a** and **b** are positional vector and slope vector of the line respectively.
Lines can be intersecting or parallel if they are coplanar and non intersecting and non parallel if they are not coplanar

Skew Lines

Skew lines are two lines that do not intersect and are not parallel.
Example: Pair of lines through opposite edges of a regular tetrahedron.

Solution Contd.

The lines L_1 and L_2 are not parallel as $\mathbf{b_1} \neq k\mathbf{b_2}$.
Let the given lines L_1 and L_2 in the form of $\mathbf{a_i} + \lambda_i \mathbf{b_i}$ be intersecting, then

$$\begin{pmatrix}1\\2\\3\end{pmatrix} + \lambda_1 \begin{pmatrix}1\\-3\\2\end{pmatrix} = \begin{pmatrix}4\\5\\6\end{pmatrix} + \lambda_2 \begin{pmatrix}2\\3\\1\end{pmatrix} \tag{4}$$

$$\begin{pmatrix}1 & -2\\-3 & -3\\2 & -1\end{pmatrix} \begin{pmatrix}\lambda_1\\\lambda_2\end{pmatrix} = \begin{pmatrix}3\\3\\3\end{pmatrix} \tag{5}$$

The augmented matrix for (5) in row reduced form becomes

$$\begin{pmatrix}1 & -2 & 3\\-3 & -3 & 3\\2 & -1 & 3\end{pmatrix} \longleftrightarrow \begin{pmatrix}1 & -2 & 3\\0 & 1 & 9\\0 & 0 & -3\end{pmatrix} \tag{6}$$

Since the rank of the augmented matrix is 3, the system of equations is inconsistent.
Hence, the lines are not intersecting.

Since the lines are neither parallel nor intersecting, the lines are said to be skew lines.

Finding shortest distance between two skew lines

Let $\mathbf{p_1}$, $\mathbf{p_2}$ be the closest points on lines L_1 and L_2 respectively.
Then the shortest distance between two skew lines will be the length of line perpendicular to both the lines L_1, L_2 and passing through $\mathbf{p_1}$ and $\mathbf{p_2}$.
The slope of line passing through $\mathbf{p_1}$ and $\mathbf{p_2}$ is along $\mathbf{p_2} - \mathbf{p_1}$, which is perpendicular to both L_1 and L_2. Thus,

$$\mathbf{b_1}^\top (\mathbf{p_2} - \mathbf{p_1}) = 0 \tag{7}$$

$$\mathbf{b_2}^\top (\mathbf{p_2} - \mathbf{p_1}) = 0 \tag{8}$$

Let $\mathbf{B} = \begin{pmatrix} \mathbf{b_2} & \mathbf{b}_1 \end{pmatrix}$, combining (7) and (8) in terms of $\mathbf{B}$ and $\mathbf{B}^\top$, we have

$$\mathbf{B}^\top \mathbf{B} \begin{pmatrix} \lambda_2 \\ -\lambda_1 \end{pmatrix} = \mathbf{B}^\top (\mathbf{a}_1 - \mathbf{a}_2) \tag{9}$$

Substituting values of a_1, a_2, b_1, b_2, in (9)

$$\begin{pmatrix} 14 & -5 \\ -5 & 14 \end{pmatrix} \begin{pmatrix} \lambda_2 \\ -\lambda_1 \end{pmatrix} = \begin{pmatrix} -18 \\ 0 \end{pmatrix} \tag{10}$$

Solving for λ_1 and λ_2,

$$\begin{pmatrix} \lambda_2 \\ -\lambda_1 \end{pmatrix} = \begin{pmatrix} -1.4736 \\ -0.5263 \end{pmatrix} \tag{11}$$

The closest points are

$$\mathbf{p_1} = \begin{pmatrix} 1.5263 \\ 0.4210 \\ 4.0526 \end{pmatrix} \qquad \mathbf{p_2} = \begin{pmatrix} 1.0526 \\ 0.5789 \\ 4.5263 \end{pmatrix} \tag{12}$$

Therefore, the shortest distance between these two skew lines is

$$d = \|\mathbf{p_2} - \mathbf{p_1}\| = 0.6882 \tag{13}$$