LINEAR SYSTEMS AND SIGNAL PROCESSING ASSIGNMENT 4

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Download latex codes from

https://github.com/VARSHITHAGANJI/EE3900_VECTORS_ASSIGNMENTS/blob/main/LINEAR_FORMS_ASSIGNMENT4/LINEAR_FORMS_ASSIGNMENT4.tex

Download all python codes from

https://github.com/VARSHITHAGANJI/EE3900_VECTORS_ASSIGNMENTS/blob/main/LINEAR_FORMS_ASSIGNMENT4/skew_lines_code.py

QUESTION

Linearforms 2.23 Find the shortest distance between the lines

$$L_1 : \mathbf{x} = \begin{pmatrix}1\\2\\3\end{pmatrix} + \lambda_1 \begin{pmatrix}1\\-3\\2\end{pmatrix} \tag{0.0.1}$$

$$L_2 : \mathbf{x} = \begin{pmatrix}4\\5\\6\end{pmatrix} + \lambda_2 \begin{pmatrix}2\\3\\1\end{pmatrix} \tag{0.0.2}$$

SOLUTION

We have,

$$L_1 : \mathbf{x} = \mathbf{a_1} + \lambda_1 \mathbf{b_1} \tag{0.0.3}$$

$$L_2 : \mathbf{x} = \mathbf{a_2} + \lambda_2 \mathbf{b_2} \tag{0.0.4}$$

where $\mathbf{a_i}, \mathbf{b_i}$ are positional vector, slope vector of line L_i respectively.
As $\mathbf{b_1} \neq k\mathbf{b_2}$, the lines are not parallel to each other.

Let us assume that L_1 and L_2 intersect at a point. Therefore,

$$\begin{pmatrix}1\\2\\3\end{pmatrix} + \lambda_1 \begin{pmatrix}1\\-3\\2\end{pmatrix} = \begin{pmatrix}4\\5\\6\end{pmatrix} + \lambda_2 \begin{pmatrix}2\\3\\1\end{pmatrix} \tag{0.0.5}$$

$$\lambda_1 \begin{pmatrix}1\\-3\\2\end{pmatrix} + \lambda_2 \begin{pmatrix}-2\\-3\\-1\end{pmatrix} = \begin{pmatrix}3\\3\\3\end{pmatrix} \tag{0.0.6}$$

$$\begin{pmatrix}1 & -2\\-3 & -3\\2 & -1\end{pmatrix} \begin{pmatrix}\lambda_1\\\lambda_2\end{pmatrix} = \begin{pmatrix}3\\3\\3\end{pmatrix} \tag{0.0.7}$$

The augmented matrix for the above equation in row reduced form

$$\begin{pmatrix}1 & -2 & 3\\-3 & -3 & 3\\2 & -1 & 3\end{pmatrix} \xleftrightarrow{R_2 \leftarrow R_2+3R_1} \begin{pmatrix}1 & -2 & 3\\0 & 3 & 9\\2 & -1 & 3\end{pmatrix}$$
$$\xleftrightarrow{R_3 \leftarrow R_3-2R_1} \begin{pmatrix}1 & -2 & 3\\0 & 3 & 9\\0 & 3 & -3\end{pmatrix} \xleftrightarrow[R_3 \leftarrow R_3-3R_2]{R_2 \leftarrow \frac{R_2}{3}} \begin{pmatrix}1 & -2 & 3\\0 & 1 & 9\\0 & 0 & -3\end{pmatrix} \tag{0.0.8}$$

$\therefore$ The rank of the matrix = 3. Hence the lines do not intersect.
L_1 and L_2 are skew lines.
Let d be the shortest distance and $\mathbf{p_1}, \mathbf{p_2}$ be positional vectors of its end points. For d to be shortest, we know that,

$$\mathbf{b_1}^\top (\mathbf{p_2} - \mathbf{p_1}) = 0 \tag{0.0.9}$$

$$\mathbf{b_2}^\top (\mathbf{p_2} - \mathbf{p_1}) = 0 \tag{0.0.10}$$

$$\mathbf{b_1}^\top ((\mathbf{a}_2 - \mathbf{a}_1)) + \begin{pmatrix}\mathbf{b_2} & \mathbf{b}_1\end{pmatrix} \begin{pmatrix}\lambda_1\\\lambda_2\end{pmatrix} = 0 \tag{0.0.11}$$

$$\mathbf{b_2}^\top ((\mathbf{a}_2 - \mathbf{a}_1)) + \begin{pmatrix}\mathbf{b_2} & \mathbf{b_1}\end{pmatrix} \begin{pmatrix}\lambda_1\\\lambda_2\end{pmatrix} = 0 \tag{0.0.12}$$

Let

$$\mathbf{B} = \begin{pmatrix} \mathbf{b_2} & \mathbf{b}_1 \end{pmatrix} \qquad \mathbf{B}^\top = \begin{pmatrix} \mathbf{b_2}^\top \\ \mathbf{b_1}^\top \end{pmatrix} \tag{0.0.13}$$

By combining equations (0.0.11) and (0.0.12) and writing in terms of $\mathbf{B}$ and $\mathbf{B}^\top$ using (0.0.13), we get

$$\mathbf{B}^\top \mathbf{B} \begin{pmatrix} \lambda_2 \\ -\lambda_1 \end{pmatrix} = \mathbf{B}^\top (\mathbf{a}_1 - \mathbf{a}_2) \tag{0.0.14}$$

By putting the values of a_1, a_2, b_1, b_2 in (0.0.14), we get

$$\begin{pmatrix} 14 & -5 \\ -5 & 14 \end{pmatrix} \begin{pmatrix} \lambda_2 \\ -\lambda_1 \end{pmatrix} = \begin{pmatrix} -18 \\ 0 \end{pmatrix} \tag{0.0.15}$$

Solving (0.0.15), we get

$$\begin{pmatrix} \lambda_2 \\ -\lambda_1 \end{pmatrix} = \begin{pmatrix} -1.4736 \\ -0.5263 \end{pmatrix} \tag{0.0.16}$$

Substituting the value of λ_1 and λ_2 in (0.0.3) and (0.0.4), we get

$$\mathbf{p_1} = \begin{pmatrix} 1.5263 \\ 0.4210 \\ 4.0526 \end{pmatrix} \qquad \mathbf{p_2} = \begin{pmatrix} 1.0526 \\ 0.5789 \\ 4.5263 \end{pmatrix} \tag{0.0.17}$$

Hence, the shortest distance between these two skew lines is

$$d = \|\mathbf{p_2} - \mathbf{p_1}\| = 0.6882 \tag{0.0.18}$$

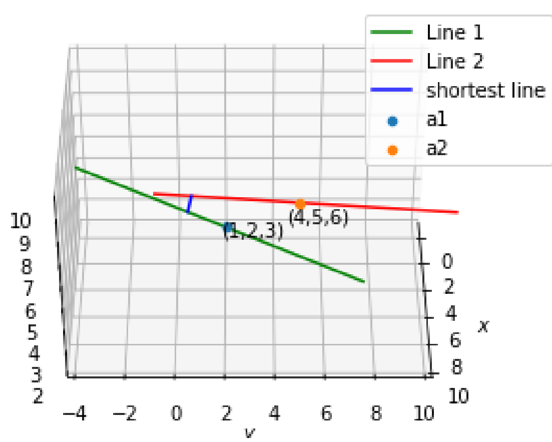

Fig. 1: Plot of skew lines