

LINEAR SYSTEMS AND SIGNAL PROCESSING

ASSIGNMENT 4

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Download latex codes from

https://github.com/VARSHITHAGANJI/EE3900_VECTORS_ASSIGNMENTS/blob/main/LINEAR_FORMS_ASSIGNMENT4/LINEAR_FORMS_ASSIGNMENT4.tex

Download all python codes from

https://github.com/VARSHITHAGANJI/EE3900_VECTORS_ASSIGNMENTS/blob/main/LINEAR_FORMS_ASSIGNMENT4/skew_lines_code.py

Let us assume that L_1 and L_2 intersect at a point. Therefore,

$$\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \lambda_1 \begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix} = \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} + \lambda_2 \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} \quad (0.0.5)$$

$$\lambda_1 \begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix} + \lambda_2 \begin{pmatrix} -2 \\ -3 \\ -1 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \\ 3 \end{pmatrix} \quad (0.0.6)$$

$$\begin{pmatrix} 1 & -2 \\ -3 & -3 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \\ 3 \end{pmatrix} \quad (0.0.7)$$

QUESTION

Linearforms 2.23 Find the shortest distance between the lines

$$L_1 : \mathbf{x} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \lambda_1 \begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix} \quad (0.0.1)$$

$$L_2 : \mathbf{x} = \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} + \lambda_2 \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} \quad (0.0.2)$$

SOLUTION

We have,

$$L_1 : \mathbf{x} = \mathbf{a}_1 + \lambda_1 \mathbf{b}_1 \quad (0.0.3)$$

$$L_2 : \mathbf{x} = \mathbf{a}_2 + \lambda_2 \mathbf{b}_2 \quad (0.0.4)$$

where $\mathbf{a}_i, \mathbf{b}_i$ are positional vector, slope vector of line L_i respectively.

As $\mathbf{b}_1 \neq k\mathbf{b}_2$, the lines are not parallel to each other.

The augmented matrix for the above equation in row reduced form

$$\begin{pmatrix} 1 & -2 & 3 \\ -3 & -3 & 3 \\ 2 & -1 & 3 \end{pmatrix} \xrightarrow{R_2 \leftarrow R_2 + 3R_1} \begin{pmatrix} 1 & -2 & 3 \\ 0 & 3 & 9 \\ 2 & -1 & 3 \end{pmatrix} \xrightarrow{R_3 \leftarrow R_3 - 2R_1} \begin{pmatrix} 1 & -2 & 3 \\ 0 & 3 & 9 \\ 0 & 3 & -3 \end{pmatrix} \xrightarrow{R_2 \leftarrow \frac{R_2}{3}} \begin{pmatrix} 1 & -2 & 3 \\ 0 & 1 & 9 \\ 0 & 0 & -3 \end{pmatrix} \xrightarrow{R_3 \leftarrow R_3 - 3R_2} \begin{pmatrix} 1 & -2 & 3 \\ 0 & 1 & 9 \\ 0 & 0 & -3 \end{pmatrix} \quad (0.0.8)$$

\therefore The rank of the matrix = 3. Hence the lines do not intersect.

L_1 and L_2 are skew lines.

Let d be the shortest distance and $\mathbf{p}_1, \mathbf{p}_2$ be positional vectors of its end points. For d to be shortest, we know that,

$$\mathbf{b}_1^\top (\mathbf{p}_2 - \mathbf{p}_1) = 0 \quad (0.0.9)$$

$$\mathbf{b}_2^\top (\mathbf{p}_2 - \mathbf{p}_1) = 0 \quad (0.0.10)$$

$$\mathbf{b}_1^\top ((\mathbf{a}_2 - \mathbf{a}_1)) + (\mathbf{b}_2 - \mathbf{b}_1) \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} \quad (0.0.11)$$

$$\mathbf{b}_2^\top ((\mathbf{a}_2 - \mathbf{a}_1)) + (\mathbf{b}_2 - \mathbf{b}_1) \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} \quad (0.0.12)$$

Let

$$\mathbf{B} = \begin{pmatrix} \mathbf{b}_2 & \mathbf{b}_1 \end{pmatrix} \quad \mathbf{B}^\top = \begin{pmatrix} \mathbf{b}_2^\top \\ \mathbf{b}_1^\top \end{pmatrix} \quad (0.0.13)$$

By combining equations (0.0.11) and (0.0.12) and writing in terms of \mathbf{B} and \mathbf{B}^\top using (0.0.13), we get

$$\mathbf{B}^\top \mathbf{B} \begin{pmatrix} \lambda_1 \\ -\lambda_2 \end{pmatrix} = \mathbf{B}^\top (\mathbf{a}_1 - \mathbf{a}_2) \quad (0.0.14)$$

By putting the values of a_1, a_2, b_1, b_2 in (0.0.14), we get

$$\begin{pmatrix} 14 & -5 \\ -5 & 14 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ -\lambda_2 \end{pmatrix} = \begin{pmatrix} -18 \\ 0 \end{pmatrix} \quad (0.0.15)$$

Solving (0.0.15), we get

$$\begin{pmatrix} \lambda_1 \\ -\lambda_2 \end{pmatrix} = \begin{pmatrix} -1.4736 \\ -0.5263 \end{pmatrix} \quad (0.0.16)$$

Substituting the value of λ_1 and λ_2 in (0.0.3) and (0.0.4), we get

$$\mathbf{p}_1 = \begin{pmatrix} -0.4736 \\ 6.4210 \\ 0.0526 \end{pmatrix} \quad \mathbf{p}_2 = \begin{pmatrix} 5.05263 \\ 6.57894 \\ 6.5263 \end{pmatrix} \quad (0.0.17)$$

Hence, the shortest distance between these two skew lines is

$$d = \|\mathbf{p}_2 - \mathbf{p}_1\| = 8.5131 \quad (0.0.18)$$

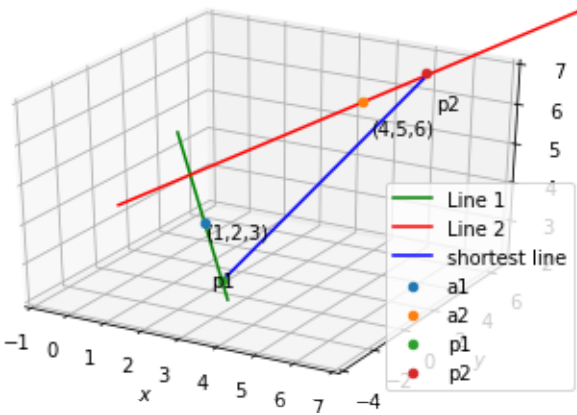


Fig. 1: Plot of skew lines