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Chapter 1

Triangle

Consider a triangle with vertices

$$\mathbf{A} = \begin{pmatrix} -3 \\ -5 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 3 \\ -5 \end{pmatrix}, \mathbf{c} = \begin{pmatrix} -4 \\ -3 \end{pmatrix}, \quad (1.1)$$

1.1. Vectors

1.2. Median

1.3. Altitude

1.4. Perpendicular Bisector

1.5. Angular Bisector

1.6. Matrix

The matrix of the vertices of the triangle is defined as

$$\mathbf{P} = \begin{pmatrix} \mathbf{A} & \mathbf{B} & \mathbf{C} \end{pmatrix} \quad (1.2)$$

$$= \begin{pmatrix} -3 & 3 & -4 \\ -5 & -5 & -3 \end{pmatrix} \quad (1.3)$$

1.6.1. Vectors

1.6.1.1. Obtain the direction matrix of the sides of $\triangle ABC$ defined as

$$\mathbf{M} = \begin{pmatrix} \mathbf{A} - \mathbf{B} & \mathbf{B} - \mathbf{C} & \mathbf{C} - \mathbf{A} \end{pmatrix} \quad (1.6.1.1.1)$$

Solution:

$$\mathbf{M} = \begin{pmatrix} \mathbf{A} - \mathbf{B} & \mathbf{B} - \mathbf{C} & \mathbf{C} - \mathbf{A} \end{pmatrix} \quad (1.6.1.1.2)$$

$$= \begin{pmatrix} \mathbf{A} & \mathbf{B} & \mathbf{C} \end{pmatrix} \begin{pmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} \quad (1.6.1.1.3)$$

$$= \begin{pmatrix} -3 & 3 & -4 \\ -5 & -5 & -3 \end{pmatrix} \begin{pmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} \quad (1.6.1.1.4)$$

Using Matrix multiplication

$$\mathbf{M} = \begin{pmatrix} -6 & 7 & -1 \\ 0 & -2 & 2 \end{pmatrix} \quad (1.6.1.1.5)$$

where the second matrix above is known as a circulant matrix. Note that the 2nd and 3rd row of the above matrix are circular shifts of the 1st row.

1.6.1.2. Obtain the normal matrix of the sides of $\triangle ABC$

Solution: Considering the rotation matrix

$$\mathbf{R} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad (1.6.1.2.1)$$

the normal matrix is obtained as

$$\mathbf{N} = \mathbf{R}\mathbf{M} \quad (1.6.1.2.2)$$

$$= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} -6 & 7 & -1 \\ 0 & -2 & 2 \end{pmatrix} \quad (1.6.1.2.3)$$

Using matrix multiplication

$$\mathbf{N} = \begin{pmatrix} 0 & 2 & -2 \\ -6 & 7 & -1 \end{pmatrix} \quad (1.6.1.2.4)$$

1.6.1.3. Obtain a, b, c .

Solution: The sides vector is obtained as

$$\mathbf{d} = \sqrt{\text{diag}(\mathbf{M}^\top \mathbf{M})} \quad (1.6.1.3.1)$$

$$\mathbf{M}^\top \mathbf{M} = \begin{pmatrix} -6 & 0 \\ 7 & -2 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} -6 & 7 & -1 \\ 0 & -2 & 2 \end{pmatrix} \quad (1.6.1.3.2)$$

Using matrix multiplication

$$\mathbf{M} = \begin{pmatrix} 36 & -42 & 6 \\ -42 & 53 & -11 \\ 6 & -11 & 5 \end{pmatrix} \quad (1.6.1.3.3)$$

$$\mathbf{d} = \sqrt{\text{diag} \left(\begin{pmatrix} 36 & -42 & 6 \\ -42 & 53 & -11 \\ 6 & -11 & 5 \end{pmatrix} \right)} \quad (1.6.1.3.4)$$

$$= \begin{pmatrix} 6 & \sqrt{53} & \sqrt{5} \end{pmatrix} \quad (1.6.1.3.5)$$

1.6.1.4. Obtain the constant terms in the equations of the sides of the triangle.

Solution: The constants for the lines can be expressed in vector form as

$$\mathbf{c} = \text{diag} \left\{ \left(\mathbf{N}^\top \mathbf{P} \right) \right\} \quad (1.6.1.4.1)$$

$$\mathbf{N}^\top \mathbf{P} = \begin{pmatrix} 0 & -6 \\ 2 & 7 \\ -2 & -1 \end{pmatrix} \begin{pmatrix} -3 & 3 & -4 \\ -5 & -5 & -3 \end{pmatrix} \quad (1.6.1.4.2)$$

$$(1.6.1.4.3)$$

Using matrix multiplication

$$= \begin{pmatrix} 30 & 30 & 18 \\ -41 & -29 & -29 \\ 11 & -1 & 11 \end{pmatrix} \quad (1.6.1.4.4)$$

$$\mathbf{c} = \text{diag} \left(\begin{pmatrix} 30 & 30 & 18 \\ -41 & -29 & -29 \\ 11 & -1 & 11 \end{pmatrix} \right) \quad (1.6.1.4.5)$$

$$= \begin{pmatrix} 30 & -29 & 11 \end{pmatrix} \quad (1.6.1.4.6)$$

1.6.2. Median

1.6.2.1. Obtain the mid point matrix for the sides of the triangle

Solution:

$$\begin{pmatrix} \mathbf{D} & \mathbf{E} & \mathbf{F} \end{pmatrix} = \frac{1}{2} \begin{pmatrix} \mathbf{A} & \mathbf{B} & \mathbf{C} \end{pmatrix} \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \quad (1.6.2.1.1)$$

$$= \frac{1}{2} \begin{pmatrix} -3 & 3 & -4 \\ -5 & -5 & -3 \end{pmatrix} \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \quad (1.6.2.1.2)$$

Using matrix multiplication

$$\begin{pmatrix} \mathbf{D} & \mathbf{E} & \mathbf{F} \end{pmatrix} = \begin{pmatrix} \frac{-1}{2} & \frac{-7}{2} & 0 \\ -4 & -4 & -5 \end{pmatrix} \quad (1.6.2.1.3)$$

1.6.2.2. Obtain the median direction matrix.

Solution: The median direction matrix is given by

$$\mathbf{M}_1 = \begin{pmatrix} \mathbf{A} - \mathbf{D} & \mathbf{B} - \mathbf{E} & \mathbf{C} - \mathbf{F} \end{pmatrix} \quad (1.6.2.2.1)$$

$$= \begin{pmatrix} \mathbf{A} - \frac{\mathbf{B}+\mathbf{C}}{2} & \mathbf{B} - \frac{\mathbf{C}+\mathbf{A}}{2} & \mathbf{C} - \frac{\mathbf{A}+\mathbf{B}}{2} \end{pmatrix} \quad (1.6.2.2.2)$$

$$= \begin{pmatrix} \mathbf{A} & \mathbf{B} & \mathbf{C} \end{pmatrix} \begin{pmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & 1 & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & 1 \end{pmatrix} \quad (1.6.2.2.3)$$

$$= \begin{pmatrix} -3 & 3 & -4 \\ -5 & -5 & -3 \end{pmatrix} \begin{pmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & 1 & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & 1 \end{pmatrix} \quad (1.6.2.2.4)$$

Using matrix multiplication

$$\mathbf{M}_1 = \begin{pmatrix} \frac{-5}{2} & \frac{13}{2} & -4 \\ -1 & -1 & 2 \end{pmatrix} \quad (1.6.2.2.5)$$

1.6.2.3. Obtain the median normal matrix.

Solution: Considering the roation matrix

$$\mathbf{R} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad (1.6.2.3.1)$$

the normal matrix is obtained as

$$\mathbf{N}_1 = \mathbf{R}\mathbf{M}_1 \quad (1.6.2.3.2)$$

$$= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \frac{-5}{2} & \frac{13}{2} & -4 \\ -1 & -1 & 2 \end{pmatrix} \quad (1.6.2.3.3)$$

$$\mathbf{N}_1 = \begin{pmatrix} 1 & 1 & -2 \\ \frac{-5}{2} & \frac{13}{2} & -4 \end{pmatrix} \quad (1.6.2.3.4)$$

1.6.2.4. Obtian the median equation constants.

$$\mathbf{c}_1 = \text{diag} \left(\left(\mathbf{N}_1^\top \begin{pmatrix} \mathbf{D} & \mathbf{E} & \mathbf{F} \end{pmatrix} \right) \right) \quad (1.6.2.4.1)$$

$$\mathbf{N}_1^\top \begin{pmatrix} \mathbf{D} & \mathbf{E} & \mathbf{F} \end{pmatrix} = \begin{pmatrix} 1 & \frac{-5}{2} \\ 1 & \frac{13}{2} \\ -2 & -4 \end{pmatrix} \begin{pmatrix} \frac{-1}{2} & \frac{-7}{2} & 0 \\ -4 & -4 & -5 \end{pmatrix} \quad (1.6.2.4.2)$$

$$(1.6.2.4.3)$$

Using matrix multiplication

$$= \begin{pmatrix} \frac{19}{2} & \frac{13}{2} & \frac{25}{2} \\ \frac{-53}{2} & \frac{-59}{2} & \frac{-65}{2} \\ 17 & 23 & 20 \end{pmatrix} \quad (1.6.2.4.4)$$

$$\mathbf{c}_1 = \text{diag} \left(\begin{pmatrix} \frac{19}{2} & \frac{13}{2} & \frac{25}{2} \\ \frac{-53}{2} & \frac{-59}{2} & \frac{-65}{2} \\ 17 & 23 & 20 \end{pmatrix} \right) \quad (1.6.2.4.5)$$

$$\mathbf{c}_1 = \begin{pmatrix} \frac{19}{2} & \frac{-59}{2} & 20 \end{pmatrix} \quad (1.6.2.4.6)$$

1.6.2.5. Obtain the centroid by finding the intersection of the medians.

Solution:

$$\left(\mathbf{N}_1^\top \mid \mathbf{c}^\top \right) = \left(\begin{array}{cc|c} 1 & \frac{-5}{2} & \frac{19}{2} \\ 1 & \frac{13}{2} & \frac{-59}{2} \\ -2 & -4 & 20 \end{array} \right) \quad (1.6.2.5.1)$$

Using Gauss-Elimination method:

$$\left(\begin{array}{cc|c} 1 & -\frac{5}{2} & \frac{19}{2} \\ 1 & \frac{13}{2} & -\frac{59}{2} \\ -2 & -4 & 20 \end{array} \right) \xleftrightarrow{R_2 \leftarrow R_2 - R_1} \left(\begin{array}{cc|c} 0 & -\frac{5}{2} & \frac{19}{2} \\ 0 & 9 & -39 \\ -2 & -4 & 20 \end{array} \right) \quad (1.6.2.5.2)$$

$$\xleftrightarrow{R_3 \leftarrow R_3 + 2R_1} \left(\begin{array}{cc|c} 1 & -\frac{5}{2} & \frac{19}{2} \\ 0 & 9 & -39 \\ 0 & -9 & 39 \end{array} \right) \quad (1.6.2.5.3)$$

$$\xleftrightarrow{R_2 \leftarrow \frac{1}{9}R_2} \left(\begin{array}{cc|c} 1 & -\frac{5}{2} & \frac{19}{2} \\ 0 & 1 & -\frac{13}{3} \\ 0 & -9 & 39 \end{array} \right) \quad (1.6.2.5.4)$$

$$\xleftrightarrow{R_1 \leftarrow R_1 + \frac{5}{2}R_2} \left(\begin{array}{cc|c} 1 & 0 & -\frac{4}{3} \\ 0 & 1 & -\frac{13}{3} \\ 0 & -9 & 39 \end{array} \right) \quad (1.6.2.5.5)$$

$$\xleftrightarrow{R_3 \leftarrow R_3 + 9R_2} \left(\begin{array}{cc|c} 1 & 0 & -\frac{4}{3} \\ 0 & 1 & -\frac{13}{3} \\ 0 & 0 & 0 \end{array} \right) \quad (1.6.2.5.6)$$

$$\text{Therefore } \mathbf{G} = \left(\begin{array}{c} -\frac{4}{3} \\ -\frac{13}{3} \\ 0 \end{array} \right) \quad (1.6.2.5.7)$$

1.6.3. Altitude

1.6.3.1. Find the normal matrix for the altitudes

Solution: The desired matrix is

$$\mathbf{M}_2 = \begin{pmatrix} \mathbf{B} - \mathbf{C} & \mathbf{C} - \mathbf{A} & \mathbf{A} - \mathbf{B} \end{pmatrix} \quad (1.6.3.1.1)$$

$$= \begin{pmatrix} \mathbf{A} & \mathbf{B} & \mathbf{C} \end{pmatrix} \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix} \quad (1.6.3.1.2)$$

$$= \begin{pmatrix} -3 & 3 & -4 \\ -5 & -5 & -3 \end{pmatrix} \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix} \quad (1.6.3.1.3)$$

Using Matrix multiplication

$$\mathbf{M}_2 = \begin{pmatrix} 7 & -1 & -6 \\ -2 & 2 & 0 \end{pmatrix} \quad (1.6.3.1.4)$$

1.6.3.2. Find the constants vector for the altitudes.

Solution: The desired vector is

$$\mathbf{c}_2 = \text{diag} \left\{ \left(\mathbf{M}^\top \mathbf{P} \right) \right\} \quad (1.6.3.2.1)$$

$$\mathbf{M}^\top \mathbf{P} = \begin{pmatrix} 7 & -2 \\ -1 & 2 \\ -6 & 0 \end{pmatrix} \begin{pmatrix} -3 & 3 & -4 \\ -5 & -5 & -3 \end{pmatrix} \quad (1.6.3.2.2)$$

$$(1.6.3.2.3)$$

Using matrix multiplication

$$\mathbf{M}^\top \mathbf{P} = \begin{pmatrix} -11 & 31 & -22 \\ -7 & -13 & -2 \\ 18 & -18 & 24 \end{pmatrix} \quad (1.6.3.2.4)$$

$$\mathbf{c}_2 = \text{diag} \left(\begin{pmatrix} -11 & 31 & -22 \\ -7 & -13 & -2 \\ 18 & -18 & 24 \end{pmatrix} \right) \quad (1.6.3.2.5)$$

$$\mathbf{c}_2 = \begin{pmatrix} -11 & -13 & 24 \end{pmatrix} \quad (1.6.3.2.6)$$

1.6.4. Perpendicular Bisector

1.6.4.1. Find the normal matrix for the perpendicular bisectors

Solution: The normal matrix is \mathbf{M}_2

$$\mathbf{M}_2 = \begin{pmatrix} 7 & -1 & -6 \\ -2 & 2 & 0 \end{pmatrix} \quad (1.6.4.1.1)$$

1.6.4.2. Find the constants vector for the perpendicular bisectors.

Solution: The desired vector is

$$\mathbf{c}_3 = \text{diag} \left\{ \mathbf{M}_2^\top \begin{pmatrix} \mathbf{D} & \mathbf{E} & \mathbf{F} \end{pmatrix} \right\} \quad (1.6.4.2.1)$$

Solution:

$$\mathbf{c}_3 = \text{diag} \left\{ \mathbf{M}_2^\top \begin{pmatrix} \mathbf{D} & \mathbf{E} & \mathbf{F} \end{pmatrix} \right\} \quad (1.6.4.2.2)$$

$$\mathbf{M}_2^\top \begin{pmatrix} \mathbf{D} & \mathbf{E} & \mathbf{F} \end{pmatrix} = \begin{pmatrix} 7 & -2 \\ -1 & 2 \\ -6 & 0 \end{pmatrix} \begin{pmatrix} \frac{-1}{2} & \frac{-7}{2} & 0 \\ -4 & -4 & -5 \end{pmatrix} \quad (1.6.4.2.3)$$

$$(1.6.4.2.4)$$

Using matrix multiplication

$$\mathbf{M}_2^\top \begin{pmatrix} \mathbf{D} & \mathbf{E} & \mathbf{F} \end{pmatrix} = \begin{pmatrix} \frac{9}{2} & \frac{-33}{2} & 10 \\ \frac{-15}{2} & \frac{-9}{2} & -10 \\ 3 & 21 & 0 \end{pmatrix} \quad (1.6.4.2.5)$$

$$\mathbf{c}_3 = \text{diag} \left(\begin{pmatrix} \frac{9}{2} & \frac{-33}{2} & 10 \\ \frac{-15}{2} & \frac{-9}{2} & -10 \\ 3 & 21 & 0 \end{pmatrix} \right) \quad (1.6.4.2.6)$$

$$\mathbf{c}_3 = \begin{pmatrix} \frac{9}{2} & \frac{-9}{2} & 0 \end{pmatrix} \quad (1.6.4.2.7)$$

1.6.5. Angle Bisector

1.6.5.1. Find the points of contact.

Solution: The points of contact are given by

$$\begin{pmatrix} \frac{n\mathbf{A}+p\mathbf{C}}{n+p} & \frac{p\mathbf{B}+m\mathbf{A}}{p+m} & \frac{m\mathbf{C}+n\mathbf{B}}{m+n} \end{pmatrix} = \begin{pmatrix} \mathbf{A} & \mathbf{B} & \mathbf{C} \end{pmatrix} \begin{pmatrix} \frac{n}{b} & \frac{m}{c} & 0 \\ 0 & \frac{p}{c} & \frac{n}{a} \\ \frac{p}{b} & 0 & \frac{m}{a} \end{pmatrix} \quad (1.6.5.1.1)$$

$$\begin{pmatrix} \mathbf{p} & \mathbf{m} & \mathbf{n} \end{pmatrix} = \frac{1}{2} \begin{pmatrix} \mathbf{a} & \mathbf{b} & \mathbf{c} \end{pmatrix} \begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix} \quad (1.6.5.1.2)$$

$$= \frac{1}{2} \begin{pmatrix} 6 & \sqrt{53} & \sqrt{5} \end{pmatrix} \begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix} \quad (1.6.5.1.3)$$

$$= \frac{1}{2} \begin{pmatrix} 6 & 7.280109889 & 2.236067977 \end{pmatrix} \begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix} \quad (1.6.5.1.4)$$

Using matrix multiplication

$$\begin{pmatrix} \mathbf{p} & \mathbf{m} & \mathbf{n} \end{pmatrix} = \begin{pmatrix} 1.758088933 & 0.4479790441 & 5.522020956 \end{pmatrix}$$

(1.6.5.1.5)

$$\begin{pmatrix} \mathbf{A} & \mathbf{B} & \mathbf{C} \end{pmatrix} \begin{pmatrix} \frac{n}{b} & \frac{m}{c} & 0 \\ 0 & \frac{p}{c} & \frac{n}{a} \\ \frac{p}{b} & 0 & \frac{m}{a} \end{pmatrix} = \begin{pmatrix} -3 & 3 & -4 \\ -5 & -5 & -3 \end{pmatrix} \begin{pmatrix} \frac{5.522020956}{\sqrt{53}} & \frac{0.4479790441}{\sqrt{5}} & 0 \\ 0 & \frac{1.758088933}{\sqrt{5}} & \frac{5.522020956}{6} \\ \frac{1.758088933}{\sqrt{53}} & 0 & \frac{0.4479790441}{6} \end{pmatrix}$$

(1.6.5.1.6)

Using matrix multiplication We get the points of contact

$$= \begin{pmatrix} -3.24149209 & 1.71744762 & 2.44235781 \\ -4.51701582 & -5 & -4.84067366 \end{pmatrix} \quad (1.6.5.1.7)$$