Contents

Chapter 1

Triangle

Consider a triangle with vertices

$$\mathbf{A} = \begin{pmatrix} -3 \\ -5 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 3 \\ -5 \end{pmatrix}, \mathbf{c} = \begin{pmatrix} -4 \\ -3 \end{pmatrix}, \tag{1.1}$$

1.1. Vectors

1.2. median

1.2.1. If $\mathbf{D}$ divides BC in the ratio $k : 1$,

$$\mathbf{D} = \frac{k\mathbf{C} + \mathbf{B}}{k+1} \tag{1.2.1.1}$$

Find the mid points $\mathbf{D}, \mathbf{E}, \mathbf{F}$ of the sides BC, CA and AB respectively.

If $\mathbf{D}$ divides BC in the ratio $k : 1$,

$$\mathbf{D} = \frac{k\mathbf{C} + \mathbf{B}}{k+1} \tag{1.2.1.2}$$

Find the mid points $\mathbf{D}, \mathbf{E}, \mathbf{F}$ of the sides BC, CA and AB respectively.

Given:

$$\mathbf{A} = \begin{pmatrix} -3 \\ -5 \end{pmatrix} \tag{1.2.1.3}$$

$$\mathbf{B} = \begin{pmatrix} 3 \\ -5 \end{pmatrix} \tag{1.2.1.4}$$

$$\mathbf{C} = \begin{pmatrix} -4 \\ -3 \end{pmatrix} \tag{1.2.1.5}$$

Solution: Since $\mathbf{D}$ is the midpoint of BC,

$$k = 1 \tag{1.2.1.6}$$

$$\implies \mathbf{D} = \frac{\mathbf{C} + \mathbf{B}}{2} \tag{1.2.1.7}$$

$$= \frac{1}{2} \begin{pmatrix} -1 \\ -8 \end{pmatrix} \tag{1.2.1.8}$$

Similarly,

$$\mathbf{E} = \frac{\mathbf{A}+\mathbf{C}}{2} \tag{1.2.1.9}$$

$$= \frac{1}{2}\begin{pmatrix} -7 \\ -8 \end{pmatrix} \tag{1.2.1.10}$$

$$\mathbf{F} = \frac{\mathbf{A}+\mathbf{B}}{2} \tag{1.2.1.11}$$

$$= \begin{pmatrix} 0 \\ -5 \end{pmatrix} \tag{1.2.1.12}$$

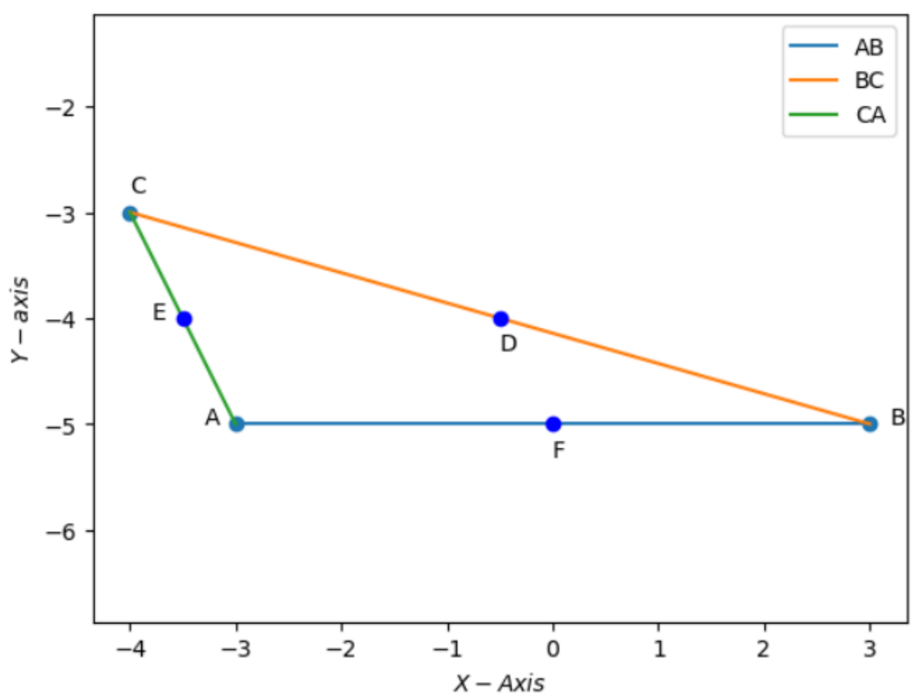

Figure 1.1: Triangle ABC with midpoints D,E and F

1.2.2. Find the equations of AD, BE and CF.

Solution: : $\mathbf{D}$,$\mathbf{E}$,$\mathbf{F}$ are the midpoints of BC,CA,AB respectively, then

$$\mathbf{D} = \begin{pmatrix} \frac{-1}{2} \\ -4 \end{pmatrix} \tag{1.2.2.1}$$

$$\mathbf{E} = \begin{pmatrix} \frac{-7}{2} \\ -4 \end{pmatrix} \tag{1.2.2.2}$$

$$\mathbf{F} = \begin{pmatrix} 0 \\ -5 \end{pmatrix} \tag{1.2.2.3}$$

(a) The normal equation for the median AD is

$$\mathbf{n}^\top \left(\mathbf{x} - \mathbf{A}\right) = 0 \tag{1.2.2.4}$$

$$\implies \mathbf{n}^\top \mathbf{x} = \mathbf{n}^\top \mathbf{A} \tag{1.2.2.5}$$

We have to find the $\mathbf{n}$ so that we can find $\mathbf{n}^\top$. Since,

$$\mathbf{n} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \mathbf{m} \tag{1.2.2.6}$$

Here $\mathbf{m} = \mathbf{D} - \mathbf{A}$ for median AD

$$\mathbf{m} = \begin{pmatrix} \frac{-1}{2} \\ -4 \end{pmatrix} - \begin{pmatrix} -3 \\ -5 \end{pmatrix} \tag{1.2.2.7}$$

$$= \begin{pmatrix} \frac{5}{2} \\ 1 \end{pmatrix} \tag{1.2.2.8}$$

Since,

$$\mathbf{n} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \mathbf{m} \tag{1.2.2.9}$$

$$\implies \mathbf{n} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \frac{5}{2} \\ 1 \end{pmatrix} \tag{1.2.2.10}$$

$$= \begin{pmatrix} 1 \\ \frac{-5}{2} \end{pmatrix} \tag{1.2.2.11}$$

Hence the normal equation of median AD is

$$\begin{pmatrix} 1 & \frac{-5}{2} \end{pmatrix} \mathbf{x} = \begin{pmatrix} 1 & \frac{-5}{2} \end{pmatrix} \begin{pmatrix} -3 \\ -5 \end{pmatrix} \tag{1.2.2.12}$$

$$\implies \begin{pmatrix} 1 & \frac{-5}{2} \end{pmatrix} \mathbf{x} = \frac{19}{2} \tag{1.2.2.13}$$

(b) The normal equation for the median BE is

$$\mathbf{n}^\top \left(\mathbf{x} - \mathbf{B}\right) = 0 \tag{1.2.2.14}$$

$$\implies \mathbf{n}^\top \mathbf{x} = \mathbf{n}^\top \mathbf{B} \tag{1.2.2.15}$$

Here $\mathbf{m} = \mathbf{E} - \mathbf{B}$ for median BE

$$\mathbf{m} = \begin{pmatrix} \frac{-7}{2} \\ -4 \end{pmatrix} - \begin{pmatrix} 3 \\ -5 \end{pmatrix} \tag{1.2.2.16}$$

$$= \begin{pmatrix} \frac{-13}{2} \\ 1 \end{pmatrix} \tag{1.2.2.17}$$

Since,

$$\mathbf{n} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \mathbf{m} \tag{1.2.2.18}$$

$$\implies \mathbf{n} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \frac{-13}{2} \\ 1 \end{pmatrix} \tag{1.2.2.19}$$

$$= \begin{pmatrix} 1 \\ \frac{13}{2} \end{pmatrix} \tag{1.2.2.20}$$

Hence the normal equation of median BE is

$$\begin{pmatrix} 1 & \frac{13}{2} \end{pmatrix} \mathbf{x} = \begin{pmatrix} 1 & \frac{13}{2} \end{pmatrix} \begin{pmatrix} 3 \\ -5 \end{pmatrix} \tag{1.2.2.21}$$

$$\implies \begin{pmatrix} 1 & \frac{13}{2} \end{pmatrix} \mathbf{x} = \frac{-59}{2} \tag{1.2.2.22}$$

(c) The normal equation for the median CF is

$$\mathbf{n}^\top \left(\mathbf{x} - \mathbf{C} \right) = 0 \tag{1.2.2.23}$$

$$\implies \mathbf{n}^\top \mathbf{x} = \mathbf{n}^\top \mathbf{C} \tag{1.2.2.24}$$

Here $\mathbf{m} = \mathbf{F} - \mathbf{C}$ for median CF

$$\mathbf{m} = \begin{pmatrix} 0 \\ -5 \end{pmatrix} - \begin{pmatrix} -4 \\ -3 \end{pmatrix} \tag{1.2.2.25}$$

$$= \begin{pmatrix} 4 \\ -2 \end{pmatrix} \tag{1.2.2.26}$$

Since,

$$\mathbf{n} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \mathbf{m} \tag{1.2.2.27}$$

$$\implies \mathbf{n} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 4 \\ -2 \end{pmatrix} \tag{1.2.2.28}$$

$$= \begin{pmatrix} -2 \\ -4 \end{pmatrix} \tag{1.2.2.29}$$

Hence the normal equation of median CF is

$$\begin{pmatrix} -2 & -4 \end{pmatrix} \mathbf{x} = \begin{pmatrix} -2 & -4 \end{pmatrix} \begin{pmatrix} -4 \\ -3 \end{pmatrix} \tag{1.2.2.30}$$

$$\implies \begin{pmatrix} -2 & -4 \end{pmatrix} \mathbf{x} = 20 \tag{1.2.2.31}$$

1.2.3. Find the intersection $\mathbf{G}$ of BE and CF

Solution: $\mathbf{A}, \mathbf{B}$ and $\mathbf{C}$ are vertices of triangle:

$$\mathbf{A} = \begin{pmatrix} -3 \\ -5 \end{pmatrix} \tag{1.2.3.1}$$

$$\mathbf{B} = \begin{pmatrix} 3 \\ -5 \end{pmatrix} \tag{1.2.3.2}$$

$$\mathbf{C} = \begin{pmatrix} -4 \\ -3 \end{pmatrix} \tag{1.2.3.3}$$

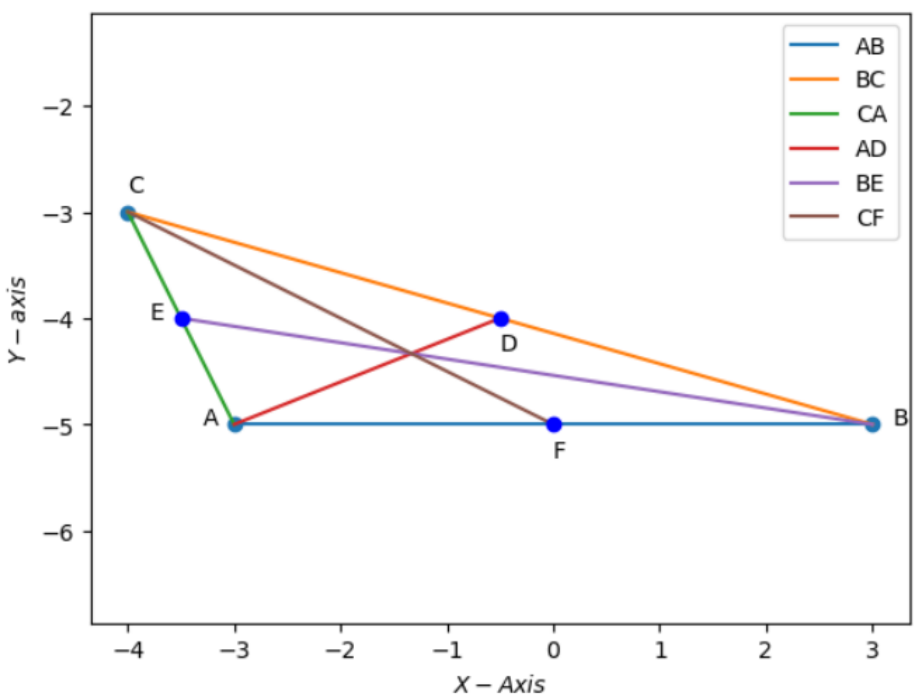

Figure 1.2: Medians AD , BE and CF

Since $\mathbf{E}$ and $\mathbf{F}$ are midpoints of CA and AB,

$$\mathbf{E} = \frac{\mathbf{A}+\mathbf{C}}{2} \tag{1.2.3.4}$$

$$= \begin{pmatrix} \frac{-7}{2} \\ -4 \end{pmatrix} \tag{1.2.3.5}$$

$$\mathbf{F} = \frac{\mathbf{B}+\mathbf{A}}{2} \tag{1.2.3.6}$$

$$= \begin{pmatrix} 0 \\ -5 \end{pmatrix} \tag{1.2.3.7}$$

The line BE in vector form is given by

$$\begin{pmatrix} 1 & \frac{13}{2} \end{pmatrix} \mathbf{x} = \begin{pmatrix} \frac{-59}{2} \end{pmatrix} \tag{1.2.3.8}$$

The line CF in vector form is given by

$$\begin{pmatrix} -2 & -4 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 20 \end{pmatrix} \tag{1.2.3.9}$$

From (**??**) and (**??**) the augmented matrix is:

$$\begin{pmatrix} 1 & \frac{13}{2} & \frac{-59}{2} \\ -2 & -4 & 20 \end{pmatrix} \tag{1.2.3.10}$$

Solve for $\mathbf{x}$ using Gauss-Elimination method:

$$\begin{pmatrix} 1 & \frac{13}{2} & \frac{-59}{2} \\ -2 & -4 & 20 \end{pmatrix} \xleftrightarrow{R_2 \leftarrow R_2 + 2R_1} \begin{pmatrix} 1 & \frac{13}{2} & \frac{-59}{2} \\ 0 & 9 & -39 \end{pmatrix} \tag{1.2.3.11}$$

$$\xleftrightarrow{R_2 \leftarrow R_2/9} \begin{pmatrix} 1 & \frac{13}{2} & \frac{-59}{2} \\ 0 & 1 & \frac{-13}{3} \end{pmatrix} \tag{1.2.3.12}$$

$$\xleftrightarrow{R_1 \leftarrow R_1 - \frac{13}{2} R_2} \begin{pmatrix} 1 & 0 & \frac{-4}{3} \\ 0 & 1 & \frac{-13}{3} \end{pmatrix} \tag{1.2.3.13}$$

Therefore,

$$\mathbf{G} = \begin{pmatrix} \frac{-4}{3} \\ \frac{-13}{3} \end{pmatrix} \tag{1.2.3.14}$$

From Fig. **??**, We can see that $\mathbf{G} = \begin{pmatrix} \frac{-4}{3} \\ \frac{-13}{3} \end{pmatrix}$ is the intersection of BE and CF

1.2.4. Verify that

$$\frac{BG}{GE} = \frac{CG}{GF} = \frac{AG}{GD} = 2 \tag{1.2.4.1}$$

Question 1.2.4:Verify that

$$\frac{BG}{GE} = \frac{CG}{GF} = \frac{AG}{GD} = 2 \tag{1.2.4.2}$$

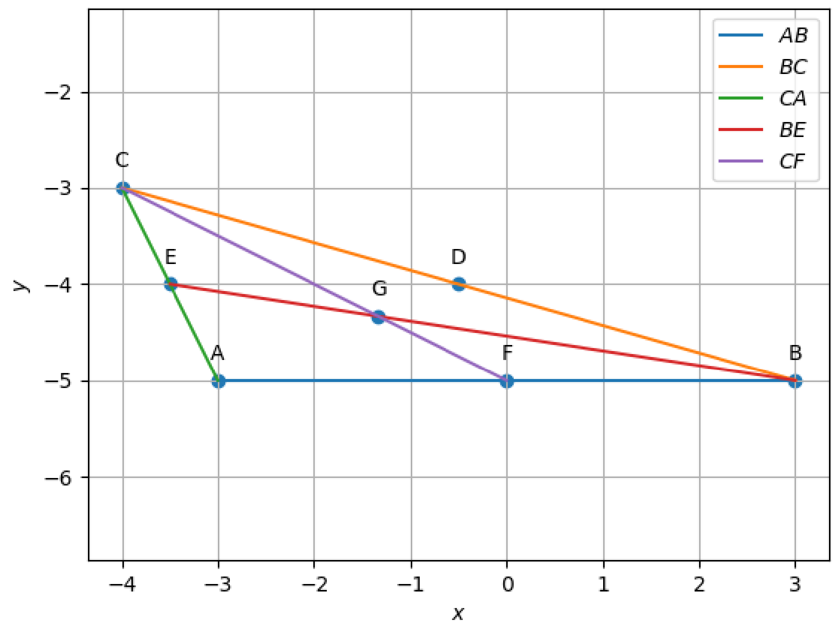

Figure 1.3: G is the centroid of triangle ABC

Solution: In order to verify the above equation we first need to find $\mathbf{G}$.$\mathbf{G}$ is the intersection of BE and CF,Using the value of $\mathbf{G}$ from (1.2.3).

$$\mathbf{G} = \begin{pmatrix} \frac{-4}{3} \\ \frac{-13}{3} \end{pmatrix} \tag{1.2.4.3}$$

Also,We know that $\mathbf{D}, \mathbf{E}$ and $\mathbf{F}$ are midpoints of BC, CA and AB

respectively from (1.2.1).

$$\mathbf{D} = \begin{pmatrix} \frac{-1}{2} \\ -4 \end{pmatrix}, \mathbf{E} = \begin{pmatrix} \frac{-7}{2} \\ -4 \end{pmatrix}, \mathbf{F} = \begin{pmatrix} 0 \\ -5 \end{pmatrix} \tag{1.2.4.4}$$

(a) Calculating the ratio of BG and GE,

$$\mathbf{G} - \mathbf{B} = \begin{pmatrix} \frac{-13}{3} \\ \frac{2}{2} \end{pmatrix} \tag{1.2.4.5}$$

$$\mathbf{E} - \mathbf{G} = \begin{pmatrix} \frac{-13}{6} \\ \frac{1}{3} \end{pmatrix} \tag{1.2.4.6}$$

$$\|\mathbf{G} - \mathbf{B}\| = \sqrt{\left(\frac{-13}{3}\right)^2 + \left(\frac{2}{3}\right)^2} = \frac{\sqrt{173}}{3} \tag{1.2.4.7}$$

$$\|\mathbf{E} - \mathbf{G}\| = \sqrt{\left(\frac{-13}{6}\right)^2 + \left(\frac{1}{3}\right)^2} = \frac{\sqrt{173}}{6} \tag{1.2.4.8}$$

$$\frac{BG}{GE} = \frac{\|\mathbf{G} - \mathbf{B}\|}{\|\mathbf{E} - \mathbf{G}\|} = \frac{\frac{\sqrt{173}}{3}}{\frac{\sqrt{173}}{6}} = 2 \tag{1.2.4.9}$$

(b) Calculating the ratio of CG and GF,

$$\mathbf{G}-\mathbf{C}=\begin{pmatrix}\frac{8}{3}\\ \frac{-4}{3}\end{pmatrix} \tag{1.2.4.10}$$

$$\mathbf{F}-\mathbf{G}=\begin{pmatrix}\frac{4}{3}\\ \frac{-2}{3}\end{pmatrix} \tag{1.2.4.11}$$

$$\|\mathbf{G}-\mathbf{C}\|=\sqrt{\left(\frac{8}{3}\right)^2+\left(\frac{-4}{3}\right)^2}=\frac{\sqrt{80}}{3} \tag{1.2.4.12}$$

$$\|\mathbf{F}-\mathbf{G}\|=\sqrt{\left(\frac{4}{3}\right)^2+\left(\frac{-2}{3}\right)^2}=\frac{\sqrt{20}}{3} \tag{1.2.4.13}$$

$$\frac{CG}{GF}=\frac{\|\mathbf{G}-\mathbf{C}\|}{\|\mathbf{F}-\mathbf{G}\|}=\frac{\frac{\sqrt{80}}{3}}{\frac{\sqrt{20}}{3}}=2 \tag{1.2.4.14}$$

(c) Calculating the ratio of AG and GD,

$$\mathbf{G}-\mathbf{A}=\begin{pmatrix}\frac{5}{3}\\ \frac{2}{3}\end{pmatrix} \tag{1.2.4.15}$$

$$\mathbf{D}-\mathbf{G}=\begin{pmatrix}\frac{5}{6}\\ \frac{1}{3}\end{pmatrix} \tag{1.2.4.16}$$

$$\|\mathbf{G}-\mathbf{A}\|=\sqrt{\left(\frac{5}{3}\right)^2+\left(\frac{2}{3}\right)^2}=\frac{\sqrt{29}}{3} \tag{1.2.4.17}$$

$$\|\mathbf{D}-\mathbf{G}\|=\sqrt{\left(\frac{5}{6}\right)^2+\left(\frac{1}{3}\right)^2}=\frac{\sqrt{29}}{6} \tag{1.2.4.18}$$

$$\frac{AG}{GD}=\frac{\|\mathbf{G}-\mathbf{A}\|}{\|\mathbf{D}-\mathbf{G}\|}=\frac{\frac{\sqrt{29}}{3}}{\frac{\sqrt{29}}{6}}=2 \tag{1.2.4.19}$$

From (??), (??), (??)

$$\frac{BG}{GE} = \frac{CG}{GF} = \frac{AG}{GD} = 2 \tag{1.2.4.20}$$

Hence verified.

1.2.5. Show that $\mathbf{A}, \mathbf{G}$ and $\mathbf{D}$ are collinear.

Solution: Given that,

$$\mathbf{A} = \begin{pmatrix} -3 \\ -5 \end{pmatrix} \quad \mathbf{B} = \begin{pmatrix} 3 \\ -5 \end{pmatrix} \quad \mathbf{C} = \begin{pmatrix} -4 \\ -3 \end{pmatrix} \tag{1.2.5.1}$$

We need to show that points $\mathbf{A}, \mathbf{D}, \mathbf{G}$ are collinear. From Problem 1.2.3 We know that, The point $\mathbf{G}$ is

$$\mathbf{G} = \begin{pmatrix} \frac{-4}{3} \\ \frac{-13}{3} \end{pmatrix} \tag{1.2.5.2}$$

And from Problem 1.2.1 We know that, The point $\mathbf{D}$ is

$$\mathbf{D} = \begin{pmatrix} \frac{-1}{2} \\ -4 \end{pmatrix} \tag{1.2.5.3}$$

In Problem 1.1.3, There is a theorem/law mentioned i.e.,

Points $\mathbf{A}, \mathbf{D}, \mathbf{G}$ are defined to be collinear if

$$\text{rank} \begin{pmatrix} 1 & 1 & 1 \\ \mathbf{A} & \mathbf{D} & \mathbf{G} \end{pmatrix} = 2 \tag{1.2.5.4}$$

Using the above law/Theorem Let

$$\mathbf{R} = \begin{pmatrix} 1 & 1 & 1 \\ -3 & \frac{-1}{2} & \frac{-4}{3} \\ -5 & -4 & \frac{-13}{3} \end{pmatrix} \tag{1.2.5.5}$$

The matrix $\mathbf{R}$ can be row reduced as follows,

$$\begin{pmatrix} 1 & 1 & 1 \\ -3 & \frac{-1}{2} & \frac{-4}{3} \\ -5 & -4 & \frac{-13}{3} \end{pmatrix} \xleftrightarrow{R_2 \leftarrow R_2 + 3R_1} \begin{pmatrix} 1 & 1 & 1 \\ 0 & \frac{5}{2} & \frac{5}{3} \\ 0 & 1 & \frac{2}{3} \end{pmatrix} \tag{1.2.5.6}$$

$$\xleftrightarrow{R_2 \leftarrow \frac{2}{5} R_2} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & \frac{2}{3} \\ 0 & 1 & \frac{2}{3} \end{pmatrix} \tag{1.2.5.7}$$

$$\xleftrightarrow{R_3 \leftarrow R_3 - R_2} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & \frac{2}{3} \\ 0 & 0 & 0 \end{pmatrix} \tag{1.2.5.8}$$

Rank of above matrix is 2.

Hence, we proved that that points $\mathbf{A}, \mathbf{D}, \mathbf{G}$ are collinear.

1.2.6. Verify that

$$\mathbf{G} = \frac{\mathbf{A} + \mathbf{B} + \mathbf{C}}{3} \tag{1.2.6.1}$$

$\mathbf{G}$ is known as the <u>centroid</u> of $\triangle ABC$.

Verify that

$$\mathbf{G} = \frac{\mathbf{A}+\mathbf{B}+\mathbf{C}}{3} \tag{1.2.6.2}$$

$\mathbf{G}$ is known as the <u>centroid</u> of $\triangle$ABC SOLUTION:

let us first evaluate the R.H.S of the equation

$$\begin{aligned} \mathbf{G} &= \frac{\begin{pmatrix}-3\\-5\end{pmatrix}+\begin{pmatrix}3\\-5\end{pmatrix}+\begin{pmatrix}-4\\-3\end{pmatrix}}{3} \\ &= \begin{pmatrix}\frac{-3+3-4}{3}\\ \frac{-5-5-3}{3}\end{pmatrix} \\ &= \begin{pmatrix}\frac{-4}{3}\\ \frac{-13}{3}\end{pmatrix} \end{aligned} \tag{1.2.6.3}$$

Hence verified.

1.2.7. Verify that

$$\mathbf{A}-\mathbf{F} = \mathbf{E}-\mathbf{D} \tag{1.2.7.1}$$

The quadrilateral $AFDE$ is defined to be a parallelogram.

Question : Verify that

$$\mathbf{A}-\mathbf{F} = \mathbf{E}-\mathbf{D} \tag{1.2.7.2}$$

The quadrilateral $AFDE$ is defined to be parallelogram

Solution: Given that,

$$\mathbf{A} = \begin{pmatrix} -3 \\ -5 \end{pmatrix} \quad \mathbf{B} = \begin{pmatrix} 3 \\ -5 \end{pmatrix} \quad \mathbf{C} = \begin{pmatrix} -4 \\ -3 \end{pmatrix} \tag{1.2.7.3}$$

From Problem 1.2.1 We know that, The point $\mathbf{D}, \mathbf{E}, \mathbf{F}$ is

$$\mathbf{D} = \begin{pmatrix} \frac{-1}{2} \\ -4 \end{pmatrix} \quad \mathbf{E} = \begin{pmatrix} \frac{-7}{2} \\ -4 \end{pmatrix} \quad \mathbf{F} = \begin{pmatrix} 0 \\ -5 \end{pmatrix} \tag{1.2.7.4}$$

Evaluating the R.H.S of the equation

$$\mathbf{A} - \mathbf{F} = \begin{pmatrix} -3 \\ -5 \end{pmatrix} - \begin{pmatrix} 0 \\ -5 \end{pmatrix} \tag{1.2.7.5}$$

$$= \begin{pmatrix} -3 \\ 0 \end{pmatrix} \tag{1.2.7.6}$$

Evaluating the L.H.S of the equation

$$\mathbf{E} - \mathbf{D} = \begin{pmatrix} \frac{-7}{2} \\ -4 \end{pmatrix} - \begin{pmatrix} \frac{-1}{2} \\ -4 \end{pmatrix} \tag{1.2.7.7}$$

$$= \begin{pmatrix} -3 \\ 0 \end{pmatrix} \tag{1.2.7.8}$$

Hence verified that, R.H.S = L.H.S i.e.,

$$\mathbf{A} - \mathbf{F} = \mathbf{E} - \mathbf{D} \tag{1.2.7.9}$$

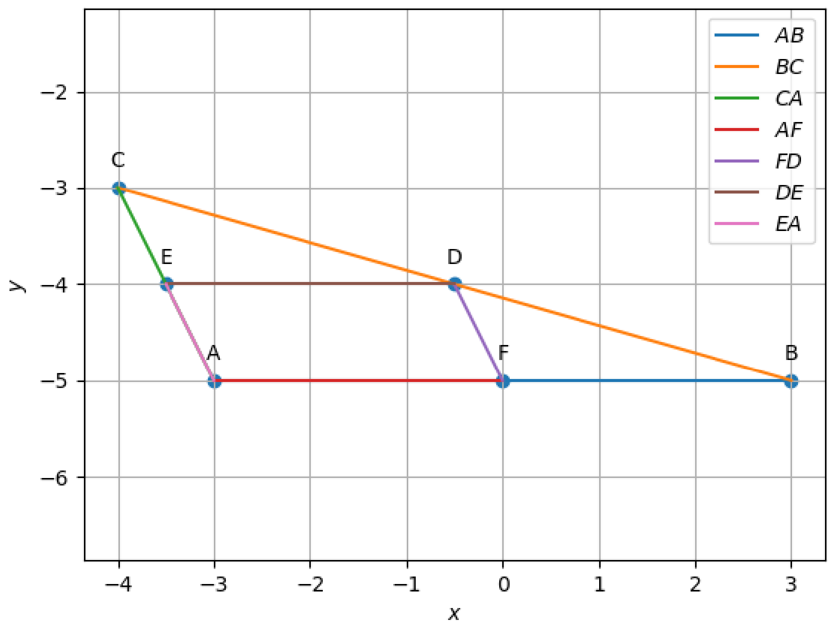

Figure 1.4: $AFDE$ form a parallelogram in triangle ABC

From the fig**??**, It is verified that $AFDE$ is a parallelogram