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# Chapter 1

## Triangle

Consider a triangle with vertices

$$\mathbf{A} = \begin{pmatrix} -3 \\ -5 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 3 \\ -5 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} -4 \\ -3 \end{pmatrix}, \quad (1.1)$$

## 1.1. Vectors

## 1.2. Median

## 1.3. Altitude

## 1.4. Perpendicular Bisector

## 1.5. Angle Bisector

1.5.1. Let  $\mathbf{D}_3, \mathbf{E}_3, \mathbf{F}_3$ , be points on  $AB, BC$  and  $CA$  respectively such that

$$AE_3 = AF_3 = m, BD_3 = BF_3 = n, CD_3 = CE_3 = p. \quad (1.5.1.1)$$

Obtain  $m, n, p$  in terms of  $a, b, c$  obtained in Question 1.1.2.

**Solution:** From Question 1.1.2

$$a = \sqrt{53} \quad (1.5.1.2)$$

$$b = \sqrt{5} \quad (1.5.1.3)$$

$$c = \sqrt{36} \quad (1.5.1.4)$$

From the given information,

$$a = m + n, \tag{1.5.1.5}$$

$$b = n + p, \tag{1.5.1.6}$$

$$c = m + p \tag{1.5.1.7}$$

which can be expressed as

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} m \\ n \\ p \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \tag{1.5.1.8}$$

$$\Rightarrow \begin{pmatrix} m \\ n \\ p \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} a \\ b \\ c \end{pmatrix} \tag{1.5.1.9}$$

Using row reduction,

$$\begin{pmatrix} 1 & 1 & 0 & | & 1 & 0 & 0 \\ 0 & 1 & 1 & | & 0 & 1 & 0 \\ 1 & 0 & 1 & | & 0 & 0 & 1 \end{pmatrix} \xleftrightarrow{R_3 \leftarrow R_3 - R_1} \begin{pmatrix} 1 & 1 & 0 & | & 1 & 0 & 0 \\ 0 & 1 & 1 & | & 0 & 1 & 0 \\ 0 & -1 & 1 & | & -1 & 0 & 1 \end{pmatrix} \quad (1.5.1.10)$$

$$\xleftrightarrow[R_1 \leftarrow R_1 - R_2]{R_3 \leftarrow R_3 + R_2} \begin{pmatrix} 1 & 0 & -1 & | & 1 & -1 & 0 \\ 0 & 1 & 1 & | & 0 & 1 & 0 \\ 0 & 0 & 2 & | & -1 & 1 & 1 \end{pmatrix} \quad (1.5.1.11)$$

$$\xleftrightarrow[R_1 \leftarrow 2R_1 + R_3]{R_2 \leftarrow 2R_2 - R_3} \begin{pmatrix} 2 & 0 & 0 & | & 1 & -1 & 1 \\ 0 & 2 & 0 & | & 1 & 1 & -1 \\ 0 & 0 & 2 & | & -1 & 1 & 1 \end{pmatrix} \quad (1.5.1.12)$$

yielding

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}^{-1} = \frac{1}{2} \begin{pmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ -1 & 1 & 1 \end{pmatrix} \quad (1.5.1.13)$$

Therefore,

$$\begin{aligned} p &= \frac{c + b - a}{2} = \frac{\sqrt{36} + \sqrt{5} - \sqrt{53}}{2} \\ m &= \frac{a + c - b}{2} = \frac{\sqrt{53} + \sqrt{36} - \sqrt{5}}{2} \\ n &= \frac{a + b - c}{2} = \frac{\sqrt{53} + \sqrt{5} - \sqrt{36}}{2} \end{aligned} \quad (1.5.1.14)$$

on solving above equations we get

$$p = 0.4479790441 \quad (1.5.1.15)$$

$$m = 50522020956 \quad (1.5.1.16)$$

$$n = 1.758088933 \quad (1.5.1.17)$$

1.5.2. Using section formula, find  $\mathbf{D}_3, \mathbf{E}_3, \mathbf{F}_3$ .

**Solution:** Given

$$\mathbf{D}_3 = \frac{m\mathbf{C} + n\mathbf{B}}{m + n}, \mathbf{E}_3 = \frac{n\mathbf{A} + p\mathbf{C}}{n + p}, \mathbf{F}_3 = \frac{p\mathbf{B} + m\mathbf{A}}{p + m} \quad (1.5.2.1)$$

Here

$$\mathbf{A} = \begin{pmatrix} -3 \\ -5 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 3 \\ -5 \end{pmatrix}, \mathbf{c} = \begin{pmatrix} -4 \\ -3 \end{pmatrix}, \quad (1.5.2.2)$$

$$p = 0.4479790441, m = 5.522020956, n = 1.758088933, \quad (1.5.2.3)$$

On substituting (1.5.2.2) and (1.5.2.3) in (??) We get

$$\mathbf{D}_3 = \frac{5.522020956 \begin{pmatrix} -4 \\ -3 \end{pmatrix} + 1.758088933 \begin{pmatrix} 3 \\ -5 \end{pmatrix}}{5.522020956 + 1.758088933} \quad (1.5.2.4)$$

$$\mathbf{E}_3 = \frac{1.758088933 \begin{pmatrix} -3 \\ -5 \end{pmatrix} + 0.4479790441 \begin{pmatrix} -4 \\ -3 \end{pmatrix}}{1.758088933 + 0.4479790441} \quad (1.5.2.5)$$

$$\mathbf{F}_3 = \frac{0.4479790441 \begin{pmatrix} 3 \\ -5 \end{pmatrix} + 5.522020956 \begin{pmatrix} -3 \\ -5 \end{pmatrix}}{0.4479790441 + 5.522020956} \quad (1.5.2.6)$$

On solving above equations We get

$$\mathbf{D}_3 = \begin{pmatrix} -2.30955539 \\ -3.48298417 \end{pmatrix} \quad (1.5.2.7)$$

$$\mathbf{E}_3 = \begin{pmatrix} -2.52202096 \\ -5 \end{pmatrix} \quad (1.5.2.8)$$

$$\mathbf{F}_3 = \begin{pmatrix} -3.21375873 \\ -4.57248255 \end{pmatrix} \quad (1.5.2.9)$$



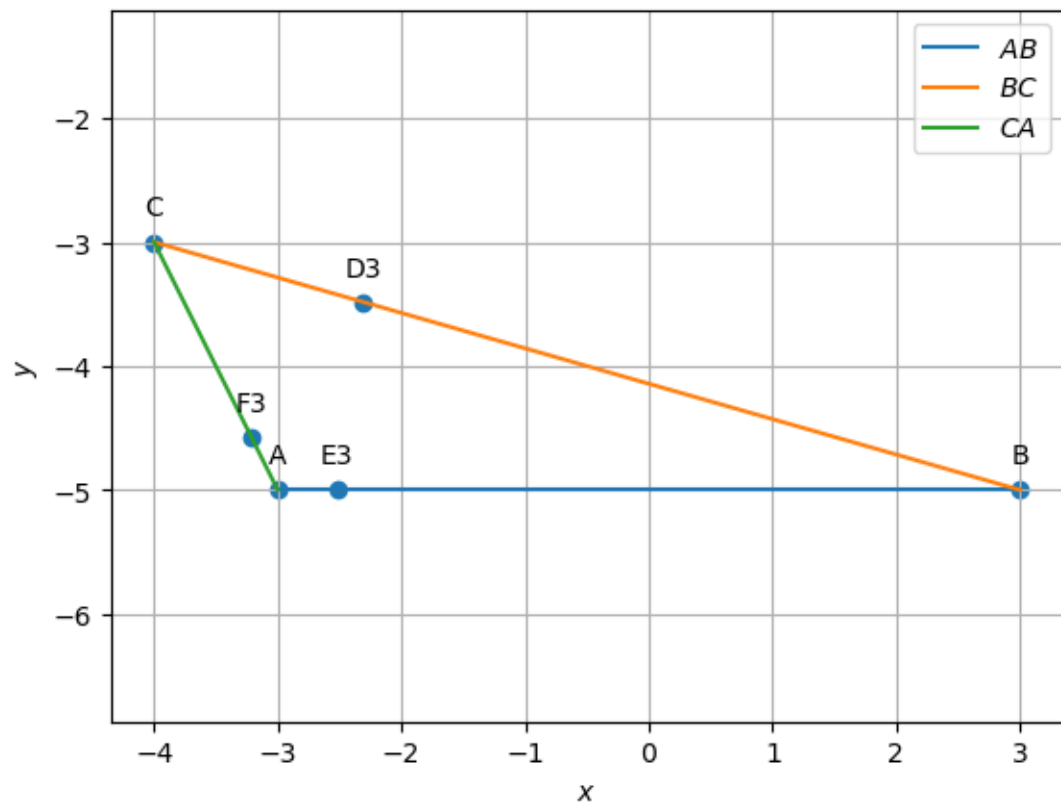


Figure 1.1: Points  $D_3$ ,  $E_3$ ,  $F_3$

1.5.3. Find the circumcentre and circumradius of  $\triangle D_3E_3F_3$ . These are the incentre and inradius of  $\triangle ABC$ .

**Solution:** Given

$$\mathbf{D}_3 = \begin{pmatrix} -2.30955539 \\ -3.48298417 \end{pmatrix} \quad (1.5.3.1)$$

$$\mathbf{E}_3 = \begin{pmatrix} -2.52202096 \\ -5 \end{pmatrix} \quad (1.5.3.2)$$

$$\mathbf{F}_3 = \begin{pmatrix} -3.21375873 \\ -4.57248255 \end{pmatrix} \quad (1.5.3.3)$$

(a) For circumcentre

Vector equation of  $\mathbf{D} - \mathbf{E}$  is

$$(\mathbf{D}_3 - \mathbf{E}_3)^\top \left( \mathbf{x} - \frac{\mathbf{D}_3 + \mathbf{E}_3}{2} \right) = 0 \quad (1.5.3.4)$$

$$(\mathbf{D}_3 - \mathbf{F}_3)^\top \left( \mathbf{x} - \frac{\mathbf{D}_3 + \mathbf{F}_3}{2} \right) = 0 \quad (1.5.3.5)$$

on Substituting the values of  $\mathbf{D}_3, \mathbf{E}_3, \mathbf{F}_3$  and solving We get,

$$\begin{pmatrix} 0.21247 & 1.51702 \end{pmatrix} \mathbf{x} = -6.94768 \quad (1.5.3.6)$$

$$\begin{pmatrix} 0.90420 & 1.08950 \end{pmatrix} \mathbf{x} = 6.88531 \quad (1.5.3.7)$$

Thus on solving (1.5.3.6) and (1.5.3.7) using gauss elimination

We get

$$\begin{pmatrix} 0.21247 & 1.15702 & -6.94768 \\ 0.90420 & 1.08950 & 6.88531 \end{pmatrix} \quad (1.5.3.8)$$

$$\therefore \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} = \begin{pmatrix} -2.52202096 \\ -4.22661366 \end{pmatrix} \quad (1.5.3.9)$$

$$\implies \mathbf{x} = \begin{pmatrix} -2.52202096 \\ -4.22661366 \end{pmatrix} \quad (1.5.3.10)$$

(b) The circumradius is obtained from  $r = \|\mathbf{I} - \mathbf{D}_3\|$

$$\mathbf{I} = \begin{pmatrix} -2.52202096 \\ -4.22661366 \end{pmatrix} \quad (1.5.3.11)$$

$$\mathbf{D}_3 = \begin{pmatrix} -2.30955539 \\ -3.48298417 \end{pmatrix} \quad (1.5.3.12)$$

$$\mathbf{I} - \mathbf{D}_3 = \begin{pmatrix} -0.21246557 \\ -0.74362949 \end{pmatrix} \quad (1.5.3.13)$$

$$r = \|\mathbf{I} - \mathbf{D}_3\| = \sqrt{(\mathbf{I} - \mathbf{D}_3)^\top (\mathbf{I} - \mathbf{D}_3)} \quad (1.5.3.14)$$

$$r = 0.7733863392795766 \quad (1.5.3.15)$$

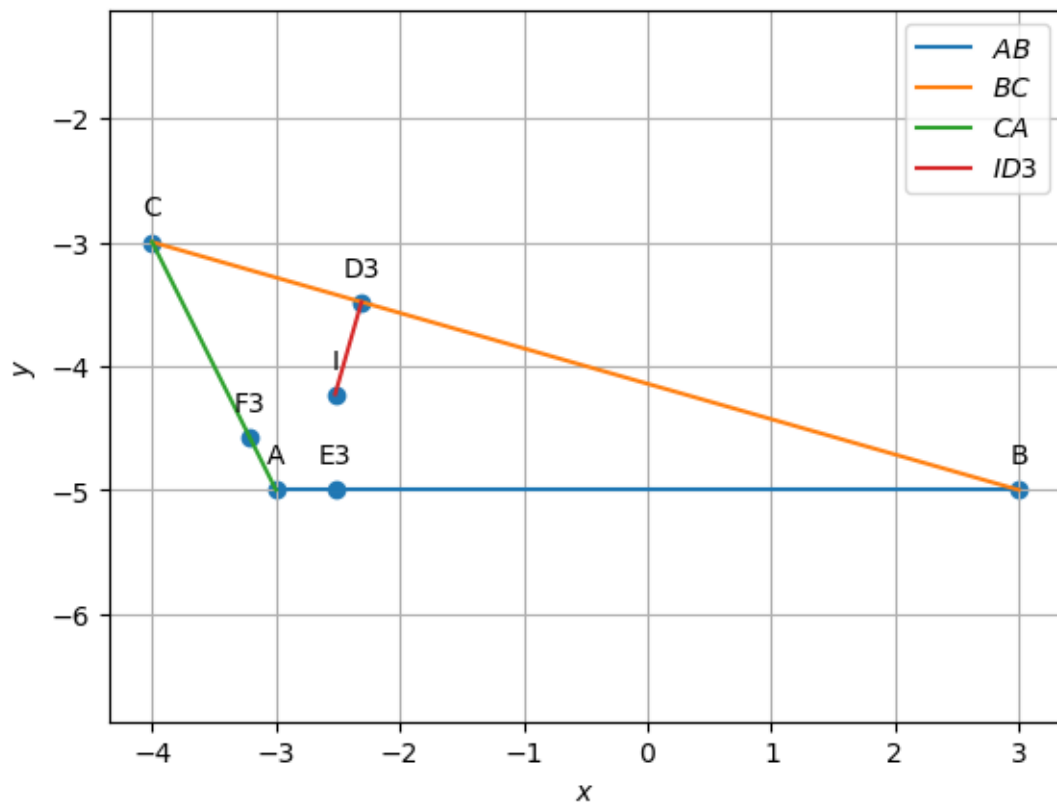


Figure 1.2: incentre and inradius of  $\triangle ABC$

1.5.4. Draw the circumcircle of  $\triangle D_3E_3F_3$ . This is known as the incircle of  $\triangle ABC$ .

**Solution:**

$$\mathbf{D}_3 = \begin{pmatrix} -2.30955539 \\ -3.48298417 \end{pmatrix} \quad (1.5.4.1)$$

$$\mathbf{E}_3 = \begin{pmatrix} -2.52202096 \\ -5 \end{pmatrix} \quad (1.5.4.2)$$

$$\mathbf{F}_3 = \begin{pmatrix} -3.21375873 \\ -4.57248255 \end{pmatrix} \quad (1.5.4.3)$$

Incentre

$$I = \begin{pmatrix} -2.52202096 \\ -4.22661366 \end{pmatrix} \quad (1.5.4.4)$$

Radius

$$r = 0.7733863392795766 \quad (1.5.4.5)$$

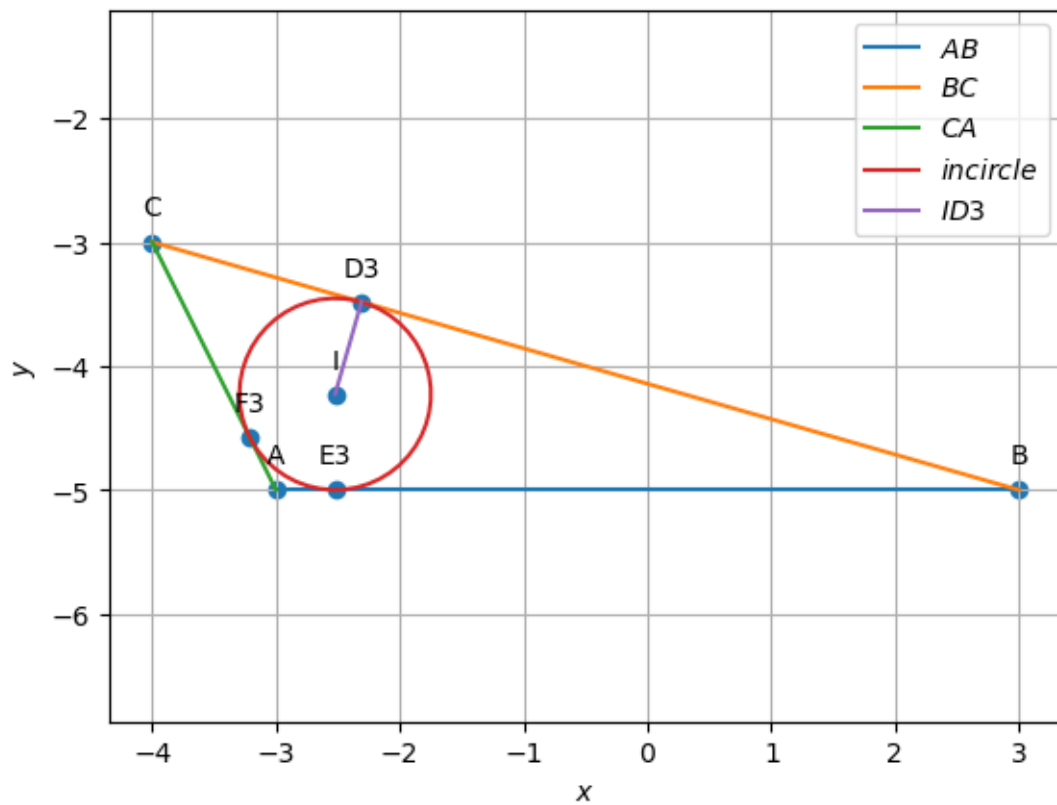


Figure 1.3: incircle of  $\triangle ABC$

1.5.5. Using (1.1.7) verify that

$$\angle BAI = \angle CAI. \quad (1.5.5.1)$$

$AI$  is the bisector of  $\angle A$ .

**Solution:**

$$\cos \angle BAI \triangleq \frac{(\mathbf{B} - \mathbf{A})^\top (\mathbf{I} - \mathbf{A})}{\|\mathbf{B} - \mathbf{A}\| \|\mathbf{I} - \mathbf{A}\|} \quad (1.5.5.2)$$

$$\cos \angle CAI \triangleq \frac{(\mathbf{C} - \mathbf{A})^\top (\mathbf{I} - \mathbf{A})}{\|\mathbf{C} - \mathbf{A}\| \|\mathbf{I} - \mathbf{A}\|} \quad (1.5.5.3)$$

From the given values of  $\mathbf{A}, \mathbf{B}, \mathbf{C}$  and  $vecI$ ,

$$\mathbf{A} = \begin{pmatrix} -3 \\ -5 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 3 \\ -5 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} -4 \\ -3 \end{pmatrix}, \quad (1.5.5.4)$$

$$\mathbf{I} = \begin{pmatrix} -2.52202096 \\ -4.22661366 \end{pmatrix} \quad (1.5.5.5)$$

$$\mathbf{B} - \mathbf{A} = \begin{pmatrix} 6 \\ 0 \end{pmatrix} \quad (1.5.5.6)$$

$$\mathbf{C} - \mathbf{A} = \begin{pmatrix} -1 \\ +2 \end{pmatrix} \quad (1.5.5.7)$$

$$\mathbf{I} - \mathbf{A} = \begin{pmatrix} 0.47797904 \\ 0.77338634 \end{pmatrix} \quad (1.5.5.8)$$

also calculating the values of norms

$$\|\mathbf{B} - \mathbf{A}\| = \sqrt{36} = 6 \quad (1.5.5.9)$$

$$\|\mathbf{C} - \mathbf{A}\| = \sqrt{5} \quad (1.5.5.10)$$

$$\|\mathbf{I} - \mathbf{A}\| = 0.9091701676 \quad (1.5.5.11)$$

$$(1.5.5.12)$$

(a) for  $\angle BAI$ :

On substituting the values in (1.5.5.2) ,We get

$$\cos \angle BAI \triangleq \frac{\begin{pmatrix} 6 & 0 \end{pmatrix} \begin{pmatrix} 0.47797904 \\ 0.77338634 \end{pmatrix}}{6 \times 0.9091701676} \quad (1.5.5.13)$$

$$(1.5.5.14)$$

On solving

$$\angle BAI = 58.282525832716665^\circ \quad (1.5.5.15)$$

(b) for  $\angle CAI$ :

On substituting the values in (1.5.5.2) ,We get

$$\cos \angle CAI \triangleq \frac{\begin{pmatrix} -1 & 2 \end{pmatrix} \begin{pmatrix} 0.47797904 \\ 0.77338634 \end{pmatrix}}{\sqrt{5} \times 0.9091701676} \quad (1.5.5.16)$$

$$(1.5.5.17)$$



On solving

$$\angle CAI = 58.282525832716665^\circ \quad (1.5.5.18)$$

Therefore  $\angle BAI = \angle CAI$ . and  $AI$  is the bisector of  $\angle A$ .

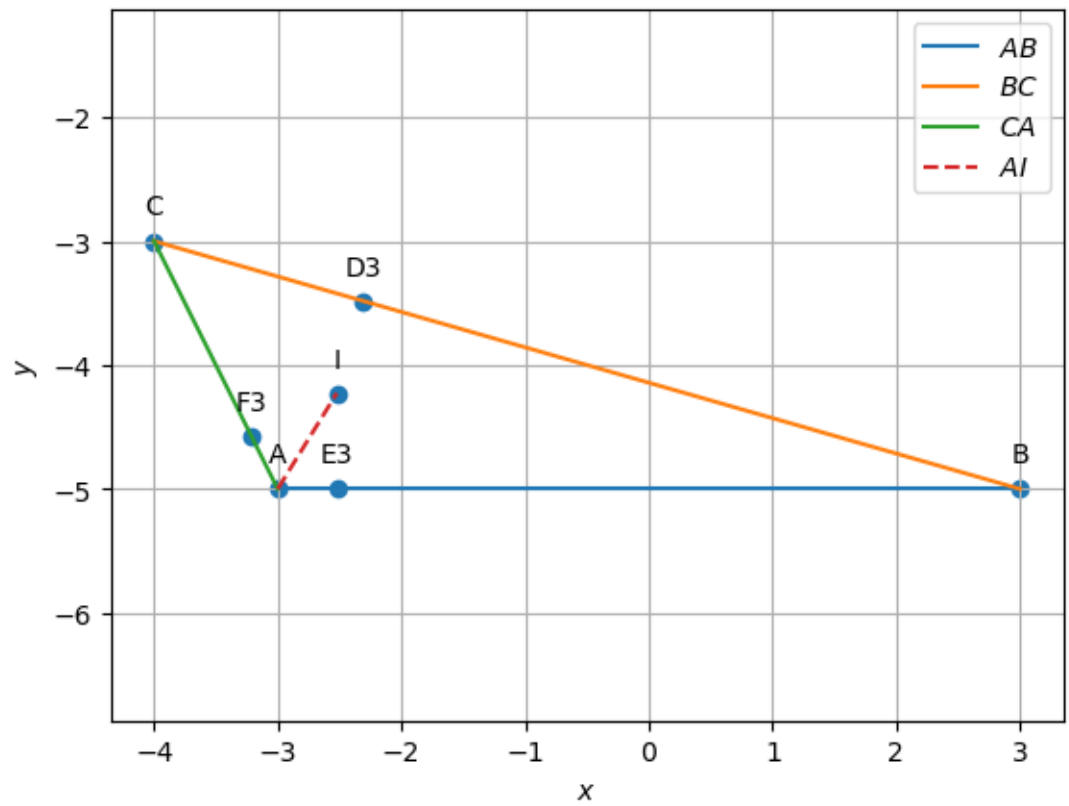


Figure 1.4: Angular bisector  $AI$

1.5.6. Verify that  $BI, CI$  are also the angle bisectors of  $\triangle ABC$ .

**Solution:**

(a) To prove  $BI$  is an angular bisector of  $\angle B$

$$\cos \angle ABI \triangleq \frac{(\mathbf{A} - \mathbf{B})^\top (\mathbf{I} - \mathbf{B})}{\|\mathbf{A} - \mathbf{B}\| \|\mathbf{I} - \mathbf{B}\|} \quad (1.5.6.1)$$

$$\cos \angle CBI \triangleq \frac{(\mathbf{C} - \mathbf{B})^\top (\mathbf{I} - \mathbf{B})}{\|\mathbf{C} - \mathbf{B}\| \|\mathbf{I} - \mathbf{B}\|} \quad (1.5.6.2)$$

From the given values of  $\mathbf{A}, \mathbf{B}, \mathbf{C}$  and  $\mathbf{I}$ ,

$$\mathbf{A} = \begin{pmatrix} -3 \\ -5 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 3 \\ -5 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} -4 \\ -3 \end{pmatrix}, \quad (1.5.6.3)$$

$$\mathbf{I} = \begin{pmatrix} -2.52202096 \\ -4.22661366 \end{pmatrix} \quad (1.5.6.4)$$

$$\mathbf{A} - \mathbf{B} = \begin{pmatrix} -6 \\ 0 \end{pmatrix} \quad (1.5.6.5)$$

$$\mathbf{C} - \mathbf{B} = \begin{pmatrix} -7 \\ +2 \end{pmatrix} \quad (1.5.6.6)$$

$$\mathbf{I} - \mathbf{B} = \begin{pmatrix} -5.52202096 \\ 0.77338634 \end{pmatrix} \quad (1.5.6.7)$$

also calculating the values of norms

$$\|\mathbf{A} - \mathbf{B}\| = \sqrt{36} = 6 \quad (1.5.6.8)$$

$$\|\mathbf{C} - \mathbf{B}\| = \sqrt{53} \quad (1.5.6.9)$$

$$\|\mathbf{I} - \mathbf{B}\| = 5.57591624 \quad (1.5.6.10)$$

$$(1.5.6.11)$$

i. for  $\angle ABI$ :

On substituting the values in (1.5.6.1) ,We get

$$\cos \angle ABI \triangleq \frac{\begin{pmatrix} -6 & 0 \end{pmatrix} \begin{pmatrix} -5.52202096 \\ 0.77338634 \end{pmatrix}}{6 \times 5.57591624} \quad (1.5.6.12)$$

$$(1.5.6.13)$$

On solving

$$\angle ABI = 7.972697951935452^\circ \quad (1.5.6.14)$$

ii. for  $\angle CBI$ :

On substituting the values in (1.5.6.1) ,We get

$$\cos \angle CBI \triangleq \frac{\begin{pmatrix} -7 & 2 \end{pmatrix} \begin{pmatrix} -5.52202096 \\ 0.77338634 \end{pmatrix}}{\sqrt{5} \times 5.57591624} \quad (1.5.6.15)$$

$$(1.5.6.16)$$

On solving

$$\angle CBI = 7.972697951935452^\circ \quad (1.5.6.17)$$

Therefore  $\angle ABI = \angle CBI$ . and  $BI$  is the bisector of  $\angle B$ .

(b) To prove  $CI$  is an angular bisector of  $\angle C$

$$\cos \angle BCI \triangleq \frac{(\mathbf{B} - \mathbf{C})^\top (\mathbf{I} - \mathbf{C})}{\|\mathbf{B} - \mathbf{C}\| \|\mathbf{I} - \mathbf{C}\|} \quad (1.5.6.18)$$

$$\cos \angle ACI \triangleq \frac{(\mathbf{A} - \mathbf{C})^\top (\mathbf{I} - \mathbf{C})}{\|\mathbf{A} - \mathbf{B}\| \|\mathbf{I} - \mathbf{C}\|} \quad (1.5.6.19)$$

From the given values of  $\mathbf{A}, \mathbf{B}, \mathbf{C}$  and  $\mathbf{I}$ ,

$$\mathbf{A} = \begin{pmatrix} -3 \\ -5 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 3 \\ -5 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} -4 \\ -3 \end{pmatrix}, \quad (1.5.6.20)$$

$$\mathbf{I} = \begin{pmatrix} -2.52202096 \\ -4.22661366 \end{pmatrix} \quad (1.5.6.21)$$

$$\mathbf{B} - \mathbf{C} = \begin{pmatrix} 7 \\ -2 \end{pmatrix} \quad (1.5.6.22)$$

$$\mathbf{A} - \mathbf{C} = \begin{pmatrix} 1 \\ -2 \end{pmatrix} \quad (1.5.6.23)$$

$$\mathbf{I} - \mathbf{C} = \begin{pmatrix} 1.47797904 \\ -1.22661366 \end{pmatrix} \quad (1.5.6.24)$$

also calculating the values of norms

$$\|\mathbf{B} - \mathbf{C}\| = \sqrt{53} \quad (1.5.6.25)$$

$$\|\mathbf{A} - \mathbf{C}\| = \sqrt{5} \quad (1.5.6.26)$$

$$\|\mathbf{I} - \mathbf{C}\| = 1.920677775 \quad (1.5.6.27)$$

$$(1.5.6.28)$$

i. for  $\angle BCI$ :

On substituting the values in (1.5.6.18) ,We get

$$\cos \angle BCI \triangleq \frac{\begin{pmatrix} 7 & -2 \end{pmatrix} \begin{pmatrix} 1.47797904 \\ -1.22661366 \end{pmatrix}}{\sqrt{53} \times 1.920677775} \quad (1.5.6.29)$$

$$(1.5.6.30)$$

On solving

$$\angle BCI = 23.74477646099958^\circ \quad (1.5.6.31)$$

ii. for  $\angle ACI$ :

On substituting the values in (1.5.6.18) ,We get

$$\cos \angle ACI \triangleq \frac{\begin{pmatrix} 1 & -2 \end{pmatrix} \begin{pmatrix} 1.47797904 \\ -1.22661366 \end{pmatrix}}{\sqrt{5} \times 1.920677775} \quad (1.5.6.32)$$

$$(1.5.6.33)$$

On solving

$$\angle ACI = 23.74477646099958^\circ \quad (1.5.6.34)$$

Therefore  $\angle BCI = \angle ACI$ . and  $CI$  is the bisector of  $\angle C$ .

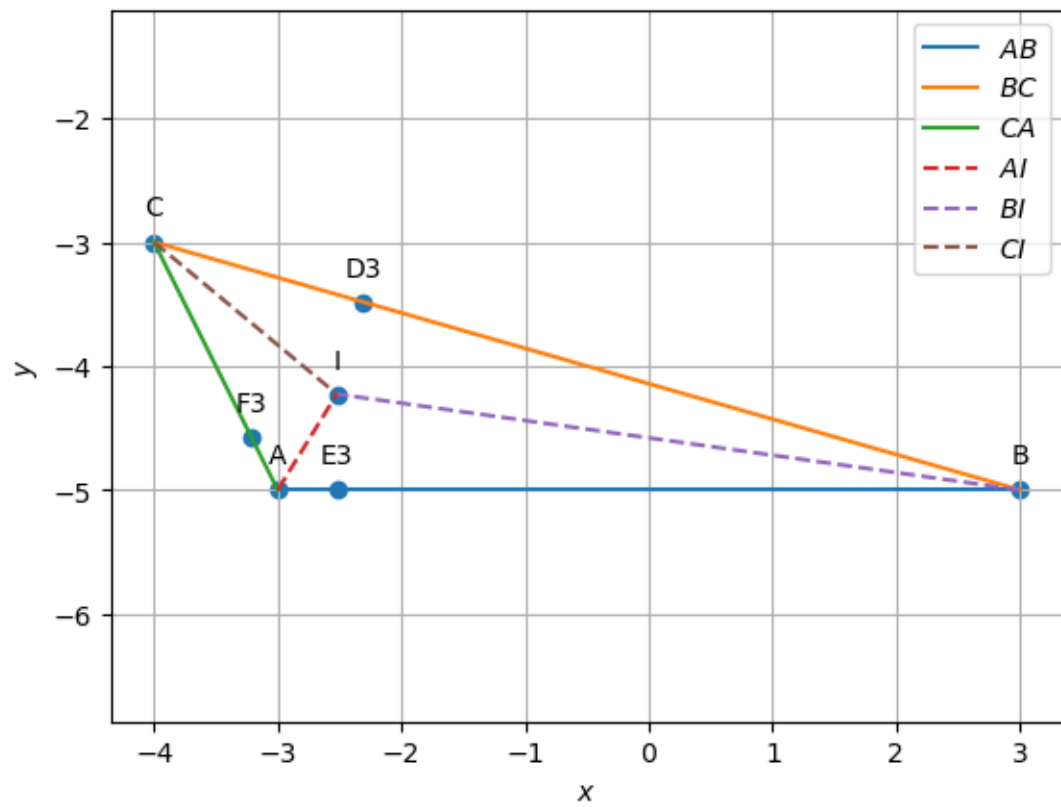


Figure 1.5: Angular bisectors  $BI$  and  $CI$