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Chapter 1

Triangle

Consider a triangle with vertices

$$\mathbf{A} = \begin{pmatrix} -3 \\ -5 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 3 \\ -5 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} -4 \\ -3 \end{pmatrix}, \quad (1.1)$$

1.1. Vectors

1.2. Median

1.3. Altitude

1.4. Perpendicular Bisector

1.4.1. The equation of the perpendicular bisector of BC is

$$\left(\mathbf{x} - \frac{\mathbf{B} + \mathbf{C}}{2} \right) (\mathbf{B} - \mathbf{C}) = 0 \quad (1.4.1.1)$$

Substitute numerical values and find the equations of the perpendicular bisectors of AB, BC and CA .

Solution:

(a) **BC**: given equation for the perpendicular bisector of **BC**:

$$\left(\mathbf{x} - \frac{\mathbf{B} + \mathbf{C}}{2} \right) (\mathbf{B} - \mathbf{C}) = 0 \quad (1.4.1.2)$$

On substituting the values,

$$\frac{\mathbf{B} + \mathbf{C}}{2} = \begin{pmatrix} \frac{-1}{2} \\ -4 \end{pmatrix} \quad (1.4.1.3)$$

$$\mathbf{B} - \mathbf{C} = \begin{pmatrix} 7 \\ -2 \end{pmatrix} \quad (1.4.1.4)$$

$$(1.4.1.5)$$

solving using matrix multiplication

$$(\mathbf{B} - \mathbf{C})^\top \left(\frac{\mathbf{B} + \mathbf{C}}{2} \right) = 0 \quad (1.4.1.6)$$

$$(\mathbf{B} - \mathbf{C})^\top = \begin{pmatrix} 7 & -2 \end{pmatrix} \quad (1.4.1.7)$$

$$(\mathbf{B} - \mathbf{C})^\top \left(\frac{\mathbf{B} + \mathbf{C}}{2} \right) = \begin{pmatrix} 7 & -2 \end{pmatrix} \begin{pmatrix} \frac{-1}{2} \\ -4 \end{pmatrix} \quad (1.4.1.8)$$

$$= \frac{9}{2} \quad (1.4.1.9)$$

Therefore perpendicular bisector of \mathbf{BC} is

$$\begin{pmatrix} 7 & -2 \end{pmatrix} \mathbf{x} = \frac{9}{2} \quad (1.4.1.10)$$

(b) \mathbf{AB} : similarly the equation for the perpendicular bisector of \mathbf{AB} :

$$\left(\mathbf{x} - \frac{\mathbf{A} + \mathbf{B}}{2} \right) (\mathbf{A} - \mathbf{B}) = 0 \quad (1.4.1.11)$$

On substituting the values,

$$\frac{\mathbf{A} + \mathbf{B}}{2} = \begin{pmatrix} 0 \\ -5 \end{pmatrix} \quad (1.4.1.12)$$

$$\mathbf{A} - \mathbf{B} = \begin{pmatrix} -6 \\ 0 \end{pmatrix} \quad (1.4.1.13)$$

$$(1.4.1.14)$$

solving using matrix multiplication

$$(\mathbf{A} - \mathbf{B})^\top \left(\frac{\mathbf{A} + \mathbf{B}}{2} \right) = 0 \quad (1.4.1.15)$$

$$(\mathbf{A} - \mathbf{B})^\top = \begin{pmatrix} -6 & 0 \end{pmatrix} \quad (1.4.1.16)$$

$$(\mathbf{A} - \mathbf{B})^\top \left(\frac{\mathbf{A} + \mathbf{B}}{2} \right) = \begin{pmatrix} -6 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ -5 \end{pmatrix} \quad (1.4.1.17)$$

$$= 0 \quad (1.4.1.18)$$

Therefore perpendicular bisector of \mathbf{AB} is

$$\begin{pmatrix} -6 & 0 \end{pmatrix} \mathbf{x} = 0 \quad (1.4.1.19)$$

(c) \mathbf{CA} : similarly the equation for the perpendicular bisector of \mathbf{CA} :

$$\left(\mathbf{x} - \frac{\mathbf{C} + \mathbf{A}}{2} \right) (\mathbf{C} - \mathbf{A}) = 0 \quad (1.4.1.20)$$

On substituting the values,

$$\frac{\mathbf{C} + \mathbf{A}}{2} = \begin{pmatrix} \frac{-7}{2} \\ -4 \end{pmatrix} \quad (1.4.1.21)$$

$$\mathbf{C} - \mathbf{A} = \begin{pmatrix} -1 \\ 2 \end{pmatrix} \quad (1.4.1.22)$$

$$(1.4.1.23)$$

solving using matrix multiplication

$$(\mathbf{C} - \mathbf{A})^\top \left(\frac{\mathbf{C} + \mathbf{A}}{2} \right) = 0 \quad (1.4.1.24)$$

$$(\mathbf{C} - \mathbf{A})^\top = \begin{pmatrix} -1 & 2 \end{pmatrix} \quad (1.4.1.25)$$

$$(\mathbf{C} - \mathbf{A})^\top \left(\frac{\mathbf{C} + \mathbf{A}}{2} \right) = \begin{pmatrix} -1 & 2 \end{pmatrix} \begin{pmatrix} \frac{-7}{2} \\ -4 \end{pmatrix} \quad (1.4.1.26)$$

$$= \frac{-9}{2} \quad (1.4.1.27)$$

Therefore perpendicular bisector of **BC** is

$$\begin{pmatrix} -1 & 2 \end{pmatrix} \mathbf{x} = \frac{-9}{2} \quad (1.4.1.28)$$

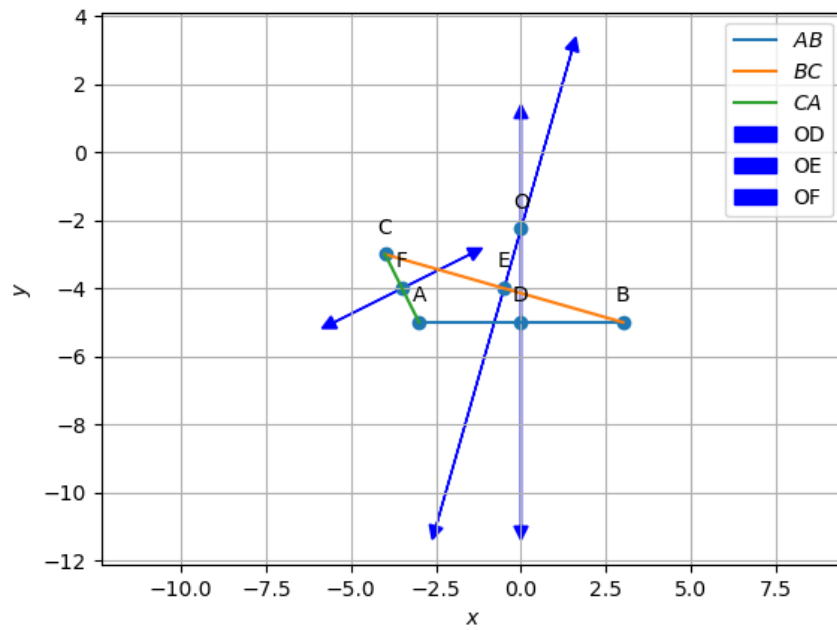


Figure 1.1: Plot of the perpendicular bisectors

1.4.2. Find the intersection **O** of the perpendicular bisectors of *AB* and *AC*.

Solution:

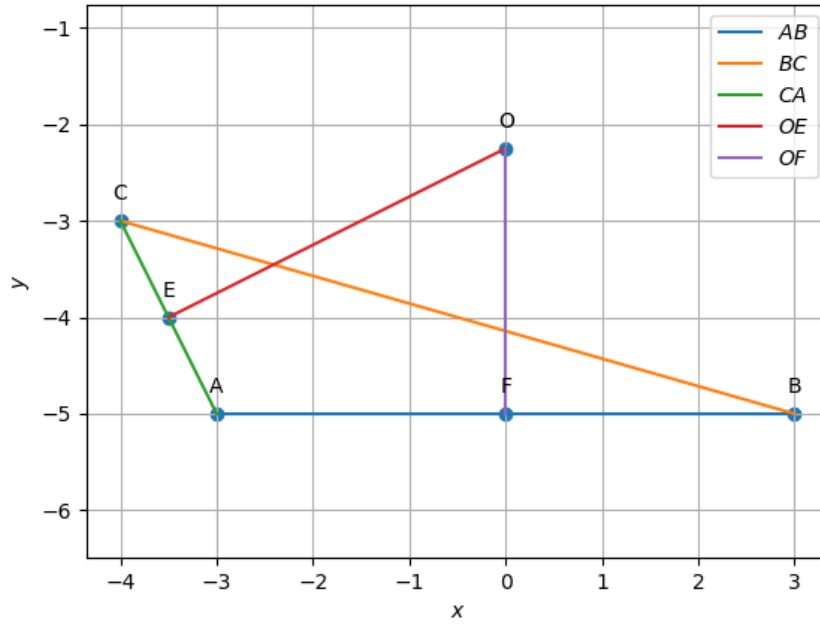


Figure 1.2: $\mathbf{O}-\mathbf{E}$ and $\mathbf{O}-\mathbf{F}$ are perpendicular bisectors of $\mathbf{A}-\mathbf{C}$ and $\mathbf{A}-\mathbf{B}$ respectively

Given,

$$\mathbf{A} = \begin{pmatrix} -3 \\ -5 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 3 \\ -5 \end{pmatrix}, \mathbf{c} = \begin{pmatrix} -4 \\ -3 \end{pmatrix}, \quad (1.4.2.1)$$

Vector equation of perpendicular bisector of $\mathbf{A}-\mathbf{B}$ is

$$(\mathbf{A}-\mathbf{B})^\top \left(\mathbf{x} - \frac{\mathbf{A}+\mathbf{B}}{2} \right) = 0 \quad (1.4.2.2)$$

where,

$$\mathbf{A} + \mathbf{B} = \begin{pmatrix} -3 \\ -5 \end{pmatrix} + \begin{pmatrix} 3 \\ -5 \end{pmatrix} \quad (1.4.2.3)$$

$$= \begin{pmatrix} 0 \\ -10 \end{pmatrix} \quad (1.4.2.4)$$

$$\mathbf{A} - \mathbf{B} = \begin{pmatrix} -3 \\ -5 \end{pmatrix} - \begin{pmatrix} 3 \\ -5 \end{pmatrix} \quad (1.4.2.5)$$

$$= \begin{pmatrix} -6 \\ 0 \end{pmatrix} \quad (1.4.2.6)$$

$$\Rightarrow (\mathbf{A} - \mathbf{B})^\top = \begin{pmatrix} -6 & 0 \end{pmatrix} \quad (1.4.2.7)$$

\therefore The vector equation of $\mathbf{O} - \mathbf{F}$ is

$$\begin{pmatrix} -6 & 0 \end{pmatrix} \left(\mathbf{x} - \begin{pmatrix} 0 \\ -5 \end{pmatrix} \right) = 0 \quad (1.4.2.8)$$

$$\Rightarrow \begin{pmatrix} -6 & 0 \end{pmatrix} \mathbf{x} = \begin{pmatrix} -6 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ -5 \end{pmatrix} \quad (1.4.2.9)$$

Performing matrix multiplication yields

$$\begin{pmatrix} -6 & 0 \end{pmatrix} \mathbf{x} = 0 \quad (1.4.2.10)$$

Vector equation of perpendicular bisector of $\mathbf{A} - \mathbf{C}$ is

$$(\mathbf{A} - \mathbf{C})^\top \left(\mathbf{x} - \frac{\mathbf{A} + \mathbf{C}}{2} \right) = 0 \quad (1.4.2.11)$$

where,

$$\mathbf{A} + \mathbf{C} = \begin{pmatrix} -3 \\ -5 \end{pmatrix} + \begin{pmatrix} -4 \\ -3 \end{pmatrix} \quad (1.4.2.12)$$

$$= \begin{pmatrix} -7 \\ -8 \end{pmatrix} \quad (1.4.2.13)$$

$$\mathbf{A} - \mathbf{C} = \begin{pmatrix} -3 \\ -5 \end{pmatrix} - \begin{pmatrix} -4 \\ -3 \end{pmatrix} \quad (1.4.2.14)$$

$$= \begin{pmatrix} 1 \\ -2 \end{pmatrix} \quad (1.4.2.15)$$

$$\implies (\mathbf{A} - \mathbf{C})^\top = \begin{pmatrix} 1 & -2 \end{pmatrix} \quad (1.4.2.16)$$

\therefore The vector equation of $\mathbf{O} - \mathbf{E}$ is

$$\begin{pmatrix} 1 & -2 \end{pmatrix} \left(\mathbf{x} - \frac{1}{2} \begin{pmatrix} -7 \\ -8 \end{pmatrix} \right) = 0 \quad (1.4.2.17)$$

$$\implies \begin{pmatrix} 1 & -2 \end{pmatrix} \mathbf{x} = \frac{1}{2} \begin{pmatrix} 1 & -2 \end{pmatrix} \begin{pmatrix} -7 \\ -8 \end{pmatrix} \quad (1.4.2.18)$$

Performing matrix multiplication yields

$$\begin{pmatrix} 1 & -2 \end{pmatrix} \mathbf{x} = \frac{9}{2} \quad (1.4.2.19)$$

Thus,

$$\begin{pmatrix} -6 & 0 & 0 \\ 1 & -2 & \frac{9}{2} \end{pmatrix} \xleftrightarrow{R_1 \leftarrow \frac{-1}{6} R_1} \begin{pmatrix} 1 & 0 & 0 \\ 1 & -2 & \frac{9}{2} \end{pmatrix} \quad (1.4.2.20)$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & -2 & \frac{9}{2} \end{pmatrix} \xleftrightarrow{R_2 \leftarrow R_2 - R_1} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & \frac{-9}{2} \end{pmatrix} \quad (1.4.2.21)$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & \frac{-9}{2} \end{pmatrix} \xleftrightarrow{R_2 \leftarrow \frac{-1}{2} R_2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \frac{-9}{4} \end{pmatrix} \quad (1.4.2.22)$$

$$\therefore \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 0 \\ \frac{-9}{4} \end{pmatrix} \quad (1.4.2.23)$$

$$\implies \mathbf{x} = \begin{pmatrix} 0 \\ \frac{-9}{4} \end{pmatrix} \quad (1.4.2.24)$$

Therefore, the point of intersection of perpendicular bisectors of $\mathbf{A} - \mathbf{B}$

and $\mathbf{A} - \mathbf{C}$ is $\mathbf{O} = \begin{pmatrix} 0 \\ \frac{-9}{4} \end{pmatrix}$

1.4.3. Verify that \mathbf{O} satisfies (1.4.1.1). \mathbf{O} is known as the circumcentre.

Solution: From the previous question we get,

$$\mathbf{O} = \begin{pmatrix} 0 \\ \frac{-9}{4} \end{pmatrix} \quad (1.4.3.1)$$

$$\left(\mathbf{x} - \frac{\mathbf{B} + \mathbf{C}}{2} \right) (\mathbf{B} - \mathbf{C}) = 0 \quad (1.4.3.2)$$

when substituted in the above equation,

$$= \left(\mathbf{O} - \frac{\mathbf{B} + \mathbf{C}}{2} \right) \cdot (\mathbf{B} - \mathbf{C}) \quad (1.4.3.3)$$

$$= \left(\begin{pmatrix} 0 \\ \frac{-9}{4} \end{pmatrix} - \frac{1}{2} \begin{pmatrix} -2 \\ -8 \end{pmatrix} \right)^\top \begin{pmatrix} 7 \\ -2 \end{pmatrix} \quad (1.4.3.4)$$

$$= \begin{pmatrix} \frac{1}{2} & \frac{7}{4} \end{pmatrix} \begin{pmatrix} 7 \\ -2 \end{pmatrix} \quad (1.4.3.5)$$

$$= 0 \quad (1.4.3.6)$$

It is hence proved that \mathbf{O} satisfies the equation (1.4.1.1)

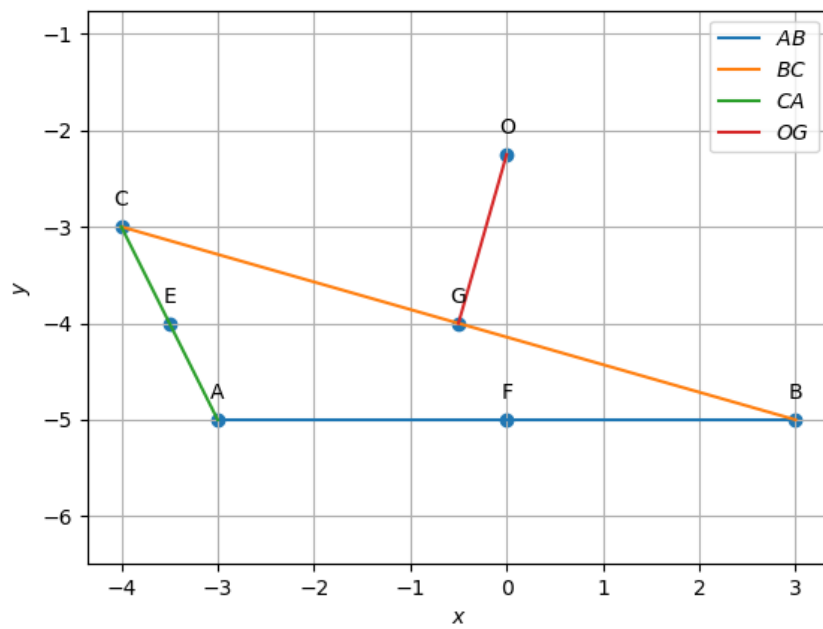


Figure 1.3: Circumcenter plotted using python

1.4.4. Verify that

$$OA = OB = OC \quad (1.4.4.1)$$

Solution: Given

$$\mathbf{A} = \begin{pmatrix} -3 \\ -5 \end{pmatrix} \quad (1.4.4.2)$$

$$\mathbf{B} = \begin{pmatrix} 3 \\ -5 \end{pmatrix} \quad (1.4.4.3)$$

$$\mathbf{C} = \begin{pmatrix} -4 \\ -3 \end{pmatrix} \quad (1.4.4.4)$$

From problem-1.4.2 :

$$O = \begin{pmatrix} 0 \\ \frac{-9}{4} \end{pmatrix} \quad (1.4.4.5)$$

$$= \begin{pmatrix} 0 \\ -2.25 \end{pmatrix} \quad (1.4.4.6)$$

(a)

$$OA = \sqrt{(\mathbf{O} - \mathbf{A})^\top (\mathbf{O} - \mathbf{A})} \quad (1.4.4.7)$$

$$= \sqrt{\begin{pmatrix} 3 & \frac{11}{4} \end{pmatrix} \begin{pmatrix} 3 \\ \frac{11}{4} \end{pmatrix}} \quad (1.4.4.8)$$

$$= \sqrt{\frac{265}{16}} \quad (1.4.4.9)$$

$$= \frac{\sqrt{265}}{4} \quad (1.4.4.10)$$

(b)

$$OB = \sqrt{(\mathbf{O} - \mathbf{B})^\top (\mathbf{O} - \mathbf{B})} \quad (1.4.4.11)$$

$$= \sqrt{\begin{pmatrix} -3 & \frac{11}{4} \end{pmatrix} \begin{pmatrix} -3 \\ \frac{11}{4} \end{pmatrix}} \quad (1.4.4.12)$$

$$= \sqrt{\frac{265}{16}} \quad (1.4.4.13)$$

$$= \frac{\sqrt{265}}{4} \quad (1.4.4.14)$$

(c)

$$OC = \sqrt{(\mathbf{O} - \mathbf{C})^\top (\mathbf{O} - \mathbf{C})} \quad (1.4.4.15)$$

$$= \sqrt{\begin{pmatrix} 4 & \frac{3}{4} \end{pmatrix} \begin{pmatrix} 4 \\ \frac{3}{4} \end{pmatrix}} \quad (1.4.4.16)$$

$$= \sqrt{\frac{265}{16}} \quad (1.4.4.17)$$

$$= \frac{\sqrt{265}}{4} \quad (1.4.4.18)$$

From above,

$$OA = OB = OC \quad (1.4.4.19)$$

Hence verified.

1.4.5. Draw the circle with centre at \mathbf{O} and radius

$$R = OA \quad (1.4.5.1)$$

This is known as the circumradius.

Solution: Given

$$\mathbf{A} = \begin{pmatrix} -3 \\ -5 \end{pmatrix} \quad (1.4.5.2)$$

$$\mathbf{B} = \begin{pmatrix} 3 \\ -5 \end{pmatrix} \quad (1.4.5.3)$$

$$\mathbf{C} = \begin{pmatrix} -4 \\ -3 \end{pmatrix} \quad (1.4.5.4)$$

From Q1.4.2, the circumcentre is

$$\mathbf{O} = \begin{pmatrix} 0 \\ \frac{-9}{4} \end{pmatrix} \quad (1.4.5.5)$$

Now we will calculate the radius,

$$R = OA \tag{1.4.5.6}$$

$$= \|\mathbf{A} - \mathbf{O}\| \tag{1.4.5.7}$$

$$= \left\| \begin{pmatrix} -3 \\ -5 \end{pmatrix} - \begin{pmatrix} 0 \\ \frac{-9}{4} \end{pmatrix} \right\| \tag{1.4.5.8}$$

$$= \left\| \begin{pmatrix} -3 \\ \frac{-11}{4} \end{pmatrix} \right\| \tag{1.4.5.9}$$

$$= \sqrt{\begin{pmatrix} -3 & \frac{-11}{4} \end{pmatrix} \begin{pmatrix} -3 \\ \frac{-11}{4} \end{pmatrix}} \tag{1.4.5.10}$$

$$= \frac{\sqrt{265}}{4} \tag{1.4.5.11}$$

see Fig. 1.4

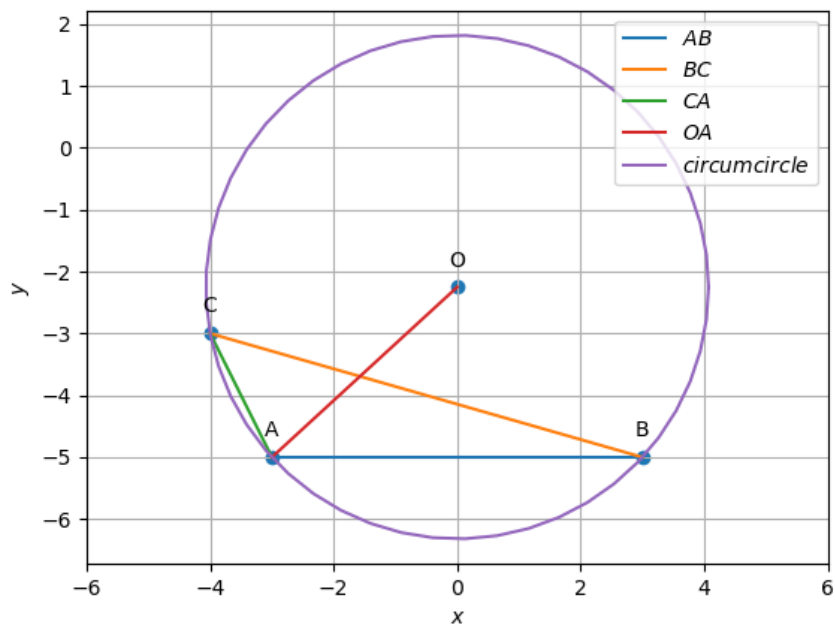


Figure 1.4: circumcircle of Triangle ABC with centre O

1.4.6. Verify that

$$\angle BOC = 2\angle BAC. \quad (1.4.6.1)$$

Solution:

(a) To find the value of $\angle BOC$:

$$\mathbf{B} - \mathbf{O} = \begin{pmatrix} 3 \\ \frac{-11}{4} \end{pmatrix} \quad (1.4.6.2)$$

$$\mathbf{C} - \mathbf{O} = \begin{pmatrix} -4 \\ \frac{-3}{4} \end{pmatrix} \quad (1.4.6.3)$$

$$\Rightarrow (\mathbf{B} - \mathbf{O})^\top (\mathbf{C} - \mathbf{O}) = \frac{-159}{16} \quad (1.4.6.4)$$

$$\Rightarrow \|\mathbf{B} - \mathbf{O}\| = \frac{\sqrt{265}}{4} \quad (1.4.6.5)$$

$$\|\mathbf{C} - \mathbf{O}\| = \frac{\sqrt{265}}{4} \quad (1.4.6.6)$$

Thus,

$$\cos BOC = \frac{(\mathbf{B} - \mathbf{O})^\top (\mathbf{C} - \mathbf{O})}{\|\mathbf{B} - \mathbf{O}\| \|\mathbf{C} - \mathbf{O}\|} = \frac{-159}{265} \quad (1.4.6.7)$$

$$\Rightarrow \angle BOC = \cos^{-1} \left(\frac{-159}{265} \right) \quad (1.4.6.8)$$

$$= 116.5650^\circ \quad (1.4.6.9)$$

Taking the reflex of above angle we get

$$\angle BOC = 360^\circ - 116.5650^\circ \quad (1.4.6.10)$$

$$= 233.130^\circ \quad (1.4.6.11)$$

(b) To find the value of $\angle BAC$:

$$\mathbf{B} - \mathbf{A} = \begin{pmatrix} 6 \\ 0 \end{pmatrix} \quad (1.4.6.12)$$

$$\mathbf{C} - \mathbf{A} = \begin{pmatrix} -1 \\ 2 \end{pmatrix} \quad (1.4.6.13)$$

$$\Rightarrow (\mathbf{B} - \mathbf{A})^\top (\mathbf{C} - \mathbf{A}) = -6 \quad (1.4.6.14)$$

$$\|\mathbf{B} - \mathbf{A}\| = \sqrt{36} = 6 \quad \|\mathbf{C} - \mathbf{A}\| = \sqrt{5} \quad (1.4.6.15)$$

Thus,

$$\cos BAC = \frac{(\mathbf{B} - \mathbf{A})^\top (\mathbf{C} - \mathbf{A})}{\|\mathbf{B} - \mathbf{A}\| \|\mathbf{C} - \mathbf{A}\|} = \frac{-1}{\sqrt{5}} \quad (1.4.6.16)$$

$$\Rightarrow \angle BAC = \cos^{-1} \left(\frac{-1}{\sqrt{5}} \right) \quad (1.4.6.17)$$

$$= 116.565^\circ \quad (1.4.6.18)$$

$$2 \times \angle BAC = 233.130 \quad (1.4.6.19)$$

From (1.4.6.18) and (??),

$$2 \times \angle BAC = \angle BOC \quad (1.4.6.20)$$

Hence Verified

1.4.7. Let

$$\mathbf{P} = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \quad (1.4.7.1)$$

Find θ if

$$\mathbf{C} - \mathbf{O} = \mathbf{P} (\mathbf{A} - \mathbf{O}) \quad (1.4.7.2)$$

Solution:

$$\mathbf{C} - \mathbf{O} = \begin{pmatrix} -4 \\ \frac{-3}{4} \end{pmatrix} \quad (1.4.7.3)$$

$$\mathbf{A} - \mathbf{O} = \begin{pmatrix} -3 \\ \frac{-11}{4} \end{pmatrix} \quad (1.4.7.4)$$

$$\mathbf{P} = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \quad (1.4.7.5)$$

$$\mathbf{C} - \mathbf{O} = \mathbf{P} (\mathbf{A} - \mathbf{O}) \quad (1.4.7.6)$$

Now from (1.4.7.6)

$$\begin{pmatrix} -4 \\ \frac{-3}{4} \end{pmatrix} = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} -3 \\ \frac{-11}{4} \end{pmatrix} \quad (1.4.7.7)$$

solving using matrix multiplication,we get

$$\begin{pmatrix} -4 \\ -\frac{3}{4} \end{pmatrix} = \begin{pmatrix} -3 \cos \theta + \frac{11}{4} \sin \theta \\ -3 \sin \theta + \frac{-11}{4} \cos \theta \end{pmatrix} \quad (1.4.7.8)$$

Comparing on Both sides ,we get

$$-3 \cos \theta + \frac{11}{4} \sin \theta = -4 \quad (1.4.7.9)$$

$$-3 \sin \theta + \frac{-11}{4} \cos \theta = \frac{-3}{4} \quad (1.4.7.10)$$

On solving equations (1.4.7.9) and (1.4.7.10)

$$\cos \theta = \frac{45}{53} \quad (1.4.7.11)$$

$$\sin \theta = \frac{-28}{53} \quad (1.4.7.12)$$

$$\theta = \cos^{-1} \frac{45}{53} \quad (1.4.7.13)$$

$$= 31.8907 \quad (1.4.7.14)$$

$$\therefore \theta = 31.8907 \quad (1.4.7.15)$$