Math computing

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NCERT 9.7.1.5

This question is from class 9 ncert chapter 7.triangles

1. Line l is the bisector of an angle $\angle A$ and B is any point on l. BP and BQ are perpendiculars from B to the arms of $\angle A$. Show that:
    (a) $\triangle APB \cong \triangle AQB$
    (b) $BP = BQ$ or B is equidistant from the arms of $\angle A$.

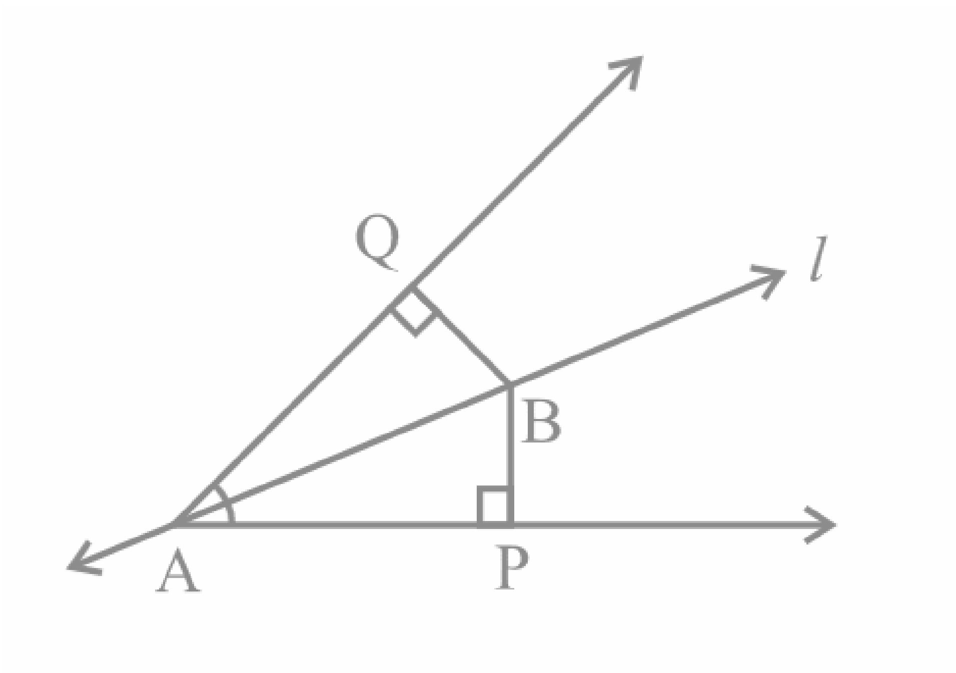

Figure 1: $\triangle AQB \quad and \quad \triangle APB$

1. **Construction steps:**
   The input parameters for construction are as follows.

Symbol	Value	Description
θ	$30°$	$\angle BAQ = \angle BAP$
x	5	Length of AB
l	4	Length of AP and AQ

Table 1: parameters

   (a) Let point A be the reference point whose coordinates are at origin.

$$\mathbf{A} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{1}$$

   (b) Let the distance between point A and B be x ,and also considering the point B on same axis .

$$\|A - B\| = x \tag{2}$$

   (c) So,the coordinates of point B be,

$$\mathbf{B} = \begin{pmatrix} x \\ 0 \end{pmatrix} \tag{3}$$

   (d) Let the coordinates of point P be,

$$\mathbf{P} = \begin{pmatrix} a \\ b \end{pmatrix} \tag{4}$$

   (e) And,let the coordinates of point Q be,

$$\mathbf{Q} = \begin{pmatrix} c \\ d \end{pmatrix} \tag{5}$$

   (f) Let assume the distance between point A and P be r ,and let the line AB makes an angle θ anticlock wise with line AP.

$$\|A - P\| = l \tag{6}$$

$$\angle PAB = \theta \tag{7}$$

   (g) Now the coordinates of point P will be,

$$\mathbf{P} = \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} l\cos\theta \\ -l\sin\theta \end{pmatrix} \tag{8}$$

(h) Similarly , let assume the distance between point A and Q also be r , and the line AB makes an angle θ clock wise with line AQ.

$$\|A - Q\| = l \tag{9}$$

$$\angle QAB = \theta \tag{10}$$

(i) Now the coordinates of point Q will be,

$$\mathbf{Q} = \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} l\cos\theta \\ l\sin\theta \end{pmatrix} \tag{11}$$

(j) Now the coordinates of A,B,P,Q are ,

$$\mathbf{A} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} x \\ 0 \end{pmatrix}, \mathbf{P} = \begin{pmatrix} l\cos\theta \\ -l\sin\theta \end{pmatrix}, \mathbf{Q} = \begin{pmatrix} l\cos\theta \\ l\sin\theta \end{pmatrix} \tag{12}$$

(k) on substituting the values from Table 1 ,

$$\mathbf{A} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 5 \\ 0 \end{pmatrix}, \mathbf{P} = \begin{pmatrix} 4\cos 30^\circ \\ -4\sin 30^\circ \end{pmatrix}, \mathbf{Q} = \begin{pmatrix} 4\cos 30^\circ \\ 4\sin 30^\circ \end{pmatrix} \tag{13}$$

(l) on calculating we get ,

$$\mathbf{A} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 5 \\ 0 \end{pmatrix}, \mathbf{P} = \begin{pmatrix} 3.464101 \\ -2 \end{pmatrix}, \mathbf{Q} = \begin{pmatrix} 3.464101 \\ 2 \end{pmatrix} \tag{14}$$

Joining these points forms the required figure

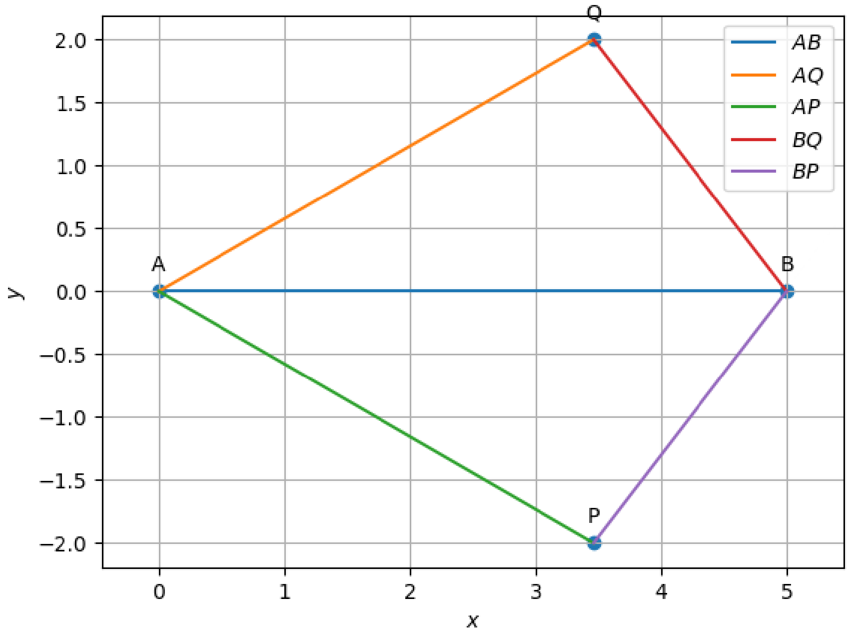

Figure 2: Foot of perpendiculars P and Q

2. **Finding foot of perpendiculars P, Q using Eigen Approach:**

   Assuming that the point A be the intersection point of two tangents of a circle with centre B and radius r. Let consider P, Q be the points of contact of two tangents to the circle.

Symbol	Value	Description
$\mathbf{A}$	$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$	coordinates of vertex A
$\mathbf{B}$	$\begin{pmatrix} 3 \\ 3 \end{pmatrix}$	centre of circle B
r	2	radius of circle

Table 2: parameters for eigen approach

   (a) let ,consider $h = A$ and $u = -B$

$$\mathbf{h} = \mathbf{A} \tag{15}$$
$$\mathbf{u} = -\mathbf{B} \tag{16}$$
$$\mathbf{h} = \begin{pmatrix} a \\ b \end{pmatrix} \tag{17}$$
$$\mathbf{u} = \begin{pmatrix} c \\ d \end{pmatrix} \tag{18}$$
$$\mathbf{V} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{19}$$
$$\mathbf{f} = \|u\|^2 - r^2 \tag{20}$$

   (b) finding the circle equation:

$$\mathbf{gh} = \mathbf{h^T V h} + 2\mathbf{u^T h} + f \tag{21}$$
$$\mathbf{gh} = \begin{pmatrix} a & b \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} + 2 \begin{pmatrix} c & d \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} + f \tag{22}$$
$$\tag{23}$$

   (c) from Table 2 and (15),

$$\mathbf{h} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{24}$$
$$\mathbf{u} = \begin{pmatrix} -3 \\ -3 \end{pmatrix} \tag{25}$$
$$\mathbf{r} = 2 \tag{26}$$

(d) on substituting these in (20) and (22) we get,

$$\mathbf{f} = 14.000000 \tag{27}$$

$$\mathbf{gh} = \begin{pmatrix} 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix} + 2 \begin{pmatrix} -3 & -3 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix} + 14.0 \tag{28}$$

$$\mathbf{gh} = 14.000000 \tag{29}$$

(e) Finding the sigma matrix :

$$\mathbf{\Sigma} = (\mathbf{Vh} + \mathbf{u})(\mathbf{Vh} + \mathbf{u})^T - \mathbf{ghV} \tag{30}$$

$$\mathbf{\Sigma} = \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} + \begin{pmatrix} c \\ d \end{pmatrix} \right) \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} + \begin{pmatrix} c \\ d \end{pmatrix} \right)^T - \mathbf{gh} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{31}$$

on substituting we get

$$\mathbf{\Sigma} = \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix} + \begin{pmatrix} -3 \\ -3 \end{pmatrix} \right) \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix} + \begin{pmatrix} -3 \\ -3 \end{pmatrix} \right)^T - 14.0 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{32}$$

$$= \begin{pmatrix} -5.000000 & 9.000000 \\ 9.000000 & -5.000000 \end{pmatrix} \tag{33}$$

(f) Finding eigen values and eigen vectors for (33) ,we get

$$\lambda = \begin{pmatrix} 4 \\ -14 \end{pmatrix} \tag{34}$$

$$\lambda_{\mathbf{1}} = 4 \tag{35}$$

$$\lambda_{\mathbf{2}} = -14 \tag{36}$$

$$\mathbf{P} = \begin{pmatrix} -0.707107 & -0.707107 \\ -0.707107 & 0.707107 \end{pmatrix} \tag{37}$$

(g) Finding The direction vectors of the tangents from a point h to the circle

$$\mathbf{m} = \mathbf{P} \begin{pmatrix} \sqrt{|\lambda_2|} \\ \pm\sqrt{|\lambda_1|} \end{pmatrix} \tag{38}$$

$$= \begin{pmatrix} -0.707107 & -0.707107 \\ -0.707107 & 0.707107 \end{pmatrix} \begin{pmatrix} \sqrt{|-14|} \\ \pm\sqrt{|4|} \end{pmatrix} \tag{39}$$

on solving we get

$$\mathbf{m_1} = \begin{pmatrix} -4.059965 \\ -1.231538 \end{pmatrix} \tag{40}$$

$$\mathbf{m_2} = \begin{pmatrix} -1.231538 \\ -4.059965 \end{pmatrix} \tag{41}$$

(h) Finding the contact points

$$\mathbf{x} = \mathbf{h} + \mu\mathbf{m} \tag{42}$$

Here

$$\mu = -\frac{\mathbf{m^T}\left(\mathbf{Vh} + u\right)}{\mathbf{m^T V m}} \tag{43}$$

$$\mu_{\mathbf{1}} = -\frac{\mathbf{m_1^T}\left(\mathbf{Vh} + u\right)}{\mathbf{m_1^T V m_1}} \tag{44}$$

$$\mu_{\mathbf{2}} = -\frac{\mathbf{m_2^T}\left(\mathbf{Vh} + u\right)}{\mathbf{m_2^T V m_2}} \tag{45}$$

$$\mu_{\mathbf{1}} = -\frac{\begin{pmatrix} -4.059965 & -1.231538 \end{pmatrix}\left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \end{pmatrix} + \begin{pmatrix} -3 \\ -3 \end{pmatrix}\right)}{\begin{pmatrix} -4.059965 & -1.231538 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} -4.059965 \\ -1.231538 \end{pmatrix}} \tag{46}$$

$$\mu_{\mathbf{2}} = -\frac{\begin{pmatrix} -1.231538 & -4.059965 \end{pmatrix}\left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \end{pmatrix} + \begin{pmatrix} -3 \\ -3 \end{pmatrix}\right)}{\begin{pmatrix} -1.231538 & -4.059965 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} -1.231538 \\ -4.059965 \end{pmatrix}} \tag{47}$$

on solving we get,

$$\mu_{\mathbf{1}} = -0.881917 \tag{48}$$

$$\mu_{\mathbf{2}} = -0.881917 \tag{49}$$

(i) now for contact points P and Q

$$\mathbf{P} = \mathbf{h} + \mu_1 \mathbf{m_1} \tag{50}$$

$$\mathbf{Q} = \mathbf{h} + \mu_2 \mathbf{m_2} \tag{51}$$

$$\mathbf{P} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} + (-0.881917) \begin{pmatrix} -4.059965 \\ -1.231538 \end{pmatrix} \tag{52}$$

$$\mathbf{Q} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} + (-0.881917) \begin{pmatrix} -1.231538 \\ -4.059965 \end{pmatrix} \tag{53}$$

$$\mathbf{P} = \begin{pmatrix} 3.580552 \\ 1.086114 \end{pmatrix} \tag{54}$$

$$\mathbf{Q} = \begin{pmatrix} 1.086114 \\ 3.580552 \end{pmatrix} \tag{55}$$

(j) As the lines BP and BQ are radius of circle ,So the lengths of BP and BQ are equal

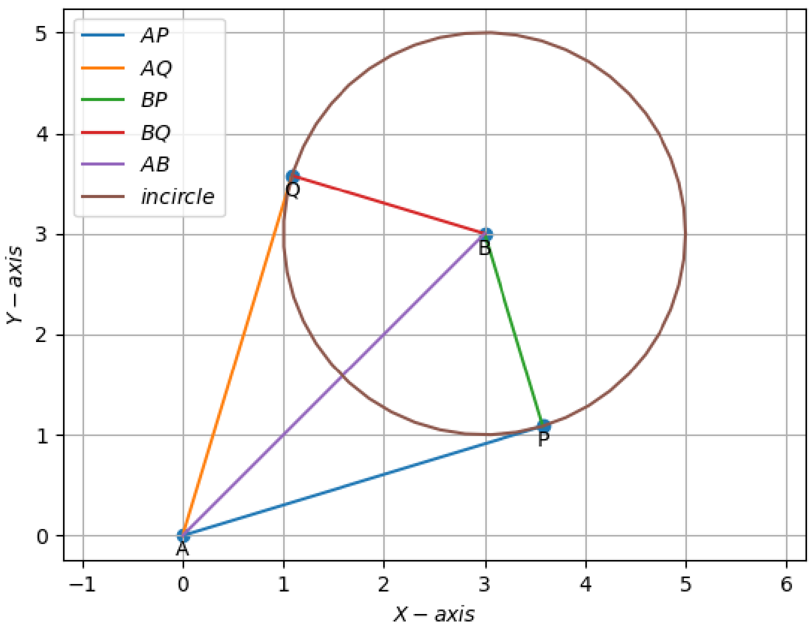

Figure 3: Contact points P and Q