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The Two-Period Travelling Salesman Problem Applied to Milk Collection in Ireland

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Abstract. We describe a new extension to the Symmetric Travelling Salesman Problem (STSP) in which some nodes are visited in both of 2 periods and the remaining nodes are visited in either 1 of the periods. A number of possible Integer Programming Formulations are given. Valid cutting plane inequalities are defined for this problem which result in an, otherwise prohibitively difficult, model of 42 nodes becoming easily solvable by a combination of cuts and Branch-and-Bound. Some of the cuts are entered in a "pool" and only used when it is automatically verified that they are violated. Other constraints which are generalisations of the subtour and comb inequalities for the single period STSP, are identified manually when needed. Full computational details of solution process are given.

Keywords: travelling salesman problem, inequalities, cutting planes, Branch-and-Bound

1. Introduction

The optimisation problem we consider originates from Irish dairy farm collection in the county of Dublin. Specifically, given some farms needing everyday pick-up and others requiring every other day pick-up, the problem is to identify two tours with a combined distance that is minimized and such that:

1. each farm requiring daily pick-up is visited exactly once by *each* tour.
2. each farm requiring every other day pick-up is visited exactly once by only *one* of the tours.

For an illustration of a feasible solution (produced by a heuristic) to the problem, see the figure 1 where everyday farms are marked "+" and every other day farms are marked "$*$". The two tours are distinguished by normal and bold lines respectively.

1. Letting n_2 and n_1 denote the number of farms requiring daily and every other day pick-up, respectively, and $n = n_2+n_1$ as the total number of farms, we see that the standard symmetric

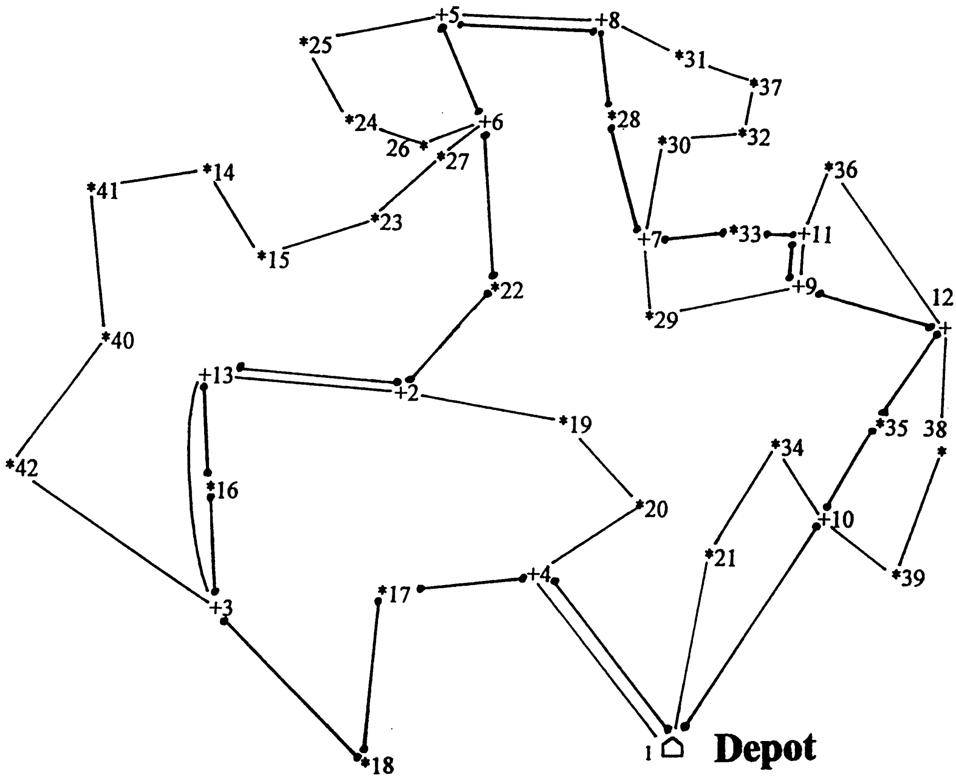

Figure 1. A feasible solution (length = 1750).

travelling salesman problem (STSP) is the special case where $n_1 = 0$. Consequently, we refer to this problem as the "two-period symmetric travelling salesman problem". We note that the natural generalisation is the k-period STSP where there are k different types of farms and one must identify k tours with minimum combined tour length and satisfying each farm's pick-up requirements exactly. Further, it is clear that for all integer $k \geq 1$, the k-period STSP is a member of the class of NP-complete problems. Although much attention has been given to the standard (single-period) STSP, to our knowledge, this two-period, much less the k-period, STSP has not previously been examined.

As a first step, intuition might lead one to consider a transformation of the problem to a standard STSP by duplication of the everyday farms and introduction of pairwise infinite costs between them. Unlike the vehicle routing and multiple travelling salesman problems, however, the transformed problem allows solutions that are not feasible to the original problem. In the next section, we introduce alternative integer linear programming formulations of the two-period STSP.

The remainder of the paper focuses on obtaining a polyhedral description of the convex hull of feasible integer solutions to a linear programming formulation of the problem. As might be expected, some of the inequalities are well-known to be facet-defining for the standard STSP polytope (see for instance, Grötschel and Padberg [5] or Lawler et al. [7]) as well as generalizations of these inequality classes proved extremely useful in a polyhedral-based solution procedure. The proofs that these inequality classes are facet-defining for the 2-period STSP polytope will be left for future research; the objective of this work is to introduce the problem and report initial computational findings. We begin with a discussion of integer programming formulations for the problem, followed by a brief presentation of the inequalities incorporated into the cutting plane solution procedure. We report computational results in Section 4 and conclude with further extensions and applications of the problem in Section 5.

The notation used throughout is consistent with that found in the literature related to polyhedral descriptions and solution approaches for the standard STSP (e.g., Grötschel and Padberg [5] and more recently, Padberg and Rinaldi [11]). Some adaptation, of course, is needed since this is a variation of the standard STSP (hereafter referred to simply as STSP). Specifically, $K_n = (V, E)$ still denotes a complete graph with a node set V having $1 \cdots n$ labelled nodes and E is an edge set consisting of $n(n-1)/2$ unordered pairs of indices. However, V can be identifiably partitioned into V_2 having n_2 nodes and V_1 having $n_1 = n - n_2$ nodes. We will assume that the nodes of V_2 correspond to farms requiring every day pick-up and are labelled $1 \cdots n_2$, with 1 signifying the depot, whereas the nodes of V_1 correspond to every other day pick-up farms and are labelled $n_2 + 1 \cdots n$. We will refer to two tours that exactly satisfy all farms pick-up requirements as a "feasible 2-tour". For distance values $c_e \in \mathcal{R}$ given for each $e \in E$, one can ascertain the length of a feasible 2-tour by adding the length of the two tours that make up the 2-tour. Thus, the problem we address in this work is that of finding a feasible 2-tour with minimum length.

2. Problem formulations

Unlike the STSP, the problem considered is a new variation of the STSP, and as such, does not have a well-known, established formulation.

There are, however, other extension and variants of the STSP which have received similar treatments such as Balas' work [1] on the Prize Collecting TSP and Bauer's work [2] on the Circuit Polytope as well as the work of Fischetti et al. [3] and [4].

We introduce the following variables:

$$x_{ek} = \begin{cases} 1 & \text{if edge } e \text{ is on tour } k, \quad k \in \{1, 2\} \\ 0 & \text{otherwise} \end{cases}$$

We are effectively duplicating each edge in E so that an optimal solution vector will offer a well-defined separation between the two tours' edge sets. We also introduce the variables:

$$y_{vk} = \begin{cases} 1 & \text{if farm } v \in V_1 \text{ is on tour } k, \quad k \in \{1, 2\} \\ 0 & \text{otherwise} \end{cases}$$

In this formulation, we have that for every possible subset of $E \times E \times V \times V$, there is a unique 0-1 incidence vector $(x, y) \in \mathcal{R}^{2(|E|+|V|)}$ (and vice versa). On a complete graph

$K_n = (V, E)$, we again let $\mathcal{T}_n$ denote the set of all feasible 2-tours. We are interested in the polytope:

$$Q_T^n = \text{conv}\big\{(x, y) \in \mathcal{R}^{2(\|E\|+\|V\|)} : (x, y) \text{ is the incidence vector for a } T \in \mathcal{T}_n\big\}.$$

Letting $x^k(F)$ now abbreviate, for $F \subseteq E$, the sum $\sum_{e\in F} x_{ek}$, we have the following formulation for the 2-period STSP:

$$\text{Minimize} \quad Z = \sum_{k=1}^{2} \sum_{e\in E} c_e x_{ek}$$

$$\text{s.t.} \quad x^k(\delta(v)) = 2 \qquad \forall v \in V_2, \quad k \in \{1, 2\} \tag{1}$$

$$x^k(\delta(v)) = 2y_{vk} \quad \forall v \in V_1, \quad k \in \{1, 2\} \tag{2}$$

$$y_{v1} + y_{v2} = 1 \qquad \forall v \in V_1 \tag{3}$$

$$(x, y) \text{ is the incidence vector for some member of } \mathcal{T}_n \tag{4}$$

$$x_{ek} \in \{0, 1\} \tag{5}$$

$$y_{vk} \in \{0, 1\} \tag{6}$$

The Eq. (1) will ensure that farms in V_2 are included on both tours, whereas (2) and (3) will force farms in V_1 to be visited by exactly one of the tours. In other words, the $2n + n_1$ equations of (1) through (3) are satisfied by all incidence vectors of feasible 2-tours. Thus, we have that

$$Q_T^n \subseteq \big\{(x, y) \in \mathcal{R}^{2(\|E\|+\|V\|)} : (x, y) \text{ satisfies (1)–(3), (5)–(6)}\big\}.$$

We note that the number of y-variables could be halved and constraints (3) omitted by replacing each y_{v2} by $1 - y_{v1}$. As will be detailed in Section 4, this substitution was not made so that we could exploit a software feature offered by the simplex code selected for use in this study.

In the LP-based solution approach discussed in the next section, we begin with the relaxation

$$Q_D^n = \big\{(x, y) \in \mathcal{R}^{2(|E|+|V|)} : (x, y) \text{ satisfies (1)–(3)}, 0 \le x \le 1, 0 \le y \le 1\big\},$$

3. Solution approach

We now describe the solution approach used to identify optimal integer solutions for twenty and forty-two node 2-period STSPs. The procedure is similar to that used in a number of computational studies on small STSPs (see for instance, Rinaldi and Yarrow [12], Land [6] and Miliotis [8] and [9]), but differs in the types of constraints generated. That is, we use a man-machine method that can be summarized by the flow chart given below in figure 2.

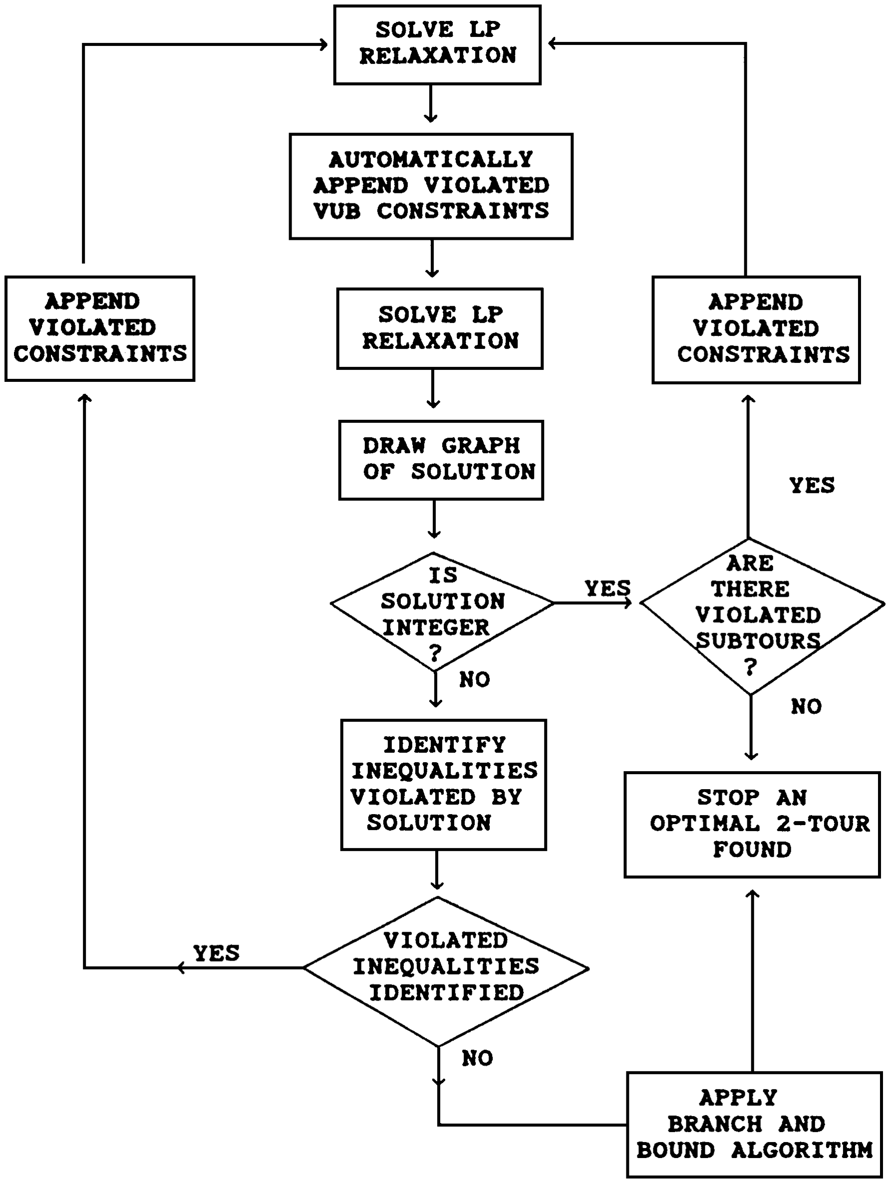

Figure 2. Solution approach.

Since the problem we considered in this preliminary study was relatively small, we did not investigate or develop a sophisticated code for a branch-and-cut strategy as is now common in polyhedral strategies for solving integer programmes. Rather, we were interested in quickly and easily testing the strength of generalisations of both the inequalities known to be facet-defining for STSPs (see Nemhauser and Wolsey [10] for full details). We now define the classes of inequalities used in the solution procedure and when it is not readily apparent, show their validity for the 2-period STSP polytope, Q_T^n, defined in the previous section.

The simplest class of inequalities arose from the logic of constraints (2). That is, for all $v \in V_1$, we have that each edge adjacent to farm v must be in at most one tour.

Disaggregating these constraints, we have:

$$x_{ek} \leq y_{vk} \qquad \forall e \in \delta(v), \qquad \forall v \in V_1, \quad k \in \{1, 2\} \tag{7}$$

We refer to these constraints as variable upper bounds (VUB's).

As will be the case for *all* constraint classes discussed hereafter, the number of different inequalities in this class is quite large (in particular, $n_1(n_1 - 1) + n_1 n_2$). Thus, they were appended to the LP-relaxation on an "as needed" basis.

The next class of inequalities considered were modified subtour elimination constraints (SECs) for single-periods. In our modification of traditional SECs, we have employed the definition of y-variables to tighten the usual SEC inequalities of the STSP. Letting $E(S)$ denote the set of edges with both endpoints in $S \subseteq V$ for a given $k \in \{1, 2\}$, we distinguish subsets $S \subset V$ to have either $S \cap V_2 \neq \emptyset$ or $S \cap V_2 = \emptyset$. In the former case, we have that all feasible 2-tours must satisfy

$$\sum_{e \in E(S)} x_{ek} \leq |S \cap V_2| + \sum_{v \in S \cap V_1} y_{vk} - 1,$$
$$\forall S \subseteq V - \{1\}, 2 \leq |S| \quad \text{and} \quad S \cap V_2 \neq \emptyset, k \in \{1, 2\}. \tag{8}$$

It should be clear that the usual SECs of the STSP,

$$\sum_{e \in E(S)} x_{ek} \leq |S| - 1,$$

are special cases of (8) where for a given $k \in \{1, 2\}$ and $S \subseteq V - 1$, we have either $S \cap V_1 = \emptyset$ or $y_{vk} = 1$ for all $v \in S \cap V_1$.

For $S \subseteq V_1$, we have the group of inequalities

$$\sum_{e \in E(S)} x_{ek} \leq \sum_{v \in S} y_{vk} - y_{lk}, \qquad \forall S \subseteq V_1, 2 \leq |S| \quad \text{and} \quad l \in S, k \in \{1, 2\} \tag{9}$$

that are also valid for the 2-period STSP polytope Q_n^T. From a computational perspective, rather than identifying for a given $S \subset V_1$, the group of inequalities violated by a solution (x, y), we identify only the inequalities that are *most* violated. That is, we find for $S \subseteq V_1$ and $k \in \{1, 2\}$ (i.e., for a *specified* tour day k) $y_{mk} = \max_v y_{vk}$, $v \in S$

$$\sum_{e \in E(S)} x_{ek} \leq \sum_{v \in S} y_{vk} - y_{mk}, \tag{10}$$

that is violated by the linear programming relaxation solution (x, y). Henceforth, inequalities of the form (8) and (10) will be referred to as SEC2 and SEC1 inequalities, respectively.

In fact the VUB inequalities (7) could be regarded as a special case of (9) when $|S| = 2$.

On the basis of these modified SEC inequalities, we considered the equivalent of comb inequalities for the 2-period STSP. Thus, as in the standard STSP, we define subsets, H(*handle*), $W_1, W_2, \ldots, W_t$(*teeth*), of the vertex set V satisfying:

1. $|H \cap W_i| \geq 1$ for $i = 1, \ldots, t$
2. $|W_i \backslash H| \geq 1$ for $i = 1, \ldots, t$
3. $W_i \cap W_j = \emptyset$ for $i \neq j$
4. t is at least 3,

and consider the subgraph, a comb, generated by these subsets comprising a handle and teeth.

For a straightforward generalisation of comb inequalities for each k we need all teeth W_i to satisfy

$$|W_i \cap H \cap V_2| \geq 1 \tag{11}$$

$$|W_i \backslash H \cap V_2| \geq 2 \tag{12}$$

and t to be odd. We then consider the degree constraints for $V \in H$ weighted by $\frac{1}{2}$ and added to give

$$\sum_{e \in E(H)} x_{ek} + \frac{1}{2} \sum_{e \in \delta(H)} x_{ek} = |H \cap V_2| + \sum_{v \in H \cap V_1} y_{vk}$$

Adding $-\frac{1}{2} x_{ek} \leq 0$ for all $e \in \delta(v) \backslash U_{i=1}^{t} E(W_i)$ gives

$$\sum_{e \in E(H)} x_{ek} + \frac{1}{2} \sum_{i=1}^{t} \sum_{e \in \delta(H) \cap E(W_i)} x_{ek} \leq |H \cap V_2| + \sum_{v \in H \cap V_1} y_{vk} \tag{13}$$

The SEC2 constraints (8) for W_i, $H \cap W_i$ and $W_i \backslash H$ are respectively

$$\sum_{e \in E(W_i)} x_{ek} \leq |W_i \cap V_2| + \sum_{v \in W_i \cap V_i} y_{vk} - 1 \tag{14}$$

$$\sum_{e \in E(W_i \cap H)} x_{ek} \leq |W_i \cap H \cap V_2| + \sum_{v \in W_i \cap H \cap V_1} y_{vk} - 1 \tag{15}$$

$$\sum_{e \in E(W_i \backslash H)} x_{ek} \leq |W_i \backslash H \cap V_2| + \sum_{v \in W_i \backslash H \cap V_1} y_{vk} - 1 \tag{16}$$

Multiplying each of (14), (15) and (16) by $\frac{1}{2}$, adding to (13) and rounding gives

$$\begin{aligned} \sum_{e \in E(H)} x_{ek} + \sum_{i=1}^{t} \sum_{e \in E(W_i)} x_{ek} &\leq |H \cap V_2| + \sum_{v \in H \cap V_1} y_{vk} \\ &+ \sum_{i=1}^{t} \left(|W_i \cap V_2| + \sum_{v \in W_i \cap V_1} y_{vk} - 1 \right) - \frac{t+1}{2} \end{aligned} \tag{17}$$

These constraints relate to a specific period k and will be referred to as COMB1 constraints.

If an odd number of teeth W_i (as well as possible other teeth, making a total of 3 or more teeth) have

$$W_i \cap H \cap V_2 = \emptyset \tag{18}$$

or

$$W_i \backslash H \cap V_2 = \emptyset \tag{19}$$

or both

then a similar argument to the above can be applied to produce *aggregated* comb inequalities, i.e., applying to the periods taken together.

Constraints (13) apply as before but some or all of (14), (15) and (16) become SEC1 constraints (9).

Multiplying each of them by $\frac{1}{2}$ and adding to (13) now gives

$$\begin{aligned}\sum_{e\in E(H)} x_{ek} + \sum_{i=1}^{t} \sum_{e\in E(W_i)} x_{ek} &\le |H \cap V_2| + \sum_{v\in(H\cap V_1)} y_{vk} \\ &+ \sum_{i=1}^{t} \left(|W_i \cap V_2| + \sum_{v\in(W_i\cap V_1)} y_{vk} - \beta_i \right)\end{aligned} \tag{20}$$

where

$$\begin{aligned}\beta_i \text{ is } & 1 + \frac{1}{2} y_{\ell_1 k} \\ \text{or } & 1 + \frac{1}{2} y_{\ell_2 k} \\ \text{or } & \frac{1}{2}(y_{\ell_1 k} + y_{\ell_2 k} + y_{\ell_e k}) \\ \text{or } & \frac{3}{2} \qquad \text{with } \ell_i \in W_i \cap H,\ \ell_2 \in W_i \backslash H,\ \ell_3 \in W_i\end{aligned}$$

depending on whether one of (18), (19), both of (18) and (19) or both (11) and (12) apply respectively.

Adding together (20) for all k, using constraints (3) and rounding gives

$$\sum_{k} \sum_{e\in E(H)} x_{ek} + \sum_{k} \sum_{i=1}^{t} \sum_{e\in E(W_i)} x_{ek} \le 2|H \cap V_2| + |H \cap V_1| + \sum_{i=1}^{t} \mu(W_i) - \frac{s+1}{2} \tag{21}$$

where

$$\begin{aligned}\mu(W_i) &= 2|W_i \cap V_2| + |W_i \cap V_1| - 2 && \text{if one of (18) or (19) apply} \\ &= 2|W_i \cap V_2| + |W_i \cap V_1| - 1 && \text{if both apply} \\ &= 2|W_i \cap V_2| + |W_i \cap V_1| - 3 && \text{if (11) and (12) apply}\end{aligned}$$

These aggregated constraints are referred to as COMB2 constraints.

It is also possible to generalise the clique-tree constraints for the standard STSP in both single period and aggregated form.

4. Computational results

We have computed the optimal solution (illustrated in figure 7) to the problem consisting of 41 farms and a depot. 12 of the farms (and the depot) are visited every day and the remaining 29 every other day. On the figures the depot and every-day farms are marked "+" and numbered 1 to 13 and the remaining farms marked "$*$" and numbered 14 to 42. Distances are measured in $\frac{1}{10}$ mile.

Using the numbering introduced we therefore have

$$n_2 = 13, \qquad n_1 = 29, \qquad n = 42.$$

For a 1-period STSP with n cities the total number of feasible solutions is

$$\frac{(n-1)!}{2} \simeq 1.67 \times 10^{49} \quad \text{when } n = 42$$

By Considering all possible partitions of the single day farms into 2 sets it can be seen that the total number of feasible solution to the 2-period STSP (only counting symmetric alternatives once) is

$$\frac{1}{2}\sum_r \frac{n_1!}{r!(n_1-r)!}\,\frac{(n_2+r-1)!}{2}\,\frac{(n_2+(n_1-r)-1)!}{2} \simeq 4.27 \times 10^{62}$$
$$\text{when } n_1 = 29, n_2 = 13$$

An alternative form of this expression can be obtained by considering all possible ways of inserting the single day farms into all possible combinations of the 2 tours round the everyday farms. This gives the simpler expression

$$\frac{[(n_2-1)!]^2(2n_2+n_1-1)!}{8(2n_2-1)!}$$

The argument to produce the above expression directly (as opposed to algebraic simplification) was suggested by Professor C.C. Chen of the National University of Singapore.

The solution approach illustrated in figure 2 was adopted. There are $2n_2$ constraints of type (6), $2n_1$ constraints of type (7), and n_1 constraints of type (8) giving a total of $2n + n_1$ constraints. The total number of variables is $n(n-1) + 2n_1$. In order to avoid generating symmetric solutions (i.e., interchanging alternate day solutions) farm $n_2 + 1$ was fixed to be visited on day 1 (and alternate days from then on). Fixing the corresponding x and y variables results in a model with 112 constraints and 1778 0-1 variables. This model is, of course, a (generalised matching) relaxation of the 2-period STSP and would be expected to produce a solution with subtours. It would also be expected to take a long time to solve.

The computation was carried out using version 7.14 of XPRESS-MP [13] on an IBM 486 PC with 33 MHz and 4 Mb RAM. Full computational results are summarized in Table 1.

Table 1. Compiutational summary (times in seconds).

Stage	Time to solve LP relaxation (secs)	Iterations	Objective value	Violated MVUB Constraints	Time to resolve	Iterations	Objective value	Violated SEC1s	Violated SEC2s	Violated COMB1s	Violated COMB2s
1	2	993	1570	61	6	2531	$1696\frac{1}{2}$	None	15	1	2
2	2 (Starting basis)	529	1628	50	10	1988	$1711\frac{1}{2}$	None	3	None	2
3	2 (Starting basis)	564	$1645\frac{1}{2}$	57	13	3399	$1720\frac{5}{6}$	None	None	None	1

From this fractional (Objective $= 1720\frac{5}{6}$) solution branch and bound took 35 nodes and 13 seconds to produce the optimal solution with objective 1725.

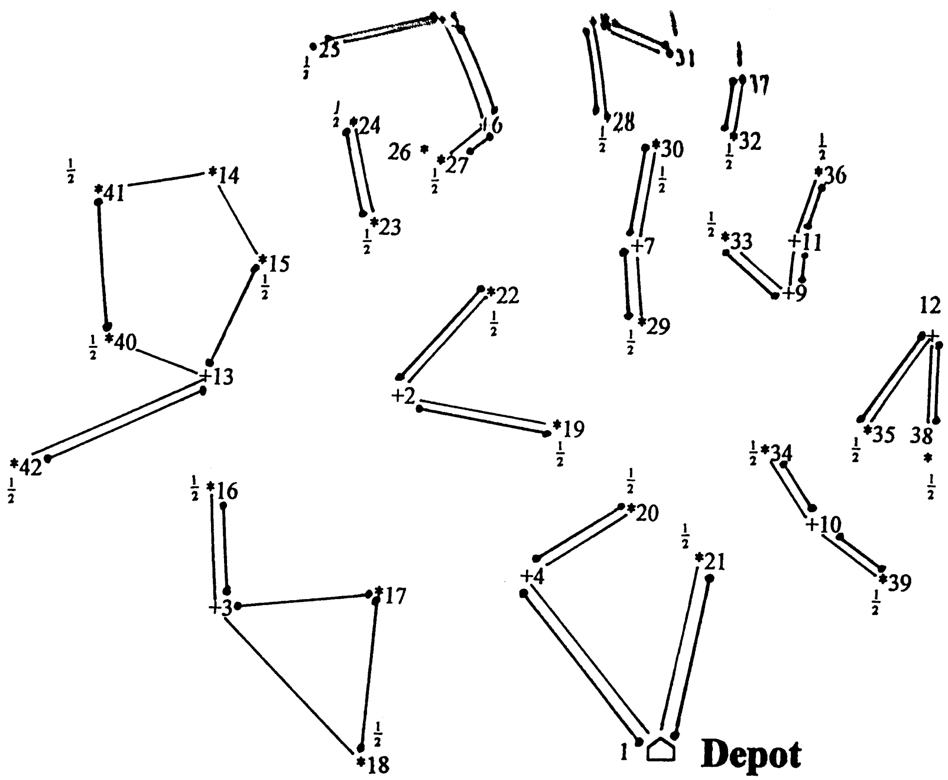

Figure 3. First LP relaxation solution (length $= 1570$).

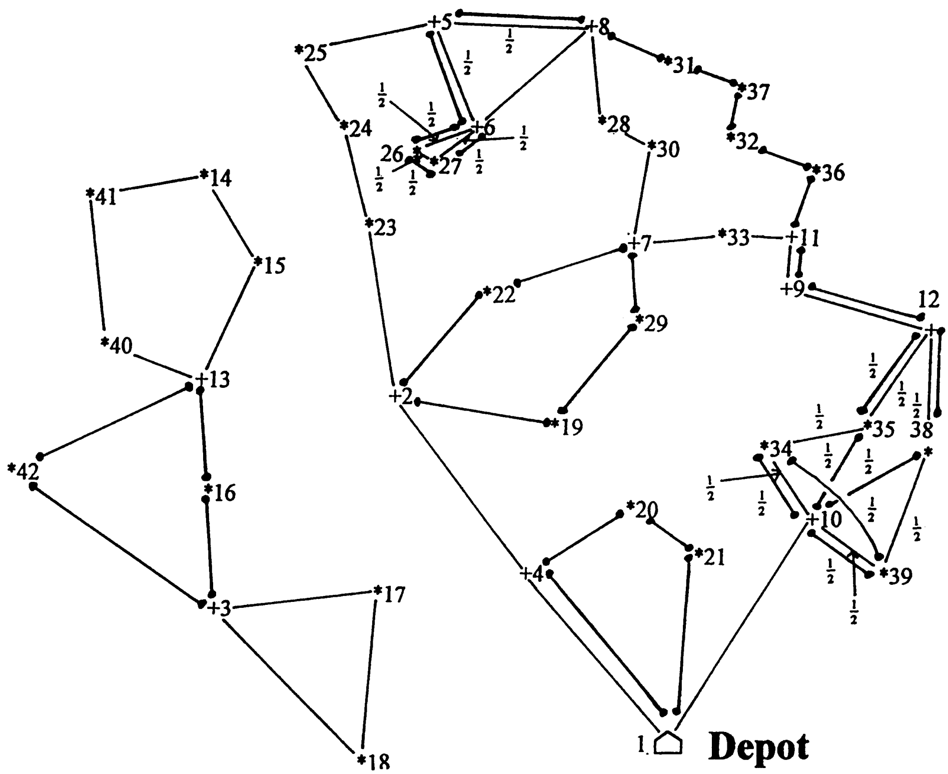

Figure 4. LP relaxation solution after VUB constraints appended (length = $1696\frac{1}{2}$).

Solving the LP relaxation of the model described above took 2 seconds and produced the solution shown in figure 3 with an objective value of 1570. It can seen that this solution is fractional. For interest the Branch-and-Bound algorithm was then applied to the model to see if it produced a solution in a reasonable period of time. As expected it was prohibitively difficult to solve and the run was terminated after 16 hours.

It can be seen that the solution in figure 3 violates some of the inequalities (12). For example $x_{14,15,1} = 1$, $x_{13,15,2} = 1$, $y_{15,1} = \frac{1}{2}$ (farm 15 is "half-visited" on day 1 and "half-visited" on day 2). Fractional values of the variables y_{v1} are marked at the nodes in figure 3. Rather than append the total of 1189 inequalities of type (12) a feature of XPRESS-MP was used in which it is possible to put all these constraints in a "pool". By using a procedure MVUB (Make Variable Upper Bounds) only those constraints which were violated were incorporated into the model which was then repeatedly reoptimized until all such inequalities (12) were satisfied. A total of 61 such constraints were so introduced at this stage. It took 6 seconds to reoptimize the LP relaxation and produce the solution shown in figure 4 with an objective value of $1696\frac{1}{2}$. Clearly this LP relaxation is considerably more constrained. Fractional values on arcs are marked.

It should be noted that, in spite of its name, the procedure MVUB incorporates explicit constraints of type (7) into the model, when violated, and does not use the variable upper bound variant of the simplex algorithm. A restriction of the current implementation of MVUB is that the violated constraints are only appended in the current run. When the model is rerun the MVUB procedure has to be repeated.

The reason for not halving the number of y-variables by replacing y_{v1} by $1 - y_{vi}$ as mentioned in Section 2, is that MVUB demands constraints in the form (7) and would not therefore accept the resultant transformation of these constraints.

Again it was realised that applying the Branch-and-Bound algorithm to this model with the VUB constraints was still likely to take a long time and result in subtours. This was indeed the case. Although the run terminated in 2282 seconds, it produced a solution with subtours and an objective of 1709.

In the solution, in figure 4, 15 violated subtour elimination constraints (SEC2s) of the form described in Section 3 are apparent. For example taking $S = \{3, 17, 18\}$ and $k = 1$ gives one such constraint. $S = \{10, 34, 39\}$ and $k = 2$ gives another. The full list of violated constraints at this stage is given in Table 2. Since only specifying such a violated constraint for a single day is likely to result in the next solution violating the constraint for the opposite day such constraints are specified for both days resulting in a total of 28 SEC2 constraints being appended.

There is also a violated COMB1 and two violated COMB2 constraints. These are also given in Table 2. As with the subtour constraints it seemed desirable to duplicate the COMB1 constraint to apply also for the opposite days and so avoid symmetric solutions.

Appending all these constraints (32 in total), resolving the LP relaxation and recalling the MVUB procedure results in the fractional solution given in figure 5 with an objective

Table 2. Violated SEC and COMB inequalities after Stage 1.

SEC1s	None	
SEC2s	$\{13, 14, 15, 40, 41\}$	$k = 1$
	$\{3, 17, 18\}$	$k = 1$
	$\{1, 4, 20, 21\}$	$k = 2$
	$\{10, 34, 39\}$	$k = 2$
	$\{6, 26, 27\}$	$k = 1$
	$\{6, 26, 27\}$	$k = 2$
	$\{3, 13, 16, 42\}$	$k = 2$
	$\{10, 12, 34, 35, 38, 39\}$	$k = 2$
	$\{2, 7, 19, 22, 29\}$	$k = 2$
	$\{5, 6, 8, 26, 27\}$	$k = 2$
	$\{10, 34, 35, 39\}$	$k = 2$
	$\{10, 34, 38, 39\}$	$k = 2$
	$\{5, 6, 26, 27\}$	$k = 2$
	$\{5, 6, 24, 25, 26, 27\}$	$k = 2$
	$\{5, 6, 8, 24, 25, 26, 27\}$	$k = 2$
COMB1	$H = \{7, 8, 11, 28, 30, 31, 32, 33, 36, 37\}$	
	$W_1 = \{5, 6, 8, 24, 25, 26, 27\}$	
	$W_2 = \{2, 7, 19, 22, 29\}$	
	$W_3 = \{9, 11\}$	$k = 2$
COMB2		
$H = \{5, 6, 8\},\ W_1 = \{8, 28\},\ W_2 = \{5, 25\},\ W_3 = \{6, 26, 27\}$		
$H = \{5, 6, 8\},\ W_1 = \{8, 31\},\ W_2 = \{5, 25\},\ W_3 = \{6, 26, 27\}$		

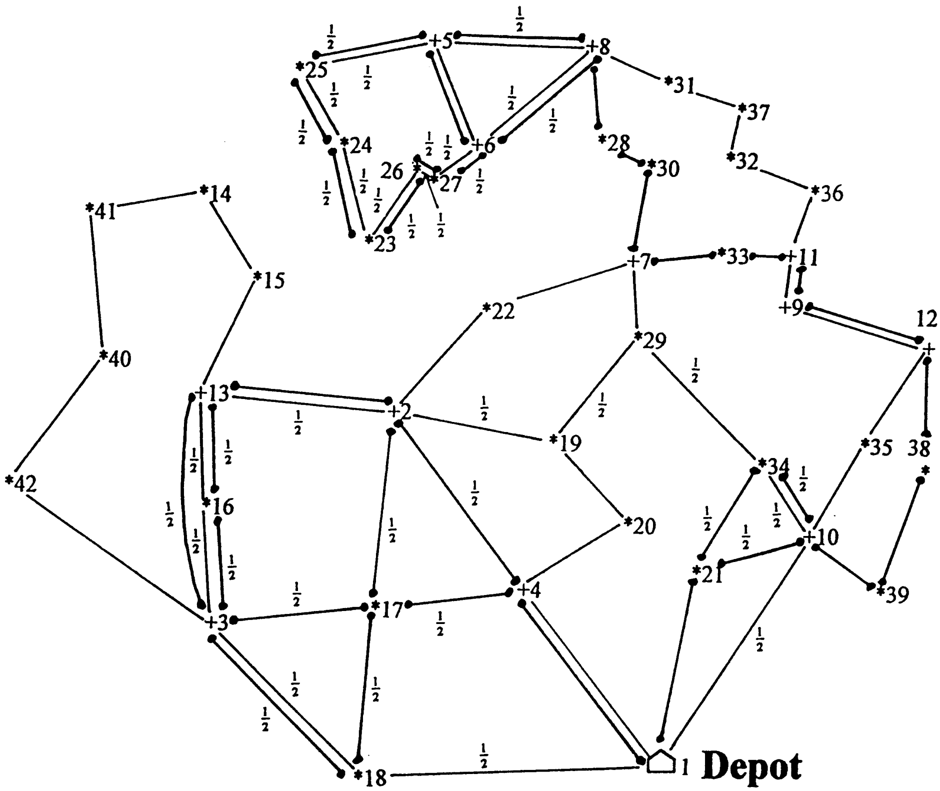

Figure 5. LP relaxation solution after 1st set of subtours and combs (length $= 1711\frac{1}{2}$).

value of $1711\frac{1}{2}$. For interest applying the Branch-and-Bound algorithm takes 428 seconds and produces a solution with subtours and an objective of 1722.

In the solution in figure 5, 3 violated SEC2 constraints are apparent together with two COMB2 constraints. The full list is given in Table 3.

Appending all these constraints, and their symmetric alternatives where appropriate (a total of 6), then resolving the LP relaxation and recalling the MVUB procedure results in the fractional solution given in figure 6 with an objective value of $1720\frac{5}{6}$. Applying the Branch-and-Bound algorithm takes 13 seconds and produces the optimal solution shown in figure 7 with an objective value of 1725.

For interest the fractional solution given in figure 6 was examined for violated constraints only one of which was found and is given in Table 4. The appending of this constraint (and VUB constraints) results in an LP relaxation with an objective value of $1721\frac{1}{2}$ and reduces the Branch-and-Bound search to 10 seconds and 33 nodes. No other violated subtour or comb constraints are apparent.

In contrast the corresponding 1-period 42 city problem was solved entirely by appending cuts to the LP relaxation. A total of 9 subtour elimination and 9 comb constraints were appended in 15 stages. The final solution was obtained in 90 iterations and less than 1 second.

Table 3. Violated SEC and COMB inequalities after Stage 2.

SEC1s	None	
SEC2s	{5, 6, 8, 23, 24, 25, 26, 27}	$k = 1$
	{5, 6, 8, 23, 24, 25, 26, 27}	$k = 2$
	{3, 13, 14, 15, 16, 40, 41, 42}	$k = 1$
COMB1s	None	
COMB2s	$H = \{7, 8, 11, 28, 30, 31, 32, 33, 36, 37\}$	
	$W_1 = \{5, 6, 8, 23, 24, 25, 26, 27\}$	
	$W_2 = \{7, 29\}$, $W_3 = \{9, 11\}$	
	$H = \{7, 8, 11, 28, 30, 31, 32, 33, 36, 37\}$	
	$W_1 = \{5, 6, 8, 23, 24, 25, 26, 27, \}$	
	$W_2 = \{7, 22\}$, $W_3 = \{9, 11\}$	

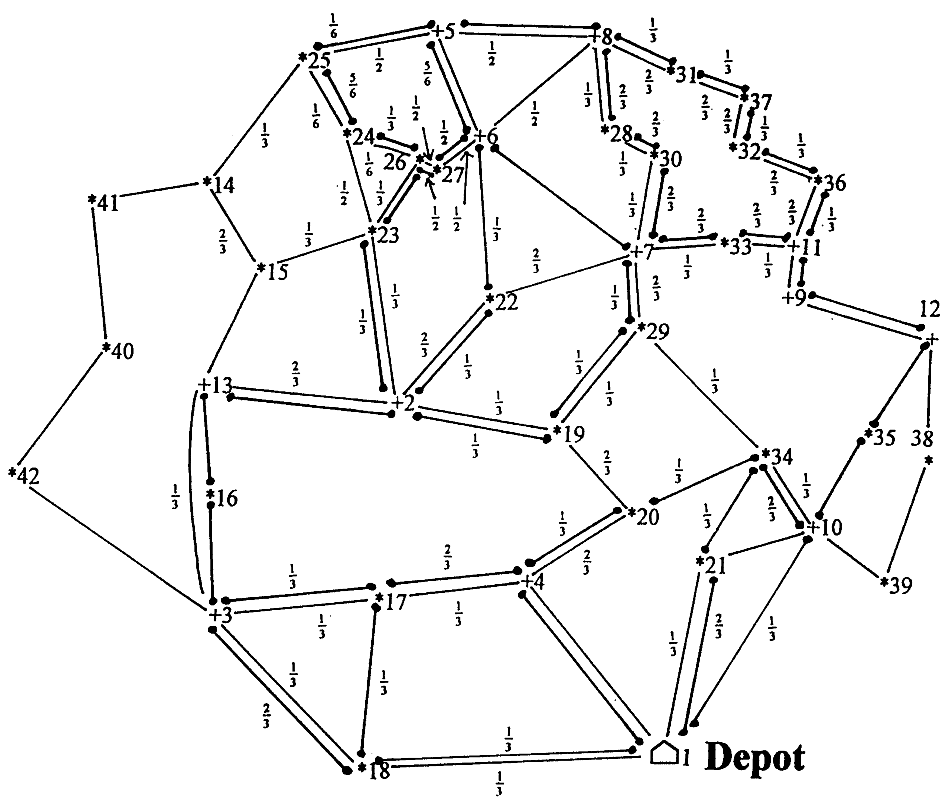

Figure 6. LP relaxation solution after 2nd set of subtours and combs (length $= 1720\frac{5}{6}$).

Table 4. Violated SEC and COMB inequalities after Stage 3.

SEC1s	None
SEC2s	None
COMB1	None
COMB2	$H = \{23, 24, 26\},\ W_1 = \{2, 23\},\ W_2 = \{24, 25\},\ W_3 = \{26, 27\}$

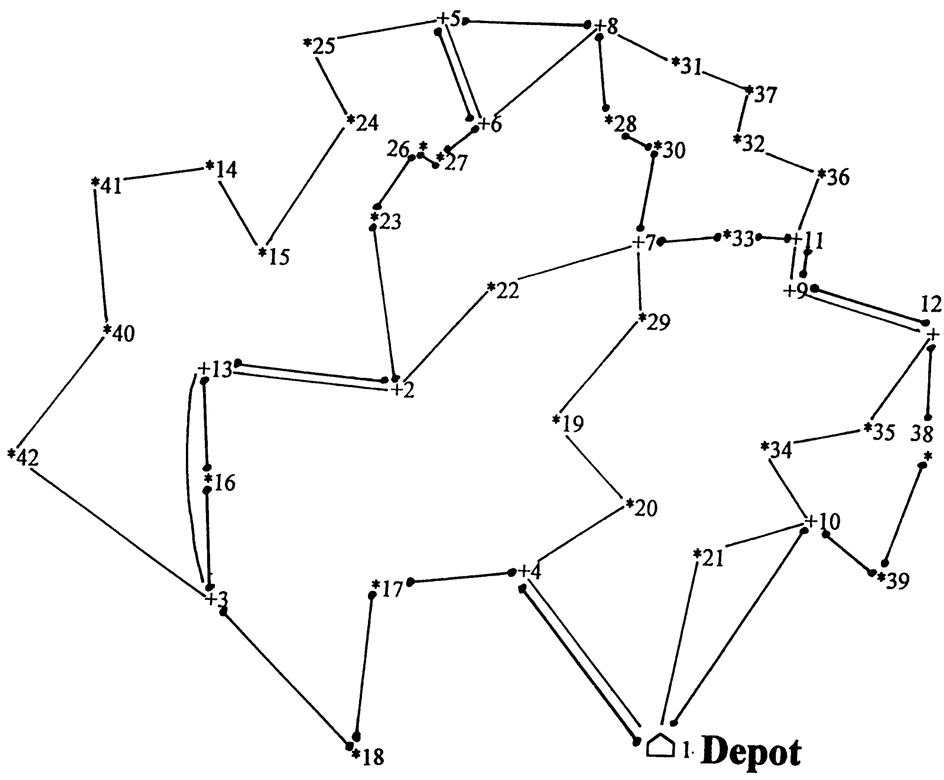

Figure 7. Optimal solution (length = 1725).

5. Conclusions and further work

This study demonstrates the value of combining polyhedral theory with Branch-and-Bound in solving difficult Combinatorial Optimization models. It also introduces a new practical variant to the TSP with generalisations of the well known subtour and comb elimination constraints.

Further work will involve (i) proving (or otherwise) new constraints are facets of the convex hull of feasible solutions, (ii) using generalisations of the clique-tree elimination constraints, (iii) Automating practical separation procedures for identifying violated constraints,

(iv) applying heuristics to the problem and (v) incorporating vehicle capacity constraints in the model.

Appendix: Solutions

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