H. PAUL WILLIAMS

Model Building in Mathematical Programming

FIFTH EDITION

WILEY

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A John Wiley & Sons, Ltd., Publication

Library of Congress Cataloging-in-Publication Data

Williams, H. P.
Model building in mathematical programming / H. Paul Williams. – 5th ed.
p. cm.
Includes bibliographical references and index.
ISBN 978-1-118-44333-0 (pbk.)
1. Programming (Mathematics) 2. Mathematical models. I. Title.
T57.7.W55 2013
519.7–dc23

2012037292

A catalogue record for this book is available from the British Library.

ISBN: 978-1-118-44333-0

Set in 10pt/12pt Times by Laserwords Private Limited, Chennai, India

To Eileen, Anna, Alexander and Eleanor

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Preface

Mathematical programmes are among the most widely used models in operational research and management science. In many cases their application has been so successful that their use has passed out of operational research departments to become an accepted routine planning tool. It is therefore rather surprising that comparatively little attention has been paid in the literature to the problems of formulating and building mathematical programming models or even deciding when such a model is applicable. Most published work has tended to be of two kinds. Firstly, case studies of particular applications have been described in the operational research journals and journals relating to specific industries. Secondly, research work on new algorithms for special classes of problems has provided much material for the more theoretical journals. This book attempts to fill the gap by, in Part I, discussing the general principles of model building in mathematical programming. In Part II, 29 practical problems are presented to which mathematical programming can be applied. By simplifying the problems, much of the tedious institutional detail of case studies is avoided. It is hoped, however, that the essence of the problems is preserved and easily understood. Finally, in Parts III and IV, suggested formulations and solutions to the problems are given together with some computational experience.

Many books already exist on mathematical programming or, in particular, linear programming. Most such books adopt the conventional approach of paying a great deal of attention to algorithms. Since the algorithmic side has been so well and fully covered by other texts, it is given much less attention in this book. The concentration here is more on the building and interpreting of models rather than on the solution process. Nevertheless, it is hoped that this book may spur the reader to delve more deeply into the often challenging algorithmic side of the subject as well. It is, however, the author's contention that the practical problems and model building aspect should come first. This may then provide a motivation for finding out how to solve such models. Although desirable, knowledge of algorithms is no longer necessary if practical use is to be made of mathematical programming. The solution of practical models is now largely automated by the use of commercial package programs that are discussed in Chapter 2.

For the reader with some prior knowledge of mathematical programming, parts of this book may seem trivial and can be skipped or read quickly. Other parts are, however, rather more advanced and present fairly new material. This is particularly true of the chapters on integer programming. Indeed, this book can be treated in a nonsequential manner. There is much cross-referencing to enable the reader to pass from one relevant section to another. This book is aimed at three types of readers:

1. It is intended to provide students in universities and polytechnics with a solid foundation in the principles of model building as well as the more mathematical, algorithmic side of the subject, which is conventionally taught. For students who finally go on to use mathematical programming to solve real problems, the model building aspect is probably the more important. The problems in Part II provide practical exercises in problem formulation. By formulating models and solving them with the aid of a computer, students learn the art of formulation in the most satisfying way possible. They can compare their numerical solutions with those of other students obtained from differently build models. In this way they learn how to validate a model.

It is also hoped that these problems will be of use to research students seeking new algorithms for solving mathematical programming problems. Very often they have to rely on trivial or randomly generated models to test their computational procedures. Such models are far from typical of those found in the real world. Moreover, they are one (or more) steps removed from practical situations. They therefore obscure the need for efficient formulations as well as algorithms.

2. This book is also intended to provide managers with a fairly nontechnical appreciation of the scope and limitations of mathematical programming. In addition, by looking at the practical problems described in Part II they may recognize a situation in their own organization to which they had not realized mathematical programming could be applied.

3. Finally, constructing a mathematical model of an organization provides one of the best methods of understanding that organization. It is hoped that the general reader will be able to use the principles described in this book to build mathematical models and thereby learn about the functioning of systems, which purely verbal descriptions fail to explain. It has been the author's experience that the process of building a model of an organization can often be more beneficial even than the obtaining of a solution. A greater understanding of the complex interconnections between different facets of an organization is forced upon anybody who realistically attempts to model that organization.

Part I of this book describes the principles of building mathematical programming models and how they may arise in practice. In particular, linear programming, integer programming and separable programming models are described. A discussion of the practical aspects of solving such models and a very full discussion of the interpretation of their solutions is included.

Part II presents each of the 29 practical problems in sufficient detail to enable the reader to build a mathematical programming model using the numerical data. In some cases the origin of the problem is mentioned.

Part III discusses each problem in detail and presents a possible formulation as a mathematical programming model.

Part IV gives the optimal solutions obtained from the formulations presented in Part III. Some computational experience is also given in order to give the reader some feel of the computational difficulty of solving the particular type of model.

It is hoped that readers will attempt to formulate and possibly solve the problems for themselves before proceeding to Parts III and IV.

By presenting 29 problems from widely different contexts the power of the technique of mathematical programming in giving a method of tackling them all should be apparent. Some problems are intentionally "unusual" in the hope that they may suggest the application of mathematical programming in rather novel areas.

Many references are given at the end of the book. The list is not intended to provide a complete bibliography of the vast number of case studies published. Many excellent case studies have been ignored. The list should, however, provide a representative sample which can be used as a starting point for a deeper search into the literature.

Many people have both knowingly and unknowingly helped in the preparation of successive editions of this book with their suggestions and opinions. In particular I would like to thank Gautam Appa, Sheena and Robert Ashford, Martin Beale, Tony Brearley, Ian Buchanan, Colin Clayman, Lewis Corner, Martyn Jeffreys, Bob Jeroslow, Clifford Jones, Bernard Kemp, Ailsa Land, Adolfo Fonseca Manjarres, Kenneth McKinnon, Gautam Mitra, Heiner Müller-Merbach, Bjorn Nygreen, Pat Rivett, Richard Thomas, Steven Vajda and Will Watkins. I must express a great debt of gratitude to Robin Day of Edinburgh University, whose deep computing knowledge and programming ability helped me immensely in building the models for early editions of this book and implementing our design of the modelling system MAGIC. This has now been replaced by the system NEWMAGIC, which has been written by George Skondras. It is available with the optimizer EMSOL. All the models in the book have been built and solved in this system. The models, formulated in Part III, have been modelled in the NEWMAGIC language and are available on the website: `www.wiley.com/go/model_building_mathematical_programming`

Since the first edition was written, computational power has increased immensely. Solution times for the models are therefore, in most cases, of little relevance and are ignored. A few of the models are still difficult to solve. In these cases, some computational experience is given.

The fourth edition has been enhanced by new sections or models on Constraint Logic Programming, Data Envelopment Analysis, Hydro Electricity Generation, Milk Distribution and Yield Management in an airline together with state-of-the-art material and extra references on a number of topics and applications. I am very grateful to Kenneth McKinnon for advice and help with these models.

It is gratifying to know how well this book has been received together with the demand for a fourth edition.

Preface to the Fifth Edition

The fifth edition includes new sections on Stochastic Programming and Column Generation, a subsection on the Vehicle Routing problem and an enhanced section on Constraint Logic Programming. In addition, the subsection on modelling non-linear functions and constraints, by integer variables, has been improved with a more versatile formulation. Many other small clarifications and improvements have also been included.

Five new problems have been added: a Car Rental and Return problem and an extension, an Airport Lost Baggage Distribution problem and two problems in Molecular Biology.

New references have been added, although it is recognized that it is impossible to mention all the excellent papers that have been written since earlier editions of this book. I apologize to the many authors of these papers who have not been cited.

A number of people have given me advice on improving and correcting the fourth edition to produce this fifth edition. In particular, I would like to mention Harvey Greenberg, John Hooker, Cormac Lucas, Andrew McGee and Ken McKinnon. They have all helped validate the new material. Also, a number of correspondents have pointed out small typos. I am grateful to them all.

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Part I

1

Introduction

1.1 The concept of a model

Many applications of science make use of *models*. The term ‘model’ is usually used for a structure that has been built with the purpose of exhibiting features and characteristics of some other objects. Generally, only some of these features and characteristics will be retained in the model depending upon the use to which it is to be put. Sometimes, such models are *concrete*, as is a model aircraft used for wind tunnel experiments. More often, in operational research, we will be concerned with *abstract* models. These models will usually be mathematical in that algebraic symbolism will be used to mirror the internal relationships in the object (often an organization) being modelled. Our attention will mainly be confined to such mathematical models, although the term ‘model’ is sometimes used more widely to include purely descriptive models.

The essential feature of a mathematical model in operational research is that it involves a set of *mathematical relationships* (such as equations, inequalities and logical dependencies) that correspond to some more down-to-earth relationships in the real world (such as technological relationships, physical laws and marketing constraints).

There are a number of motives for building such models:

1. The actual exercise of building a model often reveals relationships that were not apparent to many people. As a result, a greater understanding is achieved of the object being modelled.
2. Having built a model it is usually possible to analyse it mathematically to help suggest courses of action that might not otherwise be apparent.
3. Experimentation is possible with a model, whereas it is often not possible or desirable to experiment with the object being modelled. It would

clearly be politically difficult, as well as undesirable, to experiment with unconventional economic measures in a country if there were a high probability of disastrous failure. The pursuit of such courageous experiments would be more (though not perhaps totally) acceptable on a mathematical model.

It is important to realize that a model is really defined by the relationships that it incorporates. These relationships are, to a large extent, independent of the *data* in the model. A model may be used on many different occasions with differing data, for example, costs, technological coefficients and resource availabilities. We would usually still think of it as the same model even though some coefficients have been changed. This distinction is not, of course, total. Radical changes in the data would usually be thought of as a change in the relationships and therefore the model.

Many models used in operational research (and other areas such as engineering and economics) take standard forms. The mathematical programming type of model that we consider in this book is probably the most commonly used standard type of model. Other examples of some commonly used mathematical models are *simulation models, network planning models, econometric models* and *time series models*. There are many other types of model, all of which arise sufficiently often in practice to make them areas worthy of study in their own right. It should be emphasized, however, that any such list of standard types of model is unlikely to be exhaustive or exclusive. There are always practical situations that cannot be modelled in a standard way. The building, analysing and experimenting with such new types of model may still be a valuable activity. Often, practical problems can be modelled in more than one standard way (as well as in non-standard ways). It has long been realized by operational research workers that the comparison and contrasting of results from different types of model can be extremely valuable.

Many misconceptions exist about the value of mathematical models, particularly when used for planning purposes. At one extreme, there are people who deny that models have any value at all when put to such purposes. Their criticisms are often based on the impossibility of satisfactorily quantifying much of the required data, for example, attaching a cost or utility to a social value. A less severe criticism surrounds the lack of precision of much of the data that may go into a mathematical model; for example, if there is doubt surrounding 100 000 of the coefficients in a model, how can we have any confidence in an answer it produces? The first of these criticisms is a difficult one to counter and has been tackled at much greater length by many defenders of cost–benefit analysis. It seems undeniable, however, that many decisions concerning unquantifiable concepts, however, they are made, involve an implicit quantification that cannot be avoided. Making such a quantification explicit by incorporating it in a mathematical model seems more honest as well as scientific. The second criticism concerning accuracy of the data should be considered in relation to each specific model. Although many coefficients in a model may be inaccurate, it is still possible that the structure of the model results in little inaccuracy in the solution. This subject is mentioned in depth in Sections 4.2 and 6.3.

At the opposite extreme to the people who utter the above criticisms are those who place an almost metaphysical faith in a mathematical model for decision-making (particularly if it involves using a computer). The quality of the answers that a model produces obviously depends on the accuracy of the structure and data of the model. For mathematical programming models, the definition of the objective clearly affects the answer as well. Uncritical faith in a model is obviously unwarranted and dangerous. Such an attitude results from a total misconception of how a model should be used. To accept the first answer produced by a mathematical model without further analysis and questioning should be very rare. A model should be used as one of a number of tools for decision-making. The answer that a model produces should be subjected to close scrutiny. If it represents an unacceptable operating plan, then the reasons for unacceptability should be spelled out and if possible incorporated in a modified model. Should the answer be acceptable, it might be wise only to regard it as an *option*. The specification of another objective function (in the case of a mathematical programming model) might result in a different option. By successive questioning of the answers and altering the model (or its objective), it should be possible to clarify the options available and obtain a greater understanding of what is possible.

1.2 Mathematical programming models

It should be pointed out immediately that *mathematical programming* is very different from *computer programming*. Mathematical programming is 'programming' in the sense of 'planning'. As such, it need have nothing to do with computers. The confusion over the use of the word 'programming' is widespread and unfortunate. Inevitably, mathematical programming becomes involved with computing as practical problems almost always involve large quantities of data and arithmetic that can only reasonably be tackled by the calculating power of a computer. The correct relationship between computers and mathematical programming should, however, be understood.

The common feature that mathematical programming models have is that they all involve *optimization*. We wish to *maximize* something or *minimize* something. The quantity that we wish to maximize or minimize is known as an *objective function*. Unfortunately, the realization that mathematical programming is concerned with optimizing an objective often leads people summarily to dismiss mathematical programming as being inapplicable in practical situations where there is no clear objective or there are a multiplicity of objectives. Such an attitude is often unwarranted because, as we shall see in Chapter 3, there is often value in optimizing some aspect of a model when in real life there is no clear-cut single objective.

In this book, we confine our attention to some special types of mathematical programming model. These can most easily be classified as *linear programming*

(*LP*) *models*, *non-linear programming* (*NLP*) *models* and *integer programming* (*IP*) *models*. We begin by describing what an LP model is by means of two small examples.

Example 1.1: A Linear Programming (LP) Model (Product Mix)

An engineering factory can produce five types of product (PROD 1, PROD 2, ..., PROD 5) by using two production processes: grinding and drilling.

After deducting raw material costs, each unit of each product yields the following contributions to profit:

PROD 1	PROD 2	PROD 3	PROD 4	PROD 5
£550	£600	£350	£400	£200

Each unit requires a certain time on each process. These are given below (in hours). A dash indicates when a process is not needed.

	PROD 1	PROD 2	PROD 3	PROD 4	PROD 5
Grinding	12	20	–	25	15
Drilling	10	8	16	–	–

In addition, the final assembly of each unit of each product uses 20 hours of an employee's time.

The factory has three grinding machines and two drilling machines and works a six-day week with two shifts of 8 hours on each day. Eight workers are employed in assembly, each working one shift a day.

The problem is to find how much of each product is to be manufactured so as to maximize the total profit contribution.

This is a very simple example of the so-called 'product mix' application of LP.

In order to create a *mathematical model*, we introduce variables $x_1, x_2, \ldots, x_5$ representing the numbers of PROD 1, PROD 2, ..., PROD 5 that should be produced in a week. As each unit of PROD 1 yields £550 contribution to profit and each unit of PROD 2 yields £600 contribution to profit, etc., our total profit contribution will be represented by the expression:

$$550x_1 + 600x_2 + 350x_3 + 400x_4 + 200x_5. \tag{1.1}$$

The *objective* of the factory is to choose $x_1, x_2, \ldots, x_5$ so as to make the value of this expression as high as possible, that is, Expression (1.1) is the *objective function* that we wish to *maximize* (in this case).

Clearly, our processing and labour capacities, to some extent, limit the values that the x_j can take. Given that we have only three grinding machines working for a total of 96 hours a week each, we have 288 hours of grinding capacity available. Each unit of PROD 1 uses 12 hours grinding. x_1 units will therefore use $12x_1$ hours. Similarly, x_2 units of PROD 2 will use $20x_2$ hours. The total amount of grinding capacity that we use in a week is given by the expression on the left-hand side of Inequality (1.2):

$$12x_1 + 20x_2 + 25x_4 + 15x_5 \leq 288. \tag{1.2}$$

Inequality (1.2) is a mathematical way of saying that we cannot use up more than the 288 hours of grinding available per week. Inequality (1.2) is known as a *constraint*. It restricts (or constrains) the possible values that the variables x_j can take.

The drilling capacity is 192 hours a week. This gives rise to the following constraint:

$$10x_1 + 8x_2 + 16x_3 \leq 192. \tag{1.3}$$

Finally, the fact that we have only a total of eight assembly workers each working 48 hours a week gives us a labour capacity of 384 hours. As each unit of each product uses 20 hours of this capacity, we have the constraint

$$20x_1 + 20x_2 + 20x_3 + 20x_4 + 20x_5 \leq 384. \tag{1.4}$$

We have now expressed our original practical problem as a mathematical model. The particular form that this model takes is that of an *LP model*. This model is now a well-defined mathematical problem. We wish to find values for the variables $x_1, x_2, \ldots, x_5$ that make expression (1.1) (the objective function) as large as possible but still satisfy constraints (1.2)–(1.4). You should be aware of why the term 'linear' is applied to this particular type of problem. Expression (1.1) and the left-hand sides of constraints (1.2)–(1.4) are all linear. Nowhere do we get terms like x_1^2, x_1x_2 or $\log x$ appearing.

There are a number of implicit assumptions in this model that we should be aware of. Firstly, we must obviously assume that the variables $x_1, x_2, \ldots, x_5$ are not allowed to be negative, that is, we do not make negative quantities of any product. We might explicitly state these conditions by the extra constraints

$$x_1, x_2, \ldots, x_5 \geq 0. \tag{1.5}$$

In most LP models the non-negativity constraints (1.5) are implicitly assumed to apply unless we state otherwise. Secondly, we have assumed that the variables $x_1, x_2, \ldots, x_5$ can take fractional values, for example, it is meaningful to make 2.36 units of PROD 1. This assumption may or may not be entirely warranted. If, for example, PROD 1 represented gallons of beer, fractional quantities would be acceptable. On the other hand, if it represented numbers of motor cars, it would not be meaningful. In practice, the assumption that the variables can be fractional is perfectly acceptable in this type of model, if the errors involved in rounding to the nearest integer are not great. If this is not the case, we have to resort to *IP*.

The model above illustrates some of the essential features of an LP model:

1. There is a single linear expression (the *objective function*) to be maximized or minimized.
2. There is a series of *constraints* in the form of linear expressions, which must not exceed ($\leq$) some specified value. LP constraints can also be of the form '$\geq$' and '$=$', indicating that the value of certain linear expressions must not fall below a specified value or must exactly equal a specified value.
3. The set of coefficients 288, 192, 384, on the right-hand sides of constraints (1.2)–(1.4), is generally known as the *right-hand side* column.

Practical models will, of course, be much bigger (more variables and constraints) and more complicated but they must always have the above three essential features. The optimal solution to the above model is included in Section 6.2.

In order to give a wider picture of how LP models can arise, we give a second small example of a practical problem.

Example 1.2: A Linear Programming Model (Blending)

A food is manufactured by refining raw oils and blending them together. The raw oils come in two categories:

Vegetable oils	VEG 1
	VEG 2
Non-vegetable oils	OIL 1
	OIL 2
	OIL 3

Vegetable oils and non-vegetable oils require different production lines for refining. In any month, it is not possible to refine more than 200 tons of vegetable oil and more than 250 tons of non-vegetable oils. There is no loss of weight in the refining process and the cost of refining may be ignored.

There is a technological restriction of hardness in the final product. In the units in which hardness is measured, this must lie between 3 and 6. It is assumed that hardness blends linearly. The costs (per ton) and hardness of the raw oils are

	VEG 1	VEG 2	OIL 1	OIL 2	OIL 3
Cost	£110	£120	£130	£110	£115
Hardness	8.8	6.1	2.0	4.2	5.0

The final product sells for £150 per ton.

How should the food manufacturer make their product in order to maximize their net profit?

This is another very common type of application of LP although, of course, practical problems will be, generally, much bigger.

Variables are introduced to represent the unknown quantities. $x_1, x_2, \ldots, x_5$ represent the quantities (tons) of VEG 1, VEG 2, OIL 1, OIL 2 and OIL 3 that should be bought, refined and blended in a month. y represents the quantity of the product that should be made. Our objective is to maximize the net profit:

$$-110x_1 - 120x_2 - 130x_3 - 110x_4 - 115x_5 + 150y. \tag{1.6}$$

The refining capacities give the following two constraints:

$$x_1 + x_2 \leq 200, \tag{1.7}$$

$$x_3 + x_4 + x_5 \leq 250. \tag{1.8}$$

The hardness limitations on the final product are imposed by the following two constraints:

$$8.8x_1 + 6.1x_2 + 2x_3 + 4.2x_4 + 5x_5 - 6y \leq 0, \tag{1.9}$$

$$8.8x_1 + 6.1x_2 + 2x_3 + 4.2x_4 + 5x_5 - 3y \geq 0. \tag{1.10}$$

Finally it is necessary to make sure that the weight of the final product is equal to the weight of the ingredients. This is done by a continuity constraint:

$$x_1 + x_2 + x_3 + x_4 + x_5 - y = 0. \tag{1.11}$$

The objective function (1.6) (to be maximized) together with constraints (1.7)–(1.11) make up our LP model.

The linearity assumption of LP is not always warranted in a practical problem, although it makes any model computationally much easier to solve. When we have to incorporate non-linear terms in a model (either in the objective function or the constraints) we obtain an *non-linear programming (NLP) model*. In Chapter 7, we will see how such models may arise and a method of modelling a wide class of such problems using *separable programming*. Nevertheless, such models are usually far more difficult to solve.

Finally, the assumption that variables can be allowed to take fractional values is not always warranted. When we insist that some or all of the variables in an LP model must take integer (whole number) values we obtain an *integer programming (IP) model*. Such models are again much more difficult to solve than conventional LP models. We will see in Chapters 8–10 that IP opens up the possibility of modelling a surprisingly wide range of practical problems.

Another type of model that we discuss in Section 4.2 is known as a *stochastic programming model*. This arises when some of the data are uncertain but can be specified by a probability distribution. Although data in many LP models may be uncertain, their representation by *expected* values alone may not be sufficient. Situations in which a more explicit recognition of the probabilistic nature of data may be made, but the resultant model still converted to a linear program, are described. In Chapter 3, we mention *chance-constrained* models and in Chapter 4, *multi-staged models with recourse*, both of which fall in the category of stochastic programming. An example of the use of this latter type of model is given in Sections 12.24, 13.24 and 14.24 where it is applied to determining the price of airline tickets over successive periods in the face of uncertain demand. A good reference to stochastic programming is Kall and Wallace (1994).

2

Solving mathematical programming models

2.1 Algorithms and packages

A set of mathematical rules for solving a particular class of problem or model is known as an *algorithm*. We are interested in algorithms for solving linear programming (LP), separable programming and integer programming IP models. An algorithm can be programmed into a set of computer routines for solving the corresponding type of model assuming the model is presented to the computer in a specified format. For algorithms that are used frequently, it turns out to be worth writing very sophisticated and efficient computer programmes for use with many different models. Such programmes usually consist of a number of algorithms collected together as a '*package*' of computer routines. Many such package programmes are available commercially for solving mathematical programming models. They usually contain algorithms for solving LP models, separable programming models and IP models. These packages are written by computer manufacturers, consultancy firms and software houses. They are frequently very sophisticated and represent many person-years of programming effort. When a mathematical programming model is built, it is usually worth making use of an existing package to solve it rather than getting diverted onto the task of programming the computer to solve the model oneself.

The algorithms that are almost invariably used in commercially available packages are (i) the revised simplex algorithm for LP models; (ii) the separable extension of the revised simplex algorithm for separable programming models; and (iii) the branch and bound algorithm for IP models.

It is beyond the scope of this book to describe these algorithms in detail. Algorithms (i) and (ii) are well described in Beale (1968). Algorithm (iii) is outlined in Section 8.3 and well described in Nemhauser and Wolsey (1988). Although the above three algorithms are not the only methods of solving the corresponding models, they have proved to be the most efficient general methods. It should also be emphasized that the algorithms are not totally independent. Hence, the desirability of incorporating them in the same package. Algorithm (ii) is simply a modification of (i) and would use the same computer programme that would make the necessary changes in execution on recognizing a separable model. Algorithm (iii) uses (i) as its first phase and then performs a tree search procedure as described in Section 8.3.

One of the advantages of package programmes is that they are generally very flexible to use. They contain many procedures and options that may be used or ignored as the user thinks fit. We outline some of the extra facilities which most packages offer besides the three basic algorithms mentioned above.

2.1.1 Reduction

Some packages have a procedure for detecting and removing redundancies in a model and so reducing its size and hence time to solve. Such procedures usually go under the name REDUCE, PRESOLVE or ANALYSE. This topic is discussed further in Section 3.4.

2.1.2 Starting solutions

Most packages enable a user to specify a starting solution for a model if he/she wishes. If this starting solution is reasonably close to the optimal solution, the time to solve the model can be reduced considerably.

2.1.3 Simple bounding constraints

A particularly simple type of constraint that often occurs in a model is of the form

$$x \leq U,$$

where U is a constant. For example, if x represented a quantity of a product to be made, U might represent a marketing limitation. Instead of expressing such a constraint as a conventional constraint row in a model, it is more efficient simply to regard the variable x as having an upper bound of U. The revised simplex algorithm has been modified to cope with such a bound algorithmically (the *bounded variable* version of the revised simplex). Lower bound constraints such as

$$x \geq L$$

need not be specified as conventional constraint rows either but may be dealt with analogously. Most computer packages can deal with bounds on variables in this way.

2.1.4 Ranged constraints

It is sometimes necessary to place upper *and* lower bounds on the level of some activity represented by a linear expression. This could be done by *two* constraints such as

$$\sum_j a_j x_j \leq b_1 \quad \text{and} \quad \sum_j a_j x_j \geq b_2.$$

A more compact and convenient way to do this is to specify only the first constraint above together with a *range* of $b_1 - b_2$ on the constraint. The effect of a range is to limit the *slack* variable (which will be introduced into the constraint by the package) to have an upper bound of $b_1 - b_2$, so implying the second of the above constraints. Most commercial packages have the facility for defining such ranges (not to be confused with ranging in sensitivity analysis, discussed below) on constraints.

2.1.5 Generalized upper bounding constraints

Constraints representing a bound on a sum of variables such as

$$x_1 + x_2 + \cdots + x_n \leq M$$

are very common in many LP models. Such a constraint is sometimes referred to by saying that there is a generalized upper bound (GUB) of M on the set of variables $(x_1, x_2, \ldots, x_n)$. If a considerable proportion of the constraints in a model are of this form and each such set of variables is exclusive of variables in any other set, then it is efficient to use the so-called GUB extension of the revised simplex algorithm. When this is used, it is not necessary to specify these constraints as rows of the model but treat them in a slightly analogous way to simple bounds on single variables. The use of this extension to the algorithm usually makes the solution of a model far quicker.

2.1.6 Sensitivity analysis

When the optimal solution of a model is obtained, there is often interest in investigating the effects of changes in the objective and right-hand side coefficients (and sometimes other coefficients) on this solution. *Ranging* is the name of a method of finding limits (ranges) within which one of these coefficients can be changed to have a predicted effect on the solution. Such information is very valuable in performing a sensitivity analysis on a model. This topic is discussed at length in Section 6.3 for LP models. Almost all commercial packages have a range facility.

2.2 Practical considerations

In order to demonstrate how a model is presented to a computer package and the form in which the solution is presented, we will consider the second example given in Section 1.2. This blending problem is obviously much smaller than most realistic models but serves to show the form in which a model might be presented.

This problem was converted into a model involving five constraints and six variables. It is convenient to name the variables VEG 1, VEG 2, OIL 1, OIL 2, OIL 3 and PROD. The objective is conveniently named PROF (profit) and the constraints VVEG (vegetable refining), NVEG (non-vegetable refining), UHAR (upper hardness), LHAR (lower hardness) and CONT (continuity). The data are conveniently drawn up in the matrix presented in Table 2.1. It will be seen that the right-hand side coefficients are regarded as a column and named CAP (capacity). Blank cells indicate a zero coefficient.

Table 2.1

	VEG 1	VEG 2	OIL 1	OIL 2	OIL 3	PROD		CAP
PROF	−110	−120	−130	−110	−115	150		
VVEG	1	1					$\leq$	200
NVEG			1	1	1		$\leq$	250
UHAR	8.8	6.1	2.0	4.2	5.0	−6.0	$\leq$	
LHAR	8.8	6.1	2.0	4.2	5.0	−3.0	$\geq$	
CONT	1.0	1.0	1.0	1.0	1.0	−1.0	$=$	

The information in Table 2.1 would generally be presented to the computer through a modelling language. There is, however, a standard format for presenting such information to most computer packages, and almost all modelling languages could convert a model to this format. This is known as MPS (mathematical programming system) format. Other format designs exist, but MPS format is the most universal. The presented data would be as in Table 2.1.

These data are divided into three main sections: the ROWS section, the COLUMNS section and the RHS section. After naming the problem BLEND, the ROWS section consists of a listing of the rows in the model together with a designator N, L, G or E. N stands for a *non-constraint row* – clearly the objective row must not be a constraint; L stands for a *less-than-or-equal* ($\leq$) constraint; G stands for a *greater-than-or-equal* ($\geq$) constraint; E stands for an *equality* ($=$) constraint. The COLUMNS section contains the body of the matrix coefficients. These are scanned column by column with up to two non-zero coefficients in a statement (zero coefficients are ignored). Each statement contains the column name, row names and corresponding matrix coefficients. Finally, the RHS section is regarded as a column using the same format as the COLUMNS section. The ENDATA entry indicates the end of the data.

Clearly, it may sometimes be necessary to put in other data as well (such as bounds). The format for such data can always be found from the appropriate manual for a package.

With large models, the solution procedures used will probably be more complicated than the standard 'default' methods of the package being used. There are many refinements to the basic algorithms which the user can exploit if he or she thinks it desirable. It should be emphasized that there are few hard-and-fast rules concerning when these modifications should be used. A mathematical programming package should not be regarded as a 'black box' to be used in the same way with every model on all computers. Experience with solving the same model again and again on a particular computer with small modifications to the data should enable the user to understand what algorithmic refinements prove efficient or inefficient with the model and computer installation. Experimentation with different strategies and even different packages is always desirable if a model is to be used frequently.

```
NAME          BLEND
ROWS
N PROF
L VVEG
L NVEG
L UHRD
G LHRD
E CONT
COLUMNS
       VEG         01      PROF    −110.000000    VVEG      1.000000
       VEG         01      UHRD       8.800000    LHRD      8.800000
       VEG         01      CONT       1.000000
       VEG         02      PROF    −120.000000    VVEG      1.000000
       VEG         02      UHRD       6.100000    LHRD      6.100000
       VEG         02      CONT       1.000000
       OIL         01      PROF    −130.000000    NVEG      1.000000
       OIL         01      UHRD       2.000000    LHRD      2.000000
       OIL         01      CONT       1.000000
       OIL         02      PROF    −110.000000    NVEG      1.000000
       OIL         02      UHRD       4.200000    LHRD      4.200000
       OIL         02      CONT       1.000000
       OIL         03      PROF    −115.000000    NVEG      1.000000
       OIL         03      UHRD       5.000000    LHRD      5.000000
       OIL         03      CONT       1.000000
      PROD                 PROF     150.000000    UHRD     −6.000000
      PROD                 LHRD      −3.000000    CONT     −1.000000
RHS
    RHS00001               VVEG     200.000000    NVEG    250.000000
ENDATA
```

One computational consideration that is very important with large models is mentioned briefly. This concerns starting the solution procedure at an intermediate stage. There are two main reasons why one might wish to do this. First, one might be resolving a model with slightly modified data. Clearly, it would be desirable to exploit one's knowledge of a previous optimal solution to save computation in obtaining the new optimal solution. With linear and separable programming models, it is usually fairly easy to do this with a package, although it is much more difficult for IP models. Most packages have the facility to SAVE a solution on a file. Through the control programme, it is usually possible to RESTORE (or REINSTATE) such a solution as the starting point for a new run. A second reason for wishing to SAVE and RESTORE solutions is that one may wish to terminate a run prematurely. Possibly, the run may be taking a long time and a more urgent job has to go on the computer. Alternatively, the calculations may be running into numerical difficulty and have to be abandoned. In order not to waste the (sometimes considerable) computer time already expended, the intermediate (non-optimal) solution obtained just before termination can be saved and used as a starting point for a subsequent run. It is common to save intermediate solutions at regular intervals in the course of a run. In this way, the last such solution before termination is always available.

2.3 Decision support and expert systems

Some mathematical programming algorithms are incorporated in computer software designed for specific applications. Such systems are sometimes referred to as *decision support systems*. Often, they are incorporated into *management information systems*. They usually perform a large number of other functions as well as, possibly, solving a model. These functions probably include accessing *databases* and interacting with managers or decision makers in a 'user friendly' manner. If such systems do incorporate mathematical programming algorithms then many of the modelling and algorithmic aspects are removed from the user who can concentrate on their *specific* application. In *designing and writing* such systems, however, it is obviously necessary to automate many of the model building and interpreting procedures discussed in this book.

As decision support systems become more sophisticated, they are likely to present users with possible *choices* of decision besides the simpler tasks of storing, structuring and presenting data. Such choices may well necessitate the use of mathematical programming, although this function will be hidden from the user.

Another related concept in computer applications software is that of *expert systems*. These systems also pay great attention to the *user interface*. They are, for example, sometimes designed to accept 'informal problem definitions' and by interacting with users, to help them build up a more precise definition of the problem, possibly in the form of a model. This information is combined with 'expert' information built up in the past in order to help decision making. The computational procedures used often involve mathematical programming concepts (e.g. tree searches in IP as described in Section 8.3). While expert systems are beyond the scope of this book, the design and writing of such systems should again depend on mathematical programming and modelling concepts.

The use of mathematical programming in artificial intelligence and expert systems, in particular, is described by Jeroslow (1985) and Williams (1987).

2.4 Constraint programming (CP)

This is a different approach to solving many problems that can also be solved by IP. It is also, sometimes, known as *constraint satisfaction* or *finite domain programming*. While the methods used can be regarded as less sophisticated, in a conventional mathematical sense, they use more sophisticated computer science techniques. They have representational advantages that can make problems easier to model and, sometimes, easier to solve in view of the more concise representation. Therefore, we discuss the subject here. It seems likely that *hybrid* systems will eventually emerge in which the modelling capabilities of constraint programming (CP) are used in modelling languages for systems that use either CP or IP for solving the models. The integration of the two approaches is discussed by Hooker (2011) who has helped to design such a system SIMPL (Yunes *et al*. (2010)).

In CP, each *variable* has a finite domain of possible values. *Constraints* connect (or restrict) the possible combinations of values that the variables can take. These constraints are of a richer variety than the linear constraints of IP (although these are also included) and are usually expressed in the form of *predicates*, which must be *true* or *false*. They are also known as *global* (or *meta*) *constraints* since they are applied (if used) to the model as a whole and incorporated in the CP system used. That is in contrast to '*local*' or '*procedural*' constraints which can be constructed by the user to aid the solution process. Conventional IP models use only *declarative* constraints, so making a clear distinction between the statement of the model and the solution procedure. For illustration, we give some of the more common global constraints that are used in CP systems (often with different names).

all_different $(x_1, x_2, \ldots, x_n)$ means that all the variables in the predicate must take *distinct* values. While this condition can be modelled by a conventional IP formulation, it is more cumbersome. Once one of the variables has been set to one of the values in its domain (either temporarily or permanently), this predicate implies that the value must be taken out of the domain of the other variables (a process known as *constraint propagation*). In this way, constraint propagation is used progressively to restrict the domain of the variables until a feasible set of values can be found for each variable, or it can be shown that none exists. This has similarities to REDUCE, PRESOLVE and ANALYSE discussed in Section 3.4 when applied to bound reduction. CP, however, allows '*domain reduction*' as well as '*bound reduction*'.

Other useful predicates for modelling are

The $\neq$ predicate stipulated by a constraint of the form $\sum_j a_j x_j \neq b$.

The *cardinality* predicate written in a form such as $card_m(x_1, x_2, \ldots, x_n|v)$ meaning that exactly m of the variables $x_1, x_2, \ldots, x_n$ must take the value v.

The *cumulative* $((t_1, t_2, \ldots, t_n), (D_1, D_2, \ldots, D_n), (C_1, C_2, \ldots, C_n), C)$ predicate restricts n jobs to start at times $t_1, t_2, \ldots, t_n$ when they have durations $D_1, D_2, \ldots, D_n$ consume a resource in amounts $C_1, C_2, \ldots, C_n$ and cannot be using a total of more than C units of the resource at any time.

The *circuit* $(x_1, x_2, \ldots, x_n)$ predicate stipulates that the numbers x_1, $x_2, \ldots, x_n$ are a permutation of $1, 2, \ldots, n$ where x_i is the number, in the permutation, occurring after i. Whatsmore, the permutation must be such that it represents a complete circuit, that is, there are no sub-circuits (see Chapter 9 in relation to the travelling salesman problem).

The *element* $(j, (x_1, x_2, \ldots, x_n), z)$ predicate sets z equal to the jth element in list of variables $(x_1, x_2, \ldots, x_n)$.

The *lex − greater* predicate can be written in the form $(x_1, x_2, \ldots, x_n) >_{lex} (y_1, y_2, \ldots, y_n)$. It is true if $x_1 = y_1, x_2 = y_2, \ldots, x_r = y_r, x_{r+1} > y_{r+1}$.

While all the above predicates can be modelled (often in more than one way) using conventional IP variables and constraints, this may be cumbersome and difficult.

There are many more global constraints/predicates available with most commercial CP systems described in the associated manual.

There are a number of situations where CP is more flexible than IP.

For example, it is common, when building an IP model (Chapter 9), to wish to use '*assignment*' variables such as $x_{ij} = 1$ if and only if i is assigned to j. In CP, this would be done by a function $f(i) = j$.

Another common condition is to wish to specify the '*inverse*' function, that is, given j, find i such that $f(i) = j$. This can be done by a version of the *element* predicate, for example, *element* $(j, f(i), i)$, which specifies whether the inverse of j is i.

The problems to which CP is applied often exhibit a large degree of *symmetry*. This may be among the constraints (predicates), solutions or both. For example, $i_1, i_2, \ldots, i_n$ may be identical entities that have to be assigned to entities $j_1, j_2, \ldots, j_n$. Rather than enumerating all the equivalent solutions, computation can be radically reduced by, for example, using *lex − greater* to give a priority order to the solutions. This problem also arises with conventional IP models, and improved modelling can also be used there (Chapter 10).

CP is mainly useful where one is trying to find a solution from an often astronomic, number of possibilities. This often happens with *combinatorial* problems (Chapter 8) where one is trying to find a 'needle in a haystack', that is, a solution to a complicated set of conditions where few or none exist. It is less useful where one is seeking an *optimal* solution to a problem with many feasible solutions, for example, the travelling salesman problem (Chapter 9) or if a problem has no objective function, or all that is sought is a feasible solution. It usually lacks the ability to *prove* optimality (apart from by imposing successively weaker constraints on the objective until feasibility is obtained). Conventional IP uses the concept of a (usually LP) *relaxation* (Section 8.3) to restrict the tree search for an optimal solution. This is done since the relaxation gives a *bound* on the optimal objective value (an *upper bound* for a maximization or a *lower bound* for a minimization). This gives it great strength. However, CP uses more sophisticated branching strategies than IP. Such algorithmic considerations are beyond the scope of this book.

There is considerable interest in using predicates, such as those used in CP, for modelling in IP (and then converting to a conventional IP formulation). An early paper by McKinnon and Williams (1989) shows how all the constraints of any IP model can be formulated using the nested $at\_least_m\ (x_1, x_2, \ldots, x_n|v)$ (meaning 'at least m of $x_1, x_2, \ldots, x_n$ must take the value v') predicate. Comparisons and connections between IP and CP are discussed by Barth (1995), Bockmayr and Kasper (1998), Brailsford *et al*. (1996), Proll and Smith (1998), Darby-Dowman and Little (1998), Hooker (1998) and Wilson and Williams (1998). Different ways of formulating the *all_different* predicate using IP is discussed by Williams and Yan (2001).

3

Building linear programming models

3.1 The importance of linearity

It was pointed out in Section 1.2 that a linear programming model demands that the objective function and constraints involve *linear* expressions. Nowhere can we have terms such as x_1^3, e^{x_1} or x_1x_2 appearing. For many practical problems, this is a considerable limitation and rules out the use of linear programming. Non-linear expressions can, however, sometimes be converted into a suitable linear form. The reason why linear programming models are given so much attention in comparison with non-linear programming models is that they are much easier to solve. Care should also be taken, however, to make sure that a linear programming model is only fitted to situations where it represents a valid model or justified approximation. It is easy to be influenced by the comparative ease with which linear programming models can be solved compared with non-linear ones.

It is worth giving an indication of why linear programming models can be solved more easily than non-linear ones. In order to do this, we use a two-variable model, as it can be represented geometrically.

$$\begin{aligned} \text{Maximize} \quad & 3x_1 + 2x_2 \\ \text{subject to} \quad & x_1 + x_2 \le 4, \\ & 2x_1 + x_2 \le 5, \\ & -x_1 + 4x_2 \ge 2, \\ & x_1, x_2 \ge 0. \end{aligned}$$

The values of the variables x_1 and x_2 can be regarded as the coordinates of the points in Figure 3.1.

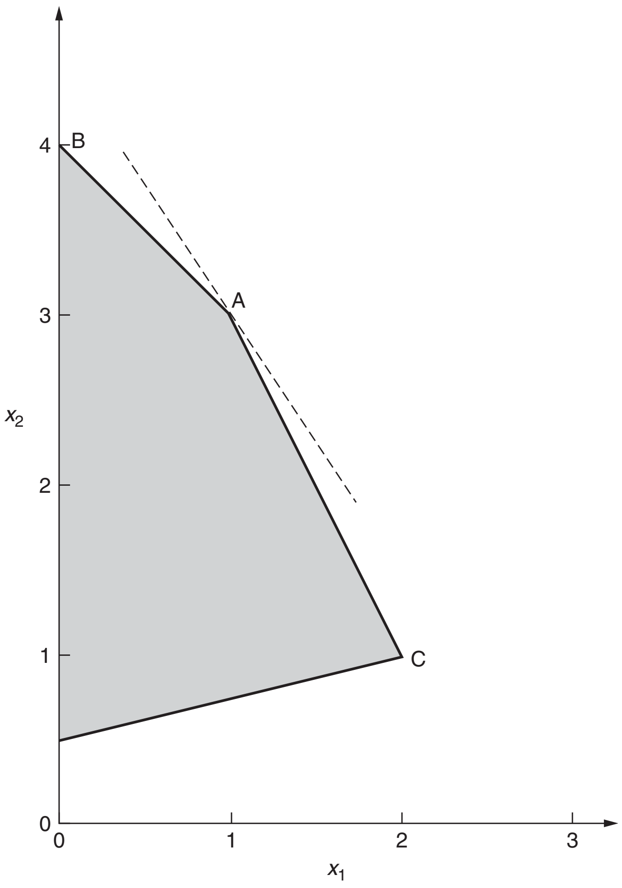


Figure 3.1

The optimal solution is represented by point A where $3x_1 + 2x_2$ has a value of 9. Any point on the broken line in Figure 3.1 will give the objective $3x_1 + 2x_2$ this value.

Other values of the objective correspond to a line parallel to this. It should be obvious geometrically that in any two-variable example, the optimal solution will always lie on the boundary of the feasible (shaded) region. Usually, it will occur at a vertex such as A. It is possible, however, that the objective lines might be parallel to one of the sides of the feasible region. For example, if the objective function in the above example were $4x_1 + 2x_2$, the objective lines would be parallel to AC in Figure 3.1. The point A would then still be an optimal solution but C would be also, and any point between A and C. We would have a case of *alternative solutions*. This topic is discussed at greater detail in Sections 6.2 and 6.3. Our focus here, however, is to show that the optimal solution (if there is one) always lies on the boundary of the feasible region. In fact, even if the case of alternative solutions does arise, *there will always be an optimal solution that lies at a vertex*. This last fact generalizes to problems with more variables (which would need more dimensions to be represented geometrically). It is this fact that makes *linear* programming models comparatively easy to solve. The simplex algorithm works by only examining vertex solutions (rather than the generally infinite set of feasible solutions).

It should be possible to appreciate that this simple property of linear programming models may not apply well to non-linear programming models. For models with a non-linear objective function, the objective lines in two dimensions would no longer be straight lines. If there were non-linearities in the constraints, the feasible region might not be bounded by straight lines either. In these circumstances, the optimal solution might well not lie at a vertex. It might even lie in the interior of the feasible region. Moreover, having found a solution, it may be rather difficult to be sure that it is optimal (so-called local optima may exist). All these considerations are described in Chapter 7. Our purpose here is simply to indicate the large extent to which linearity makes mathematical programming models easier to solve.

Finally, it should not be suggested that a linear programming model always represents an approximation to a non-linear situation. There are many practical situations where a linear programming model is a totally respectable representation.

3.2 Defining objectives

With a given set of constraints, different objectives will probably lead to different optimal solutions. Nevertheless, it should not automatically be assumed that this will always happen. It is possible that two different objectives can lead to the same operating pattern. As an extreme case of this, it can happen that the objective is irrelevant. The constraints of a problem may define a unique solution. For example, consider the following three constraints:

$$x_1 + x_2 \leq 2, \tag{3.1}$$

$$x_1 \geq 1, \tag{3.2}$$

$$x_2 \geq 1. \tag{3.3}$$

These force the solution $x_1 = x_2 = 1$ no matter what the objective is. Practical situations do arise where there is no freedom of action and only one feasible solution is possible. If the model for such a situation is at all complicated, this property may not be apparent. Should different objective functions always yield the same optimal solution the property may be suspected and should be investigated as a greater understanding of what is being modelled must surely result. In fact, such a discovery would result in there being no further need for linear programming.

We now assume, however, that we have a problem, where the definition of a suitable objective is of genuine importance. Possible objectives that might be suggested for optimization by an organization are as follows:

Maximize profit;
Minimize cost;
Maximize utility;
Maximize turnover;
Maximize return on investment;
Maximize net present value;
Maximize number of employees;
Minimize number of employees;
Minimize redundancy;
Maximize customer satisfaction;
Maximize probability of survival;
Maximize robustness of operating plan.

Many other objectives could be suggested. It could well be that there is no desire to optimize anything at all. Frequently, a number of objectives will apply simultaneously. Possibly, some of these objectives will conflict. It is, however, our contention that mathematical programming can be relevant in any of these situations, that is, in the case of optimizing *single objectives*, *multiple and conflicting objectives* or problems where there is *no optimization* of the objective.

3.2.1 Single objectives

Most practical mathematical programming models used in operational research involve either maximizing profit or minimizing cost. The 'profit' that is maximized would usually be more accurately referred to as *contribution to profit* or the cost as *variable cost*. In a cost minimization, the cost incorporated in an objective function would normally only be a *variable cost*. For example, suppose each unit of a product produced cost £C. It would only be valid to assume that x units would cost £Cx if C were a *marginal cost*, that is, the extra cost incurred for each extra unit produced. It would normally be incorrect to add in *fixed costs* such as administration or capital costs of equipment. An exception to this does arise with some integer programming models when we allow the model itself to decide whether or not a certain fixed cost should be incurred; for example, if we do not make anything we incur no fixed cost, but if we make anything at all we do incur such a cost. Normally, however, with standard linear programming models, we are interested only in variable costs. Indeed, a common mistake in building a model is to use average costs rather than marginal costs. Similarly, when profit coefficients are calculated, it is only normally correct to subtract variable costs from incomes. As a result of this, the term *profit contribution* might be more suitable.

In the common situation where a linear programming model is being used to allocate productive capacity in some sort of optimal manner, there is often a choice to be made over whether simply to minimize cost or maximize profit. Normally, a cost minimization involves allocating productive capacity to meet some specified known demand at minimum cost. Such models will probably contain constraints such as

$$\sum_j x_{ij} = D_i \text{ for all } i. \tag{3.4}$$

or something analogous, where x_{ij} is the quantity of product i produced by process j and D_i is the demand for product i.

Should such constraints be left out inadvertently, as sometimes happens, the minimum cost solution will often turn out to produce nothing! On the other hand, if a profit maximization model is built, it allows one to be more ambitious. Instead of specifying constant demands D_i, it is possible to allow the model to determine optimal production levels for each product. The quantities D_i would then become variables d_i representing these production levels. Constraints (3.4) would become

$$\sum_j x_{ij} - d_i = 0 \text{ for all } i. \tag{3.5}$$

In order that the model can determine optimal values for the variable d_i, they must be given suitable unit profit contribution coefficients P_i in the objective function. The model would then be able to weigh up the profits resulting from different production plans in comparison with the costs incurred and determine the optimal level of operations. Clearly, such a model is doing rather more than the simpler cost minimization model. In practice, it may be better to start with a cost minimization model and get it accepted as a planning tool before extending the model to be one of profit maximization.

In a profit maximization model, the unit profit contribution figure P_i may itself depend on the value of the variable d_i. The term $P_i d_i$ in the objective function would no longer be linear. If P_i could be expressed as a function of d_i, a non-linear model could be constructed. An example of this idea is described by McDonald *et al*. (1974) in a resource allocation model for the health service.

A complication may arise in defining a monetary objective when the model represents activities taking place over a period of time. Some method has to be found for valuing profits or costs in the future in comparison with the present. The most usual technique is to discount future money at some rate of interest. Objective coefficients corresponding to the future will then be suitably reduced. Models where this might be relevant are discussed in Section 4.1 and are known as *multiperiod* or *dynamic models*. A number of problems presented in Part II give rise to such models. A further complication is illustrated by the ECONOMIC PLANNING problem of Part II. Here, decisions have to be made regarding whether or not to forego profit now in order to invest in new plant so as to make larger profits in the future. The relative desirabilities of different growth patterns lead to alternative objective functions.

3.2.2 Multiple and conflicting objectives

A mathematical programming model involves a single objective function that is to be maximized or minimized. This does not, however, imply that problems with multiple objectives cannot be tackled. Various modelling techniques and solution strategies can be applied to such problems.

A first approach to a problem with multiple objectives is to solve the model a number of times with each objective in turn. The comparison of the different results may suggest a satisfactory solution to the problem or indicate further investigations. A fairly obvious example of two objectives, each of which can be optimized in turn, is given by the MANPOWER PLANNING problem of Part II. Here, either *cost* or *redundancy* can be minimized. The computational task of using different objectives in turn is eased if each solution is used as a starting solution to a run with a new objective as mentioned in Section 2.3.

Objectives and constraints can often be interchanged, for example, we may wish to pursue some desirable social objective so long as costs do not exceed a specified level, or alternatively, we may wish to minimize cost using the social consideration as a constraint on our operations. This interplay between objectives and constraints is a feature of many mathematical programming models, which is far too rarely recognized. Once a model has been built, it is extremely easy to convert an objective into a constraint or vice versa. The proper use for such a model is to solve it a number of times, making such changes. An examination and discussion of the resultant solutions should lead to an understanding of the operating options available in the practical situation, which has been modelled. We therefore have one method of coping with multiple objectives through treating all but one objective as a constraint. Experiments can then be carried out by varying the objective to be optimized and the right-hand side values of the objective/constraints.

Another way of tackling multiple objectives is to take a suitable linear combination of all the objective functions and optimize that. It is clearly necessary to attach relative weightings or utilities to the different objectives. What these weightings should be may often be a matter of personal judgment. Again, it will probably be necessary to experiment with different such composite objectives in order to ‘map out’ a series of possible solutions. Such solutions can then be presented to the decision makers as policy options. Most commercial packages allow the user to define and vary the weightings of composite objective functions very easily. It should not be thought that this approach to multiple objectives is completely distinct from the approach of treating all but one of the objectives as a constraint. When a linear programming model is solved, each constraint has associated with it a ‘value’ known as a *shadow price*. These values have considerable economic importance and are discussed at length in Section 6.2.

If these shadow prices were to be used as the weightings to be given to the objectives/constraints in the composite approach that we have described in this section, then the same optimal solution could be obtained.

When objectives are in conflict, as multiple objectives frequently will be to some extent, any of the above approaches can be adopted. Care must be taken, however, when objectives are replaced by constraints not to model conflicting constraints as such. The resultant model would clearly be infeasible. Conflicting constraints necessitate a relaxation in some or all of these constraints. Constraints become *goals*, which may, or may not, be achieved. The degree of over, or under, achievement is represented in the objective function. This way of allowing the model itself to determine how to allow a certain degree of relaxation is described in Section 3.3 and is sometimes known as *goal programming*.

There is no one obvious way of dealing with multiple objectives through mathematical programming. Some or all of the above approaches should be used in particular circumstances. Given a situation with multiple objectives in which there are no clearly defined weightings for the objectives no cut-and-dried approach can ever be possible. Rather than being a cause for regret, this is a healthy situation. It would be desirable if alternative approaches were adopted more often in the case of single objective models rather than a once only solution being obtained and implemented.

3.2.3 Minimax objectives

The following type of objective arises in some situations:

$$\begin{array}{ll} \text{Minimize} & \left(\underset{i}{\text{Maximum}} \sum_j a_{ij} x_j\right) \\ \text{subject to} & \text{conventional linear constraints.} \end{array}$$

This can be converted into a conventional linear programming form by introducing a variable z to represent the above objective. In addition to the original constraints, we express the transformed model as

$$\begin{array}{ll} \text{Minimize} & z \\ \text{subject to} & \sum_j a_{ij} x_j - z \le 0 \text{ for all } i. \end{array}$$

The new constraints guarantee that z will be *greater than or equal* to each of $\sum_j a_{ij} x_j$ for all i. By minimizing z, it will be driven down to the maximum of these expressions.

A special example of this type of formulation arises in goal programming and is discussed in Section 3.3.

It also arises in the formulation of zero-sum games as linear programmes.

Of course, a *maximin* objective can easily be dealt with similarly. It should, however, be pointed out that a *maximax* (or *minimin*) objective cannot be dealt with by linear programming and requires integer programming. This is discussed in Section 9.4.

3.2.4 Ratio objectives

In some applications, the following non-linear objective arises:

$$\begin{array}{ll}\text{Maximize} & \\ \text{(or Minimize)} & \end{array} \quad \frac{\sum_j a_j x_j}{\sum_j b_j x_j}.$$

Rather surprisingly, the resultant model can be converted into a linear programming form by the following transformations.

(i) Replace the expression $\frac{1}{\sum_j b_j x_j}$ by a variable t.

(ii) Represent the products $x_j t$ by variables w_j. The objective now becomes

$$\text{Maximize} \quad \sum_j a_j w_j.$$

(iii) Introduce a constraint

$$\sum_j b_j w_j = 1$$

in order to satisfy condition (i).
Convert the original constraints of the form

$$\sum_j d_j x_j \overset{\le}{\underset{\ge}{=}} e$$

to

$$\sum_j d_j w_j - et \overset{\le}{\underset{\ge}{=}} 0.$$

It must be pointed out that this transformation is only valid if the denominator $\sum_j b_j x_j$ is always of the same sign and non-zero. If necessary (and it is valid), an extra constraint must be introduced to ensure this. If $\sum_j b_j x_j$ always be negative the directions of the inequalities in the constraints above must, of course, be reversed.

Once the transformed model is solved, the values of the x_j variables can be found by dividing w_j by t.

An interesting application involving this type of objective arises in devising *performance measures* for certain organizations where, for example, there is no profit criterion that can be used. This is described by Charnes *et al*. (1978). The objective arises as a ratio of weighted *inputs* and *outputs* of the organization variables represent the weighting factors. Each organization is allowed to choose the weighting factors so as to maximize its performance ratio subject to certain constraints. This subject is now known as *data envelopment analysis* (DEA). An illustration and description of its use are given by the EFFICIENCY ANALYSIS problem in Part II.

3.2.5 Non-existent and non-optimizable objectives

The phrase 'non-optimizable objectives' might be regarded as self-contradictory. We will, however, take the word 'objective', in this phrase, in the non-technical sense. In many practical situations, there is no wish to optimize anything. Even if an organization has certain objectives (such as survival), there may be no question of optimizing them or possibly any meaning to be attached to the word 'optimization' when so applied. Mathematical programming is sometimes dismissed rather peremptorily as being concerned with optimization when many practical problems involve no optimization. Such a dismissal of the technique is premature. If the problem involves constraints, finding a solution that satisfies the constraints may be by no means straightforward. Solving a mathematical programming model of the situation with an arbitrary objective function will at least enable one to find a *feasible* solution if it exists, that is, a solution that satisfies the constraints. The last remark is often very relevant to certain integer programming models, where a very complex set of constraints may exist. It is, however, sometimes relevant to the constraints of a conventional linear programming model. The use of a (contrived) objective is also of value beyond simply enabling one to create a well-defined mathematical programming model. By optimizing an objective, or a series of objectives, in turn, 'extreme' solutions satisfying the constraints are obtained. These extreme solutions can be of great value in indicating the accuracy or otherwise of the model. Should any of these solutions be unacceptable from a practical point of view, then the model is incorrect and should be modified if possible. As stated before, the validation of a model in this way is often as valuable, if not more valuable, an activity as that of obtaining solutions to be used in decision making.

3.3 Defining constraints

Some of the most common types of constraint that arise in linear programming models are classified below.

3.3.1 Productive capacity constraints

These are the sorts of constraints that arose in the product mix example of Section 1.2. If resources to be used in a productive operation are in limited supply, then the relationship between this supply and the possible uses made of it by the different activities give rise to such a constraint. The resources to be considered could be processing capacity or manpower.

3.3.2 Raw material availabilities

If certain activities (such as producing products) make use of raw materials that are in limited supply, then this clearly should result in constraints.

3.3.3 Marketing demands and limitations

If there is a limitation on the amount of a product that can be sold, which could well result in less of the product being manufactured than would be allowed by the other constraints, this should be modelled. Such constraints may be of the form

$$x \leq M, \tag{3.6}$$

where x is the variable representing the quantity to be made and M is the market limitation.

Minimum marketing limitations might also be imposed if it were necessary to make at least a certain amount of a product to satisfy some demand. Such constraints would be of the form:

$$x \geq L. \tag{3.7}$$

Sometimes Inequalities (3.6) or (3.7) might be '=' constraints instead if some demand had to be met exactly.

The constraints (3.6) and (3.7) (or as equalities) are especially simple. When the simplex algorithm is used to solve a model, such constraints are more efficiently dealt with as *simple bounds* on a variable. This is discussed later in this chapter.

3.3.4 Material balance (continuity) constraints

It is often necessary to represent the fact that the sum total of the quantities going into some process equals the sum total coming out. For example, in the blending problem of Section 1.2, we had to ensure that the weight of the final product was equal to the total weight of the ingredients. Such conditions are often easily overlooked. Material balance constraints such as this will usually be of the form:

$$\sum_j x_j - \sum_k y_k = 0, \tag{3.8}$$

showing that the total quantity (weight or volume) represented by the x_j variables must be the same as the total quantity represented by the y_k variables. Sometimes, some coefficients in such a constraint will not be unity, but some value representing a loss or gain of weight or volume in a particular process.

3.3.5 Quality stipulations

These constraints usually arise in blending problems where certain ingredients have certain measurable qualities associated with them. If it is necessary that the products of blending have qualities within certain limits, then constraints will result. The blending example of Section 1.2 gave an example of this. Such constraints may involve, for example, quantities of nutrients in foods, octane values for petrols and strengths of materials.

We now turn our attention to a number of more abstract considerations concerning constraints that we feel a model builder should be aware of.

3.3.6 Hard and soft constraints

A linear programming constraint such as

$$\sum_j a_j x_j \leq b \tag{3.9}$$

obviously rules out any solutions in which the sum over j exceeds the quantity b. There are some situations where this is unrealistic. For example, if Inequality (3.9) represented a productive capacity limitation or a raw material availability, there might be practical circumstances in which this limitation would be overruled. It might sometimes be worthwhile or necessary to buy in extra capacity or raw materials at a high price. In such circumstances, (3.9) would be an unrealistic representation of the situation. Other circumstances exist in which it might be impossible to violate (3.9). For example, (3.9) might be a technological constraint imposed by the capacity of a pipe whose cross section could not be expanded. Constraints such as (3.9), which cannot be violated, are sometimes known as *hard* constraints. It is often argued that what we need are *soft* constraints that can be violated at a certain cost. If (3.9) were to be rewritten as

$$\sum_j a_j x - u \leq b \tag{3.10}$$

and u were given a suitable positive (negative) cost coefficient c for a minimization (maximization) problem, we would have achieved our desired effect. The variable b would represent a capacity or raw material availability that could be expanded to $b + u$ at a cost cu if the optimization of the model found this to be desirable. Possibly the 'surplus' variable u would be given a simple upper bound as well to prevent the increase exceeding a specified amount.

If Inequality (3.9) were a '$\geq$' constraint, an analogous effect could be achieved by a 'slack' variable. If Inequality (3.9) be an equality constraint, it would be possible to allow the right-hand side coefficient b to be overreached or underreached by modelling it as

$$\sum_j a_j x_j + u - v = b \tag{3.11}$$

and giving u and v appropriately weighted coefficients in the objective function. It should be apparent that either u or v must be zero in the optimal solution. Any solution in which u and v came out positive could be adjusted to produce a better solution by subtracting the smaller of u and v from both of them.

An alternative way of viewing such soft constraints is through *fuzzy sets*, where a *degree of membership* corresponds to the amount by which the constraint is violated. It has, however, been shown by Dyson (1980) that formulations in terms of fuzzy set theory can be reformulated as conventional LP models with *minimax* objectives.

3.3.7 Chance constraints

In some applications, it is desired to specify that a certain constraint be satisfied with a given *probability*; for example, we may wish to be 95% confident that a constraint holds. This situation is written as

$$P\left[\sum_j a_j x_j \leq b\right] \geq \beta, \tag{3.12}$$

where β is a probability. In practice, one might expect larger values of β to correspond to higher costs, which should be reflected in the objective function. This would create a more complicated model. Should no relationship between the value of β and the cost be known (as is often the case) then a rather crude, but sometimes satisfactory, approach is to replace Inequality (3.12) by a *deterministic equivalent*. We would then replace (3.12) by

$$\sum_j a_j x_j \leq b', \tag{3.13}$$

where b' is a number larger than b such that the satisfaction of (3.13) implies Inequality (3.12). This idea is due to Charnes and Cooper (1959).

3.3.8 Conflicting constraints

It sometimes happens that a problem involves a number of constraints, not all of which can be satisfied simultaneously. A conventional model would obviously be infeasible. The objective in such a case is sometimes stipulated to be to *satisfy all the constraints as nearly as possible*. We have the case of conflicting objectives referred to in Section 3.2 but postponed to this section.

The type of model that this situation gives rise to is sometimes known as a *goal programming* model. This term was invented by Charnes and Cooper (1961b), but the type of model that results is still a linear programming model.

Each constraint is regarded as a 'goal' that must be satisfied as nearly as possible. For example, if we wished to impose the following constraints:

$$\sum_j a_{ij} x_j = b_i \text{ for all } i, \tag{3.14}$$

but wanted to allow the possibility of them not all being satisfied exactly, we would replace them by the 'soft' constraints

$$\sum_j a_{ij} x_j + u_i - v_i = b_i. \tag{3.15}$$

We are clearly using the same device described before.

Our objective would be to make sure that each such constraint (3.14) is as nearly satisfied as possible. There are a number of alternative ways of making such an objective specific. Two possibilities are as follows:

1. Minimize the sum total of the deviations of the row activities $\sum_j a_{ij} x_j$ from the right-hand side values b_i.
2. Minimize the maximum such deviation over the constraints.

For many practical problems which of the above objectives is used is not very important.

Objective 1 can be dealt with by defining an objective function consisting of the sum of the slack (u) and surplus (v) variables in the constraints such as Equation (3.15). Possibly, these variables might be weighted in the objective with non-unit coefficients to reflect the relative importance of different constraints. An example of the use of such an objective is given in the MARKET SHARING problem of Part II. The fact that this is an integer programming model does not affect the argument, as such models could equally well result from linear programming models. A linear programming application arises in the CURVE FITTING problem.

Objective 2 is slightly more complicated to deal with but can surprisingly still be accomplished in a linear programming model. An extra variable z is introduced. This variable represents the maximum deviation. We, therefore, have to impose extra constraints.

$$z - u_i \geq 0, \quad \text{for all } i, \tag{3.16}$$

$$z - v_i \geq 0, \quad \text{for all } i. \tag{3.17}$$

The objective function to be minimized is simply the variable z. Clearly, the optimal value of z will be no greater than the maximum of u_i and v_i by virtue of optimality. Nor can it be smaller than any of u_i or v_i by virtue of the constraints (3.16) and (3.17). Therefore, the optimal value of z will be both as small as possible and exactly equal to the maximum deviation of $\sum_j a_{ij} x_j$ from its corresponding right-hand side b_i. This type of problem is sometimes known as a *bottleneck problem*. Both the CURVE FITTING and MARKET SHARING problems illustrate this type of objective.

An interesting example of such a *minimax* objective is described by Redpath and Wright (1981) in an application of linear programming to deciding levels and directions for irradiating cancerous tumours. As an alternative to minimizing the *variance* of radiation across a tumour, they *minimize the maximum difference* of radiation levels. This can clearly be done by computationally easier linear programming rather than quadratic programming.

Hooker and Williams (2012) show how equity, in the form of a minimax objective, can be combined with a utilitarian objective in an integer programming model.

3.3.9 Redundant constraints

Suppose we have a constraint such as

$$\sum_j a_j x_j \leq b \tag{3.18}$$

in a linear programming model. If, in the optimal solution, the quantity $\sum_j a_j x_j$ turns out to be less than b, then the constraint (3.18) will be said to be *non-binding* (and have a zero shadow price). Such a non-binding constraint could just as well be left out of the model, as this would not affect the optimal solution. There may well, however, be good reasons for including such *redundant constraints* in a model. First, the redundancy may not be apparent until the model is solved. The constraint must, therefore, be included in case it turns out to be *binding*. Second, if a model is to be used regularly with changes in the data, then the constraint might become binding with some future data. There would then be virtue in keeping the constraint to avoid a future remodelling. Third, *ranging* information, which is discussed in Section 6.3, depends on constraints that may well be redundant in the sense of not affecting the optimal solution.

It should be noted that a constraint such as (3.18) can be non-binding even if the quantity $\sum_j a_j x_j$ is equal to b. This happens if the removal of the constraint does not affect the optimal solution, that is, $\sum_j a_j x_j$ must be equal to b for reasons other than the existence of constraint (3.18). Such non-binding constraints are recognized by there being a zero *shadow price* on the constraint in the optimal solution. Shadow prices are discussed in Section 6.2.

For '$\geq$' constraints, analogous results hold. If Inequality (3.18) were such a constraint, it would be non-binding if $\sum_j a_j x_j$ exceeded b but possibly still non-binding if $\sum_j a_j x_j$ equalled b.

Finally, it should be pointed out that with integer programming, it would not be true to say that if $\sum_j a_j x_j$ were less than b in Inequality (3.18) then Inequality (3.18) would be non-binding. Inequality (3.18) could well be binding and, therefore, not redundant. This is discussed in Section 10.3.

The advisability or otherwise of including redundant constraints in a model is discussed in Section 3.3. A quick method of detecting some such redundancies is also described.

3.3.10 Simple and generalized upper bounds

It was pointed out that marketing constraints often take the particularly simple forms of Inequality (3.6) or (3.7). *Simple bounds* on a variable such as these are more efficiently dealt with through a modification of the simplex algorithm. Most commercial package programmes use this modification. Such bounds are not, therefore, specified as conventional constraints but specified as simple bounds for the appropriate variables. A generalization of simple bounds known as *generalized upper bounds* (GUBs) also proves to be of great computational value in solving a model and therefore worth recognizing. Such constraints are usually written as

$$\sum_j x_j = b. \tag{3.19}$$

The set of variables x_j is said to have a GUB of b. This means that the sum of these variables must be b. If Equation (3.19) were a '$\leq$' constraint instead of an equality, the addition of a slack variable would obviously convert it to the form (Equation 3.19). The fact that the coefficients of the variables in Equation (3.19) are unity is not very important, as scaling can always convert any constraint with all non-negative coefficients into this form. What is, however, important is that when a number of constraints such as Equation (3.19) exist, the variables in them form *exclusive sets*. A set of variables is said to belong to the GUB set, if it can belong to no others. If variables are specified as belonging to GUB sets, it is not necessary to specify the corresponding constraints such as Equation (3.19). A further modification of the simplex algorithm copes with the implied constraint in an analogous way to simple bounding constraints. The diagrammatic representation of a model in Figure 3.2 shows three GUB-type constraints. These constraints could be removed from the model and treated as GUB sets.

There is normally only virtue in using the GUB modification of the simplex algorithm if a large number of GUB sets can be isolated. For example, in the *transportation problem*, which is discussed in Section 5.3, it can be seen that at least half of the constraints can be regarded as GUB sets.

The computational advantages of recognizing GUB constraints can be very great indeed with large models. A way of detecting a large number of such sets if they exist in a model is described by Brearley *et al*. (1975).

3.3.11 Unusual constraints

In previous sections, we have concentrated on constraints that can be modelled using linear programming. It is important, however, not to dismiss 'unusual' restrictions that may arise in a practical problem as not being able to be so

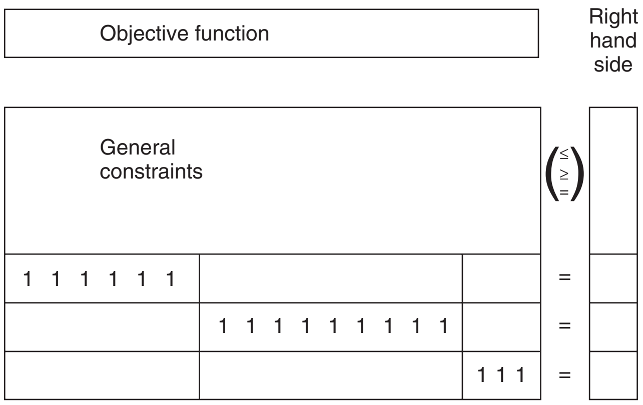


Figure 3.2

modelled. By extending a model to be an integer programming model, it is sometimes possible to model such restrictions. For example, a restriction such as

> We can only produce product 1 if product 2 is produced but neither of products 3 or 4 are produced.

could be modelled. This topic is discussed much further in Chapter 9.

3.4 How to build a good model

Possible aims that a model builder has when constructing a model are the ease of understanding the model, detecting errors in the model and computing the solution. Ways of trying to achieve and resolve these aims are described below.

3.4.1 Ease of understanding the model

It is often possible to build a compact but realistic model when quantities appear implicitly rather than explicitly, for example, instead of a non-negative quantity being represented by a variable y and being equated to an expression $f(x)$ by the constraint

$$f(x) - y = 0, \tag{3.20}$$

the variable y does not appear, but $f(x)$ is substituted into all the expressions where it would appear. Building such compact models often leads to great difficulty and extra calculation in interpreting the solution. Even though a less-compact model takes longer to solve, it is often worth the extra time. If, however, a report writer is used, this may take care of interpretation difficulties, and the use of compact models is desirable from the point of view of the third aim.

It is also desirable to use mnemonic names for the variables and constraints in a problem to ease the interpretation of the solution. The computer input to the small blending problem illustrated in Section 1.2 shows how such names can be constructed. A very systematic approach to naming variables and constraints in a model is described by Beale *et al*. (1974).

3.4.2 Ease of detecting errors in the model

This aim is clearly linked to the first. Errors can be of two types: (i) clerical errors such as bad typing and (ii) formulation errors. To avoid the first type of error, it is desirable to build any but very small models using a matrix generator (MG) or language.

There is also great value to be obtained from using a PRESOLVE or REDUCE procedure on a model for error detection. Clerical or formulation errors often result in a model being unbounded or infeasible. Such conditions can often be revealed easily using such a procedure. A simple procedure of this kind, which also simplifies models, is outlined below.

Formulation can sometimes be done with error detection in mind. This point is illustrated in Section 6.1.

3.4.3 Ease of computing the solution

LP models can use large amounts of computer time, and it is desirable to build models that can be solved as quickly as possible. This objective can conflict with the first. In order to meet the first objective, it is desirable to avoid compact models. If a PRESOLVE or REDUCE procedure is applied after the model has been built, but before solving it, then dramatic reductions in size can sometimes be achieved. An algorithm for doing this is described by Brearley *et al*. (1975). Karwan *et al*. (1983) provided a collection of methods for detecting redundancy. The reduced problem can then be solved, and this solution used to generate a solution to the original problem.

In order to illustrate the possibility of spotting redundancies in a linear programming model, we consider the following numerical example:

$$\begin{aligned}
\text{Maximize} \quad & 2x_1 + 3x_2 - x_3 - x_4 && (3.21)\\
\text{subject to} \quad & x_1 + x_2 + x_3 - 2x_4 \leq 4, && (3.22)\\
& -x_1 - x_2 + x_3 - x_4 \leq 1, && (3.23)\\
& x_1 + x_4 \leq 3, && (3.24)\\
& x_1, x_2, x_3, x_4 \geq 0. && (3.25)
\end{aligned}$$

As x_3 has a negative objective coefficient and the problem is one of maximization, it is desirable to make x_3 as small as possible, x_3 has positive coefficients in constraints (3.22) and (3.23). As these constraints are both of the '$\leq$' type, there can be no restriction on making x_3 as small as possible. Therefore, x_3 can be reduced to its lower bound of zero and hence be regarded as a redundant variable.

Having removed the variable x_3 from the model, constraint (3.23) is worthy of examination. All the coefficients in this constraint are now negative. Therefore, the value of the expression on the left-hand side of the inequality relation can never be positive. This expression must, therefore, always be smaller than the right-hand side value of 1, indicating that the constraint is redundant and may be removed from the problem.

The above model could, therefore, be reduced in size. With large models, such reductions could well lead to substantial reductions in the amount of computation needed to solve the model. *Infeasibilities* and *unboundedness* can also sometimes be revealed by such a procedure. A model is said to be *infeasible*, if there is no solution that satisfies all the constraints (including non-negativity conditions on the variables). If, on the other hand, there is no limit to the amount by which the objective function can be optimized, the model is said to be *unbounded*. Such conditions in a model usually suggest modelling errors and are discussed in detail in Section 6.1.

Some package programmes have procedures for reducing models in this way. Such procedures go under names such as REDUCE, PRESOLVE and ANALYSE. Problem reduction can be taken considerably further. Using simple bounds on variables and considering the dual model (the dual of a linear programming model is described in Section 6.2), the above example can be reduced to nothing (completely solved). A more complete treatment of reduction is beyond the scope of this book, as such reduction procedures are usually programmed and carried out automatically. It is not, therefore, always important that a model builder knows how to reduce his or her model, although the fact that it can be simply reduced must have implications for the situation being modelled. A much fuller treatment of the subject is given in Brearley *et al*. (1975) and Karwan *et al*. (1983).

Substantial reduction in computing time can also often be achieved by exploiting the special structure of a problem. One such structure that has proved particularly valuable is the GUB as described in Section 3.3. It is obviously desirable that the model builder detects such structure if it be present in a problem, although a few computer packages have facilities for doing this automatically through procedures such as those described by Brearley, Mitra and Williams.

3.4.4 Modal formulation

In large LP problems, reduction in the number of constraints can be achieved using *modal formulations*. If a series of constraints involves only a few variables, the feasible region for the space of these variables can be considered. For example, suppose that x_A and x_B are two of the (non-negative) variables in an LP model, and they occur in the following constraints:

$$x_A + x_B \leq 7, \tag{3.26}$$

$$3x_A + x_B \leq 15, \tag{3.27}$$

$$x_B \leq 5. \tag{3.28}$$

The situation can be modelled, as demonstrated in Figure 3.3, by letting the activities for the 'extreme modes' of operation be represented by variables instead of x_A and x_B. If these variables are λ_0, λ_1, λ_2, λ_3 and λ_4, we only have to specify the single constraint.

$$\lambda_0 + \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = 1. \tag{3.29}$$

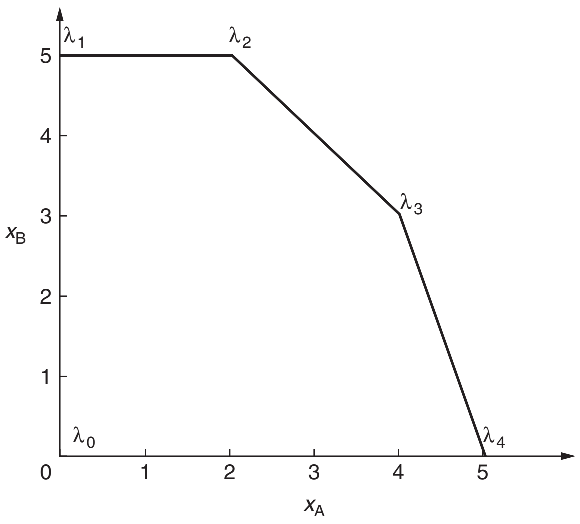


Figure 3.3

Whenever x_A and x_B occur in other parts of the model, apart from constraints (3.26), (3.27) and (3.28) that are now ignored, we substitute the following expressions:

$$\text{for } x_A, \quad 2\lambda_2 + 4\lambda_3 + 5\lambda_4; \tag{3.30}$$

$$\text{for } x_B, \quad 5\lambda_1 + 5\lambda_2 + 3\lambda_3. \tag{3.31}$$

The coefficients for x_A are the 'x_A coordinates' of $\lambda_1, \lambda_2, \ldots$, in Figure 3.3 and the coefficients of x_B are the 'x_B coordinates'.

In this case, we have saved two constraints. The ingenious use of this idea can result in substantial savings in the number of constraints in a model.

Experiments in the use of a special case of this technique are described by Knolmayer (1982). Smith (1973) discusses the use of this approach to modelling practical problems. Modal formulations are quite common for particular processes in the petroleum industry. The REFINERY OPTIMIZATION model demonstrates a very simple case of this, where a process has only one *mode* of operation with inputs and outputs in *fixed* proportions. In this case, it is better to model the level of the process as an activity to avoid having to represent the (fixed) relationships between the inputs and outputs by constraints.

A general methodology for modelling processes with different types of input–output relations is developed by Müller-Merbach (1987).

3.4.5 Units of measurement

When a practical situation is modelled, it is important to pay attention to the units in which quantities are measured. Great disparity in the sizes of the coefficients in a linear programming model can make such a model difficult, if not impossible, to solve. For example, it would be stupid to measure profit in pounds if the unit profit coefficients in the objective function were of the order of millions of pounds. Similarly, if capacity constraints allowed quantities in thousands of tons, it would be better to allow each variable to represent a quantity in thousands of tons rather than in tons. Ideally, units should be chosen so that each non-zero coefficient in a linear programming model is of a magnitude between 0.1 and 10. In practice, this may not always be possible. Most commercial package programmes have procedures for automatically *scaling* the coefficients of a model before it is solved. The solution is then automatically unscaled before being printed out.

3.5 The use of modelling languages

A number of computer software aids exist for helping users to *structure* and *input* their problem into a package in the form of a model. Programmes to do this are sometimes known as *matrix generators* (MGs). Some such systems are more usefully thought of as special purpose high level programming languages and are, therefore, known as *modelling languages*.

It is now recognized that the main hurdle to the successful application of a mathematical programming model is often no longer the problem of computing the solution. Rather, it lies with the *interface* between the user and the computer. Some of the difficulties can be overcome by freeing the modeller from the specific needs of the package used to solve the model.

An ambitious attempt to overcome the user interface problems is described by Greenberg (1986). A survey of modelling systems is given by Greenberg and Murphy (1992).

Other systems are described by Buchanan and McKinnon (1987) and Greenberg *et al*. (1987). These systems pay attention to the output interface as well as the input.

A number of quite distinct approaches to building MGs/languages are described below. Before doing this, however, we specify the main advantages to be gained from using them.

3.5.1 A more natural input format

The input formats required by most packages are designed more with the package in mind than the user. As described in Section 2.2, most commercial packages use the MPS format. This orders the model by columns in a fixed format based on the internal ordering of the matrix in the computer. This is often unnatural, tedious, and a source of clerical error. In addition, this format is far from concise, requiring a separate data statement for every two coefficients in the model. The use of a modelling language can overcome all these disadvantages.

3.5.2 Debugging is made easier

As with computer programming, the debugging and verification of a model is an important and sometimes time-consuming task. The use of a format that the modeller finds natural makes this task much easier.

3.5.3 Modification is made easier

Models are often used on a regular basis with minor modifications, or once built may be used experimentally many times with small changes in data. The separation of the data (which changes) from the structure of the model, in most modelling languages, facilitates this task.

3.5.4 Repetition is automated

Large models usually arise from the combination or repetition of smaller models, for example, multiple periods, plants or products, as described in Section 4.1. In such circumstances, there will be a lot of repetition of particular items of data and structure. Formats such as MPS demand a repetition of this data, which is both inefficient and a potential source of error. Modelling languages generally take account of such repetition very easily through the indexing of time periods, etc.

Some of the distinct approaches adopted by different modelling languages are outlined below. Fourer (1983) also provides a very good and comprehensive discussion. The methods can only be understood in detail through the manuals of the relevant systems.

3.5.5 Special purpose generators using a high level language

It is possible to write a different program, in a general language, for each model one wishes to build. This approach ignores the similarities in structure that all mathematical programming models have, however diverse their application area, as well as the standard format required (e.g. MPS), which is tedious to remember and incorporate into a general program. It seems more sensible to incorporate such similarities into a language.

3.5.6 Matrix block building systems

The often huge matrix of coefficients in a practical model always exhibits a high degree of structure. The matrix tends to consist of small dense blocks of coefficients in an otherwise very sparse matrix. These blocks are often repetition of each other and are linked together by simple matrices such as identity matrices. The systematic combination and replication of submatrices into larger matrices in this way is the method used in certain systems. Such systems have functions that, for example, concatenate and transpose matrices. A defect of such systems is that they still require users to think in terms of matrices of coefficients, even though they free them from some of the mechanics of assembling such matrices.

3.5.7 Data structuring systems

Some systems allow the user to structure a model in a diagrammatic form. Flow diagrams are often particularly helpful. For example, a problem may involve flows of different raw materials into processes to give products that are then distributed via depots to customers. The system will allow the user to translate this representation into the system directly, avoiding algebraic notation. Such systems can be useful for *specific* applications, for example, process industries. They do have the defect that the user needs to learn quite a lot about the system before he or she can use it.

3.5.8 Mathematical languages

The above approaches all avoid the use of conventional notation, that is, '$\sum$' notation and *indexed* sets. Indeed, an argument sometimes used in their favour is that mathematical notation is unnatural to certain users who prefer to think of their model in other ways. This may well be the case with certain established applications (e.g. the petroleum industry), where mathematical programming is so established and important for a conceptual methodology of its own to have developed. If more generality is required, then *conventional mathematical notation* provides a *language* that is widely understood as well as providing a concise way of defining many models. The fact that the notation is so widely understood by anyone with an elementary mathematical training makes learning to use such a system comparatively easy.

Most general modelling systems, based on mathematical notation, are very similar. We describe their common features. We then illustrate this by a formulation of one of the models in Part II using a typical language NEWMAGIC. It is easy to translate such a model into any of the other, similar, modelling languages. Such languages require the user to identify the following components of their model. It is straightforward to see how these elements are used by means of the example below (although different systems will have varied key words and syntax rules).

3.5.8.1 SETs

These usually represent *indices* for certain classes of linear or integer programming variables. For example, $PROD1, PROD2, PROD57$ might be the variables representing 57 different products that vary only in their profits and resource usages. We could define a ‘root name’ *PROD* followed by an index from a SET $A = \{1, \ldots, 57\}$. In most systems, indices can be numbers or names (e.g. towns or months of the year).

3.5.8.2 DATA

These are usually coefficients for the model, which may be defined in arrays or read from external files. The items in the arrays or files are often indexed by the indices described above.

3.5.8.3 VARIABLES

These are the linear or integer programming variables that appear in the model, typically defined using a root name together with associated indices, for example, $PROD[A]$. If the variable is of a special sort, for example, *integer*, *free*, *binary*, etc., this is usually specified with the variable.

3.5.8.4 OBJECTIVE

This is the *Objective Function* of the model given with **MAXIMIZE** or **MINIMIZE**. It is given a name and usually defined using $\sum$ notation over index sets defined above but by means of key words such as *sum* or *sigma* (as most keyboards do not support the symbol $\sum$).

3.5.8.5 CONSTRAINTS

These are the conventional *linear* constraints. They will, probably, also be *indexed*.

In addition, such languages will have many other facilities and features that are specific to the language and are not discussed in this chapter, but explained in the associated manuals. We give the example in NEWMAGIC below. As with other languages, it is convenient to include *comments* that are done here by including them between the symbols /*, or on a line preceded by !.

```
MODEL FoodB
DATA
   max t = 6, max i = 5;
SET
   A = {1..2}, B = {1..5},
   T = {1..max t}, I = {1..max i};
DATA
   cost[T,I] << " cost.dat ",
      hard[I] = [8,8,6.1,2.0,4.2,5.0];
   VARIABLES
      b[I,T], u[I,T], s[I,T], Prod[T];
   OBJECTIVE
   MAXIMIZE
      prof = sum{t in T} (150 * Prod[t] - sum{i in I} (cost[t,i] *
b[i,t] + 5 * s[i,t]));
   CONSTRAINTS
      loil{i in I, 1} : 500 + b[i,1] - u[i,1] - s[i,1] = 0,
      loil{i in I, t in T, t > 1} : s[i, t - 1] + b[i,t] - u[i,t]
- s[i,t] = 0,
      vveg{t in T} : sum{i in A} u[i,t] < = 200,
      voil{t in T} : sum{i in B} u[i,t] < = 250,
      lhrd{t in T} : 3 * Prod[t] < = sum{i in I} hard[i] * u[i,t],
      uhrd{t in T} : sum{i in I} hard[i] * u[i,t] < 6 * Prod[t],
      cont{t in T} : sum{i in I} u[i,t] = Prod[t],
   /* Specify bounds for the variables */
      for{i in I, t in T, t < maxt} s[i,t] < = 1000,
      for{i in I} s[i, maxt] = 500;
   /* This is the end of the continuous part of the model. The
   section below adds integer variables and extra constraints
   to model some logical conditions. Separate VARIABLE and
   CONSTRAINT sections are not necessary; they could be included
   in the VARIABLE and CONSTRAINT sections above. The keyword
   BINARY means an integer variable that can only take the
   values 0 or 1 */
   VARIABLES
   d[I,T] BINARY;
   CONSTRAINTS
   for{t in T}
   {
   ! Formulate second condition
   for{i in A}
   { ! UB on amount of veg. oil refined/month = 200
   20 * d[i,t] <= u[i,t], u[i,t] <= 200 * d[i,t]
   },
   for{i in B}
   {! UB on amount of non-veg. oil refined/month = 250
   20 * d[i,t] <= u[i,t], u[i,t] <= 250 * d[i,t]
   },
   ! Formulate first condition
   sum{i in I} d[i,t] <= 3,
   ! Formulate third condition
   d[1,t] <= d[5,t], d[2,t] <= d[5,t]
   };
   END MODEL
   SOLVE FoodB;
   PRINT SOLUTION FOR FoodB > > "FoodB.sol";
   QUIT;
```

4

Structured linear programming models

4.1 Multiple plant, product and period models

The purpose of this section is to show how large linear programming (LP) models can arise through the combining of smaller models. Almost all very large models arise in this way. Such models prove to be more powerful as decision making tools than the submodels from which they are constructed. In order to illustrate how a multiplant model can arise in this way, we take a very small illustrative example.

Example 4.1: A Multiplant Model

A company operates in two factories, A and B. Each factory makes two products, *standard* and *deluxe*. A unit of *standard* gives a profit contribution of £10, while a unit of *deluxe* gives a profit contribution of £15.

Each factory uses two processes, grinding and polishing, for producing its products. Factory A has a grinding capacity of 80 hours per week and polishing capacity of 60 hours per week. For factory B, these capacities are 60 and 75 hours per week, respectively.

The grinding and polishing times in hours for a unit of each type of product in each factory are given in the table below.

	Factory A		Factory B	
	Standard	Deluxe	Standard	Deluxe
Grinding	4	2	5	3
Polishing	2	5	5	6

It is possible, for example, that factory B has older machines than factory A, resulting in higher unit processing times.

In addition, each unit of each product uses 4 kg of a raw material, which we refer to as *raw*. The company has 120 kg of *raw* available per week. To start with, we will assume that factory A is allocated 75 kg of *raw* per week and factory B the remaining 45 kg per week.

Each factory can build a very simple LP model to maximize its profit contribution. This is an obvious example of the product mix application of LP mentioned in Section 1.2. The following are the resultant models:

Factory A's Model

$$\begin{array}{lll}
\text{Maximize} & \text{Profit A} & 10x_1 + 15x_2 \\
\text{subject to} & \text{Raw A} & 4x_1 + \ 4x_2 \le 75, \\
 & \text{Grinding A} & 4x_1 + \ 2x_2 \le 80, \\
 & \text{Polishing A} & 2x_1 + \ 5x_2 \le 60, \\
 & & x_1, x_2 \ge 0,
\end{array}$$

where x_1 is the quantity of standard to be produced in A and x_2 is the quantity of deluxe to be produced in A.

Factory B's Model

$$\begin{array}{lll}
\text{Maximize} & \text{Profit B} & 10x_3 + 15x_4, \\
\text{subject to} & \text{Raw B} & 4x_3 + \ 4x_4 \le 45, \\
 & \text{Grinding B} & 5x_3 + \ 3x_4 \le 60, \\
 & \text{Polishing B} & 5x_3 + \ 6x_4 \le 75, \\
 & & x_3, x_4 \ge 0,
\end{array}$$

where x_3 is the quantity of standard to be produced in B and x_4 is the quantity of deluxe to be produced in B.

These two models can easily be solved graphically. Our purpose is not, however, to concentrate on the mechanics of solving these individual models. We do, however, give the optimal solutions below as these will be discussed later.

Optimal Solution to Factory A's Model

Profit is £225 obtained from making 11.25 of standard and 7.5 of deluxe. There is a surplus grinding capacity of 20 hours.

Optimal Solution to Factory B's Model

Profit is £168.75 obtained from making 11.25 deluxe. There is a surplus grinding capacity of 26.25 hours, and a surplus polishing capacity of 7.5 hours.

Suppose now that a company model is built in order to maximize total profit. We will assume that the factories remain distinct and geographically separated. We will, however, no longer allocate 75 kg of raw to A and 45 kg to B. Instead, we will allow the model to decide this allocation. There will now be a single raw material constraint limiting the company to 120 kg per week. The resultant model is as follows:

$$\begin{array}{lllrrrrl}
\text{Maximize} & \text{Profit} & 10x_1 + & 15x_2 + & 10x_3 + & 15x_4 & \\
\text{subject to} & \text{Raw} & 4x_1 + & 4x_2 + & 4x_3 + & 4x_4 & \le 120, \\
& \text{Grinding A} & 4x_1 + & 2x_2 & & & \le 80, \\
& \text{Polishing A} & 2x_1 + & 5x_2 & & & \le 60, \\
& \text{Grinding B} & & & 5x_3 + & 3x_4 & \le 60, \\
& \text{Polishing B} & & & 5x_3 + & 6x_4 & \le 75, \\
\end{array}$$

$$x_1, x_2, x_3, x_4 \ge 0.$$

The Company Model

The fact that the constraints raw A and raw B of the factory models have been combined into a single constraint for the company model is of crucial significance. We are now asking the model to split the 120 kg of raw optimally between A and B rather than making an arbitrary allocation ourselves. As a consequence, we would expect a more efficient split resulting in a greater overall company profit. This is borne out by the optimal solution.

Optimal Solution to the Company Model

Total profit is £404.15, obtained from making 9.17 of standard in A, 8.33 of deluxe in A and 12.5 of deluxe in B. There is a surplus grinding capacity in A of 26.67 hours, and a surplus grinding capacity in B of 22.5 hours.

A number of points are worth noting in comparing this solution with those for factories A and B individually:

1. The total profit is £404.14, which is greater than the combined profit £393.75 from A and B acting independently.
2. Factory A only contributes £216.65 to the new total profit, whereas it produced a profit of £225 before. Factory B, however, now contributes £187.5 to total profit, whereas it only produced £168.75 before.
3. Factory A now uses 70 kg of raw and factory B uses 50 kg.

It is clear that the company model has biased production more toward factory B than before. This has been done by allocating B 50 kg of raw instead of 45 kg and so depriving A of 5 kg. If it had been possible to decide this 70/50 split before, it would not have been necessary to build a company model. This argument also applies to much larger, more realistic, multiplant models. Normally, however, there will be a number of scarce resources that must be shared between plants rather than the single resource *raw*, which we consider here. An optimal split would have to be found for each of these resources. Determining such splits would obviously be complex. The needs of each plant have to be balanced against how efficiently they use the scarce resources. In our example, factory B's older machinery results in it being allocated less of raw than A. To start with, however, our 75/45 split was over biased in A's favour.

The above example is intended to show how a multiplant model can arise. It is a method of using LP to cope with allocation problems *between* plants as well as help with decision making *within* plants. The model that we built was a very simple example of a common sort of structure, which arises in multiplant models. This is known as a *block angular* structure. If we detach the coefficients in the company model and present the problem in a diagrammatic form, we obtain Figure 4.1.

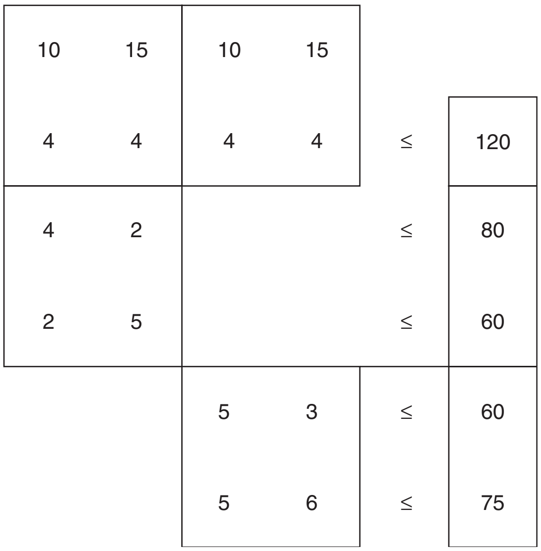


Figure 4.1

The first two rows are known as *common rows*. Obviously, one of the common rows will always be the objective row. The two diagonally placed blocks of coefficients are known as *submodels*. For a more general problem with a number of shared resources and n plants, we would obtain the general block angular structure shown in Figure 4.2.

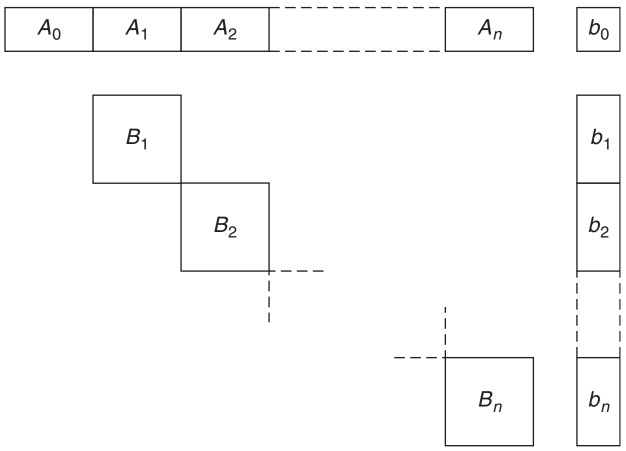


Figure 4.2

$A_0, A_1, \ldots, B_1, B_2, \ldots,$ are blocks of coefficients. $b_0, b_1, \ldots,$ are columns of coefficients forming the right-hand side. The block A_0 may or may not exist but is sometimes conveniently represented. $A_0, A_1, \ldots,$ represent the common rows. Common constraints in multiplant models usually involve allocating scarce resources (raw material, processing capacity, labour, etc.) across plants. They might sometimes represent transport relations between plants. For example, it might in certain circumstances be advantageous to transport the product of an intermediate process from one plant to another. The model could conveniently allow for this if extra variables were introduced into the plants in question to represent amounts transported. Suppose we simply wanted to allow for transport from plant 1 to plant 2. This would be accomplished by the constraint

$$x_1 - x_2 = 0, \tag{4.1}$$

where x_1 is the quantity transported from 1 into 2 and x_2 is the quantity transported into 2 from 1.

Apart from constraint (4.1), x_1 would only be involved in constraints relating to the submodel for plant 1. Similarly, x_2 would only be involved in constraints relating to plant 2. x_1 (or x_2 but not both) would probably be given cost coefficients in the objective function representing unit transport costs. Constraint (4.1) clearly gives another common row constraint.

Should a problem with a block angular structure have no common constraints, it should be obvious that optimizing it simply amounts to optimizing each subproblem with its appropriate portion of the objective. For our simple numerical example, if there had been no raw material constraint, we could have solved each factory's model separately and obtained an overall company optimum. In fact, as far as the company was concerned, it would have been perfectly acceptable to treat each factory as autonomous. Once we introduce common constraints, however, this will probably no longer be the case, as the small example demonstrated. The more common constraints there are, the more interconnected the separate plants must be. In Section 4.2, we discuss how a knowledge of the optimal solutions to the subproblems might be used to obtain an optimal solution to the total problem. This can be quite important computationally, as such structured problems can often be very large and take a long time to solve if treated as one large model. Common sense would suggest that the number of common rows would be a good measure of how close the optimal solution to the total problem was to the sum total of the optimal solutions to the subproblems. This is usually the case. For problems with only a few common rows, one might expect there to be virtue in taking account of the optimal solutions of the subproblems.

The block angular structure exhibited in Figure 4.2 does not arise only in multiplant models. *Multi-product* models arise quite frequently in blending problems. Suppose, for example, that the blending problem that we presented in Section 1.2 represented only one of a number of products (brands) that a company manufactured. If the different products used some of the same ingredients and processing capacity then it would be possible to take account of their limited supply through a structured model. The individual blending models such as those presented in Section 1.2 would be unable to help make rational allocations of such scarce shared resources between products. For example, in Figure 4.2, $B_1, B_2, \ldots$, would represent the individual blending constraints for each product. These subproblems would contain variables such as x_{ij} representing the quantity of ingredient i in product j. If ingredient i was in limited supply, we would impose a common constraint:

$$\sum_j x_{ij} \leq \text{availability of ingredient } i. \tag{4.2}$$

If ingredient i used α_{ij} units of a particular processing capacity in blending product j, we would have the common constraint:

$$\sum_j \alpha_{ij} x_{ij} \leq \text{total processing capacity available for ingredient } i. \tag{4.3}$$

As with multiplant models, multi-product models almost always arise through combining existing submodels. The submodels can be used to help make certain operational decisions. By combining such submodels into multiple models, further decisions can be brought into the realm of LP.

Another way in which the block angular structure of Figure 4.2 arises is in *multi-period* models. Suppose that in our blending problem of Section 1.2, we wanted to determine not just how to blend in a particular month but also how to purchase each month with the possibility of storing for later use. It would then be necessary to distinguish between *buying, using*, and *storing*. For each ingredient, there would be three corresponding variables. These would be linked together by the relations:

$$\text{Amount in store at end of period } (t-1) + \text{amount bought in period } t = \text{amount used in period } t + \text{ amount in store at end of period } t. \quad (4.4)$$

These relations give equality constraints. The block angular structure of Figure 4.2 arises through these equality constraints providing common rows linking consecutive time periods. Each subproblem B_1 consists of the original blending constraints involving only the 'using' variables. Such a multi-period model arises from the FOOD MANUFACTURE problem of Part 2. This problem is the blending problem of Section 1.2 taken over six months with different raw oil prices for different months. Full details of the formulation of this problem are given in Part 3.

In a multi-period model of the kind described above, each period need not necessarily be of the same duration. Some periods might be of a month, while later months might be aggregated together (with a corresponding increase in resources represented by right-hand side coefficients). It will, however, be very likely that $B_1, B_2, \ldots,$ in Figure 4.2 are the same submatrices representing the same blending constraints in each period. The corresponding rows and columns of these matrices will of course be distinguished by different names.

The way in which multi-period models should be used is important. Such a model is usually run with the first period relating to the present times and subsequent periods relating to the future. As a result, only the operating decisions suggested by the model for the present month are acted on. Operating decisions for future months will probably only be taken as provisional. After a further month (or the appropriate time period) has elapsed, the model will be rerun with updated data and the first period applying to the new present period. In this way, a multi-period model is in constant use as both an operating tool for the present and a provisional planning tool for the future.

A further point of importance in multi-period models concerns what happens at the end of the last time period in the model. If the stocks at the end of the last period, which occur in constraint (4.4), are included simply as variables, the optimal solution will almost always decide that they should be zero. From the point of view of the model, this would be sensible as it would be the minimum 'cost' or maximum 'profit' solution. In a practical situation, however, the model is unrealistic as operations will almost certainly continue beyond the end of the last period and stocks would probably not be allowed to run right down. One possible way out is to set the final stocks to constant values representing sensible final levels. It could be argued that the operating plans for the final period will be very provisional anyway and any inaccuracy that far ahead will not be serious. This is the suggestion that is made for dealing with both the multi-period FACTORY PLANNING and FOOD MANUFACTURE problems of Part 2. An alternative approach that is sometimes adopted is to 'value' the final stocks in some way, that is, give the appropriate variables positive 'profits' in a maximization model or negative 'costs' in a minimization model. In effect, such a valuation would cause the optimal solution to suggest producing final stocks to sell if it appeared profitable. Although the organization might never consider the possibility of selling off final stocks, the fact that they had been given realistic valuations would cause them to come out at sensible levels.

A description of a highly structured model that is potentially multi-period, multi-product and multiplant is given by Williams and Redwood (1974).

Another type of structure that often arises in multi-period models is the *staircase* structure. This is illustrated in Figure 4.3.

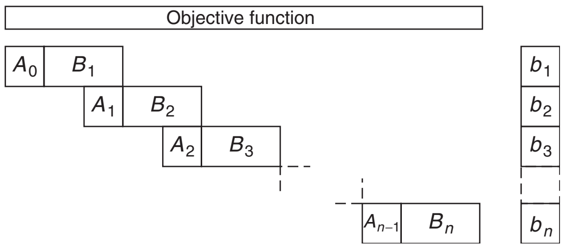


Figure 4.3

In fact, a staircase structure such as this could be converted into a block angular structure. If alternate 'steps' such as $(A_0, B_1), (A_2, B_3)$ were treated as subproblem constraints and the intermediate 'steps' as common rows, we would have a block angular structure. The block angular multi-period model, which was described above, could also easily be rearranged into a staircase structure.

Another structure that sometimes arises, although less commonly than block angular and staircase structures, is shown in Figure 4.4.

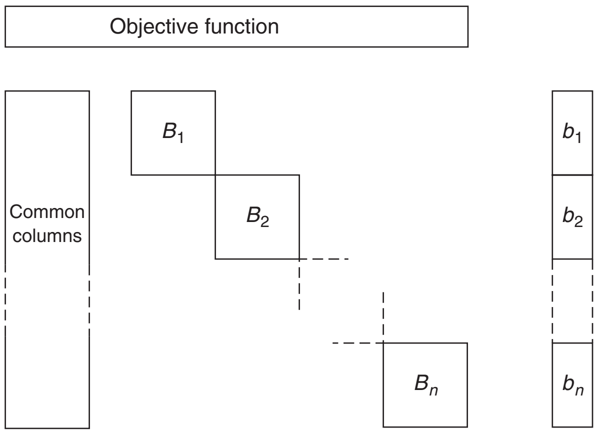


Figure 4.4

In this type of model, the subproblems are connected by common columns rather than rows. Such a structure is the dual to the block angular structure (the dual is defined in Section 6.2).

The method of Benders (1962) is applicable to solving these types of model (even if they are not integer programmes). A way in which this type of structure arises is given in the next section.

4.2 Stochastic programmes

Stochastic programmes are a special type of LP model. Although the title 'Stochastic Programming' is sometimes used, it can convey the misleading impression that this is another technique. A stochastic programme is a type of LP, which models uncertainty in a particular way. Stochastic programming is a special case of robust optimization. In many practical situations, the data in a resultant LP model is not known with certainty. This may be for a number of reasons. One is an inability to obtain totally accurate data. Another is that the model is time staged (discussed in Section 4.1) where certain events, which need to be modelled, have not yet occurred. Robust optimization encompasses both these situations, even when we cannot quantify the uncertainty or associated risks. It is concerned with obtaining solutions that remain reasonably stable in the face of the uncertainty, whether it can be quantified or not. The use of sensitivity analysis, which is discussed in Section 6.3, gives information on how a solution changes with uncertainty in the data. This is a rather limited analysis that can be misleading in some circumstances, see Kall and Wallace (1994). A totally risk-averse approach might be to seek a maximin solution (as discussed in Section 3.2) in order to make the worst possible result (however unlikely) as little bad as possible. Often, however, this extreme approach is excessively 'conservative'. A very good discussion of robust optimization is given by Greenberg and Morrison (2008).

In this section, we confine ourselves to situations where the uncertainty can be quantified. One way of modelling some such situations is by chance constraints as discussed in Section 3.3. We will confine ourselves, here, to modelling situations where some decisions (stage 1) have to be made before other information becomes available. The result of the stage 1 decisions and the subsequent availability of extra information allow stage 2 decisions to be made. Unfortunately, the relative merits of the stage 1 decisions depend on this uncertain information and the resultant stage 2 decisions. For example, it may be necessary to decide production (stage 1) before the demand (uncertain) is known. Then as a result of the production decisions and the subsequent actual demand, stage 2 decisions must be made regarding selling any excess production at a lower price or extra production to make up a shortfall at a higher cost.

If it is realistic to consider the uncertain information as taking one of a (small) finite number of possible values with known probabilities (a discrete probability distribution) then a set of distinct stage 2 variables may be introduced to correspond to each uncertain value. The objective becomes one of minimizing *expected cost*. This model then becomes a linear program.

For example, suppose that the production decisions are represented by stage 1 variables $x_1, x_2, \ldots, x_n$. The excess production or shortfall is represented by stage 2 variables $y_1, y_2, \ldots, y_n, z_1, z_2, \ldots, z_n$. These stage 2 variables will be replicated m times according to each of the possible demand levels $d_j^{(1)}, d_j^{(2)}, \ldots, d_j^{(m)}$. This gives a model of the form:

$$\text{Minimize} \quad \sum_j c_j x_j + \sum_r p_r \left(\sum_j e_j y_j^{(r)} + \sum_j f_j z_j^{(r)} \right)$$

$$\text{subject to} \quad \sum_j a_{ij} x_j \leq b_i \text{ for all production constraints } i$$

$$\left.\begin{aligned} x_j - y_j^{(r)} + z_j^{(r)} &= d_j^{(r)} \\ x_j, y_j^{(r)}, \ z_j^{(r)} &\geq 0 \end{aligned}\right. \quad \text{for all } j \text{ and } r$$

where p_r are given probabilities and c_j, e_j and f_j are production, excess production, and shortfall costs, respectively.

It is important to realize how such a model might be used. Initial decisions would be made as a result of stage 1 variables. Only the values of those stage 2 variables would ultimately be used, which corresponded to the subsequent value of the uncertain information.

Also, it is important to realize the extra information that is incorporated in such a model. While a 'conventional' model may contain future data in the form of a single 'point' estimate, a stochastic model represents such data by a series of estimates, weighted by probabilities. The result may be a 'solution' that is not as 'good' as the conventional model, in terms of optimal objective, but that is more likely to be realistic, to be realizable, and to reduce risk. Another way of viewing such models is one of 'keeping one's options open', that is, not 'putting all one's eggs in one basket'.

The coefficients of the x_j variables, above, form the common columns of the LP model, while the variables $y_j^{(r)}$ and $z_j^{(r)}$ each appear in only the rth block of the model. The combination of this structure with the block angular structure sometimes occurs giving models of the form shown in Figure 4.5.

An example of the use of a stochastic program is YIELD MANAGEMENT (selling airline tickets), which is given in Sections 12.24, 13.24 and 14.24.

References to stochastic programming are Dempster (1980), Kall and Wallace (1994) and Greenberg and Morrison (2008). Stochastic programmes can become very large because of the multiplicity of scenarios resulting from successive periods, that is, from each scenario in the second stage, there will be a number of scenarios into the third stage, etc. The resultant *tree* of scenarios into the

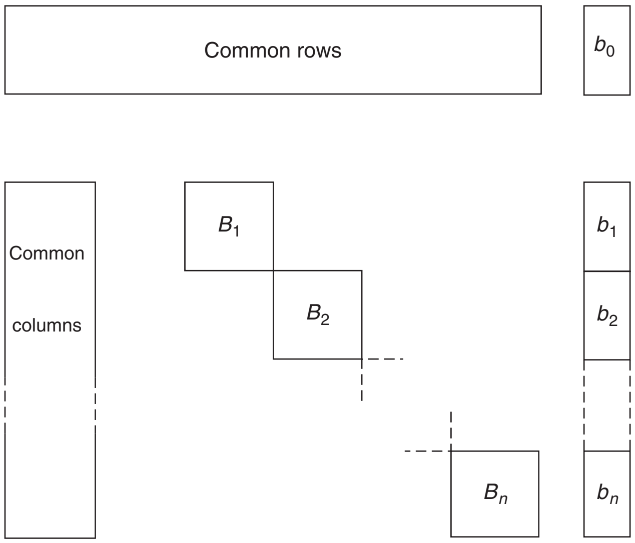


Figure 4.5

future rapidly becomes very large. The largest LP ever built and solved (to the author's belief) is a stochastic program due to Gondzio and Grothey (2006). It has 353 million constraints and 1010 million variables.

4.3 Decomposing a large model

Common sense suggests that the optimal solution to a structured model should bear some relation to the optimal solutions to the submodels from which it is constructed. This is usually the case and there may, therefore, be virtue in devising computational procedures to exploit this, particularly in view of the large size of many structured models. Ways of solving a structured model by way of solutions to the subproblems form the subject of *decomposition*. It is sometimes mistakenly thought that decomposition is the actual act of splitting a model up into submodels. Although computational procedures have been devised for doing this, decomposition concerns the solution process on a structure that is usually already known to the modeler.

The importance of decomposition is not only computational but also economic. A decomposition procedure applies to a mathematical model of a structured practical problem. If the structured model arises by way of a structured organization (such as a multiplant model), the decomposition procedure mirrors a method of *decentralized planning*. It is for this reason that decomposition is discussed in a book on model building. The subject is best considered through this analogy with decentralized planning.

We will again consider the small illustrative problem of Section 4.1, which gave rise to the following multiplant model:

$$\begin{array}{lllr}
\text{Maximize} & \text{Profit} & 10x_1 + 15x_2 + 10x_3 + 15x_4 & \\
\text{subject to} & \text{Raw} & 4x_1 + 4x_2 + 4x_3 + 4x_4 & \leq 120, \\
 & \text{Grinding A} & 4x_1 + 2x_2 & \leq 80, \\
 & \text{Polishing A} & 2x_1 + 5x_2 & \leq 60, \\
 & \text{Grinding B} & 5x_3 + 3x_4 & \leq 60, \\
 & \text{Polishing B} & 5x_3 + 6x_4 & \leq 75,
\end{array}$$
$$x_1, x_2, x_3, x_4 \geq 0.$$

It was seen that splitting the 120 kg of *raw* between A and B in the ratio 75 : 45 led to a non-optimal overall solution. The optimal overall solution showed that this ratio should be 70 : 50. Unfortunately, we had to solve the total model in order to find this optimal split. If we had a method of predetermining this optimal split, we would be able to solve the individual models for factories A and B and combine the solutions to give an optimal solution for the total model. For a general block angular model, as illustrated in Figure 4.2, we would need to find optimal splits in all the right-hand side coefficients in b_0 for the common rows. Algorithms for the decomposition of a block angular structure based on this principle do exist. Such methods are known as *decomposition by allocation*. One such algorithm is the method of Rosen (1964).

An alternative approach is *decomposition by pricing*. In a block angular structure, such as our small example above where common rows represent constraints on raw material availability, we could try to seek valuations for the limited raw material. These valuations could be used as *internal prices* to be charged to the submodels. If accurate valuations could be obtained, we might hope to get each submodel optimizing to the overall benefit of the total model. One such approach to this is the *Dantzig–Wolfe decomposition algorithm*. A full description of the algorithm is given in Dantzig (1963). We provide a less rigorous description here paying attention to the economic analogy with decentralized planning.

To illustrate the method, we again use the small two-factory example. If it were not for the raw material being in limited supply, we would have the following submodels for A and B:

$$\begin{array}{lllr}
\text{Maximize} & \text{Profit A} & 10x_1 + 15x_2 & \\
\text{subject to} & \text{Grinding A} & 4x_1 + 2x_2 & \leq 80, \\
 & \text{Polishing A} & 2x_1 + 5x_2 & \leq 60,
\end{array}$$
$$x_1, x_2 \geq 0;$$

$$\begin{array}{lll} \text{Maximize} & \text{Profit B} & 10x_3 + 15x_4 \\ \text{subject to} & \text{Grinding B} & 5x_3 + 3x_4 \leq 60, \\ & \text{Polishing B} & 5x_3 + 6x_4 \leq 75, \\ & & x_3, x_4 \geq 0. \end{array}$$

These submodels should not be confused with the submodels for the same problem in Section 4.1. The raw availability constraints were included in both submodels with a suitable allocation of raw material between them. Here, we are not including such constraints. Instead, an attempt is made to find a suitable 'internal price' for *raw* and to incorporate this into the submodels. Suppose that *raw* were to be internally priced at £p per kilogram. The objective functions for the above submodels would then become

$$\text{Profit A} \;\; (10–4p)\,x_1 + (15 - 4p)\,x_2 \tag{4.5}$$

and

$$\text{Profit B} \;\; (10–4p)\,x_3 + (15 - 4p)\,x_4. \tag{4.6}$$

In effect, we have taken multiples of p times the raw material availability constraints and subtracted them from the objectives. If p is set too low, we might find that the combined solutions to the submodels use more raw than is available in which case p should be increased. For example, if p is taken as zero (there is no internal charge for *raw*), we obtain the following optimal solutions:

Factory A's Optimal Solution with Raw Valued at 0

Profit is £250 obtained from making 17.5 standard and 5 of deluxe.

Factory B's Optimal Solution with Raw Valued at 0

Profit is £187.5 obtained from making 12.5 of deluxe.

These solutions are clearly unacceptable to the company as a whole, as they demand 140 kg of raw, which is more than the 120 kg available. We, therefore seek some way of estimating a more realistic value for the internal price p. Whatever the value of p, A and B will have optimal solutions that are vertex solutions of the submodels presented above. As these models only involve two variables each, they can be represented graphically. This is shown in Figures 4.6 and 4.7.

With the value of zero for p, we can easily verify the optimal solutions above for A and B as being (17.5, 5) in Figure 4.6 and (0, 12.5) in Figure 4.7.

Any feasible solution to the total problem must clearly be feasible with respect to both subproblems (as well as additionally satisfying raw material availability limitation). The values of x_1 and x_2 in any feasible solution to the total problem

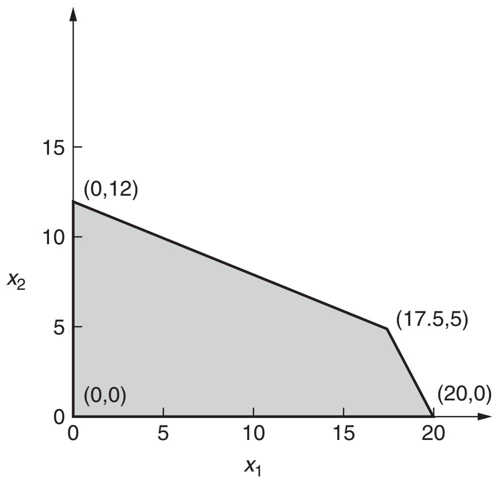


Figure 4.6

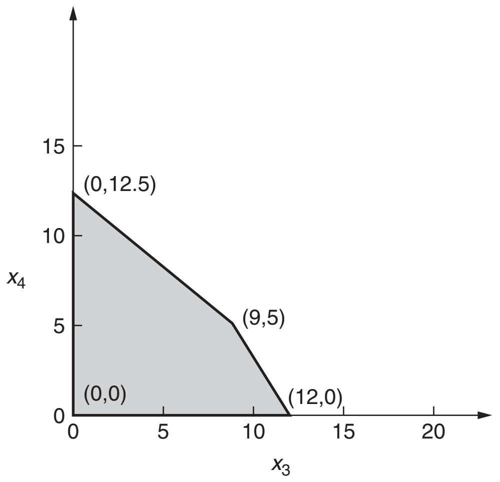


Figure 4.7

must, therefore, be a *convex linear combination* of the vertices of the feasible region shown in Figure 4.6, that is,

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \lambda_{11} \begin{pmatrix} 0 \\ 0 \end{pmatrix} + \lambda_{12} \begin{pmatrix} 20 \\ 0 \end{pmatrix} + \lambda_{13} \begin{pmatrix} 17.5 \\ 5 \end{pmatrix} + \lambda_{14} \begin{pmatrix} 0 \\ 12 \end{pmatrix}. \tag{4.7}$$

$\lambda_{11}, \lambda_{12}, \lambda_{13}$, and λ_{14} are 'weights' attached to the vertices. They must be non-negative and satisfy the convexity condition:

$$\lambda_{11} + \lambda_{12} + \lambda_{13} + \lambda_{14} = 1. \tag{4.8}$$

The vector Equation (4.7) is a way of relating x_1 and x_2 to a new set of variables λ_{ij} by the following two equations:

$$x_1 = 0\lambda_{11} + 20\lambda_{12} + 17.5\lambda_{13} + 0\lambda_{14}, \tag{4.9}$$

$$x_2 = 0\lambda_{11} + \ \ 0\lambda_{12} + \quad 5\lambda_{13} + 12\lambda_{14} \tag{4.10}$$

A similar argument can be applied to the second subproblem represented in Figure 4.7. This allows x_3 and x_4 to be related to yet more variables $\lambda_{21}, \lambda_{22}, \lambda_{23}$, and λ_{24} by the following equations:

$$x_3 = 0\lambda_{21} + 12\lambda_{22} + 9\lambda_{23} + \quad 0\lambda_{24}, \tag{4.11}$$

$$x_4 = 0\lambda_{21} + \ \ 0\lambda_{22} + 5\lambda_{23} + 12.5\lambda_{24}, \tag{4.12}$$

$$\lambda_{21} + \quad \lambda_{22} + \ \lambda_{23} + \quad \lambda_{24} = 1. \tag{4.13}$$

λ_{2j} are ‘weights’ for vertices in the second subproblem while λ_{2j} are ‘weights’ in the first subproblem.

It is worth pointing out that what we are really doing is giving *modal formulations*, as described in Section 3.4, for each subproblem.

A slight complication arises if the feasible regions of some of the submodels are ‘open’, for example, we have the situation shown in Figure 4.8. This complication is easily dealt with and fully explained in Dantzig (1963).

We can use Equations (4.9)–(4.12) to substitute for x_1, x_2, x_3 and x_4 in the objective and single common constraint raw of our total model. The grinding and polishing constraints of the two subproblems will be satisfied so long as the λ_{ij} are non-negative and satisfy the convexity constraints, Equations (4.8)

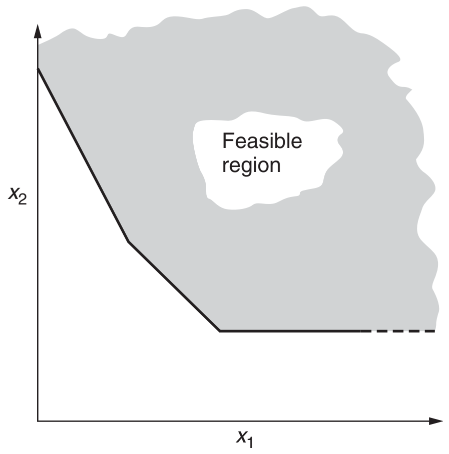


Figure 4.8

and (4.13). In this way, our multiplant model can be re-expressed as

$$\begin{array}{lrcl}
\text{Maximize} & & & \\
\text{Profit} & 200\lambda_{12} + 250\lambda_{13} + 180\lambda_{14} + 120\lambda_{22} + 165\lambda_{23} + 187.5\lambda_{24} & & \\
\text{subject to} & & & \\
\text{Raw} & 80\lambda_{12} + 90\lambda_{13} + 48\lambda_{14} + 48\lambda_{22} + 56\lambda_{23} + 50\lambda_{24} & \leq & 120 \\
\text{conv 1} & \lambda_{11} + \lambda_{12} + \lambda_{13} + \lambda_{14} & = & 1, \\
\text{conv 2} & \lambda_{21} + \lambda_{22} + \lambda_{23} + \lambda_{24} & = & 1.
\end{array}$$

The above model is known as the *master model*. It can be interpreted as a model to find the 'optimum mix' of vertex solutions of each of the submodels. For any internal price p, which is charged to A and B, they will each produce vertex solutions. Such vertex solutions give rise to what are known as *proposals*, as they represent 'proposed' solutions from the submodels, given the provisional internal price p for raw. The proposals are the columns of coefficients in the master model corresponding to a particular vertex of a submodel. For example, the proposal from the third vertex of the first subproblem is the column

$$\begin{pmatrix} 250 \\ 90 \\ 1 \\ 0 \end{pmatrix}.$$

This proposal is given a weight of λ_{13} in the master model. The role of the master model is to choose the best combination of all the proposals that have been obtained.

The master model would generally have far fewer constraints than the original model. There will be the same number of common rows as in the total model. Each submodel will, however, have been condensed down into a single convexity constraint such as conv 1 in the example above. Unfortunately, the saving-in constraints will generally be more than offset by a vast expansion in the number of variables. We will have a λ_{ij} variable for each vertex of each submodel. This could be an astronomical number in a realistic problem. In practice, the great majority of proposals corresponding to these variables will be zero in an optimal solution. For a master model with a comparatively small number of constraints, but a very large number of variables, the great majority of variables will never enter a solution. We therefore resort to a practice that is used quite widely in mathematical programming. *Columns are generated in the course of optimization*. A column (proposal) is added to the master model only when it seems worthwhile. This is an example of *column generation*, which is discussed in Section 9.6. We therefore deal only with a subset of the possible proposals. Such a truncated model is known as a *restricted master model*. Proposals are added to (and sometimes deleted from) the restricted master model in the course of optimization. In general, only a very small number of the potential proposals will ever be generated and added to the restricted master problem.

In order to describe how we proceed, we will consider our small multiplant model. Instead of using the master model, which we were fortunate enough to be able to obtain from geometrical considerations in so small an example, we will work with a restricted master problem. To start with we will take only the proposals corresponding to λ_{11} and λ_{13} from submodel A and λ_{21} and λ_{24} from submodel B. This choice is largely arbitrary. How it is made is not important to our discussion. In practice, a number of 'good' proposals from each submodel would be used to make up the first version of the restricted master model in order to have a reasonably realistic model with some substance. Our first restricted master model is therefore

$$\begin{array}{llrrrrrl}
\text{Maximize} & \text{Profit} & & 250\lambda_{13} & & +\,187.5\lambda_{24} & & \\
\text{subject to} & \text{Raw} & & 90\lambda_{13} & & +\quad 50\lambda_{24} & \leq & 120, \\
 & \text{conv 1} & \lambda_{11}+ & \lambda_{13} & & & = & 1, \\
 & \text{conv 2} & & & \lambda_{21}+ & \lambda_{24} & = & 1.
\end{array}$$

When this model is optimized, we can obtain a 'valuation' for the raw material. This valuation is the *marginal value* of the raw constraint in the optimal solution. Such marginal values for constraints are sometimes known as *shadow prices*. They are discussed much more fully together with their economic interpretation in Section 6.2. In fact, the marginal value associated with a constraint such as raw is the rate at which the optimal profit would increase for small ('marginal') increases in the right-hand side coefficient. It is sufficient for our purpose here simply to remark that such valuations for constraints are possible. With any package programs, these valuations (shadow prices) are presented with the optimal solution.

If our first restricted master model is optimized, the shadow price on the raw constraint turns out to be £2.78. This can be taken as value p and used as an internal price, which factories A and B are charged for each kilogram of *raw* that they wish to use. When A is charged, this internal price it will reform its objective function to take account of the new charge. The new objective function comes from the expression (4.5) taking p as £2.78. This gives

$$\text{Profit A} - 1.12x_1 + 3.18x_2. \tag{4.14}$$

If this objective is used with the constraints of the submodel for A, we obtain the solution

$$x_1 = 0, \qquad x_2 = 12.$$

This clearly corresponds to the vertex (0, 12) in Figure 4.6. The proposal corresponding to this is the column for λ_{14}. This is easily calculated to be

$$\begin{pmatrix} 180 \\ 48 \\ 1 \\ 0 \end{pmatrix}.$$

A new variable (λ_{14} but with a different name) is therefore added to the restricted master problem with this column of coefficients. This new proposal represents factory A's new provisional production plan given the new internal charge for *raw*.

Our attention is now turned to factory B. When they are charged £2.78 per kilogram for raw, the expression (4.6) gives their objective function as

$$\text{Profit B} - 1.12x_3 + 3.18x_4. \tag{4.15}$$

When this objective is optimized with the constraints of the submodel for factory B, we obtain the solution

$$x_3 = 0, \quad x_4 = 12.5.$$

We have the vertex (0, 12.5) in Figure 4.7. The proposal corresponding to this is the column for λ_{24}. This proposal has already been included in our first restricted master model. We, therefore, conclude that even if factory B is charged at the suggested rate of £2.78 per kilogram for raw, they would not suggest a new proposal (provisional production plan).

Having added only the proposal corresponding to λ_{14} to our restricted master model, it now becomes

$$\begin{array}{lllllll}
\text{Maximize} & \text{Profit} & & 250\lambda_{13} + & 180\lambda_{14} + & 187.5\lambda_{24} & \\
\text{subject to} & \text{Raw} & & 90\lambda_{13} + & 48\lambda_{14} + & 50\lambda_{24} & \le 120, \\
 & \text{conv 1} & \lambda_{11} + & \lambda_{13} + & \lambda_{14} & & = 1, \\
 & \text{conv 2} & & & & \lambda_{21} + \lambda_{24} & = 1.
\end{array}$$

Optimizing this model, the shadow price on *raw* turns out to be £1.67. We see that our previous valuation of £2.78 appears to have been an over estimate.

The cycle is now repeated and each factory is internally charged £1.67 per kilogram for raw. This gives the following new objectives for A and B:

$$\text{Profit A} \;\; 3.32x_1 + 8.32x_2, \tag{4.16}$$

$$\text{Profit B} \;\; 3.32x_3 + 8.32x_4. \tag{4.17}$$

When the objective (4.16) is used with the constraints of the submodel for factory A, the optimal solution obtained is

$$x_1 = 17.5, \quad x_2 = 5.$$

This is the vertex (17.5, 5) in Figure 4.6 and gives the proposal corresponding to λ_{13}. As this proposal has already been incorporated in the restricted master problem, factory A has no new proposal to offer as a result of the revised internal charge of £1.67 per kilogram for raw.

Factory B optimizes objective (4.17) subject to the constraints of its submodel. This results in the solution:

$$x_3 = 0, \quad x_4 = 12.5.$$

This is the vertex (0, 12.5) of Figure 4.7, which results in the proposal corresponding to λ_{24}. As this proposal is already present in the restricted master model, factory B also has no further useful proposal to add as a result of the revised charge for raw.

We therefore conclude that factories A and B have submitted all the useful proposals that they can. The optimal solution to the latest version of the restricted master model gives the proportions in which these proposals should be used. For our example, the optimal solution to this restricted master model is

$$\lambda_{13} = 0.52, \quad \lambda_{14} = 0.48, \quad \lambda_{24} = 1.$$

This enables us to calculate the optimal values for x_1, x_2, x_3, and x_4 by considering the vertex solutions of the submodels corresponding to λ_{13} and λ_{14}, and λ_{24}. We obtain

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0.52 \begin{pmatrix} 17.5 \\ 5 \end{pmatrix} + 0.48 \begin{pmatrix} 0 \\ 12 \end{pmatrix},$$

$$\begin{pmatrix} x_3 \\ x_4 \end{pmatrix} = 1 \begin{pmatrix} 0 \\ 12.5 \end{pmatrix}.$$

This gives us the optimal solution to the total model,

$$x_1 = 9.17, \quad x_2 = 8.33, \quad x_4 = 12.5,$$

giving an objective value of £404.15.

Notice, however, that we have obtained the optimal solution to the total model without solving it directly. Instead, we have dealt with what would generally be much smaller models. The two types of model we have used are the *submodels* and the *restricted master model*. We further discuss the significance of these models below.

4.3.1 The submodels

These contain the details relevant to the individual subproblems. For a multiplant model such as that used in our example, the coefficients in the constraints concern only the particular factory, that is, grinding and polishing times and capacities in each factory.

4.3.2 The restricted master model

This is an overall model for the whole organization but, unlike the total model, it contains none of the technological detail relating to the individual subproblems. Such detail is left to the submodels. Instead, the constraints for each subproblem are accounted for by a simple convexity constraint. In our example, we had the constraints for factories A and B reducing to convexity constraints conv 1 and conv 2, respectively. On the other hand, the restricted master model does contain the common rows in full, as its main purpose is to determine suitable valuations for the resources represented by these common rows.

By means of interactions between the submodels and the restricted master model, it is possible eventually to obtain the optimal solution to the (usually much larger) total model without ever building and solving it directly. The process that we describe here is diagrammatically represented in Figure 4.9.

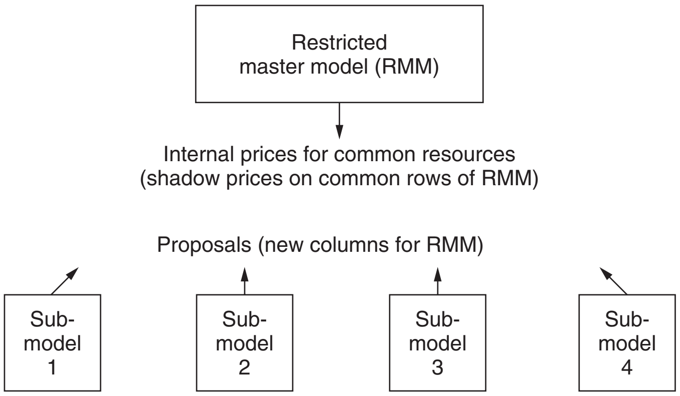

Restricted master model (RMM)
Internal prices for common resources (shadow prices on common rows of RMM)
Proposals (new columns for RMM)
Sub-model 1
Sub-model 2
Sub-model 3
Sub-model 4

Figure 4.9

There is considerable attraction in such a scheme, as it is never necessary to build, solve, and maintain the often huge total model, which would result from a large structured organization. In a multiplant organization, the individual plants would probably be geographically separated. This would make the avoidance of including all their technological details in one central model desirable. Each plant might well build and maintain their own model inside their own plant and solve it on their own computer. The head office could maintain a restricted master model on another computer, which would be linked to the computers in the individual plants. Each model could then be run independently but could supply the vital information of *proposals* and *internal prices* to the other models. It would then be possible to use a system automatically to obtain an overall optimal solution for the organization.

Decomposition has interest for economists, as it clearly represents a system of decentralized planning. The existence of decomposition algorithms such as the

Dantzig–Wolfe algorithm demonstrates that it is possible to devise a method of decentralized planning, which achieves an optimal solution for an organization considered as a whole. This is done by allowing the suborganizations to decide their own optimal policies, given limited control from the centre. In the case of the Dantzig–Wolfe algorithm, this control takes the form of internal prices. For other methods, it may take the form of allocations. An informal version of such procedures takes place in many organizations. A discussion of a large number of decomposition algorithms and their relation to decentralized planning in real life is given by Atkins (1974).

Decomposition algorithms are also of considerable computational interest, as they offer a possibility of avoiding the large amount of time needed to solve large models should those models be structured. Unfortunately, decomposition has only met with limited computational success. The most widely used algorithm is the Dantzig–Wolfe algorithm that applies to block angular structured models. There are many circumstances, however, in which it is more efficient to solve the total model rather than to use decomposition. The main advice to someone building a structured model should not be to attempt decomposition on other than an experimental basis. As experience and knowledge grow, decomposition may well become a more reliable tool, but at present, the computational experience with such methods has been disappointing. If a model is to be used very frequently, experimenting with decomposition might be considered worthwhile. The larger the model and the smaller the proportion of common rows (for block angular structures), the more valuable this is likely to be. Sometimes, other aspects of the structure can be exploited to advantage. An example of this is described by Williams and Redwood (1974). Computational experiences using Dantzig–Wolfe decomposition are given by Beale *et al*. (1965) and Ho and Loute (1981). A very full description of the computational side of decomposition is given by Lasdon (1970). Finally, decomposition may commend itself for the purely organizational considerations resulting from the desirability of decentralized planning.

The use of communication networks for linking computers together makes it possible to implement a decomposition algorithm splitting the models between different computers, e.g. the master model in the head office and the submodels at individual plants. This idea has not, to the author's knowledge, yet been tried.

5

Applications and special types of mathematical programming model

5.1 Typical applications

The purpose of this section is to create an awareness of the areas where linear programming (LP) is applicable. To categorize totally those industries and problems where LP can, or cannot, be used would be impossible. Some problems clearly lend themselves to an LP model. For other problems, the use of an LP model may not provide a totally satisfactory solution but may be considered acceptable in the absence of other approaches. The decision of when to use, and when not to use, LP is often a subjective one depending on an individual's experience.

This section can do no more than try to give a 'feel' for those areas in which LP can be applied. In order to do this, a list of industries and areas in which the technique has been applied is given. This list is by no means exhaustive but is intended to include most of the major users. A short discussion is given of the types of LP models that are of use in each area. References are given to some of the relevant published case studies. In view of the very wide use that has been made of LP, it would be almost impossible to seek out every reference to published case studies. Nor would it be helpful to submerge the reader in a mass of often superfluous literature. The intention is to give sufficient references and allowing the reader to follow up published case studies. From the references given here, it should be possible to find other references if necessary. In many cases, practical applications are illustrated by problems in Part II.

Although the intention is mainly to consider *LP* applications in this chapter, the resultant models can very often naturally be extended by *integer programming* or *non-linear programming models*. In this way, more complicated or realistic situations can often be modelled. These topics and further applications are considered more fully in Chapters 7, 8, 9 and 10.

The subject of LP does not have clearly defined boundaries. Other subjects impinge on, and merge with, LP. Two types of model that have, to some extent, been studied independently of LP are considered further in this chapter. Firstly, *economic models*, which are sometimes referred to as *input–output* or *Leontief models*, are considered in Section 5.2. Such models can often be regarded as a special type of LP model. Secondly, *network models*, which arise frequently in operational research, are considered in Section 5.3. Such models are, again, often usefully considered as special types of LP model. Another area that also has connections with LP is not considered in this book in view of its limited practical applicability to date. This is the *theory of games*. Game theory models can sometimes be converted into LP models (and vice versa).

The following list of applications should indicate the surprisingly wide applicability of LP and, in consequence, its economic importance. Many more references exist than are given. The purpose of the references is to give a 'window' into the literature.

5.1.1 The petroleum industry

This is by far the biggest user of LP. Very large models involving thousands, and occasionally tens of thousands, of constraints have been built. These models are used to help make a number of decisions starting with where and how to buy crude oil, how to ship it and which products to produce out of it. Such 'corporate models' contain elements of *distribution, resource allocation, blending* and possibly, *marketing*. A typical example of the sort of model that arises in the industry is the REFINERY OPTIMIZATION problem of Part II. Descriptions of the use of LP in the petroleum industry are given by Manne (1956), Catchpole (1962) and McColl (1969).

5.1.2 The chemical industry

The applications here are rather similar to those in the petroleum industry although the models are rarely as large. Applications usually involve *blending* or *resource allocation*. An application is described by Royce (1970).

5.1.3 Manufacturing industry

LP is frequently used here for resource allocation. The '*product mix*' example described in Section 1.2 is an example of this type of application. Resources to be allocated are usually processing capacity, raw materials and manpower. A multi-period problem of this type, considered in relation to the engineering industry, is the FACTORY PLANNING problem of Part II. Other common applications of LP in manufacturing are *blending* and *blast furnace burdening* (the steel industry). Three references to the application of LP here are Lawrence and Flowerdew (1963), Fabian (1967) and Sutton and Coates (1981).

5.1.4 Transport and distribution

Problems of distribution can often be formulated as LP models. Two classic examples are the *transportation* and *transhipment* problems, which are considered in Section 5.3 as they involve *networks*. A simple DISTRIBUTION problem of this type is presented in Part II. An extension of this problem, DISTRIBUTION 2, involves *depot location* as well and requires integer programming. *Scheduling* problems (e.g. lorries, aircraft, tankers, trains and buses) can often be tackled through integer programming. One such problem is the MILK COLLECTION problem given in Part II. Problems of *assignment* (trains to mines and power stations) arising in transport have also been tackled through mathematical programming. A comprehensive description of the use of mathematical programming in *shipping* is given in Christiansen *et al*. (2007). Applications of mathematical programming in distribution, in general, are described by Eilon *et al*. (1971) and Markland (1975).

5.1.5 Finance

A very early application of mathematical programming was in *portfolio selection*. This was due to Markowitz (1959). Given a sum of money to invest, the problem was how to spend it among a portfolio of shares and stocks. The objective was to maintain a certain expected rate of return from the investment but to minimize the variance of that return. The model that results is a quadratic programming model.

Agarwala and Goodson (1970) suggest how LP can be used by a government to design an *optimum tax package* to achieve some required aim (in particular, an improvement in the balance of payments).

LP is increasingly being used in *accountancy*. The economic information that can be derived from the solution to an LP model can provide accountants with very useful costing information. This sort of information is described in detail in Section 6.2. A description of how LP can be used in accountancy is given by Salkin and Kornbluth (1973).

Spath *et al*. (1975) describe how an LP model is applied by a mail order firm in order to minimize the total interest cost on all credits.

An interesting extension of a finance model to allow the user to set *goals* interactively (rather than objectives) and operate within the degrees of freedom permitted by the constraints is described by Jack (1985).

A very important potential area of application is yield (revenue) management, which is concerned with setting prices for goods at different times in order to maximize revenue. It is particularly applicable to the hotel, catering, airline and train industries. An example of its use is the YIELD MANAGEMENT problem in Part II.

5.1.6 Agriculture

LP has been used in agriculture for *farm management*. Such models can be used to decide what to grow where, how to rotate crops, how to expand production and where to invest. An example of such a problem is the FARM PLANNING problem of Part II. Swart *et al*. (1975) apply a multi-period LP model to planning the expansion of a large dairy farm. Other general references are Balm (1980) and Fokkens and Puylaert (1981).

Blending models are often applicable to agriculture problems. It is often desired to blend together livestock feeds or fertilizer at minimum cost. Glen (1980) describes a model for beef cattle ration formulation.

Distribution problems often arise in this area. The distribution of farm products, in particular milk, can be examined by the network type of LP model described in Section 5.3.

Models for planning the growth and harvesting of a number of agricultural products are described by Glen (1980, 1988, 1995, 1996, 1997).

Quadratic programming has been used for determining *optimal prices* for the sale of milk in the Netherlands. This is described by Louwes *et al*. (1963). The AGRICULTURAL PRICING problem of Part II is based on this study. A mixed integer programming model for irrigation in a developing country is described by Rose (1973).

5.1.7 Health

The obvious application of mathematical programming in this area is in problems of *resource allocation*. How are scarce resources, for example, doctors, nurses, hospitals, etc., to be used to best effect? In such problems, there will obviously be considerable doubt concerning the validity of the data, for example, how much of a nurse’s time does a particular type of treatment really need? In spite of doubts concerning much of the data in such problems, it has been possible to use mathematical programming models to suggest plausible policy options. McDonald *et al*. (1974) describe a non-linear programming model for allocating resources in the UK health service. Revelle *et al*. (1969) describe how a non-linear programming model can be used for controlling tuberculosis in an underdeveloped country.

Warner and Prawda (1972) describe a mathematical programming model for scheduling nurses. Redpath and Wright (1981) describe how LP is used to decide intensity and direction of beams for irradiating cancerous tumours.

5.1.8 Mining

A number of interesting applications of mathematical programming occur in mining. The straightforward applications are simply ones of *resource allocation*, that is, how should manpower and machinery be deployed to best effect?

Blending problems also occur when it is necessary to mix together ores to achieve some required quality.

Two examples of mining problems are given in Part II. The MINING problem concerns what combination of mines a company should operate in successive years. The OPENCAST MINING problem is to decide what the boundaries of an opencast mine should be. References to the application of mathematical programming in mining are Young *et al*. (1963), Meyer (1969) and Boland *et al*. (2009).

5.1.9 Manpower planning

The possible movement of people between different types of job and its control by recruitment, promotion, retraining, etc., can be examined by linear programming. An example of such a problem, MANPOWER PLANNING, is given in Part II. Applications of mathematical programming to manpower planning are described by Price and Piskor (1972), Davies (1973), Vajda (1975), Charnes *et al*. (1975) and Lilien and Rao (1975).

5.1.10 Food

The food industry makes considerable use of LP. *Blending* (sausages, meat pies, margarines, ice cream, etc.) is an obvious application often giving rise to very small and easily solved models.

Problems of *distribution* also arise in this industry giving rise to the network type models described in Section 5.3.

As in other manufacturing industries, problems of *resource allocation* arise, which can be tackled through LP.

The FOOD MANUFACTURE problem of Part II is an example of a multi-period blending problem in the food industry. A more complicated version of this problem is described by Williams and Redwood (1974). Jones and Rope (1964) describe another LP model in the food industry.

5.1.11 Energy

The electricity and gas supply industries both use mathematical programming to deal with problems of resource allocation. The TARIFF RATES (POWER GENERATION) problem of Part II involves scheduling electric generators to meet varying loads at different times of day. This problem is similar to that described by Garver (1963). This problem is extended to the HYDRO POWER model, which illustrates another important application. Archibald *et al*. (1999) describe how a similar problem can be tackled by stochastic dynamic programming.

LP can also be applied to *distribution* problems involving the design and use of supply networks.

Applications of mathematical programming in these areas are also described by Babayer (1975), Fanshel and Lynes (1964), Muckstadt and Koenig (1977) and Khodaverdian *et al*. (1986).

5.1.12 Pulp and paper

Problems of *resource allocation* in the manufacture of paper give rise to LP models. Such models frequently involve an element of *blending*. In addition, the possibility of *recycling waste paper* has also been examined by LP as described by Glassey and Gupta (1974).

A totally different type of problem arising in the paper industry (and the glass industry) is the *trimloss* problem. This is the problem of arranging orders for rolls of paper of different widths so as to minimize the wastage. This problem can be tackled by linear (or sometimes integer) programming. This problem is described by Eisemann (1957). It has also been considered by Gilmore and Gomory (1961, 1963). It is used, in Section 9.6, to demonstrate *column generation*.

5.1.13 Advertising

The problem of spreading one's advertising budget over possible advertising outlets (television commercials, newspaper advertisements, etc.) has been approached through mathematical programming. These problems are known as *media scheduling* problems.

Authors differ over the usefulness of mathematical programming in tackling this type of problem. Selected references are Engel and Warshaw (1964), Bass and Lonsdale (1966) and Charnes *et al*. (1968). The last-mentioned reference gives a very full list of references itself.

5.1.14 Defence

Problems of *resource allocation* give rise to military applications of LP. Such an application is described by Beard and McIndoe (1970).

The siting of missile sites is described by Miercort and Soland (1971).

5.1.15 The supply chain

It is apparent, from the discussion of applications above, that many aspects of an organization are amenable to being modelled using mathematical programming. This is particularly true of the manufacturing industry but is also true of other industries such as finance and telecommunications. When optimizing the operations of only one aspect, or division, of an organization, one is in danger of losing sight of the 'bigger picture'. Furthermore, there can sometimes be a danger of damaging part of an organization's operations by optimizing the operations of another part. For example, concentrating entirely on maximizing production could result in policies that are infeasible from the point of view of distribution or marketing. What are needed are *integrated* models that link together such different aspects. There is, of course, a danger that such models could become unwieldy and unusable if approached in the wrong way (the aborting of the projected huge IT system for the British National Health Service is a stark reminder). Such major considerations form the subject of *information systems* and are beyond the scope of this book. It is usually the case that such integrated models *evolve* from existing models of smaller specialist functions. Some of the discussion in Chapter 4 is relevant to this. Advances in *information technology, for example*, communications software, database technology and 'user-friendly' interfaces have made it easier to create and use such models.

Although applicable in other areas, we illustrate our discussion of supply chain models by using examples from the manufacturing sector. The following functions are amenable to optimization:

Purchasing, for example, deciding the best suppliers and quantities of raw material to buy at the right time.

Inward Distribution, for example, deciding what vehicles to hire, of what capacities and how to route them.

Inventory Planning, for example, deciding how much to store of both inward materials and intermediate or final products.

Financial Planning, for example, deciding on whether to finance some activities by loans and what debt to incur.

Production, for example, how much to make of each product using which production/manpower capacities and whether to take on extra capacity.

Marketing, for example, what marketing outlets to use and where to deploy marketing effort.

In addition other, non-mathematical programming models may be relevant, for example, *forecasting* and *accounting* models.

A *supply chain* model integrates models for all, or some, of these functions, taking outputs from some as inputs to others, and performs a *global* optimization, for example, *to maximize total profit*.

5.1.16 Other applications

There are numerous other applications of mathematical programming. A few of the less usual ones are given here as they might otherwise go unnoticed.

Heroux and Wallace (1973) describe a multi-period LP model for land development.

Souder (1973) discusses the effectiveness of a number of mathematical programming models for research and development. Feuerman and Weiss (1973) show how a knapsack integer programming model can be used to help design multiple choice–type examinations. Kalvaitis and Posgay (1974) apply integer programming to the problem of selecting the most promising kind of mailing list.

Wardle (1965) applies LP to forestry management.

Problems of pollution control have been tackled through mathematical programming. Applications are described by Loucks *et al*. (1968).

Kraft and Hill (1973) describe a 0–1 integer programming model for selecting journals for a university library.

Jünger *et al*. (1989) and Ferreira *et al*. (1993) describe integer programming models arising in computer design.

Bollapragada *et al*. (2001) apply integer programming to the civil engineering problem of *truss design*, that is, constructing an optimal structure of trusses to support a given weight, modelling the logical dependencies of trusses.

Trick has applied integer programming extensively to *sports scheduling* (see, e.g. Nemhauser and Trick (1998)).

Chang and Sahinidis (2011), among others, have applied integer programming to *DNA sequencing*. This has similarities to the (easier) *archaeological seriation* problem, which has been considered by Laporte (1976) and Wilkinson (1971), who formulate this problem as a version of the Travelling Salesman Problem (see Chapter 9).

The use of LP for performance evaluation in a number of organizations (particularly in the public sector) through the use of a type of model known as *Data Envelopment Analysis* has become important. This is described through the EFFICIENCY ANALYSIS problem in Part II. References are Charnes *et al*. (1978), Farrell (1957), Land (1991) and Thanassoulis *et al*. (1987).

A much fuller list of papers on mathematical programming applications has been compiled by Riley and Gass (1958). The special editions of the journal *Mathematical Programming Studies* Nos 9 (1975) and 20 (1982) are devoted to applications.

5.2 Economic models

A widely used type of national economic model is the *input–output model* representing the interrelationships between the different sectors of a country's economy. Such models are often referred to as *Leontief models* after their originator, who built such a model of the American economy. This is described by Leontief (1951). Input–output models are often regarded usefully as a special type of LP model.

5.2.1 The static model

The *output* from a particular industry or a sector of an economy is often used for two purposes: (i) for immediate consumption, for example, coal to be sold to domestic consumers; (ii) as an *input* to other industries or sectors of the economy, for example, coal to provide power for the steel industry. As the outputs from many industries can be split in this way between (exogenous) consumption and as (endogenous) inputs into these same industries, a complex set of interrelationships will exist. The input–output type of model provides about the simplest way of representing these relationships. A number of strong (and usually oversimplified) assumptions are made regarding the inter-industry relationships. The two major assumptions are as follows:

1. The output from each industry is directly proportional to its inputs, for example, doubling all the inputs to an industry will double its outputs.
2. The inputs to a particular industry are all in fixed proportions, for example, it is not possible to decrease one input and compensate for this by increasing another input. These fixed proportions are determined by the technology of the production process. In other words, there is *non-substitutability* of inputs.

In order to demonstrate such a model, we will consider a very simple example.

Example 5.1: A Three-industry Economy

We suppose that we have an economy made up of only three types of industry: coal, steel and transport. Part of the outputs from these industries are needed as inputs to others, for example, coal is needed to fire the blast furnaces that produce steel, steel is needed in the machinery for extracting coal, etc. The necessary inputs to produce 1 unit of output for each industry are given in the *input–output* matrix in Table 5.1.

Table 5.1 An input–output matrix

Inputs	Outputs		
	Coal	Steel	Transport
Coal	0.1	0.5	0.4
Steel	0.1	0.1	0.2
Transport	0.2	0.1	0.2
Labour	0.6	0.3	0.2

It is usual in such tables to measure all units of production in monetary terms. We then see that, for example, to produce £1 in worth of coal requires £0.1 of coal (to provide the necessary power), £0.1 of steel (the steel 'used up' in the 'wear and tear' on the machinery) and £0.2 of transport (for moving the coal from the mine). In addition, £0.6 of labour is required. Similarly, the other columns of Table 5.1 give the inputs required (£s) for each pound of steel and each pound of transport (lorries, cars, trains, etc.).

Notice that the value of each unit of output is exactly matched by the sum of the values of its inputs.

This economy is assumed to be '*open*' in the sense that some of the output from the above three industries is used for exogenous consumption. We will assume that these 'external' requirements are (in £ millions)

Coal	20
Steel	5
Transport	25

Such a set of exogenous demands is known as a *bill of goods*.

A number of questions naturally arise concerning our economy, which a mathematical model might be used to answer:

1. How much should each industry produce in total in order to satisfy a given bill of goods?
2. How much labour would this require?
3. What should the price of each product be?

If variables x_c, x_s and x_t are used to represent the total quantities of coal, steel and transport produced (in a year) we get the following relationships:

$$x_c = 20 + 0.1x_c + 0.5x_s + 0.4x_t, \tag{5.1}$$

$$x_s = 5 + 0.1x_c + 0.1x_s + 0.2x_t, \tag{5.2}$$

$$x_t = 25 + 0.2x_c + 0.1x_s + 0.2x_t. \tag{5.3}$$

For example, Equation (5.1) tells us that we must produce enough coal to satisfy external demand (£20m), input to the coal industry ($0.1x_c$), input to the steel industry ($0.5x_s$) and input to the transport industry ($0.4x_t$)

Equations (5.1), (5.2) and (5.3) can conveniently be rewritten as

$$0.9x_c - 0.5x_s - 0.4x_t = 20, \tag{5.4}$$

$$-0.1x_c + 0.9x_s - 0.2x_t = 5, \tag{5.5}$$

$$-0.2x_c - 0.1x_s + 0.8x_t = 25. \tag{5.6}$$

Such a set of equations in the same number of unknowns can generally be solved uniquely. In this case, we would obtain the solution

$$x_c = 56.1, \quad x_s = 22.4, \quad x_t = 48.1.$$

The total labour requirement can then easily be obtained as

$$0.6 \times 5.61 + 0.3 \times 22.4 + 0.2 \times 48.1 = 50.$$

Clearly Equations (5.4)–(5.6) could be regarded as the constraints of an LP model. An objective function could be constructed and we could maximize it or minimize it subject to the constraints. As the model stands, however, there would be little point in doing this as there is generally only one feasible solution. The objective function would, therefore, have no influence on the solution.

Once, however, we extend this very simple type of input–output model, we frequently obtain a genuine LP model.

The model described above is unrealistic in a number of respects. Equations (5.4)–(5.6) give no real limitation to the productive capacity of the economy. It can fairly easily be shown that so long as a particular, positive, bill of goods can be produced (the economy is a 'productive' one), then these relationships guarantee that any bill of goods, however large, can be produced. This is clearly unrealistic. Firstly, we would expect some limitation on productive capacity preventing more than a certain amount of output from each industry in a given period of time. Secondly, we would expect the output from an industry only to be effective as the input to another industry after a certain time has elapsed. This second consideration leads to *dynamic input–output models*, which we consider below. Before doing this, however, we will consider the problem of modelling limited productive capacity in the case of a *static* model.

In our small example, we assumed that once we had decided how much each industry should produce in order to meet a specified bill of goods, we could provide the labour required. If labour were in short supply, it might limit our productive capacity. There would then be interest in seeing what bills of goods are or are not producible in a particular period of time. Returning to our example, if we were to limit labour to 40 (£m per year) we could not produce our previous bill of goods. But what bill of goods could we produce? Answers to this question can be explored through LP. Variables will now represent our bill of goods:

Coal	y_c
Steel	y_s
Transport	y_t

Equations (5.4)–(5.6) will give the constraints

$$0.9x_c - 0.5x_s - 0.4x_t - y_c = 0, \tag{5.7}$$

$$-0.1x_c + 0.9x_s - 0.2x_t - y_s = 0, \tag{5.8}$$

$$-0.2x_c - 0.1x_s + 0.8x_t - y_t = 0. \tag{5.9}$$

The labour limitation gives the constraint

$$0.6x_c + 0.3x_s + 0.2x_t \leq 40. \tag{5.10}$$

Achievable bills of goods will be represented by the values of y_c, y_r and y_t in feasible solutions to Equations (5.7)–(5.10). Specific solutions can be found by introducing an objective function. For example, we might wish to maximize the total output:

$$x_c + x_s + x_t. \tag{5.11}$$

Alternatively, we might weight some outputs more heavily than others by giving x_c, x_s and x_t different objective coefficients. We might wish simply to maximize production in one particular sector of the economy, such as steel, and simply maximize x_s. This is clearly a situation of the type referred to in Section 3.2 in which it is of interest to experiment with a number of different objectives rather than simply concentrate on one.

We have considered only labour as a limiting factor in productive capacity. In practice, there could well be other resource limitations such as processing capacity, raw material, etc. Such limitations could, of course, easily be incorporated in a model by extra constraints. Limited resources of this sort are sometimes known as *primary goods*. Primary goods only provide inputs to the economy. They are not produced as outputs as well. A major advantage of treating such models as LP models is that a lot of subsidiary economic information is also obtained from solving such a model. Such information is described very fully in Section 6.2. In particular, valuations are obtained for the constraints of a model. These valuations are known as *shadow prices*. For the type of model considered here, we would obtain meaningful valuations for the primary goods. In this way, a pricing system could be introduced into our model. This would give suitable prices for the outputs from all the industries.

Although any number of primary goods can be considered in an LP formulation of an input–output model, it is quite common only to consider labour. In practice, particularly in the relatively simple economies of developing countries, it may well not be unreasonable to consider labour as the overall limitation. If this can be done, there is another less obvious advantage to be gained in the applicability of such a model. It has already been pointed out that an input–output model assumes *non-substitutability* of the inputs, that is, it is not possible to vary the relative proportions in which all the inputs are used to produce the output of a particular industry. In practice, this might well be a far from realistic assumption. For example, we might well be able to produce each unit of coal by using more power (more coal) and less machinery (less steel). To model this possibility would require a variation in the coefficients of the input–output matrix. It has been shown that if there is only one primary good (usually labour), then it will only be worthwhile to concentrate on one production process for each industry. This is the result of the *Samuelson substitution theorem*, which we will not prove. Such theoretical results and a fuller description of input–output models are given in Dorfman *et al*. (1958). Another good reference is Shapiro (1979). The importance of this result is that we need not worry about the apparent non-substitutability limitation so long as we only have one primary good. There will be one, and only one, best set of inputs (production processes) for each industry. This best production process will remain the best no matter what bill of goods we are producing. There is, of course, the problem of finding, for each industry, the production process which should be used. Once, however, this has been done, no matter what the bill of goods, we need only incorporate this one production process (column of the input–output matrix) into all future models. In fact, the finding of the best production process for each industry can be done by LP.

Table 5.2 An input–output matrix with alternative production processes

Inputs	Outputs					
	Coal		Steel		Transport	
Coal	0.1	0.2	0.5	0.6	0.4	0.6
Steel	0.1	–	0.1	0.1	0.2	0.2
Transport	0.2	0.1	0.1	–	0.2	0.05
Labour	0.6	0.7	0.3	0.3	0.2	0.15

To illustrate how this may be done as well as illuminating the import of the Samuelson substitution theorem, we will extend our small example. Table 5.2 gives two possible sets of inputs (production processes) to produce 1 unit of the three industries.

In practice, there might be many more (possibly an infinite number) than two processes for each industry.

On the face of it, we might think it advantageous to use some combination of the two processes for producing coal. The first process is more economical on coal but uses some steel as well, which the second process does not. Similarly, some mixture of the two processes for producing steel and the two processes for producing transport might seem appropriate. Moreover, which processes are used might seem likely to depend upon the particular bill of goods.

We repeat, however, that our intuitive idea would be false. There will be exactly one best process for each industry and this will be used whatever bill of goods we have. Instead of the variables x_c, x_s and x_t in our original model we can introduce variables $x_{c1}, x_{c2}, x_{s1}, x_{s2}, x_{t1}$ and x_{t2} to represent the total quantities of coal, steel and transport produced by each process. Using the same bill of goods as before, instead of constraints (5.1)–(5.3), we obtain

$$x_{c1} + x_{c2} = 20 + 0.1x_{c1} + 0.2x_{c2} + 0.5x_{s1} + 0.6x_{s2} + 0.4x_{t1} + 0.6x_{t2}, \quad (5.12)$$

$$x_{s1} + x_{s2} = 5 + 0.1x_{c1} + 0.0x_{c2} + 0.1x_{s1} + 0.1x_{s2} + 0.2x_{t1} + 0.2x_{t2}, \quad (5.13)$$

$$x_{t1} + x_{t2} = 25 + 0.2x_{c1} + 0.1x_{c2} + 0.1x_{s1} + 0.0x_{s2} + 0.2x_{t1} + 0.05x_{t2}, \quad (5.14)$$

These equations can be written as

$$0.9x_{c1} + 0.8x_{c2} - 0.5x_{s1} - 0.6x_{s2} - 0.4x_{t1} - 0.6x_{t2} = 20, \quad (5.15)$$

$$-0.1x_{c1} - 0.0x_{c2} + 0.9x_{s1} + 0.9x_{s2} - 0.2x_{t1} - 0.2x_{t2} = 5, \quad (5.16)$$

$$-0.2x_{c1} - 0.1x_{c2} - 0.1x_{s1} - 0.0x_{s2} + 0.8x_{t1} + 0.95x_{t2} = 25. \quad (5.17)$$

We are considering labour as the only primary good and will limit ourselves to 60 (£m). This gives the constraint

$$0.6x_{c1} + 0.7x_{c2} + 0.3x_{s1} + 0.3x_{s2} + 0.2x_{t1} + 0.15x_{t2} \leq 60. \quad (5.18)$$

There will generally be more than one solution to a system such as this. In order to find the 'best' solution we will define an objective function. One possible objective function would, of course, be to ignore constraint (5.18) and minimize the expression on the left-hand side representing labour usage. Alternatively, we might specify another objective function. For this example, we will do this and simply maximize total output:

$$x_{c1} + x_{c2} + x_{s1} + x_{s2} + x_{t1} + x_{t2}. \tag{5.19}$$

Our resultant optimal solution gives

$$\begin{aligned} x_{c1} &= 64.6, \\ x_{c2} &= 0, \\ x_{s1} &= 22.6, \\ x_{s2} &= 0, \\ x_{t1} &= 0, \\ x_{t2} &= 44.6. \end{aligned}$$

Notice that the Samuelson substitution theorem has worked in this case. One process in each industry is the best to the total exclusion of all the others. Moreover, it could be shown that these processes will be the best no matter what the bill of goods is. The optimal solution will be made up of only the variables x_{c1}, x_{s1} and x_{t2}, no matter what the right-hand side coefficients in constraints (5.15)–(5.18) are. We could, therefore, confine all our attention to these processes and ignore the others. It should be noted, however, that these 'best' processes are only the best because of the objective function (5.19) that we have chosen. If instead of maximizing output, we were to choose another objective, it might be preferable to switch to another process in some cases. It will never, however, be worth 'mixing' processes. Once we consider more than a single primary good such mixing may well, however, become desirable.

5.2.2 The dynamic model

We have pointed out that our static model assumed that we could ignore time lags between an output being produced and used as an input to another (or the same) industry. This unrealistic assumption can be avoided by introducing a *dynamic model*. It has already been shown in Section 4.1 that LP models can often be extended to multi-period models. We can do much the same thing with the static type of input–output model. In practice, some of the output from an economy will immediately be consumed (e.g. cars for private motoring) while some will go to increase productive capacity (e.g. factory machinery). Such alternative uses for the output will result in different possible growth patterns for the economy, that is, we can live well now but neglect to invest for the future or we can sacrifice present day consumption in the interests of future wealth-producing capacity. A simple example of such a problem, ECONOMIC PLANNING, leading to a dynamic input–output model is given in Part II. Rather than discuss Dynamic input–output models, the discussion is taken up with the formulation of this problem in Part III. A description of dynamic input–output models of this type is given by Wagner (1957).

5.2.3 Aggregation

To sum up the characteristics of a whole industry or sector of an economy in one column of an input–output matrix obviously requires a large amount of simplification of the real situation. It is necessary to group together many different industries into one. This *aggregation* is necessary in order to obtain a reasonable size of problem. Most input–output models are aggregated into less than 1000 industries. The problem of aggregation is obviously of paramount concern to the model builder. Unfortunately, very little theoretical work has been done to indicate *mathematically* when aggregation is and is not justified. Three criteria that common sense would suggest to be good grounds for aggregating particular industries are (i) substitutability, (ii) complementarity and (iii) similarity of production processes. Problems of this sort are discussed more fully by Stone (1960).

In view of their sophistication and efficiency, commercial mathematical programming packages provide a useful way of solving input–output models. Even if the model is of the simplest kind described above and only requires the solution of a set of simultaneous equations, such packages are of use. Almost all packages contain an inversion routine, which is very useful for inverting large matrices (sets of simultaneous equations). It should, however, be pointed out that input–output models are often quite dense. In this respect, they are untypical of general LP models. As already mentioned in Section 2.1, in a model with 1000 constraints, one would expect only about 1% of the coefficients to be non-zero. For an input–output model, this figure could well be as high as 50%. As a result, input–output models can take a long time to solve on a computer and run into numerical difficulties. It is sometimes worth exploiting the special structure of an input–output LP model and using a special purpose algorithm. Dantzig (1955) describes how the simplex algorithm can be adapted to this purpose.

5.3 Network models

The use of models involving networks is very widespread in operational research. Problems involving distribution, assignment and planning (critical path analysis and PERT) frequently give rise to the analysis of networks. Many of the resultant problems can be regarded as special types of LP problem. It is often more efficient to use special purpose algorithms rather than the revised simplex algorithm. Nevertheless, it is important for the model builder to be aware when he/she is dealing with a special kind of LP model. In order to solve the model it may be useful for the builder to adapt the simplex algorithm to suit the special structure. It may even, sometimes, be worth ignoring the special structure and using a general purpose package program. As such programs are often highly efficient and well-designed, their speeds outweigh the algorithmic efficiency of less well-designed but more specialized programs.

It is not intended that the coverage of this topic be comprehensive. The main aim is simply to show the connection between network models and LP models. References are given to much fuller treatments of the subject.

5.3.1 The transportation problem

This famous type of problem first described by Hitchcock (1941) is usefully regarded as one of obtaining the minimum cost flow through a special type of network.

Suppose that a number of suppliers ($S_1, S_2, \ldots, S_m$) are to provide a number of customers ($T_1, T_2, \ldots, T_n$) with a commodity. The transportation problem is how to meet each customer's requirement, while not exceeding the capacity of any supplier, at minimum cost. Costs are known for supplying 1 unit of the commodity from each S_i to each T_j. In some cases, it may not be possible to supply a particular customer T_j from a particular supplier S_i. It is sometimes useful to regard these costs as infinite in such cases. In distribution problems, these costs will often be related to the distances between S_i and T_j. It is assumed that the capacity of each supplier (over some period such as a year) is known and the requirement of each customer T_j is also known. In order to describe the problem further, we will consider a small numerical example.

Example 5.2: A Transportation Problem

Three suppliers (S_1, S_2, S_3) are used to provide four customers (T_1, T_2, T_3, T_4) with their requirements for a particular commodity over a year. The yearly capacities of the suppliers and requirements of the customers are given below (in suitable units)

Suppliers	S_1	S_2	S_3	
Capacities (per year)	135	56	93	
Customers	T_1	T_2	T_3	T_4
Requirements (per year)	62	83	39	91

The unit costs for supplying each customer from each supplier are given in Table 5.3 (in pounds per unit).

We can easily formulate this problem as a conventional LP model by introducing variables x_{ij} to represent the quantity of the commodity sent from S_i to

Table 5.3

Supplier	Customer			
	T_1	T_2	T_3	T_4
S_1	132	–[a]	97	103
S_2	85	91	–	–
S_3	106	89	100	98

[a]A dash indicates the impossibility of certain suppliers for certain depots or customers.

T_j in a year. The resultant model is

Minimize

$$132x_{11} + Mx_{12} + 97x_{13} + 103x_{14} + 85x_{21} + 91x_{22} + Mx_{23} + Mx_{24} + 106x_{31} + 89x_{32} + 100x_{33} + 98x_{34} \tag{5.20}$$

subject to

$$x_{11} + x_{12} + x_{13} + x_{14} \leq 135, \tag{5.21}$$

$$x_{21} + x_{22} + x_{23} + x_{24} \leq 56, \tag{5.22}$$

$$x_{31} + x_{32} + x_{33} + x_{34} \leq 93, \tag{5.23}$$

$$x_{11} + x_{21} + x_{31} = 62, \tag{5.24}$$

$$x_{12} + x_{22} + x_{32} = 83, \tag{5.25}$$

$$x_{13} + x_{23} + x_{33} = 39, \tag{5.26}$$

$$x_{14} + x_{24} + x_{34} = 91, \tag{5.27}$$

$$x_{ij} \geq 0, \quad \text{all } i, j.$$

This model obviously has a very special structure to which we will refer later. Notice that we have included variables for non-allowed routes in the model with objective coefficients M (some very large number). This has been done simply to preserve the pattern of the model. In practice, if we were to solve the model as an LP problem of this form, we would simply leave these variables out.

Constraints (5.21)–(5.23) are known as *availability constraints*. There is one such constraint for each of the three suppliers. These constraints ensure that the total quantity out of a supplier (in a year) does not exceed his capacity. Constraints (5.24)–(5.27) are known as *requirement constraints*. These constraints ensure that each customer obtains his requirement. In some formulations of the transportation problem, constraints (5.21)–(5.23) are treated as '$=$' instead of '$\leq$'. If the sum total of the availabilities exactly matches the sum total of the requirements, then this is acceptable as all capacities must obviously be completely exhausted. In a case such as our numerical example, however, this is not so. Total capacity (284) exceeds total demand (275). This can be coped with by introducing a dummy customer T_5 with a requirement for the excess of 9. If the cost of meeting this requirement of T_5 from each supplier S_i, is made zero we have equated total capacity to total demand with no inaccuracy in our modified model. The three constraints (5.21)–(5.23) could then be made '='. When special algorithms are used to solve the transportation problem, the employment of devices such as this is sometimes necessary. For a conventional LP formulation of the problem this is not necessary. For a general transportation problem with m suppliers $(S_1, S_2, \ldots, S_m)$ and n customers $(T_1, T_2, \ldots, T_n)$, there will be m availability constraints and n requirement constraints giving a total of $m+n$ constraints. If each supplier can be potentially used for each customer there will be mn variables in the LP model. Clearly for practical problems involving large numbers of suppliers and customers, the LP model could be very large. This is one motive for using special algorithms.

The above problem can be looked at graphically as illustrated in Figure 5.1.

In the network of Figure 5.1, we have the suppliers S_1, S_2 and S_3 and the five customers T_1, T_2, T_3, T_4, and T_5 (including the dummy customer). S_i and T_j provide the *nodes* of the network to which we have attached the (positive) capacities or (negative) requirements. The possible supply patterns S_i to T_j provide the *arcs* of the network to which we have attached the unit supply costs. Our problem can now be regarded more abstractly as one where we wish to obtain the

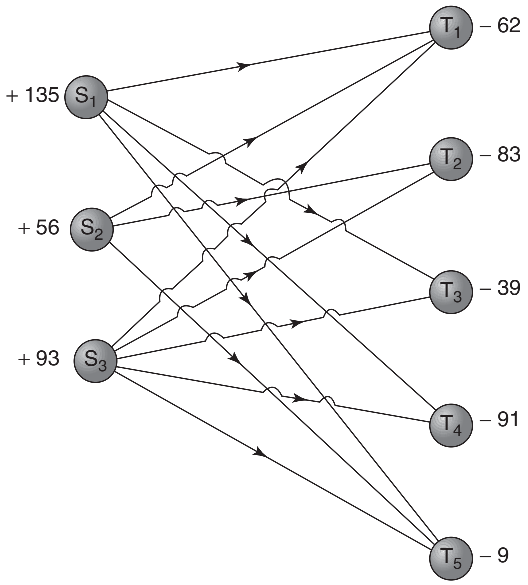


Figure 5.1

minimum cost flow through the network. The S_i nodes are 'sources' for the flow entering the system and the T_j nodes are 'sinks' where flow leaves the system. We must ensure that there is continuity of flow at each node (total flow in equals total flow out). These conditions give rise to material balance constraints of the type discussed in Section 3.3.

If x_{ij} represents the quantity of flow in the arc i to j we obtain the following constraints:

$$\begin{array}{lllll}
-x_{11} - x_{13} - x_{14} - x_{15} & & & = -135, & (5.28)\\
 & -x_{21} - x_{22} - x_{25} & & = -56, & (5.29)\\
 & & -x_{31} - x_{32} - x_{33} - x_{34} - x_{35} & = -93, & (5.30)\\
x_{11} & +x_{21} & +x_{31} & = 62, & (5.31)\\
 & x_{22} & +x_{32} & = 83, & (5.32)\\
x_{13} & & +x_{33} & = 39, & (5.33)\\
x_{14} & & +x_{34} & = 91, & (5.34)\\
x_{15} & +x_{25} & +x_{35} & = 9. & (5.35)
\end{array}$$

These constraints are clearly equivalent to the constraints (5.21) to (5.27). We have, however, added the dummy customer T_5. This has resulted in the additional variables x_{15}, x_{25} and x_{35}, and an added constraint (5.35), but allowed us to deal entirely with '=' constraints. We have also revised the signs on both sides of the availability constraints. For more general minimum cost flow problems, which we consider later, it is convenient to give negative coefficients to flows *out* of a node and positive coefficients to flows *in*. Therefore, this convention has been applied here.

The transportation problem also arises in less obvious contexts than distribution. We give a numerical example below of a production planning problem.

Example 5.3: Production Planning

A company produces a commodity in two shifts (regular working and overtime) to meet known demands for the present and future. Over the next four months, the production capacities and demands (in thousands of units producible) are

	January	February	March	April
Regular working	100	150	140	160
Overtime	50	75	70	80
Demand	80	200	300	200

The cost of production of each unit is £1 if done in regular working or £1.50 if done in overtime. Units produced can be stored before delivery at a cost of £0.30 per month per unit.

The problem is how much to produce each month to satisfy present and future demand.

It is convenient to summarize the costs in Table 5.4 (in £).

Table 5.4

Production		Demand			
		January	February	March	April
January	Regular	1	1.3	1.6	1.9
	Overtime	1.5	1.8	2.1	2.4
February	Regular	–	1	1.3	1.6
	Overtime	–	1.5	1.8	2.1
March	Regular	–	–	1	1.3
	Overtime	–	–	1.5	1.8
April	Regular	–	–	–	1
	Overtime	–	–	–	1.5

Clearly it is impossible to produce for demand of an earlier month. This is represented by a dash in the positions indicated (an infinite unit cost). The other unit costs arise from a combination of production and storage costs, for example, production in January by overtime working for delivery in March gives a unit cost of £1.50 (production) + £0.60 (storage) = £2.10. This cost matrix is of the same form as that given for the transportation problem in Table 5.3. Although the problem here is not one of distribution, it can still therefore be regarded as a transportation problem. In this case, there are eight sources and five sinks including the 'surplus' demand of 45 units.

Transportation problems are obviously expressed much more compactly in a square array such as Tables 5.3 and 5.4 rather than as an LP matrix. This is one virtue of using a special purpose algorithm. Dantzig (1951) uses the simplex algorithm but works within this compact format. The special structure results in the algorithm taking a particularly simple form. An alternative algorithm for the transportation problem is due to Ford and Fulkerson (1956). This algorithm is usefully thought of as a special case of a general algorithm for finding the minimum cost flow through a network. Such problems are considered below and described in Ford and Fulkerson (1962).

As a result of their special structure, transportation problems are particularly easy to solve in comparison with other LP problems of comparable size. They also have (together with some other network flow problems) the very important property that so long as the availabilities and requirements at the sources and sinks are integral, the values of the variables in the optimal solution will also be so. For example, so long as the right-hand side coefficients in the constraints (5.21)–(5.27) of the LP problem of Example 5.2 are integers, the variable values in the optimal solution will be as well. This rather surprising property of the transportation problem is computationally very important in many circumstances as it avoids the necessity of using *integer programming* to ensure that variables take integer values. As will be discussed in Chapters 8, 9 and 10, integer programming models are generally much more difficult to solve than LP models.

Sufficient conditions for a model to be expressible as a network flow problem are discussed in Section 10.1. The recognition of such conditions is important as it allows the use of specialized efficient algorithms and avoids the use of computationally expensive integer programming.

A further constraint that sometimes applies to transportation problems is that there are limits to the possible flow from a source to a sink. This gives rise to the *capacitated transportation problem*. There may be both lower and upper limits for the flow in each arc. For the LP formulation of the transportation problem (such as exemplified in Example 5.2 above) such limits can be accommodated by simple bounds on the variables:

$$0 \le l_{ij} \le x_{ij} \le u_{ij}.$$

Frequently, l_{ij} will be 0. Capacitated transportation problems can, like the ordinary transportation problem, be solved by straightforward extensions to the special purpose algorithms mentioned above.

Another non-distribution example of the transportation problem is described by Stanley *et al*. (1954), who describe how the problem arises in deciding how a government should award contracts.

5.3.2 The assignment problem

This is the problem of assigning n people to n jobs so as to maximize some overall level of competence. For example person i might take an average time t_{ij} to do job j. In order to assign each person to a job and to fill each job so as to minimize total time for all tasks our problem would be

$$\begin{aligned} \text{Minimize} \quad & \sum_{i,j} t_{ij} x_{ij} \\ \text{subject to} \quad & \sum_{j} x_{ij} = 1 \text{ for all } j, && (5.36) \\ & \sum_{j} x_{ij} = 1 \text{ for all } i, && (5.37) \end{aligned}$$

where

$$x_{ij} = \begin{cases} 1 & \text{if person } i \text{ is assigned to job } j, \\ 0 & \text{otherwise.} \end{cases}$$

This can obviously be regarded as a special case of the transportation problem. We can regard it as a problem with n sources and n sinks. Each source has an availability of 1 unit and each sink has a demand of 1 unit. Constraints

(5.36) impose the condition that each job be filled. Constraints (5.37) impose the condition that every person be assigned a job.

It might appear that this problem demands integer programming in order to ensure that x_{ij} can only take the values 0 or 1. Fortunately, however, because this problem is a special case of the transportation problem the integrality property mentioned above holds. If we solve an assignment problem as a conventional LP model, we can be certain that the optimal solution will give integer values to the x_{ij} (0 or 1). If marriage is regarded as an assignment problem of this kind, Dantzig has suggested that the integrality property shows that monogamy leads to greatest overall happiness!

Obviously assignment problems could be solved as LP models, although the resultant models could be very large. For example, the assigning of 100 people to 100 jobs would lead to a model with 10 000 variables. It is much more efficient to use a specialized algorithm. One of the specialized algorithms for the transportation problem could obviously be applied. The most efficient method known is one allied to the Ford and Fulkerson algorithm but more specialized. This is known as the *Hungarian method* and is described by Kuhn (1955).

5.3.3 The transhipment problem

This is an extension of the transportation problem, due to Orden (1956). In this problem, it is possible to distribute the commodity through intermediate sources and through intermediate sinks as well as from sources to sinks. In Example 5.2, we could allow flow (at a certain cost) between suppliers S_1, S_2 and S_3 as well as between customers T_1, T_2, T_3 and T_4. It might be advantageous sometimes to send a commodity from one supplier to another before dispatching it to the customer. Similarly, it might be advantageous to send a commodity to a customer via another customer first. The transhipment problem allows for these possibilities.

If we extend Example 5.2 to allow the use of certain intermediate sources and sinks, our graphical representation would be of the form of Figure 5.2.

Costs have now been attached to the arcs between sources and the arcs between sinks. Notice that it is sometimes possible to go either way between sources (or sinks), at not necessarily the same cost.

It is possible to convert a transhipment problem into a transportation problem. To do this, the sources and sinks are considered firstly as being all sources and then as all sinks. When considered as sinks, sources have no availabilities, and when considered as sources, sinks have no requirements. Flow from a sink to a source is not allowed. For the transhipment extension of Example 5.2 illustrated in Figure 5.2, we can draw up the unit cost array of Table 5.5. ‘Sources’ T_1, T_2, T_3 and T_5will have zero availabilities and ‘sinks’ S_1, S_2 and S_3 zero requirements.

Transhipment problems can obviously be formulated as LP models just as the transportation problem can. Again, it is often desirable to use a specialized algorithm such as that described by Dantzig (1951) or by Ford and Fulkerson (1962).

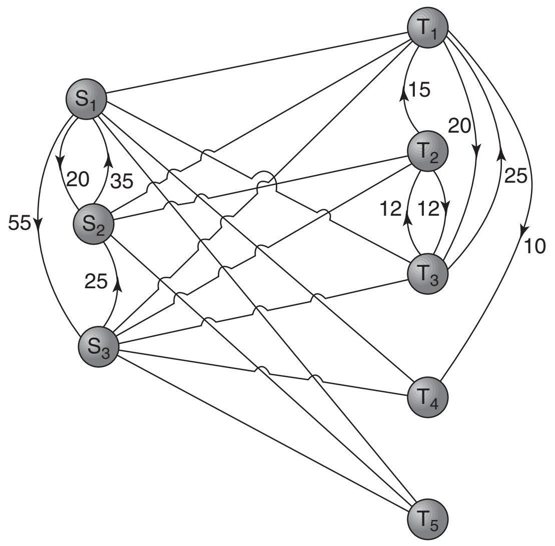


Figure 5.2

Table 5.5

Sources	Sinks							
	S_1	S_2	S_3	T_1	T_2	T_3	T_4	(T_5)
S_1	–	20	55	132	–	97	103	0
S_2	35	–	–	85	91	–	–	0
S_3	–	25	–	106	89	100	98	0
T_1	–	–	–	–	–	20	10	–
T_2	–	–	–	15	–	12	–	–
T_3	–	–	–	25	12	–	–	–
T_4	–	–	–	–	–	–	–	–
(T_5)	–	–	–	–	–	–	–	–

As with transportation problems, transhipment problems can be extended to capacitated transhipment problems where the arcs have upper and lower capacity limitations. These can also be solved by specialized algorithms.

An application of the transhipment problem outside the field of distribution is described by Srinivason (1974).

5.3.4 The minimum cost flow problem

The transportation, transhipment and assignment problems are all special cases of the general problem of finding a minimum cost flow through a network.

Such problems may have upper and lower capacities attached to the arcs in the capacitated case. The uncapacitated case will be considered here.

Example 5.4: Minimum Cost Flow

The network in Figure 5.3 has two sources 0 and 1 with availabilities of 10 and 15. There are three sinks 5, 6 and 7 with requirements 9, 10 and 6 respectively. Each arc has a unit cost of flow associated with it.

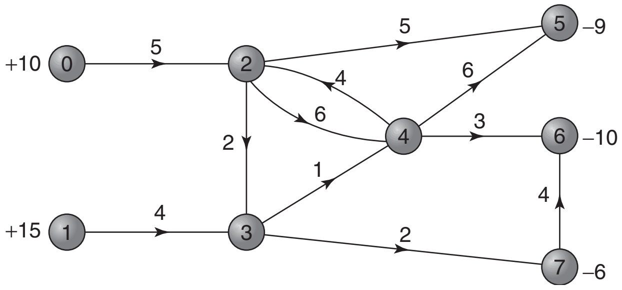


Figure 5.3

The arcs are 'directed' in the sense that only flow in the direction marked by the arrow is allowed. If flow is allowable in the opposite direction as well, this is indicated by another arc in the reverse direction. This happens in the case of the two arcs between node 2 and node 4.

The problem is simply to satisfy the requirements at the sinks by flow through the network from the sources at total minimum cost. In this case, the total availability exactly equals the total requirement. This can always be made possible by the use of a dummy sink if necessary as described for the transportation problem in Example 5.2.

The LP formulation of Example 5.4 is

Minimize

$$5x_{02} + 4x_{13} + 2x_{23} + 6x_{24} + 5x_{25} + x_{34} + 2x_{37} + 4x_{42} + 6x_{45} + 3x_{46} + 4x_{76}$$

subject to

$$
\begin{aligned}
-x_{02} &= -10, && (5.38)\\
-x_{13} &= -15, && (5.39)\\
x_{02} - x_{23} - x_{24} - x_{25} + x_{42} &= 0, && (5.40)\\
x_{13} + x_{23} - x_{34} - x_{37} &= 0, && (5.41)\\
x_{24} + x_{34} - x_{42} - x_{45} - x_{46} &= 0, && (5.42)\\
x_{25} + x_{45} &= 9, && (5.43)\\
x_{46} + x_{76} &= 10, && (5.44)\\
x_{37} - x_{76} &= 6. && (5.45)
\end{aligned}
$$

In order to be systematic about this formulation, it is convenient to regard each constraint as arising from the *material balance requirement* at each node. For example, at node 2 it is necessary to ensure that the total flow in ($x_{02} + x_{42}$) is the same as the total flow out ($x_{23} + x_{24} + x_{25}$). This is achieved by constraint (5.40). At node 7, which is a sink the total flow in (x_{37}) must again be the same as the total flow out ($x_{76} + 6$). This gives constraint (5.45).

The matrix of coefficients in constraints (5.38)–(5.45) of the model above is known as the *incidence matrix* of the network in Figure 5.3. It clearly has a very special structure. This structure is further discussed in Section 10.1 as, like the transportation problem, the minimum cost flow problem (whether capacitated or not) can be guaranteed to yield an optimal *integer* solution so long as the availabilities, requirements and arc capacities are integer values.

As with the other types of model so far discussed in this section, it is generally more efficient to use specialized algorithms. Those due to Dantzig (1951) and Ford and Fulkerson (1962) are also applicable here.

A comprehensive survey of applications of the minimum cost network flow problem is given by Bradley (1975). Other useful references are Glover and Klingman (1977) and Jensen and Barnes (1980).

It is sometimes possible to convert an LP model into a form that is immediately convertible into a network flow model. A procedure for doing this, or showing such a conversion to be impossible, is given in Section 5.4.

If arcs have lower or upper bounds (or both) on their capacities, then (as with the special case of the transportation problem) it is possible to adapt the special algorithms to cope with this. It is, however, worth pointing out that such models can be converted to the uncapacitated case. This might be necessary if a program was being used which could not deal with such bounds.

Suppose the flow from node i to node j had a lower bound of l (and cost c_{ij}). An extra node i' can be added with a new arc from i to i'. If there is an external flow l *out* of i and *into* i', as shown in Figure 5.4, this provides the necessary restriction. Similarly, Figure 5.5 demonstrates how an upper bound of u can be imposed on the flow in arc $i - j$.

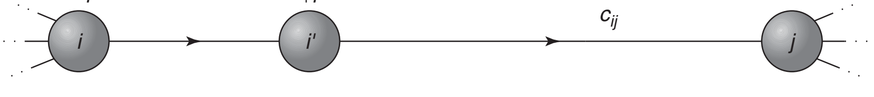


Figure 5.4

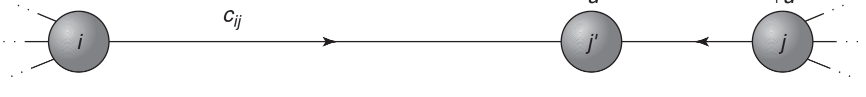


Figure 5.5

It is important to ensure that a minimum cost network flow problem is well defined. For example the unit flow costs are generally non-negative. If negative costs are allowed it is important to ensure that cost cannot be minimized indefinitely (giving an *unbounded* problem). This could happen, for example, if arc 2–4 were given in a unit cost of −6 instead of +6. Going round the loop indefinitely would continuously reduce the cost.

A minimum cost network flow problem, DISTRIBUTION, is given in Part II.

An extension of the problem of finding the minimum cost flow of a single commodity through a network is the problem of minimizing the cost of the flows of several commodities through a network. This is the *minimum* cost *multi-commodity network flow problem*. There will be capacity limitations on the flows of individual commodities through certain arcs as well as capacity limitations on the total flow of all commodities through individual arcs. For example, in the network of Figure 5.3, one commodity might flow between source 0 and sink 5 and a second commodity flow between source 1 and sinks 6 and 7. This type of problem can again be formulated as an LP model. The resultant model has a block angular structure of the type discussed in Section 4.1. The block angular structure makes the decomposition procedure of Dantzig and Wolfe, which is discussed in Section 4.2, applicable. In fact this leads to another LP formulation of the problem. This aspect of the minimum cost multi-commodity network flow problem is discussed by Tomlin (1966).

Apart from decomposition there are no special algorithms applicable to the general minimum cost multi-commodity network flow problem. The (often large) LP model resulting from such a problem is best solved by the standard revised simplex algorithm using a package programme.

Charnes and Cooper (1961a) formulate a traffic flow problem as a minimum cost multi-commodity network flow model.

This extension of the single-commodity network flow LP model to more than one commodity destroys the property that guarantees an integral optimal solution. Fractional values for the flows may result from the optimum solution to the LP model even if all capacities, availabilities and requirements are integral. If the nature of the problem requires an optimal integer solution it is necessary to resort to *integer programming*.

Another important extension of the minimum cost network flow model is the *generalized network flow model*. This is sometimes known as the *network flow with gains model*. In this extension, the flow in an arc may alter between the two nodes. A multiplier is then associated with each arc, which gives the factor by which flow is altered. Situations that require this modification result from, for example, evaporation, wastage or application of interest rates. Glover and Klingman (1977) give applications. If it is necessary that the flows be *integer* values, then this can no longer be guaranteed from an LP solution. It is necessary to use integer programming methods. Nevertheless, it is possible to exploit this simple structure to good effect in the algorithms used. Glover *et al*. (1978) describe such a method. In fact, any 0–1 integer programming problem can be converted into such a generalized network model where flows must be integer values. This is shown by Glover and Mulvey (1980).

5.3.5 The shortest path problem

This is the problem of finding a shortest path between two nodes through a network. Rather surprisingly, this problem can be regarded as a special case of the minimum cost flow problem.

Example 5.5: Finding the Shortest Path Through a Network

In the network in Figure 5.6, we wish to find the shortest path between node 0 and node 8. The lengths of each arc are marked.

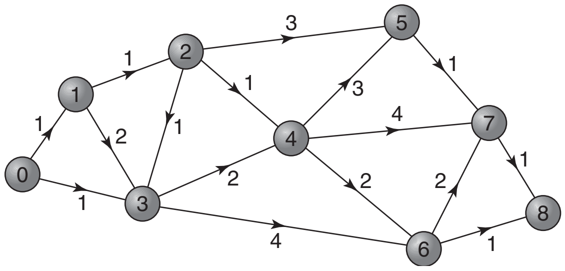


Figure 5.6

We can reduce this problem to one of finding a minimum cost flow through the network by giving node 0 an availability of 1 unit (a source) and giving node 8 a requirement of 1 unit (a sink). Because of the property that minimum cost flow (as with transportation, transhipment and assignment) problems have of guaranteeing integral optimal flows, when solved as LP models, we can be sure that this minimal cost flow through each arc in Figure 5.6 will be 0 or 1. Exactly one of the arcs out of node 0 will therefore have a flow of 1 and exactly one of the flows into node 8 will have a flow of 1. Similarly, intermediate nodes on the flow path will have exactly one arc with flow in and one with flow out. The 'cost' of the optimal flow path will give the shortest route between 0 and 8.

Although it is possible to use conventional LP to solve shortest path problems it would be more efficient to use a specialized algorithm. One of the most efficient such algorithms is due to Dijkstra (1959).

5.3.6 Maximum flow through a network

When a network has capacity limitations on the flow through arcs there is often interest in finding the maximum flow of some commodity between sources and sinks. We will again consider the network of Example 5.4 but our objective will

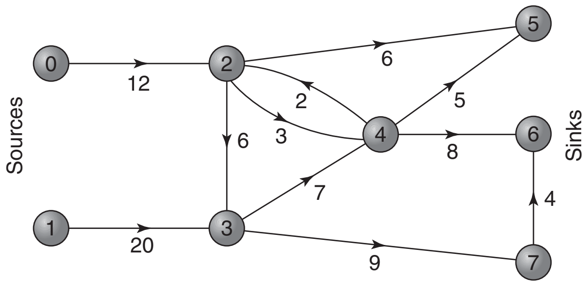


Figure 5.7

now be to maximize the flows into the sources and out of the sinks rather than these quantities being given. Each arc has now been given an upper capacity, which is the figure attached to it in Figure 5.7.

Example 5.6: Maximizing the Flow Through a Network

This problem can again be formulated as an LP model. The variables and constraints will be the same as those in Example 5.4 apart from the introduction of five new variables $x_{S0}, x_{S1}, x_{5T}, x_{6T}$ and x_{7T} representing the flows into sources 0 and 1 and out of sinks 5, 6 and 7. The resultant model is

Maximize $\quad x_{S0} + x_{S1}$

subject to

$$x_{S0} - x_{02} = 0, \tag{5.46}$$
$$x_{S1} - x_{13} = 0, \tag{5.47}$$
$$x_{02} - x_{23} - x_{24} - x_{25} + x_{42} = 0, \tag{5.48}$$
$$x_{13} + x_{23} - x_{34} - x_{37} = 0, \tag{5.49}$$
$$x_{24} + x_{34} - x_{42} - x_{45} - x_{46} = 0, \tag{5.50}$$
$$x_{25} + x_{45} - x_{5T} = 0, \tag{5.51}$$
$$x_{46} + x_{76} - x_{6T} = 0, \tag{5.52}$$
$$x_{37} - x_{76} - x_{7T} = 0, \tag{5.53}$$

$$x_{02} \le 12, x_{13} \le 20, x_{23} \le 6, x_{24} \le 3, x_{25} \le 6, x_{34} \le 7, x_{37} \le 9, x_{42} \le 2, x_{45} \le 5, x_{46} \le 8, x_{76} \le 4.$$

Again, this type of model has the property that the optimal solution will give integer flows so long as the capacities are integer.

It is again more efficient to use a specialized algorithm for this type of problem. Such an algorithm is described by Ford and Fulkerson (1962).

5.3.7 Critical path analysis

This is a method of planning projects (often in the construction industry) that can be represented by a network. The arcs of the network represent *activities* occupying a duration of time, for example, building the walls of a house, and the nodes are used to indicate the termination and beginning of activities. Once a project has been represented by such a network model the network can be analysed to answer a number of questions such as

1. How long will it take to complete the project?
2. Which activities can be delayed if necessary and by how long without delaying the overall project?

Such a mathematical analysis of this kind of network is known as *critical path analysis*. The arcs for those activities in the network that cannot be delayed without affecting the overall completion time of the project can be shown to lie on a path. This *critical path* is in fact the *longest path* through the network. The problem of finding the critical path is a special kind of LP problem although the special structure of the problem makes a specialized algorithm appropriate.

Example 5.7: Finding the Critical Path in a Network

The network in Figure 5.8 represents a project of building a house. Each arc represents some activity forming part of the project. The durations (days) of the activities are attached to the corresponding arcs. The arc 4–2 marked with a broken line is a dummy activity having no duration. Its only purpose is to prevent activity 2–5 starting before activity 3–4 has finished.

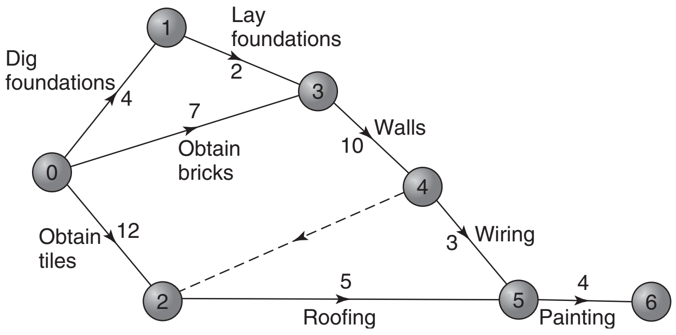


Figure 5.8

In order to formulate this problem as an LP model, we can introduce the following variables:

t_0	start time for activities 0–1, 0–3 and 0–2
t_1	start time for activity 1–3
t_2	start time for activity 2–5
t_3	start time for activity 3–4
t_4	start time for activities 4–2 and 4–5
t_5	start time for activity 5–6
z	finish time for the project.

Our model is then

$$
\begin{array}{llr}
\text{Minimize } z & & \\
\text{subject to} & -t_0 + t_1 \geq 4, & (5.54) \\
& -t_0 + t_2 \geq 12, & (5.55) \\
& -t_0 + t_3 \geq 7, & (5.56) \\
& -t_1 + t_3 \geq 2, & (5.57) \\
& -t_3 + t_4 \geq 10, & (5.58) \\
& t_2 - t_4 \geq 0, & (5.59) \\
& -t_2 + t_5 \geq 5, & (5.60) \\
& -t_4 + t_5 \geq 3, & (5.61) \\
& -t_5 + z \geq 4. & (5.62)
\end{array}
$$

Each constraint represents a *sequencing relation* between certain activities. For example activity 3–4 cannot start before activity 1–3 has finished. This gives $t_3 \leq t_1 + 2$, which leads to constraint (5.57). Finally, as the project cannot be completed before activity 5–6 is finished, we get constraint (5.62).

On solving this model, we obtain the following results:

$$
\begin{aligned}
\text{Project completion time } (z) &= 26 \text{ days} \\
t_0 &= 0, \\
t_1 &= 4, \\
t_2 &= 17, \\
t_3 &= 7, \\
t_4 &= 17, \\
t_5 &= 22.
\end{aligned}
$$

The critical path is clearly 0–3–4–2–5–6.

Building and conventionally solving the above LP model would be an inefficient method of finding the critical path. Special algorithms exist and there are widely used package programs for doing critical path analysis. Many extensions of the problem of scheduling a project in this way can be considered but are beyond the scope of this book. A full discussion of this subject is contained in Lockyer (1967).

One very practical extension of the problem is that of *allocating resources to the activities* in a project network. For example, in the network of Figure 5.8 both activity 4–5 (wiring) and activity 2–5 (roofing) may require people (although in this contrived example they would probably have different skills). If activity 4–5 requires three people and activity 2–5 requires six and there are only eight available, the optimal schedule given above is unattainable and one of those activities will have to be delayed or extended. The problem is then how to reschedule to achieve some objective. For example, the objective might well be to delay the overall completion time as little as possible. Alternatively, there might be a desire to 'smooth' the usage of this and other resources over time. This extension to the problem is mentioned again in Section 9.5 as it gives rise to an integer programming extension to the LP problem of the type above. Nevertheless, integer programming would be generally far too costly in computer time to justify solving this type of problem in this way. A problem that gives rise to a very simple network is job-shop scheduling. This problem of scheduling jobs on machines can be regarded as a problem of allocating resources to activities in a network. The operations in the job-shop (e.g. machining) give the activities. Sequencing relations between those operations give a (simple) network structure. The resources to be allocated are the limited machines.

All the problems described in this section (apart from the minimum cost multi-commodity network flow problem) are best tackled through specialized algorithms rather than the revised simplex algorithm available on commercial package programs. The reason for describing these problems and showing how they can, if necessary, be modelled as linear programs is that many practical problems are made up, in part, of network problems. Such problems often, however, contain additional complications that make it impossible to use a pure network model. This is where the conventional LP formulation becomes important. Many practical LP models have a very large network component. Such a feature usually makes very large models easy to solve using package programs. Some package programs have special features to take advantage of some of the network structure within a model. An example of this is the *generalized upper bound* (GUB) type of constraint, which was mentioned in Section 3.3. In the LP formulation of the transportation problem in Example 5.2 constraints (5.21)–(5.23) could be regarded as GUB constraints and not explicitly represented as constraints if a package with this facility was used. Alternatively (and preferably because there are more of them), constraints (5.24)–(5.27) could be represented as GUB constraints. The use of the GUB facility makes the solution of many network flow problems (or problems with a network flow component) particularly easy by conventional LP.

Another virtue in recognizing a network flow component in an LP model is that many variables are likely to come out at integer values in the optimal solution. It has been pointed out that this happens for all the variables in most of the network flow problems described in this section so long as the right-hand side coefficients are integers. If a model is 'not quite' of a network flow kind it is probable that the great majority of variables will still take integer values in the optimal LP solution. The computational difficulties of forcing all these variables to be integer values by integer programming will be much reduced. This topic is discussed much more fully in Chapter 10.

There is also great virtue to be gained from remodelling a problem in order to get it into the form of a network flow model. This then opens up the possibility of using a special purpose algorithm. An example of how this can sometimes be done for a practical problem is given by Veinott and Wagner (1962). Dantzig (1969) shows how a hospital admissions scheduling program can be remodelled to give an LP model with a large network flow component. He then exploits this structure by use of the GUB facility. Other examples of reformulations into network flow models are Daniel (1973), Wilson and Willis (1983) and Cheshire *et al*. (1984).

An automatic way of either converting an LP model to a network flow model or showing such a conversion to be impossible is given in Section 5.4. The recognition or conversion of a model as a network structure relieves the need to use the computationally much more costly methods of integer programming.

In Section 6.2, the concept of the dual of an LP model is described. Every LP model has a corresponding model known as the *dual model*. The optimal solution to the dual model is very closely related to the optimal solution of the original model. In fact the optimal solution to either one can be derived very easily from the optimal solution to the other. It turns out that many practical problems give rise to an LP model that is the dual of a network flow model. In such circumstances, it could well be worth using a specialized algorithm on the corresponding network flow model. Moreover, the dual of any of the types of network flow mode mentioned here (apart from the minimum cost multi-commodity network flow model) also has the property of guaranteeing optimal integer solutions (so long as the objective coefficients of the original model are integers). The recognition of this type of model can, again, be of great practical importance for this reason. This topic is further discussed in Sections 10.1 and 10.2.

In Part II, the OPENCAST MINING problem can be formulated as the dual of a network flow problem. The formulation is discussed in Part III. The MINING problem of Part II can be formulated as an integer programming model, a large proportion of which is the dual of a network flow model. Some examples of such models are given in Williams (1982).

One famous network problem that has not been discussed in this section is the *travelling salesman problem*. This is the problem of finding a minimum distance (cost) route round a given set of cities. This problem cannot generally be solved by an LP model, in spite of its apparent similarity to the assignment problem. It can, however, be modelled as an integer programming extension to the assignment problem and is fully discussed in Section 9.5.

5.4 Converting linear programs to networks

The advantages of converting linear programs to minimum cost network flow models, if possible, have already been discussed in Section 5.3. They are further discussed in Section 10.1 as minimum cost network flow models have *integer* optimal solutions (so long as external flows in and out are integer values). This relieves the need to use the much more expensive procedures of integer programming.

We outline a method described by Baston *et al*. (1991) for converting linear programs to network flow models. Another procedure, but expressed in the more abstract language of matroid theory, is given by Bixby and Cunningham (1980).

In order to illustrate the method, we will take a numerical example.

$$\begin{array}{lllllllllll}
\text{Minimize} & c_1x_1 + c_2x_2 + c_3x_3 + c_4x_4 \\
\text{subject to} & 2x_1 & & & & +x_5 & & & & & = b_1, \\
 & 6x_1 & & +9x_3 & & & +x_6 & & & & = b_2, \\
 & -8x_1 & +4x_2 & & -8x_4 & & & +x_7 & & & = b_3, \\
 & & x_2 & +3x_2 & -2x_4 & & & & +x_8 & & = b_4, \\
 & & x_2 & +3x_3 & & & & & & +x_9 & = b_5, \\
 & \multicolumn{9}{l}{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9} & \geq 0.
\end{array}$$

As the conversion does not depend on the objective or right-hand side coefficients, we give these in a general form. In this example, we assume all the original constraints were of the '≤' form and that *slack* variables have been added to make them equations. For '≤' constraints *surplus* variables would be subtracted. If any of the original constraints were equations then we would add *artificial* variables (variables constrained to take the value zero). These *logical* (slack, surplus or artificial) variables will represent arcs in the network created. For the case of artificial variables, these arcs will finally be deleted.

We carry out the following transformations:

1. Scale the rows and columns in order to make the constraint coefficients 0 or ±1 if possible. This may not be possible, in which case, the conversion to a network is not possible. In most practical problems (such as those referenced in Section 5.3), for which it is worth attempting a conversion these coefficients will already be 0 or ±1.

   In this example, the resultant scaled coefficients are given below.

6	7	8	9	1	2	3	4	5	
$\frac{1}{2}c_1$	c_2	$\frac{1}{3}c_3$	$\frac{1}{2}c_4$						
1				1					$=b_1$
1		1			1				$=\frac{1}{3}b_2$
−1	1		−1			1			$=\frac{1}{4}b_3$
	1	1	−1				1		$=b_4$
	1	1						1	$=b_5$

   It is also convenient to number the variables. The logical variables are numbered 1 to 5 and the original variables 6 to 9.

2. In this step, the signs of the non-zero coefficients (±1) are ignored. The arcs corresponding to the logical variables are arranged in the form of a

spanning tree of the network. This spanning tree must be *compatible* with the original variables of the model in the following sense: the original variables each form a *polygon* with some of the arcs of the spanning tree. Figure 5.9 illustrates how this is possible with the example. The arc corresponding to variable 6 forms a polygon with the arcs of the tree corresponding to variables 1, 2 and 3 as variable 6 has non-zero entries in rows 1, 2 and 3. Similarly, as variable 7 has non-zero entries in rows 3, 4 and 5, arc 7 forms a polygon with arcs 3, 4 and 5. Arcs 8 and 9 are similarly compatible with the tree.

It will not always be possible to find an arrangement of the logical arcs in the form of a spanning tree, which is compatible with the other arcs in the manner demonstrated above. In such a case, the network conversion is not possible. A systematic way of investigating whether it is possible to construct such a spanning tree, or showing it to be impossible, is described by Baston *et al*. (1991).

Having created such an *undirected* network, the arcs are orientated in the following manner:

3. For each polygon the arcs of the tree in the polygon are given a direction *opposite* to that of the non-tree arc, when going round the polygon, if the entry in the column corresponding to this arc is $+1$. If the entry is -1 the arcs are given the *same* direction. For this example, the resulting orientations are shown in Figure 5.10. Arc 6, for example, has an opposite orientation to arcs 1 and 2 (variable 6 has $+1$ coefficients in rows 1 and 2) and the same orientation to arc 3 (variable 6 has a -1 coefficient in rows 3) when going round the polygon formed by arcs 1, 2, 3 and 6. Other arcs are orientated similarly according to this rule. It may not be possible to orientate the arcs in any manner compatible with the signs of the coefficients. In such a case, a (directed) network is not constructible.

4. The (non-tree) arcs in the network are given unit costs equal to the scaled objective coefficients of corresponding variables.

5. Each node is given an external flow *in* equal to the sum of the scaled right-hand side coefficients of the rows corresponding to those tree-arcs

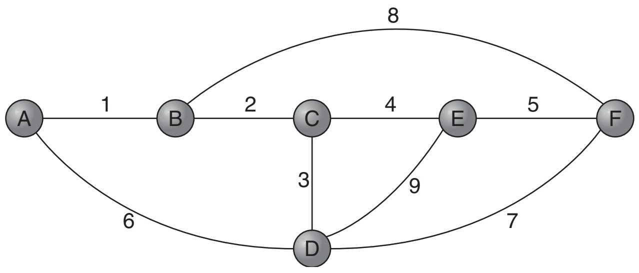


Figure 5.9

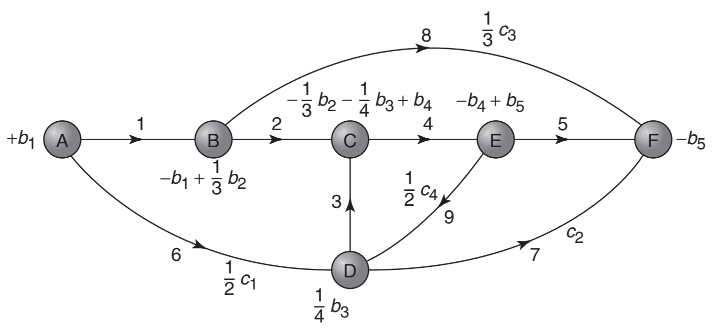


Figure 5.10

leaving the node less the sum of the scaled right-hand side coefficients of the rows corresponding to those tree-arcs entering the node.

In the example, in Figure 5.10, node C has tree-arc 4 leaving it (row 4 has scaled right-hand side b_4) and tree-arcs 2 and 3 entering it (rows 2 and 3 have scaled right-hand sides of $(1/3)b_2$ and $(1/4)b_3$, respectively). Hence the external flow into node C is $b_4 - (1/3)b_2 - (1/4)b_3$. (A negative external flow would, of course, be regarded as a positive flow out.) The other nodes have external flows calculated in a similar manner.

The resulting directed network with its external flows gives the required minimum cost network flow model equivalent to the original linear program. When solved the values of the flows in the arcs may have to be unscaled according to the scaling factors applied.

6

Interpreting and using the solution of a linear programming model

6.1 Validating a model

Having built a linear programming model we should be very careful before we rely too heavily on the answers it produces. Once a model has been built and converted into the format necessary for the computer program we will wish to attempt to solve it. Assuming that there are no obvious clerical or keying errors (which are usually detected by package programs) there are three possible outcomes: (i) the model is infeasible; (ii) the model is unbounded; (iii) the model is solvable.

6.1.1 Infeasible models

A linear programming model is infeasible if the constraints are self-contradictory. For example, a model which contained the following two constraints would be infeasible:

$$x_1 + x_2 \leq 1, \tag{6.1}$$

$$x_1 + x_2 \geq 2. \tag{6.2}$$

In practice, the infeasibility would probably be more disguised (unless it arose through a simple keying error). The program will probably go some way towards trying to solve the model until it detects that it is infeasible. Most package programs will print out the infeasible solution obtained at the point when the program gives up.

In most situations an infeasible model indicates an error in the mathematical formulation of the problem. It is, of course, possible that we are trying to devise some plan that is technologically impossible, but more usually we are modelling a situation where we know there should be a feasible solution. The detection of why a model is infeasible can be very difficult. If we have got the infeasible solution where the program gave up we can see which constraints are unsatisfied or which variables are negative in this solution. This may enable us to find the cause of infeasibility fairly easily. It is quite possible, however, that it will not help. The cause of infeasibility may be fairly subtle. For example, it is impossible to satisfy each of the following constraints in the presence of the other two:

$$x_1 - x_2 \geq 1, \tag{6.3}$$

$$x_2 - x_3 \geq 1, \tag{6.4}$$

$$-x_1 + x_3 \geq 1. \tag{6.5}$$

It might, however, be wrong to single out any one of Inequalities (6.3)–(6.5) as being an infeasible constraint. What is wrong is the mutual incompatibility of all three constraints.

Assuming that we are modelling a situation which we know to be realizable, it should be possible to construct a feasible (but probably non-optimal) solution to the situation. For example, in a product mix model such as that presented in Section 1.2 we could take a previous week's mix of products (assuming no capacity has since been reduced). If our model were a correct representation of the situation, it should be possible to substitute this constructed (or known) solution into all the constraints without breaking them. As our model admits no feasible solution, this substitution must violate some of the constraints in the model. Any constraints violated in this way must have been modelled wrongly. They are too restrictive and should be reconsidered.

6.1.2 Unbounded models

A linear programming model is said to be unbounded if its objective function can be optimized without limit, that is, for a maximization problem the objective function can be made as large as one wishes or, for a minimization problem, as small as one wishes. For example, the following trivial linear programming problem is unbounded as $x_1 + x_2$ can be made as large as we like without violating the constraint:

$$\text{Maximize} \quad x_1 + x_2 \tag{6.6}$$

$$\text{subject to} \quad 2x_1 + x_2 \leq 1. \tag{6.7}$$

While a correctly formulated infeasible model is just possible as we might be trying to attain the unattainable, a correctly formulated unbounded model is very unlikely.

As with infeasible models, most package programs print a solution to an unbounded model before the optimization is terminated on it being seen that the model is unbounded. This solution can be of help in detecting what is wrong with the model. On the other hand, detection of unboundedness may well be as difficult as the detection of infeasibility. An infeasible model has constraints which are over restrictive whereas an unbounded model either has vital constraints unrepresented or insufficiently restrictive. Usually, certain physical constraints have not been modelled. These constraints may well be so obvious as to be forgotten. A very common type of constraint to omit is a *material balance constraint* such as that described in Section 3.3. This sort of constraint is easily overlooked. If such a constraint has been inadvertently overlooked, the solution which the package program prints before giving up can be of use in detecting it. A common sense examination of this solution may reveal that things do not 'add up', for example, we may find we are producing something without using any raw material.

In Section 6.2 we define an associated linear programming model for every model known as the *dual model*. This model can have important economic interpretations. If a model is unbounded the corresponding dual model is infeasible. It is therefore conceivable that a practical unbounded model might arise as the dual to an infeasible model where we were testing whether a certain course of action was or was not possible. Should a model be infeasible we should not assume that its dual will necessarily be unbounded. It is possible that the dual will also be infeasible.

6.1.3 Solvable models

Should a linear programming model be neither infeasible nor unbounded we will refer to it as *solvable*. When we obtain the optimal solution to such a model we clearly want to know if the answer is sensible. If it is not sensible then there must be something wrong with our model. The first approach should be to examine the optimal solution critically simply using common sense. This may well reveal an obvious nonsense which should enable us to detect and correct a modelling error.

Should the solution still appear sensible we could compare the optimal objective value with what we might expect in practice. If, in a maximization problem, this value is lower than we expect, we might suspect that our model is over restrictive, that is, some constraints are too severe. On the other hand, if the optimal objective is higher than we expect, we might suspect that our model is insufficiently restrictive, that is, that some constraints are too weak or have been left out. For minimization problems these conclusions would obviously be reversed. To give an example, suppose we were considering a product mix application such as that considered in Section 1.2. On solving the model we would obtain a maximum profit contribution. Suppose this profit contribution was lower than what we already knew we could attain, for example, suppose it was lower than what we obtained in a previous week (assuming there had been no subsequent cut-back in productive capacity). In this circumstance we would obviously suspect our model to be over restrictive. To find where the over severe constraints lay we could substitute our known better solution into the constraints of our model. Some of these constraints will obviously be violated and must therefore have been modelled incorrectly. Possibly we may find that there is some productive capacity which we were unaware of but is being used by the workforce. Our model, even though not yet correct, will already be proving valuable in helping to discover things we were unaware of.

Thinking again about our product mix model, suppose we find the reverse situation that the optimal objective value was greater than anything we knew we could possibly achieve. In this situation our model would obviously be under-restrictive. The approach here would be to subject the optimal solution proposed by the model to a very critical examination. This solution could be presented in an entirely non-technical manner as a proposed operating policy and the appropriate management asked to spell out why it was impossible. It should then be possible to use this information to modify constraints or add new constraints.

For situations such as the above, where we are modelling an existing situation, we obviously have the great advantage of the ‘feasible’ solutions suggested by the existing mode of operation. These solutions can be used to test and modify our model. For situations where we are using linear programming (or any other sort of model) to design a new situation, for example, decide where to set up new plant, there may be no obvious ‘feasible’ solutions to use. Testing the model may therefore be more difficult. It is still a good approach to try to obtain good common sense solutions by rule of thumb methods. These solutions may well reveal errors in our model and show how they may be corrected.

The value of optimizing an objective should be apparent in this discussion on the validation of linear programming models. By optimizing some quantity we would expect to be using certain resources (processing capacity, raw materials, manpower, etc.) to their limit. The resultant optimal solution suggested by the model would then be fairly likely to violate certain physical restrictions which had been ignored or modelled incorrectly and thereby highlight them. It is often desirable to solve the model a number of times with different (possibly contrived) objectives in order to test out as many constraints as possible. The value of optimizing an objective (and so using mathematical programming) where there is no real life objective should be apparent in *validating* and modifying a model.

It should be obvious from the foregoing discussion that the building and validation of a model should be a two-way process gradually converging on a more and more accurate representation of the situation being modelled. Unfortunately, this process is often ignored or neglected. It can be an extremely valuable activity leading to a much clearer understanding of what is being modelled. In many situations this greater understanding may be more valuable than even the optimal solution to the (validated) model.

It is often possible to build a model with error detection in mind. For example, suppose we wished to define the following constraint:

$$\sum_j a_j x_j \leq b. \tag{6.8}$$

If there were any danger of this constraint being too severe and making the model infeasible, we could allow it to be violated at a certain high cost by rewriting it as

$$\sum_j a_j x_j - u \leq b \tag{6.9}$$

and giving u a negative ('cost') coefficient in the objective function, assuming the problem to be a maximization. For a minimization problem u would be given a positive ('profit') coefficient. In this way the constraint could no longer cause the model to be infeasible but the violation of the original constraint (6.8) would be indicated by u appearing in the solution. For certain applications it might be argued that (6.9) was a more correct formulation of the constraint anyway by, for example, allowing us to 'buy in' more resources if really needed at a certain cost rather than restricting us absolutely. This topic was discussed in more detail in Section 3.3.

Another device can be used to avoid unboundedness occurring in a model. Each variable can be given a (possibly very large) finite simple upper bound in the formulation. No variable could then exceed its upper bound but any variable which rose to its bound would be open to suspicion. This might then lead one more quickly to the cause of unboundedness in the model.

Finally the desirability of using matrix generators should be re-emphasized here. By automatically generating a model the possibility of error is greatly reduced. The validation process itself is greatly simplified as the input to the matrix generator is usually presented in a much more physically meaningful form than for a linear programming package.

6.2 Economic interpretations

It will be useful to relate the discussion in this section to a specific type of model rather than present the material more abstractly. We will use the small product mix example of Section 1.2 for this purpose. The model was

$$\begin{array}{lll}
\text{Maximize} & 550x_1 + 600x_2 + 350x_3 + 400x_4 + 200x_5 & \\
\text{subject to} & \text{Grinding} \quad 12x_1 + 20x_2 \qquad\quad + 25x_4 + 15x_5 & \leq 288, \\
& \text{Drilling} \quad 10x_1 + 8x_2 + 16x_3 & \leq 192, \\
& \text{Manpower} \quad 20x_1 + 20x_2 + 20x_3 + 20x_4 + 20x_5 & \leq 384,
\end{array}$$

$$x_1, x_2, x_3, x_4, x_5 \geq 0.$$

This problem was that of finding how much to make of five products (PROD 1, PROD 2, ..., PROD 5) subject to two processing capacity limitations (grinding and drilling) and a manpower limitation. Practical problems will, of course, usually be much bigger and more complex. The interpretation of the solution and the derivation of extra 'economic' information is, however, well illustrated by this example. Other types of application (e.g. blending) will, of course, result in different interpretations being placed on this information. It should, however, be possible to relate this information to any real life application after following the discussion applied to the small example above. The practical problems presented in Part II provide an excellent means of relating such information to real life situations.

If the model above is solved (by, for example, the simplex algorithm) the optimal solution turns out to be

$$x_1 = 12, \qquad x_2 = 7.2, \qquad x_3 = x_4 = x_5 = 0,$$

giving an objective value of £10.920, that is, we should make 12 of PROD 1, 7.2 of PROD 2 and none of the other products. This results in a total profit contribution (over a week) of £10.920.

It can easily be verified that the grinding and manpower capacities are fully exhausted but that there is slack drilling capacity. This information is usually given along with the rest of the solution when a package program is used.

It can fairly easily be shown (although it is beyond the scope of this book) that in a model with m linear constraints (including possibly simple upper bounds) and a linear objective one would never expect more than m of the variables to be non-zero in the optimal solution. In the example above one could therefore be sure that one need never produce more than three products.

In a problem of the above kind there is a considerable amount of extra *economic information*, which might be of interest. For example, we can obtain answers to the following questions:

1. Presumably, products 3–5 are underpriced in comparison with products 1 and 2. How much more expensive should we make them in order for it to be worth manufacturing them?

2. What is the value of an extra hour of grinding, drilling, or manpower capacity? Strictly speaking we are interested in the marginal values of each of these capacities, that is, the effect of very small increases or decreases in capacity.

This extra information is usually presented with the ordinary solution when a package program is used. The variables have, associated with them, quantities known as *reduced costs* which can be interpreted (in this application) as the necessary price increases. Each constraint has associated with it a quantity known as the *shadow price* which can be interpreted as the marginal effect of increases (or decreases) in the capacities.

It should be emphasized that when the simplex algorithm (or one of its variants) is used to solve a model reduced costs and shadow prices arise naturally out of the optimal solution and most package programs print this information. There is, however, an alternative, and very illustrative, way of deriving this economic information which helps to clarify its true meaning. This is through another associated linear programming model known as the dual model which will now be described.

6.2.1 The dual model

We will again use the product mix problem above to illustrate this discussion.

Suppose that an accountant were trying to value each of the resources of this problem (grinding, drilling and manpower capacities) in some way so as to give a minimal overall valuation to the factory compatible with the optimal production plan. Let us suppose that the valuations for each hour of each of the capacities were y_1, y_2 and y_3 (measured in £). The objective would be

$$\text{Minimize} \qquad 288y_1 + 192y_2 + 384y_3. \tag{6.10}$$

The accountant wishes to obtain values for y_1, y_2 and y_3 which totally explain the optimal production pattern. It should be possible to impute the profit contribution obtained from each product produced to its usage of the three resources. We must ensure that the profit contribution for each unit of each product is totally 'covered' by its imputed value. For example each unit of PROD 1 has a profit contribution of £550. This must be totally accounted for by the 'value' of the 12 hours grinding capacity, 10 hours drilling capacity and 20 hours manpower capacity used in making this unit. As the values of an hour of each of these capacities are y_1, y_2 and y_3 (in £) we have

$$12y_1 + 10y_2 + 20y_3 \geq 550 \tag{6.11}$$

The reason why '$\geq$' is used rather than '$=$' in the above constraint will not be totally obvious to start with. It should, however, become apparent later.

Similar arguments will relate the hourly values y_1, y_2 and y_3 to the unit profit contributions of each of the other products. This gives constraints (4.12)–(4.15):

$$20y_1 + 8y_2 + 20y_3 \geq 600, \tag{6.12}$$

$$16y_2 + 20y_3 \geq 350, \tag{6.13}$$

$$25y_1 \qquad + 20y_3 \geq 400, \tag{6.14}$$

$$15y_1 \qquad + y_3 \geq 200. \tag{6.15}$$

The objective (6.10) together with the constraints (6.11)–(6.15) give us another linear programming model. As with most linear programming models we will implicitly assume that the variables y_1, y_2 and y_3 can only be non-negative. This new linear programming model is referred to as the *dual* of the original product mix model. In contrast the original model is usually referred to as the *primal* model.

The derivation of this model may appear somewhat contrived at first but should become more plausible once we examine its solution.

Solving this dual model by a suitable algorithm we obtain the optimal solution

$$y_1 = 6.25, \qquad y_2 = 0, \qquad y_3 = 23.75,$$

giving an objective value of 10 920; that is, we should value each hour of grinding capacity at £6.25, each hour of drilling capacity at nothing and each hour of manpower capacity at £23.75. The total valuation of the factory is then £10 920.

We can immediately see some connections between this result and the optimal production plan for the original product mix model:

1. The total valuation of the factory (over a week) is the same as the optimal objective value of the original model. This seems plausible. The total 'value' of a factory is equal to the value of its optimal productive output. It follows from the *duality theorem* of linear programming that this result will always be true.
2. Drilling capacity was not totally utilized in the optimal solution to the original primal model. We see that it has been given a zero valuation. This again seems plausible. As we do not use all the capacity we have, we are not likely to place much value on it. The result here is another consequence of the duality theorem of linear programming. If a constraint is not 'binding' in the optimal primal solution the corresponding dual variable is zero in the optimal solution to the dual model. Economists would refer to the drilling capacity as a 'free good', that is, in one sense it is not worth anything.

Let us examine the optimal solution to the dual problem further and see what it might suggest to an accountant about the production policy which the factory should pursue.

Each unit of PROD 1 contributes £550 to profit. It uses up, however, 12 hours of grinding capacity (valued at £6.25 per hour), 10 hours of drilling capacity (valued at nothing) and 20 hours of manpower capacity (valued at £23.75 per hour). The total value imputed to each unit of PROD 1 is therefore

$$\pounds\,(12 \times 6.25 + 10 \times 0 + 20 \times 23.75) = \pounds 550,$$

that is, the profit of £550 which each unit of PROD 1 contributes is exactly explained by the value imputed to it by virtue of its usage of resources. If we regarded the dual variables y_1, y_2 and y_3 as 'costs', that is, we charged PROD 1 for its usage of scarce resources then we would come to the conclusion that PROD 1 produced zero profit. In accounting terms this does not matter as these 'costs' are purely internal accounting devices.

Similarly we find that each unit of PROD 2 has an imputed value (or extra 'cost') of

$$\pounds\,(20 \times 6.5 + 8 \times 0 + 20 \times 23.75) = \pounds 600,$$

showing that the £600 contribution to profit is exactly accounted for.

For each unit of PROD 3 we get an imputed value (or extra ‘cost’) of

$$£\,(0 \times 6.25 + 16 \times 0 + 20 \times 23.75) = £475.$$

This ‘cost’ exceeds its profit contribution by £125. An accountant would conclude that PROD 3 would ‘cost’ more (in terms of usage of scarce resources) than it would contribute to profit. He or she would therefore suggest that PROD 3 not be manufactured. We came to the same conclusion using our original primal model.

It can easily be verified that PROD 4 and PROD 5 have ‘costs’ which exceed their unit profit contributions by £231.25 and £368.75 respectively and should therefore not be produced.

We are now in a position to see why the ‘$\geq$’ rather than ‘$=$’ constraints in the dual model are acceptable. If the total activity in the left-hand side (the ‘cost’) of a constraint in the dual model strictly exceeds the right-hand side coefficient (the profit contribution) we do not incur this excess cost by simply not manufacturing the corresponding product. This gives us a third connection between the optimal solutions to the dual and primal models:

3. If a product has a negative resultant ‘profit’ after subtracting its imputed ‘costs’, we do not make it. This is again a consequence of the duality theorem in linear programming. If a constraint in the dual model is not ‘binding’ in the optimal solution to the dual model, then the corresponding variable is zero in the optimal solution to the primal model. The symmetry between this property and property 2 should be obvious. This result is sometimes referred to as the *equilibrium theorem*. Economically it simply means that non-binding constraints have zero valuation.

Returning to our accountant’s analysis of the problem using the valuations derived from the dual model we conclude that

(a) At most PROD 1 and PROD 2 should be manufactured (at zero ‘internal profit’).

(b) Drilling capacity is not a binding constraint (having a zero valuation).

(Strictly speaking the deduction of (b) is not quite valid. It is certainly true that a non-binding constraint implies a zero valuation. The converse cannot immediately be concluded, although it presents no difficulty here. This complication is referred to later.)

The accountant could therefore eliminate x_3, x_4 and x_5 from the primal problem, ignore the second (drilling) constraint and treat the remaining two constraints as equations. This gives

$$12x_1 + 20x_2 = 288, \tag{6.16}$$

$$20x_1 + 20x_2 = 384. \tag{6.17}$$

Solving this pair of simultaneous equations gives us

$$x_1 = 12 \quad \text{and} \quad x_2 = 7.2,$$

that is, the accountant deduces the same production plan using his or her valuations for the three resources as we do from our ordinary (primal) linear programming model. In practice the derivation of these valuations (values of the dual variables) by the method we have suggested (building and solving the dual model) might be just as difficult as, or more difficult than, building and solving the original (primal) model. Our purpose is, however, to explain a useful concept. In practice one would not normally build or solve the dual model (although in some circumstances this model is easier to solve computationally and might be used for this reason).

The product mix model for which we constructed a dual model was a maximization with all constraints '$\leq$'. For completeness we should define the dual corresponding to a more general model. It is convenient to regard all problems as maximizations. In order to cope with a minimization we can negate the objective function and maximize it. It is also possible to convert all the constraints to '$\leq$'. We do not choose to do this as it is helpful to keep the original model close to its original form. The dual variable corresponding to a '$\leq$' constraint was a conventional *non-negative* linear programming variable. A '$\geq$' constraint can be dealt with by only allowing the dual variable to be *non-positive*. For an '$=$' constraint we will allow the dual variable to be *unrestricted* in sign. Such a variable is sometimes known as a *free variable*.

There is a symmetry concerning duality which should be mentioned although its main interest is purely mathematical. The dual of a dual model gives the original model.

6.2.2 Shadow prices

The valuations (values of the dual variables) which we have obtained by this roundabout means are in fact the *shadow prices* which we referred to in the earlier part of this section. They arise naturally, as subsidiary information, out of the optimal solution to the primal model if the simplex algorithm is used. It would, often, in fact be possible to deduce the shadow prices from simply the optimal solution values of the variables by using an argument similar to the accountancy argument which we used above. We will not, however, discuss the derivation of the shadow prices any longer as most package programs present these values in the output of the optimal solution.

It can fairly easily be seen (although we do not do so in this book) that the values of the dual variables (the shadow prices) represent the effects of small changes on the right-hand side coefficients, that is, they are *marginal valuations*. For example, if we were to increase grinding capacity by a small amount Δ then the resultant increase in total profit (after rearranging our optimal production plan) would be £$(6.25 \times \Delta)$. Similarly the decrease in total profit by reducing grinding capacity would be £$(6.25 \times \Delta)$. There will usually be limits within which Δ can lie. These limits (*ranges*) are discussed in the next section. (Strictly speaking it is possible that one or both of these limits be zero. This complication is discussed later). It will also only be valid to interpret the shadow prices as referring to the effect of small changes on *one of the right-hand side coefficients at a time*, that is, we could not make small changes in two right-hand side coefficients simultaneously and conclude that the effect on total profit will be the sum of the shadow prices.

Shadow prices can be of considerable value in making investment decisions. For example, each extra hour of grinding capacity is worth £6.25 per week to the factory. This interpretation remains valid as long as we are permitted to increase our grinding capacity by a sufficient amount. In the next section we will see that this capacity can be increased up to 384 hours per week (its *upper range*) with each extra unit resulting in an extra £6.25 per week. The effect of increasing capacity beyond this upper range will result in a smaller (though not immediately predictable) extra profit per unit of increase. As we can increase grinding capacity up to 384 hours per week, enabling us to make an extra £600 of profit, we could decide whether it is worth investing in (or hiring) more grinding machines. We might compare this with the £23.75 which would result from each extra hour of manpower capacity (within the permitted range) and decide where limited funds might be invested to best effect.

The shadow price on a non-binding constraint such as that representing the drilling capacity is zero. As we might expect there is no value in increasing this capacity as we do not use all we have already.

Shadow prices are an example of '*opportunity costs*'. This is a concept which accountants are increasingly (although still too rarely) using. For example an increase in grinding capacity results in an increased *opportunity* to make more profit. Similarly a decrease in grinding capacity loses us some opportunity to make a profit. The shadow price represents the *cost* of the lost *opportunity*. Unlike some of the other costs which accountants use (such as average costs), opportunity costs are quite a sophisticated concept. They result from a careful weighing up of the demands which each product makes on the scarce resources and contributes to profit in return. As a consequence they take into account the *alternative uses* to which the resources may be put and the comparative values of these alternative uses. One would obviously expect such costs to be of more value than less sophisticated costs. Accountants are becoming increasingly interested in linear programming as a result. A good description of the application of linear programming to accountancy is given by Salkin and Kornbluth (1973).

Our discussion of the interpretation of shadow prices has been confined to our product mix application. For other applications it should be possible to deduce the correct interpretation by relating small changes in the right-hand side coefficients to the physical situation represented by the model. To help with any such interpretation we suggest meanings which might be attached to the different types of constraint described in Section 3.3.

6.2.3 Productive capacity constraints

These are the sort of constraints we have discussed above where the capacity may represent limited processing or manpower. The interpretation of the shadow prices on such constraints has been fully dealt with above.

6.2.4 Raw material availabilities

Suppose we have modelled limited raw material availabilities by constraints. Those raw materials which the model suggests be used to their limit will be represented by constraints generally (but not always) having a non-zero shadow price. This shadow price will indicate the value of acquiring more of the raw material (within the permitted ranges). Similarly it will represent the cost of cutting back on raw material. This shadow price may be very useful in helping to decide whether to purchase more of the raw material or not (at a certain cost).

6.2.5 Marketing demands and limitations

The interpretation of the shadow prices here will be the effect on the value of the objective of altering demands or limitations of the market, for example, forcing the factory to produce more or less, or increasing or decreasing the maximum market size. Such a figure can often usefully be compared with the cost of extra sales effort. Frequently such constraints will take the form of simple upper bounds, as discussed in Section 3.3. If such simple upper bounds are treated as such in the model, they will not appear as constraints and therefore have no shadow price. The desired interpretation can, however, be obtained from the reduced cost of the bounded variable as described below.

6.2.6 Material balance (continuity) constraints

The shadow price on such constraints may well have no useful interpretation. For example, the small blending problem (Example 1.2) of Section 1.2 has a material balance constraint to make sure that the weight of the final product equals the total weight of the ingredients. The right-hand side value is zero. The shadow price predicts the effect of altering this zero value. It is hard to see a useful interpretation for this. In some circumstances the shadow price on a material balance constraint may be of interest. For example we may be equating our initial (or final) stocks to some expression involving the variables of the model. The shadow price will then indicate the effect on the optimal objective value of changing these stocks. A good example of this is the FOOD MANUFACTURE model of Part II.

6.2.7 Quality stipulations

Any model which involves blending as part of the total model usually involves quality stipulations, for example, the proportion of vitamins must not fall below a certain value or the octane rating of the petrol must not fall below some value. As an example, the blending problem in Section 1.2 has two quality constraints indicating a limitation on 'hardness'. The shadow prices of these constraints can be used to predict the effect on total revenue of relaxing or tightening up on these hardness stipulations. In some models where the right-hand side coefficient is itself a quality parameter, the interpretation is straightforward. For our small example we had a right-hand side value of 0. The upper hardness constraint is

$$8.8x_1 + 6.1x_2 + 2x_3 + 4.2x_4 + 5x_6 - 6y \leq 0. \tag{6.18}$$

Suppose this right-hand side coefficient were not 0 but were Δ. We could rewrite the constraint as

$$8.8x_1 + 6.1x_2 + 2x_3 + 4.2x_4 + 5x_5 - \left(6 + \frac{\Delta}{y}\right) y \leq 0. \tag{6.19}$$

In order to interpret the effect of a relaxing or tightening of the upper hardness limitation of 6, we would have to take into account the value of y in the optimal solution. A unit increase or decrease in the hardness 6 would result in an increase or decrease of y in the right-hand side and the interpretation of the shadow price would have to be adjusted accordingly. The derivation of ranges within which the hardness parameter could validly be altered with this interpretation would be difficult as the value of y would probably alter as well.

To decide whether a small change in a right-hand side coefficient results in an *increase* or *decrease* in the optimal objective value we should decide whether the change results in a *relaxation* or a *tightening* of the problem. If we relax a problem slightly, for example, increase the right-hand side coefficient on a '$\leq$' constraint or decrease the right-hand side coefficient on a '$\geq$' constraint, we would expect to at worse have no effect on the optimal objective value and perhaps improve it. Therefore the optimal objective value in a maximization problem would possibly get larger and in a minimization problem possibly get smaller, that is, we would expect to do no worse, possibly to do better in view of the lessening of the restrictions. For a tightening of the constraints, for example, reducing the right-hand side coefficients in a '$\leq$' constraint and increasing them in a '$\geq$' constraint, we would expect to do worse if anything, that is, possibly degrade the optimal objective value. In the case of '$=$' constraints the improving or degrading effect of small changes in the right-hand side coefficient will probably be apparent from the meaning of the model.

Rather than formulate mathematical rules for interpreting whether the shadow price should be added or subtracted from the objective for each unit change in the corresponding right-hand coefficient it is probably better to follow the above scheme as different package programs vary in their sign conventions regarding shadow prices.

The suggested formulation of the TARIFF RATES (POWER GENERATION) problem in Part III causes the required rates for the sale of electricity to arise as shadow prices on demand constraints.

6.2.8 Reduced costs

In our small product mix example we saw that the unit profit contributions of PROD 1 and PROD 2 are exactly accounted for by the imputed 'costs' derived from the shadow prices on each constraint. For PROD 3, PROD 4 and PROD 5, however, the imputed costs exceed the unit profit contribution. The amount by which these unit profit contributions are exceeded in each case is of interest. These quantities are the *reduced costs* of these appropriate variables. Note that any variable which comes out at a non-zero level in the optimal solution has a zero reduced cost (this result is modified if the variable has a simple upper bound; the modification is dealt with later).

For our example the reduced costs which we derived earlier for PROD 3, and could also derive for PROD 4 and PROD 5 are

PROD 3	£125
PROD 4	£231.25
PROD 5	£368.75

If we wanted to make PROD 3 we would have to increase its unit price by £125 before it was (just) worth making. This price increase would then allow the profit contribution of PROD 3 to balance the 'costs' imputed to it by way of its usage of scarce resources. Similarly, the reduced costs of PROD 4 and PROD 5 indicate price increases necessary for those products. The reduced costs are usually printed out with the solution values of the variables when a package program is used and there is no necessity to calculate them by way of the shadow prices as we have done. Our purpose in using this derivation has been to indicate the correct interpretation which should be placed on reduced costs. For other applications the interpretation of reduced costs will be different. In a blending problem for example (such as that in Section 1.2) the reduced costs might represent price decreases necessary before it became worth buying and incorporating an ingredient in a blend.

The above interpretation of reduced costs can be viewed the other way round. Suppose we insisted on making a small amount of PROD 3. As this will deny processing and manpower capacity to some of the other products (who could use it to better effect) we would expect to degrade our total profit. The amount by which this profit will be degraded for each unit of PROD 3 we make will be given by the reduced cost (£125) of PROD 3. In fact there will be a limit to the level at which PROD 3 can be made with this interpretation holding. This limit is another type of *range* which will be discussed in the next section. It is therefore possible to cost a non-optimal decision such as making PROD 3. This cost results from the lost opportunity to make other, more profitable, products. It is again an *opportunity cost*.

We should mention the slight variation in the interpretation of the reduced costs on variables with a finite *simple upper bound* as discussed in Section 3.3.

If, in the optimal solution to a model, such a variable comes out at a value below its simple upper bound, there is no difficulty. The interpretation of the reduced cost will be exactly the same as that above. Suppose, however, that the variable were to come out at a value equal to its simple upper bound. This simple upper bound could well, in this case, be acting as a binding constraint. If the simple upper bound had been modelled as a conventional constraint, it would then have a non-zero shadow price which would be interpreted in the usual way. The variable would, being at a non-zero level, have a zero reduced cost. By modelling this constraint as a simple upper bound we will have lost the shadow price but it will appear as the reduced cost of the variable. The correct interpretation of this reduced cost will be the effect on the objective of forcing the variable down below its upper bound (degrading the objective) or increasing the upper bound (improving the objective).

We showed above how the shadow prices (values of the dual variables) could be used as 'accounting costs' in order to derive the optimal production plan in the product mix model. It is important to point out that such a procedure will not always work. This discussion also leads us on to an important feature of some linear programming models: *alternative optimal solutions*. Let us consider the following small linear programming model:

$$\begin{aligned}
\text{Maximize}\quad & 3x_1 + 1.5x_2 && (6.20)\\
\text{subject to}\quad & x_1 + x_2 \le 4, && (6.21)\\
& 2x_1 + x_2 \le 5, && (6.22)\\
& -x_1 + 4x_2 \ge 2, && (6.23)\\
& x_1, x_2 \ge 0
\end{aligned}$$

If the dual model is formulated with dual variables y_1, y_2 and y_3 corresponding to the three constraints, and then solved we obtain the following result:

$$y_1 = 0, \qquad y_2 = 1.5, \qquad y_3 = 0$$

An accountant could then use the values of y_1, y_2 and y_3 as valuations for the constraints. This would lead the accountant to the following conclusions:

1. The 'accounting cost' attributed to each unit of x_1 is 3, exactly equalling its objective coefficient. Similarly the 'accounting cost' attributed to x_2 is 1.5 exactly equalling its objective coefficient. Therefore x_1 and x_2 should both be allowed to be in the solution of the original (primal) model.

2. The second constraint (6.22) has a non-zero valuation and must therefore be binding (i.e. can be treated as an equation). It would be wrong, however (as was mentioned in applying the same procedure to the product mix example), to immediately conclude that the constraints (6.21) and (6.23) are non-binding as they have zero dual variables (shadow prices).

Suppose for the moment we did ignore constraints (6.21) and (6.23). We deduce that x_1 and x_2 must satisfy the equation

$$2x_1 + x_2 = 5. \tag{6.24}$$

This equation obviously does not determine x_1 and x_2 uniquely. It is by no means obvious how the accountant should proceed from here. In fact it is impossible to deduce a unique solution to the above model from the solution to the dual model as the original (primal) model does not possess a unique solution. This is easily seen geometrically in Figure 6.1.

Different values of the objective function (6.20) correspond to different lines parallel to PQ. It so happens that PQ is parallel to the edge of AC of the feasible region created by constraint (6.22). We therefore see that any point on the line

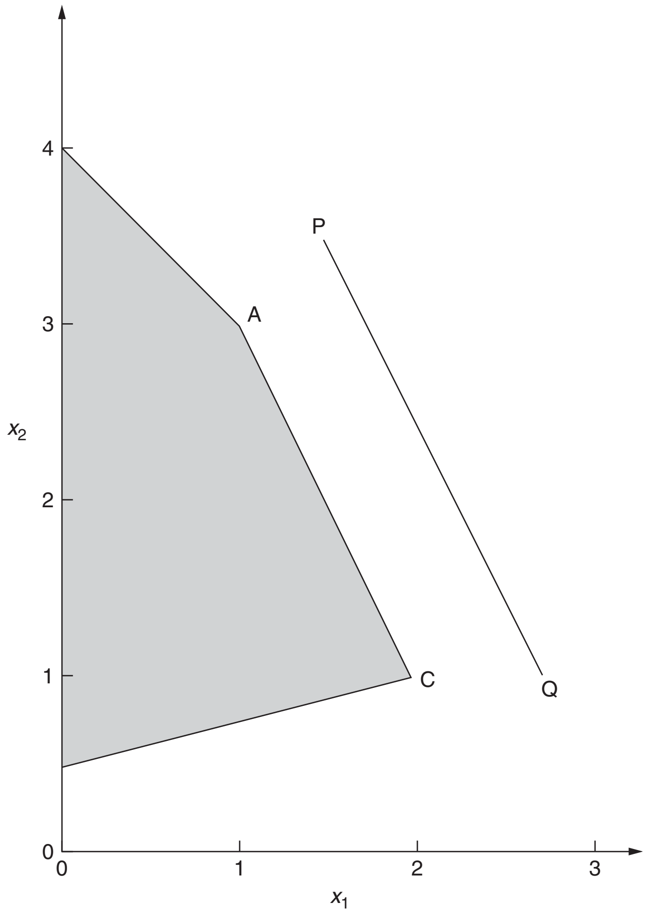


Figure 6.1

AC between (and including) A and C, gives an optimal solution to our model (objective $= 7.5$).

All that our accounting information has told us (if we ignore constraints (6.21) and (6.23)) is that $2x_1 + x_2 = 5$, that is, that we must choose points lying on (or beyond) the line AC. We still have to pay some attention to the constraints (6.21) and (6.23). If we were to treat constraint (6.21) as binding we would arrive at the point A. Treating (6.23) as binding would give us point C. As mentioned in Section 2.2, the simplex algorithm only examines vertex solutions and would therefore choose *either* the solution at A ($x_1 = 1, x_2 = 3$) *or* the solution at C ($x_1 = 2, x_2 = 1$). Most package programs indicate when there are alternative solutions. They recognize this result by noticing one of the following properties of the optimal solutions:

1. A binding constraint with a zero shadow price.
2. A variable at zero level with a zero reduced cost.

In our example, if the optimal solution at A had been selected, we would note that constraint (6.21) was binding but had a zero shadow price. At C it would be constraint (6.23) that was binding with a zero shadow price.

It might seem rather unlikely that alternative solutions would arise in a practical linear programming model. In fact this is quite a common occurrence. For problems with more than two variables the situation will obviously be more complex. The characterization of all the optimal solutions (all vertex optimal solutions and various combinations of them) is often very difficult. It is, however, important to be able to recognize the phenomenon as if one optimal solution is unacceptable for some reason we have a certain flexibility to look for another solution without degrading our objective.

One of the purposes of this discussion has been to indicate that applying valuations (shadow prices) to the constraints of a linear programming model sometimes does not lead us to a unique solution (operating plan). The converse situation can also happen. We can have a unique solution (operating plan) determined by more than one set of valuations. For example, consider the following small problem:

$$\text{Maximize} \quad 3x_1 + 2x_2 \tag{6.25}$$

$$\text{subject to} \quad x_1 + x_2 \leq 3, \tag{6.26}$$

$$2x_1 + x_2 \leq 4, \tag{6.27}$$

$$4x_1 + 3x_2 \leq 10, \tag{6.28}$$

$$x_1, x_2 \geq 0.$$

Let us attach valuations y_1, y_2, and y_3 to each of the constraints (6.26)–(6.28). There are a number of valuations which will lead us to the optimal solution. For example,

1. $y_1 = 1, \quad y_2 = 1, \quad y_3 = 0;$
2. $y_1 = 0, \quad y_2 = \frac{1}{2}, \quad y_3 = \frac{1}{2}.$

Both these sets of valuations will lead us to the unique optimal solution to this problem:

$$x_1 = 1, \qquad x_2 = 2,$$

giving an objective value of 7.

In a practical problem the question that would naturally arise is how should we derive a shadow price for our scarce resources given this ambiguity, for example, what would the value be of increasing the right-hand side value of 3 to 4? Would it be 1 or 0? This problem is again best illustrated by a diagram (Figure 6.2).

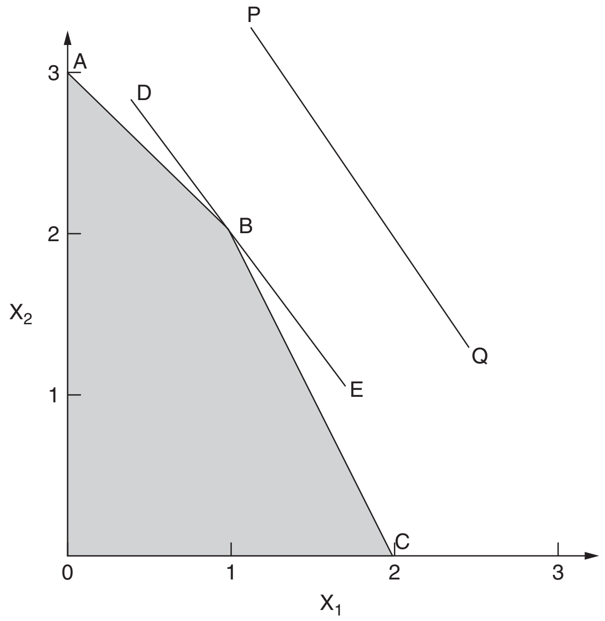


Figure 6.2

The constraints (6.26)–(6.28) give rise to the lines AB, BC and DBE respectively. Different objective values give rise to lines parallel to PQ, showing that B represents the optimal solution. The reason for the ambiguity in shadow price is that we have the 'accidental' result of three constraints going through the same point. Normally in two dimensions we would only expect two constraints to go through B. We can therefore regard either one of the constraints (6.26) or (6.28) as non-binding (having a zero valuation) in the presence of the other constraint. This leads to the ambiguity in our shadow prices.

In a situation such as this it would not be correct to increase the right-hand side value of 3 in constraint (6.26) at all using the shadow price of 1. The *upper range* of this right-hand side value (with this shadow price) will be 3. The coefficient is already at its upper range and any further increase will not alter the objective at the rate suggested by the valuation.

In these situations the shadow price is usually defined as being the rate of change of value of the objective function with respect to the right-hand side value. Then it can be shown that the *upper shadow price* (for increases in right-hand side value) is the *minimum* of *all* the possible valuations and the *lower shadow price* (for decreases in right-hand side value) is the maximum of all the possible valuations. Defining y_1, y_2, and y_3 as shadow prices we have

$$\begin{array}{llll} \text{upper shadow prices} & y_1 = 0 & y_2 = \frac{1}{2}, & y_3 = 0; \\ \text{lower shadow prices} & y_1 = 1, & y_2 = 1, & y_3 = \frac{1}{2}. \end{array}$$

When a computer package is used, the ‘shadow prices’ printed out are usually only the values of the dual variables corresponding to one dual solution. They are therefore misleading if interpreted as shadow prices according to the above definition. In fact to evaluate all the alternative dual solutions, and therefore obtain the true shadow price, would generally be computationally prohibitive. It is, however, possible to evaluate *ranges* within which the valuations associated with any particular dual solution chosen are valid. For example, the upper shadow price on constraint (6.26) is not 1. If we take the dual solution (i) above, then the associated *upper range* on the right-hand side coefficient of 3 is also 3, showing there is no range within which this coefficient can be increased to reflect a rate-of-change increase of 1 in the objective. This example is considered again in Section 6.3, when ranges are discussed. A good discussion of the problem is given by Aucamp and Steinberg (1982).

It should be pointed out that the two complications we have just discussed are dual situations. On the one hand we had a unique dual solution leading to more than one primal solution. The first phenomenon is referred to as *alternative solutions* (in the primal model) while the second phenomenon is referred to as *degeneracy* (in the primal model).

6.3 Sensitivity analysis and the stability of a model

6.3.1 Right-hand side ranges

In the last section we described how shadow prices can be useful in predicting the effect of small changes in the right-hand side coefficients on the optimal value of the objective function. It was pointed out, however, that this interpretation is only valid in a certain *range*. In this case the range is known as a *right-hand side range*. We will again use the product mix problem of Section 1.2 to illustrate our discussion. This gave the model

$$\begin{aligned}
\text{Maximize} \quad & 550x_1 + 600x_2 + 350x_3 + 400x_4 + 200x_5 & & (6.29)\\
\text{subject to} \quad & 12x_1 + 20x_2 + 25x_4 + 15x_5 & \leq 288, & \quad (6.30)\\
& 10x_1 + 8x_2 + 16x_3 & \leq 192, & \quad (6.31)\\
& 20x_1 + 20x_2 + 20x_3 + 20x_4 + 20x_5 & \leq 384. & \quad (6.32)
\end{aligned}$$

We saw that the optimal solution told us to make 12 of PROD 1 and 7.2 of PROD 2 and nothing of PROD 3, PROD 4 and PROD 5. This resulted in a profit of £10.920 and exhausted the grinding capacity (constraint (6.30)) and manpower capacity (constraint (6.32)).

The shadow prices on constraints (6.30)–(6.32) turned out to be 6.25, 0 and 23.75. Increasing the grinding capacity by Δ hours per week would therefore result in a weekly increase in a total profit of £$(6.25 \times \Delta)$. Similarly, cutting back on grinding capacity by Δ hours per week would result in a weekly decrease in the total profit of £$(6.25 \times \Delta)$. We are interested in the limits within which the change Δ can take place for this interpretation to apply. For this example these ranges are as follows:

lower range 230.4

upper range 384,

that is, we can increase grinding capacity by up to 96 hours a week or decrease it by up to 57.6 hours per week with the predicted effect on profit. Changes outside these ranges are unpredictable in their effect and require further analysis. The information that we have got here is, however, of some, if limited, usefulness. It tells us, for instance, that adding one grinding machine would improve profit by £$(96 \times 6.25) = $£600 per week. We could not, however, predict the effect on total profit of taking away a grinding machine as this would take us below the lower range of the grinding capacity. It would nevertheless be correct to say that £600 is, if anything, an *underestimate* for the resultant decrease in profit.

It is important to restate that this interpretation of the effect on the objective of decreasing or increasing a right-hand side coefficient is only valid if one coefficient is changed at a time within the permitted ranges. The effect of changing more than one coefficient at a time, as well as changes outside the permitted ranges, are efficiently examined by *parametric programming*, which is discussed in the next section.

It is beyond the scope of this book to describe how the right-hand side ranges are calculated. Most computer packages provide these ranges with their solution output.

For completeness we will give the upper and lower ranges on the other two capacities: drilling and manpower.

The ranges on the drilling capacity of 192 hours per week are fairly trivial to obtain. Remember that we are not using all our drilling capacity. In fact we have a slack capacity of 14.4 hours. This immediately enables us to calculate the lower range by subtracting this figure from the existing capacity. As we are not using all our drilling capacity, increasing it without limit can have no effect on the solution. The ranges are therefore

lower range 177.6

upper range ∞.

Within these ranges there will be no change at all in the objective (or optimal solution) as the shadow price on the (non-binding) drilling constraint is zero.

The ranges on the manpower capacity of 384 hours per week are again non-trivial to calculate. They turn out to be as follows:

lower range 288

upper range 406.1.

For changes within these ranges the objective changes by £23.75 for each unit of change. For example an extra worker (48 hours per week) improves profit by $(23.75 \times 48) = £1140$ per week. This might prove quite valuable information when compared with the fact that an extra grinding machine is worth only £600 per week.

The meaning of right-hand side ranges is well illustrated geometrically using a two-variable example:

$$\begin{aligned}
\text{Maximize} \quad & 3x_1 + 2x_2 && (6.33)\\
\text{subject to} \quad & x_1 + x_2 \le 4, && (6.34)\\
& 2x_1 + x_2 \le 5, && (6.35)\\
& -x_1 + 4x_2 \ge 2, && (6.36)\\
& x_1, x_2 \ge 0.
\end{aligned}$$

Geometrically we have the situation in Figure 6.3. The optimal solution is represented by the point A, giving an objective value of 9. The shadow price on constraint (6.34) turns out to be 1.

If constraint (6.34) is relaxed by increasing the right-hand side coefficient from 4 to 4.5 we move the boundary AB of the feasible region to A′B′. The resultant increase in the objective is $1 \times 0.5 = 0.5$. We can continue to increase the right-hand side coefficient to 5 when this boundary of the feasible region shrinks to the point D. There is clearly no virtue in further increasing the right-hand side coefficient as the point D will continue to represent the optimal solution. The *upper range* on the right-hand side coefficient in constraint (6.34) therefore is obviously 5.

If constraint (6.34) is tightened by decreasing the right-hand side coefficient, the boundary AB of the feasible region progressively moves down until

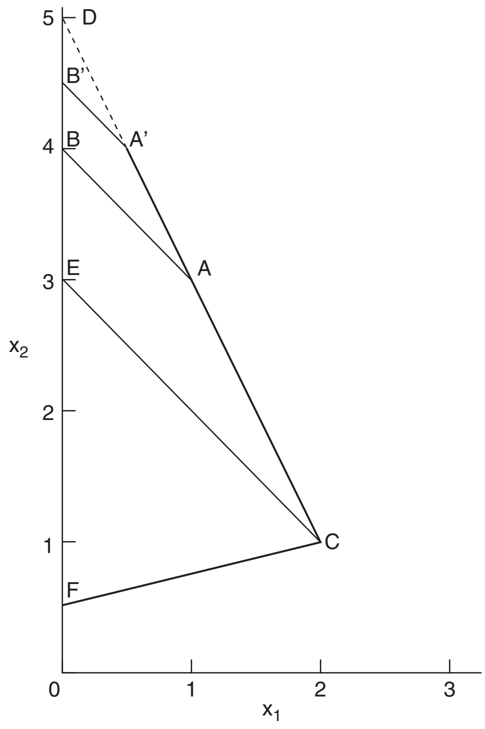


Figure 6.3

it becomes CE. This happens when the right-hand side coefficient is 3. Any decreases in this coefficient between 4 and 3 result in a degradation of the objective by 1 (the shadow price) for each unit of decrease. The value 3 is the *lower range*. For decreases in coefficient values below 3 the optimal objective value will decrease at a greater rate. This rate will not immediately be predictable from our knowledge of the solution at A. For decreases down to 3 the optimal objective value is represented by points on the line AC. Decreases below 3 give points on the line CF.

So far we have only suggested using right-hand side ranges together with shadow prices to investigate the effect which changes on the right-hand side coefficients have on the optimal objective value. Such investigations sometimes give rise to the expression *post-optimal analysis*. Another purpose to which such information can be put is in investigating the sensitivity of the model to the right-hand side data. This forms part of a subject known as *sensitivity analysis*.

We will consider this new use to which right-hand side ranging information can be put in relation to the product mix model. Suppose we were rather doubtful about the accuracy of the grinding capacity of 288 hours per week (constraint (6.30)). In practice it is quite likely that such a figure would be open to doubt as machines could well be down for maintenance or repair for durations which were not wholly predictable. How confident should we be in suggesting that the factory dispenses with making PROD 3, PROD 4 and PROD 5 and only concentrates on PROD 1 and PROD 2? From our range information we can immediately say that this should remain the optimal policy as long as the true capacity figure does not lie outside the limits 230.4–384. Although the policy (in terms of what is, and is not, made) does not change within these limits, the levels at which PROD 1 and PROD 2 are made (and the resultant profit) obviously will change. This information is therefore of limited, but nevertheless sometimes significant, usefulness. Obviously if there is doubt about one capacity figure there will probably also be doubt about others and ranging information can only strictly be applied when one change only is made. Nevertheless such information does give some indication of the sensitivity of the solution to accuracy in the right-hand side data. If, for example, the ranges on the right-hand side coefficient in constraint (6.30) had been very close, say 287–288.5, we would have to be very careful in applying our suggested solution as it clearly would depend very critically on the accuracy of the grinding capacity figure.

It should be pointed out that the sensitivity analysis interpretation of the easily obtained ranges of 177.6 and ∞ on the non-binding drilling constraint is rather stronger. As long as this capacity lies within these figures not only will our optimal solution be no different, but the levels of the variables in that optimal solution will be unchanged as well.

Changing the right-hand side coefficient on a binding constraint within the permitted ranges does of course alter the values of the variables in the optimal solution (as well as the objective value). The rates at which the values of the variables change are quite easily obtained using most package programs. Such values are known as *marginal rates of substitution* and discussed below. Before doing this, however, we will consider other types of ranging information to be obtained from a linear programming model.

6.3.2 Objective ranges

It is often useful to know the effects of changes in the objective coefficients on the optimal solution. Suppose we change an objective coefficient (e.g. a unit profit contribution or cost). How will this affect the value of the objective function? In an analogous manner to right-hand side ranges we can define *objective ranges*. If a single objective coefficient is changed within these ranges then the optimal solution values of the variables will not change (although the optimal value of the objective may change). This is a rather stronger result than in the right-hand side ranging case where the solution values could change. The result is not altogether intuitive. If a particular item became more profitable or costly we might expect to bias our solution towards or away from that item. In fact our solution will not change at all as long as we remain within the permitted ranges. The situation is well described geometrically by means of the two-variable model we used before:

$$\text{Maximize} \quad 3x_1 + 2x_2 \tag{6.37}$$

$$\text{subject to} \quad x_1 + x_2 \leq 4, \tag{6.38}$$

$$2x_1 + x_2 \leq 5, \tag{6.39}$$

$$-x_1 + 4x_2 \geq 2, \tag{6.40}$$

$$x_1, x_2 \geq 0.$$

This gives rise to Figure 6.4. The optimal solution ($x_1 = 1, x_2 = 3$, objective = 9) is represented by the point A. The line PQ represents the points giving an objective value of 9.

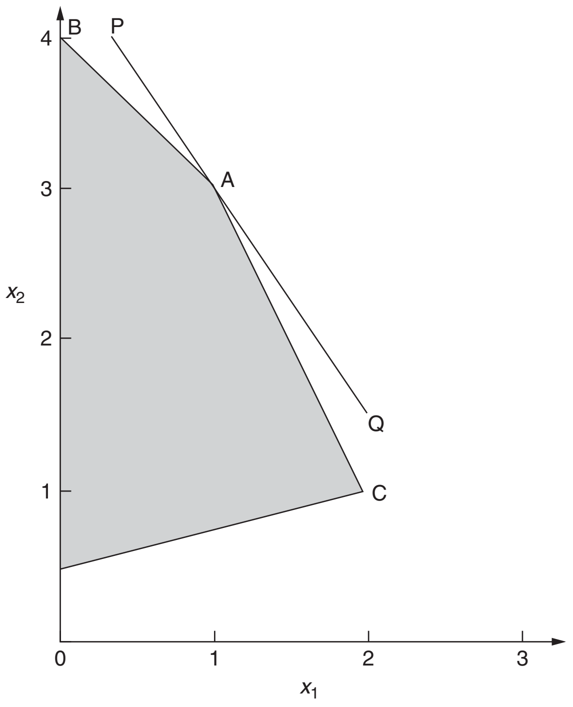


Figure 6.4

Suppose that the objective coefficient 3 of variable x_1 were to increase. The steepness of the line PQ would increase (in a negative sense) but A would still represent the optimal solution as long as PQ did not rotate (around A) in a clockwise direction beyond the direction AC. When the objective coefficient of x_1 became 4 the objective line would coincide with the line AC. This provides us with an upper range of 4 for this coefficient. Similarly a lower range is provided by decreasing this coefficient until the objective line coincides with AB. This gives a lower range of 2. Note that as long as the objective coefficient of x_1 remains within the range 2–4, the optimal solution values of the variables will be unchanged and represented by point A. This is the slightly unintuitive result mentioned before. The optimal objective value does of course change if we alter this coefficient within its ranges. As x_1 has a value of 1 in the optimal solution, each unit of increase or decrease in its objective coefficient will obviously provide an increase or decrease of 1 in the objective value.

Ranges on the objective coefficient of 2 for x_2 can be discovered in a similar manner.

We will now return to our product mix example and present the objective ranges for that problem. To describe how these ranges are calculated in a problem which cannot be represented geometrically is beyond the scope of this book. Most package programs do, however, provide this information. Our main concern is with the correct interpretation rather than the derivation.

For PROD 1 the ranges on the objective coefficient are

lower range £500

upper range £600.

This means that if we vary the objective coefficient of PROD 1 within these limits the values of the variables in our optimal solution will not change. We should still continue to produce 12 of PROD 1, 7.2 of PROD 2 and nothing else. As remarked before, such a result is often difficult for people with little understanding to accept. We might expect there to be a gradual and continuous shift towards making more of PROD 1, the more expensive we make it. In fact the production plan should not alter at all until PROD 1 produces a unit profit contribution of more than £600. When this happens there will be a new optimal production plan (almost certainly involving more of PROD 1) the investigation of which would require further analysis. The same restrictions apply to the interpretation of objective ranges as applied in the case of right-hand side ranges. Our interpretation is only valid within the permitted ranges and if only one change is made in an objective coefficient. This clearly limits the value of this information somewhat. *Parametric programming* provides an efficient means of investigating the effects of more than one change, possibly outside the ranges.

Although the optimal solution values of the variable do not change within the ranges, the objective value obviously will. For each £1 that we add to the price of PROD 1 we obviously improve the objective by £12 as the optimal solution involves producing 12 of PROD 1.

Similarly objective ranges can be obtained for PROD 2. These turn out to be

lower range £550

upper range £683.3.

For PROD 3, PROD 4 and PROD 5, which should not be made in the optimal production plan, the derivation of objective ranges is simple. We have already seen that PROD 3 must be increased in price by £125 (its reduced cost) before it is worth making. This gives us its upper range. As it is already too cheap to be worth making a reduction in price cannot affect the solution. The objective ranges for PROD 3 are therefore

lower range $-\infty$

upper range £475.

For PROD 4 we have

lower range $-\infty$

upper range £631.25.

For PROD 5 we have

lower range $-\infty$

upper range £568.75.

Objective ranges can be put to a similar use to that which right-hand side ranges are put in sensitivity analysis. If there were doubt surrounding the true profit contribution of £550 from PROD 1, we could be sure of our production plan as long as we knew that this figure was between £500 and £600. The width or narrowness of the objective ranges enables one to obtain a good feel for the sensitivity of the solution to changes (or errors in the objective data). In fact this interpretation is rather stronger than is the case with right-hand side ranges. Not only will the optimal production policy not change if a coefficient is altered within its permitted ranges; the values of the variable quantities in that plan will not alter either.

6.3.3 Ranges on interior coefficients

It is also possible to obtain ranges for other coefficients in a linear programming model. Sometimes these may provide valuable information in much the same way that right-hand side and objective ranges do. Generally, however, such information is far less useful. When a model is built often the only data which are changed are the objective and right-hand side coefficients (taken together these are sometimes referred to as the *rim* of the model). In consequence many package programs do not provide the facility for obtaining ranges on coefficients other than in the rim. When such information is obtainable its interpretation and use is very similar to that in the case of right-hand side and objective ranges.

We must now consider the complication which we described in Section 6.2 when there was ambiguity concerning the valuations (shadow prices) of the constraints. This ambiguity also extends to the ranging information. Alternative sets of valuations corresponding to a unique optimal solution lead to alternative sets of ranges. We will repeat the small example we used before:

$$\begin{aligned} \text{Maximize} \quad & 3x_1 + 2x_2 && (6.41)\\ \text{subject to} \quad & x_1 + x_2 \le 3, && (6.42)\\ & 2x_1 + x_2 \le 4, && (6.43)\\ & 4x_1 + 3x_2 \le 10, && (6.44)\\ & x_1, x_2 \ge 0. \end{aligned}$$

There are a number of possible valuations for the constraints which give rise to the optimal solution. We presented the following two where y_1, y_2 and y_3 are the dual values on constraints (6.42)–(6.44), respectively:

$$\begin{array}{llll} \text{(i)} & y_1 = 1, & y_2 = 1, & y_3 = 0; \\ \text{(ii)} & y_1 = 0, & y_2 = \frac{1}{2}, & y_3 = \frac{1}{2}. \end{array}$$

(i) arises when we regard constraints (6.42) and (6.43) only as binding; (ii) arises when we regard constraints (6.43) and (6.44) only as binding.

Table 6.1 gives the right-hand side ranges within which the set (i) can be used as shadow prices.

Table 6.1

Constraint	Lower Range	Upper Range
(6.42)	2	3
(6.43)	3	4
(6.44)	10	∞

Each range extends on one side of the right-hand side coefficient only. For example, constraint (6.42) has an upper range of 3, indicating that it would be incorrect to use a shadow price of 1 on this constraint to predict the effect of *increasing* the right-hand side coefficient. It would, however, be correct to use this as a shadow price to predict the effect of *decreasing* this coefficient. Similar arguments apply to the other two constraints. For constraint (6.44) we can increase the right-hand side coefficient of 10 indefinitely without affecting the optimal solution or objective value (shadow price of 0). We cannot, however, decrease this coefficient at all without altering the objective. This last result is obvious from studying Figure 6.2.

The clue to changes in the right-hand side coefficients in the other directions is given by the set of valuations (ii) together with their corresponding ranges. These ranges are given in Table 6.2. For example, increasing the right-hand side coefficient of 3 has no effect (shadow price of 0) on the objective whereas

Table 6.2

Constraint	Lower Range	Upper Range
(6.42)	3	∞
(6.43)	4	5
(6.44)	8	10

decreasing this coefficient does have an effect (shadow price of 1). Similarly the second set of ranges on the other constraints are in opposite directions indicating the ranges of interpretation of the second set as shadow prices.

We have demonstrated how *two-valued shadow prices* can exist for constraints with different marginal values for decreasing or increasing a right-hand side coefficient. In solving a practical model using a package program the user will only be presented with one set of 'shadow prices' corresponding to his or her optimal solution. Which set of dual values this is will be very arbitrary and depend upon the manner in which the model was solved. Solving the same model with exactly the same data could well result in a different set of dual values. The clue to the limited (one-sided) interpretation which can be placed on these shadow prices lies in the right-hand side ranges. If some of these ranges lie on one side of their corresponding right-hand side coefficient only, then this limits the shadow price to being interpreted for changes in one direction only. To investigate the effects of changes in the opposite direction alternate sets of dual values and their corresponding ranges should be sought. Alternatively *parametric programming* could be used as discussed in the next section.

It may well seem to the reader that a disproportionate amount of attention has been paid, in both this and the last section, to the apparently rather trivial complication concerning alternative valuations (shadow prices) for the constraints of a linear programming model. As has already been pointed out, however, this complication is a very common occurrence in practical linear programming models. It is well known in the computational side of linear programming as the phenomenon of *degeneracy*. We have here been concerned with the economic interpretations concerning degenerate solutions. In view of the difficulty which many people have in understanding this, we have given it considerable attention.

A further, rather different, final point should be made concerning the uniqueness of the ranging information from a model. The inclusion or exclusion of redundant constraints from a model will obviously not affect the optimal solution. This could well, however, affect the values of the ranges. Figure 6.3 and its associated model illustrate this point. Constraint (6.36) gives rise to the edge CF of the feasible region. The ignoring of constraint (6.36) would obviously not affect the optimal solution represented by the point A. In this sense constraint (6.36) is redundant and might never have been modelled in the first place in a practical situation if its redundancy had been obvious. The presence or absence of constraint (6.36) will, however, obviously affect the value of the lower right-hand side range on constraint (6.34). With constraint (6.36) present this lower range is 3. If constraint (6.36) were removed it would be 2.5.

6.3.4 Marginal rates of substitution

We have seen that there is a certain flexibility in altering a right-hand side coefficient of a model. As long as we remain with the permitted range the effect on the objective value is governed (for each unit of change) by the shadow price of the constraint. This change in the objective value occurs because the values of the variables in the optimal solution change. The relative rates at which they change are given by the *marginal rates of substitution*. It is beyond the scope of this book to describe how these figures can be calculated but they are obtainable from the solution output of most package programs. A small geometric example will illustrate the meaning which should be attached to these figures. Our example is the same as that used before:

$$\begin{aligned}
\text{Maximize}\quad & 3x_1 + 2x_2 & \\
\text{subject to}\quad & x_1 + x_2 \le 4, & (6.45)\\
& 2x_1 + x_2 \le 5, & (6.46)\\
& -x_1 + 4x_2 \ge 2, & (6.47)\\
& x_1, x_2 \ge 0. &
\end{aligned}$$

The feasible region is represented by Figure 6.5. The optimal solution is represented by point A ($x_1 = 1$, $x_2 = 3$ objective $= 9$).

We have already seen that the right-hand side coefficient of 4 on constraint (6.45) can be changed within the range 3–5, causing the objective to change by 1 (the shadow price on constraint (6.45)) for each unit of change. The values of the variables (x_1 and x_2) will also obviously change. Their rates of change, per unit increase in this right-hand side coefficient, are

x_1	−1
x_2	+2

that is, x_1 will get smaller by one and x_2 will get larger by two for each unit of increase. Decreasing the right-hand side coefficient will obviously produce the opposite effect.

Such figures can be very valuable in many applications. For example, they might indicate how a production plan should alter as a certain resource becomes scarcer or more plentiful. In a blending application they could indicate the rate at which one ingredient should be substituted for another as it became more plentiful.

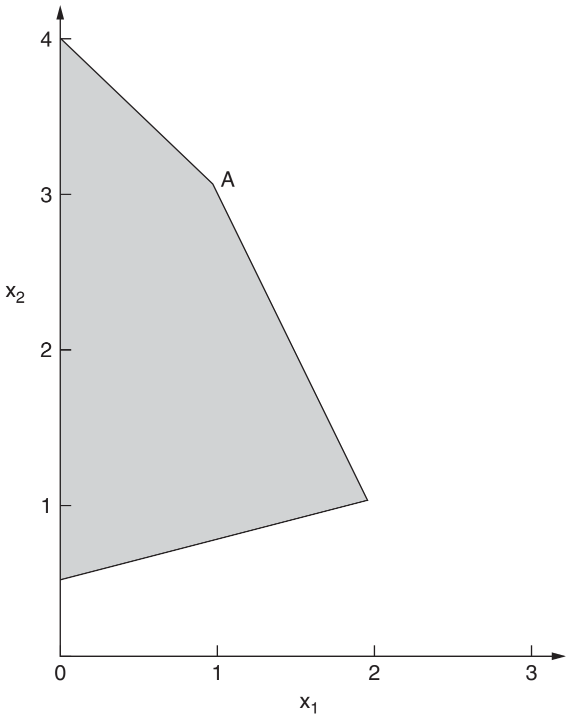


Figure 6.5

The same limitations apply to the interpretation of marginal rates of substitution as apply to shadow prices. We can only consider one change at a time within the permitted ranges. If a range is one-sided we can only apply the marginal rates of substitution for changes in one direction. A different set of marginal rates of substitution will exist (although it will require more computation to derive) for changes in the opposite direction.

The subjects of sensitivity and post-optimal analysis are fully covered in the papers of Greenberg (1993a, 1993b, 1993c, 1994).

6.3.5 Building stable models

We have shown that quite a lot of useful information can be obtained to suggest how sensitive the optimal solution to a linear programming model is to changes (or inaccuracies) in the data. In many practical situations *stable solutions* are more valuable than *optimal solutions*. Having obtained an optimal solution to a model we might be very reluctant to change our operating plan if only small changes occurred in certain parameters, for example, capacities or costs. To some extent we can incorporate this reluctance in our model.

Suppose we were to run out of a particular resource (e.g. processing capacity, raw material, and manpower). It might well be that in practice we would buy in more in this resource at a cost until this increased cost became prohibitive. As was pointed out in Section 6.1, this situation can easily be modelled. For example, suppose the following constraint represented a resource limitation:

$$\sum_j a_j x_j \leq b. \tag{6.48}$$

The alternative representation

$$\sum_j a_j x_j - u = b \tag{6.49}$$

would allow the resource's availability to be expanded at a cost per unit of c if u was given this objective cost (profit of $-c$) in the objective function. Sometimes (6.49) is referred to as a 'soft constraint' in contrast to the 'hard constraint' (6.48) as (6.49) does not provide the total barrier that (6.48) does.

6.4 Further investigations using a model

The previous two sections described how a lot of useful information could be derived concerning the effects on the optimal solution of changes in the coefficients of a linear programming model. Such information was, however, limited in value by the fact that (i) only one coefficient could be changed at a time and (ii) that coefficient could only be changed within certain ranges. *Parametric programming* is a convenient method of investigating the effects of more radical changes. It is beyond the scope of this book to describe how the method works. There is virtue, however, in describing how such investigations can be organized and presented to a computer package. Parametric programming is computationally quick and therefore provides a cheap means of determining how the optimal solution changes with alterations in objective and right-hand side coefficients. It is useful to illustrate the description again by means of a numerical example. We will again use the product mix model introduced in Section 1.2. In the last section we saw how the right-hand side coefficients could be varied individually within their permitted ranges. Suppose that instead we wished to investigate the possibility of increasing both grinding capacity and manpower simultaneously. Parametric programming on the right-hand side allows us to carry out such an investigation as long as the coefficients increase (or decrease) by constant relative amounts. For example, we could investigate the effect of employing an extra worker on assembly for each new grinding machine bought. Each increase of 96 hours per week in grinding capacity will therefore be accompanied by an increase of 48 hours per week in manpower capacity. If θ new grinding machines are bought, the right-hand side vector of coefficients would become

$$\begin{pmatrix} 288 \\ 192 \\ 384 \end{pmatrix} + \theta \begin{pmatrix} 96 \\ 0 \\ 48 \end{pmatrix}$$

Parametric programming allows one to specify a *change column* such as

$$\begin{pmatrix} 96 \\ 0 \\ 48 \end{pmatrix}$$

and *limits* within which θ can vary. The procedure will then allow θ to vary between these limits and calculate the value of the objective function at various intervals. For example, the parameter θ was allowed to vary between 0 (the original capacities) and 10 (when the new capacities become 1248, 192 and 864 respectively).

Such information can be very valuable in examining the effect of expansions or contractions in the availability of different resources. The result is well expressed graphically, as is done for the example we have just discussed in Figure 6.6.

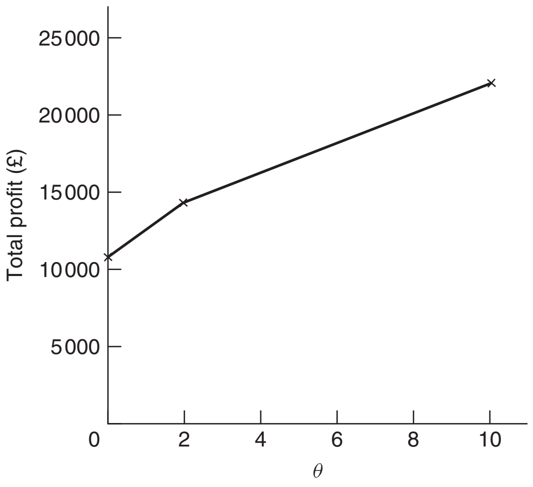


Figure 6.6

Analogous to parametric programming on the right-hand side is parametric programming on the objective function. For example, in the product mix model, if we wished to increase the prices of all products by the same amount simultaneously, we would specify a *change row*:

$$(1, 1, 1, 1, 1)\,.$$

A parameter θ would be allowed to vary between specified limits adding θ times this row to the row of objective coefficients. The effect on the optimal objective view would again be mapped out.

6.5 Presentation of the solutions

This chapter has described a lot of useful information which can be derived from linear programming models. It is often easy to lose sight of the difficulty which non-specialists may have in relating such information to the real life situation which has been modelled. Obscure presentation of information derived from a linear programming model can lead to complete rejection of the technique. Computer output is often obscure as a package program is designed to solve any model no matter what the application is. The output must obviously, therefore, relate to the mathematical aspects of the model. One way of overcoming the difficulty is for an intermediate person (possibly the model builder) to convert the computer output into a written report which can easily be understood. For models which are run frequently this can be a cumbersome and slow process. Many organizations have computer programs which read the solution output from the linear programming package and convert it into a form which relates to the application. Computer programs which do this are known as *report writers*. Clearly a report writer program must be related to the application. As a consequence such programs are usually purpose written for the type of model solved. Frequently they are written by the organization using the model rather than by the producer of the package. A high level language is usually used. It is necessary for the program to read the solution output from the package. With most packages it is quite easy to do this by getting the package to write its output on to a file. Many practitioners would argue in favour of such purpose-built report writers. There do, however, exist general purpose report writers which sometimes accompany particular packages. Sometimes these report writers can be used satisfactorily for the application required.

Report writers are often considered in conjunction with matrix generators which were discussed in Section 3.5. A matrix generator relates the *input* for a linear programming model to the practical problem while a report writer so relates the *output*.

In order to show the advantage of a report writer, the information is presented through a purpose-built report writer for the application. The output should be self-explanatory.

```
DATE:    1 JANUARY   1999
               OPTIMAL SOLUTION TO BLEND PROBLEM
        TOTAL REVENUE £17 593
        MANUFACTURE   450   TONS OF PROD
                USE   159   TONS OF VEG1
                USE    41   TONS OF VEG2
                USE   250   TONS OF OIL2
        HARDNESS OF PROD 6.0 UNITS
        ************************************************
        PRICE REDUCTION NECESSARY BEFORE PURCHASE
                   OIL1              £12/TON
                   OIL3               £8/TON
        ************************************************
        MARGINAL VALUE OF EXTRA REFINING CAPACITY
           VEGETABLE OILS                £30/TON
           NON-VEGETABLE OILS            £47/TON
        ************************************************
```

7

Non-linear models

7.1 Typical applications

It has already been pointed out that many mathematical programming models contain variables representing activities that compete for the use of limited resources. For a linear programming model to be applicable the following must apply:

1. there must be constant returns to scale;
2. use of a resource by an activity is proportional to the level of the activity;
3. the total use of a resource by a number of activities is the sum of the uses by the individual activities.

These conditions are clearly applied in the product mix example of Section 1.2 and the result was the linear programming model given there. All the expressions in that model are *linear*. Nowhere do we get expressions such as x_1^2, x_1x_2 and log x_1. Suppose, however, that the first of the above conditions did not apply. Instead of each unit of PROD 1 produced contributing £550 to profit, we suppose that the unit profit contribution depends on the quantity of PROD 1 produced. If this unit profit contribution *increases* with the quantity produced we are said to have *increasing returns to scale*. For *decreases* in the unit profit contribution there is said to be *decreasing returns to scale*. These two situations together with the case of a *constant return to scale* are illustrated in Figures 7.1–7.3 respectively.

In our product mix model each unit of PROD 1 produced contributes £550 to profit. This gives rise to a term $550x_1$, in the objective function. The whole objective function is made up of the sum of similar terms and is said to be *linear*. As an example of increasing returns to scale we could suppose that the unit profit contribution depended on x_1 and was $550 + 2x_1$. This would give

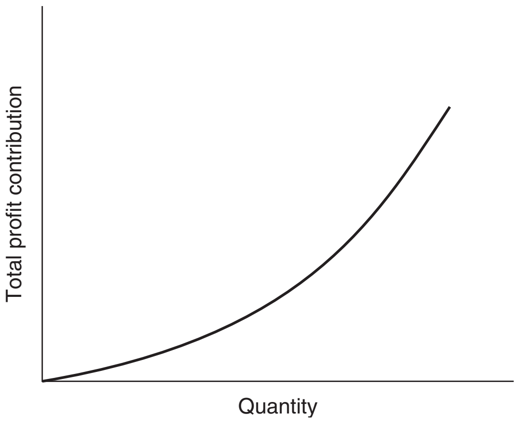


Figure 7.1

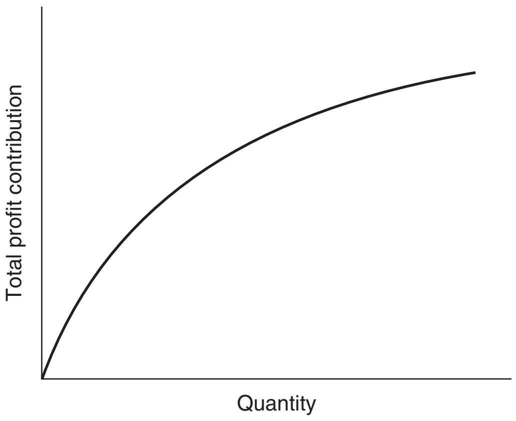


Figure 7.2

rise to the *non-linear* term $2x_1^2$ in the objective function. We would now have a nonlinear model, although the constraints would still be linear expressions. In practice the non-linear term might not be known explicitly and might simply be represented graphically as shown in Figure 7.1 or 7.2.

As an example of decreasing returns to scale we could suppose that the unit profit contribution was $550/(1 + x_1)$. This would give rise to the *non-linear* term $550x_1/(1 + x_1)$ in the objective function.

The above two cases of non-linearities in the objective function arose through profit margins being affected by increasing or decreasing unit costs arising from increased production. In a similar way unit profit margins will be altered if unit selling price is affected by the volume of production. It frequently happens, in practice, that the unit price, which a product can command, increases with the

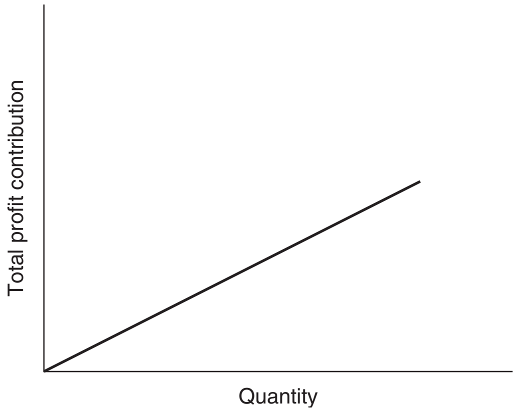


Figure 7.3

demand for it. In this case it may be convenient to treat unit selling price as a variable to be determined by the model. For example, if the quantity to be produced is x, the unit cost is c and the unit selling price is $p(x)$, which depends on x, we have

$$\text{profit contribution} = (p\,(x) - c)\,x = p\,(x)\,x - cx.$$

The term $p(x)x$ introduces a non-linearity into the objective function. An obvious first approximation is to take $p(x)$ as a linear function of x. If this is done quadratic terms are introduced into the objective function, resulting in a *quadratic programming* model. An example of such a model results from the AGRICULTURAL PRICING problem of Part II.

A, sometimes, rather more accurate way to reflect the relationship between $p(x)$ and x is through a *price elasticity*. We have the following definition of the *price elasticity of demand for a good x*:

$$E_x = \frac{\text{percentage change in quantity of } x \text{ demanded}}{\text{percentage change in } p\,(x)}.$$

It is possible to regard E_x as constant for the range of values of x and $p(x)$ considered. The above definition then leads to

$$p\,(x) = \frac{k}{x^{1/E_x}},$$

where k is the price above which the demand is reduced to unity.

For a particular volume of production, $x, p(x)$ gives the unit price at which this level will exactly balance the demand. The resultant non-linear term in the objective function will be

$$p\,(x)\,x = kx^{1-(1/E_x)}.$$

A way in which price elasticities are incorporated in a non-linear programming model of the British National Health Service is described by McDonald *et al*. (1974).

So far, we have described how mathematical programming models with non-linear objective functions can arise. Non-linear terms also, sometimes, arise in the constraints of a mathematical programming model.

For example, in a blending problem (such as that described in Example 1.2), a quality (such as hardness) might not depend linearly on the ingredient proportions. Restrictions on such qualities would then give rise to non-linear constraints.

One type of model that has received a certain amount of attention, where non-linearities occur in the constraints, is the *geometric programming* type of model. Here, the non-linear expressions are all polynomials. For example, we might have the following constraint:

$$x_1x_2x_3 + 2x_2^2x_3 \leq 32.$$

Such models do arise often in engineering problems. They are, however, fairly rare in managerial applications. A full treatment of the subject is given by Duffin *et al*. (1968).

Non-linear programming models are usually far more difficult to solve than correspondingly sized linear models. It is, however, often possible to solve an approximation to the model through either linear programming or an extension of linear programming known as *separable programming*. Which case is applicable depends upon an important classification of non-linear models into *convex* and *non-convex* problems. This distinction is explained in the next section.

7.2 Local and global optima

Non-linear programming can be divided usefully into *convex programming* and *non-convex programming*.

Definition

A *region* of space is said to be convex if the portion of the straight line between any two points in the region also lies in the region.

For example, the area shaded in Figure 7.4 is a convex region in two-dimensional space. On the other hand, the area shaded in Figure 7.5 is a non-convex region since A and B are both points within the region yet the line AB between them does not lie entirely in the region.

Definition

A *function* $f(x)$ is said to be convex if the set of points (x, y) where $y \geq f(x)$ forms a convex region.

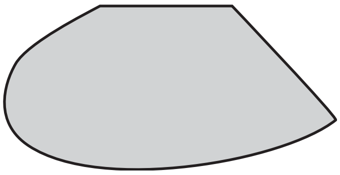


Figure 7.4

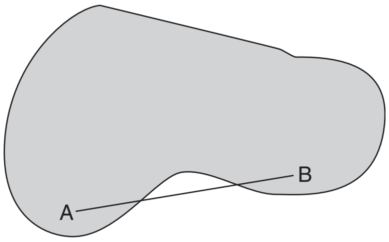


Figure 7.5

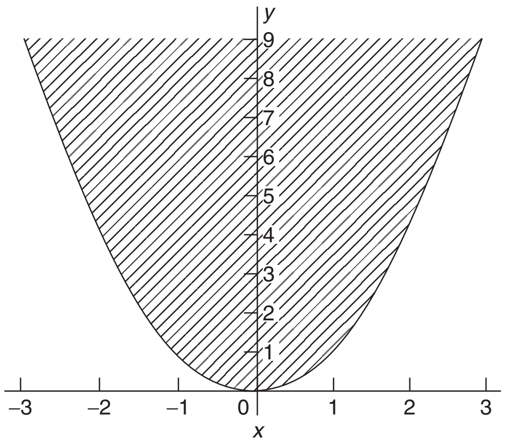


Figure 7.6

For example, the function x^2 is convex, as is demonstrated in Figure 7.6 because the shaded area is a convex region. On the other hand, the function $2-x^2$ is non-convex, as is demonstrated in Figure 7.7.

It should be intuitive that the concepts of convex and non-convex regions and functions apply in as many dimensions as required.

Definition

A *mathematical programming* model is said to be convex if it involves the *minimization of a convex function* over a convex feasible region.

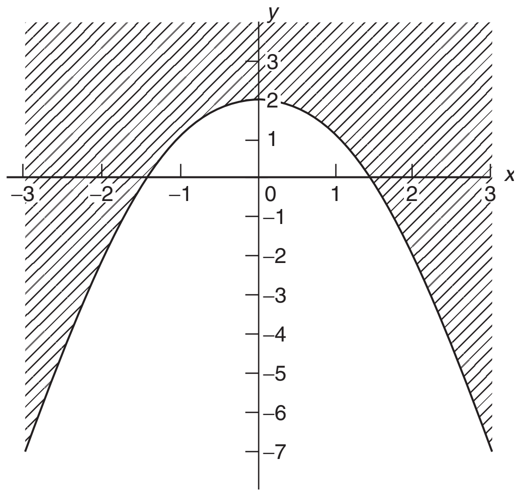


Figure 7.7

Clearly minimizing a convex function is equivalent to maximizing the negation of a convex function. Such a maximization problem will also therefore be convex.

Example 7.1: A Convex Programming Model

$$\begin{aligned} \text{Minimize} \quad & x_1^2 - 4x_1 - 2x_2 \\ \text{subject to} \quad & x_1 + x_2 \leq 4, \\ & 2x_1 + x_2 \leq 5, \\ & -x_1 + 4x_2 \geq 2, \\ & x_1, x_2 \geq 0. \end{aligned}$$

The function $x_1^2 - 4x_1 - 2x_2$ is represented in Figure 7.8 and easily seen to be convex.

This model is represented geometrically in Figure 7.9 with different objective values represented by the curved lines, which arise from contours of the surface in Figure 7.8. Clearly, the optimal solution is represented by point A.

It should be obvious that linear programming (LP) is a special case of convex programming. All LP models can be expressed as minimizations, if necessary, of linear functions. A linear function obviously satisfies the definition of a convex function. Moreover, the feasible region defined by a set of linear constraints can easily be shown to be convex.

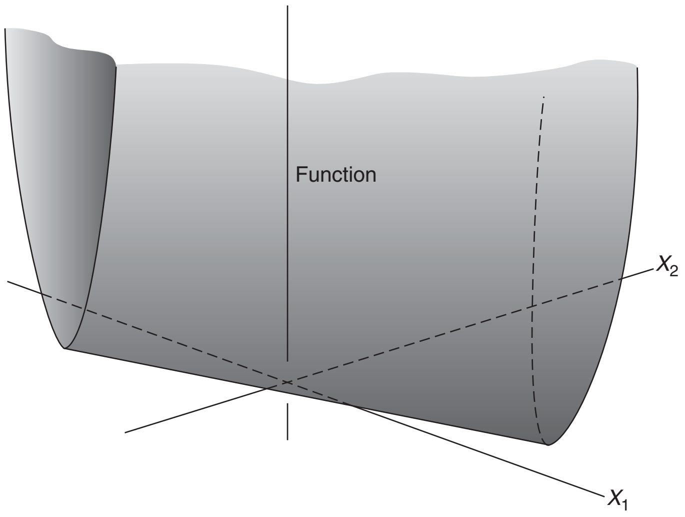


Figure 7.8

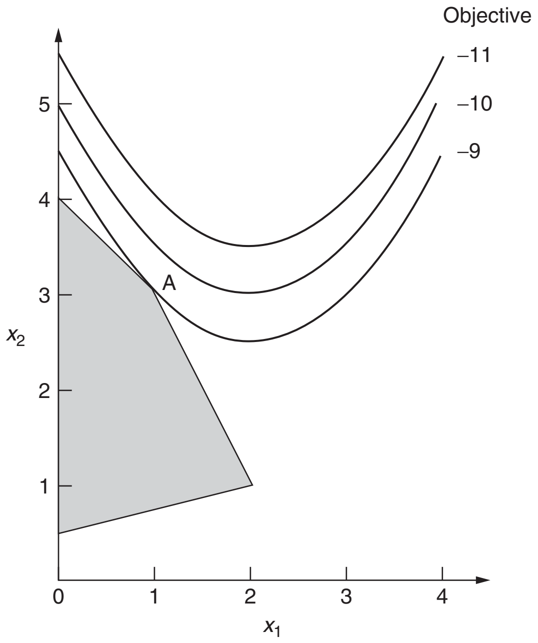


Figure 7.9

Example 7.2: A Non-convex Programming Model

$$\begin{aligned}
\text{Minimize} \quad & -4x_1^3 + 3x_1 - 6x_2 \\
\text{subject to} \quad & x_1 + x_2 \le 4, \\
& 2x_1 + x_2 \le 5, \\
& -x_1 + 4x_2 \ge 2, \\
& x_1, x_2 \ge 0.
\end{aligned}$$

The function $-4x_1^3 + 3x_1 - 6x_2$ is represented in Figure 7.10 and seen to be non-convex, although, of course, the feasible region of the model in Figure 7.9 is still convex.

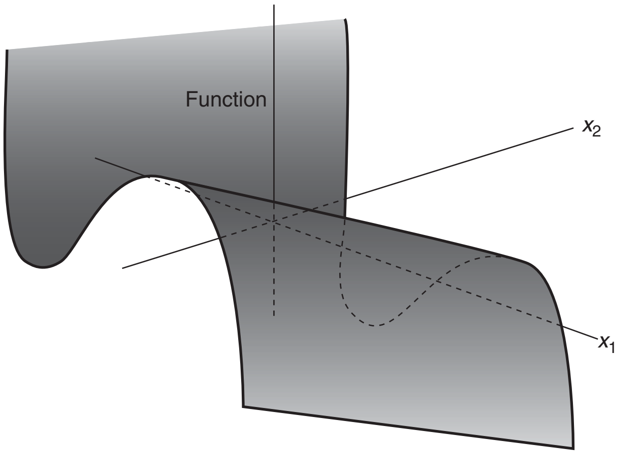


Figure 7.10

The model is represented geometrically in Figure 7.11.

Again different objective values give rise to the curved lines, which are contours of the surface in Figure 7.10. Clearly, the optimal solution is at C. For a larger problem, however, we would not have the geometrical intuition which we have here. As far as many algorithms (including the separable programming algorithm) are concerned, the point A would also appear as an optimal solution. At A the curved line representing this objective value of −19 deviates away from the feasible region in both directions. This would be taken as evidence in a convex (including linear) programming model that A were the optimal solution. Figure 7.9 demonstrates how the objective contours for a convex problem curve away from the feasible region. For our non-convex example, however, Figure 7.11 demonstrates how the objective contours curve in towards

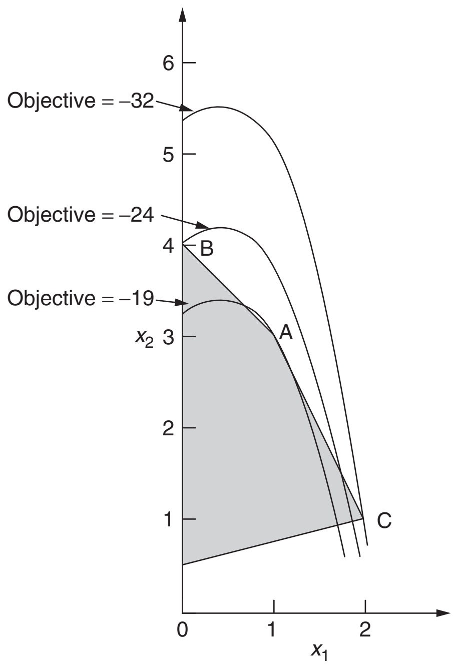


Figure 7.11

the feasible region and may re-enter it. This, in fact, happens in the example here. The fact that the objective contour passing through A deviates away from the feasible region at A in both directions cannot, therefore, be taken as evidence that it will not re-enter it. In fact, as we can see, it is possible to move out to a better (smaller) objective contour passing through B. Even then we can move out to a still better objective contour passing through C. Points A and B represent what are known as *local optima*. Many computational procedures (such as separable programming) can guarantee no more than local optima. Clearly, what is really required is a true *global optimum* such as that represented by point C in Figure 7.11. A fairly graphic way of describing the situation is to consider the problem of a team of mountaineers on a range of mountains in a thick fog. It is easy for them to determine when they are at a local optimum, that is, a mountain peak. The ground will begin to drop in height whichever way they walk. This does not guarantee, however, that there is not another higher mountain peak hidden in the fog. The situation of the mountaineers is similar to that of many non-linear programming algorithms.

The possibility of local optima arising from non-convex programming models when certain algorithms are used is what makes such models much more difficult to solve than convex programming models. With a convex programming model any optimum found must be a global optimum. To find a guaranteed global optimum to a non-convex model requires more sophisticated algorithms than the separable programming algorithm described in the next section. A satisfactory, though often computationally expensive, way of tackling such problems is through *integer programming* as described in Section 9.3.

A good illustration of the distinction between convex and non-convex programming models is given by the non-linear programming models mentioned in Section 7.1, which arise when there are *diseconomies of scale* and *economies of scale*. The former type of model is convex and the latter non-convex. To see why this is so we first consider the type of cost function that arises when we have diseconomies of scale. This is illustrated in Figure 7.12. If x_1 and x_2 represent quantities of two products to be made, diseconomies of scale will result in the total cost rising faster and faster with increasing values of x_1 and x_2. The result is clearly a convex cost function (to be minimized).

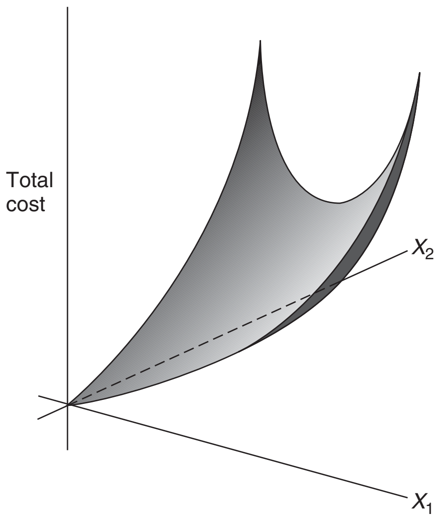


Figure 7.12

For the case of economies of scale we have the situation represented in Figure 7.13. x_1 and x_2 again represent quantities of two products to be made. As x_1 and x_2 increase, decreasing unit costs result in the total cost rising less and less quickly. The result is clearly a non-convex cost function (to be minimized).

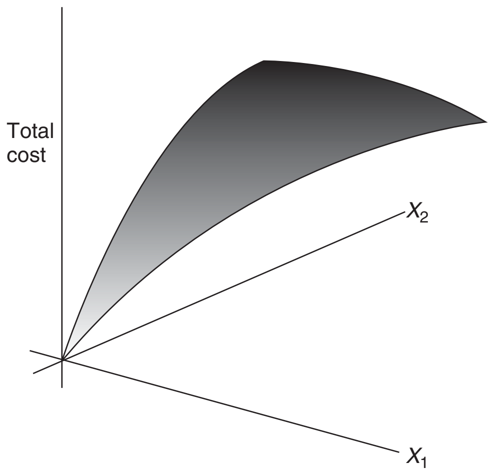


Figure 7.13

7.3 Separable programming

Definition

A *separable* function is a function that can be expressed as the sum of functions of a single variable.

For example, the function

$$x_1^2 + 2x_2 + e^{x_3}$$

is separable because each of terms x_1^2, $2x_2$ and e^{x_3} is a function of a single variable. On the other hand, the function

$$x_1x_2 + \frac{x_2}{1 + x_1} + x_3$$

is not separable because the terms x_1x_2 and $x_2/(1 + x_1)$ are each functions of more than one variable.

The importance of separable functions in a mathematical programming model lies in the fact that they can be approximated to by *piecewise linear functions*. It is then possible to use separable programming to obtain a global optimum to a convex problem or possibly only a local optimum for a non-convex problem.

Although the class of separable functions might seem to be a rather restrictive one, it is often possible to convert mathematical programming models with non-separable functions into ones with only separable functions. Ways in which this may be done are discussed in Section 7.4. In this way a surprisingly wide class of non-linear programming problems can be converted into separable programming models.

In order to convert a non-linear programming model into a suitable form for separable programming, it is necessary to make piecewise linear approximations to each of the non-linear functions of a single variable. As will become apparent, it does not matter whether the non-linearities are in the objective function or the constraints or both. To illustrate the procedure we will consider the non-linear model given in Example 7.1. The only non-linear term occurring in the model is x_1^2. A piecewise linear approximation to this function is illustrated in Figure 7.14.

It is easy to see that x_1 can never exceed 2.5 from the second constraint of the problem. The piecewise linear approximation to x_1^2 need, therefore, only be considered for values of x_1 between 0 and 2.5. The curve between 0 and C has been divided into three straight line portions. This inevitably introduces some inaccuracy into the problem. For example, when x_1, is 1.5 the transformed model will regard x_1^2 as 2.5 instead of 2.25. Such inaccuracy can obviously be reduced by a more refined grid involving more straight line portions. For our purpose, however, we will content ourselves with the grid indicated in Figure 7.14. If such an inaccuracy is considered serious, one approach would be to take the value of x, obtained from the optimal solution, refine the grid in the neighbourhood of this value, and re-optimize. Some package programs allow one to do this automatically a number of times, all within one computer run.

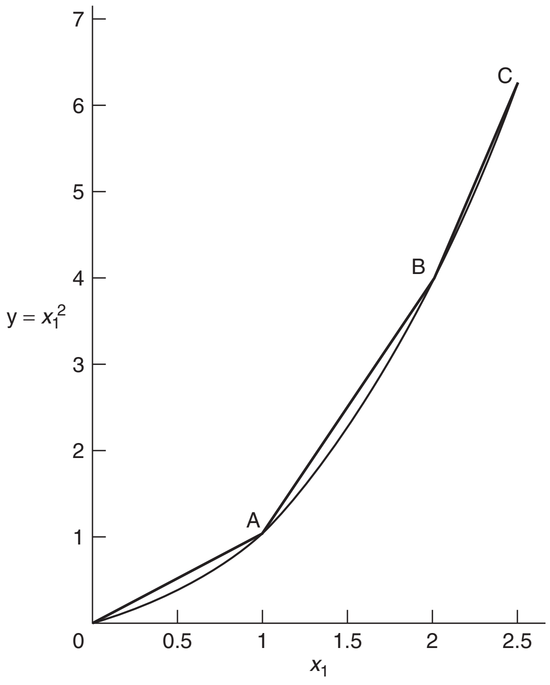


Figure 7.14

Our aim is to eliminate the non-linear term x_1^2 from our model. We can do this by replacing it by the single (linear) term y. It is now possible to relate y to x_1 by the following relationships:

$$x_1 = 0\lambda_1 + 1\lambda_2 + 2\lambda_3 + 2.5\lambda_4, \tag{7.1}$$

$$y = 0\lambda_1 + 1\lambda_2 + 4\lambda_3 + 6.25\lambda_4, \tag{7.2}$$

$$\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = 1. \tag{7.3}$$

The λ_i are new variables which we introduce into the model. They can be interpreted as 'weights' to be attached to the vertices 0, A, B and C. It is, however, necessary to add another stipulation regarding the λ_i:

$$\text{at most two adjacent } \lambda_i \text{ can be non-zero.} \tag{7.4}$$

The stipulation (7.4) guarantees that corresponding values of x_1 and y lie on one of the straight line segments 0A, AB, or BC. For example if $\lambda_2 = 0.5$ and $\lambda_3 = 0.5$ (other λ_i being zero) we could get $x_1 = 1.5$ and $y = 2.5$. Clearly, ignoring stipulation (7.4) would incorrectly allow the possibility of values x_1 and y off the piecewise straight line 0ABC.

Equations (7.1)–(7.3) give rise to constraints that can be added to our original model (Example 7.1). The term x_1^2 is replaced by y. This results in the following model:

$$\begin{array}{lllr}
\text{Minimize} & y - 4x_1 - 2x_2 & & \\
\text{subject to} & x_1 + x_2 & & \leq 4, \\
 & 2x_1 + x_2 & & \leq 5, \\
 & -x_1 + 4x_2 & & \geq 2, \\
 & -x_1 & + \lambda_2 + 2\lambda_3 + 2.5\lambda_4 & = 0, \\
 & -y & + \lambda_2 + 4\lambda_3 + 6.25\lambda_4 & = 0, \\
 & & \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 & = 1, \\
 & & y, x_1, x_2, \lambda_1, \lambda_2, \lambda_3, \lambda_4 \geq 0. &
\end{array}$$

It is important to remember that stipulation (7.4) must apply to the set of variables λ_i. A solution in which, for example, we had $\lambda_1 = \frac{1}{3}$ and $\lambda_3 = \frac{2}{3}$ would not be acceptable because this results in the wrong relationship between x_1 and $y\left(x_1^2\right)$. In general stipulation (7.4) cannot be modelled using linear programming constraints. It can, however, be regarded as a 'logical condition' on the variables λ_i and be modelled using integer programming. This is described in Section 9.3. Fortunately, in our example here, no difficulty arises over stipulation (7.4). This is because the original model was *convex*. Suppose, for example, we were to take as the set of values $\lambda_1 = 0.5$, $\lambda_2 = 0.25$, $\lambda_3 = 0.25$ and $\lambda_4 = 0$. This clearly breaks stipulation (7.4). From the relations (7.1) and (7.2) it clearly leads to the point $x_1 = 0.75$ and $y = 1.25$ in Figure 7.14. This is above the piecewise straight line. As our objective involves minimizing $y\left(x_1^2\right)$, we would expect to get a better solution by taking $x_1 = 0.75$ and $y = 0.75$ when we drop on to the piecewise straight line. In view of the (convex) shape of the graph we cannot obtain values for the λ_i, which give us points below the piecewise straight line. Therefore, we must always obtain corresponding values of x_1 and y, which lie on one of the line segments by virtue of optimality. Stipulation (7.4) is therefore guaranteed in this case. We can therefore solve our transformed model by linear programming and obtain a satisfactory optimal solution. It is not even necessary to resort to the separable extension of the simplex algorithm that is discussed below. This happens, however, only because our problem is convex.

For a non-convex problem stipulation (7.4) would generally not be satisfied automatically. In order to guarantee that it be satisfied, we could resort to the separable programming modification of the simplex algorithm. In order to demonstrate the difficulty that a non-convex problem presents, we will make a piecewise linear approximation to the non-linear term x_1^3 in the non-convex model of Example 7.2. This is demonstrated in Figure 7.15.

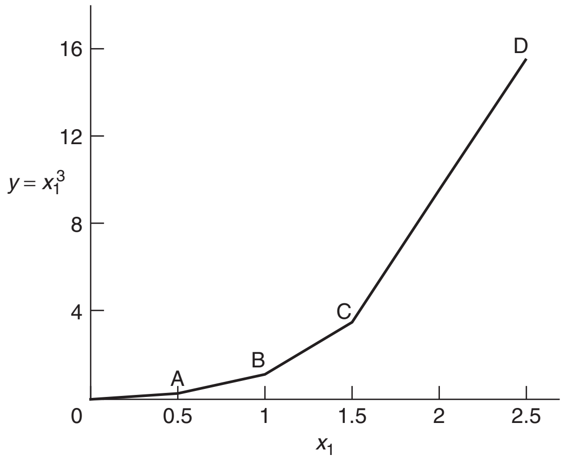


Figure 7.15

This gives us the relationships

$$x_1 = 0\lambda_1 + \quad 0.5\lambda_2 + 1\lambda_3 + \quad 1.5\lambda_4 + \quad 2.5\lambda_5,$$
$$y = 0\lambda_1 + 0.125\lambda_2 + 1\lambda_3 + 3.375\lambda_4 + 15.625\lambda_5.$$

As before, the λ_i variables can be interpreted as 'weights' attached to the vertices in Figure 7.15.

The λ_i variables must again satisfy the stipulation that at most two adjacent λ_i are non-zero. This time this stipulation is not automatically guaranteed by optimality. Suppose, for example, that we were to obtain the set of values $\lambda_2 = 0.4$, $\lambda_3 = 0.5$ and $\lambda_4 = 0.1$. This would give $x_1 = 0.85$ and $y = 1.3375$. The point with these coordinates lies above the piecewise line in Figure 7.15. As our objective function, to be minimized, is dominated by the term $-4x_1^3$, our optimization will tend to maximize y. This will tend to take us away from the piecewise line rather than down on to it. In this (non-convex) case it is therefore necessary to use an algorithm which does not allow more than two adjacent λ_i to be non-zero.

The *separable programming* extension of the simplex algorithm due to Miller (1963) never allows more than two adjacent λ_i into the solution; as a result, it restricts corresponding values of x_i and y to the coordinates of points lying on the desired piecewise straight line.

Unfortunately, with non-convex problems, such as Example 7.2, which was modelled above, restricting the values of λ_i to at most two adjacent ones being non-zero does not guarantee any more than a local optimum. We could easily end up at point A or B in Figure 7.11 rather than C.

In both our examples the non-linear functions of a single variable appeared in the objective function. Should such non-linear functions also appear in the constraints the analysis is just the same. They are replaced by linear terms and a piecewise linear approximation made to the non-linear function. Variables λ_i are then introduced in order to relate the new term to the old.

It may not always be easy to decide whether a problem is convex or not. For known convex problems (such as, problems that are linear apart from diseconomies of scale) consisting only of separable functions, piecewise linear approximations are all that are needed. It is not necessary here to use the separable programming algorithm. For problems that are non-convex (such as problems with economies of scale) or where it is not known whether or not they are convex, linear programming is not sufficient. Separable programming can be used but no more than a local optimum can be guaranteed. It is often possible to solve such a model a number of times using different strategies to obtain different local optima. The best of these may then have some chance of being a global optimum. Such computational considerations are, however, beyond the scope of this book, but are sometimes described in manuals associated with particular packages. The only really satisfactory way of being sure of avoiding local optima when a problem is not known to be convex is to resort to integer programming, which is generally much more costly in computer time. This is discussed in Sections 9.2 and 9.3.

Before the end of this section an alternative way of modelling a piecewise linear approximation to a separable function will be described. The formulation method just described is usually known as the *λ-form* for separable programming where variables λ_i are interpreted as weights attached to the vertices in the piecewise straight line. There is an alternative formulation known as the *δ-form*. In order to demonstrate the δ-form we reconsider the piecewise linear approximation to the function $y = x_1^2$ shown in Figure 7.4. This approximation is redrawn in Figure 7.16.

Variables δ_1, δ_2, and δ_3 are introduced to represent *proportions* of the intervals 0P, PQ, and QR, which are used to make up the value of x_1. We then get

$$x_1 = \delta_1 + \delta_2 + 0.5\delta_3, \tag{7.5}$$

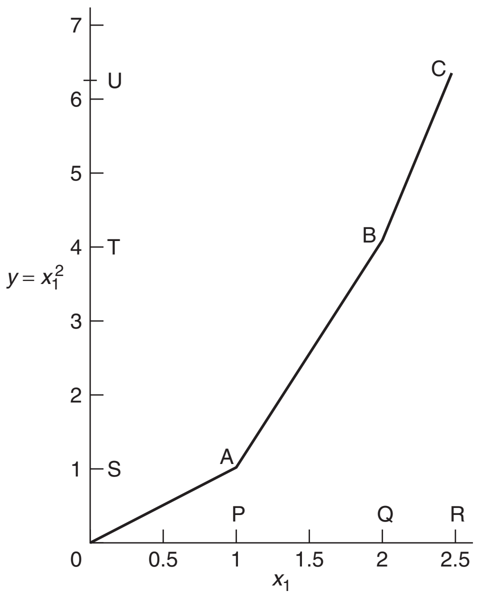


Figure 7.16

where

$$0 \leq \delta_1, \delta_2, \delta_3 \leq 1.$$

As 0P and PQ are each of length 1, the coefficients of δ_1 and δ_2 in Equation (7.5) are 1. The coefficient of δ_3 is 0.5, reflecting the length of the interval QR.

Similarly,

$$y = \delta_1 + 3\delta_2 + 2.25\delta_3, \tag{7.6}$$

where the coefficients of δ_1, δ_2, and δ_3 are now the lengths of the intervals 0S, ST and TU.

In order to ensure that x_1 and y are the coordinates of points on the piecewise lines 0C, we must make the following stipulation:

if any δ_i is non-zero, all the preceding δ_i must take the value 1 and all the succeeding δ_i must take the value 0. (7.7)

Stipulation (7.7) clearly ensures that x_1 and y truly represent distances along their respective axes.

It can be shown that the relaxation associated with the λ-formulation is more constrained than that associated with the δ-formulation. This is the relaxation obtained by dropping the 'logical' conditions, (7.4) and (7.7), on the λ and δ variables, as opposed to the LP relaxation obtained by relaxing the IP formulations (see Section 8.3) making it, generally, easier to solve. This was pointed out early on by Williams (1985). When the extra, logical conditions are imposed by integer variables as described in Section 9.3 the λ-formulation can be improved in a way described there.

7.4 Converting a problem to a separable model

The restriction of only allowing non-linear functions to be separable might seem to impose a severe limitation on the class of problems that can be tackled by separable programming. Rather, surprisingly, it is possible to convert a very large class of non-linear programming problems into separable models. When non-separable functions occur in a model, it is often possible to transform the model into one with only separable functions.

A very common non-separable function that occurs in practice is the product of two or more variables. For example, if a term such as x_1x_2 occurs, the model is not immediately a separable one because this is a non-linear function of more than one variable. The model is easily converted into a separable form, however, by the following transformation:

1. introduce two new variables u_1 and u_2 into the model;
2. relate u_1 and u_2 to x_1 and x_2 by the relations

$$u_1 = \frac{1}{2}\left(x_1 + x_2\right), \tag{7.8}$$

$$u_2 = \frac{1}{2}\left(x_1 - x_2\right); \tag{7.9}$$

3. replace the term x_1x_2 in the model by

$$u_1^2 - u_2^2.$$

It is easy to see by elementary algebra that $u_1^2 - u_2^2$ is the same as the product x_1x_2 as long as (7.8) and (7.9) are added to the model in the form of equality constraints. The model now contains non-linear functions u_1^2 and u_2^2 of single variables and is therefore separable. These non-linear terms can be dealt with by piecewise linear approximations. It is important to remember that u_2 might need to take negative values. When the possible ranges of values for u_1 and u_2 are considered it may be necessary to either translate u_2 by an appropriate amount or treat it as a 'free' variable. A 'free' variable in linear programming is one which is not restricted to non-negative values.

Should the product of more than two variables occur in a model (as might happen, for example, in a geometric programming model), then the above procedure could be repeated successively to reduce the model to a separable form.

An alternative way of dealing with product terms in a non-linear model is to use logarithms. We will again consider the example where a product x_1x_2 of two variables occurs in a model, although the method clearly generalizes to larger products. The following transformation can be carried out:

1. replace x_1x_2 by a new variable y;
2. relate y to x_1 and x_2 by the equation

$$\log y = \log x_1 + \log x_2. \tag{7.10}$$

Equation (7.10) gives a non-linear equality constraint to be added to the model. The expression in this constraint is, however, separable as we only have non-linear functions $\log y$, $\log x_1$ and $\log x_2$ of single variables.

Care must be taken, however, when this transformation is made to ensure that neither x_1 nor x_2 (and consequently y) ever take the value 0. If this were to happen their logarithms would become $-\infty$. It may be necessary to translate x_1 and x_2 by certain amounts to avoid this ever happening.

The use of logarithms to convert a non-linear model to a suitable form does, however, sometimes lead to computational problems. Beale (1975) suggests that this can happen if both a variable and its logarithm occur together in the same model. If these are likely to be of widely different orders of magnitude, numerical inaccuracy may lead to computational difficulties.

Non-linear functions of more than one variable (such as product terms) can be dealt with in yet another way by generalizing a piecewise linear approximation to more dimensions. A more complex relationship then has to be specified between the λ_i. As with non-convex separable models, this is only really satisfactorily dealt with through *integer programming* and is described in Section 9.3.

Many other non-linear functions of more than one variable can be reduced to non-linear functions of a single variable by the addition of extra variables and constraints. Ingenuity is often required but the range of non-linear programming problems that can be made separable in this way is vast. Such transformations do, however, often greatly increase the size of a model and hence the time to solve it.

8

Integer programming

8.1 Introduction

A surprisingly wide class of practical problems can be modelled using integer variables and linear constraints. Sometimes such a model consists solely of integer variables. That is a *pure integer programming* (PIP) *model*. More commonly, there are both conventional continuous variables together with integer variables present. Such a model is said to be a *mixed integer programming* (MIP) *model*.

The wide applicability of integer programming (IP) (sometimes known as *discrete programming*) as a method of modelling is not obvious. Clearly, we can think of situations where it is only meaningful to make integral quantities of certain goods, for example, cars, aeroplanes or houses, or use integral quantities of some resource, for example, employees. In these cases we might use an IP model instead of an LP model. Although such obvious applications of IP do occur they are not common. In fact, in such situations, it is often more desirable to use conventional LP and round off the optimal solution values to the nearest integers.

The obvious type of application described above obscures the real power of IP as a method of modelling. Most practical IP models restrict the integer variables to two values, 0 or 1. Such 0–1 variables are used to represent 'yes or no' decisions. Logical connections between such decisions can often be imposed using linear constraints. Such methods of modelling are described in the next chapters.

Before discussing the building of IP models something must be said about the way in which they are solved. Mathematically, IP models involve many times as much calculation in solution as similar-sized LP models. The difficulty of integer problems compared with (continuous) problems involving real or rational numbers is well known in other branches of mathematics. While an LP model involving thousands of constraints and variables can almost certainly be solved in a reasonable amount of time using a modern computer and package program, a similar situation does not hold for IP models. There is a considerable danger in building an IP model only to find no way of solving it in a reasonable time. Ways of avoiding this unfortunate and embarrassing experience are described in the next two chapters. In view of this danger as well as the computational difficulty of IP, Section 8.3 is devoted to methods of solving IP models. This section purposely outlines in any detail only the most successful method of solving IP models, the branch and bound method. It is felt necessary that a model builder should have, at least, this minimum level of understanding as he or she can often influence the exact way in which the calculation proceeds to great advantage. Moreover, this most successful method to date of solving IP models receives surprisingly little attention in theoretical mathematical programming books, possibly because of its lack of mathematical sophistication.

8.2 The applicability of integer programming

The purpose of this section is to classify loosely the different types of problem for which IP models may be built. Inevitably, there is a certain amount of overlap in this classification where particular applications do not fit neatly into any category. Many practical problems will combine aspects from a number of categories. By this loose classification, it is hoped to convey a feeling for the type of problem to which IP is applicable. In some ways the name 'discrete programming' conveys what this sort of problem is rather better. One of the reasons why IP has not been applied anywhere near as widely as it might to practical situations is the failure to recognize when a problem can be cast in this mould. This section is purposely rather superficial. Little indication is given of the way in which particular problems may be formulated as IP models. This is left to the next chapter or in particular cases to the detailed formulations of Part III of this book.

A precise definition of those situations, which can be formulated by IP models, is given by Meyer (1975) and Jeroslow (1987).

8.2.1 Problems with discrete inputs and outputs

This class of problems includes the most obvious IP applications mentioned earlier where it is only possible to make *whole numbers of products* or *use integral units of a resource*. Economists sometimes refer to such problems as having 'lumpy' inputs or outputs. To see why IP is sometimes necessary here, consider the following very small (and contrived) model:

$$\begin{aligned}
\text{Maximize} \quad & x_1 + x_2 \\
\text{subject to} \quad & -2x_1 + 2x_2 \geq 1, \\
& -8x_1 + 10x_2 \leq 13, \\
& x_1, x_2 \geq 0.
\end{aligned}$$

If the variables represent quantities of two different goods to be made, it is important to be clear if these outputs should be integral, for example, represent indivisible goods such as aeroplanes. Should this not be the case and they represent divisible goods such as gallons of beer, we would be content with treating the problem as an LP model and taking the (continuous) optimum, which is

$$x_1 = 4, \qquad x_2 = 4\frac{1}{2}. \tag{8.1}$$

On the other hand, restricting the variables to be integer forces us to accept the integer optimum, which is

$$x_1 = 1, \qquad x_2 = 2. \tag{8.2}$$

It is very difficult to see how one could arrive at the solution (8.2) from the continuous optimum (8.1). Rounding the values in (8.1) to the nearest integers gives an infeasible solution. In some circumstances such as this, it is therefore necessary to solve such a problem as an IP model. This is obviously likely to happen when the values of the variables will be small (say less than 5). For most problems of this type, however, the values of the variables are likely to be much larger than this and the errors involved in rounding the LP fractional solution will not be serious. Indeed, solving such a problem as an IP model could well take a great deal of time in view of the many combinations of integer solutions that could be considered. An extreme case of this was once seen where a national agricultural IP model was built. On examination this model was found to be an IP model only because there were an integral number of cows, hens, pigs, etc., in the country considered!

Similar considerations to those above apply to problems where the inputs rather than (or as well as) the outputs are discrete. Frequently, such an input will be manpower capacity which, if sufficiently large, may be treated as continuous (infinitely divisible).

An application that is quite common occurs when an input (usually a processing capacity or a resource) only occurs at certain discrete values. Apart from this the model may be a conventional LP model. For example, processing capacity might be measured in machine hours per week. By buying extra machines it might be possible to expand processing capacity. It will, however, only be correct to allow this capacity to expand in steps equal to the machine hours per week resulting from extra whole machines. IP can be used to model this type of situation. A related situation to this is presented in the FACTORY PLANNING 2 problem in Part II.

Another particular type of problem in this category where IP must be used is the *knapsack problem*. This is an IP problem with a single constraint. A particular instance where it might arise is in stocking a warehouse. The problem is as follows: given a limited warehouse capacity, stock the warehouse with goods (of different volumes) to maximize some objective (such as the total value of goods in the warehouse). It will generally not be possible to use the LP solution to this problem, which is trivial and simply involves stacking the warehouse to capacity with the most valuable good per unit volume, thereby ignoring the discrete nature of the goods. Extensions of the knapsack problem arise where extra simple upper bounding constraints also apply to the variables of the problem. It is again usually necessary to use IP rather than LP to obtain a meaningful solution. Knapsack problems do arise in practice but they occur most commonly as subproblems that have to be solved as part of a much larger LP or IP problem. An example of this arises in the *cutting stock* problem, discussed in Section 9.6.

8.2.2 Problems with logical conditions

It frequently happens that it is desired to impose extra conditions on an LP model. These conditions are sometimes of a logical nature that cannot be modelled by conventional LP. For example, an LP model might be used to decide how much to make of each of the possible products in a factory subject to capacity limitations (the *product mix* application). It might be desirable to add an extra condition such as 'If product A is made then product B or C must be made as well'. By introducing some extra integer variables into the model together with extra constraints, conditions such as this easily can be imposed. The resultant model is a mixed integer problem. Any such *logical condition* as that above can be imposed on an LP model using IP. This is illustrated in the FOOD MANUFACTURE 2 problem. In addition, that problem also illustrates another common use of IP to extend an LP model, *limiting the number of ingredients in a blend*.

The correct formulation of logical conditions sometimes involves considerable ingenuity and can be done in a number of different ways. Methods of approaching such a formulation systematically are described in Chapter 9. Williams (2009) and Hooker (2000) cover the relationship between logic and IP at length.

8.2.3 Combinatorial problems

Many operational research problems have the characteristic of a very large number of feasible solutions (often an astronomical number) arising from different orders of carrying out operations or of allocating items or people to different positions. Such problems are loosely referred to as *combinatorial*. It is useful to further subdivide this category into *sequencing problems* and *allocation problems*. A particularly difficult type of sequencing problem arises in *job-shop scheduling* where an optimal ordering of operations on different machines in a job-shop is desired. IP gives a method of modelling this type of situation. There are a number of possible formulations. Unfortunately, IP has not proved a very successful way of tackling this problem to date.

Another very well-known sequencing problem is the *travelling salesman problem*. This is the problem of finding the optimal order in which to visit a set of cities and return home covering a minimum distance. There are other problems which take the same form. For example, the problem of sequencing operations on a machine so as to minimize total set-up cost takes this form.

Obviously, the set-up cost for an operation will depend on the preceding operation and can be regarded as the 'distance' between operations. The travelling salesman problem is again a very difficult type of problem for which different IP formulations have been attempted.

A very straightforward allocation problem is given in Part II. This is the MARKET SHARE problem. The problem involves allocating customers to divisions in a company for their services. Although the formulation is comparatively straightforward, problems of this type are not always easy to solve. Problems of a very similar form arise in *project selection* and *capital budget allocation*.

The class of allocation problems includes two problems already mentioned for which IP is not needed. It has been pointed out in Section 5.3 that in the *transportation problem* it is not necessary to impose an integrality requirement. The LP optimal solution will automatically be integer because of the structure of the problem. As the *assignment problem* can be regarded as a special case of the transportation problem, this property holds for it also. Fortunately, these two problems, although apparently IP problems, can therefore be treated as LP problems and solved fairly easily. Other apparently IP problems also have this property or can be formulated to have this property with great computational advantage. This topic is treated further in Sections 10.1 and 10.2. A complicated extension of the assignment problem is the *quadratic assignment problem*. This problem occurs where the 'cost of an assignment' is not independent of other assignments. The resulting problem can be regarded as an assignment problem with a quadratic objective function. The quadratic terms can be converted into linear expressions reducing the problem to an IP problem. The quadratic assignment problem is one of the most difficult combinatorial problems known in mathematical programming. Fairly small problems can be tackled by reducing the problem to a linear IP model. The DECENTRALIZATION example of Part II is a special sort of quadratic assignment problem. The quadratic assignment problem is discussed further in Section 9.5.

A practical problem that arises and can be regarded as falling into the allocation category is the *assembly line balancing* problem. This is the problem of assigning workers to tasks on a production line to achieve a given production rate. It is possible to formulate this problem as an IP problem and solve it fairly easily. The usual formulation results in a special sort of IP model known as a *set partitioning problem*. This type of problem is discussed further in Section 9.5.

Another set partitioning problem is the *aircrew scheduling problem*. This is the problem of assigning aircrews to sets of flights (rotations or rosters). In practice, this type of problem frequently involves an enormous number of potential rosters and is difficult to solve as an IP problem in consequence. This is discussed further in Section 9.5.

Problems of *logical design* involving switching circuits or logical gates can be tackled through IP. Unfortunately, such problems often turn out to be very large when formulated in this way and so difficult to solve. A small problem of this kind, LOGICAL DESIGN, is given in Part II.

The *political districting* problem is also a set partitioning problem when regarded as an IP problem. This is the problem of designing constituencies or electoral districts in order to equalize, as near as possible, political representation. IP has been used in the United States to solve practical problems of this nature.

A fairly common application of IP is to the *depot location problem*. This is the problem of deciding where to locate depots (or warehouses or even factories) in order to supply customers. Two types of costs may enter the problem, the capital costs of building the depots and the distribution costs resulting from particular sites of the depots. This type of problem can be modelled using IP. Frequently, the resultant model is a mixed integer problem. The DEPOT LOCATION problem of Part II is an example.

8.2.4 Non-linear problems

As was mentioned in Chapter 7, non-linear problems can sometimes be treated as IP problems with advantage. If the problem can be expressed in a separable programming form, it can be solved using either separable programming as described in Section 7.3 or by IP. Should the problem be convex (this term is explained in Section 7.2) it may be treated by LP and no difficulty arises. On the other hand, special methods have to be used for non-convex problems where separable programming has the disadvantage of producing possibly local optima (this is again explained in Section 7.2). IP overcomes this difficulty and produces a true (global) optimal solution, although possibly with considerably more expenditure in computer time. Problems to which this method is relevant have already been mentioned in Chapter 7. They include problems involving *economies of scale*, *quadratic programming problems* and *geometric programming problems* as well as much more general non-linear programming problems.

The way in which such problems can be converted into a separable form is described in Chapter 7. How to convert the resultant separable problems into an IP form is described in Chapter 9. There is, however, an alternative way of approaching such problems by IP. This is through the use of special ordered sets of variables. A number of package programs have facilities for dealing with IP problems in this way to considerable computational advantage. The concept of special ordered sets and how they may be applied to non-linear problems (as well as other types of problem) is described by Beale and Tomlin (1969).

A very common application of IP is the *fixed charge problem*. This occurs when the cost of an activity involves a threshold set-up cost as well as the usual costs that rise in proportion to the level of the activity. In this way the problem can be regarded as non-linear. For example, if it is decided to produce any amount of a product at all it may be necessary to set up a piece of machinery. This set-up cost is independent of the quantity produced. It is not possible to model this situation using conventional LP but it can be modelled very easily using IP. This is described in Example 9.4.

8.2.5 Network problems

Many problems in operational research involve networks. A lot of these problems can be modelled using LP or IP. Those problems, which give rise to LP models, have already been considered in Section 5.3.

It has already been pointed out in Section 5.3 that the problem of finding the critical path in a PERT network can be viewed as an LP problem. A secondary problem often arises in practice. This is the problem of *resource allocation on a PERT network*. It may be necessary to alter the order in which certain activities (arcs) are carried out in view of the limited resources available, for example, we cannot simultaneously build the walls and lay the floors in a house if there are not enough workers available. The problem of optimally allocating these limited resources to the arcs of the PERT network so as to (for example) minimize the total completion time for a project can be formulated as an IP problem. Although an IP model is a method of tackling this problem, it is not to be recommended except in very simple cases. The computational difficulties of solving a complex problem of this kind by IP can be very great. In practice, non-rigorous heuristic means are used to obtain useful but non-optimal solutions.

Many IP problems arise in *graph theory*. A well-known problem to which IP is relevant is the *four-colour problem*. It has recently been proved that at most four colours are needed to colour every country on a map differently from its neighbouring countries. For any map, IP could be used to find the minimum number of colours necessary. The above problem can be represented as the problem of colouring the vertices of a graph so that vertices jointed by an edge are coloured differently. Other colouring problems arise in graph theory. For example, it is possible to devise problems involving colouring the edges of a graph. Although many such problems exist and can be solved using IP, such considerations are largely beyond the scope of this book. The concern here is mainly with practical problems. Graph theory and IP has been extensively treated elsewhere, for example, by Christofides (1975).

The above five categories encompass most of the different types of problem, which arise to which IP is applicable. In practice, most problems that one meets fall into the second category. They are LP problems on which it is desired to impose extra conditions. These extra conditions are frequently of a logical type. One therefore extends the LP model by adding integer variables and extra constraints. The extra constraints applied to the integer variables sometimes have a combinatorial flavour. Sometimes, these extra constraints serve the purpose of modelling non-linearities in an otherwise LP model.

PIP models arise less frequently in practice. They are usually combinatorial problems. The comparatively large number of combinatorial problems listed above should not disguise the fact that the majority of practical IP models are in the second category and may arise as extensions to almost any application of LP. Combinatorial problems do arise in practice, however, and are sometimes satisfactorily solved through IP. Great care must be exercised, however, when applying IP to such problems. While IP offers an apparently attractive way of modelling a combinatorial problem, experience has shown that such models can be very difficult to solve. It is often desirable to experiment with small-scale versions of the problem before embarking on a large model. There are also good and bad ways of formulating such problems from the point of view of ease of solution; this is discussed in Section 10.1. Also it is hoped that the comparative solution times given with the solutions to the models in Part IV will be indicators of the difficulty of certain types of model.

8.3 Solving integer programming models

This section is in no way intended to be a full description of IP algorithms. A much fuller description is given in Williams (1993). Instead, it is an attempt to indicate different ways in which IP models may be solved and to suggest how a model builder may use existing packages in an efficient manner. Some references to fuller expositions of the algorithmic side of IP are given.

The main approaches to solving IP problems are categorized below. Unlike LP with the simplex algorithm, no one good IP algorithm has emerged. Different algorithms prove better with different types of problem, often by exploiting the structure of special classes of problem. It seems unlikely that a universal IP algorithm will ever emerge. If it did, it would open up the possibility of solving a very wide class of problems. Some of these problems (such as the travelling salesman problem and the quadratic assignment problem) have defied many attempts to find powerful algorithms for their solution. There is now even some theoretical evidence resulting from the theory of computational complexity to suggest that a ‘universal’ IP algorithm is unlikely. The most successful algorithm so far found to solve practical general IP problems is the branch and bound method described below. Considering its apparent crudeness the success of this method is surprising. Almost all commercial packages offering an MIP facility use the algorithm. In fact, the algorithm is little more than an approach to solving IP problems. There is great flexibility in the way it can be used. This is one of the reasons why it is briefly described in a book on model building. Using the branch and bound method in a way suited to the problem can show dramatic improvements over less intelligent strategies.

Most methods of solving IP problems fall into one of four broad categories. There is some overlap between the categories and some particularly successful approaches to large problems have exploited features of a number of methods.

8.3.1 Cutting planes methods

These methods can be applied to general MIP problems. They usually start by solving an IP problem as if it were an LP problem by dropping the integrality requirements (known as the LP *relaxation*). If the resultant LP solution (the continuous optimum) is integer, this solution will also be an integer optimum; otherwise, extra constraints (cutting planes) are systematically added to the problem, further constraining it. The new solution to the further constrained problem may or may not be an integer. By continuing the process until an integer solution is found or the problem is shown to be infeasible, the IP problem can be solved.

Although cutting plane methods may appear mathematically fairly elegant, they have not proved very successful on large problems, although combined with Branch and Bound methods they can prove very powerful.

The original method of this type is described by Gomory (1958). Further references are given with the exposition in Chapter 5 of Garfinkel and Nemhauser (1972).

8.3.2 Enumerative methods

These are generally applied to the special class of 0–1 PIP problems. In theory there are only a finite (though extremely large) number of possible solutions to a problem in this class. Although it would be prohibitive to examine all these possibilities, by use of a tree search it is possible to examine only some solutions and systematically rule out many others as being infeasible or non-optimal. Such methods together with their variants and extensions have proved very successful with certain types of problem and not very successful on others. Commercial package programs do exist for such methods but are not widely used.

The best known of these methods is Balas's additive algorithm described by Balas (1965). Other methods are given by Geoffrion (1969). A good overall exposition is given in Chapter 4 of Garfinkel and Nemhauser (1972).

8.3.3 Pseudo-Boolean methods

Attempts have been made to exploit the obvious analogy between Boolean algebra and 0–1 PIP problems. A number of algorithms have been developed. As with other algorithms they work well on some types of problem but less well on others. This approach is entirely unlike any other for solving IP problems. The constraints of a problem are expressed not as equations or inequalities but through Boolean algebra. In some cases, this can give a very concise statement of the constraints, but in others it is large and unwieldy. As far as the author is aware no commercial packages capable of accepting practical problems use any of these methods.

These approaches have been developed by Hammer and are described in Hammer and Rudeanu (1968), Granot and Hammer (1972) and Hammer and Peled (1972).

It should not be thought that the specialized 0–1 PIP algorithms are only of relevance to 0–1 PIP problems. Methods have been developed for partitioning a MIP problem with 0–1 variables into its continuous and integer portions. It then becomes necessary, at stages in the optimization, to solve a

0–1 PIP problem as a subproblem. Clearly, these specialized algorithms might be applicable. Any discussion of this topic is beyond the scope of this book. The main reference to this partitioning method is Benders (1962).

8.3.4 Branch and bound methods

It is these methods that have proved most successful, in general, on practical MIP problems. Such methods are also sometimes classed as enumerative but we choose to distinguish them from the enumerative methods described earlier.

As with cutting plane methods, the IP problem is first solved as an LP problem by *relaxing* the integrality conditions. If the resultant solution (the continuous optimum) is an integer, the problem is solved; otherwise, we perform a tree search. Full details are given in Williams (1993).

A very good discussion of efficient solution strategies to use with the branch and bound method on practical models is given by Forrest *et al*. (1974). Geoffrion and Marsten (1972) put the branch and bound method, together with enumeration methods, which also use a tree search, into a general framework that makes the basic principles easy to understand. References to the various forms of the branch and bound method are given in that paper. More recent incorporation of Cutting Plane methods into Branch and Bound ('Branch and Cut') has proved very powerful.

A comprehensive survey of IP packages is given by Land and Powell (1979).

9

Building integer programming models I

9.1 The uses of discrete variables

When integer variables are used in a mathematical programming model, they may serve a number of purposes. These are described below.

9.1.1 Indivisible (discrete) quantities

This is the obvious use mentioned at the beginning of Chapter 8 where we wish to use a variable to represent a quantity that can only come in whole numbers such as aeroplanes, cars, houses or people.

9.1.2 Decision variables

Variables are frequently used in integer programming (IP) to indicate which of a number of possible decisions should be made. Usually, these variables can only take the two values, zero or one. Such variables are known as *zero–one* (0–1) (or binary) variables. For example, $\delta = 1$ indicates that a depot should be built and $\delta = 0$ indicates that a depot should not be built. We usually adopt the convention of using the Greek letter 'δ' for $0 - 1$ variables and reserve Latin letters for continuous (real or rational) variables.

It is easy to ensure that a variable, which is also specified to be integer, can only take the two values 0 or 1 by giving the variable a simple upper bound (SUB) of 1. (All variables are assumed to have a simple lower bound of 0 unless it is stated to the contrary).

Although decision variables are usually $0 - 1$, they need not always be. For example we might have $\gamma = 0$ indicates that no depot should be built; $\gamma = 1$

indicates that a depot of type A should be built; $\gamma = 2$ indicates that a depot of type B should be built.

9.1.3 Indicator variables

When extra conditions are imposed on a linear programming (LP) model, $0 - 1$ variables are usually introduced and 'linked' to some of the continuous variables in the problem to indicate certain states. For example, suppose that x represents the quantity of an ingredient to be included in a blend. We may well wish to use an indicator variable δ to distinguish between the state where $x = 0$ and the state where $x > 0$. By introducing the following constraint, we can force δ to take the value 1 when $x > 0$:

$$x - M\delta \leq 0, \tag{9.1}$$

where M is a constant coefficient representing a known upper bound for x.

Logically, we have achieved the condition

$$x > 0 \rightarrow \delta = 1, \tag{9.2}$$

where '$\rightarrow$' stands for 'implies'.

In many applications, Equation (9.2) provides a sufficient link between x and δ (e.g. Example 9.1 below). There are applications (e.g. Example 9.2 below), however, where we also wish to impose the condition

$$x = 0 \rightarrow \delta = 0. \tag{9.3}$$

Equation (9.3) is another way of saying

$$\delta = 1 \rightarrow x > 0. \tag{9.4}$$

Together, Equations (9.2) and (9.3) (or Equation (9.4)) impose the condition

$$\delta = 1 \leftrightarrow x > 0, \tag{9.5}$$

where '$\leftrightarrow$' stands for 'if and only if'.

It is not possible totally to represent Equation (9.3) (or Equation (9.4)) by a constraint. On reflection, this is not surprising. Equation (9.4) gives the condition 'if $\delta = 1$, the ingredient represented by x must appear in the blend'. Would we really want to distinguish in practice between no usage of the ingredient and, say, one molecule of the ingredient? It would be much more realistic to define some threshold level m below which we regard the ingredient as unused. Equation (9.4) can now be rewritten as

$$\delta = 1 \rightarrow x \geq m. \tag{9.6}$$

This condition can be imposed by the constraint

$$x - m\delta \geq 0. \tag{9.7}$$

Example 9.1: The Fixed Charge Problem

x represents the quantity of a product to be manufactured at a marginal cost of C_1 per unit. In addition, if the product is manufactured at all there is a set-up cost of C_2. The position is summarized as follows:

$$x = 0, \qquad \text{total cost} = 0,$$
$$x > 0, \qquad \text{total cost} = C_1x + C_2.$$

The situation can be represented graphically as in Figure 9.1.

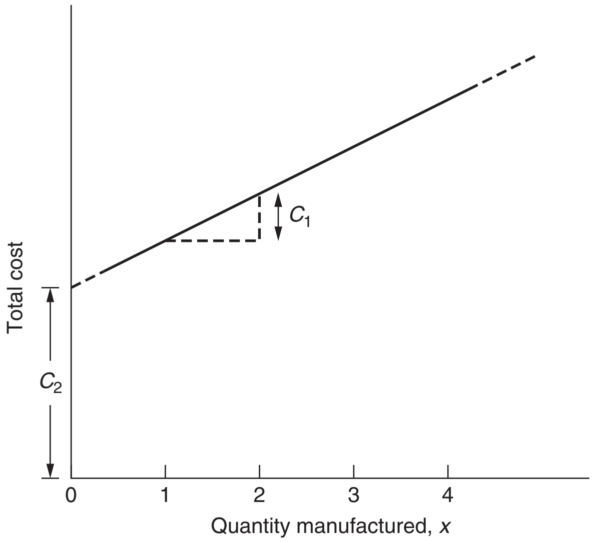


Figure 9.1

Clearly, the total cost is not a linear function of x. It is not even a continuous function as there is a discontinuity at the origin. Conventional LP is not capable of handling this situation.

In order to use IP, we introduce an indicator variable δ so that if any of the product is manufactured $\delta = 1$. This can be achieved by constraint (9.1) above. The variable δ is given a cost of C_2 in the objective function giving the following expression for the total cost:

$$\text{Total cost} = C_1x + C_2\delta.$$

By the introduction of 0–1 variables such as δ and extra constraints such as (9.1), to link these variables to the continuous variables such as x, fixed charges can be introduced into a model if the δ variables are given objective coefficients equal to the fixed charges.

It is worth pointing out that this is a situation where it is not generally necessary to model the condition (9.3). This condition will automatically be satisfied at optimality if the objective of the model has the effect of minimizing cost. Although a solution $x = 0$, $\delta = 1$ does not violate the constraints, it is clearly non-optimal as $x = 0$, $\delta = 0$ will not violate the constraints either but will result in a smaller total cost.

It would certainly not be invalid to impose condition (9.3) explicitly by a constraint such as in Equation (9.6) (as long as m was sufficiently small). In certain circumstances it might even be computationally desirable.

Example 9.2: Blending

(This example is relevant to the FOOD MANUFACTURE 2 problem in Part II.)

x_A represents the proportion of ingredient A to be included in a blend; x_B represents the proportion of ingredient B to be included in a blend.

In addition to the conventional quality constraints (for which LP can be used) connecting these and other variables in the model, it is wished to impose the following extra condition: *if A is included in the blend*, *B must be included also*.

IP must be used to model this extra condition. A 0–1 indicator variable δ is introduced, which will take the value 1 if $x_A > 0$. This is linked to variable x_A by the following constraint of type (1.4):

$$x_A - \delta \leq 0. \tag{9.8}$$

Here the coefficient M of constraint (9.1) can conveniently be taken as 1 as we are dealing with proportions.

We are now in a position to use the new 0–1 variable δ to impose the condition

$$\delta = 1 \rightarrow x_B > 0. \tag{9.9}$$

In order to impose this condition of Equation (9.4), we must choose some proportionate level m (say 1/100) below which we regard B as out of the blend. This gives us the following constraint:

$$x_B - 0.01\delta \geq 0. \tag{9.10}$$

We have now imposed the extra condition on the LP model by introducing a 0–1 variable δ with two extra constraints (9.8) and (9.10).

Notice that here (unlike Example 9.1) it was necessary to introduce a constraint to represent a condition of type (9.4). The satisfaction of such a condition could not be guaranteed by optimality. An extension of the extra condition that we have imposed might be the following: *if A is included in the blend, B must be included also and vice versa*. This requires two extra constraints that the reader might like to formulate.

It should be pointed out that any constant coefficient M can be chosen in constraints of type in Equation (9.1) so long as M is sufficiently big not to restrict the value of x to an extent not desired in the problem being modelled. In practical situations, it is usually possible to specify such a value for M. Although theoretically any sufficiently large value of M will suffice, there is computational advantage in making M as realistic as possible. This point is explained further in Section 10.1 of Chapter 10. Similar considerations apply to the coefficient m in constraints of type in Inequality (9.7).

It is possible to use indicator variables in a similar way to show whether an inequality holds or does not hold. First, suppose that we wish to indicate whether the following inequality holds by means of an indicator variable δ

$$\sum_j a_j x_j \leq b.$$

The following condition is fairly straightforward to formulate. We therefore model it first:

$$\delta = 1 \rightarrow \sum_j a_j x_j \leq b. \tag{9.11}$$

Equation (9.11) can be represented by the constraint

$$\sum_j a_j x_j + M\delta \leq M + b, \tag{9.12}$$

where M is an upper bound for the expression $\sum_j a_j x_j - b$. It is easy to verify that Inequality (9.12) has the desired effect, that is, when $\delta = 1$ the original constraint is forced to hold and when $\delta = 0$, no constraint is implied.

A convenient way of constructing Inequality (9.12) from condition (9.11) is to pursue the following train of reasoning. If $\delta = 1$ we wish to have $\sum_i a_j x_j - b \leq 0$, that is, if $(1-\delta) = 0$, we wish to have $\sum_i a_j x_j - b \leq 0$. This condition is imposed if

$$\sum_j a_j x_j - b \leq M\left(1 - \delta\right),$$

where M is a sufficiently large number. In order to find how large M must be, we consider the case $\delta = 0$, giving $\sum_j a_j x_j - b \leq M$.

This shows that we must choose M sufficiently large that this does not give an undesired constraint. Clearly, M must be chosen to be an upper bound for the expression $\sum_j a_j x_j - b$. Rearranging the constraint we have obtained with the variables on the left, we obtain Inequality (9.12).

We now consider how to model the reverse of constraint (9.11), that is,

$$\sum_j a_j x_j \leq b \rightarrow \delta = 1. \tag{9.13}$$

This is conveniently expressed as

$$\delta = 0 \rightarrow \sum_j a_j x_j \nleq b, \tag{9.14}$$

that is,

$$\delta = 0 \rightarrow \sum_j a_j x_j > b. \tag{9.15}$$

In dealing with the expression $\sum_j a_j x_j > b$, we run into the same difficulties that we met with the expression $x > 0$. We must rewrite

$$\sum_j a_j x_j > b \quad \text{as} \quad \sum_j a_j x_j \geq b + \varepsilon,$$

where ε is some small tolerance value beyond which we will regard the constraint as having been broken. Should the coefficients a_j be integers as well as the variables x_j, as often happens in this type of situation, there is no difficulty as ε can be taken as 1.

Equation (9.15) may now be rewritten as

$$\delta = 0 \rightarrow -\sum_j a_j x_j + b + \varepsilon \leq 0. \tag{9.16}$$

Using an argument similar to that above, we can represent this condition by the constraint

$$\sum_j a_j x_j - (m - \varepsilon)\,\delta \geq b + \varepsilon, \tag{9.17}$$

where m is a lower bound for the expression

$$\sum_j a_j x_j - b.$$

Should we wish to indicate whether a '$\geq$' inequality such as

$$\sum_j a_j x_j \geq b$$

holds or not by means of an indicator variable δ, the required constraints can easily be obtained by transforming the above constraint into a '$\leq$' form. The corresponding constraints to Inequalities (9.12) and (9.17) above are

$$\sum_j a_j x_j + m\delta \geq m + b, \tag{9.18}$$

$$\sum_j a_j x_j - (M + \varepsilon)\,\delta \leq b - \varepsilon, \tag{9.19}$$

where m and M are again lower and upper bounds respectively on the expression

$$\sum_j a_j x_j - b_i.$$

Finally, to use an indicator variable δ for an '=' constraint such as

$$\sum_j a_j x_j = b$$

is slightly more complicated. We can use $\delta = 1$ to indicate that the '$\leq$' and '$\geq$' cases hold simultaneously. This is done by stating both the constraints (9.12) and (9.18) together.

If $\delta = 0$, we want to force *either* the '$\leq$' or the '$\geq$' constraint to be broken. This may be done by expressing Inequalities (9.17)–(9.19) with two variables δ' and δ'' giving

$$\sum_j a_j x_j - (m - \varepsilon)\,\delta' \geq b + \varepsilon, \tag{9.20}$$

$$\sum_j a_j x_j - (M + \varepsilon)\,\delta'' \leq b - \varepsilon. \tag{9.21}$$

The indicator variable δ forces the required condition by the extra constraint

$$\delta' + \delta'' - \delta \leq 1. \tag{9.22}$$

In some circumstances, we wish to impose a condition of the type (9.11). Alternatively, we may wish to impose a condition of the type (9.13) or impose both conditions together. These conditions can be dealt with by the linear constraints (9.12) or (9.17) taken individually or together.

Example 9.3

Use a 0–1 variable δ to indicate whether or not the following constraint is satisfied:

$$2x_1 + 3x_2 \leq 1.$$

(x_1 and x_2 are non-negative continuous variables that cannot exceed 1.)

We wish to impose the following conditions:

$$\delta = 1 \rightarrow 2x_1 + 3x_2 \leq 1, \tag{9.23}$$

$$2x_1 + 3x_2 \leq 1 \rightarrow \delta = 1. \tag{9.24}$$

Using Inequality (9.12), M may be taken as 4 ($= 2 + 3 - 1$). This gives the following constraint representation of condition (9.23):

$$2x_1 + 3x_2 + 4\delta \leq 5. \tag{9.25}$$

Using Inequality (9.17), m may be taken as $-1(=0+0-1)$. We take ε as 0.1. This gives the following constraint representation of Equation (9.24):

$$2x_1 + 3x_2 + 1.1\delta \geq 1.1. \tag{9.26}$$

The reader should verify that Inequalities (9.25) and (9.26) have the desired effect by substituting 0 and 1 for δ.

In all the constraints derived in this section it is computationally desirable to make m and M as realistic as possible.

9.2 Logical conditions and 0–1 variables

In Section 9.1, it was pointed out that 0–1 variables are often introduced into an LP (or sometimes an IP) model as decision variables or indicator variables. Having introduced such variables, it is then possible to represent logical connections between different decisions or states by linear constraints involving these 0–1 variables. It is at first sight rather surprising that so many different types of logical condition can be imposed in this way.

Some typical examples of logical conditions that can be so modelled are given below. Further examples are given by Williams (1977).

1. If no depot is sited here, then it will not be possible to supply any of the customers from the depot.
2. If the library's subscription to this journal is cancelled, then we must retain at least one subscription to another journal in this class.
3. If we manufacture product A, we must also manufacture product B or at least one of products C and D.
4. If this station is closed, then both branch lines terminating at the station must also be closed.
5. No more than five of the ingredients in this class may be included in the blend at any one time.
6. If we do not place an electronic module in this position, then no wires can connect into this position.
7. Either operation A must be finished before operation B starts or vice versa.

It will be convenient to use some notation from Boolean algebra in this section. This is the so-called set of *connectives* given below:

'$\vee$' means 'or' (this is inclusive, i.e. A or B or both).

'$\cdot$' means 'and'.

'$\sim$' means 'not'.

'$\rightarrow$' means 'implies' (or 'if ... then').

'$\leftrightarrow$' means 'if and only if'.

These connectives are used to connect propositions denoted by P, Q, R, etc., $x > 0$, $x = 0, \delta = 1$, etc.

For example, if P stands for the proposition 'I will miss the bus' and Q stands for the proposition 'I will be late', then $P \rightarrow Q$ stands for the proposition 'If I miss the bus, then I will be late'. $\sim P$ stands for the proposition 'I will not miss the bus'.

As another example suppose that X_i stands for the proposition 'Ingredient i is in the blend' (i ranges over the ingredients A, B and C). Then $X_\mathrm{A} \rightarrow (X_\mathrm{B} \vee X_\mathrm{C})$ stands for the proposition 'If ingredient A is in the blend then ingredient B or C (or both) must also be in the blend'. This expression could also be written as $(X_\mathrm{A} \rightarrow X_\mathrm{B}) \vee (X_\mathrm{A} \rightarrow X_\mathrm{C})$.

It is possible to define all these connectives in terms of a subset of them. For example, they can all be defined in terms of the set $\{\vee, \sim\}$ Such a subset is known as a *complete set* of connectives. We do not choose to do this and will retain the flexibility of using all the connectives listed above. It is important, however, to realize that certain expressions are equivalent to expressions involving other connectives. We give all the equivalences below that are sufficient for our purpose.

To avoid unnecessary brackets, we consider the symbols '$\sim$', '$\cdot$', '$\vee$' and '$\rightarrow$' as each being more binding than their successor when written in this order.

For example,

$$(P \cdot Q) \vee R \text{ can be written as } P \cdot Q \vee R,$$

$$P \rightarrow (Q \vee R) \text{ can be written as } P \rightarrow Q \vee R,$$

$$\sim\sim P \text{ is the same as } P, \tag{9.27}$$

$$P \rightarrow Q \text{ is the same as } \sim P \vee Q, \tag{9.28}$$

$$P \rightarrow Q \cdot R \text{ is the same as } (P \rightarrow Q) \cdot (P \rightarrow R), \tag{9.29}$$

$$P \rightarrow Q \vee R \text{ is the same as } (P \rightarrow Q) \vee (P \rightarrow R), \tag{9.30}$$

$$P \cdot Q \rightarrow R \text{ is the same as } (P \rightarrow R) \vee (Q \rightarrow R), \tag{9.31}$$

$$P \vee Q \rightarrow R \text{ is the same as } (P \rightarrow R) \cdot (Q \rightarrow R), \tag{9.32}$$

$$\sim (P \vee Q) \text{ is the same as } \sim P \cdot \sim Q, \tag{9.33}$$

$$\sim (P \cdot Q) \text{ is the same as } \sim P \vee \sim Q. \tag{9.34}$$

Expressions (9.33) and (9.34) are sometimes known as *De Morgan's laws*.

Although Boolean algebra provides a convenient means of expressing and manipulating logical relationships, our purpose here is to express these relationships in terms of the familiar equations and inequalities of mathematical programming. (In one sense, we are performing the opposite process to that used in the pseudo-Boolean approach to 0–1 programming mentioned in Section 8.3.)

We will suppose that indicator variables have already been introduced in the manner described in Section 9.1 to represent the decisions or states that we want logically to relate.

It is important to distinguish propositions and variables at this stage. We use X_i to stand for the proposition $\delta_i = 1$ where δ_i is a 0–1 indicator variable. The following propositions and constraints can easily be seen to be equivalent:

$$X_1 \vee X_2 \text{ is equivalent to } \delta_1 + \delta_2 \geq 1; \tag{9.35}$$

$$X_1 \cdot X_2 \text{ is equivalent to } \delta_1 = 1, \delta_2 = 1; \tag{9.36}$$

$$\sim X_1 \text{ is equivalent to } \delta_1 = 0 \left(\text{or } 1 - \delta_1 = 1\right); \tag{9.37}$$

$$X_1 \rightarrow X_2 \text{ is equivalent to } \delta_1 - \delta_2 \leq 0; \tag{9.38}$$

$$X_1 \leftrightarrow X_2 \text{ is equivalent to } \delta_1 - \delta_2 = 0. \tag{9.39}$$

To illustrate the conversion of a logical condition into a constraint we consider an example.

Example 9.4: Manufacturing

If either of products A or B (or both) are manufactured, then at least one of products C, D or E must also be manufactured.

Let X_i stand for the proposition 'Product i is manufactured' (i is A, B, C, D or E). We wish to impose the logical condition

$$\left(X_{\mathrm{A}} \vee X_{\mathrm{B}}\right) \rightarrow \left(X_{\mathrm{C}} \vee X_{\mathrm{D}} \vee X_{\mathrm{E}}\right). \tag{9.40}$$

Indicator variables are introduced to perform the following functions: $\delta_i = 1$ if and only if product i is manufactured; $\delta = 1$ if the proposition $X_{\mathrm{A}} \vee X_{\mathrm{B}}$ holds. The proposition $X_{\mathrm{A}} \vee X_{\mathrm{B}}$ can be represented by the inequality

$$\delta_{\mathrm{A}} + \delta_{\mathrm{B}} \geq 1. \tag{9.41}$$

The proposition $X_{\mathrm{C}} \vee X_{\mathrm{D}} \vee X_{\mathrm{E}}$ can be represented by the inequality

$$\delta_{\mathrm{C}} + \delta_{\mathrm{D}} + \delta_{\mathrm{E}} \geq 1. \tag{9.42}$$

Firstly, we wish to impose the following condition:

$$\delta_{\mathrm{A}} + \delta_{\mathrm{B}} \geq 1 \rightarrow \delta = 1. \tag{9.43}$$

Using Inequality (9.19) of Section 9.1, we impose this condition by the constraint

$$\delta_{\mathrm{A}} + \delta_{\mathrm{B}} - 2\delta \leq 0. \tag{9.44}$$

Secondly, we wish to impose the condition

$$\delta = 1 \rightarrow \delta_C + \delta_D + \delta_E \geq 1. \tag{9.45}$$

Using Inequality (9.18) of Section 9.1, this is achieved by the constraint

$$-\delta_C - \delta_D - \delta_E + \delta \leq 0. \tag{9.46}$$

Hence the required extra condition can be imposed on the original model (LP or IP) by the following:

1. Introduce 0–1 variables $\delta_A, \delta_B, \delta_C, \delta_D$ and δ_E and link them to the original (probably continuous) variables by constraints of type (9.1) and (9.7) of Section 9.1. It is not strictly necessary to include constraints of type (9.7) for the variables δ_A and δ_B as it is not necessary to have the conditions (9.4) of Section 9.1 in these cases.
2. Add the additional constraints (9.44) and (9.46) above.

This is not the only way to model this logical condition. Using the Boolean identity (9.32) above, it is possible to show that condition (9.40) can be re-expressed as

$$\left[X_A \rightarrow \left(X_C \vee X_D \vee X_E\right)\right] \cdot \left[X_B \rightarrow \left(X_C \vee X_D \vee X_E\right)\right]. \tag{9.47}$$

The reader should verify that an analysis similar to that above results in the constraint (9.46) together with the following two constraints in place of Inequality (9.44):

$$\delta_A - \delta \leq 0; \tag{9.48}$$

$$\delta_B - \delta \leq 0. \tag{9.49}$$

Both ways of modelling the condition are correct. There are computational advantages in Inequalities (9.48) and (9.49) over (9.44). This is discussed further in Section 10.1 of Chapter 10.

It is sometimes suggested that polynomial expressions in 0–1 variables are useful for expressing logical conditions. Such polynomial expressions can always be replaced by linear expressions with linear constraints, possibly with a considerable increase in the number of 0–1 variables. For example, the constraint

$$\delta_1\delta_2 = 0 \tag{9.50}$$

represents the condition

$$\delta_1 = 0 \vee \delta_2 = 0. \tag{9.51}$$

More generally, if a product term such as $\delta_1\delta_2$ were to appear anywhere in a model, the model could be made linear by the following steps:

1. Replace $\delta_1\delta_2$ by a 0–1 variable δ_3.
2. Impose the logical condition

$$\delta_3 = 1 \leftrightarrow \delta_1 = 1 \cdot \delta_2 = 1 \tag{9.52}$$

by means of the extra constraints

$$\begin{aligned} -\delta_1 \qquad + \delta_3 &\le 0, \\ -\delta_2 + \delta_3 &\le 0, \\ \delta_1 + \delta_2 - \delta_3 &\le 1. \end{aligned} \tag{9.53}$$

An example of the need to linearize products of 0–1 variables in this way arises in the DECENTRALIZATION problem in Part III. Products involving more than two variables can be progressively reduced to single variables in a similar manner.

A major defect of the above formulation is that, if there are n original 0–1 variables, there can be up to $n(n-1)$ new 0–1 variables and $3n(n-1)$ extra constraints. A much more compact formulation is due to Glover (1975). Suppose we consider all the terms involving δ_1 in an expression and group them together, for example, $\delta_1(2\delta_2 + 3\delta_3 - \delta_4)$. We represent this expression by a new continuous variable w. Then the relations between w and the δ_i are as follows:

$$\begin{aligned} &\delta_1 = 1 \rightarrow w = 2\delta_2 + 3\delta_3 - \delta_4, \\ &\delta_1 = 0 \rightarrow w = 0. \end{aligned}$$

We model this by

$$\begin{aligned} &w \ge 5\delta_1 + 2\delta_2 + 3\delta_3 - \delta_4 - 5, \\ &w \le 2\delta_2 + 3\delta_3 - \delta_4, \\ &w \le 5\delta_1. \end{aligned}$$

It should be noted, however, that if there relatively few ‘connections’ between the variables, that is, each variable forms few products with more than one other variable, there may be little saving in the size of model.

It is even possible to linearize terms involving a product of a 0–1 variable with a continuous variable. For example the term $x\delta$, where x is continuous and δ is 0–1, can be treated in the following way:

1. Replace $x\delta$ by a continuous variable y.
2. Impose the logical conditions

$$\begin{aligned} \delta = 0 &\rightarrow y = 0, \\ \delta = 1 &\rightarrow y = x \end{aligned} \tag{9.54}$$

by the extra constraints

$$\begin{aligned} y - M\delta &\le 0, \\ -x + y \qquad &\le 0, \\ x - y + M\delta &\le M, \end{aligned} \tag{9.55}$$

where M is an upper bound for x (and hence also y).

Other non-linear expressions (such as ratios of polynomials) involving 0–1 variables can also be made linear in similar ways. Such expressions tend to occur fairly rarely and are not therefore considered further. They do, however, provide interesting problems of logical formulation using the principles described in this section and can provide useful exercises for the reader.

The purpose of this section together with Example 9.4 has been to demonstrate a method of imposing logical conditions on a model. This is by no means the only way of approaching this kind of modelling. Different rule-of-thumb methods exist for imposing the desired conditions. Experienced modellers may feel that they could derive the constraints described here by easier methods. It has been the author's experience, however, that the following are true:

1. Many people are unaware of the possibility of modelling logical conditions with 0–1 variables.
2. Among those people who realize that this is possible, many find themselves unable to capture the required restrictions by 0–1 variables with logical constraints.
3. It is very easy to model a restriction incorrectly. By using concepts from Boolean algebra and approaching the modelling in the above manner, it should be possible satisfactorily to impose the desired logical conditions.

A system for automating the formulation of logical conditions within standard predicates and implementing them within the language PROLOG is described by McKinnon and Williams (1989).

Logical conditions are sometimes expressed within a constraint logic programming language as discussed in Section 2.4. The 'tightest' way (this concept is explained in Section 8.3) of expressing certain examples using linear programming constraints is described by Hooker and Yan (1999) and Williams and Yan (2001). There is a close relationship between logic and 0–1 integer programming, which is explained in Williams (1995, 2009), Williams and Brailsford (1999) and Chandru and Hooker (1999).

9.3 Special ordered sets of variables

Two very common types of restriction arise in mathematical programming problems for which the concept special ordered set of type 1 (SOS1) and special ordered set of type 2 (SOS2) have been developed. This concept is due to Beale and Tomlin (1969).

An SOS1 is a set of variables (continuous or integer) within which exactly one variable must be non-zero.

An SOS2 is a set of variables within which at most two can be non-zero. The two variables must be adjacent in the ordering given to the set.

It is perfectly possible to model the restrictions that a set of variables belongs to an SOS1 set or an SOS2 set using integer variables and constraints. The way in which this can be done is described below. There is great computational advantage to be gained, however, from treating these restrictions algorithmically. The way in which the branch and bound algorithm can be modified to deal with SOS1 and SOS2 sets is beyond the scope of this book. It is described in Beale and Tomlin.

Some examples are given on how SOS1 sets and SOS2 sets can arise.

Example 9.5: Depot Siting

A depot can be sited at any one of the positions A, B, C, D or E. Only one depot can be built.

If 0–1 indicator variables δ_i are used to perform the following purpose: $\delta_i = 1$ if and only if the depot is sited at i (i is A, B, C, D or E), then the set of variables $(\delta_1, \delta_2, \delta_3, \delta_4, \delta_5)$ can be regarded as an SOS1 set.

The SOS1 condition together with the constraint

$$\delta_1 + \delta_2 + \delta_3 + \delta_4 + \delta_5 = 1 \tag{9.56}$$

guarantees integrality and it is not necessary to stipulate that the δ_i be integral. Only if the sites have a natural ordering is there great advantage to be gained in the SOS formulation.

Example 9.6: Capacity Extension

The capacity C of a plant can be extended in *discrete* amounts by increasing levels of investment I.

If the set of variables $(\delta_0, \delta_1, \delta_2, \delta_3, \delta_4, \delta_5)$ is regarded as an SOS1 set, then we can model

$$C = C_1\delta_1 + C_2\delta_2 + C_3\delta_3 + C_4\delta_4 + C_5\delta_5, \tag{9.57}$$

$$I = I_1\delta_1 + I_2\delta_2 + I_3\delta_3 + I_4\delta_4 + I_5\delta_5, \tag{9.58}$$

$$\delta_0 + \delta_1 + \delta_2 + \delta_3 + \delta_4 + \delta_5 = 1. \tag{9.59}$$

It is not necessary to treat the δ_i as integer variables as the SOS1 condition together with Equation (9.59) forces integrality. Conceptually, it is important to regard an SOS set as an *entity*. We can then regard C as a quantity that is a *discrete* function of I. This can be regarded as a *generalization of a* 0–1 *variable* to more than two discrete values. Such a generalization is often more useful than the conventional general integer variable.

Although the most common application of SOS1 sets is to modelling what would otherwise be 0–1 integer variables with a constraint such as in Equation (9.59), there are other applications.

The most common application of SOS2 sets is to modelling non-linear functions as described by the following example.

Example 9.7: Non-linear Functions

In Section 7.3 the concept of a separable set was introduced in order to make a piecewise linear approximation to a non-linear function of a single variable. Using the λ-convention for such a separable formulation, we obtained the following convexity constraint:

$$\lambda_1 + \lambda_2 + \cdots + \lambda_n = 1. \tag{9.60}$$

In addition, in order that the coordinates of x and y should lie on the piecewise linear curve in Figure 7.15, it was necessary to impose the following extra restriction:

$$\textit{At most two adjacent } \lambda\textit{s can be non-zero.} \tag{9.61}$$

Instead of approaching this restriction through separable programming with the danger of local rather than global optima as described in Section 7.2, we can use an SOS2. The restriction in (9.61) need not be modelled explicitly. Instead, we can say that the set of variables $(\lambda_1, \lambda_2, \ldots, \lambda_n)$ is an SOS2.

The formulation of a non-linear function in Example 9.7 demands that the non-linear function be *separable*, that is, the sum of non-linear functions of a single variable. It was demonstrated in Section 7.4 how models with non-separable functions may sometimes be converted into models where the non-linearities are all functions of a single variable. While this is often possible, it can be cumbersome, increasing the size of the model considerably as well as the computational difficulty. An alternative is to extend the concept of an SOS set to that of a *chain of linked SOS sets*. This has been done by Beale (1980). The idea is best illustrated by a further example.

Example 9.8: Non-linear Functions of Two or More Variables

Suppose $z = g(x, y)$ is a non-linear function of x and y.

We define a grid of values of (x, y) (not necessarily equidistant) and associate non-negative 'weightings' λ_{ij} with each point in the grid as shown in Figure 9.2.

If the values of (x, y) at the grid points are denoted by (X_s, Y_k) we can approximate the function $z = g(x, y)$ by means of the following relations:

$$x = \sum_s \sum_k X_s \lambda_{sk}, \tag{9.62}$$

$$y = \sum_s \sum_k Y_k \lambda_{sk}, \tag{9.63}$$

$$z = \sum_s \sum_k g\left(X_s, Y_k\right) \lambda_{sk}, \tag{9.64}$$

$$\sum_s \sum_k \lambda_{sk} = 1. \tag{9.65}$$

In addition, it is necessary to impose the following restriction on the λ variables:

$$\text{At most four neighbouring } \lambda s \text{ can be non} - \text{zero.} \tag{9.66}$$

This last condition is clearly a generalization of an SOS2 set. We can impose condition (9.66) in the following way. Let

$$\xi_s = \sum_k \lambda_{sk}, \quad \eta_k = \sum_s \lambda_{sk}$$

for all s, k. $(\xi_1, \xi_2, \xi_3, \ldots)$ and $(\eta_1, \eta_2, \eta_3, \ldots)$ with each taken as SOS2 sets. The SOS2 condition for the first set allows λ_s to be non-zero in at most two neighbouring rows in Figure 9.2. For the second set, the SOS2 condition allows λ_s to be non-zero in at most two neighbouring columns. For example, we might have $\xi_2 = 1/3, \xi_3 = 2/3, \eta_5 = 1/4, \eta_6 = 3/4$.

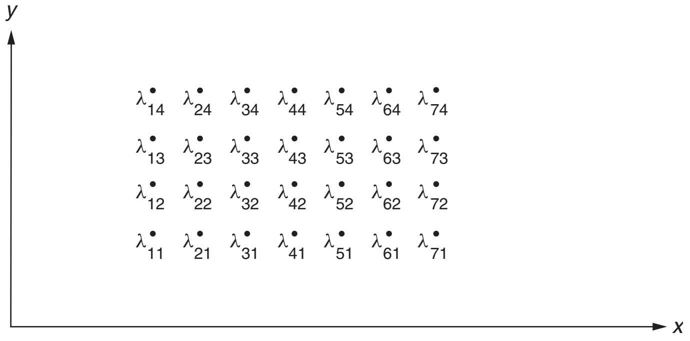

y
x
λ14 λ24 λ34 λ44 λ54 λ64 λ74
λ13 λ23 λ33 λ43 λ53 λ63 λ73
λ12 λ22 λ32 λ42 λ52 λ62 λ72
λ11 λ21 λ31 λ41 λ51 λ61 λ71

Figure 9.2

The values of ξ and η above could arise from $\lambda_{25} = 1/6, \lambda_{26} = 1/6, \lambda_{35} = 1/12, \lambda_{36} = 7/12$, all other λ_s being zero. They could, however, also arise from other values of the λ_s, for example, $\lambda_{25} = 1/4, \lambda_{26} = 1/12, \lambda_{36} = 2/3$, with all other λ_s being zero.

In order to get round this non-uniqueness, we can restrict the non-zero λ_s to vertices of a triangle (such as in the second instance above). A lengthy way of doing this is to impose the extra constraints:

$$\zeta_t = \sum_s \lambda_{s,t+s} \tag{9.67}$$

and treat the ζ_t as a further SOS2 set.

If, however, we are content to restrict the x (or y) to grid values (i.e. not interpolate in that direction) then the problem does not arise. Indeed we can also avoid introducing the sets ξ_s so long as within each set λ_{sk}, with the same s, the member of which is non-zero has the same index k. The sets λ_{sk} are then known as a *chain* of *linked SOS* sets as described by Beale (1980) and the restriction can be dealt with algorithmically.

Some of the problems presented in Part II can be formulated to take advantage of special ordered sets. In particular, DECENTRALIZATION and LOGIC DESIGN can exploit SOS1.

While it is desirable to treat SOS sets algorithmically if this facility exists in the package being used, the restrictions that they imply can be imposed using 0–1 variables and linear constraints. This is now demonstrated.

Suppose $(x_1, x_2, \ldots, x_n)$ is an SOS1 set. If the variables are not 0–1, we introduce 0–1 indicator variables $\delta_1, \delta_2, \ldots, \delta_n$ and link them to the x_i variables in the conventional way by constraints:

$$x_i - M_i\delta_i \leq 0, \quad i = 1, 2, \ldots, n, \tag{9.68}$$

$$x_i - m_i\delta_i \leq 0, \quad i = 1, 2, \ldots, n, \tag{9.69}$$

where M_i and m_i are constant coefficients being upper and lower bounds respectively for x_i.

The following constraint is then imposed on the δ_i variables:

$$\delta_1 + \delta_2 + \ldots + \delta_n = 1. \tag{9.70}$$

If the x_i variables are 0–1, we can immediately regard them as the δ_i variables above and only need impose the constraint (9.70).

To model an SOS2 set using 0–1 variables is more complicated. Suppose $(\lambda_1, \lambda_2, \ldots, \lambda_n)$ is an SOS2 set. We introduce 0–1 variables $\delta_1, \delta_2, \ldots, \delta_{n-1}$ together with the following constraints:

$$\begin{array}{llllllll}
\lambda_1 & & & & & -\delta_1 & & \leq 0, \\
& \lambda_2 & & & & -\delta_1 - \delta_2 & & \leq 0, \\
& & \lambda_3 & & & & -\delta_2 - \delta_3 & \leq 0, \\
& & & \ddots & & & \ddots & \vdots \\
& & & & \lambda_{n-1} & & -\delta_{n-2} - \delta_{n-1} & \leq 0, \\
& & & & & \lambda_n & -\delta_{n-1} & \leq 0.
\end{array} \tag{9.71}$$

and

$$\delta_1 + \delta_2 + \ldots + \delta_{n-1} = 1. \tag{9.72}$$

This formulation suggests the relationship between SOS1 and SOS2 sets as Equation (9.72) could be dispensed with by regarding the δ_i as belonging to an SOS1 set as long as the δ_i each have an upper bound of 1.

An improvement to the IP modelling of the λ-formulation has been proposed by Sherali (2001). The formulation gives a tighter LP relaxation and also allows the modelling of discontinuous piecewise linear expressions such as that illustrated in Figure 9.3. In order to define the function uniquely at the point of discontinuity, we must make it upper or lower semi-continuous at the point of discontinuity, that is, define the left or right-hand interval to be open at this point.

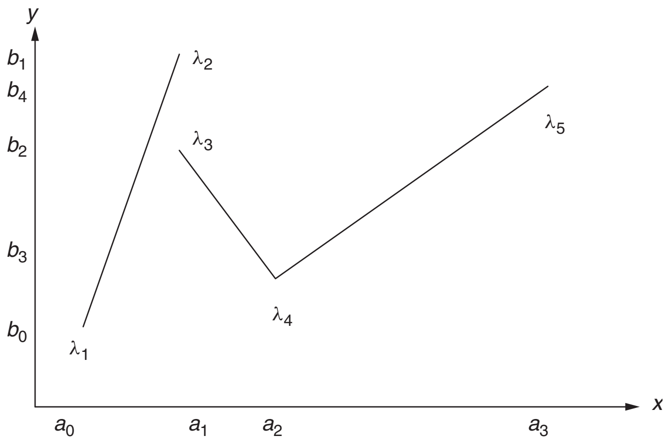


Figure 9.3

In order to model a piecewise linear function, we associate two (weighting) variables λL, λR, which must sum to 1, with each end of each section. Then we stipulate that exactly one pair of weights is positive, confining the function to exactly one of the sections. The IP formulation of the function illustrated in Figure 9.3 is

$$\delta_1 + \delta_2 + \delta_3 = 1, \delta_i \in \{0, 1\}, \tag{9.73}$$

$$\lambda_1 + \lambda_2 = \delta_1, \quad \lambda_3 + \lambda_4 = \delta_2, \quad \lambda_4 + \lambda_5 = \delta_3, \tag{9.74}$$

$$x = a_0\lambda_1 + a_1\left(\lambda_2 + \lambda_3\right) + a_2\lambda_4 + a_3\lambda_5, \tag{9.75}$$

$$y = b_0\lambda_1 + b_1\lambda_2 + b_2\lambda_3 + b_3\lambda_4 + b_4\lambda_5. \tag{9.76}$$

9.4 Extra conditions applied to linear programming models

As the majority of practical applications of IP give rise to mixed integer programming models where extra conditions have been applied to an otherwise LP model, this subject is considered further in this section. A number of the commonest applications are briefly outlined.

9.4.1 Disjunctive constraints

Suppose that for an LP problem we do not require all the constraints to hold simultaneously. We do, however, require at least one subset of constraints to hold. This could be stated as

$$R_1 \vee R_2 \vee \ldots \vee R_N, \tag{9.77}$$

where R_i is the proposition 'The constraints in subset i are satisfied' and constraints $1, 2, \ldots, N$ form the subset in question. Expression (9.77) is known as a *disjunction of constraints*.

Following the principles of Section 9.1, we introduce N indicator variables δ_i to indicate whether the R_i are satisfied. In this case, it is only necessary to impose the conditions

$$\delta_i = 1 \rightarrow R_i. \tag{9.78}$$

This may be done by constraints of type (9.12) or (9.18) (Section 9.1) taken singly or together according to whether R_i are '$\leq$', '$\geq$', or '$=$' constraints. We can then impose condition (9.77) by the constraint

$$\delta_1 + \delta_2 + \cdots + \delta_N \geq 1. \tag{9.79}$$

An alternative formulation of Expression (9.77) is possible. This is discussed in Section 10.2 and is due to Jeroslow and Lowe (1984), who report promising computational results in Jeroslow and Lowe (1985).

A generalization of Expression (9.77) that can arise is the condition

$$\text{At least } k \text{ of } \left(R_1, R_2, \ldots, R_N\right) \text{ must be satisfied.} \tag{9.80}$$

This is modelled in a similar way but using the constraint below in place of Inequality (9.79):

$$\delta_1 + \delta_2 + \cdots + \delta_N \geq k. \tag{9.81}$$

A variation of expression (9.80) is the condition

$$\text{At most } k \text{ of } \left(R_1, R_2, \ldots, R_N\right) \text{ must be satisfied.} \tag{9.82}$$

To model Expression (9.82) using indicator variables δ_i, it is only necessary to impose the conditions

$$R_i \rightarrow \delta_i = 1. \tag{9.83}$$

This may be done by constraints of type (9.17) or (9.19) (Section 9.1) taken together or singly according to whether R_i is a '$\leq$', '$\geq$', or '$=$' constraint. Condition (9.82) can then be imposed by the constraint

$$\delta_1 + \delta_2 + \cdots + \delta_N \geq k. \tag{9.84}$$

Disjunctions of constraints involve the logical connective '$\vee$' ('or') and necessitate IP models.

It is worth pointing out that the connective '·' ('and') can obviously be coped with through conventional LP as a conjunction of constraints simply involves a series of constraints holding simultaneously. In this sense, one can regard 'and' as corresponding to LP and 'or' as corresponding to IP.

9.4.2 Non-convex regions

As an application of disjunctive constraints, we show how restrictions corresponding to a non-convex region may be imposed using IP. It is well known that the feasible region of an LP model is convex. (Convexity is defined in Section 7.2.) There are circumstances, however, in non-linear programming problems where we wish to have a non-convex feasible region. For example, we consider the feasible region ABCDEFGO of Figure 9.4. This is a non-convex region bounded by a series of straight lines. Such a region may have arisen through the problem considered or represent a piecewise linear approximation to a non-convex region bounded by curves.

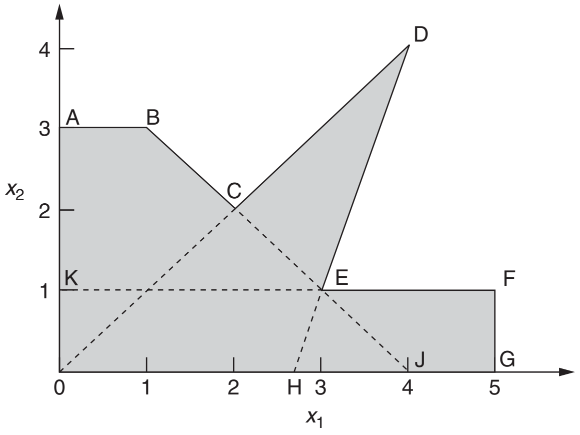


Figure 9.4

We may conveniently think of the region ABCDEFGO as made up of the union of the three convex regions ABJO, ODH and KFGO, as shown in Figure 9.4. The fact these regions overlap does not matter.

Region ABJO is defined by the constraints

$$\begin{aligned} x_2 &\le 3, \\ x_1 + x_2 &\le 4. \end{aligned} \tag{9.85}$$

Region ODH is defined by the constraints

$$\begin{aligned} -x_1 + x_2 &\le 0, \\ 3x_1 - x_2 &\le 8. \end{aligned} \tag{9.86}$$

Region KFGO is defined by the constraints

$$\begin{aligned} x_2 &\leq 1, \\ x_1 &\leq 5. \end{aligned} \tag{9.87}$$

We introduce indicator variables δ_1, δ_2 and δ_3 to use in the following conditions:

$$\delta_1 = 1 \rightarrow \left(x_2 \leq 3\right) \cdot \left(x_1 + x_2 \leq 4\right), \tag{9.88}$$

$$\delta_2 = 1 \rightarrow \left(-x_1 + x_2 \leq 0\right) \cdot \left(3x_1 - x_2 \leq 8\right), \tag{9.89}$$

$$\delta_3 = 1 \rightarrow \left(x_2 \leq 1\right) \cdot \left(x_1 \leq 5\right). \tag{9.90}$$

Equations (9.88)–(9.90) are respectively imposed by the following constraints:

$$\begin{aligned} x_2 + \delta_1 &\leq 4, \\ x_1 + x_2 + 5\delta_1 &\leq 9, \end{aligned} \tag{9.91}$$

$$\begin{aligned} -x_1 + x_2 + 4\delta_2 &\leq 4, \\ 3x_1 - x_2 + 7\delta_2 &\leq 15, \end{aligned} \tag{9.92}$$

$$\begin{aligned} x_2 + 3\delta_3 &\leq 4, \\ x_1 &\leq 5. \end{aligned} \tag{9.93}$$

It is now only necessary to impose the condition that at least one of the set of Inequalities (9.85), (9.86) or (9.87) must hold. This is done by the constraint

$$\delta_1 + \delta_2 + \delta_3 \geq 1. \tag{9.94}$$

It would also be possible to cope with a situation in which the feasible region was disconnected in this way.

There is an alternative formulation for a connected non-convex region, such as that above, so long as the line joining the origin to any feasible point lies entirely within the feasible region. Seven 'weighting' variables $\lambda_A, \lambda_B, \ldots, \lambda_G$ are associated with the vertices $A, B, \ldots, G$ and incorporated in the following constraints:

$$\lambda_B + 2\lambda_C + 4\lambda_D + 3\lambda_E + 5\lambda_F + 5\lambda_G - x_1 = 0, \tag{9.95}$$

$$3\lambda_A + 3\lambda_B + 2\lambda_C + 4\lambda_D + \lambda_E + \lambda_F - x_2 = 0, \tag{9.96}$$

$$\lambda_A + \lambda_B + \lambda_C + \lambda_D + \lambda_E + \lambda_E + \lambda_G \leq 1. \tag{9.97}$$

The λ variables are then restricted to form an SOS2. Notice that this is a generalization of the use of an SOS2 set to model a piecewise linear function such as that represented by the line ABCDEFG. In that case, constraint (9.97) would become an equation, that is, the λs would sum to 1. For the example here, we simply relax this restriction to give a '$\leq$' constraint.

9.4.3 Limiting the number of variables in a solution

This is another application of disjunctive constraints. It is well known that, in LP, the optimal solution need never have more variables at a non-zero value than there are constraints in the problem. Sometimes it is required, however, to restrict this number still further (to k). To do this requires IP. Indicator variables δ_i are introduced to link with each of the n continuous variables x_i in the LP problem by the condition

$$x_i > 0 \rightarrow \delta_1 = 1. \tag{9.98}$$

As before, this condition is imposed by the constraint

$$x_i - M_i\delta_i \leq 0, \tag{9.99}$$

where M_i is an upper bound on x_i.

We then impose the condition that at most k of the variables x_i can be non-zero by the constraint

$$\delta_1 + \delta_2 + \ldots + \delta_n \leq k.$$

A very common application of this type of condition is in limiting the number of ingredients in a blend. The FOOD MANUFACTURE 2 example of Part II is an example of this. Another situation in which the condition might arise is where it is desired to limit the range of products produced in a product mix type LP model.

9.4.4 Sequentially dependent decisions

It sometimes happens that we wish to model a situation in which decisions made at a particular time will affect decisions made later. Suppose, for example, that in a multi-period LP model (n periods) we have introduced a decision variable γ_t into each period to show how a decision should be made in each period. We let γ_t represent the following decisions: $\gamma_t = 0$ means the depot should be permanently closed down; $\gamma_t = 1$ means the depot should be temporarily closed (this period only); $\gamma_t = 2$ means the depot should be used in this period. Clearly, we would wish to impose (among others) the conditions

$$\gamma_t = 0 \rightarrow (\gamma_{t+1} = 0) \cdot (\gamma_{t+2} = 0) \ldots (\gamma_n = 0). \tag{9.100}$$

This may be done by the following constraints:

$$\begin{array}{llll} -2\gamma_1 + \gamma_2 & & \leq & 0, \\ & -2\gamma_2 + \gamma_3 & \leq & 0, \\ & \ddots \quad \vdots & & \\ & & -2\gamma_{n-1} + \gamma_n \leq 0. & \end{array} \tag{9.101}$$

In this case, the decision variable γ_t can take three values. More usually, it will be a 0–1 variable.

A case of sequentially dependent decisions arises in the MINING problem of Part II.

9.4.5 Economies of scale

It was pointed out in Chapter 7 that economies of scale lead to a non-linear programming problem where the objective is equivalent to minimizing a non-convex function. In this situation, it is not possible to reduce the problem to an LP problem by piecewise linear approximations alone. Nor is it possible to rely on separable programming as local optima might result.

Suppose, for example, that we have a model where the objective is to minimize cost. The amount to be manufactured of a particular product is represented by a variable x. For increasing x, the unit marginal costs decrease.

Diagrammatically, we have the situation shown in Figure 9.5. This may be the true cost curve or be a piecewise linear approximation.

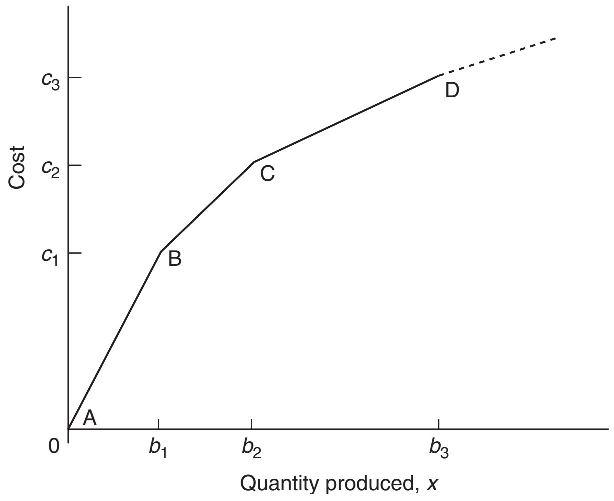


Figure 9.5

The unit marginal costs are successively

$$\frac{c_1}{b_1} > \frac{c_2 - c_1}{b_2 - b_1} > \frac{c_3 - c_2}{b_3 - b_2} \cdots .$$

Using the λ-formulation of separable programming as described in Chapter 7, we introduce $n + 1$ variables λ_i $(i = 0, 1, 2, \ldots, n)$ that may be interpreted as 'weights' attached to the vertices A, B, C, D, etc. We then have

$$x = b_1\lambda_1 + b_2\lambda_2 + \ldots + b_n\lambda_n, \tag{9.102}$$

$$\text{cost} = c_1\lambda_1 + c_2\lambda_2 + \ldots + c_n\lambda_n. \tag{9.103}$$

The set of variables $(\lambda_0, \lambda_1, \ldots, \lambda_n)$ is now regarded as a special ordered set of type 2. If IP is used, a global optimal solution can be obtained.

It is also, of course, possible to model this situation using the δ-formulation of separable programming.

9.4.6 Discrete capacity extensions

It is sometimes unrealistic to regard an LP constraint (usually a capacity constraint) as applying in all circumstances. In real life, it is often possible to violate the constraint at a certain cost. This topic has already been mentioned in Section 3.3. There, however, we allowed this constraint to be relaxed continuously. Often this is not possible. Should the constraint be successively relaxed, it may be possible that it can only be done in finite jumps, for example, we buy in whole new machines or whole new storage tanks.

Suppose the initial right-hand side (RHS) value is b_0 and that this may be successively increased to $b_1, b_2, \ldots, b_n$.

We have

$$\sum_j a_j x_j \leq b_i; \tag{9.104}$$

also

$$\text{cost} = \begin{cases} 0 & \text{if } i = 0, \\ c_i & \text{otherwise,} \end{cases}$$

where $0 < c_1 < c_2 < \cdots < c_n$.

This situation may be modelled by introducing 0–1 variables. $\delta_0, \delta_1, \delta_2$, etc., to represent the successive possible RHS values applying. We then have

$$\sum_j a_j x_j - b_0\delta_0 - b_1\delta_1 - \cdots - b_n\delta_n \leq 0. \tag{9.105}$$

The following expression is added to the objective function:

$$c_1\delta_1 + c_2\delta_2 + \cdots + c_n\delta_n. \tag{9.106}$$

The set of variables $(\delta_0, \delta_1, \ldots, \delta_n)$ may be treated as an SOS1 set. If this is done, then the δ_i can be regarded as continuous variables having a generalized upper bound of 1, that is, the integrality requirement may be ignored.

9.4.7 Maximax objectives

Suppose we had the following situation:

$$\begin{array}{ll} \text{Maximize} & \left(\underset{i}{\text{Maximum}} \left(\sum_j a_{ij} x_j \right) \right) \\ \text{subject to} & \text{conventional linear constraints.} \end{array}$$

This is analogous to the *minimax* objective discussed in Section 3.2 but unlike that case, it cannot be modelled by linear programming.

We can, however, treat it as a case of a disjunctive constraint and use integer programming. The model can be expressed as

$$\begin{aligned} &\text{Maximize} && z \\ &\text{subject to} && \sum_j a_{1j}x_j - z = 0 \text{ or } \sum_j a_{2j}x_j - z = 0 \text{ or } \ldots \end{aligned}$$

9.5 Special kinds of integer programming model

The purpose of this section is to describe some well-known special types of IP model that have received considerable theoretical attention. By describing the structure of these types of model, it is hoped that the model builder may be able to recognize when he or she has such a model. Reference to the extensive literature and computational experience on such models may then be of value.

It should, however, be emphasized that most practical IP models do not fall into any of these categories but arise as MIP models often extending an existing LP model. The considerable attention that has been paid to the IP problems described below should not disguise their limited but nevertheless sometimes significant practical importance.

9.5.1 Set covering problems

These problems derive their name from the following type of abstract problem.

We are given a set of objects that we number as the set $S = (1, 2, 3, \ldots, m)$. We are also given a class $\mathcal{S}$ of subsets of S. Each of these subsets has a cost associated with it.

The problem is to 'cover' all members of S at minimum cost using members of $\mathcal{S}$.

For example, suppose $S = (1, 2, 3, 4, 5)$ and $\mathcal{S} = ((1, 2), (1, 3, 5), (2, 4, 5), (3), (1), (4, 5))$ and all members of $\mathcal{S}$ have cost 1.

A cover for S would be $(1, 2)$, $(1,3,5)$ and $(2, 4, 5)$.

In order to obtain the minimum cost cover of S in this example, we could build a 0–1 PIP model. The variables δ_i have the following interpretation:

$$\delta_i = \begin{cases} 1 & \text{if the } i\text{th member of } \mathcal{S} \text{ is in the cover,} \\ 0 & \text{otherwise.} \end{cases}$$

Constraints are introduced to ensure that each member of S is covered. For example, in order to cover the member 1 of S, we must have at least one of the members $(1, 2), (1, 3, 5)$, or (1) of $\mathcal{S}$ in the cover. This condition is imposed by constraint (9.107) below. The other constraints ensure a similar condition for the other members of S.

If the objective is simply to minimize the number of members of S used in the cover, the resultant model is as follows:

$$\begin{aligned}
\text{Minimize}\quad & \delta_1 + \delta_2 + \delta_3 + \delta_4 + \delta_5 + \delta_6, && \\
\text{subject to}\quad & \delta_1 + \delta_2 + \delta_5 \geq 1, && (9.107)\\
& \delta_1 + \delta_3 \geq 1, && (9.108)\\
& \delta_2 + \delta_4 \geq 1, && (9.109)\\
& \delta_3 + \delta_6 \geq 1, && (9.110)\\
& \delta_2 + \delta_3 + \delta_6 \geq 1. && (9.111)
\end{aligned}$$

Alternatively, the variables could be given non-unit cost coefficients in the objective.

This model has a number of important properties:

Property 9.1

The problem is a minimization and all constraints are '$\geq$'.

Property 9.2

All RHS coefficients are 1.

Property 9.3

All other matrix coefficients are 0 or 1.

As a result of the abstract set covering application described above, all 0–1 PIP models with the above three properties are known as *set covering problems*.

Generalizations of the problem exist. If property 9.2 is relaxed and we allow larger positive integers than 1 for some RHS coefficients, we obtain the *weighted set covering problem*. An interpretation of this is that some of the members of S are to be given greater 'weight' in the covering than others, that is, they must be covered a certain number of times.

Another generalization that sometimes occurs is when we relax property 9.2 above but also relax property 9.3 to allow matrix coefficients 0 or ± 1. This gives rise to the *generalized set covering problem*.

The best known application of set covering problems is to aircrew scheduling. Here, the members of S can be regarded as 'legs' that an airline has to cover and the members of $\mathcal{S}$ are possible 'rosters' (or rotations) involving particular combinations of flights. A common requirement is to cover all legs over some period of time using the minimum number of crews. Each crew is assigned to a roster.

A comprehensive list of applications of set covering problems is given by Balas and Padberg (1975).

Special algorithms do exist for set covering problems. Two such algorithms are described in Chapter 4 of Garfinkel and Nemhauser (1972).

Set covering problems have an important property that often makes them comparatively easy to solve by the branch and bound method. It can be shown that the optimal solution to a set covering problem must be a vertex solution in the same sense as for LP problems. Unfortunately, this vertex solution will not generally be (but sometimes is) the optimal vertex solution to the corresponding LP model. It is, however, often possible to move from this continuous optimum to the integer optimum in comparatively few steps.

The difficulty of solving set covering problems usually arises not from their structure but from their size. In practice, models frequently have comparatively few constraints but an enormous number of variables. There is often considerable virtue in generating columns for the model in the course of optimization. Column generation forms the subject of Section 9.6.

9.5.2 Set packing problems

These problems are closely related to set covering problems. The name again derives from an abstract problem that we first describe.

We are given a set of objects which we number as the set $S = (1, 2, 3, \ldots, m)$. We are also given a class $\mathscr{S}$ of subsets of S each of which has a certain value associated with it.

The problem is to 'pack' as many of the members of $\mathscr{S}$ into S as possible to maximize total value but without any overlap.

For example, suppose $S = (1, 2, 3, 4, 5, 6)$ and $\mathscr{S} = ((1, 2, 5), (1, 3), (2, 4), (3, 6), (2, 3, 6))$. A pack for S would be $(1, 2, 5)$ and $(3, 6)$.

Again we could build a $0 - 1$ PIP model to help solve the problem. The variables δ_i have the following interpretation:

$$\delta_i = \begin{cases} 1 & \text{if the } i\text{th member of } \mathscr{S} \text{ is in the pack,} \\ 0 & \text{otherwise.} \end{cases}$$

Constraints are introduced to ensure that no member of S is included in more than one member of $\mathscr{S}$ in the pack, that is, there shall be no overlap. For example, in order that the member 2 shall not be included more than once we cannot have more than one of $(1, 2, 5), (2, 4)$ and $(2, 3, 6)$ in the pack. This condition gives rise to the constraint (9.113) below. The other constraints ensure similar conditions for the other members of S.

The objective here is to maximize the number of members of S we can 'pack in'. For this example the model is

$$\begin{aligned} \text{Maximize} \quad & \delta_1 + \delta_2 + \delta_3 + \delta_4 + \delta_5, \\ \text{subject to} \quad & \delta_1 + \delta_2 \le 1, && (9.112) \\ & \delta_1 + \delta_3 + \delta_5 \le 1, && (9.113) \\ & \delta_2 + \delta_4 + \delta_5 \le 1, && (9.114) \end{aligned}$$

$$\delta_3 \le 1, \quad (9.115)$$

$$\delta_1 \le 1, \quad (9.116)$$

$$\delta_4 + \delta_5 \le 1. \quad (9.117)$$

(Constraints (9.115) and (9.116) are obviously redundant.)

Like the set covering problem, this model has a number of important properties:

Property 9.4

The problem is a maximization and all constraints are '$\le$'.

Property 9.5

All RHS coefficients are 1.

Property 9.6

All other matrix coefficients are 0 or 1.

An interesting observation is that the LP problem associated with a set packing problem with objective coefficients of 1 is the dual of the LP problem associated with a set covering problem with objective coefficients of 1. This result is of little obvious significance as far as the optimal solutions to the IP problems are concerned as there may be a 'duality gap' between these solutions. This term is explained in Section 10.3. Although a set packing problem with objective coefficients of 1 is the 'dual' of a set covering problem with objective coefficients of 1 in this sense, it should be realized that the set S that we wish to cover with members of $\mathscr{S}$ is not the same as the set S that we wish to pack with members of another class of subsets $\mathscr{S}$. The two examples used above to illustrate set covering and set packing problems are so chosen as to be dual problems in this sense, but the set S and class of subsets $\mathscr{S}$ are different.

As with the set covering problem there are generalizations of the set packing problem obtained by relaxing some of the properties 9.4–9.6 above. If property 9.5 is relaxed and we allow positive integral RHS coefficients greater than 1, we obtain the *weighted set packing problem*. If we relax property 9.5 as above together with property 9.6 to allow matrix coefficients of 0 or ±1, we obtain the *generalized set packing problem*.

A special sort of packing problem is known as the *matching problem*. This problem can be represented by a graph in which the objects S to be packed are *nodes*. Each subset from S consists of two members of S and is represented by an *arc* of the graph whose end points represent the two members of S. The problem then becomes one of matching (or pairing) as many vertices as possible together. Extensive work on this problem has been done by Edmonds (1965).

Set packing problems are equivalent to the class of set partitioning problems, which we consider next. Applications and references are described there.

9.5.3 Set partitioning problems

In this case, we are, as before, given a set of objects that we number as the set $S = (1,2,3, \dots, m)$ and a class $\mathcal{S}$ of subsets of S.

The problem this time is both to cover all the members of S using members of $\mathcal{S}$ and also to have no overlap. We, in this sense, have a combined covering and packing problem. There is no useful purpose in distinguishing between maximization and minimization problems here. Both may arise.

As an example, we consider the same S and $\mathcal{S}$ used in the example to describe the set covering problem. The difference is that we must now impose stricter conditions to ensure that each member of S is in exactly one member of the partition (or cover and pack combined). This is done by making the constraints (9.107)–(9.111)'=' instead of '≥', giving

$$\begin{aligned}
\delta_1 + \delta_2 + + \delta_5 &= 1, && (9.118)\\
\delta_1 + \delta_3 &= 1, && (9.119)\\
\delta_2 + \delta_4 &= 1, && (9.120)\\
\delta_3 + \delta_6 &= 1, && (9.121)\\
\delta_2 + \delta_3 + \delta_6 &= 1. && (9.122)
\end{aligned}$$

For example, a feasible partition of S consists of $(1,2), (3)$ and $(4,5)$.

An easily understood way in which the set partitioning problem can arise is again in aircrew scheduling. Suppose that we did not allow aircrews to travel as passengers on other flights. It would then be necessary that each member of S (a leg) be covered by exactly one member of $\mathcal{S}$ (a roster). We would then have a set partitioning problem in place of the set covering problem.

Another application of the set partitioning problem is to *political districting*. In this problem, there is usually an extra constraint fixing the total number of political districts (or constituencies). The objective is usually to *minimize the maximum deviation* of the electorate in a district from the average electorate in a district. This *minimax* objective can be dealt with in the way described in Section 3.3. A reference to tackling the problem by IP is Garfinkel and Nemhauser (1970).

We now show the essential equivalence between set partitioning and set packing problems. If we introduce slack variables into each of the '≤' constraints of a set packing problem, we obtain '=' constraints. These slack variables can only take the values 0 or 1 and could therefore be regarded with all the other variables in the problems as $0-1$ variables. The result would clearly be a set partitioning problem.

To show the converse is slightly more complicated. Suppose we have a set partitioning problem. Each constraint will be of the form

$$a_1\delta_1 + a_2\delta_2 + \cdots + a_n\delta_n = 1. \qquad (9.123)$$

We introduce an extra $0-1$ variable δ into constraint (9.123) giving

$$a_1\delta_1 + a_2\delta_2 + \cdots + a_n\delta_n + \delta = 1. \qquad (9.124)$$

If the problem is one of minimization, δ can be given a sufficiently high positive cost M in the objective to force δ to be zero in the optimal solution. For a maximization problem, making M a negative number with a sufficiently high absolute value has the same effect. Instead of introducing S explicitly into the objective function, we can substitute the expression below derived from Equation (9.124):

$$1 - a_1\delta_1 - a_2\delta_2 - \cdots - a_n\delta_n. \tag{9.125}$$

There is no loss of generality in replacing Equation (9.124) by the constraint

$$a_1\delta_1 + a_2\delta_2 + \cdots + a_n\delta_n \leq 1. \tag{9.126}$$

If similar transformations are performed for all the other constraints, the set partitioning problem can be transformed into a set packing problem.

Hence we obtain the result that set packing problems and set partitioning problems can easily be transformed from one to the other. Both types of problem possess the property mentioned in connection with the set covering problems, that is, the optimal solution is always a vertex solution to the corresponding LP problem.

The set partitioning (or packing) problem is generally even easier to solve than the set covering problem. It is easy to see from the small numerical example that a set partitioning problem by using '$=$' constraints is more constrained than the corresponding set covering problem. Likewise, the corresponding LP problem is more constrained. It will be seen in Section 10.1 that constraining the corresponding LP problem to an IP problem as much as possible is computationally very desirable.

A comprehensive list of applications and references to the set packing and partitioning problems is given by Balas and Padberg (1975). Chapter 4 of Garfinkel and Nemhauser (1972) treats these problems very fully and gives a special purpose algorithm. Ryan (1992) describes how the set partitioning problem can be applied to aircrew scheduling.

In view of its close similarity to set partitioning and packing problems it is rather surprising that the set covering problem has some important differences. Although the set covering problem is an essentially easy type of problem, to solve it is distinctly more difficult than the former problems. It is possible to transform a set partitioning (or packing) problem into a set covering problem by introducing an extra $0-1$ variable with a negative coefficient in Equation (9.123) and performing analogous substitutions to those described above. The reverse transformation of a set covering problem to a set partitioning (or packing) problem is not, however, generally possible, revealing the distinctness of the problems.

9.5.4 The knapsack problem

A PIP model with a single constraint is known as a *knapsack problem*. Such a problem can take the form

$$\begin{aligned} \text{Maximize} \quad & p_1\gamma_1 + p_2\gamma_2 + \cdots + p_n\gamma_n \\ \text{subject to} \quad & a_1\gamma_1 + a_2\gamma_2 + \cdots + a_n\gamma_n \leq b, \end{aligned} \tag{9.127}$$

where $\gamma_1, \gamma_2, \ldots, \gamma_n \geq 0$ and take integer values.

The name 'knapsack' arises from the rather contrived application of a hiker trying to fill her knapsack to maximum total value. Each item she considers taking with her has a certain value and a certain weight. An overall weight limitation gives the single constraint.

An obvious extension of the problem is where there are additional upper bounding constraints on the variables. Most commonly, these upper bounds will all be 1 giving a $0-1$ *knapsack problem*.

Practical applications occasionally arise in *project selection* and *capital budgeting allocation* problems if there is only one constraint. The problem of *stocking a warehouse* to maximum value given that the goods stored come in indivisible units also gives rise to an obvious knapsack problem.

The occurrence of such immediately obvious applications is, however, fairly rare. Much more commonly, the knapsack problem arises in LP and IP problems where one generates the columns of the model in the course of optimization. As such techniques are intimately linked with the algorithmic side of mathematical programming, they are beyond the scope of this book. The original application of the knapsack problem in this way was to the *cutting stock problem*. This is described by Gilmore and Gomory (1963, 1965) (and in Section 9.6).

In practice, knapsack problems are comparatively easy to solve. The branch and bound method is not, however, an efficient method to use. If it is desired to solve a large number of knapsack problems (e.g. when generating columns for an LP or IP model), it is better to use a more efficient method. Dynamic programming proves an efficient method of solving knapsack problems. This method and further references are given in Chapter 6 of Garfinkel and Nemhauser (1972).

9.5.5 The travelling salesman problem

This problem has received a lot of attention largely because of its conceptual simplicity. The simplicity with which the problem can be stated is in marked contrast to the difficulty of solving such problems in practice.

The name 'travelling salesman' arises from the following application.

A salesman has to set out from home to visit a number of customers before finally returning home. The problem is to find the order in which he or she should visit all the customers in order to minimize the total distance covered.

Generalizations and special cases of the problem have been considered. For example, sometimes it is not necessary that the salesman return home. Often, the problem itself has special properties, for example, the distance from A to B is the same as the distance from B to A. The *vehicle routing problem* can be reduced to a travelling salesman problem. This is the problem of organizing deliveries to customers using a number of vehicles of different sizes. The reverse problem of picking up deliveries (e.g. letters from post office boxes) given a number of vehicles (or postal workers) can also be treated in this way. This problem is considered below.

Job sequencing problems in order to minimize set-up costs can be treated as travelling salesman problems where the 'distance between cities' represents the 'set-up cost between operations'. A not very obvious (nor probably practical) application of this sort is to sequencing the colours that are used for a single brush in a series of painting operations. Obviously, the transition between certain colours will require a more thorough cleaning of the brush than a transition between other colours. The problem of sequencing the colours to minimize the total cleaning time gives rise to a travelling salesman problem.

The travelling salesman problem can be formulated as an IP model in a number of ways. We present one such formulation here. Any solution (not necessarily optimal) to the problem is referred to as a 'tour'.

Suppose the cities to be visited are numbered $0, 1, 2, \ldots, n$. 0–1 integer variables δ_{ij} are introduced with the following interpretation:

$$\delta_{ij} = \begin{cases} 1 & \text{if the tour goes from } i \text{ to } j \text{ direct,} \\ 0 & \text{otherwise.} \end{cases}$$

The objective is simply to minimize $\sum_{i,j} c_{ij}\delta_{ij}$, where c_{ij} is the distance (or cost) between i and j.

There are two obvious conditions that must be fulfilled:

Exactly one city must be visited immediately after city i. (9.128)

Exactly one city must be visited immediately before city j. (9.129)

Condition (9.128) is achieved by the constraints

$$\sum_{\substack{j=0 \\ i\neq j}}^{n} \delta_{ij} = 1, \qquad i = 0, 1, \ldots, n. \tag{9.130}$$

Condition (9.129) is achieved by the constraints

$$\sum_{\substack{i=0 \\ i\neq j}}^{n} \delta_{ij} = 1, \qquad j = 0, 1, \ldots, n. \tag{9.131}$$

As the model has so far been presented, we have an assignment problem as described in Section 5.3. Unfortunately, the constraints (9.130) and (9.131) are not sufficient. Suppose, for example, we were considering a problem with eight cities ($n = 7$). The solution drawn in Figure 9.6 would satisfy all the conditions (9.130) and (9.131).

Clearly, we cannot allow subtours such as those shown here.

The problem of adding extra constraints so as to avoid subtours proves quite difficult. In practice, it is often desirable to add these constraints in the course of optimization as subtours arise. For example, as a result of the solution to the 'relaxed problem' exhibited by Figure 9.6, we would add the extra constraint

$$x_{46} + x_{64} + x_{47} + x_{74} + x_{67} + x_{76} \leq 2. \tag{9.132}$$

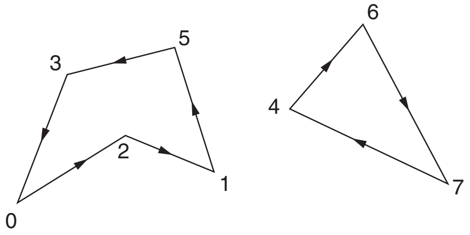


Figure 9.6

This would rule out any subtour around the cities 4, 6 and 7 (and implicitly around 0, 1, 2, 3 and 5 if these were the only other cities). Potentially, there are an exponential number of such subtour elimination constraints, and it would be impractical to state them all explicitly. Therefore, they are added on an 'as needed' basis as above. In practice, one rarely needs to add more than a very small fraction of this exponential number before a solution without subtours is obtained. When this happens, we clearly have the optimal solution.

There are ingenious ways of producing non-exponential sized formulations by the introduction of new variables or replacing the variables by new ones with a different interpretation.

However, all except one of these formulations have weaker LP relaxations than the conventional formulation, which is due to Dantzig *et al*. (1954). Orman and Williams (2006) give eight different formulations and show (surprisingly) that one can rank the strengths of their LP relaxations, independently of problem instance. They classify the non-exponential formulations into three distinct forms. A sequential formulation is due to Miller *et al*. (1960) who introduce extra variables representing the sequence numbers in which cities are visited and use these variables in constraints that prevent subtours. Gavish and Graves (1978) introduce flow variables, which are only allowed in arcs that are used in the tour. These, together with the material balance constraints (Section 5.3) of the network flow problem, prevent subtours. There are a number of variants of this formulation, for example, Finke *et al*. (1983). In particular, Wong (1980) and Claus (1984) give a multi-commodity network flow formulation, which, remarkably, has the same strength LP relaxation as the conventional formulation. However it has, of the order of n^3 variables and constraints, which still makes it impractical to solve the full model in one optimization. Finally time-staged formulations are given by Vajda (1961) and Fox *et al*. (1980) who use a third index t so as to let the new variables δ_{ijt} represent whether the salesman goes from city i to city j at stage ('time') t .These variables can then be incorporated in constraints to rule out subtours.

A comprehensive survey of the travelling salesman problem is Lawler *et al*. (1995). There have been dramatic advances in solving the problem fairly recently. Applegate *et al*. (2006) discuss the computational side. Special algorithms have been developed. One of the most successful is that of Held and Karp (1971).

There are also variants and extensions of the travelling salesman problem. One such is the MILK COLLECTION problem described in Part II. This is based on a paper by Butler *et al*. (1997). Another is the LOST BAGGAGE DISTRIBUTION problem also described in Part II.

9.5.6 The vehicle routing problem

This is, practically, the most important extension of the travelling salesman problem where one has to schedule a number of vehicles, of limited capacity, around a number of customers. Hence, in addition to the sequencing of the customers to be visited (the travelling salesman problem) one has to decide which vehicles visit which customers. Each customer has a known demand (assumed to be of one commodity, although it is possible, easily to extend to more than one commodity). Hence the limited vehicle capacities have to be taken into account. In addition, there are often *time window* conditions requiring that certain customers can only receive deliveries between certain times. We give a formulation of this problem as an extension of that of the travelling salesman problem mentioned above. As with that formulation, it is generally necessary to add extra constraints to break subtours in the course of optimization.

Let $\delta_{ijk} = 1$ iff vehicle k goes directly from customer i to customer j,
$\gamma_{ik} = 1$ iff vehicle k visits customer i, (apart from the depot),
$\tau_i =$ time at which customer i is visited.

We assume that each customer (apart from the depot, regarded as customer 0) is visited by exactly one vehicle, that is, there are no 'split' deliveries to a customer. For data, we assume there are m vehicles and that vehicle k has capacity Q_k and customer i has requirement q_i. Also we assume that the time taken between customers i and j is d_{ij} and that customer i must take delivery between times a_i and b_i.

There are a number of possible objectives, for example, minimize the number of vehicles needed, minimize total cost or minimize maximum time taken by a vehicle. For illustration, we consider the second objective and assume that the cost (not necessarly proportional to the time) of going from i to j directly is c_{ij}.

The model is

Minimize $\sum_{ijk} c_{ij}\delta_{ijk}$,

subject to $\sum_j \delta_{ijk} = \sum_j \delta_{jik} = \gamma_{ik}$, if a vehicle k visits customer i, it is entered once and left once by k.

$\sum_k \gamma_{ik} = 1$, if customer i (not 0) is visited by exactly one vehicle.

$\sum_i \gamma_{ik} \leq \gamma_{1k}$, if vehicle k is used, then it must visit town 1.

$\sum_i q_i\gamma_{ik} \leq Q_k$, the combined requirements of customers visited by vehicle k must lie within the capacity of k.

$\tau_i - \tau_j \leq M(1 - \delta_{ijk}) - d_{ij}$, if customer j is visited immediately after customer i, then this must be at a time at least d_{ij} greater than the time at which customer i is visited.

$a_i \leq \tau_i \leq b$, each customer must be visited within its time window.

Such a formulation could prove very difficult to solve for reasonable-sized problem instances. Another approach is to use *column generation*, which is discussed in Section 9.6. (see also Desrosiers *et al*. (1984)). In this approach, feasible *routes* for a vehicle, that is, obeying the capacity and time window constraints, are generated separately from the main model, which then selects and combines them in a feasible and optimal manner.

Variants of this problem are the MILK COLLECTION and LOST BAGGAGE DISTRIBUTION problems given in Part II.

9.5.7 The quadratic assignment problem

This is a generalization of the assignment problem described in Section 5.3. Whereas that problem could be treated as an LP problem and was therefore comparatively easy to solve, the quadratic assignment problem is a genuine IP problem and often very difficult to solve.

As with the assignment problem, we consider the problem as related to two sets of objects S and T. S and T both have the same number of members, which are be indexed 1 to n. The problem is to assign each member of S to exactly one member of T in order to achieve some objective. There are two sorts of conditions we must fulfil:

Each member of S must be assigned to exactly one member of T. (9.133)

Each member of T must have exactly one member of S assigned to it. (9.134)

$0-1$ variables δ_{ij} can be introduced with the following interpretations:

$$\delta_{ij} = \begin{cases} 1 & \text{if } i \text{ (a member of } S\text{)is assigned to } j \text{ (a member of } T\text{)}, \\ 0 & \text{otherwise.} \end{cases}$$

Conditions (9.133) and (9.134) are imposed by the following two types of constraint:

$$\sum_{j=1}^{n} \delta_{ij} = 1, \quad i = 1, 2, \ldots, n. \tag{9.135}$$

$$\sum_{i=1}^{n} \delta_{ij} = 1, \quad j = 1, 2, \ldots, n. \tag{9.136}$$

The objective is more complex than with the assignment problem. We have cost coefficients c_{ijkl}, which have the following interpretations. c_{ijkl} is the cost incurred by assigning i (a member of S) to j (a member of T) at the same time as assigning k (a member of S) to l (a member of T). This cost will clearly be incurred only if $\delta_{ij} = 1$ and $\delta_{kl} = 1$, that is, if the product $\delta_{ij}\delta_{kl} = 1$. The objective then becomes a quadratic expression in $0-1$ variables:

$$\text{Minimize} \quad \sum_{\substack{i,j,k,l=1 \\ k>i}}^{n} c_{ijkl}\delta_{ij}\delta_{kl}. \tag{9.137}$$

The condition $k > i$ on the indices prevents the cost of each pair of assignments being counted twice. It is very common for the coefficients c_{ijkl} to be derived from the product of other coefficients t_{ik} and d_{jl} so that

$$c_{ijkl} = t_{ik}d_{ji}. \tag{9.138}$$

In order to understand this rather complicated model, it is worth considering two applications.

Firstly, we consider S to be a set of n factories and T to be a set of n cities. The problem is to locate one factory in each city and to minimize total communication costs between factories. The communication costs depend on (i) the frequency of communication between each pair of factories and (ii) the distances between the two cities where each pair of factories is located.

Clearly, some factories will have little to do with each other and can be located far apart at little cost. On the other hand, some factories may need to communicate a lot. The cost of communication will depend on the distance apart. In this application, we can interpret the coefficients t_{ik} and d_{jl} in Equation (9.138) as follows: t_{ik} is the frequency of communication between factories i and k (measured in appropriate units); d_{jl} is the cost per unit of communication between cities j and l (clearly, this will be related to the distance between j and l). Obviously, the cost of communication between the factories i and k, located in cities j and l, will be given by Equation (9.138). The total cost is therefore represented by the objective function (9.137).

Another application concerns the placement of n electronic modules in n predetermined positions on a backplate. S represents the set of modules and T represents the set of positions on the backplate. The modules have to be connected to each other by a series of wires. If the objective is to minimize the total length of wire used, we have a quadratic assignment problem similar to the above model. The coefficients t_{ik} and d_{jl} have the following interpretations: t_{ik} is the number of wires that must connect module i to the module k, and d_{jl} is the distance between position j and position l on the backplate. Many modifications of the quadratic assignment problem described above exist. Frequently conditions (9.133) and (9.134) are relaxed, to allow more than one member of S (or possibly none) to be assigned to a member of T, where S and T may not have the same number of members. A modified problem of this sort is the DECENTRALIZATION example in Part II.

One way of tackling the quadratic assignment problem is to reduce the problem to a linear $0-1$ PIP problem. It is necessary to remove the quadratic terms in the objective function. Two ways of transforming products of $0-1$ variables into linear expressions in an IP model were described in Section 9.2. For fairly small models, the first transformation is possible but for larger models, the result can be an enormous expansion in the number of variables and constraints in the model. The solution of a practical problem of this sort by using a similar type of transformation is described by Beale and Tomlin (1972).

A survey of practical applications, as well as other methods of tackling the problem, is given by Lawler (1974). He also shows how a different formulation of the travelling salesman problem, to that given here, leads to a quadratic assignment problem. It should be pointed out, however, that although a travelling salesman problem is often difficult to solve, a quadratic assignment problem of comparable dimensions is usually even harder.

9.6 Column generation

The generation of columns in the course of optimization has already been referred to in the context of *decomposition*, in Section 4.3 and the *set covering* and *vehicle routing* problems in Section 9.5. All the relevant models there are characterized by a very large (often astronomic) number of variables, only a small proportion of which will feature in the optimal solution. Therefore, it seems more efficient only to append them to the model, in the course of optimization, if it seems likely they will be in the optimal solution. Although the method of generation will depend on the nature of the model and the optimization method used, it is still appropriate to consider this general approach in a book on model building. In principle, but not in practice, all the possible variables could be created at the beginning. Column generation is frequently applied to special types of integer programming models. Hence its inclusion in this chapter.

Possible solutions, or components of solutions, are generated separately from the main (master) model (sometimes by optimizations and sometimes by heuristics) and appended to it in the course of optimization. In the case of decomposition, these components are the solutions of subproblems. In the case of the set covering problem, applied to aircrew scheduling, they might be possible rosters generated as paths through a space–time diagram of an airline schedule. For the vehicle routing problem, they would be routes covered by individual vehicles (respecting capacity constaints and time windows).

We illustrate column generation by means of the *cutting stock* problem, which was one of the original applications decribed by Gilmore and Gomory (1961 and 1963). This can be applied to cutting ordered widths of wallpaper from standard widths to minimize waste or ordered lengths of pipe from standard lengths in order to minimize the number of standard lengths that need to be cut. The cutting stock problem can also be regarded as a special case of the *bin packing* problem where it is desired to pack a number of items of different sizes into standard size bins so as to minimize the number of bins used.

Suppose a plumber stocks standard lengths of pipe, all of length 19 m. He has orders for

12 lengths of 4 m
15 lengths of 5 m
22 lengths of 6 m

How should he cut these lengths out of his standard lengths so as to minimize the number of standard lengths used?

We firstly give a *bad* formulation of this problem, as an integer programme, in order to illustrate the advantage of one based on column generation.

Let x_{ij} = number of pipes of length i (4, 5 or 6 m) cut from standard pipe j numbering the standard, stocked pipes $1, 2, 3, \ldots$ etc.

$$\delta_j = 1 \text{ if and only if standard pipe } j \text{ used}$$

The constraints are

$x_{1j} \le 4\delta_j, x_{2j} \le 3\delta_j, x_{3j} \le 3\delta_j$ (standard pipe j only used if it appears in an order),

$4x_{1j} + 5x_{2j} + 6x_{3j} \le 19$ for $j = 1, 2, 3, \ldots$ (orders must be within length of each standard pipe used),

$$\sum_j x_{1j} \geq 12, \sum_j x_{2j} \geq 15, \sum_j x_{3j} \geq 22 \text{ (orders must be met)}.$$

The objective is

$$\text{Minimize} \quad \sum_j \delta_j.$$

A major defect of this model is that there will be many symmetrically equivalent solutions, as the standard pipes are indistinguable. Even if symmetry breaking constraints (Section 10.2) are appended, this model would take a very long time to solve (the reader might like to try solving this tiny example using a computer package). We therefore present a much better model based on column generation.

The main constraint above is the second (knapsack) one. The solutions to this constraint are the possible patterns that can be made up out of the standard lengths. We give some of the possible patterns below (including the waste, W) There are obviously more patterns and in a larger example, there will be an astronomic number, most of which will not be used.

Pattern 1

4	4	5	5	W

Pattern 2

4	4	5	6

Pattern 3

4	5	5	5

A *better* formulation, using column generation, which finds the minimum number of such patterns needed is

$$\gamma_j = \text{number of times pattern } j \text{ is used.}$$

$$\begin{array}{lrl} \text{Minimize} & \gamma_1 + \gamma_2 + \gamma_3, & \\ \text{subject to} & 2\gamma_1 + 2\gamma_2 + \gamma_3 & \geq 12, \\ & 2\gamma_1 + \gamma_2 + 3\gamma_3 & \geq 15, \\ & \gamma_2 & \geq 22. \end{array}$$

Note that each variable has a column of constraint coefficients corresponding to the number of times each of the order lengths occurs in the pattern. The RHS coefficients are the demands for each length. Solving this model (as an IP) yields $\gamma_1 = 0, \gamma_2 = 22, \gamma_3 = 0$, requiring 22 standard pipes. Clearly there are better solutions if other patterns are used as well. It is possible to devise other 'useful' patterns using the *shadow prices* from the optimal LP solution to the current model. To start with these shadow prices turn out to be $0, 0, 1$. If we were to have a pattern with a lengths of 4 m, b lengths of 5 m and c lengths of 6 m its *reduced cost* ('value' in reducing the objective) would be $c - 1$. We therefore seek a pattern that maximizes this quantity subject to fitting the pattern into the standard length, that is, we wish to

$$\begin{array}{ll} \text{Maximize} & c - 1 \\ \text{subject to} & 4a + 5b + 6c \leq 19 \end{array}$$

Clearly a, b, c are integer variables, giving a *knapsack problem*. This instance has the optimal solution $a = b = 0, c = 3$, with objective 2, indicating that this pattern will reduce the number of pipes used (in the LP relaxation) giving

Pattern 4

6	6	6	W

Hence we have *generated a new column* that can be appended to the model to give

$$\begin{array}{lrcl} \text{Minimize} & \gamma_1 + \gamma_2 + \gamma_3 + \gamma_4 & & \\ \text{subject to} & 2\gamma_1 + 2\gamma_2 + \gamma_3 & \geq & 12, \\ & 2\gamma_1 + \gamma_2 + 3\gamma_3 & \geq & 15, \\ & \gamma_2 + 3\gamma_4 & \geq & 22. \end{array}$$

This model has optimal IP solution $\gamma_1 = 0, \gamma_2 = 4, \gamma_3 = 4, \gamma_4 = 6$. requiring 14 standard pipes. The shadow prices are $1/3, 0, 1/3$ which, in order to find another pattern with maximum reduced cost, leads to

$$\begin{array}{ll} \text{Maximize} & (1/3)\,a + (1/3)\,c - 1, \\ \text{subject to} & 4a + 5b + 6c \leq 19. \end{array}$$

This has the optimal solution $a = 4, b = 0, c = 0$ with objective value 1/3, giving

Pattern 5

4	4	4	4	W

Appending the column corresponding to this pattern gives

$$\begin{aligned}
\text{Minimize} \quad & \gamma_1 + \gamma_2 + \gamma_3 + \gamma_4 + \gamma_5, \\
\text{subject to} \quad & 2\gamma_1 + 2\gamma_2 + \gamma_3 + 4\gamma_5 \geq 12, \\
& 2\gamma_1 + \gamma_2 + 3\gamma_3 \geq 15, \\
& \gamma_2 + 3\gamma_4 \geq 22.
\end{aligned}$$

This model gives the same solution as before. In fact, this is the overall optimal solution which has, therefore, been obtained without considering many of the potential patterns.

10

Building integer programming models II

10.1 Good and bad formulations

Most of the considerations of Section 3.4 concerning linear programming (LP) models will also apply to integer programming (IP) models and will not be reconsidered here. There are, however, some important additional considerations that must be taken into account when building IP models. The primary additional consideration is the much greater computational difficulty of solving IP models over LP models. It is fairly common to build an IP model only to find the cost of solving it prohibitive. Frequently, it is possible to reformulate the problem, giving another model that is easier to solve. Such reformulations must often be considered in conjunction with the solution strategy to be employed. It will be assumed throughout that the branch and bound method described in Section 8.3 is to be used.

In some respects, much greater flexibility is possible in building IP models than in building LP models. The flexibility results in a greater divergence between good and bad models of a practical problem. The purpose of this chapter is to suggest ways in which good models may be constructed.

It is convenient to consider variables and constraints separately. There is often the possibility of using many or few variables and many or few constraints in a model. The considerations governing this will be considered.

10.1.1 The number of variables in an IP model

We confine our attention here to the number of integer variables in an IP model as this is often regarded as a good indicator of the computational difficulty.

Suppose we had a 0–1 IP model (either mixed integer or pure integer). If the model had n 0–1 variables, this would indicate 2^n possible settings for the variables and hence 2^n potential nodes hanging at the bottom of the solution tree. In total there would be $2^{n+1} - 1$ nodes in such a tree. One might therefore expect the solution time to go up exponentially with the number of 0–1 variables. For quite modest values of n, 2^n is very large, for example, 2^{100} is greater than one million raised to the power 5. The situation is not of course anywhere near as bad as this, as many of the 2^n potential nodes will never be examined. The branch and bound method rules out large sections of the potential tree from examination as being infeasible or worse than solutions already known. It is, however, worth pausing to consider the fact that one may sometimes solve 100 0–1 variable IP problems in a few hundred nodes. This represents only about 0.00 ... 001 per cent of the potential total where there are 28 zeros after the decimal point. In view of this very surprising efficiency that the branch and bound method exhibits over the potential amount of computation, the number of 0–1 variables is often a very poor indicator of the difficulty of an IP model. We, however, suggest one circumstance in which the number of such variables might be usefully reduced, later in this section. Before doing that, we indicate ways in which the number of integer variables in a model might be increased to good effect.

It is convenient here to describe a well-known device for expanding any general integer variable in a model to a number of 0–1 variables. Suppose γ is a general (non-negative) integer variable with a known upper bound of u (an upper bound is required for all integer variables in an IP model if the branch and bound method is to be used), that is,

$$0 \leq \gamma \leq u.$$

γ may be replaced in this model by the expression

$$\delta_0 + 2\delta_1 + 4\delta_2 + 8\delta_3 + \cdots + 2^r\delta_r, \tag{10.1}$$

where the δ_i are 0–1 variables and 2^r is the smallest power of 2 greater than or equal to u.

It is easy to see that the expression (10.1) can take any possible integral value between 0 and u by different combinations of values for the δ_i variables. Clearly, the number of 0–1 variables required in an expansion like this is roughly $\log_2 u$. In practice, u will probably be fairly small and the number of 0–1 variables produced not too large. If, however, this device were employed on a large number of the variables in the model, the result might be a great expansion in model size. Generally, there will be little virtue in an expansion of this sort except to facilitate the use of some specialized algorithm applying only to 0–1 problems. This is beyond the scope of this book.

Although there is some virtue in keeping an LP model compact, any such advantages that this may imply for the corresponding IP model are usually drowned by other much more important considerations. Using the branch and bound method there is sometimes virtue in introducing extra 0–1 variables as useful variables in the branching process. Such 0–1 variables represent ‘dichotomies’ in the system being modelled. To make such dichotomies explicit can be valuable, as is demonstrated in the following example due to Jeffreys (1974).

Example 10.1

One new factory is to be built. The possible decisions are represented by 0–1 variables $\delta_{\mathrm{n,b}}$, $\delta_{\mathrm{n,c}}$, $\delta_{\mathrm{s,b}}$ and $\delta_{\mathrm{s,c}}$:

$$\delta_{\mathrm{n,b}} = \begin{cases} 1 & \text{if the factory is in the north and uses a batch process,} \\ 0 & \text{otherwise,} \end{cases}$$

$$\delta_{\mathrm{n,c}} = \begin{cases} 1 & \text{if the factory is in the north and uses a continuous process,} \\ 0 & \text{otherwise,} \end{cases}$$

$$\delta_{\mathrm{s,b}} = \begin{cases} 1 & \text{if the factory is in the south and uses a batch process,} \\ 0 & \text{otherwise,} \end{cases}$$

$$\delta_{\mathrm{s,c}} = \begin{cases} 1 & \text{if the factory is in the south and uses a continuous process,} \\ 0 & \text{otherwise,} \end{cases}$$

The condition that only one factory be built can be represented by the constraint

$$\delta_{\mathrm{n,b}} + \delta_{\mathrm{n,c}} + \delta_{\mathrm{s,b}} + \delta_{\mathrm{s,c}} = 1. \tag{10.2}$$

It is not possible, using constraint (10.2), to express the dichotomy ‘either we site the factory in the north or we site it in the south’ by a single 0–1 variable. As this is clearly an important decision, it would be advantageous to have a 0–1 variable indicating the decision. By adding an extra 0–1 variable δ to represent this decision, together with the extra constraints

$$\delta_{\mathrm{n,b}} + \delta_{\mathrm{n,c}} - \delta = 0, \tag{10.3}$$

$$\delta_{\mathrm{s,b}} + \delta_{\mathrm{s,c}} + \delta = 1, \tag{10.4}$$

this is possible. δ is a valuable variable to have at our disposal as use can be made of it as a variable to branch on in the tree search. The dichotomy ‘either we use a batch process or we use a continuous process’ could also be represented by another 0–1 decision variable in a similar way.

Another use of extra integer variables in a model is to specify the slack variable in a constraint, made up of only integer variables, as itself being integer. For example, if all the variables, coefficients and right-hand side are integers in the constraint

$$\sum_j a_j x_j \leq b, \tag{10.5}$$

we can put in a slack variable u and specify this variable to be an integer, giving

$$\sum_j a_j x_j + u = b. \tag{10.6}$$

Normally, such a slack variable would be inserted by the mathematical programming package used but treated only as a continuous variable. There is advantage in treating u as an integer variable and giving it priority in the branching process. When u is the variable branched on, constraint (10.6) will have the effect of a cutting plane and restrict the feasible region of the corresponding LP problem. This idea is due to Mitra (1973).

To summarize, there is often advantage in increasing rather than decreasing the number of integer variables in a model especially if these extra variables are made use of in the tree search strategy. Such ideas can be used to advantage in some of the IP problems given in Part II.

In some circumstances, however, there is advantage to be gained in reducing the number of integer variables. One such case is illustrated in the following example. Here the problem exhibits a symmetry, which can be computationally undesirable.

Example 10.2

As part of a larger IP model, the following variables are introduced.

$$\delta_{ij} = \begin{cases} 1 & \text{if lorry } i \text{ is sent on trip } j \\ 0 & \text{otherwise,} \end{cases}$$

where $i = \{1, 2, 3\}, j = \{1, 2\}$. The lorries are indistinguishable in terms of running costs, capacities, etc.

Clearly, corresponding to each possible integer solution, for example,

$$\delta_{11} = 1, \delta_{22} = 1, \delta_{32} = 1, \tag{10.7}$$

there will be symmetric integer solutions, for example,

$$\delta_{12} = 1, \delta_{21} = 1, \delta_{32} = 1. \tag{10.8}$$

As the branch and bound tree search progresses, each symmetric solution may be obtained at a separate node.

A better formulation involving fewer integer variables and avoiding the symmetry could be devised using the following integer variables:

$$n_i = \text{number of lorries sent on trip } j.$$

Solutions (10.7) and (10.8) would now be indistinguishable as the solution

$$n_1 = 1, \quad n_2 = 2. \tag{10.9}$$

10.1.2 The number of constraints in an IP model

It was pointed out in Chapter 3 that the difficulty of an LP model is very dependent on the number of constraints. Here we show that in an IP model this effect is often completely drowned by other considerations. In fact, an IP model is often made easier to solve by expanding the number of constraints.

In an LP model, we search for vertex solutions on the boundary of the feasible region. For the corresponding IP model, we are interested in integer points that may well lie in the interior of the feasible region. This is illustrated in Figure 10.1. ABCD is the feasible region of the LP problem but we must confine our attention to the lattice of integer points. For an IP model, the corresponding LP model is known as the LP *relaxation*.

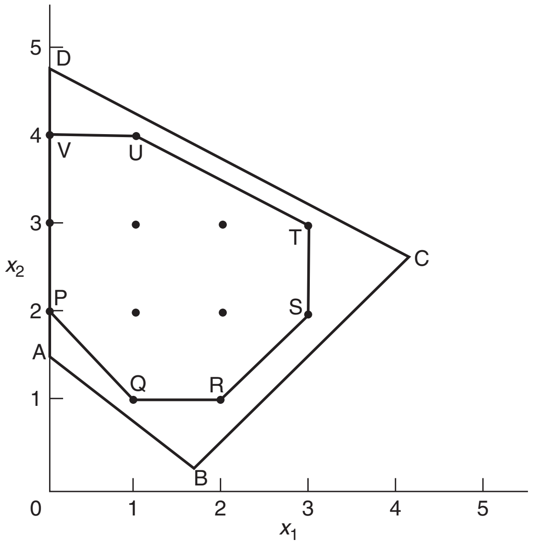


Figure 10.1

In this diagram, we assume both x_1 and x_2 to be integer variables. For mixed integer problems, we are interested in points in an diagram analogous to Figure 10.1, some of whose coordinates are integers, but where other coordinates are allowed to be continuous. Clearly, this situation is difficult to picture in a few dimensions. Suppose, however, that there were a further continuous variable x_3 to be considered in Figure 10.1. This would give rise to a coordinate coming out at right angles to the page. Feasible solutions to the mixed integer problem would consist of lines parallel to the x_3-axis coming out of the page from the integer points in Figure 10.1. These lines would have, of course, to lie inside the three-dimensional feasible region of the corresponding LP problem.

Ideally, we would like to reformulate the IP model so that the feasible region of the corresponding LP model becomes PQRSTUV. This is known as the *convex hull of feasible integer points* in ABCD. It is the smallest convex set containing all the feasible integer points. If it were possible to reformulate the IP problem in this way, we could solve the problem as an LP problem as the integrality requirement would automatically be satisfied. Such a formulation, where the LP relaxation gives the convex hull of IP solutions, is known as *sharp*.

Each vertex (and hence optimal solution) of the new feasible region PQRSTUV is an integer point. Unfortunately, in many practical problems, the effort required to obtain the convex hull of integer points would be enormous and far outweigh the computation needed to solve the original formulation of the IP problem. There are, however, important classes of problems where one of the following can happen:

(i) The straightforward formulation results in an IP model where the feasible region is already the convex hull of integer points.

(ii) The problem can fairly easily be reformulated to give a feasible region corresponding to the convex hull of integer points.

(iii) By reformulating, it is possible to reduce the feasible region of the LP problem to nearer that of the convex hull of integer points.

We consider each of these classes of problems in turn.

Case (i) concerns some problems that have already been considered in Section 5.3. Although superficially these problems might appear to give rise to PIP (pure integer programming) models, the optimal solution of the corresponding LP problem always results in integer values for the integer variables. The model may therefore be treated as an LP problem. Problems falling into this category include the *transportation problem*, the *minimum cost network flow problem* and the *assignment problem*. It is sometimes possible to recognize when an IP model has a structure that guarantees that the corresponding LP model will have an integer optimal solution. Clearly, it is very useful to be able to recognize this property as the high computational cost of IP need not be incurred. Consider the following LP model: Maximize $\mathbf{c}'\mathbf{x}$ subject to $A\mathbf{x} = \mathbf{b}$ and $\mathbf{x} \geq 0$.

Here we assume that slack variables have been added to the constraints, if necessary, to make them all equalities. For the above model to yield an optimal solution with all variables integer for every objective coefficient vector **c** and integer right-hand side **b**, the matrix A must have a property known as *total unimodularity*.

Definition

A matrix A is totally unimodular if every square sub-matrix of A has its determinant equal to 0 or ± 1.

The fact that this property guarantees that there will be an integer optimal solution to the LP model (for any **c** and integer **b**) is proved in Garfinkel and Nemhauser (1972, p. 67). Unfortunately, the above definition of total unimodularity is of little help in detecting the property. To evaluate the determinant of every square sub-matrix would be prohibitive. There is, however, a property (which we call P) that is easier to detect, which guarantees total unimodularity. It is, however, only a sufficient condition, not a necessary condition. Matrices without property P may still be totally unimodular.

Property P

1. Each element of A is 0, 1 or -1.
2. No more than two non-zero elements appear in each column.
3. The rows can be partitioned into two subsets P_1, and P_2 such that
   (a) if a column contains two non-zero elements of the same sign, one element is in each of the subsets;
   (b) if a column contains two non-zero elements of opposite sign, both elements are in the same subset.

A particular case of the property P is when subset P_1 is empty and P_2 consists of all the rows of A. Then for the property to hold, we must have all columns consisting of either one non-zero element ± 1 or two non-zero elements $+1$ and -1.

As an example of this property holding, we can consider the small transportation problem of Section 5.3:

$$\begin{array}{rrrrrrrrrrrrcr}
-x_{11} & -x_{13} & -x_{14} & -x_{15} & & & & & & & & & = & -135, \\
 & & & & -x_{21} & -x_{22} & -x_{25} & & & & & & = & -56, \\
 & & & & & & & -x_{31} & -x_{32} & -x_{33} & -x_{34} & -x_{35} & = & -93, \\
x_{11} & & & & +x_{21} & & & +x_{31} & & & & & = & 62, \\
 & & & & & +x_{22} & & & +x_{32} & & & & = & 84, \\
 & x_{13} & & & & & & & & +x_{33} & & & = & 39, \\
 & & x_{14} & & & & & & & & +x_{34} & & = & 91, \\
 & & & x_{15} & & & x_{25} & & & & & +x_{35} & = & 9.
\end{array} \tag{10.10}$$

Each column clearly contains a $+1$ and a -1, showing the property to hold and hence guaranteeing total unimodularity. There is often virtue in trying to reformulate a model in order to try to capture this easily detected property. The automation of such reformulation (where possible) was described in Section 5.4. In case (ii) below, the dual situation is illustrated.

Although the property above refers to a partitioning of the rows of a matrix A into two subsets, the partitioning could equally well apply to the columns. If A is totally unimodular, its transpose A' must also be totally unimodular. Again, a particular instance of this property guaranteeing total unimodularity is when each row of the matrix A of an LP model contains a $+1$ and a -1.

Finally, it should be pointed out that total unimodularity is a strong property that guarantees integer optimal solutions to an LP problem for all **c** and integer **b**. Many IP models for which the matrix A is not totally unimodular frequently (although not always) produce integer solutions to the optimal solution of the corresponding LP problem. In particular, this often happens with the set packing, partitioning and covering problems discussed in Section 9.5. There are good reasons why this is likely to happen for these types of problems. Such considerations are, however, fairly technical and beyond the scope of this book. They are discussed in Chapter 8 of Garfinkel and Nemhauser (1972). Properties of A that guarantee integer LP solutions for a specific right-hand side **b** are discussed by Padberg (1974) for the case of the set partitioning problem.

The discussion of total unimodularity above applies only to PIP models. Clearly, there are corresponding considerations for MIP (mixed integer programming) models, where integer values for the integer variables in the optimal LP solution are guaranteed. Such considerations are, however, very difficult and there is little theoretical work as yet that is of value to practical model builders.

Case (ii) above concerns problems where, with a little thought, a reformulation can result in a model with the total unimodularity property. Consider a generalization of constraint (9.44):

$$\delta_1 + \delta_2 + \cdots + \delta_n - n\delta \leq 0, \tag{10.11}$$

where δ_i and δ are 0–1 variables.

This kind of constraint arises fairly frequently in IP models and represents the logical condition

$$\delta_1 = 1 \vee \delta_2 = 1 \vee \cdots \vee \delta_n = 1 \rightarrow \delta = 1. \tag{10.12}$$

Sometimes, this condition is more easily thought of as the logical equivalent condition

$$\delta = 0 \rightarrow \delta_1 = 0.\ \delta_2 = 0,\ \ldots,\ \delta_n = 0. \tag{10.13}$$

It was shown in Section 9.2 that by a different argument, constraint (9.44) of that section can be reformulated using two constraints. A similar reformulation can be applied here, giving the n constraints

$$\begin{aligned} \delta_1 - \delta &\leq 0,\\ \delta_2 - \delta &\leq 0,\\ &\vdots\\ \delta_n - \delta &\leq 0. \end{aligned} \tag{10.14}$$

Should all the constraints in the model be similar to the constraints of Inequality (10.14) then the dual problem has the property P described above, which guarantees total unimodularity. There is, therefore, great virtue in such a reformulation as the high computational costs associated with an IP problem over an LP problem is avoided. An example of a reformulation of a problem in this way is described by Rhys (1970). He also demonstrates another advantage of the reformulation as yielding a more meaningful economic interpretation of the shadow prices. This topic is considered in Section 10.3. A practical example of a formulation such as that described above is given in Part III where the formulation of the OPENCAST MINING problem is discussed.

Constraint (10.11) above shows the possibility of sometimes reformulating a PIP problem that is not totally unimodular in order to make it totally unimodular. There is also virtue in reformulating an already totally unimodular problem, which we do not know to be totally unimodular, if by so doing we convert it into a form where property *P* applies. Examples of this are given by Veinott and Wagner (1962) and Daniel (1973). As with case (i), the above discussion of case (ii) only applies to PIP problems. Again, it must be possible (although difficult) to generalize these ideas to MIP problems.

Case (iii) concerns problems where there is either no obvious totally unimodular reformulation or where the problem gives an MIP model. In cases (i) and (ii), we reduced the feasible region to the convex hull of feasible integer points, even though this was not obvious from the algebraic treatment given. It is sometimes possible to go part way towards this aim. Suppose, for example (as might frequently happen in a MIP model), that only some of the constraints were of the form (10.11). By expanding these constraints into the series of constraints (10.14) we would reduce the size of the feasible region of the LP. Even though the existence of other constraints in the problem might result in some integer variables taking fractional values in the LP optimal solution, this solution should be 'nearer' the integer optimal solution than would be the case with the original model. The term 'nearer' is purposely vague. A reformulation such as this might result in the objective value at node 1 of the solution tree (e.g. Figure 8.1) being closer to the objective value of the optimal integer solution when found. On the other hand, it might result in there being less fractional solution values in the LP optimum. Whatever the result of the reformulation, one would normally expect the solution time for the reformulated model to be less than for the original model. Constraints involving just two coefficients $+1$ and -1 also arise in models involving sequentially dependent decisions as described in Section 9.4. Such constraints are always to be desired even if their derivation results in an expansion of the constraints of a model. An example of this is the suggested formulation in Part III for the MINING problem.

It is worth indicating in another way why a series of constraints such as in Inequalities (10.14) is preferable to the single constraint (10.11). Although constraints (10.11) and (10.14) are exactly equivalent in an IP sense, they are certainly not equivalent in an LP sense. In fact Inequality (10.11) is the sum of all the constraints in Inequalities (10.14). By adding together constraints in an LP problem, one generally weakens their effect. This is what happens here. Inequality (10.11) admits fractional solutions, which Inequality (10.14) would not admit. For example, the solution

$$\delta_1 = \frac{1}{2}, \delta_2 = \delta_3 = \frac{1}{4}, \quad \delta = \frac{1}{n}, \text{ all other } \delta_i = 0 \tag{10.15}$$

satisfies Inequality (10.11) but breaks Inequalities (10.14) (for $n \geq 3$).

Hence, Inequalities (10.14) are more effective at ruling out unwanted fractional solutions.

The ideas discussed above are relevant to the FOOD MANUFACTURE 2 problem, which gives rise to an MIP model and to the DECENTRALIZATION and LOGICAL DESIGN problems, which give rise to PIP models.

Some of the material so far presented in this section was first published by Williams (1974). A discussion of some very similar ideas applied to a more complicated version of the DECENTRALIZATION problem is given in Beale and Tomlin (1972).

It is also relevant here to discuss the value of the coefficient 'M' when linking indicator variables to continuous variables by constraints such as in Equations (9.1), (9.12), (9.19) and (9.21). These types of constraints usually (but not always) arise in MIP models.

We will consider the simplest way in which such a constraint arises when we are using a 0–1 variable δ to indicate the condition below on the continuous variable x:

$$x > 0 \rightarrow \delta = 1. \tag{10.16}$$

This condition is represented by the constraint

$$x - M\delta \leq 0. \tag{10.17}$$

So long as M is a true upper bound for x condition (10.16) is imposed, however large we make M. There is virtue, however, in making M as small as possible without imposing a spurious restriction on x. This is because by making M smaller we reduce the size of the feasible region of the LP problem corresponding to the MIP problem. Suppose, for example, we took M as 1000 when it was known that x would never exceed 100. The following fractional solution would satisfy Inequality (10.17):

$$x = 70, \qquad \delta = \frac{1}{2}, \tag{10.18}$$

but would violate it if M were taken as 100. There are other good reasons for making M as realistic as possible. For example, if M were again taken as 1000, the following fractional solution would satisfy Inequality (10.17):

$$x = 5, \qquad \delta = 0.005. \tag{10.19}$$

A small value of δ, such as this, might well fall below the tolerance that indicates whether a variable is integer or not. If it did, δ would be taken as 0, giving the spurious integer solution

$$x = 5, \qquad \delta = 0. \tag{10.20}$$

If, however, M were made smaller, this would be less likely to happen. Finally, the inadvisability of having coefficients of widely differing magnitudes, as mentioned in Section 3.4, makes a small value of M desirable.

It is also sometimes possible to split up a constraint using a coefficient M in an analogous fashion to the way in which Inequality (10.11) was split up into Inequalities (10.14). This is demonstrated by the following example.

Example 10.3

$$\delta = \begin{cases} 1 & \text{if the depot is built,} \\ 0 & \text{otherwise.} \end{cases}$$

If the depot is built it can supply customer i with a quantity up to $M_i, i = 1, 2, \ldots, n$. If the depot is not built, none of these customers can be supplied with anything.

$$x_i = \text{quantity supplied to customer } i.$$

These conditions can be imposed by the following constraint:

$$x_1 + x_2 + \cdots + x_n - M\delta \leq 0, \tag{10.21}$$

where $M = M_1 + M_2 + \cdots + M_n$.

On the other hand, the following constraints are superior as the corresponding LP problem is more constrained:

$$\begin{aligned} x_1 - M_1\delta &\leq 0, \\ x_2 - M_2\delta &\leq 0, \\ &\vdots \\ x_n - M_n\delta &\leq 0. \end{aligned} \tag{10.22}$$

To summarize this section, the main objectives of an IP formulation should be as follows:

1. To use integer variables which can be put to a good purpose in the branching process of the branch and bound method. If necessary, introduce extra 0–1 variables to create meaningful dichotomies.

2. To make the LP problem corresponding to the IP problem as constrained as possible.
3. To use special ordered sets as described in Section 9.3 if it is possible and the computer package used is capable of dealing with them. This is a final objective not yet mentioned in this section.

Further ways of reformulating IP models in order to ease their solution are described in the next section.

Before a large IP model is built, it is often a very good idea to build a small version of the model first. Experimentation with different solution strategies, possibly with reformulation, can give one valuable experience before one embarks on the much larger model.

Sometimes, by examining the structure of a model, it is possible to make observations that lead one to a tightening of the constraints. A dramatic example of this (even when the application was unknown) is described by Daniel (1978), resulting in the solution of a reformulated model in 171 nodes where previously the tree search had been abandoned after 4757 nodes.

The automatic reformulation of IP models in order to tighten the LP relaxation is described by Crowder *et al*. (1983) and Van Roy and Wolsey (1984). The derivation of facet-defining constraints is discussed by Wolsey (1976, 1989).

10.2 Simplifying an integer programming model

In Section 10.1, it was shown that it is often possible to reformulate an IP model in order to create another model that is easier to solve. This is sometimes made possible by considering the practical situation being modelled. In this section, we are concerned with rather less obvious transformations of an IP model. Again, the aim is to make the model easier to solve.

10.2.1 Tightening bounds

In Section 3.4, part of the procedure of Brearley *et al*. (1975) for simplifying LP models was outlined. The full application of that procedure involves removing redundant simple bounds in an LP model. It is not, however, generally worthwhile removing redundant bounds on an integer variable. Instead it is better to tighten the bounds if possible. The argument for doing this is similar to some of the reformulation arguments used in Section 10.1. By tightening bounds, the corresponding LP problem may be made more constrained resulting in the optimal solution to the LP problem being closer to the optimal IP solution. In order to illustrate the procedure, a small example from Balas (1965) is used. This example was also used in the description of the procedure given by Brearley, Mitra and Williams.

Example 10.4

$$\begin{aligned} \text{Minimize} \quad & 5\delta_1 + 7\delta_2 + 10\delta_3 + 3\delta_4 + \delta_5 \\ \text{subject to} \quad & \delta_1 - 3\delta_2 + 5\delta_3 + \delta_4 - \delta_5 \geq 2, && \text{(R1)} \\ & -2\delta_1 + 6\delta_2 - 3\delta_3 - 2\delta_4 + 2\delta_5 \geq 0, && \text{(R2)} \\ & - \delta_2 + 2\delta_3 - 2\delta_4 - \delta_5 \geq 1. && \text{(R3)} \end{aligned}$$

The δ_i are all 0–1 variables.

1. By constraint (R3)

   $$2\delta_3 \geq 1 + \delta_2 + \delta_4 + \delta_5 \geq 1.$$

   Hence

   $$\delta_3 \geq \frac{1}{2}.$$

   As δ_3 is an integer variable this implied lower bound may be tightened. In this case (as δ_3 is 0–1), δ_3 may be set to 1 and removed from the problem.

2. By constraint (R2)

   $$6\delta_2 \geq 3 + 2\delta_1 + 2\delta_4 - 2\delta_5 \geq 1.$$

   Hence

   $$\delta_2 \geq \frac{1}{6}.$$

   Similarly, the lower bound of δ_2 may be tightened to 1, so fixing δ_2 at 1.

3. By constraint (R3)

   $$\delta_4 \leq -\delta_5 \leq 0.$$

   Hence

   $$\delta_4 \leq 0.$$

   Therefore δ_4 can be fixed at 0.

4. By constraint (R3), $\delta_5 \leq 0$. Therefore δ_5 can be fixed at 0.

5. All the constraints now turn out to be redundant and may be removed. The only remaining variable is δ_1, which must obviously be set to 0.

This example is obviously an extreme case of the effect of tightening bounds in an IP model as this procedure alone completely solves the problem.

10.2.2 Simplifying a single integer constraint to another single integer constraint

Consider the integer constraint

$$4\gamma_1 + 6\gamma_2 \leq 9, \tag{10.23}$$

where γ_1 and γ_2 are general integer variables. By looking at this constraint geometrically in Figure 10.2, it is easy to see that it may also be written as

$$\gamma_1 + 2\gamma_2 \leq 2. \tag{10.24}$$

In Figure 10.2, the original constraint (10.23) indicates the feasible points must lie to the left of AB. By shifting the line AB to CD no integer points are excluded from, and no new integer points are included in, the feasible region. CD gives rise to the new constraint (10.24).

Clearly, there are advantages in using constraint (10.24) rather than (10.23) as the feasible region of the corresponding LP problem has been reduced. While constraints such as (10.23) involving general integer variables do not arise very frequently such constraints can occur involving only 0–1 variables. We therefore confine our attention to the 0–1 case. Should more general integer variables be involved, it is, of course, always possible to expand them into 0–1 variables as described in Section 10.1. Our problem is now given a constraint such as

$$a_1\delta_1 + a_2\delta_2 + \cdots + a_n\delta_n \leq a_0, \tag{10.25}$$

where the δ_i are 0–1 variables, to re-express it as an equivalent constraint

$$b_1\delta_1 + b_2\delta_2 + \cdots + b_n\delta_n \leq b_0, \tag{10.26}$$

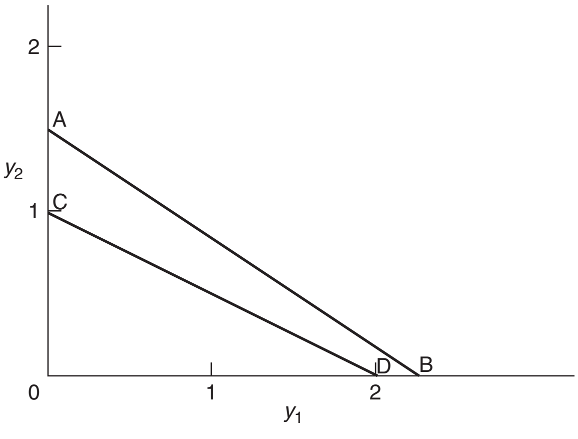


Figure 10.2

where Inequality (10.26) is more constrained than Inequality (10.25) in the corresponding LP problem. There is no loss of generality in assuming all the coefficients of Inequalities (10.25) and (10.26) to be non-negative as should a negative coefficient a_i occur in Inequality (10.25), the corresponding variable δ_i may be complemented by the substitution

$$\delta_i = 1 - \delta_i',$$

making the new coefficient of δ_i' positive.

Clearly '$\geq$' constraints can be converted to '$\leq$'. Equality constraints are most conveniently dealt with by converting them into a '$\leq$' together with a '$\geq$' constraint. The result will be two simplified constraints in place of the original single '=' constraint.

The simplification of a pure 0–1 constraint such as in Inequality (10.25) in order to produce the one in Inequality (10.26) can itself be formulated and solved as an LP problem. Rather than present the technical details of the procedure here, a specific problem, OPTIMIZING A CONSTRAINT is given in Part II. The formulation and discussion in Part III should make the general procedure clear.

If the original constraint to be simplified involved general integer variables, rather than 0–1 integer variables, and it is desired to replace it by a constraint in the same variables, then it will be necessary to restrict the new coefficients of the 0–1 form. For example, suppose we had a general integer variable γ ($\leq$7). If its original coefficient were a in the 0–1 form of the constraint, we would have the term

$$a\gamma_1 + 2a\gamma_2 + 4a\gamma_3, \tag{10.27}$$

with $\gamma_1 + 2\gamma_2 + 4\gamma_3$ representing variable γ, where γ_1, γ_2 and γ_3 are 0–1 variables. The resultant simplification would produce an equivalent term

$$b_1\gamma_1 + b_2\gamma_2 + b_3\gamma_3. \tag{10.28}$$

For this term to be replaceable by a term $b\gamma$, we would have to ensure that

$$b_3 = 2b_2, \quad b_2 = 2b_1, \tag{10.29}$$

giving $b = b_1$.

Using the LP formulation described for the 0–1 example in Part II, it would be necessary to impose conditions such as (10.29) as *extra* LP constraints for the general integer case.

The effort of trying to simplify single integer constraints in this way is often not worthwhile. In many models, these constraints represent logical conditions and are already in their simplest form as single constraints. Applications where such simplification could possibly prove worthwhile are project selection and capital budgeting problems. It also proves worthwhile to simplify single constraints in this way in the MARKET SHARING problem presented in Part II.

The simplification of a single 0–1 constraint into another single 0–1 constraint considered here has been described by Bradley *et al*. (1974).

10.2.3 Simplifying a single integer constraint to a collection of integer constraints

We again confine our attention to pure 0–1 constraints given that general integer variables can be expanded, if necessary, into 0–1 variables.

It may often be advantageous to express a single 0–1 constraint as a collection of 0–1 constraints. We have already seen this in Section 10.1, where constraint (10.11) was re-expressed as the constraints (10.14). Here, we present a general procedure for expanding any pure 0–1 constraint into a collection of constraints. Ideally, we would like to be able to re-express the constraint by a collection of constraints defining the convex hull of feasible 0–1 solutions. In order to make this clear, we present an example.

Example 10.5

$$3\delta_1 + 3\delta_2 - 2\delta_3 + 2\delta_4 + 2\delta_5 \leq 4. \tag{10.30}$$

δ_i are 0–1 variables.

The convex hull of 0–1 solutions that are feasible according to Expression (10.27) is given by the constraints

$$\delta_1 + \delta_2 - \delta_3 + \delta_4 \leq 1, \tag{10.31}$$

$$\delta_1 + \delta_2 - \delta_3 + \delta_5 \leq 1, \tag{10.32}$$

$$\delta_1 + \delta_2 + \delta_4 + \delta_5 \leq 2, \tag{10.33}$$

$$2\delta_1 + \delta_2 - \delta_3 + \delta_4 + \delta_5 \leq 2, \tag{10.34}$$

$$\delta_1 + 2\delta_2 - \delta_3 + \delta_4 + \delta_5 \leq 2, \tag{10.35}$$

together with the trivial constraints, $\delta_i \geq 0$ and $\delta_i \leq 1$.

Unfortunately, no practical procedure has yet been devised for obtaining constraints defining the convex hull of integer solutions corresponding to a single 0–1 constraint. The faces that form the boundary of the feasible region defined by an LP problem are known as '*facets*'. For two-variable problems, these facets are one-dimensional lines and can easily be visualized in a diagram such as Figure 10.2. With three-variable problems, the facets are two-dimensional planes. For n-variable problems, these facets are $(n-1)$-dimensional hyperplanes. Hammer *et al*. (1975) give a table of the facets of the convex hulls for all 0–1 constraints involving up to five variables. In order to use this table, it is first necessary to simplify the constraint to another single constraint using the procedure mentioned above. It is also possible to obtain the convex hull for particular types of constraint involving more than five variables. The expansion of constraint (10.11) into constraints (10.14) in Section 10.1 is obviously an instance of this. Although a general procedure for producing the convex hull constraints for a single 0–1 constraint does not yet exist, Balas (1975) gives a procedure for producing some of the facets of the convex hull. Hammer *et al*. (1975) and

Wolsey (1975) also give similar procedures. They are able to obtain those facets represented by constraints containing only coefficients 0 and ± 1. For example, the procedure would obtain the constraints (10.31)–(10.33) of Example 10.5.

We now describe this procedure of Balas. Consider the following pure 0–1 constraint:

$$a_1\delta_1 + a_2\delta_2 + a_3\delta_3 + \cdots + a_n\delta_n \le a_0. \tag{10.36}$$

As before, there is no loss of generality in considering a '$\le$' constraint with all coefficients non-negative.

Definition

A subset $\{i_1, i_2, \ldots, i_r\}$ of the indices $1, 2, \ldots, n$ of the coefficients in Inequality (10.33) will be called a *cover* if $a_{i_1} + a_{i_2} + \cdots + a_{i_r} > a_0$.

Clearly, it is not possible for all the δ_i corresponding to indices i within a cover to be 1 simultaneously. This condition may be expressed by the constraint

$$\delta_{i_1} + \delta_{i_2} + \cdots + \delta_{i_r} \le r - 1. \tag{10.37}$$

Definition

A cover $\{i_1, i_2, \ldots, i_r\}$ is said to be a *minimal cover* if no proper subset of $\{i_1, i_2, \ldots, i_r\}$ is also a cover.

Definition

A minimal cover $\{i_1, i_2, \ldots, i_r\}$ can be extended in the following way:

1. Choose the largest coefficient a_{i_j} where i_j is a member of the minimal cover.
2. Take the set of indices $\{i_{r+1}, i_{r+2}, \ldots, i_{r+s}\}$ not in the minimal cover corresponding to coefficients $a_{i_{r+k}}$ such that $a_{i_{r+k}} \ge a_{i_j}$.
3. Add these set of indices to the minimal cover giving $\{i_1, i_2, \ldots, i_{r+s}\}$. This new cover is known as an *extended cover*.

If $\{i_1, i_2, \ldots, i_r\}$ is a minimal cover, it can be augmented to give an extended cover $i_1, i_2, \ldots, i_{r+s}$. The constraint (10.37) corresponding to the minimal cover can correspondingly be extended to

$$\delta_{i_1} + \delta_{i_2} + \cdots + \delta_{i_{r+s}} \le r - 1. \tag{10.38}$$

Definition

If a minimal cover gives rise to an extended cover that is not a proper subset of any other extended cover arising from a minimal cover with the same number of indices, the original minimal cover is known as a *strong cover*.

Balas shows that if an extended cover arises from a strong cover then the constraints such as in Inequality (10.38) encompass all the facets of the convex hull of feasible 0–1 points of Inequality (10.33) that have coefficients 0 or 1.

It is obviously fairly easy to devise a systematic and computationally quick method of generating all strong covers corresponding to a 0–1 constraint and therefore obtaining facet constraints such as Inequality (10.38). In this way, any facet of the convex hull of feasible 0–1 points of a constraint can be defined when the facet is represented by a constraint involving only coefficients 0 and ±1. Two examples are presented in order to make the process clear.

Example 10.6

This is the same as Example 10.5, only we here show how the constraints (10.31)–(10.33) involving only coefficients 0 and ±1 are obtained. It is more convenient to write constraint (10.30) with positive coefficients as

$$3\delta_1 + 3\delta_2 + 2\delta'_3 + 2\delta_4 + 2\delta_5 \leq 6, \tag{10.39}$$

where $\delta'_3 = 1 - \delta_3$.

The minimal covers of constraint (10.39) are

$$\{1, 2, 3\}, \tag{10.40}$$

$$\{1, 2, 4\}, \tag{10.41}$$

$$\{1, 2, 5\}, \tag{10.42}$$

$$\{1, 3, 4\}, \tag{10.43}$$

$$\{1, 3, 5\}, \tag{10.44}$$

$$\{1, 4, 5\}, \tag{10.45}$$

$$\{2, 3, 4\}, \tag{10.46}$$

$$\{2, 3, 5\}, \tag{10.47}$$

$$\{2, 4, 5\}. \tag{10.48}$$

These minimal covers can be augmented to produce the extended covers

$$\{1, 2, 3\} \text{ from the cover } (10.40), \tag{10.49}$$

$$\{1, 2, 4\} \text{ from the cover } (10.41), \tag{10.50}$$

$$\{1, 2, 5\} \text{ from the cover } (10.42), \tag{10.51}$$

$$\{1, 2, 3, 4\} \text{ from the covers } (10.43) \text{ and } (10.46), \tag{10.52}$$

$$\{1, 2, 3, 5\} \text{ from the covers } (10.44) \text{ and } (10.47), \tag{10.53}$$

$$\{1, 2, 4, 5\} \text{ from the covers } (10.46) \text{ and } (10.48), \tag{10.54}$$

Since (10.49)–(10.51) are proper subsets of the covers (10.52)–(10.54), respectively, the first three minimal covers, (10.40)–(10.42), are not strong covers. The remaining minimal covers (10.43)–(10.48) are, however, strong covers and their extensions, covers (10.52)–(10.54) give rise to the following facet constraints:

$$\delta_1 + \delta_2 + \delta_3' + \delta_4 \qquad \leq 2, \tag{10.55}$$

$$\delta_1 + \delta_2 + \delta_3' \qquad + \delta_5 \leq 2, \tag{10.56}$$

$$\delta_1 + \delta_2 \qquad + \delta_4 + \delta_5 \leq 2. \tag{10.57}$$

Replacing δ_3' by $1 - \delta_3$ gives the three facet constraints (10.31)–(10.33) in Example 10.5.

Example 10.7

$$\delta_{n+1} = \begin{cases} 1 & \text{if a depot is built,} \\ 0 & \text{otherwise,} \end{cases}$$

$$\delta_i = \begin{cases} 1 & \text{if customer } i \text{ is supplied from the depot,} \\ 0 & \text{otherwise,} \end{cases}$$

where $i = 1, 2, \ldots, n$.

At most r $(r < n)$ customers may be supplied from the depot if it is built. If the depot is not built, clearly nobody can be supplied from it.

The conditions above may be expressed by the constraint

$$\delta_1 + \delta_2 + \cdots + \delta_n - r\delta_{n+1} \leq 0. \tag{10.58}$$

This is obviously a generalization of the constraint (10.16).

It is convenient to express Inequality (10.58) with positive coefficients as

$$\delta_1 + \delta_2 + \cdots + \delta_n + r\delta_{n+1}' \leq r, \tag{10.59}$$

where

$$\delta_{n+1}' = 1 - \delta_{n+1}. \tag{10.60}$$

The minimal covers of (10.59) are

$$\{i, n+1\}, \quad i = 1, 2, \ldots, n, \tag{10.61}$$

and all subsets of $\{1, 2, \ldots, n\}$ such as

$$\left\{i_1, i_2, \ldots, i_{r+1}\right\} \tag{10.62}$$

containing $r + 1$ indices.

Covers (10.61) cannot be further extended.

All the covers (10.62) do extend to the same extended cover:

$$\{1, 2, \ldots, n, n+1\}. \tag{10.63}$$

The minimal covers (10.61) and (10.62) in general are of a different size (if $r \neq 1$). Therefore, covers (10.61) and (10.63) are all extensions of strong covers and give rise (after substituting $1 - \delta_{n+1}$ for δ'_{n+1}) to the constraints

$$\delta_i - \delta_{n+1} \leq 0, \quad i = 1, 2, \ \ldots, \ n, \tag{10.64}$$

and

$$\delta_1 + \delta_2 + \cdots + \delta_n - \delta_{n+1} \leq r - 1. \tag{10.65}$$

These constraints do not necessarily all represent facets, but they are particularly restrictive constraints on the corresponding LP problem and include all the facets with coefficients 0 or 1. It would therefore be advantageous to append constraints (10.64) and (10.65) to the original constraint (10.58) in a model.

This way of obtaining particularly 'strong' extra constraints to add to an IP model could prove of value in the MARKET SHARING problem of Part II.

10.2.4 Simplifying collections of constraints

So far, we have described ways of simplifying individual constraints involving 0–1 variables. What we would ideally like to do would be to simplify the collection of all the constraints into a collection of constraints defining the convex hull of feasible integer points. It should be pointed out that simplifying constraints individually will not generally suffice although it may be computationally helpful towards the ultimate aim of solving the IP problem more easily. This is demonstrated by an example.

Example 10.8

$$\begin{aligned} \text{Maximize} \quad & \delta_1 + 2\delta_2 + \delta_3 \\ \text{subject to} \quad & 2\delta_1 + 3\delta_2 + 2\delta_3 \leq 3, \qquad (10.66) \\ & \delta_1 + \delta_2 - 2\delta_3 \leq 0. \qquad (10.67) \end{aligned}$$

where $\delta_1, \delta_2, \delta_3$ are 0–1 variables.

The optimal solution to the associated LP problem is

$$\delta_1 = 0, \quad \delta_2 = \frac{3}{4}, \quad \delta_3 = \frac{3}{8}, \quad \text{giving an objective of } \frac{15}{8}.$$

If constraint (10.66) is replaced by the constraints representing its facets, we obtain constraint (10.68) in the model below. Constraints (10.69) and (10.70) come from the two facets of constraint (10.67). The simplified model is then

$$\begin{aligned} \text{Maximize} \quad & \delta_1 + 2\delta_2 + \delta_3 \\ \text{subject to} \quad & \delta_1 + \delta_2 + \delta_3 \leq 1, && (10.68) \\ & \delta_1 - \delta_3 \leq 0, && (10.69) \\ & \delta_2 - \delta_3 \leq 0. && (10.70) \end{aligned}$$

The optimal solution to the associated LP problem for this reformulated model is

$$\delta_1 = 0, \qquad \delta_2 = \frac{1}{2}, \qquad \delta_3 = \frac{1}{2}, \qquad \text{giving an objective of } \frac{3}{2}.$$

Clearly simplifying constraints individually has not constrained the whole model sufficiently to guarantee an integer solution to the LP model although the objective value is closer to the integer optimum:

$$\delta_1 = 0, \qquad \delta_2 = 0, \qquad \delta_3 = 1, \qquad \text{giving an objective of } 1.$$

Unfortunately, no practical procedure is known for producing the convex hull of the feasible 0–1 points corresponding to a general collection of pure 0–1 constraints. In fact, if such a computationally efficient procedure were known we could reduce all PIP problems to LP problems and dispense with PIP. Procedures of this sort do exist for special restricted classes of PIP problems. The best known is for the matching problem that was mentioned in Section 9.5. The main reference to the problem is found in Edmonds (1965).

Partial results exist enabling one to obtain some of the facets of the convex hull for some types of PIP problems. Hammer *et al*. (1975) have a procedure for generating the facet constraints involving only coefficients 0 and 1 for a class of PIP problems they call '*regular*'. Within this class of problems are the set covering problem and the knapsack problem.

Clearly, by being able to cope with the knapsack problem they can also obtain the facets obtained by Balas for a single constraint. Apart from what has already been described, none of these partial results seems as yet to give a valuable formulation tool for practical problems and they will not therefore be described further.

It should also be pointed out that procedures have been devised for doing the reverse of what we have described here. A collection of pure integer equality constraints can be progressively combined with one another to obtain a single equality constraint giving a knapsack problem. Bradley (1971) describes such a procedure. Unfortunately, the resulting coefficients are often enormous. There is little interest in such a procedure if the corresponding LP problem is to be used as a starting point for solving the IP problem. The general effect of aggregating constraints in this way will be to weaken rather than restrict the corresponding LP problem. Chvátal and Hammer (1975) also describe a procedure for combining pure 0–1 inequality constraints. A more recent paper is by Bienstock and McClosky (2012).

Most of the discussion in this section has concerned adding further restrictions to an IP model in order to restrict the corresponding LP model. Such extra restrictions can be viewed in the context of cutting planes algorithms for IP. We are adding cuts to a model in order to eliminate some possible fractional solutions. Most algorithms which make use of cutting planes generate the extra constraints in the course of optimization. Our interest here, in a book on model building, is only in adding cuts to the initial model.

10.2.5 Discontinuous variables

It is sometimes necessary to restrict a continuous variable to segments of continuous values, for example,

$$x = 0 \ \text{ or } \ a \leq x \leq b \ \text{ or } \ x = c, \tag{10.71}$$

where $0 < a < b < c$.

A straightforward approach to follow is that described for disjunctive constraints in Section 9.4, where 0–1 variables are used to indicate each of the three (or more) possibilities. A constraint of type (9.79) forces x to satisfy the condition.

There is an alternative formulation that has been suggested by Brearley (1975), following a more conventional formulation of a blending problem with logical restrictions by Thomas *et al*. (1978). This is

$$x = ay_1 + by_2 + c\delta_2, \tag{10.72}$$

$$\delta_1 + y_1 + y_2 + \delta_2 = 1, \tag{10.73}$$

where δ_1 and δ_2 are 0–1 integer variables and y_1 and y_2 are (non-negative) continuous variables.

Condition (10.71) can clearly be generalized or specialized. A common special case is that of a *semi-continuous* variable, that is,

$$x = 0 \ \text{ or } \ x \geq a \quad (a > 0)\,. \tag{10.74}$$

In order to model this, an upper bound (M) must be specified for x, giving the formulation

$$x = ay_1 + My_2, \tag{10.75}$$

$$\delta + y_1 + y_2 = 1, \tag{10.76}$$

where δ is a 0–1 variable and y_1 and y_2 are (non-negative) continuous variables.

10.2.6 An alternative formulation for disjunctive constraints

An alternative formulation for a *disjunction* of constraints has been given by Jeroslow and Lowe (1984). In addition, they construct a 'Theory of Mixed Integer Programming Representability' which begins to put the whole subject on systematic foundations. The use of Jeroslow's 'disjunctive formulations' has a considerable theoretical advantage as well as manifesting a practical advantage in large models. Computational experience is reported in Jeroslow and Lowe (1985).

A comprehensive discussion of the subject is given in the lecture notes of Jeroslow (1989).

Suppose we have a disjunction of constraints such as (9.77) where each R_k represents a set of constraints:

$$\sum_j a_{ijk} x_j \leq b_{ik}, \quad i = 1, 2, \ldots, m_k. \tag{10.77}$$

We will suppose that each set of constraints (10.77) in this disjunction has a *closed* feasible region. If necessary, this may be achieved by using known bounds on the quantities in the constraints as is also required in the conventional formulation. In fact, even if the feasible region is not closed, it is not always necessary to use such bounds in this new mode of formulation. The exact condition is given in Jeroslow and Lowe (1984).

Each variable x_j is split into separate variables x_{jk} with the following constraints:

$$x_j = x_{j1} + x_{j2} + \cdots + x_{jN}. \tag{10.78}$$

The new variables replace the original variables in the set of constraints R_k to which they correspond giving the following constraints:

$$\sum_j a_{ijk} x_{jk} - b_{ik}\delta_k \leq 0, \quad i - 1, 2, \ldots, m_k, \tag{10.79}$$

$$\sum_k \delta_k = 1 \tag{10.80}$$

where δ_k are 0–1 integer variables.

Constraint (10.80) forces exactly one δ_k to be 1 and the others to be 0. If δ_k is 0, then constraints (10.79) (having a closed feasible region) force the corresponding x_{jk} all to be zero. Hence for each j only one component x_{jk} can be non-zero making it, by constraint (10.78), equal to x_j and so guaranteeing the constraints corresponding to R_k.

It can be shown that if each set of constraints (10.77) is a *sharp* formulation of R_k, then this resultant formulation of the disjunction is also sharp, that is, the LP relaxation gives the convex hull of feasible integer solutions. If, for example, the variables in Inequality (10.77) are all continuous variables, then the property holds.

In Section 9.2, it was shown that logical conditions often can be specified in more than one way. A well-known result in Boolean algebra is that any proposition can be expressed in a standard form known as *disjunctive normal form*, which uses only the *and* (.), *or* ($\vee$) and *not* ($\sim$) connectives, for example,

$$\left(R_{11} \,.\, \mathrm{R}_{12} \cdots \mathrm{R}_{1m_1}\right) \vee \left(R_{21} \,.\, \mathrm{R}_{22} \cdots \mathrm{R}_{2m_2}\right) \vee \cdots \vee \left(R_{N1} \,.\, \mathrm{R}_{N2} \cdots \mathrm{R}_{Nm_2}\right) \tag{10.81}$$

where there is a disjunction of *clauses*, each of which is made up of a *conjunction* of statements R_{ij} (some of which may be negated statements). Jeroslow suggests that it is generally better to express a model using this disjunctive normal form and then use his disjunctive formulation. It is theoretically possible to arrive at a sharp formulation for any IP this way, by taking account of the fact that any (bounded) integer variable represents a disjunction of possibilities, for example,

$$x = 0 \vee x = 1 \vee x = 2 \vee \cdots \vee \; x = m. \tag{10.82}$$

In practice, the number of variables created by such a formulation can be prohibitively large as the disjunctive formulation splits variables into components in the manner described above. A compromise must generally be adopted, which will not be sharp but is often tighter than a conventional formulation. In practice, many simplifications of the resultant formulation are often possible (using, e.g. a reduction procedure).

The possibility of obtaining tighter constraints by introducing extra variables is discussed by Martin (1987) and Williams and Brailsford (1997).

Two other observations of Jeroslow are worth making here. He points out that it is desirable, from the sharpness point of view, to apply the *and* connective before specifying an IP formulation. This is in contrast to creating IP formulations of components of a problem and then applying the *and* connective (i.e. putting all the resulting constraints together). This is because (in set notation)

$$\mathrm{Con}\,(S \cap T) \subseteq \mathrm{Con}\,(S) \cap \mathrm{Con}\,(T) \tag{10.83}$$

where S and T are sets and Con is the operation of taking the convex hull.

It is also desirable to aim for formulations that are *hereditarily sharp*, that is, when certain integer variables are fixed (by the branch and bound algorithm), the resulting submodels remain sharp.

10.2.7 Symmetry

Sometimes, a model can have a large number of equivalent solutions. For example, in a distribution problem, we might have variables of the form $\delta_{ij} = 1$ if and only if vehicle i visits customer j. If the vehicles were identical, a solution in which $\delta_{12} = 1$ and $\delta_{22} = 0$ would be equivalent to one in which $\delta_{12} = 0$ and $\delta_{22} = 1$. In order to avoid the computational cost of generating both solutions, we could impose the following extra constraints:

$$\delta_{1j} \geq \delta_{2j} \geq \cdots \geq \delta_{nj} \;\text{ for all }\; j \tag{10.84}$$

An alternative would be to collect these variables together as n_j representing the number of vehicles that visit customer j. However, in some circumstances, such an *aggregation* of variables can weaken the LP relaxation. Such methods of reducing the generation of symmetrically equivalent solutions will generally be dependent on the particular model, but it is important to recognize the computational disadvantages of symmetry in IP (unlike many other branches of mathematics).

10.3 Economic information obtainable by integer programming

We saw in Section 6.2 that in addition to the optimal solution values of an LP problem, important additional economic information can also be obtained from such quantities as the shadow prices and reduced costs. The dual LP model was also shown to have an important economic interpretation in many situations. In addition, a close relationship between the solution to the original model and its dual exist.

It is worth just briefly pointing out how the duality relationship fails in IP. Suppose we have an IP maximization problem P. Corresponding to P we have the LP problem P'. As long as P' is feasible and not unbounded, we have a solvable dual problem Q'. From duality in LP, we know that

$$\text{maximum objective of } P' = \text{minimum objective of Q}'.$$

By imposing extra integrality requirements on P', we obtain the IP problem P. Clearly, as P is more constrained than P', we have

$$\text{maximum objective of } \; P \leq P' = \text{minimum objective of } \; Q'.$$

The minimum objective of Q' is the *smallest upper bound* we can obtain for the objective of P by a set of valuations on the constraints of P. This contrasts with the LP case where the dual values provide a *strict upper bound* (i.e. a bound that is obtained by the optimum) for the objective. In consequence, the difference between the maximum objective value of P and the maximum objective of P' (or minimum objective of Q') is sometimes known as a *duality gap*. It can be regarded (rather loosely) as a measure of how inadequate any dual values will be when used as shadow prices.

In this section, we attempt to obtain corresponding economic information from an IP model to that obtainable from an LP model. It will be seen that this information is much more difficult to come by in the case of IP and is, in some cases, rather ambiguous. In order to demonstrate the difficulties, we will consider a ‘product mix’ problem where the variables in the model represent quantities of different products to be made and the constraints represent limitations on productive capacity. It is only meaningful to make integral numbers of each product.

Example 10.9

$$\begin{aligned}
\text{Maximize} \quad & 12\gamma_1 + 5\gamma_2 + 15\gamma_3 + 10\gamma_4 \\
\text{subject to} \quad & 5\gamma_1 + \gamma_2 + 9\gamma_3 + 12\gamma_4 \leq 15, && (10.85) \\
& 2\gamma_1 + 3\gamma_2 + 4\gamma_3 + \gamma_4 \leq 10, && (10.86) \\
& 3\gamma_1 + 2\gamma_2 + 4\gamma_3 + 10\gamma_4 \leq 8, && (10.87) \\
& \gamma_1, \gamma_2, \gamma_3, \gamma_4 \geq 0.
\end{aligned}$$

The γ_1 are general integer variables.

The optimal integer solution is $\gamma_1 = 2, \gamma_2 = 1, \gamma_3 = 0$ and $\gamma_4 = 0$, giving an objective value of 29.

For comparison, the optimal solution to the corresponding LP problem is $\gamma = 2\frac{2}{3}$, $\gamma_2 = 0, \gamma_3 = 0$ and $\gamma_4 = 0$, giving an objective value of 32.

In addition to the fractional solution of the LP problem, we would be able to obtain answers to such questions as the following:

(Q1) What is the marginal value of increasing fully utilized capacities?

(Q2) How much should a non-manufactured product be increased in price to make it worth manufacturing?

We saw in Section 6.2 that the answers to Q1 came from the shadow prices on the corresponding constraints. These shadow prices represented valuations that could be placed on the capacities. Once these optimal valuations have been obtained, the optimal manufacturing policy can be deduced by simple accounting. Moreover, the total valuation for all the capacities implied by these shadow prices is the same as the profit obtainable by the optimal manufacturing policy.

Unfortunately, there are no such neat values that may be placed on capacities in the case of an IP model. For example, the capacity represented by constraint (10.85) is not fully used up in the IP optimal solution. In an LP problem, if a constraint has slack capacity, as in this case, it represents a *free good* as described in Section 6.2 and has a zero shadow price. Such a constraint could be omitted from the LP model and the optimal solution would be unchanged. In this case, constraint (10.85) cannot be omitted without changing the optimal solution. We would therefore like to give the constraint some economic valuation. This gives the first important difference that must exist between any valuations of constraints in IP and shadow prices in LP:

(A) If a constraint has positive slack it does not necessarily represent a free good and may therefore have a positive economic value.

Why this should be so is fairly easy to see as, although there is no virtue in slightly increasing the right-hand side value of 15 in constraint (10.85), there clearly is virtue in increasing it by at least 3 as we could then bring two of γ_3 into the solution in place of two of γ_1 and one of γ_2.

Even if we admit positive valuations on unsatisfied as well as satisfied capacities, it may still be impossible to arrive at a method of decision-making through pricing in a similar way to that described in Section 6.2 for LP. This is demonstrated by the following example.

Example 10.10

$$\begin{aligned} \text{Maximize} \quad & 4\gamma_1 + 3\gamma_2 + \gamma_3 \\ \text{subject to} \quad & 2\gamma_1 + 2\gamma_2 + \gamma_3 \le 7, \end{aligned} \tag{10.88}$$

where $\gamma_1, \gamma_2, \gamma_3 \ge 0$ and take integer values.

The optimal solution is $\gamma_1 = 3$, $\gamma_2 = 0, \gamma_3 = 1$. We will attempt to find a valuation for the constraint that will produce this answer.

Suppose we give the constraint a 'shadow price' (or accounting value) of π, then to make γ_3 profitable we must have $\pi \le 1$. This, however, implies $2\pi < 3$ making γ_3 also profitable. We know from the optimal solution that γ_3 is worth making but that γ_2 is not.

This example demonstrates a second difference between any economic valuation of constraints in IP in comparison with shadow prices in LP:

(B) For general IP problems, no valuations will necessarily exist for the constraints that allow the optimal solution to be obtained in a similar manner to the LP case.

By 'constraints' in (B) we must, as usual, exclude the feasibility constraints $\gamma_1 \ge 0$.

One way out of the dilemma posed by the IP model above is to obtain constraints representing the convex hull of feasible integer solutions. (For MIP models we would of course consider the convex hull of points with integer coordinates in the dimensions representing integer variables as mentioned in Section 10.1). The model can then be treated as an LP problem and shadow prices obtained with desirable properties. Although a procedure for obtaining the convex hull of integer solutions is computationally often impractical, as discussed in the last section, it can be applied to certain models making useful economic information possible. This is demonstrated on a particular type of problem, the *shared fixed cost* problem, in Example 10.12 below. In spite of the general computational difficulties, it is still worth considering this as a theoretical solution to our dilemma. For the purposes of explanation, we have reformulated the IP model in Example 10.9, using constraints for the convex hull of feasible integer points. This gives the model below.

Example 10.11

$$\text{Maximize} \quad 12\gamma_1 + 5\gamma_2 + 15\gamma_3$$

$$\text{subject to} \quad \gamma_1 + \quad + \gamma_3 \le 2, \tag{10.89}$$

$$\gamma_1 + \gamma_2 + \gamma_3 \le 3, \tag{10.90}$$

$$2\gamma_1 + \gamma_2 + 3\gamma_3 \le 5, \tag{10.91}$$

$$\gamma_3 \le 1, \tag{10.92}$$

where $\gamma_1, \gamma_2, \gamma_3 \geq 0$ and take integer values. (When it takes an integer value, γ_4 is clearly forced to be zero by constraint (10.87).)

The shadow prices on the new constraints are

constraint (10.89)	shadow price 2
constraint (10.90)	shadow price 0
constraint (10.91)	shadow price 5
constraint (10.92)	shadow price 0

Note that as Example 10.11 is degenerate, there are alternative shadow prices, as explained in Section 6.2.

Unfortunately, it is not clear how the new constraints (10.89)–(10.92) defining the convex hull of Example 10.9 can be related back to the original constraints (10.85)–(10.87). It is therefore difficult to apply the shadow prices above to give meaningful valuations for the physical constraints of the original model. An attempt has been made to do this by Gomory and Baumol (1960). They apply a cutting planes algorithm to the IP model, successively adding constraints until an integer solution is obtained. Then shadow prices are obtained for the original constraints and for the added constraints. The shadow prices for the added constraints are imputed back to the original constraints from which they were derived. Unfortunately, the valuations they obtain for the original constraints are not unique and depend on the way in which the cutting planes algorithm is applied. In addition, they have to include the feasibility conditions $\gamma_i \geq 0$ among their original constraints. As a result, these constraints may end up being given non-zero economic valuations. In LP, such valuations could be regarded as the reduced costs on the variables γ_i and no difficulty would arise. Variables with positive reduced costs would be out of the optimal solution. With IP using this procedure, it would be perfectly possible for the feasibility condition $\gamma_i \geq 0$ to have non-zero economic values (suggesting that in some sense $\gamma_i \geq 0$ was a 'binding' constraint) and for γ_i to be in the optimal solution at a positive level. One way of justifying this would be to regard the economic valuation given to the feasibility constraint as a cost associated with the indivisibility of γ_i. Not only should γ_i be charged according to the use it makes of scarce capacities, it should also be charged an extra amount in view of the fact it can only come in integral quantities.

The fact that a constraint such as $\gamma_i \geq 0$ may have a non-zero valuation in the Gomory–Baumol system, yet γ_i may not be at zero level is a special case of a more general difference between the Gomory–Baumol prices in IP and the shadow prices in LP:

(C) A free good as represented by a constraint in IP does not necessarily have a zero Gomory–Baumol price attached to it.

This difference is not as serious as it might first seem as a similar situation can happen in LP. We saw in Section 6.2 that with degeneracy there are alternate dual solutions, some of which may give non-zero shadow prices to (alternatively) redundant constraints. This also indicates that the problem of non-uniqueness in the Gomory–Baumol prices is not confined to IP. It clearly also happens, although to a much less serious extent, with degenerate problems in LP.

The exact way of obtaining limited economic information from an MIP model that is used quite widely in practice should be mentioned. This is simply to take this information from the LP sub-problem at the node in the solution tree that gave the optimal integer solution. Such information may well be unreliable as the integer variables should only change by discrete amounts while the economic information results from the effect of marginal changes. Other integer variables (particularly 0–1 variables) will have become fixed by the bounds imposed in the course of evaluating the solution tree and it will not be possible to evaluate the effect of any changes on these variables.

A variation of this way of obtaining economic information from an MIP model is to 'fix' all integer variables at their optimal values and only consider the effect of marginal changes on the continuous variables. This procedure has something to recommend it as the integer variables usually represent major operating decisions. Given that these decisions have been accepted, the economic effects of marginal changes within the basic operating pattern may be of interest.

An example of the need to 'value' the constraints of an MIP model is provided by the TARIFF RATES (POWER GENERATION) problem in Part II. The rates at which electricity is sold on different tariffs implicitly value the constraints. Different ways of doing this are discussed with the solution of the model in Part IV.

In spite of the difficulties in getting meaningful subsidiary economic information out of an IP model, there are circumstances where useful information can be obtained by reformulation of the model. This is discussed by Williams (1981). We illustrate this in the example below by reformulating a model using the ideas contained in Section 10.2. The problem considered involves *shared fixed costs* and is described by Rhys (1970), to whom these ideas are due.

Example 10.12

In the network of Figure 10.3, the nodes represent capital investments with the costs associated with them. The arcs represent money-making activities with the estimated revenues associated with them. To carry out any activity (arc), it is necessary to use both the resources represented by the nodes at either end of the arc. The problem is to share the capital investment (fixed) cost associated with a node in some optimal way among the activities associated with the arcs joining the node, for example, how should the capital cost of node D be shared among the arcs AD, BD, CD, and DE? An illustrative way of viewing this problem is to think of the nodes as stations with the arcs as railways between those stations.

In order to show how the required economic information can arise through reformulating an IP model, we first consider a rather different problem.

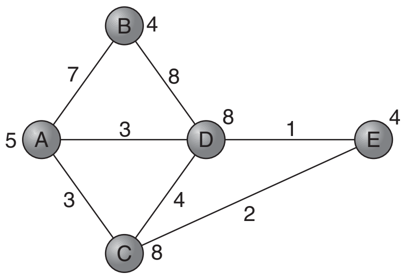


Figure 10.3

Suppose we wish to decide which nodes (stations) to cut in order to make the whole network (railway system) as profitable as possible. It must be borne in mind that cutting out a node (station) necessitates cutting out all the arcs (lines) leading into it. This problem can easily be formulated as an IP problem using the following 0–1 variables:

$$\delta_i = \begin{cases} 1 & \text{if node } i \text{ is kept,} \\ 0 & \text{if node } i \text{ is cut out;} \end{cases}$$

$$\delta_{ij} = \begin{cases} 1 & \text{if arc } (ij) \text{ is kept,} \\ 0 & \text{if arc } (ij) \text{ is cut;} \end{cases}$$

i,j are A, B, C, D, and E.

The objective to be maximized is

$$-5\delta_{\text{A}} - 4\delta_{\text{B}} - 8\delta_{\text{C}} - 8\delta_{\text{D}} - 4\delta_{\text{E}} + 7\delta_{\text{AB}} + 3\delta_{\text{AC}} + 3\delta_{\text{AD}} + 8\delta_{\text{BD}} + 4\delta_{\text{CD}} + \delta_{\text{DE}} + 2\delta_{\text{CE}}. \quad (10.93)$$

The conditions to be modelled are that certain arcs require certain nodes, that is,

$$\delta_{ij} = 1 \rightarrow \delta_i = 1, \ \ \delta_j = 1. \quad (10.94)$$

We saw in Section 10.1 that such a condition may be modelled two ways using either one or two constraints as follows:

$$-\delta_i - \delta_j + 2\delta_{ij} \leq 0 \quad (10.95)$$

or

$$\begin{aligned} -\delta_i \quad\quad + \delta_{ij} &\leq 0, \\ -\delta_i + \delta_{ij} &\leq 0. \end{aligned} \quad (10.96)$$

The second formulation has the advantage that the model will be totally unimodular and can be solved as an LP problem yielding an integer optimal solution.

Geometrically, we have specified the convex hull of feasible integer points by the constraints (10.96). As we now have an LP problem, we obtain well-defined shadow prices on the constraints. In this example, the shadow prices have the following interpretation.

The shadow price on $-\delta_i + \delta_{ij} \leq 0$ is the amount of the capital cost of node i that should be met by revenue from arc (ij).

Similarly the shadow price on $-\delta_i + \delta_{ij} \leq 0$ is the amount of the capital cost of node j that should be met by revenue from arc (ij).

We have clearly found a way of sharing the capital costs of the nodes among the arcs. Should any activity not be able to meet the capital cost demanded of it, it should be cut out. This allocation of capital costs will be such as to lead to the most profitable network. Using the numbers given on the network in Figure 10.3, the following shadow prices result as shown in Table 10.1.

Table 10.1

Constraint	Shadow Price
$-\delta_{\mathrm{A}} + \delta_{\mathrm{AB}} \leq 0$	5
$-\delta_{\mathrm{B}} + \delta_{\mathrm{AB}} \leq 0$	2
$-\delta_{\mathrm{A}} + \delta_{\mathrm{AC}} \leq 0$	0
$-\delta_{\mathrm{C}} + \delta_{\mathrm{AC}} \leq 0$	3
$-\delta_{\mathrm{A}} + \delta_{\mathrm{AD}} \leq 0$	0
$-\delta_{\mathrm{D}} + \delta_{\mathrm{AD}} \leq 0$	3
$-\delta_{\mathrm{B}} + \delta_{\mathrm{BD}} \leq 0$	2
$-\delta_{\mathrm{D}} + \delta_{\mathrm{BD}} \leq 0$	6
$-\delta_{\mathrm{C}} + \delta_{\mathrm{CD}} \leq 0$	4
$-\delta_{\mathrm{D}} + \delta_{\mathrm{CD}} \leq 0$	0
$-\delta_{\mathrm{C}} + \delta_{\mathrm{CE}} \leq 0$	0
$-\delta_{\mathrm{E}} + \delta_{\mathrm{CE}} \leq 0$	2
$-\delta_{\mathrm{D}} + \delta_{\mathrm{DE}} \leq 0$	0
$-\delta_{E} + \delta_{\mathrm{DE}} \leq 0$	1

It will be seen that, for example, node C will receive 3 from AC, 4 from CD and 0 from CE. Clearly node C is no longer viable and must be cut. Applying similar arguments to all the other nodes, we are left with the optimal network shown in Figure 10.4.

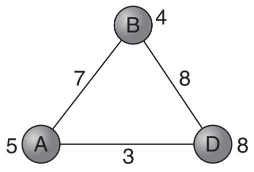


Figure 10.4

An interesting observation following from duality in LP is that dividing up the capital costs of the nodes in other ways among the arcs could not lead to a more profitable network and could well lead to a less profitable one.

It should have become apparent from all the discussion in this section that there is no generally satisfactory way of getting the subsidiary economic information from an IP model that often proves so valuable in the case of LP. This topic represents a considerable gap in mathematical programming theory. The subject is more fully discussed in Williams (1979, 1997). Appa (1997) shows that after fixing the values of integer variables in a mixed integer programming model, the structural duality of the resulting LP model can sometimes be used. Greenberg (1998) gives a bibliography.

10.4 Sensitivity analysis and the stability of a model

We saw in Section 6.3 that having built and solved an LP model, it was very important to see how sensitive the answer was to changes in the data from which the model was constructed. Using ranging procedures on the objective and right-hand side coefficients, some insight into this could be obtained. In addition, it was shown how a model could, to some extent, be built in order to behave in a 'stable' fashion. Our considerations here are exactly the same as those in Section 6.3, only they concern IP models. As in the Section 10.3, we will see that IP models present considerable extra difficulties. This section falls into two parts. Firstly, we consider ways of testing the sensitivity of the solution of an IP model. Secondly, we consider how models may be built in order that they may exhibit stability.

10.4.1 Sensitivity analysis and integer programming

A theoretical way of doing sensitivity analysis on the objective coefficients of a model would be to replace the constraints by constraints representing the convex hull of feasible integer points. The model could then be treated as an LP model and objective ranging performed as described in Section 6.3.

For those PIP models where a reformulation easily yields the constraints for the convex hull, this is fairly straightforward. Otherwise, it is not a practical way of approaching the problem. Nor does it give a way of performing right-hand side ranging.

For MIP models solved by the branch and bound method, a sensitivity analysis can be performed on the LP sub-problem at the node giving the optimal integer solution. Alternatively, the integer variables can be fixed at their optimal values and a sensitivity analysis performed on the continuous part of the problem. These approaches clearly have the same drawbacks as those apparent when using similar approaches to derive economic information from an IP model as described in Section 10.3.

The only really satisfactory method of sensitivity analysis in IP involves solving the model again with changed coefficients and comparing optimal solutions. Obviously, the subsequent time to solve the model should be able to be reduced by exploiting the knowledge of the previous solution.

10.4.2 Building a stable model

In an LP model, the optimal value of the objective function varies continuously as the right-hand side and objective coefficients are changed. In an IP model, this may not happen. We consider the following very simple example.

Example 10.13

$$\begin{aligned} \text{Maximize} \quad & 40\delta_1 + 35\delta_2 + 15\delta_3 + 8\delta_4 + 9\delta_5 \\ \text{subject to} \quad & 8\delta_1 + 8\delta_2 + 5\delta_3 + 4\delta_4 + 3\delta_5 \leq 16. \end{aligned} \tag{10.97}$$

where δ_1, δ_2, δ_3, δ_4, δ_5 are 0–1 variables.

The optimal solution to this model is $\delta_1 = 1$ and $\delta_2 = 1$, giving an objective value of 75.

If, however, the right-hand side value of 16 is reduced by a small amount, the optimal solution changes to $\delta_1 = 1, \delta_4 = 1$ and $\delta_5 = 1$, giving an objective value of 57.

The optimal value of the objective function is obviously not a continuous function of the right-hand side coefficient. In many practical situations, this is unrealistic. Suppose constraint (10.97) represented a budgetary limitation and the 0–1 variables referred to capital investments. It is very unlikely that a small decrease in the budget would cause us radically to alter our plans. A more likely occurrence is that the budget would be stretched slightly at some increased cost or the cost of one of the capital investments would be trimmed slightly. It is important that if this is the case this should be represented in the model. As it stands, the above example represents a poor model of the situation.

One way to remodel constraint (10.97) is to use the device described in Section 3.3 in order to allow constraints to be violated at a certain cost. A surplus (continuous) variable u is added into constraint (10.97) and given a cost (say 20) in the objective. This results in the following model:

$$\begin{array}{ll} \text{Maximize} & 40\delta_1 + 35\delta_2 + 15\delta_3 + 8\delta_4 + 9\delta_5 - 20u \\ \text{subject to} & 8\delta_1 + 8\delta_2 + 5\delta_3 + 4\delta_4 + 3\delta_5 - u \leq 16. \end{array} \tag{10.98}$$

For a right-hand side of 16, the optimal solution is still $\delta_1 = \delta_2 = 1$, giving an objective value of 75.

If the right-hand side value of 16 is reduced slightly, the effect of u will be to 'top the budget up' to 16 at a unit cost of 20. For example, if the right-hand side falls to $15\frac{1}{2}$, u will become $\frac{1}{2}$ We will retain the same optimal solution but the objective will fall by 10 to 65. As the right-hand side is further reduced, the optimal value of the objective will continue gradually to fall until we reach a right-hand side value of $15\frac{1}{10}$. We will then have the solution $\delta_1 = 1$, $\delta_4 = 1$ and $\delta_5 = 1$ as an alternative optimum. If the right-hand side is further reduced, we will transfer to this alternative optimum. It can be seen that this device of adding a surplus variable to the problem with a certain cost has two desirable effects on the model:

1. The optimal objective value becomes a continuous function of the right-hand side coefficient.
2. The optimal solution values do not change 'suddenly' as the right-hand side coefficient changes. They are said to be 'semi-continuous' functions of the right-hand side.

In some applications, we might wish to add a slack (continuous) variable v as well; giving v a cost in the objective (of say 8). This would result in the following model:

$$\begin{array}{ll} \text{Maximize} & 40\delta_1 + 35\delta_2 + 15\delta_3 + 8\delta_4 + 9\delta_5 - 20u - 8v \\ \text{subject to} & 8\delta_1 + 8\delta_2 + 5\delta_3 + 4\delta_4 + 3\delta_5 - u + v = 16. \end{array} \tag{10.99}$$

This topic is not pursued further here as it has been fully covered for LP problems in Section 6.3. It is important to notice, however, the desirability of forcing the optimal solution of an IP model to vary 'continuously' with the data coefficients. Clearly, for many logical type constraints which appear in MIP models, it is meaningless to use a device such as that above. There are some classes of MIP models where the continuity property can be shown to hold without further reformulation. This subject is discussed in greater depth by A. C. Williams (1989).

10.5 When and how to use integer programming

In this section we try, very briefly, to summarize some of the points made in the preceding three chapters as a quick guide to using IP.

1. If a practical problem has any of the characteristics described in Section 8.2, it is worth considering the use of an IP model.

2. Before a decision is made to build an IP model, an estimate should be made of the potential size. If the number of integer variables is more than a few hundred, then, unless the problem has a special structure, it is probable that IP will be computationally too costly.

3. A close examination of the structure of the IP model that would result from the problem is always worthwhile. Should the model be a PIP model and have a totally unimodular structure, then LP can be used and models involving thousands of constraints and variables solved in a reasonable period of time. If the structure is a PIP model but not totally unimodular it is worth seeing if it can easily be transformed into a known totally unimodular structure, as described in Section 10.2. For MIP models or models where there is no apparent way of producing a unimodular structure, it is often possible to constrain the corresponding LP problem more tightly. If it is apparent that the IP model will have one of the other special structures mentioned in Section 9.5, it is worth examining the literature and computational experience with the class of problem to get an impression of the difficulty.

4. Before embarking on the full-scale model, it is worth building a model for a much smaller version of the problem. Experiments should be performed on this model to find out how easy it is to solve. If necessary, reformulations such as those described in Section 10.2 should be tried. Different solution strategies as mentioned in Section 8.3 should also be experimented with.

5. If the problem appears too difficult to solve as an IP model after carrying out the above investigations, some *heuristic* method will have to be used. Much literature exists on different heuristic algorithms for operational research problems but this topic is beyond the scope of this book. For an apparently difficult problem where an IP model still seems worthwhile, it may also be worth some time being spent on a heuristic approach to get a fairly good, though probably not optimal, solution. This good solution can then be exploited in the tree search as a cut-off value for the objective function as described in Section 8.3.

6. Having built an IP model, it is very important to use an intelligent solution strategy, using, if possible, one's knowledge of the practical problem. This has been mentioned briefly in Section 8.3 but is, in the main, beyond the scope of a book on model building. An extremely good description of this subject is given by Forrest *et al*. (1974).

Finally, it should be pointed out that theoretical and computational progress in IP is being made all the time, making it possible to solve larger and more complex models.

11

The implementation of a mathematical programming system of planning

11.1 Acceptance and implementation

Most of this book has been concerned with the problems of formulating and interpreting the solution of mathematical programming models. There is, however, generally another phase to be gone through before the solution of a model influences the making of real decisions. This final phase is that of gaining acceptance for, and implementing the solution. Many people who have been involved with all the phases of formulating a model, solving it, interpreting the solution, gaining acceptance for the solution, and then implementing it have found the last two phases to be the most difficult. In some cases, they may have stumbled fatally at this point. There are a number of lessons to be learnt from such experiences that are considered in this chapter. Obviously, the problem of acceptance and implementation will depend on the type of organization involved as well as the type of application. It is useful here to classify mathematical programming models for planning into short-, medium- and long-term planning models.

Short-term planning models may be simply 'one off' models used to decide the answer to a specific question, for example, do we build a new factory or not, what is the optimum design for this communications network? If these questions are unlikely to recur then there is generally little to be said about the acceptance and implementation. The solution produced by the model is either used or not used. For other short-term planning questions that do arise regularly at daily or weekly intervals, there may be an implementation problem. Firstly, it should be pointed out that in some cases a mathematical programming model may be applicable but unworkable. For example, a complicated distribution or job-shop scheduling problem may be essentially 'dynamic'. Changes may be taking place all the time, new orders may be coming in and machines may be breaking down. It might be impossible to define a once and for all schedule. To adapt such a schedule to each change might involve a rerun of the model. Unless the changes were infrequent and the model comparatively quick to solve, such an approach would be impossible. In such cases, special purpose adaptive (and probably non-optimal) quick decision rules would probably be used. Some short-term decision problems that occur regularly can often be tackled through a mathematical programming model. Day to day blending problems, for example, may be of this nature. Sometimes fertilizer suppliers or food manufacturers run small linear programming models for individual orders or blends. In such cases acceptance of the method must have already been achieved. An organizational problem will also have had to be solved probably by automating the conversion of the data into a standard type of model for quick solution by a computer (probably through a terminal). The use of such short-term operational models creates few other implementation problems because once accepted their use is fairly straightforward.

Medium-term planning is usually considered to involve periods of a month up to one or two years. It is for these problems that mathematical programming has been most widely used to date. Once a medium-term planning model has been implemented and is being used regularly, it may be incorporated into, or form a starting point for, a *longer term planning* model typically looking up to about six years ahead. The problems of getting acceptance of such models are similar but usually more acute in the case of longer term planning. Both are considered together.

Much has been written on the more general problem of how to gain acceptance for and ensure the implementation of the results of any operational research study. A clear and useful account of the considerations that should be made is given by Rivett (1968). The main lesson to be learnt is to involve the potential decision makers in a model building project at an early stage. If they have lived with the problems of defining and building a model they will be much more likely to accept and understand its final use than if it is sprung on them at a late stage. Early involvement by top management can have its dangers. It is possible that the development of a model may get bogged down in detailed technical arguments between managers who would normally never be concerned with such technicalities. More seriously, a model building project might be rejected at an early stage through disagreement over detail. These risks are, however, normally worth taking in comparison with the risk of rejection at the final hurdle. Involving top management will often be by no means easy. They will need to understand the potentialities of a mathematical programming model but will probably lack both detailed technical knowledge as well as detailed departmental knowledge of some aspects of their company. Some of this detail may well need to be incorporated into the model. The solution is probably to involve one or two such managers in the model building project as well as give regular presentations to the others. There is then of course another danger to be avoided. It is important not to oversell the model. If it is thought that the model will answer every question, then later, disillusion may be in store.

Many people have found that a new mathematical programming model has gained acceptance when the answers which it produces confirm a decision that has already been made. For example, an important investment decision may have been made in some other way. If the answer to the model confirms this, the ultimate regular use of mathematical programming may be assured.

It has already been suggested that sometimes the very exercise of building a model leads to the explicit recognition of relationships that were not realized before. In such circumstances the modelling exercise may be as valuable as the answers produced. When this happens the effect on management may well be to make the use of mathematical programming more acceptable.

The ultimate acceptance of a regular mathematical programming system of planning usually requires organizational changes that are considered in the next section.

A very full and useful description of the experiences in developing and gaining acceptance for a large long-term planning mathematical programming model in British Petroleum is described by Stewart (1971).

An analysis of the factors that make for success or failure in implementing the results of corporate models is given by Harvey (1970). On the result of a survey he isolates the characteristics of both management and decision problems that lead to success or failure in the implementation of a model. Problems of implementation are also discussed very fully in the book by Miller and Starr (1960).

A perceptive analysis of three industrial problems where linear programming has yielded disappointing results is given by Corner (1979). He draws conclusions and presents advice on the basis of these experiences.

Finally, as was mentioned in Section 2.3, it seems quite likely that mathematical programming methods will be incorporated in much larger pieces of software known as *decision support systems*. The resultant systems will usually be tailor-made to specific applications. In this way much of the effort of model building will be transferred to the computer system. It will then be necessary to include model building expertise in the designing and writing of such systems.

11.2 The unification of organizational functions

One off the virtues of a corporate planning model is that many interconnections between different departments and functions in an organization have to be represented explicitly. The obvious example is production and marketing. In many manufacturing industries these two functions are kept very separate. The result is sometimes a divergence of objectives. Production may be trying to satisfy orders with a reasonable level of productivity. Marketing may be trying to maximize the total volume of sales rather than concentrating on those that make the greatest contribution to profit. One of the greatest virtues of a product mix type of model is that it forces marketing to take account of production costs. The result is almost always a reduction in the range of products produced to those that can be produced most efficiently. In practice with this type of model the incorporation of the marketing function can present difficulties. As was suggested in Section 3.2 it may sometimes be politically more acceptable to first of all leave out the marketing aspect. A model is built simply to meet all market estimates at minimum production cost. When this type of model has come into regular use it can fairly easily be extended also to decide quantities to be marketed, taking into account their profit contributions. The marketing division of a company is often less able to quantify its data and more reluctant to accept the answers produced by a model. Indeed, their individual objective may be at variance with that of the company. Stewart (1971) gives a case history of a linear programming model built for the Gas Council, which was considered to be of little use for marketing.

In order to give an idea of the different company functions that can be incorporated in a mathematical programming model, Figure 11.1 demonstrates how purchasing, operating and planning functions can become involved in a model.

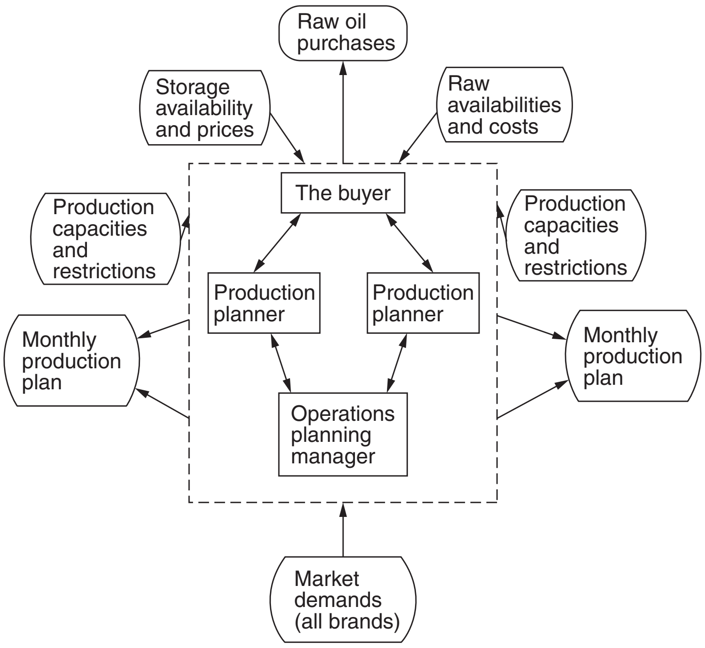

Raw oil purchases
Storage availability and prices
Raw availabilities and costs
The buyer
Production capacities and restrictions
Production capacities and restrictions
Production planner
Production planner
Monthly production plan
Monthly production plan
Operations planning manager
Market demands (all brands)

Figure 11.1

The model here is described by Williams and Redwood (1974) and takes in all the functions in the box enclosed by a dotted line. It is a multi-period, multibrand model for aiding decisions of the purchase of edible oils on the commodity market and their blending together into foods. The involvement of different functions and departments in the building and use of a model such as this can force some degree of centralization on an organization. This aspect is considered in Section 11.3.

While the explicit recognition of interrelationships in an organization through a model is desirable, it usually makes extra coordination necessary for the supply of data to, and the use of, a model. Often, this requires the creation of a special job or even a small department. Such a coordinating role is often effectively done through the management accounting department. They are one of the few departments with an overview of the whole organization.

One of the possibly contentious results of decisions, which may be reached through a corporate model, should be recognized. Because such a model will have a corporate objective, this may conflict within the objectives of individual departments. It has already been pointed out that a marketing division may force them to reduce their product range and therefore their total volume in the interests of greater company profit. The disputes that may result in consequence are added reason for obtaining top management backing for this method of planning. In Section 4.1 it was demonstrated through a very simple example how a corporate objective could result in an individual factory having its individual profit reduced in the interests of overall company profit. Again the possible political implications are obvious.

Although there may be obvious difficulties in getting different departments to work more closely together, there can be advantages resulting from a planning model. The combination of a large amount of data and information in one computer model can lead to greater convenience. Each department can be given the relevant portions of a computer run. Problems of communication and incompatibility of information will be reduced in consequence. One of the by-products of a corporate model is usually greater contact between departments. Each department will want to know what information other departments have fed into the model, because this may affect the suggested plans for their own department.

The correct use of a multi-period model for planning has already been mentioned in Section 4.1. Such a model may well provide operating information for the current time period as well as provisional information for future periods. Such a model would generally be rerun at least once in each period, giving the same combination of operating information and planning information based on the best available future estimates that have been fed into the model.

11.3 Centralization versus decentralization

A corporate model obviously brings into its orbit many aspects of an organization, which might normally be separated into different divisions. Sometimes these divisions may be separated geographically. In some respects the resulting greater centralization of planning may seem desirable. The small example of Section 4.1 demonstrated how individual factories in a company might produce sub-optimal plans when working independently. In the last section, it was pointed out how the recognition of the interdependence of different departments in an organization often results from a model. Although many of these features are to some degree desirable, when carried to excess they may be far from desirable. In fact, the increasing disadvantages of greater and greater centralization have been a major factor in discouraging the continual growth in the size of mathematical programming models.

In theory, decomposition, which was discussed in Section 4.2, offers a way out by avoiding the need to include all the details of an organization in one total model. Unfortunately, the computational problems of using a decomposition method in general have not yet been fully resolved. At present, it cannot be considered to be a sufficiently developed technique to be always usable in practice. Moreover, mathematical methods of decomposition do not always correspond to acceptable decentralized planning methods. This aspect of the subject has been discussed by Atkins (1974).

Stewart (1971) rather surprisingly suggests that the use of a long-term planning model in British Petroleum led to some degree of decentralization. In particular, it was then possible for a division to obtain complete information before making a plan themselves rather than the information being restricted to head office. This enabled the division to take on more responsibility. On the other hand, the division now had to work within some centrally defined objective rather than choose its own objective. People now, however, felt that they were working to an independent system, in the form of a model, rather than being simply dictated to by head office. This, to some extent, made centralization more acceptable.

As was pointed out in Chapter 6, a wealth of information can be obtained from the solution to a linear programming model. For large corporate models this excess of information can create difficulties. It is very important to ensure that departments only get relevant information; otherwise, they can be submerged. The use of an automatic report writer as discussed in Section 6.5 is obviously desirable. Such a report writer can be designed to produce the information in different sections for the use of different departments.

It is necessary to strike a correct balance between the model builders, the contributors of data and the managers who use the results. There is a great danger, in a large organization, of them losing touch with one another. Relevant data may not be used at the right time and inaccurate results may be obtained. Ideally, management will become accustomed to using a model on a 'what if ... ' basis. They will ask questions which the model builder will be able to answer quickly by new runs or post-optimal analysis.

The degree of centralization or decentralization, which the use of a model should be allowed to impose on an organization, must obviously depend very much on the organization and its ultimate objectives. In consequence, few general guidelines can be given. The recognition of the problem of centralization is, however, important. It is interesting that mathematical programming was used quite widely in the former communist countries for national economic planning. In the West, this is far less common. Models are usually limited to the level of individual companies.

11.4 The collection of data and the maintenance of a model

The collection of data is a major task in building a large mathematical programming model for the first time. Once this has been done, the organization should be geared to providing and updating the data regularly. Inevitably, much work will be involved in collecting suitable data. Coming from different departments, it will often need to be standardized. Again such standardization and coordination will ultimately need to be the responsibility of a special person or department if the model is to be used regularly. As has already been pointed out, a management accounting department is often a suitable place to which to entrust this responsibility.

The person entrusted with the regular collection and standardization of the data must be in a position where he/she is kept up to date with all changes. These changes will come from different departments. There may be marketing estimates, changes in productive capacity, technological changes in production processes, and changes in raw materials costs. Ensuring that all this information gets incorporated in the model and it remains an up-to-date representation is no mean task. It may sometimes be necessary to do preliminary processing of the data. For example, sales estimates may be the result of a regular forecasting exercise. This may require preliminary computation. It may be necessary to convert costing data to a suitable form, for example, to ensure that only variable costs are used. With some packages it is now possible to prepare, manipulate and present the data though *spreadsheets*. The solutions are sometimes conveniently presented in this form also.

The use and regular updating of *databases* to include all relevant statistical information is of value when used in conjunction with a mathematical programming model. Such databases may also be used for other purposes as well within an organization. It is important, however, to ensure that their organization and design are compatible with the mathematical programming model used. Most commercial package programs for mathematical programming can be incorporated as part of a much larger computer software system (e.g. a *decision support system*) using a database.

As with so many computer applications their use does not reduce the need for employees; instead, it changes the type of employee required and sometimes increases the total number of people. It should be obvious that the problems of collecting data and maintaining a model deserve considerable attention and effort if its use is to be effective. The gain should come from the wider range of policy options that can be considered as well as their greater reliability.

Part II

12

The problems

There is no significance in the order in which the following 29 problems are presented. Some are easy to formulate and present no computational difficulties in solution, while others are more difficult in either or both of these respects. Some can be solved with linear programming; others require integer programming or separable programming.

It is suggested that readers attempt to formulate those problems that interest them before consulting the formulations and solutions proposed in Parts III and IV. A computer package may be used on the model, or an intuitive (heuristic) approach to some of the problems may be attempted, using original methods of the reader's own choosing; answers can then be compared with the optimal solutions given in Part IV.

Readers wishing to follow the recommended course of solving the models using a computer package are strongly advised to use a matrix generator/language, as discussed in Sections 3.5 and 4.3. This will enable them to concentrate on the *structure* of the model as well as facilitating error detection and greatly reducing data preparation.

12.1 Food manufacture 1

A food is manufactured by refining raw oils and blending them together. The raw oils are of two categories:

Vegetable oils	VEG 1
	VEG 2
Non-vegetable oils	OIL 1
	OIL 2
	OIL 3

Each oil may be purchased for immediate delivery (January) or bought on the futures market for delivery in a subsequent month. Prices at present and in the futures market are given below in (£/ton):

	VEG 1	VEG 2	OIL 1	OIL 2	OIL 3
January	110	120	130	110	115
February	130	130	110	90	115
March	110	140	130	100	95
April	120	110	120	120	125
May	100	120	150	110	105
June	90	100	140	80	135

The final product sells at £150 per ton.

Vegetable oils and non-vegetable oils require different production lines for refining. In any month, it is not possible to refine more than 200 tons of vegetable oils and more than 250 tons of non-vegetable oils. There is no loss of weight in the refining process, and the cost of refining may be ignored.

It is possible to store up to 1000 tons of each raw oil for use later. The cost of storage for vegetable and non-vegetable oil is £5 per ton per month. The final product cannot be stored, nor can refined oils be stored.

There is a technological restriction of hardness on the final product. In the units in which hardness is measured, this must lie between 3 and 6. It is assumed that hardness blends linearly and that the hardnesses of the raw oils are

VEG 1	8.8
VEG 2	6.1
OIL 1	2.0
OIL 2	4.2
OIL 3	5.0

What buying and manufacturing policy should the company pursue in order to maximise profit?

At present, there are 500 tons of each type of raw oil in storage. It is required that these stocks will also exist at the end of June.

This problem, and the subsequent problem, is based on a larger model built for the margarine producer Van den Bergs and Jurgens and discussed in Williams and Redwood (1974).

12.2 Food manufacture 2

It is wished to impose the following extra conditions on the food manufacture problem:

1. The food may never be made up of more than three oils in any month.
2. If an oil is used in a month, at least 20 tons must be used.
3. If either of VEG 1 or VEG 2 are used in a month then OIL 3 must also be used.

Extend the food manufacture model to encompass these restrictions and find the new optimal solution.

12.3 Factory planning 1

An engineering factory makes seven products (PROD 1 to PROD 7) on the following machines: four grinders, two vertical drills, three horizontal drills, one borer and one planer. Each product yields a certain contribution to profit (defined as £/unit selling price minus cost of raw materials). These quantities (in £/unit) together with the unit production times (hours) required on each process are given below. A dash indicates that a product does not require a process.

	PROD 1	PROD 2	PROD 3	PROD 4	PROD 5	PROD 6	PROD 7
Contribution to profit	10	6	8	4	11	9	3
Grinding	0.5	0.7	–	–	0.3	0.2	0.5
Vertical drilling	0.1	0.2	–	0.3	–	0.6	–
Horizontal drilling	0.2	–	0.8	–	–	–	0.6
Boring	0.05	0.03	–	0.07	0.1	–	0.08
Planing	–	–	0.01	–	0.05	–	0.05

In the present month (January) and the five subsequent months, certain machines will be down for maintenance. These machines will be as follows:

January	1 Grinder
February	2 Horizontal drills
March	1 Borer
April	1 Vertical drill
May	1 Grinder and 1 Vertical drill
June	1 Planer and 1 Horizontal drill

There are marketing limitations on each product in each month. These are given in the following table:

	1	2	3	4	5	6	7
January	500	1000	300	300	800	200	100
February	600	500	200	0	400	300	150
March	300	600	0	0	500	400	100
April	200	300	400	500	200	0	100
May	0	100	500	100	1000	300	0
June	500	500	100	300	1100	500	60

It is possible to store up to 100 of each product at a time at a cost of £0.5 per unit per month. There are no stocks at present, but it is desired to have a stock of 50 of each type of product at the end of June.

The factory works a six days a week with two shifts of 8 h each day.

No sequencing problems need to be considered.

When and what should the factory make in order to maximise the total profit? Recommend any price increases and the value of acquiring any new machines.

N.B. It may be assumed that each month consists of only 24 working days.

This problem, and the subsequent problem, is based on a larger model built for the Cornish engineering company of Holman Brothers (which no longer exists).

12.4 Factory planning 2

Instead of stipulating when each machine is down for maintenance in the factory planning problem, it is desired to find the best month for each machine to be down.

Each machine must be down for maintenance in one month of the six apart from the grinding machines, only two of which need be down in any six months.

Extend the model to allow it to make these extra decisions. How much is the extra flexibility of allowing down times to be chosen worth?

12.5 Manpower planning

A company is undergoing a number of changes that will affect its manpower requirements in future years. Owing to the installation of new machinery, fewer unskilled but more skilled and semi-skilled workers will be required. In addition to this, a downturn in trade is expected in the next year, which will reduce the need for workers in all categories. The estimated manpower requirements for the next three years are as follows:

	Unskilled	Semi-skilled	Skilled
Current strength	2000	1500	1000
Year 1	1000	1400	1000
Year 2	500	2000	1500
Year 3	0	2500	2000

The company wishes to decide its policy with regard to the following over the next three years:

1. Recruitment
2. Retraining
3. Redundancy
4. Short-time working.

There is a natural wastage of labour. A fairly large number of workers leave during their first year. After this, the rate is much smaller. Taking this into account, the wastage rates can be taken as follows:

	Unskilled (%)	Semi-skilled (%)	Skilled (%)
Less than one year's service	25	20	10
More than one year's service	10	5	5

There has been no recent recruitment and all workers in the current labour force have been employed for more than one year.

12.5.1 Recruitment

It is possible to recruit a limited number of workers from outside. In any one year, the numbers that can be recruited in each category are as follows:

Unskilled	Semi-skilled	Skilled
500	800	500

12.5.2 Retraining

It is possible to retrain up to 200 unskilled workers per year to make them semi-skilled. This costs £400 per worker. The retraining of semi-skilled workers to make them skilled is limited to no more than one quarter of the skilled labour force at the time as some training is done on the job. Retraining a semi-skilled worker in this way costs £500.

Downgrading of workers to a lower skill is possible but 50% of such workers leave, although it costs the company nothing. (This wastage is additional to the 'natural wastage' described above).

12.5.3 Redundancy

The redundancy payment to an unskilled worker is £200 and to a semi-skilled or skilled worker is £500.

12.5.4 Overmanning

It is possible to employ up to 150 more workers over the whole company than are needed, but the extra costs per employee per year are as follows:

Unskilled	Semi-skilled	Skilled
£1500	£2000	£3000

12.5.5 Short-time working

Up to 50 workers in each category of skill can be put on short-time working. The cost of this (per employee per year) is as follows:

Unskilled	Semi-skilled	Skilled
£500	£400	£400

An employee on short-time working meets the production requirements of half a full-time employee.

The company's declared objective is to minimise redundancy. How should they operate in order to do this?

If their policy were to minimise costs, how much extra would this save? Deduce the cost of saving each type of job each year.

12.6 Refinery optimisation

An oil refinery purchases two crude oils (crude 1 and crude 2). These crude oils are put through four processes: distillation, reforming, cracking and blending, to produce petrols and fuels that are sold.

12.6.1 Distillation

Distillation separates each crude oil into fractions known as *light naphtha*, *medium naphtha*, *heavy naphtha*, *light oil*, *heavy oil* and *residuum* according to their boiling points. Light, medium and heavy naphthas have octane numbers of 90, 80 and 70, respectively. The fractions into which one barrel of each type of crude splits are given in the following table:

	Light naphtha	Medium naphtha	Heavy naphtha	Light oil	Heavy oil	Residuum
Crude 1	0.1	0.2	0.2	0.12	0.2	0.13
Crude 2	0.15	0.25	0.18	0.08	0.19	0.12

N.B. There is a small amount of wastage in distillation.

12.6.2 Reforming

The naphthas can be used immediately for blending into different grades of petrol or can go through a process known as *reforming*. Reforming produces a product known as *reformed gasoline* with an octane number of 115. The yields of reformed gasoline from each barrel of the different naphthas are given as follows:

1 barrel of light naphtha yields 0.6 barrels of reformed gasoline;
1 barrel of medium naphtha yields 0.52 barrels of reformed gasoline;
1 barrel of heavy naphtha yields 0.45 barrels of reformed gasoline.

12.6.3 Cracking

The oils (light and heavy) can either be used directly for blending into *jet fuel* or *fuel oil* or be put through a process known as *catalytic cracking*. The catalytic cracker produces *cracked oil* and *cracked gasoline*. Cracked gasoline has an octane number of 105.

1 barrel of light oil yields 0.68 barrels of cracked oil and 0.28 barrels of cracked gasoline;
1 barrel of heavy oil yields 0.75 barrels of cracked oil and 0.2 barrels of cracked gasoline.

Cracked oil is used for blending *fuel oil* and *jet fuel*; cracked gasoline is used for blending *petrol*.

Residuum can be used for either producing *lube-oil* or blending into *jet fuel* and *fuel oil*:

1 barrel of residuum yields 0.5 barrels of lube-oil.

12.6.4 Blending

12.6.4.1 Petrols (motor fuel)

There are two sorts of petrol, *regular* and *premium*, obtained by blending the naphtha, reformed gasoline and cracked gasoline. The only stipulations concerning them are that regular must have an octane number of at least 84 and that premium must have an octane number of at least 94. It is assumed that octane numbers blend linearly by volume.

12.6.4.2 Jet fuel

The stipulation concerning jet fuel is that its vapour pressure must not exceed $1\,\mathrm{kg\,cm}^2$. The vapour pressures for light, heavy, cracked oils and residuum are 1.0, 0.6, 1.5 and $0.05\,\mathrm{kg\,cm}^2$, respectively. It may again be assumed that vapour pressures blend linearly by volume.

12.6.4.3 Fuel oil

To produce fuel oil, light oil, cracked oil, heavy oil and residuum must be blended in the ratio 10:4:3:1.

There are availability and capacity limitations on the quantities and processes used as follows:

1. The daily availability of crude 1 is 20 000 barrels.
2. The daily availability of crude 2 is 30 000 barrels.
3. At most 45 000 barrels of crude can be distilled per day.
4. At most 10 000 barrels of naphtha can be reformed per day.
5. At most 8000 barrels of oil can be cracked per day.
6. The daily production of lube oil must be between 500 and 1000 barrels.
7. Premium motor fuel production must be at least 40% of regular motor fuel production.

The profit contributions from the sale of the final products are (in pence per barrel) as follows:

Premium petrol	700
Regular petrol	600
Jet fuel	400
Fuel oil	350
Lube-oil	150

How should the operations of the refinery be planned in order to maximise total profit?

12.7 Mining

A mining company is going to continue operating in a certain area for the next five years. There are four mines in this area, but it can operate at most three in any one year. Although a mine may not operate in a certain year, it is still necessary to keep it 'open', in the sense that royalties are payable, if it be operated in a future year. Clearly, if a mine is not going to be worked again, it can be permanently closed down and no more royalties need be paid. The yearly royalties payable on each mine kept 'open' are as follows:

Mine 1	£5 million
Mine 2	£4 million
Mine 3	£4 million
Mine 4	£5 million

There is an upper limit to the amount of ore, which can be extracted from each mine in a year. These upper limits are as follows:

Mine 1	2×10^6 tons
Mine 2	2.5×10^6 tons
Mine 3	1.3×10^6 tons
Mine 4	3×10^6 tons

The ore from the different mines is of varying quality. This quality is measured on a scale so that blending ores together results in a linear combination of the quality measurements, for example, if equal quantities of two ores were combined, the resultant ore would have a quality measurement half way between that of the ingredient ores. Measured in these units the qualities of the ores from the mines are given as follows:

Mine 1	1.0
Mine 2	0.7
Mine 3	1.5
Mine 4	0.5

In each year, it is necessary to combine the total outputs from each mine to produce a blended ore of exactly some stipulated quality. For each year, these qualities are as follows:

Year 1	0.9
Year 2	0.8
Year 3	1.2
Year 4	0.6
Year 5	1.0

The final blended ore sells for £10 ton each year. Revenue and expenditure for future years must be discounted at a rate of 10% per annum.

Which mines should be operated each year and how much should they produce?

This problem is based on a larger one arising in deciding which pits to work in the firm of English China Clays. In that problem (in the 1970s), it was wished to work up to 4 mines out of 20 in each year. The model proved very difficult to solve.

12.8 Farm planning

A farmer wishes to plan production on his 200 acre farm over the next five years.

At present, he has a herd of 120 cows. This is made up of 20 heifers and 100 milk-producing cows. Each heifer needs 2/3 acre to support it and each dairy cow 1 acre. A dairy cow produces an average of 1.1 calves per year. Half of these calves will be bullocks, which are sold almost immediately for an average of £30 each. The remaining heifers can be either sold almost immediately for £40 or reared to become milk-producing cows at two years old. It is intended that all dairy cows be sold at 12 years old for an average of £120 each, although there will probably be an annual loss of 5% per year among heifers and 2% among dairy cows. At present, there are 10 cows each aged from newborn to 11 years old. The decision of how many heifers to sell in the current year has already been taken and implemented.

The milk from a cow yields an annual revenue of £370. A maximum of 130 cows can be housed at the present time. To provide accommodation for each cow beyond this number will entail a capital outlay of £200 per cow. Each milk-producing cow requires 0.6 tons of grain and 0.7 tons of sugar beet per year. Grain and sugar beet can both be grown on the farm. Each acre yields 1.5 tons of sugar beet. Only 80 acres are suitable for growing grain. They can be divided into four groups whose yields are as follows:

Group 1	20 acres	1.1 tons per acre
Group 2	30 acres	0.9 tons per acre
Group 3	20 acres	0.8 tons per acre
Group 4	10 acres	0.65 tons per acre

Grain can be bought for £90 per ton and sold for £75 per ton. Sugar beet can be bought for £70 per ton and sold for £58 per ton.

The labour requirements are as follows:

Each heifer	10 h per year
Each milk-producing cow	42 h per year
Each acre put to grain	4 h per year
Each acre put to sugar beet	14 h per year

Other costs are as follows:

Each heifer	£50 per year
Each milk-producing cow	£100 per year
Each acre put to grain	£15 per year
Each acre put to sugar beet	£10 per year

Labour costs for the farm are at present £4000 per year and provide 5500 h labour. Any labour needed above this will cost £1.20 per hour.

How should the farmer operate over the next five years to maximise profit? Any capital expenditure would be financed by a 10-year loan at 15% annual interest. The interest and capital repayment would be paid in 10 equally sized yearly instalments. In no year can the cash flow be negative. Finally, the farmer would neither wish to reduce the total number of dairy cows at the end of the five-year period by more than 50% nor wish to increase the number by more than 75%.

12.9 Economic planning

An economy consists of three industries: coal, steel and transport. Each unit produced by one of the industries (a unit will be taken as £1's worth of value of production) requires inputs from possibly its own industry as well as other industries. The required inputs as well as the manpower requirements (also measured in £) are given in Table 12.1. There is a time lag in the economy so that the output in year $t+1$ requires an input in year t.

Output from an industry may also be used to build productive capacity for itself or other industries in future years. The inputs required to give unit increases (capacity for £1's worth of extra production) in productive capacity are given in Table 12.2. Input from an industry in year t results in a (permanent) increase in productive capacity in year $t+2$.

Stocks of goods may be held from year to year. At present (year 0), the stocks and productive capacities (per year) are given in Table 12.3 (in £m). There is a limited yearly manpower capacity of £470 m.

Table 12.1

Inputs (year t)	Outputs (year $t+1$), production		
	Coal	Steel	Transport
Coal	0.1	0.5	0.4
Steel	0.1	0.1	0.2
Transport	0.2	0.1	0.2
Manpower	0.6	0.3	0.2

Table 12.2

Inputs (year t)	Outputs (year $t+2$), productive capacity		
	Coal	Steel	Transport
Coal	0.0	0.7	0.9
Steel	0.1	0.1	0.2
Transport	0.2	0.1	0.2
Manpower	0.4	0.2	0.1

Table 12.3

	Year 0	
	Stocks	Productive capacity
Coal	150	300
Steel	80	350
Transport	100	280

It is wished to investigate different possible growth patterns for the economy over the next five years. In particular, it is desirable to know the growth patterns that would result from pursuing the following objectives:

1. Maximising total productive capacity at the end of the five years while meeting an exogenous consumption requirement of £60 m of coal, £60 m of steel and £30 m of transport in every year (apart from year 0).
2. Maximising total production (rather than productive capacity) in the fourth and fifth years, but ignoring exogenous demand in each year.
3. Maximising the total manpower requirement (ignoring the manpower capacity limitation) over the period while meeting the yearly exogenous demands of (1).

12.10 Decentralisation

A large company wishes to move some of its departments out of London. There are benefits to be derived from doing this (cheaper housing, government incentives, easier recruitment, etc.), which have been costed. Also, however, there will be greater costs of communication between departments. These have also been costed for all possible locations of each department.

Where should each department be located so as to minimise overall yearly cost?

The company comprises five departments (A, B, C, D and E). The possible cities for relocation are Bristol and Brighton, or a department may be kept in London. None of these cities (including London) may be the location for more than three of the departments.

Benefits to be derived from each relocation are given (in thousands of pounds per year) as follows:

	A	B	C	D	E
Bristol	10	15	10	20	5
Brighton	10	20	15	15	15

Communication costs are of the form $C_{ik}D_{jl}$, where C_{ik} is the quantity of communication between departments i and k per year and D_{jl} is the cost per unit of communication between cities j and l. C_{ik} and D_{jl} are given by the following tables:

Quantities of communication C_{ik} (in thousands of units)

	A	B	C	D	E
A	–	0.0	1.0	1.5	0.0
B	–	–	1.4	1.2	0.0
C	–	–	–	0.0	2.0
D	–	–	–	–	0.7

Costs per unit of communication D_{jl} (in £)

	Bristol	Brighton	London
Bristol	5	14	13
Brighton	–	5	9
London	–	–	10

12.11 Curve fitting

A quantity y is known to depend on another quantity x. A set of corresponding values has been collected for x and y and is presented in Table 12.4.

1. Fit the 'best' straight line $y = bx + a$ to this set of data points. The objective is to minimise the sum of *absolute deviations* of each observed value of y from the value predicted by the linear relationship.
2. Fit the 'best' straight line where the objective is to minimise the *maximum deviation* of all the observed values of y from the value predicted by the linear relationship.
3. Fit the 'best' quadratic curve $y = cx^2 + bx + a$ to this set of data points using the same objectives as in (1) and (2).

Table 12.4

x	0.0	0.5	1.0	1.5	1.9	2.5	3.0	3.5	4.0	4.5
y	1.0	0.9	0.7	1.5	2.0	2.4	3.2	2.0	2.7	3.5
x	5.0	5.5	6.0	6.6	7.0	7.6	8.5	9.0	10.0	
y	1.0	4.0	3.6	2.7	5.7	4.6	6.0	6.8	7.3	

12.12 Logical design

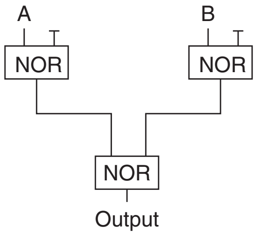
Logical circuits have a given number of inputs and one output. Impulses may be applied to the inputs of a given logical circuit, and it will respond by giving either an output (signal 1) or no output (signal 0). The input impulses are of the same kind as the outputs, that is, 1 (positive input) or 0 (no input).

In this example, a logical circuit is to be built up of NOR gates. A NOR gate is a device with two inputs and one output. It has the property that there is positive output (signal 1) if and only if *neither* input is positive, that is, both inputs have the value 0. By connecting such gates together with outputs from one gate possibly being inputs into another gate, it is possible to construct a circuit to perform any desired logical function. For example, the circuit illustrated in Figure 12.1 will respond to the inputs A and B in the way indicated by the truth table.

The problem here is to construct a circuit using the *minimum number* of NOR gates that will perform the logical function specified by the truth table in Figure 12.2. This problem, together with further references to it, is discussed in Williams (1974).

'Fan-in' and 'fan-out' are not permitted. That is, more than one output from a NOR gate cannot lead into one input nor can one output lead into more than one input.

It may be assumed throughout that the optimal design is a 'subnet' of the 'maximal' net shown in Figure 12.3.



A	B	Output
0	0	0
0	1	0
1	0	0
1	1	1

Figure 12.1

Inputs		Output
A	B	
0	0	0
0	1	1
1	0	1
1	1	0

Figure 12.2

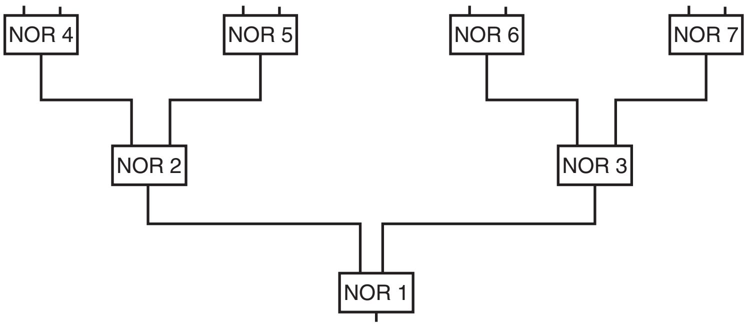


Figure 12.3

12.13 Market sharing

A large company has two divisions, D1 and D2. The company supplies retailers with oil and spirit. This is a much smaller version of the problem British Petroleum and Shell faced when they were forced to demerge – one of the largest demergers in history. The original model proved impossible to solve in 1972.

It is desired to allocate each retailer to either division D1 or division D2. This division will be the retailer's supplier. As far as possible, this division must be made so that D1 controls 40% of the market and D2 the remaining 60%. The retailers are listed below as M1 to M23. Each retailer has an estimated market for oil and spirit. Retailers M1 to M8 are in region 1; retailers M9 to M18 are in region 2 and retailers M19 to M23 are in region 3. Certain retailers are considered to have good growth prospects and categorised as group A and the others are in group B. Each retailer has a certain number of delivery points as given below. It is desired to make the 40/60 split between D1 and D2 in each of the following respects:

1. Total number of delivery points
2. Control of spirit market
3. Control of oil market in region 1
4. Control of oil market in region 2
5. Control of oil market in region 3
6. Number of retailers in group A
7. Number of retailers in group B.

There is a certain flexibility in that any share may vary by ±5%. That is, the share can vary between the limits 35/65 and 45/55.

The primary aim is to find a feasible solution. If, however, there is some choice then possible objectives are (i) to minimise the sum of the percentage deviations from the 40/60 split and (ii) to minimise the maximum such deviation.

Build a model to see if the problem has a feasible solution and if so find the optimal solutions.

The numerical data are given in Table 12.5.

Table 12.5

	Retailer	Oil market (10^6 gallons)	Delivery points	Spirit market (10^6 gallons)	Growth category
Region 1	M1	9	11	34	A
	M2	13	47	411	A
	M3	14	44	82	A
	M4	17	25	157	B
	M5	18	10	5	A
	M6	19	26	183	A
	M7	23	26	14	B
	M8	21	54	215	B
Region 2	M9	9	18	102	B
	M10	11	51	21	A
	M11	17	20	54	B
	M12	18	105	0	B
	M13	18	7	6	B
	M14	17	16	96	B
	M15	22	34	118	A
	M16	24	100	112	B
	M17	36	50	535	B
	M18	43	21	8	B
Region 3	M19	6	11	53	B
	M20	15	19	28	A
	M21	15	14	69	B
	M22	25	10	65	B
	M23	39	11	27	B

12.14 Opencast mining

A company has obtained permission to opencast mine within a square plot 200 ft × 200 ft. The angle of slip of the soil is such that it is not possible for the sides of the excavation to be steeper than 45°. The company has obtained estimates for the value of the ore in various places at various depths. Bearing in mind the restrictions imposed by the angle of slip, the company decides to consider the problem as one of the extracting of rectangular blocks. Each block has horizontal dimensions 50 ft × 50 ft and a vertical dimension of 25 ft. If the blocks are chosen to lie above one another, as illustrated in vertical section in Figure 12.4, then it is only possible to excavate blocks forming an upturned pyramid. (In a three-dimensional representation, Figure 12.4 would show four blocks lying above each lower block.)

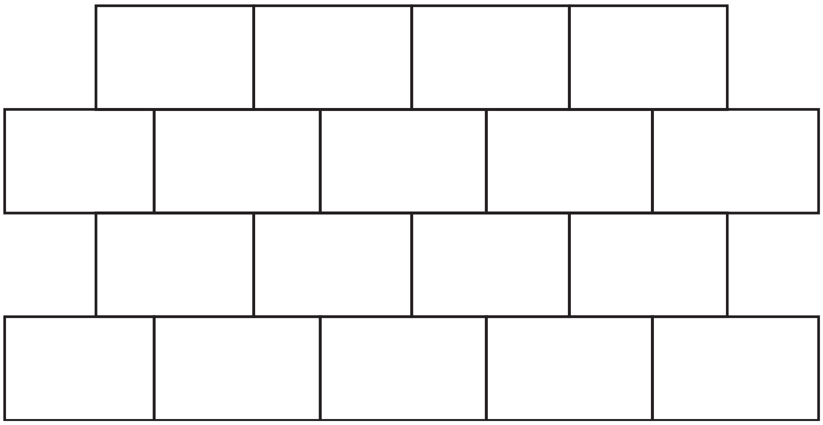


Figure 12.4

If the estimates of ore value are applied to give values (in percentage of pure metal) for each block in the maximum pyramid, which can be extracted, then the following values are obtained:

Level 1 (surface)

1.5	1.5	1.5	0.75
1.5	2.0	1.5	0.75
1.0	1.0	0.75	0.5
0.75	0.75	0.5	0.25

Level 2 (25 ft depth)

4.0	4.0	2.0
3.0	3.0	1.0
2.0	2.0	0.5

Level 3 (50 ft depth)

12.0	6.0
5.0	4.0

Level 4 (75 ft depth) 6.0

The cost of extraction increases with depth. At successive levels, the cost of extracting a block is as follows:

Level 1	£3000
Level 2	£6000
Level 3	£8000
Level 4	£10 000

The revenue obtained from a '100% value block' would be £200 000. For each block here, the revenue is proportional to ore value.

Build a model to help decide the best blocks to extract. The objective is to maximise revenue−cost.

The larger version of this problem arose with open-cast iron mining in South Africa.

12.15 Tariff rates (power generation)

A number of power stations are committed to meeting the following electricity load demands over a day:

12 p.m. to 6 a.m.	15 000 MW
6 a.m. to 9 a.m.	30 000 MW
9 a.m. to 3 p.m.	25 000 MW
3 p.m. to 6 p.m.	40 000 MW
6 p.m. to 12 p.m.	27 000 MW

There are three types of generating unit available: 12 of type 1, 10 of type 2 and five of type 3. Each generator has to work between a minimum and a maximum level. There is an hourly cost of running each generator at minimum level. In addition, there is an extra hourly cost for each megawatt at which a unit is operated above the minimum level. Starting up a generator also involves a cost. All this information is given in Table 12.6 (with costs in £).

In addition to meeting the estimated load demands there must be sufficient generators working at any time to make it possible to meet an increase in load of up to 15%. This increase would have to be accomplished by adjusting the output of generators already operating within their permitted limits.

Table 12.6

	Minimum level	Maximum level	Cost per hour at minimum	Cost per hour per megawatt above minimum	Cost
Type 1	850 MW	2000 MW	1000	2	2000
Type 2	1250 MW	1750 MW	2600	1.30	1000
Type 3	1500 MW	4000 MW	3000	3	500

Which generators should be working in which periods of the day to minimise total cost?

What is the marginal cost of production of electricity in each period of the day; that is, what tariffs should be charged?

What would be the saving of lowering the 15% reserve output guarantee; that is, what does this security of supply guarantee cost?

12.16 Hydro power

This is an extension of the Tariff Rates (Power Generation) problem of Section 12.15. In addition to the thermal generators, a reservoir powers two hydro generators: one of type A and one of type B. When a hydro generator is running, it operates at a fixed level and the depth of the reservoir decreases. The costs associated with each hydro generator are a fixed start-up cost and a running cost per hour. The characteristics of each type of generator are shown in Table 12.7.

For environmental reasons, the reservoir must be maintained at a depth of between 15 and 20 m. Also, at midnight each night, the reservoir must be 16 m deep. Thermal generators can be used to pump water into the reservoir. To increase the level of the reservoir by 1 m, it requires 3000 MWh of electricity. You may assume that rainfall does not affect the reservoir level.

At any time, it must be possible to meet an increase in demand for electricity of up to 15%. This can be achieved by any combination of the following: switching on a hydro generator (even if this would cause the reservoir depth to fall below 15 m); using the output of a thermal generator, which is used for pumping water into the reservoir; and increasing the operating level of a thermal generator to its maximum. Thermal generators cannot be switched on instantaneously to meet increased demand (although hydro generators can be).

Table 12.7

	Operating level	Cost per hour	Reservoir depth reduction per hour (m)	Start-up cost
Hydro A	900 MW	£90	0.31	£1500
Hydro B	1400 MW	£150	0.47	£1200

Which generators should be working in which periods of the day, and how should the reservoir be maintained to minimise the total cost?

12.17 Three-dimensional noughts and crosses

Twenty-seven cells are arranged $3 \times 3 \times 3$ in a three-dimensional array as shown in Figure 12.5.

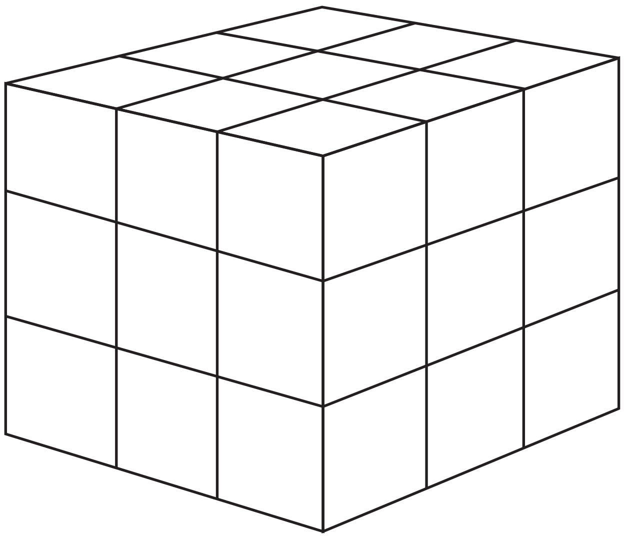


Figure 12.5

Three cells are regarded as lying in the same line if they are on the same horizontal or vertical line or the same diagonal. Diagonals exist on each horizontal and vertical section and connecting opposite vertices of the cube. (There are 49 lines altogether.)

Given 13 white balls (noughts) and 14 black balls (crosses), arrange them, one to a cell, so as to minimise the number of lines with balls all of one colour.

12.18 Optimising a constraint

In an integer programming problem, the following constraint occurs:

$$9x_1 + 13x_2 - 14x_3 + 17x_4 + 13x_5 - 19x_6 + 23x_7 + 21x_8 \leq 37.$$

All the variables occurring in this constraint are 0–1 variables, that is, they can only take the value of 0 or 1.

Find the 'simplest' version of this constraint. The objective is to find another constraint involving these variables that is logically equivalent to the original constraint but that has the smallest possible absolute value of the right-hand side (with all coefficients of similar signs to the original coefficients).

If the objective were to find an equivalent constraint where the sum of the absolute values of the coefficients (apart from the right-hand side coefficient) were a minimum what would be the result?

12.19 Distribution 1

A company has two factories, one at Liverpool and one at Brighton. In addition, it has four depots with storage facilities at Newcastle, Birmingham, London and Exeter. The company sells its product to six customers C1, C2, ..., C6. Customers can be supplied from either a depot or the factory directly (see Figure 12.6; Table 12.8).

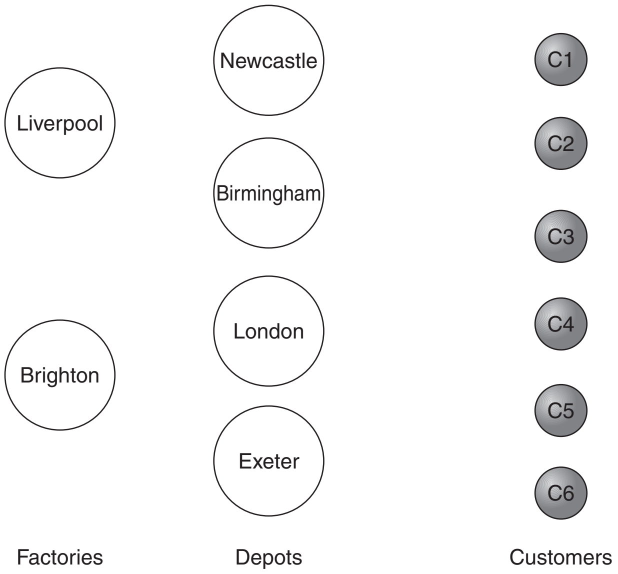


Figure 12.6

Table 12.8

Supplied to	Supplier					
	Liverpool factory	Brighton factory	Newcastle depot	Birmingham depot	London depot	Exeter depot
Depots						
Newcastle	0.5	–				
Birmingham	0.5	0.3				
London	1.0	0.5				
Exeter	0.2	0.2				
Customers						
C1	1.0	2.0	–	1.0	–	–
C2	–	–	1.5	0.5	1.5	–
C3	1.5	–	0.5	0.5	2.0	0.2
C4	2.0	–	1.5	1.0	–	1.5
C5	–	–	–	0.5	0.5	0.5
C6	1.0	–	1.0	–	1.5	1.5

[a] A dash indicates the impossibility of certain suppliers for certain depots or customers.

The distribution costs (which are borne by the company) are known; they are given in Table 12.8 (in £ per ton delivered).

Certain customers have expressed preferences for being supplied from factories or depots, which they are used to. The preferred suppliers are as follows:

C1	Liverpool (factory)
C2	Newcastle (depot)
C3	No preferences
C4	No preferences
C5	Birmingham (depot)
C6	Exeter or London (depots)

Each factory has a monthly capacity given as follows, which cannot be exceeded:

Liverpool	150 000 tons
Brighton	200 000 tons

Each depot has a maximum monthly throughput given as follows, which cannot be exceeded:

Newcastle	70 000 tons
Birmingham	50 000 tons
London	100 000 tons
Exeter	40 000 tons

Each customer has a monthly requirement given as follows, which must be met:

C1	50 000 tons
C2	10 000 tons
C3	40 000 tons
C4	35 000 tons
CS	60 000 tons
C6	20 000 tons

The company would like to determine the following:

1. What distribution pattern would minimise overall cost?
2. What the effect of increasing factory and depot capacities would be on distribution costs?
3. What the effects of small changes in costs, capacities and requirements would be on the distribution pattern?
4. Would it be possible to meet all customer preferences regarding suppliers, and if so what would the extra cost of doing this be?

12.20 Depot location (distribution 2)

In the distribution problem, there is a possibility of opening new depots at Bristol and Northampton, as well as of enlarging the Birmingham depot.

It is not considered desirable to have more than four depots and if necessary Newcastle or Exeter (or both) can be closed down.

The monthly costs (in interest charges) of the possible new depots and expansion at Birmingham are given in Table 12.9 together with the potential monthly throughputs.

Table 12.9

	Cost (£1000)	Throughput (1000 tons)
Bristol	12	30
Northampton	4	25
Birmingham (expansion)	3	20

The monthly savings of closing down the Newcastle and Exeter depots are given in Table 12.10.

The distribution costs involving the new depots are given in Table 12.11 (in £ per ton delivered).

Table 12.10

	Saving (£1000)
Newcastle	10
Exeter	5

Table 12.11

Supplied to	Supplier			
	Liverpool factory	Brighton factory	Bristol depot	Northampton depot
New depots				
Bristol	0.6	0.4		
Northampton	0.4	0.3		
Customers				
C1	–	–	1.2	–
C2	–	–	0.6	0.4
C3	As given for distribution problem		0.5	–
C4			–	0.5
C5	–	–	0.3	0.6
C6	–	–	0.8	0.9

Which new depots should be built? Should Birmingham be expanded? Should Exeter or Newcastle be closed down? What would be the best resultant distribution pattern to minimise overall costs?

12.21 Agricultural pricing

The government of a country wants to decide what prices should be charged for its dairy products, milk, butter and cheese. All these products arise directly or indirectly from the country's raw milk production. This raw milk is usefully divided into the two components as fat and dry matter. After subtracting the quantities of fat and dry matter, which are used for making products for export or consumption on the farms, there is a total yearly availability of 600 000 tons of fat and 750 000 tons of dry matter. This is all available for producing milk, butter and two kinds of cheese for domestic consumption.

The percentage compositions of the products are given in Table 12.12.

For the previous year, the domestic consumption and prices for the products are given in Table 12.13.

Table 12.12

	Fat	Dry matter	Water
Milk	4	9	87
Butter	80	2	18
Cheese 1	35	30	35
Cheese 2	25	40	35

Table 12.13

	Milk	Butter	Cheese 1	Cheese 2
Domestic consumption (1000 tons)	4820	320	210	70
Price (£/ton)	297	720	1050	815

Price elasticities of demand, relating consumer demand to the prices of each product, have been calculated on the basis of past statistics. The price elasticity E of a product is defined by

$$E = \frac{\text{Percentage decrease in demand}}{\text{Percentage increase in price}}.$$

For the two makes of cheese, there will be some degree of substitution in consumer demand depending on relative prices. This is measured by cross-elasticity of demand with respect to price. The cross-elasticity E_{AB} from a product A to a product B is defined by

$$E_{\text{AB}} = \frac{\text{Percentage increase in demand for A}}{\text{Percentage increase in price of B}}.$$

The elasticities and cross-elasticities are given in Table 12.14.

Table 12.14

Milk	Butter	Cheese 1	Cheese 2	Cheese 1 to Cheese 2	Cheese 2 to Cheese 1
0.4	2.7	1.1	0.4	0.1	0.4

The objective is to determine what prices and resultant demand will maximise the total revenue.

It is, however, politically unacceptable to allow a certain price index to rise. As a result of the way this index is calculated, this limitation simply demands that the new prices must be such that the total cost of last year's consumption would not be increased. A particularly important additional requirement is to quantify the economic cost of this political limitation.

12.22 Efficiency analysis

A car manufacturer wants to evaluate the efficiencies of different garages, who have received a franchise to sell its cars. The method to be used is data envelopment analysis (DEA). References to this technique are given in Section 3.2. Each garage has a certain number of measurable 'inputs'. These are taken to be *Staff*, *Showroom Space*, *Catchment Population* in different economic categories and annual *Enquiries* for different brands of car. Each garage also has a certain number of measurable 'outputs'. These are taken to be *Number Sold* of different brands of car and annual *Profit*. Table 12.15 gives the inputs and outputs for each of the 28 franchised garages.

A central assumption of DEA (although modified models can be built to alter this assumption) is that constant returns to scale are possible, that is, doubling a garage's inputs should lead to a doubling of all its outputs. A garage is deemed to be efficient if it is not possible to find a mixture of proportions of other garages, whose combined inputs do not exceed those of the garage being considered, but whose outputs are equal to, or exceed, those of the garage. Should this not be possible then the garage is deemed to be inefficient and the comparator garages can be identified.

A linear programming model can be built to identify efficient and inefficient garages and their comparators.

12.23 Milk collection

A small milk processing company is committed to collecting milk from 20 farms and taking it back to the depot for processing. The company has one tanker lorry with a capacity for carrying 80 000 litres of milk. Eleven of the farms are small and need a collection only every other day. The other nine farms need a collection every day. The positions of the farms in relation to the depot (numbered 1) are given in Table 12.16 together with their collection requirements.

Find the optimal route for the tanker lorry on each day, bearing in mind that it has to (i) visit all the 'every day' farms, (ii) visit some of the 'every other day' farms and (iii) work within its capacity. On alternate days, it must again visit the 'every day' farms and also visit the 'every other day' farms not visited on the previous day.

For convenience, a map of the area considered is given in Figure 12.7.

Table 12.15

Garage		Inputs						Outputs		
		Staff	Show room space (100 m^2)	Population in category 1 (1000 s)	Population in category 2 (1000 s)	Enquiries Alpha model (100 s)	Enquiries Beta model (100 s)	Alpha sales (1000 s)	Beta sales (1000 s)	Profit (millions)
1	Winchester	7	8	10	12	8.5	4	2	0.6	1.5
2	Andover	6	6	20	30	9	4.5	2.3	0.7	1.6
3	Basingstoke	2	3	40	40	2	1.5	0.8	0.25	0.5
4	Poole	14	9	20	25	10	6	2.6	0.86	1.9
5	Woking	10	9	10	10	11	5	2.4	1	2
6	Newbury	24	15	15	13	25	1.9	8	2.6	4.5
7	Portsmouth	6	7	50	40	8.5	3	2.5	0.9	1.6
8	Alresford	8	7.5	5	8	9	4	2.1	0.85	2
9	Salisbury	5	5	10	10	5	2.5	2	0.65	0.9
10	Guildford	8	10	30	35	9.5	4.5	2.05	0.75	1.7
11	Alton	7	8	7	8	3	2	1.9	0.70	0.5
12	Weybridge	5	6.5	9	12	8	4.5	1.8	0.63	1.4
13	Dorchester	6	7.5	10	10	7.5	4	1.5	0.45	1.45
14	Bridport	11	8	8	10	10	6	2.2	0.65	2.2
15	Weymouth	4	5	10	10	7.5	3.5	1.8	0.62	1.6
16	Portland	3	3.5	3	20	2	1.5	0.9	0.35	0.5
17	Chichester	5	5.5	8	10	7	3.5	1.2	0.45	1.3
18	Petersfield	21	12	6	6	15	8	6	0.25	2.9
19	Petworth	6	5.5	2	2	8	5	1.5	0.55	1.55

(*continued overleaf*)

Table 12.15 (*continued*)

Garage		Inputs						Outputs		
		Staff	Show room space (100 m^2)	Population in category 1 (1000 s)	Population in category 2 (1000 s)	Enquiries Alpha model (100 s)	Enquiries Beta model (100 s)	Alpha sales (1000 s)	Beta sales (1000 s)	Profit (millions)
20	Midhurst	3	3.6	3	3	2.5	1.5	0.8	0.20	0.45
21	Reading	30	29	120	80	35	20	7	2.5	8
22	Southampton	25	16	110	80	27	12	6.5	3.5	5.4
23	Bournemouth	19	10	90	22	25	13	5.5	3.1	4.5
24	Henley	7	6	5	7	8.5	4.5	1.2	0.48	2
25	Maidenhead	12	8	7	10	12	7	4.5	2	2.3
26	Fareham	4	6	1	1	7.5	3.5	1.1	0.48	1.7
27	Romsey	2	2.5	1	1	2.5	1	0.4	0.1	0.55
28	Ringwood	2	3.5	2	2	1.9	1.2	0.3	0.09	0.4

Table 12.16

Farm	Position 10 miles		Collection frequency	Collection requirement (1000 l)
	East	North		
1 (Depot)	0	0	–	–
2	−3	3	Every day	5
3	1	11	Every day	4
4	4	7	Every day	3
5	−5	9	Every day	6
6	−5	−2	Every day	7
7	−4	−7	Every day	3
8	6	0	Every day	4
9	3	−6	Every day	6
10	−1	−3	Every day	5
11	0	−6	Every other day	4
12	6	4	Every other day	7
13	2	5	Every other day	3
14	−2	8	Every other day	4
15	6	10	Every other day	5
16	1	8	Every other day	6
17	−3	1	Every other day	8
18	−6	5	Every other day	5
19	2	9	Every other day	7
20	−6	−5	Every other day	6
21	5	−4	Every other day	6

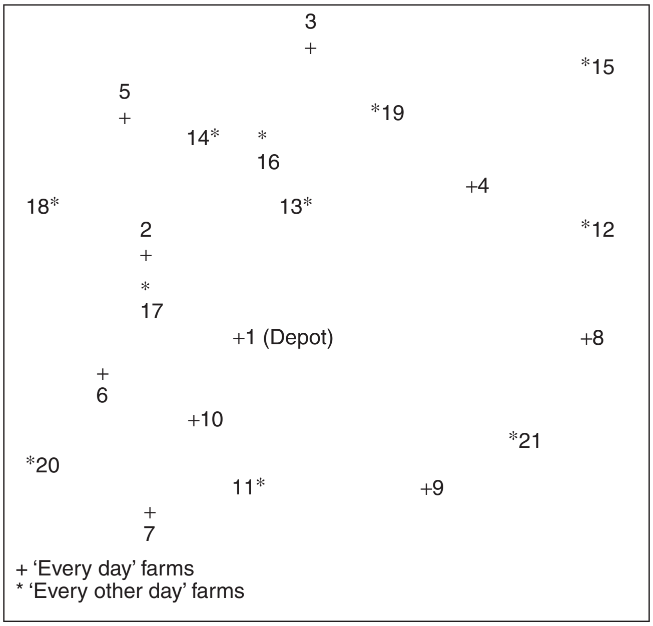


Figure 12.7

12.24 Yield management

An airline is selling tickets for flights to a particular destination. The flight will depart in three weeks' time. It can use up to six planes each costing £50 000 to hire. Each plane has the following:

37 First Class seats

38 Business Class seats

47 Economy Class seats.

Up to 10% of seats in any one category can be transferred to an adjacent category.

It wishes to decide a price for each of these seats. There will be further opportunities to update these prices after one week and two weeks. Once a customer has purchased a ticket, there is no cancellation option.

For administrative simplicity, three price level options are possible in each class (one of which must be chosen). The same option need not be chosen for each class. These are given in Table 12.17 for the current period (period 1) and two future periods.

Table 12.17

	Option 1	Option 2	Option 3	
First	£1200	£1000	£950	Period 1
Business	£900	£800	£600	
Economy	£500	£300	£200	
First	£1400	£1300	£1150	Period 2
Business	£1100	£900	£750	
Economy	£700	£400	£350	
First	£1500	£900	£850	Period 3
Business	£820	£800	£500	
Economy	£480	£470	£450	

Demand is uncertain but will be affected by price. Forecasts have been made of these demands according to a probability distribution that divides the demand levels into three scenarios for each period. The probabilities of the three scenarios in each period are as follows:

Scenario 1	0.1
Scenario 2	0.7
Scenario 3	0.2

The forecast demands are shown in Table 12.18.

Table 12.18

	Price option 1	Price option 2	Price option 3	
First	10	15	20	Period 1
Business	20	25	35	Scenario 1
Economy	45	55	60	
First	20	25	35	Period 1
Business	40	42	45	Scenario 2
Economy	50	52	63	
First	45	50	60	Period 1
Business	45	46	47	Scenario 3
Economy	55	56	64	
First	20	25	35	Period 2
Business	42	45	46	Scenario 1
Economy	50	52	60	
First	10	40	50	Period 2
Business	50	60	80	Scenario 2
Economy	60	65	90	
First	50	55	80	Period 2
Business	20	30	50	Scenario 3
Economy	10	40	60	
First	30	35	40	Period 3
Business	40	50	55	Scenario 1
Economy	50	60	80	
First	30	40	60	Period 3
Business	10	40	45	Scenario 2
Economy	50	60	70	
First	50	70	80	Period 3
Business	40	45	60	Scenario 3
Economy	60	65	70	

Decide price levels for the current period, how many seats to sell in each class (depending on demand), the provisional number of planes to book and provisional price levels and seats to sell in future periods in order to maximise expected yield. You should schedule to be able to meet commitments under all possible combinations of scenarios.

With hindsight (i.e. not known until the beginning of the next period), it turned out that demand in each period (depending on the price level you chose) was as shown in Table 12.19.

Table 12.19

	Price option 1	Price option 2	Price option 3	
First	25	30	40	Period 1
Business	50	40	45	
Economy	50	53	65	
First	22	45	50	Period 2
Business	45	55	75	
Economy	50	60	80	
First	45	60	75	Period 3
Business	20	40	50	
Economy	55	60	75	

Use the actual demands that resulted from the prices you set in period 1 to rerun the model at the beginning of period 2 to set price levels for period 2 and provisional price levels for period 3.

Repeat this procedure with a rerun at the beginning of period 3. Give the final operational solution.

Contrast this solution to one obtained at the beginning of period 1 by pricing to maximise yield based on expected demands.

12.25 Car rental 1

A small ('cut price') car rental company, renting one type of car, has depots in Glasgow, Manchester, Birmingham and Plymouth. There is an estimated demand for each day of the week except Sunday when the company is closed. These estimates are given in Table 12.20. It is not necessary to meet all demand.

Table 12.20

	Glasgow	Manchester	Birmingham	Plymouth
Monday	100	250	95	160
Tuesday	150	143	195	99
Wednesday	135	80	242	55
Thursday	83	225	111	96
Friday	120	210	70	115
Saturday	230	98	124	80

Cars can be rented for one, two or three days and returned to either the depot from which rented or another depot at the start of the next morning. For example, a 2-day rental on Thursday means that the car has to be returned on Saturday morning; a 3-day rental on Friday means that the car has to be returned on Tuesday morning. A 1-day rental on Saturday means that the car has to be returned on Monday morning and a 2-day rental on Tuesday morning.

Table 12.21

From	To			
	Glasgow	Manchester	Birmingham	Plymouth
Glasgow	60	20	10	10
Manchester	15	55	25	5
Birmingham	15	20	54	11
Plymouth	8	12	27	53

Table 12.22

From	To			
	Glasgow	Manchester	Birmingham	Plymouth
Glasgow	–	20	30	50
Manchester	20	–	15	35
Birmingham	30	15	–	25
Plymouth	50	35	25	–

The rental period is independent of the origin and destination. From past data, the company knows the distribution of rental periods: 55% of cars are hired for one day, 20% for two days and 25% for three days. The current estimates of percentages of cars hired from each depot and returned to a given depot (independent of day) are given in Table 12.21.

The marginal cost, to the company, of renting out a car ('wear and tear', administration etc.) is estimated as follows:

1-Day hire	£20
2-Day hire	£25
3-Day hire	£30

The 'opportunity cost' (interest on capital, storage, servicing, etc.) of owning a car is £15 per week.

It is possible to transfer undamaged cars from one depot to another depot, irrespective of distance. Cars cannot be rented out during the day in which they are transferred. The costs (£), per car, of transfer are given in Table 12.22.

Ten percent of cars returned by customers are damaged. When this happens, the customer is charged an excess of £100 (irrespective of the amount of damage that the company completely covers by its insurance). In addition, the car has to be transferred to a repair depot, where it will be repaired the following day. The cost of transferring a damaged car is the same as transferring an undamaged one (except when the repair depot is the current depot, when it is zero). Again the transfer of a damaged car takes a day, unless it is already at a repair depot. Having arrived at a repair depot, all types of repair (or replacement) take a day.

Only two of the depots have repair capacity. These are (cars/day) as follows:

Manchester	12
Birmingham	20

Having been repaired, the car is available for rental at the depot the next day or may be transferred to another depot (taking a day). Thus, a car that is returned damaged on a Wednesday morning is transferred to a repair depot (if not the current depot) during Wednesday, repaired on Thursday and is available for hire at the repair depot on Friday morning.

The rental price depends on the number of days for which the car is hired and whether it is returned to the same depot or not. The prices are given in Table 12.23 (in £).

Table 12.23

	Return to Same Depot	Return to Another Depot
1-Day hire	50	70
2-Day hire	70	100
3-Day hire	120	150

There is a discount of £20 for hiring on a Saturday so long as the car is returned on Monday morning. This is regarded as a 1-day hire.

For simplicity, we assume the following at the beginning of each day:

1. Customers return cars that are due that day
2. Damaged cars are sent to the repair depot
3. Cars that were transferred from other depots arrive
4. Transfers are sent out
5. Cars are rented out
6. If it is a repair depot, then the repaired cars are available for rental.

In order to maximise weekly profit, the company wants a ‘steady state’ solution in which the same expected number will be located at the same depot on the same day of subsequent weeks.

How many cars should the company own and where should they be located at the start of each day?

This is a case where the integrality of the cars is not worth modelling. Rounded fractional solutions are acceptable

12.26 Car rental 2

In the light of the solution to the problem stated in Section 12.25, the company wants to consider where it might be most worthwhile to expand repair capacity. The weekly fixed costs, given below, include interest payments on the necessary loans for expansion.

The options are as follows:

1. Expand repair capacity at Birmingham by 5 cars per day at a fixed cost per week of £18 000.
2. Further expand repair capacity at Birmingham by 5 cars per day at a fixed cost per week of £8000.
3. Expand repair capacity at Manchester by 5 cars per day at a fixed cost per week of £20 000.
4. Further expand repair capacity at Manchester by 5 cars per day at a fixed cost per week of £5000.
5. Create repair capacity at Plymouth of 5 cars per day at a fixed cost per week of £19 000.

If any of these options is chosen, it must be carried out in its entirety, that is, there can be no partial expansion. Also, a further expansion at a depot can be carried out only if the first expansion is also carried out, so for example option (2) at Birmingham cannot be chosen unless option (1) is also chosen. If option (2) is chosen, thereby also choosing option (1), these count as two options. Similar stipulations apply regarding the expansions at Manchester. At most three of the options can be carried out.

12.27 Lost baggage distribution

A small company with six vans has a contract with a number of airlines to pick up lost or delayed baggage, belonging to customers in the London area, from Heathrow airport at 6 p.m. each evening. The contract stipulates that each customer must have their baggage delivered by 8 p.m. The company requires a model, which they can solve quickly each evening, to advise them what is the minimum number of vans they need to use and to which customers each van should deliver and in what order. There is no practical capacity limitation on each van. All baggage that needs to be delivered in a two-hour period can be accommodated in a van. Having ascertained the minimum number of vans needed, a solution is then sought, which minimises the maximum time taken by any van.

On a particular evening, the places where deliveries need to be made and the times to travel between them (in minutes) are given in Table 12.24. No allowance

Table 12.24

Heathrow	20	25	35	65	90	85	80	86	25	35	20	44	35	82
	Harrow	15	35	60	55	57	85	90	25	35	30	37	20	40
		Ealing	30	50	70	55	50	65	10	25	15	24	20	90
			Holborn	45	60	53	55	47	12	22	20	12	10	21
				Sutton	46	15	45	75	25	11	19	15	25	25
					Dartford	15	15	25	45	65	53	43	63	70
						Bromley	17	25	41	25	33	27	45	30
							Greenwich	25	40	34	32	20	30	10
								Barking	65	70	72	61	45	13
									Hammersmith	20	8	7	15	25
										Kingston	5	12	45	65
											Richmond	14	34	56
												Battersea	30	40
													Islington	27
														Woolwich

is made for drop off times. For convenience, Heathrow will be regarded as the first location.

Formulate optimisation models that will minimise the number of vans that need to be used, and within this minimum, minimise the time taken for the longest time delivery.

12.28 Protein folding

This problem is based on one in the paper by Forrester and Greenberg (2008). It is a simplification of a problem in molecular biology. We take a protein as consisting of a chain of amino acids. For the purpose of this problem, the amino acids come in two forms: *hydrophilic* (waterloving) *and hydrophobic* (water hating). An example of such a chain is given in Figure 12.8. with the hydrophobic acids marked in bold.

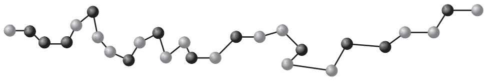


Figure 12.8

Such a chain naturally folds so as to bring as many hydrophobic acids, as possible, close together. An optimum folding for the chain, in two dimensions, is given in Figure 12.9, with the new matches marked by dashed lines. The problem is to predict the optimum folding. (Forrester and Greenberg also impose a condition that the resultant protein be confined to a given lattice of points. We do not impose that condition here). This problem can be modelled by a number of integer programming formulations. Some of these are discussed in the above reference. Another formulation is suggested in section 13.28. The problem posed here is to find the optimum folding for a chain of 50 amino acids with hydrophobic acids at positions 2, 4, 5, 6, 11, 12, 17, 20, 21, 25, 27, 28, 30, 31, 33, 37, 44 and 46 as shown in Figure 12.10.

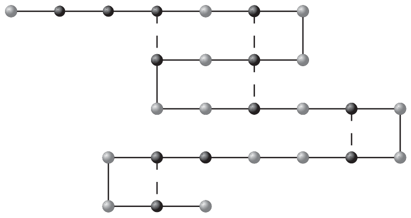


Figure 12.9

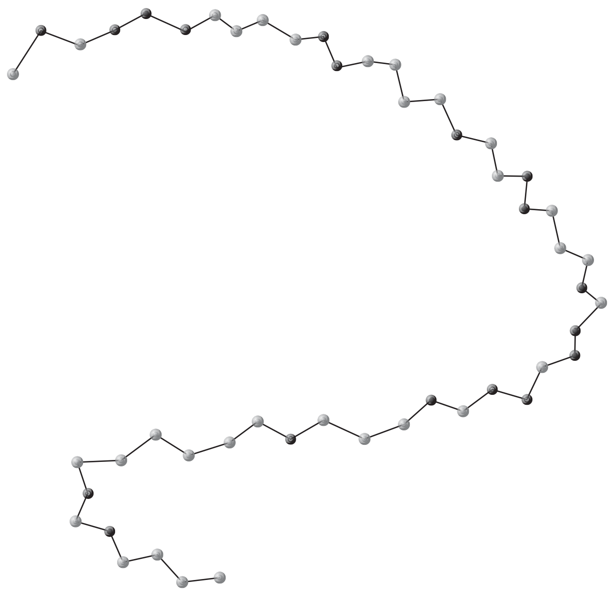


Figure 12.10

12.29 Protein comparison

This problem is also based on one in the paper by Forrester and Greenberg (2008). It is concerned with measuring the similarities of two proteins. A protein can be represented by an (undirected) graph with the acids represented by the nodes and the edges being present when two acids are within a threshold distance of each other. This graphical representation is known as the *contact map* of the protein. Given two contact maps, representing proteins, we would like to find the largest (measured by number of corresponding edges) isomorphic subgraphs in each graph. The acids in each of the proteins are ordered. We need to preserve this ordering in each of the subgraphs, which implies that there can be no *crossovers* in the comparison. This is illustrated in Figure 12.11. If $i < k$ in the contact map

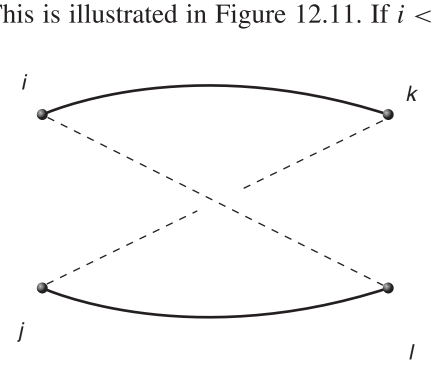


Figure 12.11

for the first protein then we cannot have $l < j$ in the second protein, if i is to be associated with j and k with l in the comparison.

This problem is well known for being very difficult to solve for even modestly sized proteins.

In Figure 12.12, we give an optimal comparison between two small contact maps leading to 5 corresponding edges.

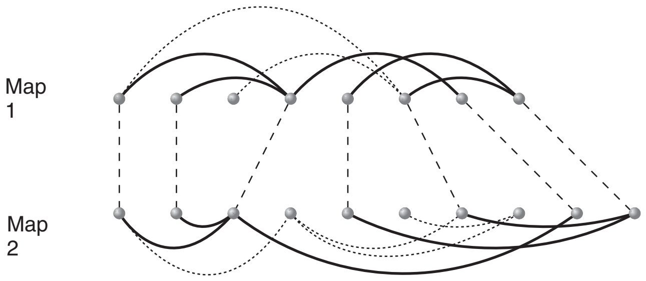


Figure 12.12

The problem, we present here, is to compare the contact maps given in Figures 12.13 and 12.14.

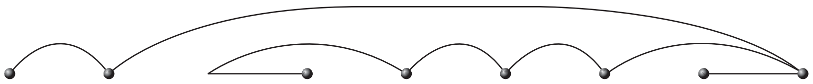


Figure 12.13

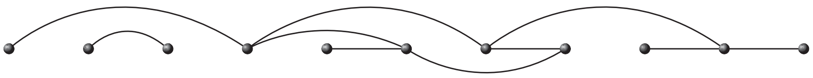


Figure 12.14

Part III

13

Formulation and discussion of problems

A suggested way of formulating each of the problems of Part II as a mathematical programming model is described. Each of these formulations is 'good' in the sense that it proved possible to solve the resultant model in a reasonable time on the computer system used. With some of the problems, other formulations were tried only to show that computation times were prohibitive. It must be emphasized that although the formulations presented here are the best known to the author, there are probably better formulations possible for some of the problems. Indeed, one of the purposes of the latter part of this book is to present concrete problems that may help advance the art of formulation.

All the models described here assume that a computer program is to be employed using one of the following algorithms:

1. The revised simplex algorithm for linear programming problems.
2. The branch and bound algorithm for integer programming problems.
3. The separable extension to the revised simplex algorithm for separable programming problems.

These are the three algorithms that are most widely incorporated into commercial package programs.

For some of the problems, more specialized algorithms might be more efficient. In practice, the employment of such algorithms is often made difficult by the absence of efficient computer packages for handling large models. Where there is, however, obvious advantage to be gained from a specialized algorithm, this is pointed out in the discussion associated with the formulation description.

This is particularly true where the model (or its dual) admits a network formulation. In such cases, it should still, however, be possible to solve the model reasonably efficiently using one of the above three methods. Sometimes, it is desirable to build a model in a particular way to suit a particular algorithm (particularly for integer programming problems). In the formulations here the models are all built on the assumption that one of the above three algorithms will be used.

Again, especially with integer programming models, it is often desirable to build a model with a particular branch and bound solution strategy in mind. When this is done the suggested strategy is explained.

Most of the models are very easy to solve. For the more difficult ones there is some computational discussion.

13.1 Food manufacture 1

Blending problems are frequently solved using linear programming. Linear programming has been used to find 'minimum cost' blends of fertilizer, metal alloys, clays and many food products, to name only a few. Applications are described in Fisher and Schruben (1953) and Williams and Redwood (1974), for example.

The problem presented here has two aspects. Firstly, it is a series of simple blending problems. Secondly, there is a purchasing and storing problem. To understand how this problem may be formulated, it is convenient to consider first the blending problem for only one month. This is the single-period problem that has already been presented as the second example in Section 1.2.

13.1.1 The single-period problem

If no storage of raw oils were allowed, the problem of what to buy and how to blend in January could be formulated as follows:

$$\begin{array}{llllllll}
\text{Maximize} & & & & & & & \\
\text{PROFIT} & -110x_1 & -\,120x_2 & -\,130x_3 & -\,110x_4 & -\,115x_5 & +\,150y & \\
\text{subject to} & & & & & & & \\
\text{VVEG} & x_1 + & x_2 & & & & & \leq 200, \\
\text{NVEG} & & & x_3 + & x_4 + & x_5 & & \leq 250, \\
\text{UHRD} & 8.8x_1 + & 6.1x_2 + & 2x_3 + & 4.2x_4 + & 5x_5 & -\,6y & \leq \quad 0, \\
\text{LHRD} & 8.8x_1 + & 6.1x_2 + & 2x_3 + & 4.2x_4 + & 5x_5 & -\,3y & \geq \quad 0, \\
\text{CONT} & x_1 + & x_2 + & x_3 + & x_4 + & x_5 & -\;\, y & = \quad 0.
\end{array}$$

The variables x_1, x_2, x_3, x_4, x_5 represent the quantities of the raw oils that should be bought respectively, that is, VEG 1, VEG 2, OIL 1, OIL 2 and OIL 3. y represents the quantity of PROD that should be made.

The objective is to maximize profit, which represents the income derived from selling PROD minus the cost of the raw oils.

The first two constraints represent the limited production capacities for refining vegetable and non-vegetable oils.

The next two constraints force the hardness of PROD to lie between its upper limit of 6 and its lower limit of 3. It is important to model these restrictions correctly. A frequent mistake is to model them as

$$8.8x_1 + 6.1x_2 + 2x_3 + 4.2x_4 + 5x_5 \leq 6$$

and

$$8.8x_1 + 6.1x_1 + 2x_3 + 4.2x_4 + 5x_5 \geq 3.$$

Such constraints are clearly dimensionally wrong. The expressions on the left have the dimension of hardness multiplied by quantity, whereas the figures on the right have the dimensions of hardness. Instead of the variables x_i in the above two inequalities, expressions x_i/y are needed to represent *proportions* of the ingredients rather than the absolute quantities x_i. When such replacements are made, the resultant inequalities can easily be re-expressed in a linear form as the constraints UHRD and LHRD.

Finally, it is necessary to make sure that the weight of the final product PROD is equal to the weight of the ingredients. This is done by the last constraint CONT, which imposes this continuity of weight.

The single-period problems for the other months would be similar to that for January apart from the objective coefficients representing the raw oil costs.

13.1.2 The multi-period problem

The decisions of how much to buy each month with a view to storing for use later can be incorporated into a linear programming model. To do this, a 'multi-period' model is built. It is necessary, each month, to distinguish the quantities of each raw oil bought, used and stored. These quantities must be represented by different variables. We suppose the quantities of VEG 1 bought, used and stored in each successive month are represented by variables with the following names:

BVEG 11,	BVEG 12,	and so on,
UVEG 11,	UVEG 12,	and so on,
SVEG 11,	SVEG 12,	and so on.

It is necessary to link these variables together by the relation

$$\text{quantity stored in month } (t-1) + \text{quantity bought in month } t$$
$$= \text{quantity used in month } t + \text{quantity stored in month } t.$$

Initially (month 0) and finally (month 6), the quantities in store are constants (500). The relation above involving VEG 1 gives rise to the following constraints:

$$\begin{aligned}
&\text{BVEG 11} - \text{UVEG 11} - \text{SVEG 11} = -500,\\
\text{SVEG 11} &+ \text{BVEG 12} - \text{UVEG 12} - \text{SVEG 12} = 0,\\
\text{SVEG 12} &+ \text{BVEG 13} - \text{UVEG 13} - \text{SVEG 13} = 0,\\
\text{SVEG 13} &+ \text{BVEG 14} - \text{UVEG 14} - \text{SVEG 14} = 0,\\
\text{SVEG 14} &+ \text{BVEG 15} - \text{UVEG 15} - \text{SVEG 15} = 0,\\
\text{SVEG 15} &+ \text{BVEG 16} - \text{UVEG 16} = 500.
\end{aligned}$$

Similar constraints must be specified for the other four raw oils.

It may be more convenient to introduce variables SVEG 10, and so on, and SVEG 16, and so on, into the model and *fix* them at the value 500.

In the objective function, the 'buying' variables will be given the appropriate raw oil costs in each month. The storage variables will be given the cost of £5 (or 'profit' of –£5). Separate variables PROD 1, PROD 2, and so on, must be defined to represent the quantity of PROD to be made in each month. These variables will each have a profit of £150.

The resulting model will have the following dimensions as well as the single objective function:

$6 \times 5 = 30$	buying variables
$6 \times 5 = 30$	using variables
$5 \times 5 = 25$	storing variables
6	product variables
Total 91	variables

$6 \times 5 = 30$	blending constraints (as in the single-period model)
$6 \times 5 = 30$	storage linking constraints
Total 60	constraints

It is also important to realize the use to which a model such as this might be put for medium-term planning. By solving the model in January, buying and blending plans could be determined for January together with provisional plans for the succeeding months. In February, the model would probably be resolved with revised figures to give firm plans for February together with provisional plans for succeeding months up to and including July. By this means, the best use is made of the information for succeeding months to derive an operating policy for the current month.

13.2 Food manufacture 2

The extra restrictions stipulated are quite common in blending problems. It is often desired to (i) limit the number of ingredients in a blend; (ii) rule out small quantities of one ingredient and (iii) impose 'logical conditions' on the combinations of ingredients.

These restrictions cannot be modelled by conventional linear programming. Integer programming is the obvious way of imposing the extra restrictions. In order to do this, it is necessary to introduce 0–1 integer variables into the problem as described in Section 9.2. For each 'using' variable in the problem, a corresponding 0–1 variable is also introduced. This variable is used as an indicator of whether the corresponding ingredient appears in the blend or not. For example, corresponding to variable UVEG 11, a 0–1 variable DVEG 11 is introduced. These variables are linked together by two constraints. Supposing x_1 represents UVEG 11 and δ_1 represents DVEG 11, the following extra constraints are added to the model:

$$x_1 - 200\delta_1 \le 0,$$
$$x_1 - \ \ 20\delta_1 \ge 0.$$

As δ_1 is only allowed to take the integer values 0 and 1, x_1 can only be non-zero (i.e. VEG 1 in the blend in month 1) if $\delta_1 = 1$ and then it must be at a level of at least 20 tons. The constant 200 in the first of the above inequalities is a known upper limit to the level of UVEG 11 (the combined quantities of vegetable oils used in a month cannot exceed 200). Similar 0–1 variables and corresponding 'linkage' constraints are introduced for the other ingredients. Suppose x_2, x_3, x_4 and x_5 represent UVEG 21, UOIL 11; UOIL 21 and UOIL 31, then the following constraints and 0–1 variables are also introduced:

$$\begin{aligned} x_2 - 200\delta_2 &\le 0, & x_2 - 20\delta_2 &\ge 0,\\ x_3 - 250\delta_3 &\le 0, & x_3 - 20\delta_3 &\ge 0,\\ x_4 - 250\delta_4 &\le 0, & x_4 - 20\delta_4 &\ge 0,\\ x_5 - 250\delta_5 &\le 0, & x_5 - 20\delta_5 &\ge 0. \end{aligned}$$

All these variables and constraints are repeated for all six months.

In this way, condition 2 in the statement of the problem is automatically imposed. Condition 1 can be imposed by the constraint

$$\delta_1 + \delta_2 + \delta_3 + \delta_4 + \delta_5 \le 3,$$

and the corresponding constraints for the other five months.

Condition 3 can be imposed in two possible ways: by the single constraints,

$$\delta_1 + \delta_2 - 2\delta_5 \le 0,$$

or by the pairs of constraints,

$$\delta_1 - \delta_5 \le 0,$$

$$\delta_2 - \delta_5 \le 0.$$

There is computational advantage to be gained by using the second pair of constraints as they are 'tighter' in the continuous problem (see Section 10.1). Similar constraints are, of course, imposed for the other five months.

The model has now been augmented in the following way:

$6 \times 5 =$	30	0–1 variables
Total	30	extra variables (all integer)
$2 \times 6 \times 5 =$	60	linking constraints
	6	constraints for condition 1
$2 \times 6 =$	12	constraints for condition 3
Total	78	extra constraints

It is also necessary to impose upper bounds of 1 on all the 30 integer variables.

There is probably advantage to be gained from the user specifying a priority order for the variables in order to control the tree search in the branch and bound algorithm. The six δ_5 variables (one for each month) were given priority in choosing branching variables.

13.3 Factory planning 1

This example is typical of some very common applications of linear programming. The objective is to find the optimum 'product mix' subject to the production capacity and the marketing limitations. As with the food manufacture problem, there are two aspects. Firstly, there is the single-period problem, then the extension to six months.

13.3.1 The single-period problem

If storage of finished products is not allowed, the model for January can be formulated as follows:

$$\begin{array}{ll}
\text{Maximize} & \\
\text{PROFIT} & 10x_1 + 6x_2 + 8x_3 + 4x_4 + 11x_5 + 9x_6 + 3x_7 \\
\text{subject to} & \\
\text{GR} & 0.5x_1 + 0.7x_2 + 0.3x_5 + 0.2x_6 + 0.5x_7 \le 1152 \\
\text{VD} & 0.1x_1 + 0.2x_2 + 0.3x_4 + 0.6x_6 \le 768 \\
\text{HD} & 0.2x_1 + 0.8x_3 + 0.6x_7 \le 1152
\end{array}$$

BR	$0.05x_1 +$	$0.03x_2 +$		$0.07x_4 +$	$0.1x_5 +$		$0.08x_7$	$\leq$	384
PL			$0.01x_3 +$		$0.05x_5 +$		$0.05x_7$	$\leq$	384
Market (upper bounds)	500	1000	300	300	800	200	100		

where GR, VD, HD, BR and PL stand for grinding, vertical drilling, horizontal drilling, boring and planing, respectively.

The variables x_i represent the quantities of PROD i to be made.

The single-period problems for the other months would be similar apart from different market bounds, and different capacity figures for the different types of machine.

13.3.2 The multi-period problem

It is necessary each month to distinguish the quantities of each product manufactured from the quantities sold and held over in storage. These quantities must be represented by different variables. Suppose the quantities of PROD 1 manufactured, sold and held over in successive months are represented by variables with the following names:

MPROD 11,	MPROD 12,	and so on,
SPROD 11,	SPROD 12,	and so on,
HPROD 11,	HPROD 12,	and so on.

It is necessary to link these variables together by the relation

$$\begin{aligned}&\text{quantity held in month } (t-1) + \text{quantity manufactured in month } t\\&\qquad = \text{quantity sold in month } t + \text{quantity held in month } t.\end{aligned}$$

Initially (month 0), there is nothing held but finally (month 6) there are 50 of each product held. This relation involving PROD 1 gives rise to the following constraints:

$$\begin{aligned}\text{MPROD 11} - \text{SPROD 11} - \text{HPROD 11} &= 0,\\ \text{HPROD 11} + \text{MPROD 12} - \text{SPROD 12} - \text{HPROD 12} &= 0,\\ \text{HPROD 12} + \text{MPROD 13} - \text{SPROD 13} - \text{HPROD 13} &= 0,\\ \text{HPROD 13} + \text{MPROD 14} - \text{SPROD 14} - \text{HPROD 14} &= 0,\\ \text{HPROD 14} + \text{MPROD 15} - \text{SPROD 15} - \text{HPROD 15} &= 0,\\ \text{HPROD 15} + \text{MPROD 16} - \text{SPROD 16} &= 50.\end{aligned}$$

Similar constraints must be specified for the other six products.

It may be more convenient to define also variables HPROD 16, HPROD 26, and so on, and 'fix' them at the value 50.

In the objective function, the 'selling' variables are given the appropriate 'unit profit' figures and the 'holding' variables coefficients of −0.5.

The resulting model has the following dimensions:

$6 \times 7 = 42$	manufacturing variables
$6 \times 7 = 42$	selling variables
$6 \times 7 = \underline{42}$	holding variables
Total 126	variables
$6 \times 5 = 30$	capacity constraints
$6 \times 7 = \underline{42}$	monthly linking constraints
Total 72	constraints

In addition, in each month, there are market bounds on the seven products and bounds on the holding quantities. This gives a total of 84 upper bounds.

13.4 Factory planning 2

The extra decisions that this problem requires over the factory planning problem requires the use of integer programming. This is a clear case of extending a linear programming model by adding integer variables with extra constraints.

It is convenient to number the different types of machine as below:

Type 1	Grinders
Type 2	Vertical drills
Type 3	Horizontal drills
Type 4	Borer
Type 5	Planer

13.4.1 Extra variables

Integer variables γ_{it} are introduced with the following interpretations:

$$\gamma_{it} = \text{number of machines of type } i \text{ down from maintenance in month } t;$$

$$\text{for } i = \begin{cases} 4, 5 & \gamma_{it} \text{ will have upper bounds of 1,} \\ 1, 2 & \gamma_{it} \text{ will have upper bounds of 2,} \\ 3 & \gamma_{it} \text{ will have upper bounds of 3.} \end{cases}$$

There are 30 such integer variables.

13.4.2 Revised constraints

The machining capacity constraints in the original model must be changed as capacity will now depend on the values of γ_{it}. For example, the grinding capacity in month t will now be (in hours)

$$1536 - 384\gamma_{it}.$$

There will be similar expressions representing the capacities of other machines. The integer variables γ_{it} can be transferred to the left-hand side of the inequalities. As a result the single-period model for January would become

$$\begin{array}{ll}
\text{Maximize} & \\
\text{PROFIT} & 10x_1 + 6x_2 + 8x_3 + 4x_4 + 11x_5 + 9x_6 + 3x_7 \\
\text{subject to} & \\
\text{GR} & 0.5x_1 + 0.7x_2 + 0.3x_5 + 0.2x_6 + 0.5x_7 + 384\gamma_{11} \leq 1536, \\
\text{VD} & 0.1x_1 + 0.2x_2 + 0.3x_4 + 0.6x_6 + 384\gamma_{21} \leq 768, \\
\text{HD} & 0.2x_1 + 0.8x_3 + 0.6x_7 + 384\gamma_{31} \leq 1152, \\
\text{BR} & 0.05x_1 + 0.03x_2 + 0.07x_4 + 0.1x_5 + 0.08x_7 + 384\gamma_{41} \leq 384, \\
\text{PL} & 0.01x_3 + 0.05x_5 + 0.05x_7 + 384\gamma_{51} \leq 384.
\end{array}$$

The upper market bounds would still apply together with upper bounds on the new integer variables.

The extension to a multi-period model would be similar to that described from the original problem.

It is necessary to ensure that each machine (apart from the grinders) is down for maintenance once in six months. This is achieved by the following constraints:

$$\sum_{t=1}^{6} \gamma_{it} = \begin{cases} 2 & \text{for} \quad i = 1, 2, \\ 3 & \text{for} \quad i = 3, \\ 1 & \text{for} \quad i = 4, \\ 1 & \text{for} \quad i = 5. \end{cases}$$

The new model therefore has five extra constraints.

Clearly, the solution to the original problem implies a feasible (though probably non-optimal) solution to the new problem. The optimal objective value obtained there can usefully be used as a cut-off value in the tree search.

An alternative formulation is possible using a 0–1 variable to indicate for *each* machine whether it is down for maintenance in a particular month or not. Such a formulation would have more variables and suffer the drawback, mentioned in Section 10.1, of producing equivalent alternate solutions in the tree search.

13.5 Manpower planning

A number of applications of linear programming to manpower planning have been published. Selected references are Davies (1973), Price and Piskor (1972), who apply goal programming, and Vajda (1975).

In order to formulate the problem presented here, it will be assumed that everything happens on the first day of each year. Clearly, this assumption is far from the truth. It is necessary to make some such assumption as it is only possible to represent quantities at discrete points of time if linear programming is to be applied.

On the first day of each year the following changes will take place simultaneously:

1. Workers will be recruited into all categories.
2. A certain proportion of these will leave immediately (less than one year's service).
3. A certain proportion of last year's labour force will leave (more than one year's service).
4. A certain number of workers will be (simultaneously) retrained.
5. A certain number of workers will be declared redundant.
6. A certain number of workers will be put on short time.

13.5.1 Variables

Strength of Labour Force

$t_{\mathrm{SK}i}$ = number of skilled workers employed in year i
$t_{\mathrm{SS}i}$ = number of semi-skilled workers employed in year i
$t_{\mathrm{US}i}$ = number of unskilled workers employed in year i

Recruitment

$u_{\mathrm{SK}i}$ = number of skilled workers recruited in year i
$u_{\mathrm{SS}i}$ = number of semi-skilled workers recruited in year i
$u_{\mathrm{US}i}$ = number of unskilled workers recruited in year i

Retraining

$v_{\mathrm{USSS}i}$ = number of unskilled workers retrained to semi-skilled in year i
$v_{\mathrm{SSSK}i}$ = number of semi-skilled workers retrained to skilled in year i

Downgrading

$v_{\mathrm{SKSS}i}$ = number of skilled workers downgraded to semi-skilled in year i
$u_{\mathrm{SKUS}i}$ = number of skilled workers downgraded to unskilled in year i
$v_{\mathrm{SSUS}i}$ = number of semi-skilled workers downgraded to unskilled in year i

Redundancy

$w_{\mathrm{SK}i}$ = number of skilled workers made redundant in year i

$w_{\mathrm{SS}i}$ = number of semi-skilled workers made redundant in year i
$w_{\mathrm{US}i}$ = number of unskilled workers made redundant in year i

Short-time Working

$x_{\mathrm{SK}i}$ = number of skilled workers on short-time working in year i
$x_{\mathrm{SS}i}$ = number of semi-skilled workers on short-time working in year i
$x_{\mathrm{US}i}$ = number of unskilled workers on short-time working in year i

Overmanning

$y_{\mathrm{SK}i}$ = number of superfluous skilled workers employed in year i
$y_{\mathrm{SS}i}$ = number of superfluous semi-skilled workers employed in year i
$y_{\mathrm{US}i}$ = number of superfluous unskilled workers employed in year i

13.5.2 Constraints

Continuity

$$\begin{aligned}
t_{\mathrm{SK}i} &= 0.95 t_{\mathrm{SK}i-1} t + 0.9 u_{\mathrm{SK}i} + 0.95 v_{\mathrm{SSSK}i} - v_{\mathrm{SKSS}i} - v_{\mathrm{SKUS}i} - w_{\mathrm{SK}i},\\
t_{\mathrm{SS}i} &= 0.95 t_{\mathrm{SS}i-1} + 0.8 u_{\mathrm{SS}i} + 0.95 v_{\mathrm{USSS}i} - v_{\mathrm{SSSK}i} + 0.5 v_{\mathrm{SKSS}i}\\
&\quad - v_{\mathrm{SSUS}i} - w_{\mathrm{SS}i},\\
t_{\mathrm{US}i} &= 0.9 t_{\mathrm{US}i-1} + 0.75 u_{\mathrm{US}i} - v_{\mathrm{USSS}i} + 0.5 v_{\mathrm{SKUS}i} + 0.5 v_{\mathrm{SSUS}i} - w_{\mathrm{US}i}.
\end{aligned}$$

Retraining Semi-skilled Workers

$$v_{\mathrm{SSSK}i} - 0.25 t_{\mathrm{SK}i} \le 0.$$

Overmanning

$$y_{\mathrm{SK}i} + y_{\mathrm{SS}i} + y_{\mathrm{US}i} \le 150.$$

Requirements

$$\begin{aligned}
t_{\mathrm{SK}i} - y_{\mathrm{SK}i} - 0.5 x_{\mathrm{SK}i} &= 1000, 1500, 2000 \ \ (i = 1, 2, 3)\,,\\
t_{\mathrm{SS}i} - y_{\mathrm{SS}i} - 0.5 x_{\mathrm{SS}i} &= 1400, 2000, 2500 \ \ (i = 1, 2, 3)\,,\\
t_{\mathrm{US}i} - y_{\mathrm{US}i} - 0.5 x_{\mathrm{US}i} &= 1000, 500, 0 \qquad (i = 1, 2, 3)\,.
\end{aligned}$$

13.5.3 Initial conditions

The initial conditions are $t_{\mathrm{SK}0} = 1000$, $t_{\mathrm{SS}0} = 1500$, $t_{\mathrm{US}0} = 2000$.

Some variables have upper bounds. These are (for $i = 1, 2, 3$)

Recruitment	Short-time working	Retraining
$u_{\mathrm{SK}i} \leq 500$	$x_{\mathrm{SK}i} \leq 50$	$v_{\mathrm{USSS}i} \leq 200$
$u_{\mathrm{SS}i} \leq 800$	$x_{\mathrm{SS}i} \leq 50$	
$u_{\mathrm{US}i} \leq 500$	$x_{\mathrm{US}i} \leq 50$	

To minimize *redundancy*, the objective function is

$$\sum_i (w_{\mathrm{SK}i} + w_{\mathrm{SSS}i} + w_{\mathrm{US}i}).$$

To minimize *cost*, the objective function is

$$\begin{aligned}\sum_i (&400v_{\mathrm{USS}i} + 500v_{\mathrm{SSSK}i} + 200w_{\mathrm{US}i} + 500w_{\mathrm{SS}i} + 500w_{\mathrm{SK}i}\\ &+ 500x_{\mathrm{US}i} + 400x_{\mathrm{SS}i} + 400x_{\mathrm{SK}i} + 1500y_{\mathrm{US}i} + 2000y_{\mathrm{SS}i}\\ &+ 3000y_{\mathrm{SK}i}).\end{aligned}$$

This formulation has 24 constraints and 60 variables as well as simple upper bounds on 21 variables.

With most packages, it is convenient to incorporate both objectives into the model as 'non-constraint' rows. It is then possible to optimize both objectives within one computer run by means of the control program. In some packages, it is possible to form a composite objective through the control program, taking a certain linear combination of the original objectives. Alternatively, one of the objectives can be made a constraint. For a model such as this, with only two objectives, the most efficient way of investigating their effect is to treat one as a constraint and to perform parametric programming on its right-hand side.

13.6 Refinery optimization

The petroleum industry is the major user of linear programming models. This is a very small version of a typical application. Generally, the models used will consist of thousands of constraints, linking together possibly more than one oil refinery, giving a structured model as described in Section 4.1. The application of linear programming in the petroleum industry is described by Manne (1956).

13.6.1 Variables

In view of the many different sorts of variables in a model of this sort, it is convenient to use mnemonic names in this description of the formulation. The following variables are used to represent quantities of the materials (measured in barrels):

CRA	crude 1
CRB	crude 2
LN	light naphtha
MN	medium naphtha
HN	heavy naphtha
LO	light oil
HO	heavy oil
R	residuum
LNRG	light naphtha used to produce reformed gasoline
MNRG	medium naphtha used to produce reformed gasoline
HNRG	heavy naphtha used to produce reformed gasoline
RG	reformed gasoline
LOCGO	light oil used to produce cracked oil and cracked gasoline
HOCGO	heavy oil used to produce cracked oil and cracked gasoline
CG	cracked gasoline
CO	cracked oil
LNPMF	light naphtha used to produce premium motor fuel
LNRMF	light naphtha used to produce regular motor fuel
MNPMF	medium naphtha used to produce premium motor fuel
MNRMF	medium naphtha used to produce regular motor fuel
HNPMF	heavy naphtha used to produce premium motor fuel
HNRMF	heavy naphtha used to produce regular motor fuel
RGPMF	reformed gasoline used to produce premium motor fuel
RGRMF	reformed gasoline used to produce regular motor fuel
CGPMF	cracked gasoline used to produce premium motor fuel
CGRMF	cracked gasoline used to produce regular motor fuel
LOJF	light oil used to produce jet fuel
HOJF	heavy oil used to produce jet fuel
RJF	residuum used to produce jet fuel
COJF	cracked oil used to produce jet fuel
RLBO	residuum used to produce lube-oil
PMF	premium motor fuel
RMF	regular motor fuel
JF	jet fuel

FO	fuel oil
LBO	lube-oil

There are 36 such variables.

13.6.2 Constraints

Availabilities

The limited availability of the crude oils gives simple upper bounding constraints:

$$\text{CRA} \le 20\,000,$$

$$\text{CRB} \le 30\,000.$$

Capacities

The distillation capacity constraint is

$$\text{CRA} + \text{CRB} \le 45\,000.$$

The reforming capacity constraint is

$$\text{LNRG} + \text{MNRG} + \text{HNRG} \le 10\,000.$$

The cracking capacity constraint is

$$\text{LOCGO} + \text{HOCGO} \le 8000.$$

The stipulation concerning production of lube-oil gives the following lower and upper bounding constraints:

$$\text{LBO} \ge 500,$$

$$\text{LBO} \le 1000.$$

Continuities

The quantity of light naphtha produced depends on the quantities of the crude oil used, taking into account the way in which each crude splits under distillation. This gives

$$-0.1\text{CRA} - 0.15\text{CRB} + \text{LN} = 0.$$

Similar constraints exist for MN, HN, LO, HO and R.

The quantity of reformed gasoline produced depends on the quantities of the naphthas used in the reforming process. This gives the constraint

$$-0.6\ \text{LNRG} - 0.52\ \text{MNRG} - 0.45\ \text{HNRG} + \text{RG} = 0.$$

The quantities of cracked oil and cracked gasoline produced depend on the quantities of light and heavy oil used. This gives the constraints

$$-0.68\ \mathrm{LOCGO} - 0.75\ \mathrm{HOCGO} + \mathrm{CO} = 0,$$

$$-0.28\ \mathrm{LOCGO} - 0.2\ \mathrm{HOCGO} + \mathrm{CG} = 0.$$

The quantity of lube-oil produced (and sold) is 0.5 times the quantity of residuum used. This gives

$$-0.5\mathrm{RLBO} + \mathrm{LBO} = 0.$$

The quantities of light naphtha used for reforming and blending are equal to the quantities available. This gives

$$-\mathrm{LN} + \mathrm{LNRG} + \mathrm{LNPMF} + \mathrm{LNRMF} = 0.$$

Similar constraints exist for MN and HN.

The quantities of light oil used for cracking and blending are equal to the quantities available.

For the blending of fuel oil, the proportion of light oil is *fixed* at 10/18. Therefore separate variables have not been introduced for this proportion as it is determined by the variable LO. This gives

$$-\mathrm{LO} + \mathrm{LOCGO} + \mathrm{LOJF} + 0.55\ \mathrm{FO} = 0.$$

Similar constraints exist for HO, CO and R, also involving *fixed proportions* of FO, and for CG and RG.

The quantity of premium motor fuel produced is equal to the total quantity of its ingredients. This gives

$$-\mathrm{LNPMF} - \mathrm{MNPMF} - \mathrm{HNPMF} - \mathrm{RGPMF} - \mathrm{CGPMF} + \mathrm{PMF} = 0.$$

Similar constraints exist for RMF and JF.

Premium motor fuel production must be at least 40% of regular motor fuel production, giving

$$\mathrm{PMF} - 0.4\ \mathrm{RMF} \geq 0.$$

Qualities

It is necessary to stipulate that the octane number of premium motor fuel does not drop below 94. This is done by the constraint

$$-90\ \mathrm{LNPMF} - 80\ \mathrm{MNPMF} - 70\ \mathrm{HNPMF} - 115\ \mathrm{RGPMF}$$
$$- 105\ \mathrm{CGPMF} + 94\ \mathrm{PMF} \leq 0.$$

There is a similar constraint for RMF.

For jet fuel we have the constraint imposed by vapour pressure. This is

$$-\text{LOJF} - 0.6\ \text{HOJF} - 1.5\ \text{COJF} - 0.05\ \text{RJF} + \text{JF} \geq 0.$$

This model has 29 constraints together with simple bounds on three variables.

Some comment should be made concerning the blending of fuel oil where the ingredients (light, heavy, and cracked oil and residuum) are taken in *fixed* proportions. Here it might be preferable to think of the production of FO as an *activity*. It is common in the oil industry to think in terms of activities rather than quantities. In Section 3.4, *modal formulations* are discussed where activities represent the extreme *modes* of operation of a process. Here we have a special case of a process with one mode of operation. The level of this activity then fixes the proportions of the ingredients, in a case such as this, automatically.

13.6.3 Objective

The only variables involving a profit (or cost) are the final products. This gives an objective (in pounds) to be maximized of

$$7\ \text{PMF} + 6\ \text{RMF} + 4\ \text{JF} + 3.5\ \text{FO} + 1.5\ \text{LBO}.$$

13.7 Mining

This problem has a combinatorial character. In each year a choice of up to three out of the four possible mines must be chosen for working. There are 15 ways of doing this each year, giving a total of 15^5 possible ways of working over five years. For larger problems with, say, 15 mines being considered over 20 years the number of possibilities will be astronomical. By using integer programming, with the branch and bound method, only a fraction of these possibilities need be investigated.

0–1 variables are introduced to represent decisions whether to work or not to work a particular mine in a certain year. The different sorts of variables are described below.

13.7.1 Variables

$$\delta_{it} = \begin{cases} 1 & \text{if mine } i \text{ is worked in year } t, \\ 0 & \text{otherwise.} \end{cases}$$

There are 20 such 0–1 integer variables.

$$\gamma_{it} = \begin{cases} 1 & \text{if mine } i \text{ is ‘open’ in year } t \quad \text{(i.e. royalities are payable)} \\ 0 & \text{otherwise.} \end{cases}$$

There are 20 such 0–1 integer variables.

$$x_{it} = \text{output from mine } i \text{ in year } t \text{ (millions of tons)}.$$

There are 20 such continuous variables.

$$q_t = \text{quantity of blended ore produced in year } t \text{ (millions of tons)}.$$

There are five such continuous variables.

In total, there are therefore 65 variables, of which 40 are integer.

13.7.2 Constraints

$$x_{it} - M_i\delta_{it} \leq 0 \text{ for all } i, t.$$

M_i is the maximum yearly output from mine i. This constraint implies that if mine i is not worked in year t there can be no output from it in that year. There are 20 such constraints.

$$\sum_{i=1}^{4} \delta_{it} \leq 3 \quad \text{for all } t.$$

This constraint allows no more than three mines to be worked in any year. There are five such constraints.

$$\delta_{it} - \gamma_{it} \leq 0 \quad \text{for all } i, t.$$

According to this constraint, if mine i is 'closed' in year t, it cannot be worked in that year. There are 20 such constraints.

$$\gamma_{it+1} - \gamma_{it} \leq 0 \quad \text{for all } i, t < 5.$$

This forces a mine to be closed in all years subsequent to that in which it is first closed. There are 16 such constraints.

$$\sum_{i=1}^{4} Q_i x_{it} - P_t q_t = 0 \quad \text{for all } t.$$

Q_i is the quality of the ore from mine i and P_t the quality required in year t. There are five such blending constraints.

$$\sum_{i=1}^{4} x_{it} - q_t = 0 \quad \text{for all } t.$$

This constraint ensures that the tonnage of blended ore in each year equals the combined tonnage of the constituents. There are five such constraints.

In total there are 71 constraints.

13.7.3 Objective

The total profit consists of the income from selling the blended ore minus the royalties payable. This is to be maximized. It can be written

$$-\sum_{\substack{i=1,4\\t=1,5}} R_{it}\gamma_{it} + \sum_{t=1,5} I_t q_t.$$

R_{it} is the royalty payable on mine i in year t discounted at a rate of 10% per annum. I_t is the selling price of each ton of blended ore in year t discounted at a rate of 10% per annum. The advantage of this formulation of the problem over an alternative one is discussed by Williams (1978).

There would seem to be advantage in this model in working mines early in the hope that they may be closed permanently later. In addition, the discounting of revenue gives an advantage to working mines early. The suggested solution strategy is therefore to branch on the variables δ_{it}, giving priority to low values of t.

13.8 Farm planning

This problem is based on that of Swart *et al*. (1975). By only considering five years, the model can be kept reasonably small but inevitably much of the realism is lost.

There are a number of different ways of formulating this problem. In the formulation suggested here, a large number of variables are introduced whose values are effectively fixed by the constraints of the model. These variables represent the numbers of cows of different ages in each year. For example, the number of cows of ages 1, 2, and so on, in year 1 will be determined by the initial numbers of cows given in year 0. Similarly, the numbers of cows of ages 2, 3 and so on, in year 2 will be fixed. It would therefore be possible to calculate the values of all these variables and not introduce them into the model. While this would make for a more compact model, it would not be as easy to understand. Nor does it seem worthwhile to carry out manual calculation best done by a computer. The most satisfactory course of action is to *reduce* or *presolve* the model in order to determine the *fixed* values of these variables. Experience with doing this is reported with the solution in Part IV.

13.8.1 Variables

x_{it} = number of tons of grain grown on group i land in year t
y_t = number of tons of sugar beet grown in year t
z_t = number of tons of grain bought in year t

s_t	= number of tons of grain sold in year t
u_t	= number of tons of sugar beet bought in year t
v_t	= number of tons of sugar beet sold in year t
l_t	= extra labour recruited in year t (in units of 100 hours)
m_t	= capital outlay in year t (in units of £200)
n_t	= number of heifers sold at birth in year t
q_{jt}	= number of cows of age j years in year t
r_t	= number of cows of age 0 in year t
p_t	= profit in year t

$i = 1, 2, 3, 4; t = 1, 2, 3, 4, 5$ and $j = 1, 2, \ldots, 12$. (A variable n_6 is also defined to allow heifers to be sold at the beginning of the sixth year.)

As with other planning models, it is necessary to consider discrete intervals of time and assume changes occur once in each interval. In this case, we assume changes occur once each year.

13.8.2 Constraints

Continuity

$$\begin{aligned}
q_{1,t+1} &= 0.95r_t, && \text{for} \quad t = 1, 2, 3,\\
q_{2,t+1} &= 0.95q_{1t}, && \text{for} \quad t = 1, 2, 3,\\
q_{j+1,t+1} &= 0.98q_{jt}, && \text{for all} \quad j > 1, t = 1, 2, 3,\\
r_t &= \frac{1.1}{2}\sum_{j=2,11} q_{jt} - n_t, && \text{for all} \quad t.
\end{aligned}$$

Initial Conditions (Fixed Variables)

$$\begin{aligned}
q_{j1} &= 9.5, \text{ for } \quad j = 1,\ 2,\\
q_{j1} &= 9.8, \text{ for } \quad j = 3,\ 4,\ \ldots, 12.
\end{aligned}$$

Accommodation

$$r_t + \sum_{j=1,11} q_{jt} \leq 130 + \sum_{k \leq t} m_k, \quad \text{for all } t.$$

Grain Consumption

$$\sum_{j=2,11} q_{jt} \leq \frac{1}{0.6}\left(\sum_{i=1,4} x_{it} + z_t - s_t\right), \quad \text{for all } t.$$

Sugar Beet Consumption

$$\sum_{j=2,11} q_{it} \leq \frac{1}{0.7}\left(y_t + u_t - v_t\right), \quad \text{for all } t.$$

Grain Growing

$$\begin{aligned} x_{1t} &\leq 1.1 \times 20, \qquad \text{for all } t. \\ x_{2t} &\leq 0.9 \times 30, \\ x_{3t} &\leq 0.8 \times 20, \\ x_{4t} &\leq 0.65 \times 10, \end{aligned}$$

Acreage

$$\begin{aligned} &\frac{1}{1.1}x_{1t} + \frac{1}{0.9}x_{2t} + \frac{1}{0.8}x_{3t} + \frac{1}{0.65}x_{4t} + \frac{1}{1.5}y_t + \frac{2}{3}r_t + \frac{2}{3}q_{1t} \\ &\quad + \sum_{j=2,11} q_{jt} \leq 200, \quad \text{for all } t. \end{aligned}$$

Labour (in 100 hours)

$$\begin{aligned} &0.1r_t + 0.1q_{1t} + 0.42\sum_{j=2,11} q_{jt} + 0.04\left(\frac{1}{1.1}x_{1t} + \frac{1}{0.9}x_{2t} + \frac{1}{0.8}x_{3t}\right. \\ &\qquad\qquad \left. + \frac{1}{0.65}x_{4t}\right) + \frac{0.14}{1.5}y_t \leq 55 + l_t \quad \text{for all } t. \end{aligned}$$

End Total

$$\sum_{j=2,11} q_{j5} \leq 175,$$

$$\sum_{j=2,11} q_{j5} \geq 50.$$

(This constraint may be specified by a *range* of 125 on the previous constraint.)

Profit

$$p_t = 30 \times \frac{1.1}{2} \sum_{j=2,11} q_{jt} + 40n_t \text{ (selling heifers)}$$

$$+ 120q'_{12,t} \text{ (selling 12-year-old cows)}$$

$$+ 370 \sum_{j=2,11} q_{jt} \text{ (selling milk)} + 75s_t \text{ (selling grain)}$$

$$+ 58v_t \text{ (selling sugar beet)} - 90z_t \text{ (buying grain)}$$

$$- 70u_t \text{ (buying sugar beet)} - 120l_t - 4000 \text{ (labour)}$$

$$- 50r_t - 50q_{1t} \text{ (heifer costs)} - 100 \sum_{j=2,11} q_{jt} \text{ (dairy cow costs)}$$

$$- 15\left(\frac{1}{1.1}x_{1t} + \frac{1}{0.9}x_{2t} + \frac{1}{0.8}x_{3t} + \frac{1}{0.65}x_{4t}\right) \text{ (grain costs)}$$

$$- \frac{10}{1.5}y_t \text{ (sugar beet costs)} - 39.71\sum_{k\le t} m_k \text{ (capital costs),} \quad \text{for all } t.$$

(The annual repayment on a £200 loan is £39.71.)

Profit Can Never be Negative

$$p_t \ge 0, \text{ for all } t.$$

(The main effect of this constraint is to limit capital expenditure to cash available.)

13.8.3 Objective function

In order to make capital expenditure as 'costly' in latter years as in former ones, it is necessary to take account of repayments beyond the five years. This gives an objective function (to be maximized):

$$\sum_{t=1,5} p_t - 39.71 \sum_{t=1,5} (4+t)\, m_t.$$

The costs incurred by the repayments beyond the five years must be credited back when the final profit over the five years is obtained.

This model has 84 constraints and 130 variables.

13.9 Economic planning

The model resulting from this problem is a dynamic Leontief model of the type mentioned in Section 5.2. A rather similar model has been considered by Wagner (1957).

13.9.1 Variables

x_{it} = total output of industry i in year $t(i =$ C (coal), S(steel), T(transport), $t = 1, 2, \ldots, 5)$
s_{it} = stock level of industry i at the beginning of year t
y_{it} = extra productive capacity for industry i becoming effective in year $t(t = 2, 3, \ldots, 6)$

13.9.2 Constraints

Total Input

$$\sum_{j=1,3} c_{ij} x_{jt+1} + \sum_{j=1,3} d_{ij} y_{jt+2} + s_{it} - s_{it+1}$$
$$- \text{ exogenous demand for industry } i, \text{ for all } i, t$$

(s_{i0} are initial stocks given), where c_{ij} and d_{ij} are the input/output coefficients in the first three rows of Tables 12.1 and 12.2 respectively in the statement of the problem.

Manpower

$$0.6x_{\mathrm{C}t+1} + 0.3x_{\mathrm{S}t+1} + 0.2x_{\mathrm{T}t+1} + 0.4y_{\mathrm{C}t+2} + 0.2y_{\mathrm{S}t+2} + 0.1y_{\mathrm{T}t+2} \le 470 \text{ for all } t.$$

Productive Capacity

$$x_{it} \le \underset{\text{of } i \text{ (year 0)}}{\text{initial capacity}} + \sum_{l \le t} y_{il} \quad \text{for all } i, t.$$

In order to build a realistic model, it is necessary to think beyond the end of the five-year period. To ignore exogenous demand in the sixth and subsequent years would result in no inputs being accounted for in the fifth year. We therefore assume that exogenous demand remains constant up to and beyond year 5, the stock level remains constant, and that there is no increase in productive capacity after year 5. In order to find the inputs to each industry in year 5, we simply have to solve a *static Leontief model*:

$$\sum_j c_{ij} x_j = x_i - \text{endogenous demand for industry } i, \text{ for all } i.$$

x_i is the (static) output from industry i in year 5 and beyond.

This set of three equations gives lower limits to the variables, giving

$$x_{\mathrm{C}} \geq 116.4,$$
$$x_{\mathrm{S}} \geq 105.7,$$
$$x_{\mathrm{T}} \geq 92.3.$$

In the total output constraint above, x_{it} will be set greater than or equal to these values for $t \geq 6$. y_{it} will be set to 0 for $t \geq 6$.

13.9.3 Objective function

1. Maximize

$$\sum_{\substack{i \\ l \leq 5}} y_{il}$$

2. Maximize

$$\sum_{\substack{i \\ t=4,5}} x_{it}$$

(Exogenous demand is set to 0 with this objective.)

3. Maximize

$$\sum_t \left(0.6x_{\mathrm{C}t+1} + 0.3x_{\mathrm{S}t+1} + 0.2x_{\mathrm{T}t+1} + 0.4y_{\mathrm{C}t+2} + 0.2y_{\mathrm{S}t+2} + 0.1y_{\mathrm{T}t+2}\right).$$

(The manpower constraint is ignored with this objective.)

This model has 45 variables and 42 constraints.

13.10 Decentralization

This is a modified form of the quadratic assignment problem described in Section 9.5. Methods of solving such problems are described by Lawler (1974). The problem presented here is based on the problem described by Beale and Tomlin (1972). They treat their problem by linearizing the quadratic terms and reducing the problem to a 0–1 integer programming problem. We adopt the same approach here.

13.10.1 Variables

$$\delta_{ij} = \begin{cases} 1 & \text{if department } i \text{ is located in city } j \ (i = \text{A, B, C, D, E, } j = \text{L (London), S (Bristol), G (Brighton)}), \\ 0 & \text{otherwise} \end{cases}$$

There are 15 such 0–1 variables.

$$\gamma_{ijkl} = \begin{cases} 1 & \text{if } \delta_{ij} = 1 \text{ and } \delta_{kl} = 1, \\ 0 & \text{otherwise.} \end{cases}$$

γ_{ijkl} is only defined for $i < k$ and $C_{ik} \neq 0$. There are 54 such 0–1 variables.

13.10.2 Constraints

Each department must be located in exactly one city. This gives the constraints

$$\sum_j \delta_{ij} = 1, \quad \text{for all } i.$$

There are five such constraints. These constraints can be treated as special ordered sets of type 1 as described in Section 9.3.

No city may be the location for more than three departments. This gives the constraints

$$\sum_j \delta_{ij} \leq 3, \quad \text{for all } j.$$

There are three such constraints.

Using the variables δ_{ij} together with the two types of constraint above, we could formulate a model with an objective function involving some quadratic terms $\delta_{ij}\delta_{kl}$. Instead, these terms are replaced by the 0–1 variables γ_{ijkl} giving a linear objective function. It is, however, necessary to relate these new variables to the δ_{ij} variables correctly. To do this we model the relations

$$\gamma_{ijkl} = 1 \rightarrow \delta_{ij} = 1, \quad \delta_{kl} = 1$$

and

$$\delta_{ij} = 1, \ \delta_{kl} = 1 \rightarrow \gamma_{ijkl} = 1$$

Following the discussion in Section 9.2, the first conditions can be achieved by the following constraints:

$$\gamma_{ijkl} - \delta_{ij} \leq 0 \ \text{ for all } i, j, k > i, i,$$

$$\gamma_{ijkl} - \delta_{kl} \leq 0 \ \text{ for all } i, j, k > i, l.$$

There are 108 such constraints.

The second conditions are achieved by the constraints

$$\delta_{ij} + \delta_{kl} - \gamma_{ijkl} \leq 1 \text{ for all } i, j, k > i, l.$$

There are 54 such constraints.

13.10.3 Objective

The objective is to minimize

$$-\sum_{i,j} B_{ij}\delta_{ij} + \sum_{\substack{i,j,k,l\\ l<k}} C_{ik} D_{jl}\gamma_{ijkl},$$

where B_{ij} is the benefit to be gained from locating department i in city j as given in Part II (for $j = \mathrm{L}$ (London), $B_{ij} = 0$). C_{ik} and D_{ji} are given in the tables in Part II, Section 12.10.

This model has 162 constraints and 69 variables (all 0–1).

Beale and Tomlin formulate their model more compactly. As a consequence, the corresponding linear programming problem is much less constrained than it might be (see Section 10.1). They then expand some of the constraints. Use is then made of the branching strategy to avoid expanding other constraints.

13.11 Curve fitting

This is an application of the goal programming type of formulation discussed in Section 3.3. Each pair of corresponding data values (x_i, y_i) gives rise to a constraint. For cases (1) and (2), these constraints are

$$bx_i + a + u_i - v_i = y_i, \qquad i = 1, 2, \ldots, 19.$$

x_i and y_i are constants (the given values); b, a, u_i and v_i are variables. u_i and v_i give the amounts by which the values of y_i proposed by the linear expression differ from the observed values. It is important to allow a and b to be 'free' variables, that is, they can be allowed to take negative as well as positive values.

In case (1) the objective is to minimize

$$\sum_i u_i + \sum_i v_i.$$

This model has 19 constraints and 40 variables.

In case (2) it is necessary to introduce another variable z together with 38 more constraints:

$$z - u_i \geq 0, \qquad z - v_i \geq 0, \qquad i = 1, 2, \ldots, 19.$$

The objective, in this case, is simply to minimize z. This minimum value of z will clearly be exactly equal to the maximum value of v_i and u_i.

In case (3), it is necessary to introduce a new (free) variable c into the first set of constraints to give

$$cx_i^2 + bx_i + a + u_i - v_i = y, \qquad i = 1, 2, \ldots, 19.$$

The same objective functions as in (1) and (2) will apply.

It is much more usual in statistical problems to minimize the sum of squares of the deviations as the resultant curve often has desirable statistical properties. There are, however, some circumstances in which a sum of absolute deviations is acceptable or even more desirable. Moreover, the possibility of solving this type of problem by linear programming makes it computationally easy to deal with large quantities of data.

Minimizing the maximum deviation has certain attractions from the point of view of presentation. The possibility of a single data point appearing a long way off the fitted curve is minimized.

13.12 Logical design

In order to simplify the formulation, the optimal circuit can be assumed to be a subnet of the maximum shown in Figure 13.1. The following 0–1 integer variables are used:

$$s_i = \begin{cases} 1 & \text{if NOR gate } i \text{ exists, } i = 1, 2, \ldots, 7, \\ 0 & \text{otherwise;} \end{cases}$$

$$t_{i1} = \begin{cases} 1 & \text{if external input } A \text{ is an input to gate } i, \\ 0 & \text{otherwise;} \end{cases}$$

$$t_{i2} = \begin{cases} 1 & \text{if external input } B \text{ is an input to gate } i, \\ 0 & \text{otherwise.} \end{cases}$$

x_{ij} is the output from gate i for the combination of external input signals specified in the jth row of the truth table.

The following constraints are imposed.

A NOR gate can only have an external input if it exists. These conditions are imposed by the constraints

$$s_i - t_{i1} \geq 0, \qquad s_i - t_{i2} \geq 0, \qquad i = 1, 2, \ldots, 7.$$

If a NOR gate has one (or two) external inputs leading into it, only one (or no) NOR gates can feed into it. These conditions are imposed by the constraints

$$s_j + s_k + t_{i1} + t_{i2} \leq 2, \qquad i = 1, 2, 3$$

where j and k are the two NOR gates leading into i in Figure 13.1.

The output signal from NOR gate i must be the correct logical function (NOR) of the input signals into gate i if gate i exists. Let α_{1j} (a constant) be the value of the external input signal A in the jth row of the truth table. Similarly

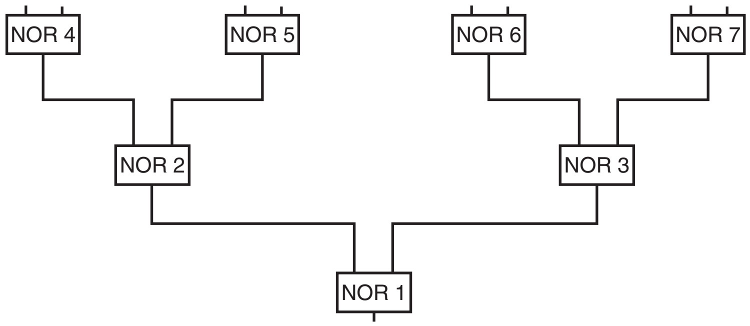


Figure 13.1

α_{2j} corresponds to the external input signal B. These conditions give

$$\left.\begin{aligned} x_{jl} + x_{il} &\le 1, \\ x_{kl} + x_{il} &\le 1, \end{aligned}\right. \quad i = 1, 2, 3$$

$$\left.\begin{aligned} \alpha_{1l}t_{i1} + x_{il} &\le 1, \\ \alpha_{2l}t_{i2} + x_{il} &\le 1, \end{aligned}\right. \quad i = 1, 2, \ldots, 7;$$

$$\alpha_{1l}t_{i1} + \alpha_{2l}t_{i2} + x_{il} - s_i \ge 0, \quad i = 4, 5, 6, 7;$$

$$\alpha_{1l}t_{i1} + \alpha_{2l}t_{i2} + x_{jl} + x_{kl} + x_{il} - s_i \ge 0, \quad i = 1, 2, 3.$$

The above six classes of constraints are defined for $l = 1, 2, \ldots, 4$, where j and k are the NOR gates leading into gate i in Figure 13.1. As the α_{ij} are constants, some of the above constraints are redundant for particular values of l and may be ignored.

For NOR gate 1, the x_{1l} variables are fixed at the values specified in the truth table, that is,

$$x_{11} = 0, \qquad x_{12} = 1, \qquad x_{13} = 1, \qquad x_{14} = 0.$$

If there is an output signal of 1 from a particular NOR gate for any combination of the input signals, then that gate must exist.

$$s_i - x_{il} \ge 0, \qquad i = 1, 2, \ldots, 7, \qquad l = 1, 2, 3, 4.$$

In order to avoid a trivial solution containing no NOR gates, it is necessary to impose a constraint such as

$$s_1 \ge 1$$

The objective is to minimize $\sum_i s_i$.

This model has 154 constraints and 49 variables (all 0−1), including the four fixed variables. The redundant constraints are included in this total.

13.13 Market sharing

This problem can be formulated as an integer programming model where each of the 23 retailers is represented by a 0–1 variable δ_i. If $\delta_i = 1$ retailer i is assigned to division D1. Otherwise, the retailer is assigned to division D2. Slack and surplus variables can be introduced into each constraint to provide the required objective, which is to minimize the sum of the proportionate deviations from the desired 'goals' of the 40/60 splits. For example, in the oil market in region 3, we have a total market of 100 (10^6 gallons). In order to split this 40/60, we have the constraint

$$6\delta_{19} + 15\delta_{20} + 15\delta_{21} + 25\delta_{22} + 39\delta_{23} + y_1 - y_2 = 40.$$

y_1 and y_2 are slack and surplus variables which are given a cost of 0.01 in the first objective function. y_1 and y_2 are also given simple upper bounds of 5. The other 'goal' constraints can be treated in a similar fashion.

In order to define the second 'minimax' objective, we need to introduce a variable z together with the constraints

$$z - \frac{y_i}{w_i} \geq 0.$$

for all slack and surplus variables y_i, where w_i is the total quantity for the 40% goal constraint with which y_i is associated. The second objective is then simply to minimize z.

The resultant model has 18 constraints and 36 variables, of which 23 are 0–1.

As it stands, this model is a correct representation of the problem but could prove difficult to solve as one feasible integer solution need bear no resemblance to another one. In consequence, the tree search for the optimal integer solution could be fairly random. As the results in Section 14.13 suggest, this model did prove difficult to solve optimally. One approach to reformulating the problem is to replace some of the constraints by 'facet' constraints in the manner described in Section 10.2. For example, the goal constraint above is equivalent to

$$\begin{aligned} 6\delta_{19} + 15\delta_{20} + 15\delta_{21} + 25\delta_{22} + 39\delta_{23} &\geq 35, \\ 6\delta_{19} + 15\delta_{20} + 15\delta_{21} + 25\delta_{22} + 39\delta_{23} &\leq 45. \end{aligned}$$

The first of these constraints can be augmented by the following constraints obtained from strong covers:

$$\begin{aligned} \delta_{20} \phantom{{}+\delta_{21}} + \delta_{22} + \delta_{23} &\geq 1, \\ \delta_{21} + \delta_{22} + \delta_{23} &\geq 1, \\ \delta_{19} \phantom{{}+\delta_{20}+\delta_{21}} + \delta_{22} + \delta_{23} &\geq 1, \\ \delta_{20} + \delta_{21} \phantom{{}+\delta_{22}} + \delta_{23} &\geq 1. \end{aligned}$$

The second of the constraints can be augmented by the constraints

$$\begin{aligned}
\delta_{20} \qquad\qquad\quad + \delta_{23} &\leq 1,\\
\delta_{21} + \qquad \delta_{23} &\leq 1,\\
\delta_{22} + \delta_{23} &\leq 1,\\
\delta_{19} + \delta_{20} \qquad + \delta_{22} + \delta_{23} &\leq 2,\\
\delta_{19} \qquad + \delta_{21} + \delta_{22} + \delta_{23} &\leq 2,\\
\delta_{20} + \delta_{21} + \delta_{22} + \delta_{23} &\leq 2.
\end{aligned}$$

All the other constraints could be treated in a similar fashion. Unfortunately, the number of such 'facet' constraints derivable from each of the goal constraints for (1) and (2) in the statement of the problem in Section 12.13 proved prohibitively large. There are, however, a large but manageable number which can be derived from (3) and (4) together with those above for (5). In total, the number of such constraints turned out to be 228. They were enumerated in a trivial amount of time, using the method described in Section 10.2.

It is still necessary to retain the original constraints, with their slack and surplus variables providing an objective function, as we are only adding in some of the 'facet' constraints for each goal constraint.

Unfortunately, this augmented formulation proved difficult to solve largely as a result of its increased size.

A third possibility is to replace some of the goal constraints by *single* tighter constraints. How this may be done for a general 0–1 constraint is described by Bradley *et al*. (1974). An example of such a single constraint simplification is provided by the problem OPTIMIZING A CONSTRAINT, in Section 13.18.

The three goal constraints for (3)–(5) in Section 12.13 can be written as three pairs of constraints:

$$\begin{gathered}
9\delta_1 + 13\delta_2 + 14\delta_3 + 17\delta_4 + 18\delta_5 + 19\delta_6 + 23\delta_7 + 21\delta_8 \leq 60,\\
9\delta_1 + 13\delta_2 + 14\delta_3 + 17\delta_4 + 18\delta_5 + 19\delta_6 + 23\delta_7 + 21\delta_8 \geq 46,\\
9\delta_9 + 11\delta_{10} + 17\delta_{11} + 18\delta_{12} + 18\delta_{13} + 17\delta_{14} + 22\delta_{15} + 24\delta_{16} + 36\delta_{17} + 43\delta_{18} \leq 96,\\
9\delta_9 + 11\delta_{10} + 17\delta_{11} + 18\delta_{12} + 18\delta_{13} + 17\delta_{14} + 22\delta_{15} + 24\delta_{16} + 36\delta_{17} + 43\delta_{18} \geq 75,\\
6\delta_{19} + 15\delta_{20} + 15\delta_{21} + 25\delta_{22} + 39\delta_{23} \leq 45,\\
6\delta_{19} + 15\delta_{20} + 15\delta_{21} + 25\delta_{22} + 39\delta_{23} \geq 35.
\end{gathered}$$

Using the method referenced above, it is possible to give six constraints that have equivalent sets of 0–1 solutions but are 'tighter' in a *linear programming* sense. These are

$$\begin{gathered}
5\delta_1 + 7\delta_2 + 8\delta_3 + 9\delta_4 + 10\delta_5 + 10\delta_6 + 13\delta_7 + 12\delta_8 \leq 33,\\
2\delta_1 + 3\delta_2 + 3\delta_3 + 4\delta_4 + 4\delta_5 + 4\delta_6 + 5\delta_7 + 5\delta_8 \geq 11,\\
8\delta_9 + 8\delta_{10} + 13\delta_{11} + 14\delta_{12} + 14\delta_{13} + 13\delta_{14} + 17\delta_{15} + 19\delta_{16} + 28\delta_{17} + 34\delta_{18} \leq 75,
\end{gathered}$$

$$8\delta_9 + 8\delta_{10} + 14\delta_{11} + 15\delta_{12} + 15\delta_{13} + 14\delta_{14} + 18\delta_{15} + 20\delta_{16} + 30\delta_{17} + 36\delta_{18} \geq 63,$$
$$\delta_{19} + 2\delta_{20} + 2\delta_{21} + 3\delta_{22} + 4\delta_{23} \leq 5,$$
$$\delta_{19} + 2\delta_{20} + 2\delta_{21} + 3\delta_{22} + 5\delta_{23} \geq 5.$$

Each of these constraints is the equivalent one, to the corresponding one above, which has the smallest right-hand side coefficient. These tighter constraints were derived in a trivial amount of time using linear programming. Details of the method are described with the example, OPTIMIZING A CONSTRAINT, in Section 13.18.

It is still necessary to retain the original constraints with their slack and surplus variables needed in the objective function. This reformulation proved the most successful. Computational experience is given with the solution in Section 14.13.

13.14 Opencast mining

This problem can be formulated as a pure 0–1 programming problem by introducing 0–1 variables δ_i and numbering the blocks:

$$\delta_i = \begin{cases} 1 & \text{if block } i \text{ is extracted,} \\ 0 & \text{otherwise.} \end{cases}$$

If a block is extracted, then the four blocks above it must also be extracted. Suppose, for example, that the numbering were such that blocks 2– 5 were directly above 1. The condition could then be imposed by the four constraints

$$\delta_2 - \delta_1 \geq 0,$$
$$\delta_4 - \delta_1 \geq 0,$$
$$\delta_4 - \delta_1 \geq 0,$$
$$\delta_5 - \delta_1 \geq 0.$$

Similar constraints can be imposed for all other blocks (apart from those on the surface).

The objective is to maximize

$$\sum_i p_i\delta_i,$$

where

$$p_i = \text{profit from block}\, i \;\; (\text{revenue} - \text{cost of extraction})\,.$$

This formulation has the extremely important property described in Section 10.1. Each constraint contains one coefficient +1 and one coefficient −1. This guarantees a *totally unimodular matrix*. If the model is solved as a continuous linear programming model, the optimal solution will be integer automatically. There is therefore no need to use, computationally much more costly, integer programming.

This property makes it possible to solve much larger versions of this problem in a reasonable amount of time. As described in Section 10.1, the dual of this linear programming model is a network flow problem and could be solved very efficiently by a specialized network flow algorithm, see Williams (1982).

The model has 56 constraints and 30 variables. Each variable has an upper bound of 1.

13.15 Tariff rates (power generation)

This problem is based on a model described by Garver (1963). The following formulation is suggested here.

13.15.1 Variables

n_{ij} = number of generating units of type i working in period j (where $j = 1, 2, 3, 4,$ and 5 are the five periods of the day listed in the question)
s_{ij} = number of generators of type i started up in period j
x_{ij} = total output rate from generators of type i in period j

x_{ij} are continuous variables; n_{ij} and s_{ij} are general integer variables.

13.15.2 Constraints

Demand must be met in each period:

$$\sum_i x_{ij} \geq D_j \quad \text{for all } j,$$

where D_j is demand given in period j.

Output must lie within the limits of the generators working:

$$x_{ij} \geq m_i n_{ij}, \qquad \text{for all } i \text{ and } j,$$
$$x_{ij} \leq M_i n_{ij}, \qquad \text{for all } i \text{ and } j,$$

where m_i and M_i are the given minimum and maximum output levels for generators of type i.

The extra guaranteed load requirement must be able to be met without starting up any more generators:

$$\sum_i M_i n_{ij} \geq \frac{115}{100} D_j \quad \text{for all } j.$$

The number of generators started in period j must equal the increase in number:

$$s_{ij} \geq n_{ij} - n_{ij-1}, \quad \text{for all } i \text{ and } j,$$

where n_{ij} is number of generators started in period j (when $j = 1$, period $j-1$ is taken as 5).

In addition, all the integer variables have simple upper bounds corresponding to the total number of generators of each type.

13.15.3 Objective function (to be minimized)

$$\text{Cost} = \sum_{i,j} C_{ij}\left(x_{ij} - m_i n_{ij}\right) + \sum_{i,j} E_{ij} n_{ij} + \sum_{i,j} F_i s_{ij},$$

where C_{ij} are costs per hour per megawatt above minimum level multiplied by the number of hours in the period; E_{ij} are costs per hour for operating at minimum level multiplied by the number of hours in the period and F_i are start-up costs.

This model has a total of 55 constraints and 30 simple upper bounds. There are 45 variables, of which 30 are general integer variables.

13.16 Hydro power

The model described in Section 13.15 can be extended by additional variables and constraints.

13.16.1 Variables

h_{ij} $= 1$ if hydro of type i working in period j
$= 0$ otherwise
where $i = 1, 2$
t_{ij} $= 1$ if hydro of type i started in period j
$= 0$ otherwise
l_j $=$ height of reservoir at beginning of period j
p_j $=$ number of megawatts of pumping in period j

13.16.2 Constraints

The demand constraint becomes

$$\sum_i x_{ij} + \sum_i L_i h_{ij} - p_j \geq D_j \quad \text{for all } j,$$

where L_i is the operating level of hydro i.

Reservoir level changes resulting from pumping and generation give

$$l_j - l_{j-1} - \frac{T_j p_j}{3000} + \sum_i T_j R_i h_{ij} = 0 \quad \text{for all } j,$$

where T_j is number of hours in period j and R_i is height reduction per hour caused by hydro j.

The extra load guarantee requirement becomes

$$\sum_i M_i n_{ij} \geq \frac{115}{100} D_j - \sum_i L_i \quad \text{for all } j.$$

The number of hydros started in period j must equal the increase in number:

$$t_{ij} \geq h_j - h_{j-1} \quad \text{for all } i \text{ and } j.$$

13.16.3 Objective function (to be minimized)

This needs to be augmented by the terms

$$\sum_{i,j} K_i T_j h_{ij} + \sum_{i,j} G_i t_{ij}$$

where K_i is the cost per hour of hydro i and G_i is the start-up cost of hydro i.

This model has 55 constraints, 25 upper bounds and 75 variables, of which 50 are integer.

13.17 Three-dimensional noughts and crosses

This ‘pure’ problem is included as it typifies the combinatorial character of quite a lot of integer programming problems. Clearly, there are an enormous number of ways of arranging the balls in the three-dimensional array. Such problems often prove difficult to solve as integer programming models. There is advantage to be gained from using a heuristic solution first. This solution can then be used to obtain a cut-off value for the branch and bound tree search as described in Section 8.3. A discussion of two possible ways of formulating this problem is given in Williams (1974). The better of those formulations is described here.

13.17.1 Variables

The cells are numbered from 1 to 27. It is convenient to number sequentially row by row and section by section. Associated with each cell a 0–1 variable δ_j is introduced with the following interpretation:

$$\delta_j = \begin{cases} 1 & \text{if cell } j \text{ contains a black ball,} \\ 0 & \text{if cell } j \text{ contains a white ball.} \end{cases}$$

There are 27 such 0–1 variables.

There are 49 possible lines in the cube. With each of these lines, we associate a 0–1 variable γ_i with the following interpretations:

$$\gamma_i = \begin{cases} 1 & \text{if all the balls in the line } i \text{ are of the same colour,} \\ 0 & \text{if there are a mixture of colours of ball in line } i. \end{cases}$$

There are 49 such 0–1 variables.

13.17.2 Constraints

We wish to ensure that the values of the variables γ_i truly represent the conditions above. In order to do this, we have to model the condition

$$\gamma_i = 0 \rightarrow \delta_{i1} + \delta_{i2} + \delta_{i3} \geq 1 \quad \text{and} \quad \delta_{i1} + \delta_{i2} + \delta_{i3} \leq 2,$$

where $i1$, $i2$, and $i3$ are the numbers of the cells in line i.

This condition can be modelled by the constraints

$$\begin{gathered} \delta_{i1} + \delta_{i2} + \delta_{i3} - \gamma_i \leq 2, \\ \delta_{i1} + \delta_{i2} + \delta_{i3} + \gamma_i \geq 1, \\ i = 1, 2, \ldots, 49. \end{gathered}$$

In fact, these constraints do not ensure that if $\gamma_i = 1$ all balls will be of the same colour in the line. When the objective is formulated, it will be clear that this condition will be guaranteed by optimality.

In order to limit the black balls to 14, we impose the constraint

$$\sum_j \delta_j = 14.$$

There are a total of 99 constraints.

13.17.3 Objective

In order to minimize the number of lines with balls of a similar colour, we minimize

$$\sum_i \gamma_i.$$

In total, this model has 99 constraints and 76 0–1 variables.

13.18 Optimizing a constraint

A procedure for simplifying a single 0–1 constraint has been described by Bradley *et al*. (1974). We adopt their procedure of using a linear programming model. It is convenient to consider the constraint in a standard form with positive coefficients in descending order of magnitude. This can be achieved by the transformation

$$\begin{aligned}
&y_1 = x_7, \quad y_2 = x_8, \quad y_3 = 1 - x_6, \quad y_4 = x_4,\\
&y_5 = 1 - x_3, \quad y_6 = x_5, \quad y_7 = x_2, \quad y_8 = x_1
\end{aligned}$$

giving

$$23y_1 + 21y_2 + 19y_3 + 17y_4 + 14y_5 + 13y_6 + 13y_7 + 9y_8 \le 70.$$

We wish to find another equivalent constraint of the form

$$a_1y_1 + a_2y_2 + a_3y_3 + a_4y_4 + a_5y_5 + a_6y_6 + a_7y_7 + a_8y_8 \le a_0.$$

The a_i coefficients become variables in the linear programming model. In order to capture the total logical import of the original constraint, we search for subsets of the indices known as *roofs* and *ceilings*. Ceilings are 'maximal' subsets of the indices of the variables for which the sum of the corresponding coefficients does not exceed the right-hand side coefficient. Such a subset is maximal in the sense that no subset properly containing it, or to the left in the implied lexicographical ordering, can also be a ceiling. For example, the subset $\{1, 2, 4, 8\}$ is a ceiling, $23 + 21 + 17 + 9 \le 70$, but any subset properly containing it (e.g. $\{1, 2, 4, 7, 8\}$) or to the 'left' of it (e.g. $\{1, 2, 4, 7\}$) is not a ceiling. Roofs are 'minimal' subsets of the indices for which the sum of the corresponding coefficients exceeds the right-hand side coefficient. Such a subset is 'minimal' in the same sense as a subset is 'maximal'. For example $\{2, 3, 4, 5\}$ is a roof, $21 + 19 + 17 + 14 > 70$, but any subset properly contained in it (e.g. $\{3, 4, 5\}$) or to the 'right' of it (e.g. $\{2, 3, 4, 6\}$) is not a roof.

If $\{i_1, i_2, \ldots, i_r\}$ is a ceiling, the following condition among the new coefficients a_i is implied:

$$a_{i1} + a_{i2} + \cdots + a_{ir} \le a_0.$$

If $\{i_1, i_2, \ldots, i_r\}$ is a roof, the following condition among the new coefficients a_i is implied:

$$a_{i1} + a_{i2} + \cdots + a_{ir} \ge a_0 + 1.$$

It is also necessary to guarantee the ordering of the coefficients. This can be done by the series of constraints

$$a_1 \ge a_2 \ge a_3 \ge \cdots \ge a_8.$$

If these constraints are given together with each constraint corresponding to a roof or ceiling, then this is a sufficient set of conditions to guarantee that the new 0–1 constraint has exactly the same set of feasible 0–1 solutions as the original 0–1 constraint.

In order to pursue the first objective, we minimize $a_0 - a_3 - a_5$ subject to these constraints.

For the second objective, we minimize $\sum_{i=1}^{8} a_i$.

For this example, the set of ceilings is

$$\{1, 2, 3\}, \{1, 2, 4, 8\}, \{1, 2, 6, 7\}, \{1, 3, 5, 6\}, \{2, 3, 4, 6\}, \{2, 5, 6, 7, 8\}.$$

The set of roofs is

$$\{1, 2, 3, 8\}, \{1, 2, 5, 7\}, \{1, 3, 4, 7\}, \{1, 5, 6, 7, 8\}, \{2, 3, 4, 5\}, \{3, 4, 6, 7, 8\}.$$

The resultant model has 19 constraints and 9 variables.

If the constraint were to involve *general integer* rather than 0–1 variables, then we could still formulate the simplification problem in a similar manner after first converting the constraint to one involving 0–1 variables in the way described in Section 10.1. It is, however, necessary to ensure, by extra constraints in our LP model, the correct relationship between the coefficients in the simplified 0–1 form. How this may be done is described in Section 10.2.

13.19 Distribution 1

This problem can be regarded as one of finding the minimum cost flow through a network. Such network flow problems have been extensively treated in the mathematical programming literature. A standard reference is Ford and Fulkerson (1962). Specialized algorithms exist for solving such problems and are described in Ford and Fulkerson (1962), Jensen and Barnes (1980), Glover and Klingman (1977) and Bradley (1975).

It is, however, always possible to formulate such problems as ordinary linear programming models. Such models have the total unimodularity property described in Section 10.1. This property guarantees that the optimal solution to the LP problem will be integer as long as the right-hand side coefficients are integer.

We choose to formulate this problem as an ordinary LP model in order that we may use the standard revised simplex algorithm. There would be virtue in using a specialized algorithm. The special features of this sort of problem, which also make the use of a specialized algorithm worthwhile, fortunately, make the problem fairly easy to solve as an ordinary LP problem. Sometimes, however, when formulated in this way, the resultant model is very large. The use of a specialized algorithm then also becomes desirable as it results in a compact representation of the problem. As the example presented is very small, such considerations do not arise here.

The factories, depots and customers will be numbered as below:

Factories	1	Liverpool
	2	Brighton
Depots	1	Newcastle
	2	Birmingham
	3	London
	4	Exeter
Customers	C1, C2, ..., C6	

13.19.1 Variables

x_{ij} = quantity sent from factory i to depot j, $i = 1, 2$, $j = 1, 2, 3, 4$
y_{ik} = quantity sent from factory i to customer k, $i = 1, 2$, $k = 1, 2, \ldots, 6$
z_{jk} = quantity sent from depot j to customer k, $j = 1, 2, 3, 4$, $k = 1, 2, \ldots, 6$

There are 44 such variables.

13.19.2 Constraints

Factory Capacities

$$\sum_{j=1}^{2} x_{ij} + \sum_{k=1}^{6} y_{ik} \leq \text{capacity}, \quad i = 1, 2.$$

Quantity into Depots

$$\sum_{i=1}^{2} x_{ij} \leq \text{capacity}, \quad j = 1, 2, 3, 4.$$

Quantity out of Depots

$$\sum_{k=1}^{6} z_{jk} = \sum_{i=1}^{2} x_{ij}, \quad j = 1, 2, 3, 4.$$

Customer Requirements

$$\sum_{i=1}^{2} y_{ik} + \sum_{j=1}^{4} z_{jk} = \text{requirement}, \quad k = 1, 2, \ldots, 6.$$

The capacity, quantity and requirement figures are given with the statement of the problem in Part II.

There are 16 such constraints.

13.19.3 Objectives

The first objective is to minimize cost. This is given by

$$\sum_{\substack{i=1\\j=1}}^{\substack{i=2\\j=4}} c_{ij}x_{ij} + \sum_{\substack{i=1\\k=1}}^{\substack{i=2\\k=6}} d_{ik}y_{ik} + \sum_{\substack{j=1\\k=1}}^{\substack{i=2\\k=6}} e_{jk}z_{jk},$$

where the coefficients c_{ij}, d_{ik} and e_{jk} are given with the problem in Part II.

The second objective will take the same form as that above, but this time, the c_{ij}, d_{ik} and e_{jk} will be defined as below:

$$d_{ik} = \begin{cases} 0 & \text{if customer } k \text{ prefers factory } i, \\ 1 & \text{otherwise;} \end{cases}$$

$$e_{jk} = \begin{cases} 0 & \text{if customer } k \text{ prefers depot } j, \\ 1 & \text{otherwise;} \end{cases}$$

$$c_{ij} = 0 \quad \text{for all } i, j.$$

This objective is to be minimized.

13.20 Depot location (distribution 2)

The linear programming formulation of the distribution problem can be extended to a *mixed integer* model to deal with the extra decisions of whether to build or close down depots. Extra 0–1 integer variables are introduced with the following interpretations:

$$\delta_1 = \begin{cases} 1 & \text{if the Newcastle depot is retained,} \\ 0 & \text{otherwise;} \end{cases}$$

$$\delta_2 = \begin{cases} 1 & \text{if the Birmingham depot is expanded,} \\ 0 & \text{otherwise;} \end{cases}$$

$$\delta_4 = \begin{cases} 1 & \text{if the Exeter depot is retained,} \\ 0 & \text{otherwise;} \end{cases}$$

$$\delta_5 = \begin{cases} 1 & \text{if a depot is built at Bristol,} \\ 0 & \text{otherwise;} \end{cases}$$

$$\delta_6 = \begin{cases} 1 & \text{if a depot is built at Northampton,} \\ 0 & \text{otherwise.} \end{cases}$$

In addition, extra continuous variables x_{i5}, x_{i6}, z_{5k} and z_{6k} are introduced to represent quantities sent to and from the new depots.

The following constraints are added to the model.

If a depot is closed down or not built, then nothing can be supplied to it or from it:

$$\sum_{i=1}^{2} x_{ij} \leq T_j \delta_j,$$

where T_j is the capacity of depot j.

From Birmingham the quantity supplied to and from the depot must lie within the extension:

$$\sum_{i=1}^{2} x_{i2} \leq 50 + 20\delta_2.$$

There can be no more than four depots (including Birmingham and London):

$$\delta_1 + \delta_4 + \delta_5 + \delta_6 \leq 2.$$

In the objective function, the new x_{ij} and z_{jk} variables are given their appropriate costs. The additional expression involving the δ_j variables is added to the objective function:

$$10\delta_1 + 3\delta_2 + 5\delta_4 + 12\delta_5 + 4\delta_6 - 15.$$

This model has 21 constraints and 65 variables (five are integer and 0–1).

13.21 Agricultural pricing

This problem is based on that described by Louwes *et al*. (1963).

Let x_M, x_B, x_{C1} and x_{C2} be the quantities of milk, butter, cheese 1 and cheese 2 consumed (in thousands of tons) and p_M, p_B, p_{C1} and p_{C2} their respective prices (in £1000 per ton).

The limited availabilities of fat and dry matter give the following two constraints:

$$0.04x_M + 0.8\ x_B + 0.35x_{C1} + 0.25x_{C2} \leq 600,$$

$$0.09x_M + 0.02x_B + 0.3\ x_{C1} + 0.4\ x_{C2} \leq 750.$$

The price index limitation gives (measured in £1000)

$$4.82p_M + 0.32p_B + 0.21p_{C1} + 0.07p_{C2} \leq 1.939.$$

The objective is to maximize $\sum_i x_i p_i$.

In addition the x variables are related to the p variables through the price elasticity relationships:

$$\frac{\mathrm{d}x_{\mathrm{M}}}{x_{\mathrm{M}}} = -E_{\mathrm{M}}\frac{\mathrm{d}p_{\mathrm{M}}}{p_{\mathrm{M}}}, \qquad \frac{\mathrm{d}x_{\mathrm{B}}}{x_{\mathrm{B}}} = -E_{\mathrm{B}}\frac{\mathrm{d}p_{\mathrm{B}}}{p_{\mathrm{B}}},$$

$$\frac{\mathrm{d}x_{\mathrm{C1}}}{x_{\mathrm{C1}}} = -E_{\mathrm{C1}}\frac{\mathrm{d}p_{\mathrm{C1}}}{p_{\mathrm{C1}}} + E_{\mathrm{C1C2}}\frac{\mathrm{d}p_{\mathrm{C2}}}{p_{\mathrm{C2}}}, \qquad \frac{\mathrm{d}x_{\mathrm{C2}}}{x_{\mathrm{C2}}} = -E_{\mathrm{C2}}\frac{\mathrm{d}p_{\mathrm{C2}}}{p_{\mathrm{C2}}} + E_{\mathrm{C2C1}}\frac{\mathrm{d}p_{\mathrm{C1}}}{p_{\mathrm{C1}}}.$$

These differential equations can easily be integrated to give the x variables as expressions involving the p variables. If these expressions are substituted in the above constraints and the objective function, non-linearities are introduced into the first two constraints as well as the objective function. The non-linearities could be separated and approximated to by piecewise linear functions as described in Section 7.4.

In order to reduce the number of non-linearities in the model, the relationships implied by the differential equations above can be approximated to by the linear relationships:

$$\frac{x_{\mathrm{M}} - \bar{x}_{\mathrm{M}}}{\bar{x}_{\mathrm{M}}} = -E_{\mathrm{M}}\frac{p_{\mathrm{M}} - \bar{p}_{\mathrm{M}}}{\bar{p}_{\mathrm{M}}}, \qquad \frac{x_{\mathrm{B}} - \bar{x}_{\mathrm{B}}}{\bar{x}_{\mathrm{B}}} = -E_{\mathrm{B}}\frac{p_{\mathrm{B}} - \bar{p}_{\mathrm{B}}}{\bar{p}_{\mathrm{B}}},$$

$$\frac{x_{\mathrm{C1}} - \bar{x}_{\mathrm{C1}}}{\bar{x}_{\mathrm{C1}}} = -E_{\mathrm{C1}}\frac{p_{\mathrm{C1}} - \bar{p}_{\mathrm{C1}}}{\bar{p}_{\mathrm{C1}}} + E_{\mathrm{C1C2}}\frac{p_{\mathrm{C2}} - \bar{p}_{\mathrm{C2}}}{\bar{p}_{\mathrm{C2}}},$$

$$\frac{x_{\mathrm{C2}} - \bar{x}_{\mathrm{C2}}}{\bar{x}_{\mathrm{C2}}} = -E_{\mathrm{C2}}\frac{p_{\mathrm{C2}} - \bar{p}_{\mathrm{C2}}}{\bar{p}_{\mathrm{C2}}} + E_{\mathrm{C2C1}}\frac{p_{\mathrm{C1}} - \bar{p}_{\mathrm{C1}}}{\bar{p}_{\mathrm{C1}}}.$$

$\bar{x}$ and $\bar{p}$ are the known quantities consumed with their prices for the previous year. The approximation can be regarded as warranted if the resultant values of x and p do not differ significantly from $\bar{x}$ and $\bar{p}$.

Using the above relationships to substitute for the x variables in the first two constraints and the objective function gives the model

$$\begin{aligned}
\text{Maximize} \quad & -6492p_{\mathrm{M}}^2 - 1200p_{\mathrm{B}}^2 - 220p_{\mathrm{C1}}^2 - 34p_{\mathrm{C2}}^2 + 53p_{\mathrm{C1}}p_{\mathrm{C2}} \\
& \quad + 6748p_{\mathrm{M}} + 1184p_{\mathrm{B}} + 420p_{\mathrm{C1}} + 70p_{\mathrm{C2}} \\
\text{subject to} \quad & 260p_{\mathrm{M}} + 960p_{\mathrm{B}} + 70.25p_{\mathrm{C1}} - 0.6p_{\mathrm{C2}} \geq 782, \\
& 584p_{\mathrm{M}} + \ \ 24p_{\mathrm{B}} + 55.2p_{\mathrm{C1}} + 5.8p_{\mathrm{C2}} \geq 35, \\
& 4.82p_{\mathrm{M}} + 0.32p_{\mathrm{B}} + 0.21p_{\mathrm{C1}} + 0.07p_{\mathrm{C2}} \leq 1.939.
\end{aligned}$$

In addition, it is necessary to represent explicitly the non-negativity conditions on the x variables. These give

$$p_{\mathrm{M}} \leq 1.039, \quad p_{\mathrm{B}} \leq 0.987,$$

$$220p_{\mathrm{C1}} - 26p_{\mathrm{C2}} \leq 420,$$

$$-27p_{\mathrm{C1}} + 34p_{\mathrm{C2}} \leq 70.$$

This is a *quadratic programming model* as there are quadratic terms in the objective function. Special algorithms exist for obtaining a, possibly, *local optimum* such as that of Beale (1959). In order to solve the model with a standard package (if this does not have a quadratic programming facility), we can convert it into a separable form and approximate the non-linear terms by piecewise linear expressions.

In order to put this model into a separable form, it is necessary to remove the term $p_{C1}p_{C2}$. This may be done by introducing a new variable q together with the constraint

$$p_{C1} - p_{C2} - 0.194q = 0.$$

(It is important to allow q to be negative, if necessary, by incorporating it in the model as a 'free' variable.)

The objective function can then be written in the separable form (a sum of non-linear functions of *single* variables):

$$-6492p_M^2 - 1200p_B^2 - 194p_{C1}^2 - 8p_{C2}^2 - q^2 + 6748p_M + 1184p_B + 420p_{C1} + 70p_{C2}.$$

This transformation also demonstrates that the model is *convex* (as described in Section 7.2). Using a piecewise linear approximation to the non-linearities, there is therefore no danger of obtaining a local optimum with separable programming. In fact, it is sufficient to use the conventional simplex algorithm. This will enable one to obtain a true (global) optimum.

Nevertheless, there is no guarantee that models of this sort (if the price elasticities had been different) will always be convex. Then it would be necessary to use integer programming to guarantee a global optimum or separable programming to produce a local optimum.

The solution given in Part IV is based on grids where p_i, q and p_i^2 and q^2 are defined in intervals of width 0.05. It is only necessary to define the grids within the possible ranges of values which p_i and q can take. These ranges can be found by examining the constraints of the model or by using linear programming to successively maximize and minimize the individual variables p_i and q subject to the constraints.

13.22 Efficiency analysis

There are a number of linear programming formulations of the DEA problem. Fuller coverage of the subject can be found in Farrell (1957), Charnes *et al.* (1978) and Thanassoulis *et al.* (1987). The formulation we give here is well described in Land (1991). It should be pointed out that there is also a *dual* formulation that is commonly used and relies on finding weighted ratios of outputs to inputs using the technique mentioned in Section 3.2. This formulation can be found in the above references.

For garage j suppose its six inputs are a_{1j} to a_{6j} and its three outputs are a_{7j} to a_{9j}. If we choose a specific garage k, we will seek to find a mixture of the 28 garages whose combined inputs do not exceed those of k but whose combined output does. Let us choose x_j units of each of the garages. Our constraints are then

$$\sum_j a_{ij}x_j \le a_{ik} \quad i = 1, 2, \dots, 6;$$

$$\sum_j a_{ij}x_j \ge a_{ik}w \quad i = 7, 8, 9;$$

$$x_j, w \ge 0 \quad j = 1, 2, \dots, 28.$$

If we choose to maximize w then, so long as we can make w larger than 1, we will have chosen a mixture of garages whose combined inputs do not exceed that of the garage under consideration but whose combined outputs do exceed some of the outputs, demonstrating its *inefficiency* compared with this mixture. $1/w$ is usually referred to as the *efficiency number*.

Should it not be possible to find such a mixture, then the result will be to use one unit of the garage itself, resulting in a value of 1 for w.

In order to solve this problem, it is necessary to solve the model 28 times for each value of k. With some modelling/optimization systems, it is possible to do this automatically.

13.23 Milk collection

This problem is an extension of the travelling salesman problem whose formulation is discussed in Section 9.5. We extend the formulation given there.

13.23.1 Variables

$$\begin{aligned} x_{ijk} &= 1 \text{ if the tour on day } k \text{ goes directly between farms } i \text{ and } j \\ &\quad \text{(in either direction) for } i < j, k = 1, 2, \\ &= 0 \text{ otherwise;} \\ y_{ik} &= 1 \text{ if farm } i \text{ is visited on the tour on day } k \text{ for } i = 11 \text{ to } 21, k = 1, 2, \\ &= 0 \text{ otherwise.} \end{aligned}$$

13.23.2 Constraints

The limited tanker capacity gives

$$\sum_i K_i y_{ik} \le C \quad \text{for } k = 1, 2,$$

where K_i is the daily pickup requirement from farm i and C is the tanker capacity.

The limit on visiting some farms only every other day gives

$$y_{i1} + y_{i2} = 1 \quad \text{for } i = 11 \text{ to } 21.$$

The need to visit the 'every day' farms on each tour gives

$$\sum_{j:j>i} x_{ijk} + \sum_{j:j<i} x_{jik} = 2 \quad \text{for } i = 1 \text{ to } 10,\ k = 1, 2.$$

The need to visit the 'every other day' farms on the chosen day gives

$$\sum_{j:j>i} x_{ijk} + \sum_{j:j<i} x_{jik} - 2y_{ik} = 0 \quad \text{for } i = 11 \text{ to } 21,\ k = 1, 2.$$

Taking into account the considerations discussed in Chapter 10, these constraints imply those below for which the associated linear programming relaxation is more constrained, making the model much easier to solve:

$$x_{ijk} - y_{ik} \leq 0 \quad \text{for } i = 11 \text{ to } 21, j = 11 \text{ to } 21, \quad j > i, k = 1, 2;$$
$$x_{jik} - y_{ik} \leq 0 \quad \text{for } i = 11 \text{ to } 21, j = 1 \text{ to } 21, \quad i > j, k = 1, 2.$$

In order to prevent unnecessary (and computationally more costly) symmetric alternative solutions (switching days of visiting farms), it is convenient to set $y_{11,1}$ to 1, forcing farm 11 to be visited on the first day.

13.23.3 Objective

$$\text{Minimize} \qquad \sum_{\substack{i,\ j \\ i<j}} c_{ij} x_{ij}$$

where c_{ij} is the distance between farm i and farm j.

This model has 65 constraints and 442 variables (all 0–1 integer).

As it is only a *relaxation* of the true model (the subtour elimination constraints have been ignored), it will almost certainly be necessary to add these on an 'as needed' basis during the course of optimization in a similar manner as for the travelling salesman problem, as described in Section 9.5.

13.24 Yield management

In order to solve this problem, a *stochastic program* (as mentioned in Section 1.2) will be built. This will be a three-period model. Solving the model for the first time will give recommended price levels and sales three weeks from departure and recommended price levels and sales for subsequent weeks under all possible scenarios. Account will be taken of the probabilities of these scenarios in order to *maximize expected yield*. A week later the model will be rerun, taking into account the committed sales and revenue in the first week, to redetermine recommended prices and sales for the second week (i.e. with ‘recourse’) and the third week under all possible scenarios. The procedure will be repeated again a week later.

13.24.1 Variables

p_{1ch}	$=1$ if price option h chosen for class c in week 1
	$=0$ otherwise ($c=1,2,3,\ h=1,2,3$)
p_{2ich}	$=1$ if price option h chosen for class c in week 2 as a result of scenario i in week 1
	$=0$ otherwise ($c=1,2,3,\ h=1,2,3,\ i=1,2,3$)
p_{3ijch}	$=1$ if price option h chosen for class c in week 3 as a result of scenario i in week 1 and scenario j in week 2
	$=0$ otherwise
s_{1ich}	$=$ number of tickets sold in week 1 for class c under price option h and scenario i
s_{2ijch}	$=$ number of tickets sold in week 2 for class c under price option h if scenario i holds in week 1 and scenario j in week 2
s_{3ijkch}	$=$ number of tickets sold in week 3 for class c under price option if scenario i holds in week 1, scenario j in week 2 and scenario k in week 3
r_{1ich}	$=$ revenue in week 1 from class c under price option h and scenario i
r_{2ijch}	$=$ revenue in week 2 from class c under price option h if scenario i holds in week 1 and scenario j in week 2
r_{3ijkch}	$=$ revenue in week 3 from class c under price option h if scenario i holds in week 1, scenario j in week 2 and scenario k in week 3
u_{ijkc}	$=$ slack capacity in class c under scenarios i,j,k in successive weeks
v_{ijkc}	$=$ excess capacity in class c under scenarios i,j,k in successive weeks
n	$=$ number of planes to fly

While it is necessary to make the p variables $0-1$ integer and n integer, the s variables can be treated as continuous and their values rounded.

13.24.2 Constraints

If a particular price option is chosen (under certain scenarios), then the sales cannot exceed the estimated demand and the revenue must be the product of the price and sales. These conditions can be imposed using the device described in

Section 9.2 for modelling the product of a continuous and 0–1 integer variable.

$$r_{1ich} - P_{1ch}s_{1ich} \le 0,$$

$$P_{1ch}s_{1ich} - r_{1ich} + P_{1ch}D_{1ich}P_{1ch} \le P_{1ch}D_{1ich} \quad \text{for all } i, c, h,$$

$$r_{2ijch} - P_{2ch}s_{2ijch} \le 0,$$

$$P_{2ch}s_{2ijch} - r_{2ijch} + P_{2ch}D_{2jch}P_{2ich} \le P_{2ch}D_{2jch} \quad \text{for all } i, j, c, h,$$

$$r_{3ijkch} - P_{3ch}s_{3ijkch} \le 0,$$

$$P_{3ch}s_{3ijkch} - r_{3ijkch} + P_{3ch}D_{3kch}p_{3ijch} \le P_{3ch}D_{3kch} \quad \text{for all } i, j, k, c, h,$$

where P and D are the given prices and demands for the corresponding periods, scenarios, classes and options.

The seat capacities must be abided by for all scenarios:

$$s_{1ich} + s_{2ijch} + s_{3ijkch} + u_{ijkc} - v_{ijkc} - C_c n \le 0 \quad \text{for all } i, j, k, c,$$

where C_c is the given capacity per plane for class c.

Adjustment is possible between classes:

$$u_{ijkc} - 0.1C_c \le 0,$$

$$v_{ijkc} - 0.1C_c \le 0 \qquad \text{for all } i, j, k, c,$$

$$\sum_c u_{ijkc} - \sum_c v_{ijkc} = 0 \qquad \text{for all } i, j, k.$$

Exactly one price option must be chosen in each class under each set of scenarios:

$$\sum_h p_{1ch} = 1 \quad \text{for all } c,$$

$$\sum_h p_{2ich} = 1 \quad \text{for all } c,$$

$$\sum_h p_{3ijxh} = 1 \quad \text{for all } c.$$

The above set of constraints could be replaced by SOS1 sets without the need to declare the p variables integer.

Numbers sold cannot exceed demand:

$$s_{1ich} \le D_{1ich}p_{1ch} \quad \text{for all } i, c, h,$$

$$s_{2ijch} \le D_{2jch}p_{2ich} \quad \text{for all } i, j, c, h,$$

$$s_{3ijkch} \le D_{3kch}p_{3ijch} \quad \text{for all } i, j, k, c, h.$$

Up to six planes can be flown:

$$n \leq 6.$$

13.24.3 Objective

$$\text{Maximize} \quad \sum_{i,c,h} Q_i r_{1ich} + \sum_{i,j,c,h} Q_i Q_j r_{2ijch} + \sum_{i,j,k,c,h} Q_i Q_j Q_k r_{3ijkch} - 50n$$

(measuring in £1000) where Q_i is the probability of scenario i.

This model has 1200 constraints, one bound and 982 variables, of which 117 are 0–1 integer and one general integer.

In defining the data, it is desirable that the demands in the scenarios cover the *extremes* as well as the most likely cases. The recommended sales will not exceed those of the most extreme scenario in the solution to this model. In this example, it can be seen that final demands (known with hindsight) exceed those of all scenarios in a few cases. As a result, the solution will not result in sales to meet all of these demands.

Models for subsequent weeks (with recourse) are obtained from this model by fixing prices and sales for weeks that have elapsed.

13.25 Car rental 1

We model this problem as a deterministic linear programme. There would be advantage to be gained from modelling it as a stochastic programme. In order to do this, we would need to obtain data to quantify the uncertain demand.

13.25.1 Indices

i, j	= Glasgow, Manchester, Birmingham, Plymouth
t	= Monday, Tuesday, Wednesday, Thursday, Friday, Saturday
k	= 1, 2, 3 (days hired)

Although this is a ‘dynamic’ problem, we seek a steady-state solution and can therefore regard the set of days as ‘circular’, that is, Monday of a week ‘follows’ after the subsequent Saturday; that is, if $t =$ Monday then $t - 1 =$ Saturday.

13.25.2 Given data

D_{it}	= estimated rental demand at depot i on day t as given in Table 12.19
P_{ij}	= proportion of cars rented at depot i to be returned to depot j as given in Table 12.21
C_{ij}	= cost of transfer of a car from depot i to depot j as given in Table 12.22
Q_k	= proportion of cars hired for k days

R_i	= repair capacity of depot i
RCA_k	= rental cost for k days with return to same depot as given in Table 12.23
RCB_k	= rental cost for k days with return to another depot as given in Table 12.23
RCC	= rental cost for Saturday with return to same depot on Monday
RCD	= rental cost for Saturday with return to another depot on Monday
CS_k	= marginal cost to company of k day hire of a car for all i, t

13.25.3 Variables

n	= total number of cars owned
nu_{it}	= number of undamaged cars at depot i at the beginning of day t for all i, t
nd_{it}	= number of damaged cars at depot i at the beginning of day t for all i, t
tr_{it}	= number of cars rented out from depot i at beginning of day t for all i, t
eu_{it}	= undamaged cars left at depot i during day t for all i, t
ed_{it}	= damaged cars left at depot i at the beginning of day t for all i, t
tu_{ijt}	= number of undamaged cars at depot i at the beginning of day t, to be transferred to depot j for all i, j, t
td_{ijt}	= number of damaged cars at depot i at the beginning of day t, transferred to depot t for all i, j, t
rp_{it}	= number of damaged cars to be repaired at depot i on day for all i, t

13.25.4 Constraints

Total number of undamaged cars into depot i on day t

$$\sum_{jk} 0.9 P_{ji} Q_k tr_{jt-k} + \sum_j tu_{jit-1} + rp_{it-1} + eu_{it-1} = nu_{it} \quad \text{for all } i, t.$$

Total number of damaged cars into depot i on day t

$$\sum_{jk} 0.1 P_{ji} Q_k tr_{jt-k} + \sum_j td_{jit-1} + ed_{it-1} = nd_{it} \quad \text{for all } i, t.$$

Total number of undamaged cars out of depot i on day t

$$tr_{it} + \sum_j tu_{ijt} + rp_{it-1} + eu_{it} = nu_{it} \quad \text{for all } i, t.$$

Total number of damaged cars out of depot i on day t

$$rp_{it-1} + \sum_j td_{ijt} + ed_{it} = nd_{it} \quad \text{for all } i, t.$$

Repair capacity of depot i on all days

$$rp_{it} \le R_i \quad \text{for all } i, t.$$

Demand at depot i on day t

$$tr_{it} \le D_{it} \quad \text{for all } i, t.$$

Total number of cars equals number hired out from all depots on Monday for 3 days, plus those on Tuesday for 2 or 3 days, plus all damaged and undamaged cars in depots at the beginning of Wednesday.

$$\sum_i \left(0.25tr_{i1} + 0.45tr_{i2} + nu_{i2} + nd_{i2}\right) = n \quad \text{for all } i.$$

13.25.5 Objective

Note that £10 has been added to the profit on each rented car to reflect the surcharge of £100 charged on the 10% of cars that are returned damaged.

$$\begin{aligned}
\text{Profit} = & \sum_{itk, t \neq \text{SATURDAY}} P_{ii} Q_k (RCA_k - CS_k + 10) tr_{it} \\
& + \sum_{ijtk, t \neq \text{SATURDAY}} P_{ij} Q_k (RCB_k - CS_k + 10) tr_{it} \\
& + \sum_{i\text{SATURDAY}} P_{ii} Q_1 (RCC - CS_1 + 10) tr_{it} \\
& + \sum_{ij\text{SATURDAY}} P_{ij} Q_1 (RCD - CS + 10) tr_{it} \\
& + \sum_{i\text{SATURDAY}k} P_{ii} Q_k (RCA_k - CS_k + 10) tr_{it} + \sum_{i\text{SATURDAY}k} P_{ij} Q_k (RCB_k - CS_k \\
& + 10) tr_{it} \\
& - \sum_{ijt} C_{ij} tu_{ijt} - \sum_{ijt} C_{ij} td_{ijt} - 15n.
\end{aligned}$$

This model has a total of 84 constraints and 337 variables.

It is not necessary to constrain the w_{ijkl} to be non-negative. They can be regarded as 'free' variables. Avoiding these two stipulations helps the model to solve more easily.

13.26 Car rental 2

We introduce the following 0–1 integer variables with the following interpretations:

δ_{B1} = 1, if the Birmingham repair capacity is expanded by 5 cars per day.
δ_{B2} = 1, if the Birmingham repair capacity is expanded by a further 5 cars per day.
δ_{M1} = 1, if the Manchester repair capacity is expanded by 5 cars per day.
δ_{M2} = 1, if the Manchester repair capacity is expanded by a further 5 cars per day.
δ_P = 1, if the Plymouth repair capacity is expanded by 5 cars per day.

The following expression is added to the objective function:

$$18\,000\delta_{\text{B1}} + 8\,000\delta_{\text{B2}} + 20\,000\delta_{\text{M1}} + 5\,000\delta_{\text{M2}} + 19\,000\delta_{\text{P}}$$

together with the following extra constraints:

$$\delta_{\text{B1}} \geq \delta_{\text{B2}}, \quad \delta_{\text{M1}} \geq \delta_{\text{M2}}, \quad \delta_{\text{B1}} + \delta_{\text{B2}} + \delta_{\text{M1}} + \delta_{\text{M2}} + \delta_{\text{P}} \leq 3$$

and the repair capacity constraints for Birmingham, Manchester and Plymouth have $5\delta_{\text{B1}} + 5\delta_{\text{B2}}, 5\delta_{\text{M1}} + 5\delta_{\text{M2}}$ and $5\delta_P$, respectively added.

13.27 Lost baggage distribution

We formulate this problem as an integer programming model. It is an extension of the travelling salesman problem, discussed in section 9.5. It can be regarded as a (simple) case of the vehicle routing problem. In the problem here, there are no vehicle capacites and no time windows for the delivery to each location (apart from the overall 2 hours guarantee). Nevertheless, it is potentially a very difficult problem to solve, for more than a modest number of locations. Also it can give rise to a very large model. There are a number of different ways of formulating a model, which vary greatly in their size and difficuly of solution. We suggest a model that is practicable in both size and computational tractibility. Although the problem is 'symmetric' in the sense that the distance from X to Y is the same as that from Y to X, we extend the asymmetric formulation of the travelling salesman problem in order to give the 'time' to return to Heathrow, after a van has completed all its deliveries, as 0. In this way, only the total time taken to reach the last delivery is counted within the 2 hours time limit. We are therefore seeking a 'Hamiltoniam path' (as opposed to a circuit) for each van, starting from Heathrow.

13.27.1 Variables

All variables are 0–1 and integer

$x_{ijk} = 1$, iff van k visits, and goes directly from location i to j for all i, j, and k.

$y_{ik} = 1$, iff location i is visited by van k for all i, k.

$\delta_k = 1$, iff van k is used.

13.27.2 Objective

Minimize $\sum_k \delta_k$, that is, minimize the number of vans used.

13.27.3 Constraints

$y_{ik} \leq \delta_k$ for all $i > 1, j, k$ if a location is visited by van k, then it is (obviously) used

$\sum_{ij:i<j} c_{ij}x_{ijk} + \sum_{ij:i>j:j\neq 1} c_{ji}x_{jik} \leq 120$ for all k, no van travels for more than 2 hours. The time from any location back to Heathrow does not count

$\sum_k y_{ik} = 1$ for all i (apart from $i = 1 =$ Heathrow), i.e. each location is visited by exactly one van

$\sum_k y_{1k} \geq \sum_k \delta_k$, Heathrow is visited by every van

$\sum_i x_{ijk} = y_{ik}$ for all j, k. if location j is visited by van k, then there is exactly one link in

$\sum_{i:} x_{jik} = y_{ik}$ for all j, k. if location j is visited by van k, then there is exactly one link out.

In order to avoid a lot of symmetric solutions (i.e. interchanging identical vans), we can stipulate

$$\sum_i y_{i1} \geq \sum_i y_{i2} \geq \cdots \geq \sum_i y_{i6}.$$

This model has 290 constraints and 1356 variables (all integer).

This is a *relaxation* of the true model as the subtour elimination constraints have been ignored. It will, almost certainly be necessary to append some of them, possibly automatically during the course of optimization.

Having determined the minimum number of vans needed, we fix the values of the δ_k variables, introduce slack variables into the time limit constraints and minimize the maximum of these.

13.28 Protein folding

We define the following binary variables

$x_{ij} = 1$, iff hydrophobic acid i is matched with acid j $(i < j)$. (This does not include those matchings that are predefined by virtue of the acids being contiguous in the chain.)

$y_i = 1$, iff a fold occurs between the ith and $(i + 1)$st acids in the chain.

For each pair of hydrophobic acids i and j, we can match them if (i) they are not contiguous, that is, already matched, (ii) they have an even number of acids between them in the chain and (iii) there is exactly one fold between i and j. This gives rise to the constraints

$y_k + x_{ij} \leq 1$ for all k and i, j such that $i \leq k < j$ and $k \neq (i + j - 1)/2$
$x_{ij} = y_{(i+j-1)/2}$ for all i, j such that $(i + j - 1)$ is even.

The objective is

$$\text{Maximize} \quad \sum_{ij} x_{ij}.$$

This model has 1190 constraints and 196 binary variables.

13.29 Protein comparison

There are a number of possible integer programming formulations of this problem (discussed in Forrester and Greenberg (2008)). The formulation we give below results in a large model but is solvable for the instance we are considering.

We define the following variables:

$x_{ij} = 1$, iff node i in G_1 is paired with node j in G_2
$w_{ijkl} = 1$, iff (i, k) is an edge in G_1 which is paired with edge (j, l) in G_2.

The objective is

$$\text{Maximize} \sum_{ijkl} w_{ijkl}$$

subject to $\sum_i x_{ij} \leq 1$ for all j. No node in G_1 can be paired with more than one in G_2,

$\sum_j x_{ij} \leq 1$ for all i no node in G_2 can be paired with more than one in G_1,

$w_{ijkl} \leq x_{ij}, w_{ijkl} \leq x_{kl}$ if edges (i, k) and (j, l)
are paired then so are the corresponding nodes,

$x_{ij} + x_{kl} \leq 1$ if $i < k$ but $l < j$ there can be no crossovers.

This model has 2160 constraints and 179 variables. It is not necessary to stipulate that the 80 w_{ijkl} variables be integer as this will be satisfied automatically. Also it is not necessary to constrain the w_{ijkl} to be non-negative. They can be regarded as 'free' variables Avoiding these two stipulations helps the model to solve more easily.

Part IV

14

Solutions to problems

Optimal solutions to the problems presented in Part II are given here. The main purpose in giving the solutions is to allow the reader to check solutions he/she may have obtained to the same problems. All the solutions given here result from the formulations presented in Part III. Sometimes, alternative optimal solutions are possible, as described in Section 6.2, although the optimal value of the objective function will be unique.

If the reader obtains a different solution to a problem from that given here, he/she should attempt to validate his/her solution using the principles described in Section 6.1. In many cases, it is realistic to take the solution given here to represent the current operating pattern in the situation being modelled. The different solution obtained can then be validated relative to this solution.

14.1 Food manufacture 1

The optimal policy is given in Table 14.1.

The profit (income from sales − cost of raw oils) derived from this policy is £107 843. This figure includes storage costs for the last month.

There are alternative optimal solutions.

14.2 Food manufacture 2

This model proved comparatively hard to solve. The optimal solution found is given in Table 14.2.

The profit (income from sales − cost of raw oils) derived from this policy is £100 279. There are alternative, equally good, solutions. This solution was obtained after 645 nodes. A total of 2007 nodes were examined to prove optimality.

Table 14.1

	Buy	Use	Store
January	Nothing	22.2 tons VEG 1 177.8 tons VEG 2 250 tons OIL 3	477.8 tons VEG 1 322.2 tons VEG 2 500 tons OIL 1 500 tons OIL 2 250 tons OIL 3
February	250 tons OIL 2	200 tons VEG 2 250 tons OIL 3	477.8 tons VEG 1 122.2 tons VEG 2 500 tons OIL 1 750 tons OIL 2
March	Nothing	159.3 tons VEG 1 40.7 tons VEG 2 250 tons OIL 2	318.5 tons VEG 1 81.5 tons VEG 2 500 tons OIL 1 500 tons OIL 2
April	Nothing	159.3 tons VEG 1 40.7 tons VEG 2 250 tons OIL 2	159.3 tons VEG 1 40.7 tons VEG 2 500 tons OIL 1 250 tons OIL 2
May	500 tons OIL 3	159.3 tons VEG 1 40.7 tons VEG 2 250 tons OIL 2	500 tons OIL 1 500 tons OIL 3
June	659.3 tons VEG 1 540.7 tons VEG 2 750 tons OIL 2	159.3 tons VEG 1 40.7 tons VEG 2 250 tons OIL 2	500 tons each oil (stipulated)

14.3 Factory planning 1

The optimal policy is given in Table 14.3.

This policy yields a total profit of £93 715.

Information on the value of acquiring new machines can be obtained from the *shadow prices* on the appropriate constraints. The value of an extra hour in the particular month when a particular type of machine is used to capacity is given below:

	Machines used to capacity	Value
January	Grinders	£8.57
February	Horizontal drills	£0.625
March	Borer	£200[a]

[a]The borer is not in use in March. This figure indicates the high value of putting it into use. As explained in Section 6.2, these figures only give the *marginal value* of increases in capacity and can only therefore be used as indicators.

Table 14.2

	Buy	Use	Store
January	Nothing	85.2 tons VEG 1 114.8 tons VEG 2 250 tons OIL 3	414.8 tons VEG 1 385.2 tons VEG 2 500 tons OIL 1 500 tons OIL 2 250 tons OIL 3
February	190 tons OIL 2	200 tons VEG 2 230 tons OIL 2 20 tons OIL 3	414.8 tons VEG 1 185.2 tons VEG 2 500 tons OIL 1 460 tons OIL 2 230 tons OIL 3
March	580 tons OIL 3	85.2 tons VEG 1 114.8 tons VEG 2 250 tons OIL 3	329.6 tons VEG 1 70.4 tons VEG 2 500 tons OIL 1 460 tons OIL 2 560 tons OIL 3
April	Nothing	155 tons VEG 1 230 tons OIL 2 20 tons OIL 3	174.6 tons VEG 1 70.4 tons VEG 2 500 tons OIL 1 230 tons OIL 2 540 tons OIL 3
May	Nothing Nothing	250 tons VEG 1 230 tons OIL 2 20 tons OIL 3	19.6 tons VEG 1 70.4 tons VEG 2 500 tons OIL 1 520 tons OIL 3
June	480.4 tons VEG 1 629.6 tons VEG 2 730 tons OIL 2	200 tons VEG 2 230 tons OIL 2 20 tons OIL 3	500 tons each oil (stipulated)

14.4 Factory planning 2

There are a number of alternative optimal maintenance schedules. One such is to put the following machines down for maintenance in the following months:

January	–
February	One vertical drill, one horizontal drill
March	One grinder, two horizontal drills
April	One grinder, one borer, one planer
May	One vertical drill
June	–

Table 14.3

	Manufacture	Sell	Hold
January	500 PROD 1	500 PROD 1	
	888.6 PROD 2	888.6 PROD 2	
	382.5 PROD 3	300 PROD 3	82.5 PROD 3
	300 PROD 4	300 PROD 4	
	800 PROD 5	800 PROD 5	
	200 PROD 6	200 PROD 6	
February	700 PROD 1	600 PROD 1	100 PROD 1
	600 PROD 2	500 PROD 2	100 PROD 2
	117.5 PROD 3	200 PROD 3	100 PROD 5
	500 PROD 5	400 PROD 5	100 PROD 7
	300 PROD 6	300 PROD 6	
	250 PROD 7	150 PROD 7	
March	400 PROD 6	100 PROD 1	Nothing
		100 PROD 2	
		100 PROD 5	
		400 PROD 6	
		100 PROD 7	
April	200 PROD 1	200 PROD 1	Nothing
	300 PROD 2	300 PROD 2	
	400 PROD 3	400 PROD 3	
	500 PROD 4	500 PROD 4	
	200 PROD 5	200 PROD 5	
	100 PROD 7	100 PROD 7	
May	100 PROD 2	100 PROD 2	100 PROD 3
	600 PROD 3	500 PROD 3	100 PROD 5
	100 PROD 4	100 PROD 4	100 PROD 7
	1100 PROD 5	1000 PROD 5	
	300 PROD 6	300 PROD 6	
	100 PROD 7		
June	550 PROD 1	500 PROD 1	50 of every
	550 PROD 2	500 PROD 2	product
	350 PROD 4	50 PROD 3	(stipulated)
	550 PROD 6	300 PROD 4	
		50 PROD 5	
		500 PROD 6	
		50 PROD 7	

This results in the production and marketing plans are shown in Table 14.4. These plans yield a total profit of £108 855.

There is, therefore, a gain of £15 140 over the six months by seeking an optimal maintenance schedule instead of the one imposed in the FACTORY PLANNING solution.

Table 14.4

	Manufacture	Sell	Hold
January	500 PROD 1	500 PROD 1	Nothing
	1000 PROD 2	1000 PROD 2	
	300 PROD 3	300 PROD 3	
	300 PROD 4	300 PROD 4	
	800 PROD 5	800 PROD 5	
	200 PROD 6	200 PROD 6	
	100 PROD 7	100 PROD 7	
February	600 PROD 1	600 PROD 1	Nothing
	500 PROD 2	500 PROD 2	
	200 PROD 3	200 PROD 3	
	400 PROD 5	400 PROD 5	
	300 PROD 6	300 PROD 6	
	150 PROD 7	150 PROD 7	
March	400 PROD 1	300 PROD 1	100 PROD 1
	700 PROD 2	600 PROD 2	100 PROD 2
	100 PROD 3		100 PROD 3
	100 PROD 4		100 PROD 4
	600 PROD 5	500 PROD 5	100 PROD 5
	400 PROD 6	400 PROD 6	
	200 PROD 7	100 PROD 7	100 PROD 7
April	Nothing	100 PROD 1	Nothing
	100 PROD 2	100 PROD 2	
	100 PROD 3	100 PROD 3	
	100 PROD 4	100 PROD 4	
	100 PROD 5	100 PROD 5	
	100 PROD 7	100 PROD 7	
May	100 PROD 2	100 PROD 2	Nothing
	500 PROD 3	500 PROD 3	
	100 PROD 4	100 PROD 4	
	1000 PROD 5	1000 PROD 5	
	300 PROD 6	300 PROD 6	
June	550 PROD 1	500 PROD 1	50 PROD 1
	550 PROD 2	500 PROD 2	50 PROD 2
	150 PROD 3	100 PROD 3	50 PROD 3
	350 PROD 4	300 PROD 4	50 PROD 4
	1150 PROD 5	1100 PROD 5	50 PROD 5
	550 PROD 6	500 PROD 6	50 PROD 6
	110 PROD 7	60 PROD 7	50 PROD 7

This solution was obtained in 191 nodes. The optimal objective value of £93 715 for the factory planning solution can be used as an objective cut-off because this clearly corresponds to a known integer solution. A total of 3320 nodes were needed to prove optimality.

14.5 Manpower planning

With the objective of minimizing redundancy, the optimal policies to pursue are given below:

Recruitment

	Unskilled	Semi-skilled	Skilled
Year 1	0	0	0
Year 2	0	649	500
Year 3	0	677	500

Retraining and downgrading

	Unskilled to semi-skilled	Semi-skilled to skilled	Semi-skilled to unskilled	Skilled to unskilled	Skilled to semi-skilled
Year 1	200	256	0	0	168
Year 2	200	80	0	0	0
Year 3	200	132	0	0	0

Redundancy

	Unskilled	Semi-skilled	Skilled
Year 1	445	0	0
Year 2	164	0	0
Year 3	232	0	0

Short-time working

	Unskilled	Semi-skilled	Skilled
Year 1	50	50	50
Year 2	50	0	0
Year 3	50	0	0

Overmanning

	Unskilled	Semi-skilled	Skilled
Year 1	130	18	0
Year 2	150	0	0
Year 3	150	0	0

These policies result in a total redundancy of 842 workers over the three years. The total cost of pursuing these policies is £1 438 383.

In order to obtain this total cost, it is important to recognize that there are alternative solutions and we should choose that with minimum cost.

If the objective is to minimize the cost, the optimal policies are those given below:

Recruitment

	Unskilled	Semi-skilled	Skilled
Year 1	0	0	55
Year 2	0	800	500
Year 3	0	800	500

Retraining and downgrading

	Unskilled to semi-skilled	Semi-skilled to skilled	Semi-skilled to unskilled	Skilled to unskilled	Skilled to semi-skilled
Year 1	0	0	25	0	0
Year 2	142	105	0	0	0
Year 3	96	132	0	0	0

Redundancy

	Unskilled	Semi-skilled	Skilled
Year 1	812	0	0
Year 2	258	0	0
Year 3	354	0	0

Short-time working

	Unskilled	Semi-skilled	Skilled
Year 1	0	0	0
Year 2	0	0	0
Year 3	0	0	0

Overmanning

	Unskilled	Semi-skilled	Skilled
Year 1	0	0	0
Year 2	0	0	0
Year 3	0	0	0

These policies cost £498 677 over the three years and result in a total redundancy of 1424 workers. Again, alternative solutions should be considered if necessary to ensure that this redundancy is the minimum possible for this (minimum) level of cost.

Clearly, minimizing costs instead of redundancy saves £939 706 but results in 582 extra redundancies.

The cost of saving each job (when minimizing redundancy) could, therefore, be regarded as £1615.

14.6 Refinery optimization

The optimal solution results in a profit of £211 365.

The optimal values of the variables defined in Part III are given below:

CRA	15 000	MNPMF	3537
CRB	30 000	MNRMF	6962
LN	6000	HNPMF	0
MN	10 500	HNRMF	2993
HN	8400	RGPMF	1344
LO	4200	RGRMF	1089
HO	8700	CGPMF	1936
R	5550	CGRMF	0
LNRG	0	LOJF	0
MNRG	0	HOJF	4900
HNRG	5407	RJF	4550

RG	2433	COJF	5706
LOCGO	4200	RLBO	1000
HOCGO	3800	PMF	6818
CG	1936	RMF	17 044
CO	5706	JF	15 156
LNPMF	0	FO	0
LNRMF	6000	LBO	500

14.7 Mining

The optimal solution is as follows: work the following mines in each year

Year 1	Mines 1, 3, 4
Year 2	Mines 2, 3, 4
Year 3	Mines 1, 3
Year 4	Mines 1, 2, 4
Year 5	Mines 1, 2, 3

Keep every mine open each year apart from mine 4 in year 5.

Produce the following quantities of ore (in millions of tons) from each mine every year:

	Mine 1	Mine 2	Mine 3	Mine 4
Year 1	2.0	–	1.3	2.45
Year 2	–	2.5	1.3	2.2
Year 3	1.95	–	1.3	–
Year 4	0.12	2.5	–	3.0
Year 5	2.0	2.17	1.3	–

Produce the following quantities of blended ore (in millions of tons) each year:

Year 1	5.75
Year 2	6.00
Year 3	3.25
Year 4	5.62
Year 5	5.47

This solution results in a total profit of £146.84 millions.

14.8 Farm planning

The optimal plan results in a total profit of £121 719 over the five years. Each year's detailed plan should be as follows:

Year 1	Grow 22 tons of grain on group 1 land
	Grow 91 tons of sugar beet
	Buy 37 tons of grain
	Sell 23 tons of sugar beet
	Sell 31 heifers (leaving 23)

The profit on the year will be £21 906.

Year 2	Grow 22 tons of grain on group 1 land
	Grow 94 tons of sugar beet
	Buy 35 tons of grain
	Sell 27 tons of sugar beet
	Sell 41 heifers (leaving 12)

The profit on the year will be £21 888.

Year 3	Grow 22 tons of grain on group 1 land
	Grow 3 tons of grain on group 2 land
	Grow 98 tons of sugar beet
	Buy 38 tons of grain
	Sell 25 tons of sugar beet
	Sell 57 heifers (leaving none)

The profit on the year will be £25 816.

Year 4	Grow 22 tons of grain on group 1 land
	Grow 115 tons of sugar beet
	Buy 40 tons of grain
	Sell 42 tons of sugar beet
	Sell 57 heifers (leaving none)

The profit on the year will be £26 826.

Year 5	Grow 22 tons of grain on group 1 land
	Grow 131 tons of sugar beet
	Buy 33 tons of grain
	Sell 67 tons of sugar beet
	Sell 51 heifers (leaving none)

The profit on the year will be £25 283.

Only in year 1, the accommodation limits the total number of cows. No investment should be made in further accommodation.

At the end of the five-year period, there will be 92 dairy cows.

As pointed out in Part III, the numbers of cows of successively increasing ages in successive years are effectively fixed by the continuity constraints. This allows the model to be reduced using the procedures mentioned in Section 3.4. When this is done, a further 18 variables can be fixed (besides the 20 in the original model) and 38 constraints can be removed.

14.9 Economic planning

With the first objective (maximizing total productive capacity in year 5), the growth pattern shown in Table 14.5 results in a total productive capacity of £2142 million in year 5. Quantities are in million pounds.

Table 14.5

	Year 1	Year 2	Year 3	Year 4	Year 5
Coal capacity	300	300	300	489	1512
Steel capacity	350	350	350	350	350
Transport capacity	280	280	280	280	280
Coal output	260.4	293.4	300	17.9	166.4
Steel output	135.3	181.7	193.1	105.7	105.7
Transport output	140.7	200.6	267.2	92.3	92.3
Manpower requirement	270.6	367	470	150	150
End of the year					
Coal stock	0	0	0	148.4	0
Steel stock	0	0	0	0	0
Transport stock	0	0	0	0	0

With the second objective (maximizing total production in the fourth and fifth years), the growth pattern shown in Table 14.6 results in a total output of £2619 millions in those years.

Table 14.6

	Year 1	Year 2	Year 3	Year 4	Year 5
Coal capacity	300	430.5	430.5	430.5	430.5
Steel capacity	350	350	350	359.4	359.4
Transport capacity	280	280	280	519.4	519.4
Coal output	184.8	430.5	430.5	430.5	430.5
Steel output	86.7	153.3	182.9	359.4	359.4
Transport output	141.3	198.4	225.9	519.4	519.4
Manpower requirement	344.6	384.2	470	470	150
End of the year					
Coal stock	31.6	16.4	0	0	0
Steel stock	11.5	0	0	0	176.5
Transport stock	0	0	0	0	0

With the third objective (maximizing manpower requirements), the growth pattern shown in Table 14.7 results in a total manpower usage of £2450 millions.

Table 14.7

	Year 1	Year 2	Year 3	Year 4	Year 5
Coal capacity	300	316	320	366	859.4
Steel capacity	350	350	350	350	350
Transport capacity	280	280	280	280	280
Coal output	251.8	316	319.8	366.3	859.4
Steel output	134.8	179	224.1	223.1	220
Transport output	143.6	181.7	280	279.1	276
Manpower requirement	281.1	333.2	539.7	636.8	659.7
End of the year					
Coal stock	0	0	0	0	0
Steel stock	11	0	0	0	0
Transport stock	4.2	0	0	0	0

The first objective obviously results in effort being concentrated on building up capacity in the coal industry. This happens because the production of extra coal capacity uses comparatively little output from the other industries.

With the second objective, more effort is put onto the transport industry. This results, in part, from the fact that transport uses less manpower than other industries.

With the third objective, the coal industry is again boosted in view of its heavy manpower requirement.

14.10 Decentralization

The optimal solution is

locate departments A and D in Bristol;
locate departments B, C and E in Brighton.

This results in a yearly benefit of £80 000 but communications costs of £65 100. It is interesting to note that communication costs are also reduced by moving out of London in this problem because they would have been £78 000 if each department had remained in London.

The net yearly benefit (benefits less communication costs) is therefore £14 900.

14.11 Curve fitting

1. The 'best' straight line that minimizes the *sum of absolute deviations* is

$$y = 0.6375x + 0.5812.$$

This is line 1 shown in Figure 14.1. The sum of absolute deviations resulting from this line is 11.46.

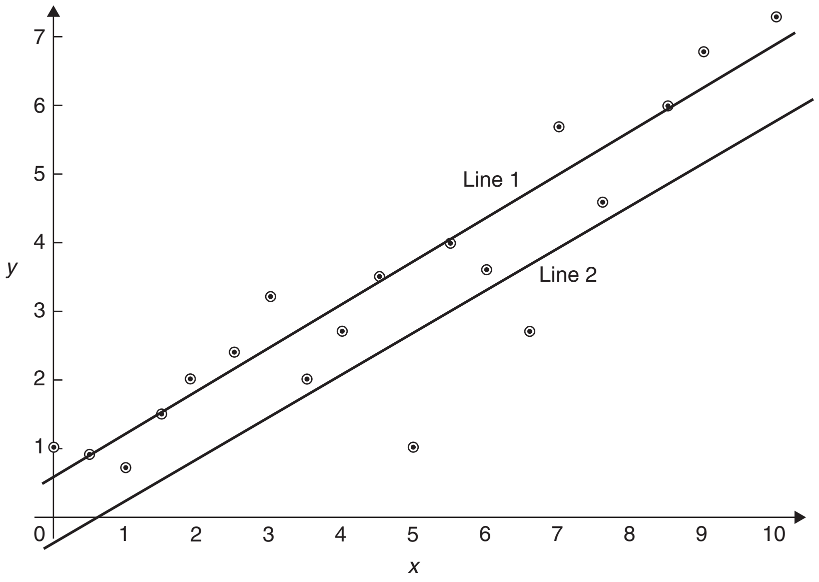


Figure 14.1

2. The 'best' straight line that minimizes the *maximum absolute deviation* is

$$y = 0.625x - 0.4.$$

This is line 2 shown in Figure 14.1. The maximum absolute deviation resulting from this line is 1.725. (Points (3.0, 3.2), (5.0, 1.0) and (7.0, 5.7) all have this absolute deviation from the line.) In contrast, line 1 allows point (5.0, 1.0) to have an absolute deviation of 2.77. On the other hand, although line 2 allows no point to have an absolute deviation of more than 1.725, the sum of absolute deviations is 19.95 compared with the 11.47 resulting from line 1.

3. The 'best' quadratic curve that minimizes the *sum of absolute deviations* is

$$y = 0.0337x^2 + 0.2945x + 0.9823.$$

This is curve 1 shown in Figure 14.2. The sum of absolute deviations resulting from this curve is 10.45.

4. The 'best' quadratic curve that minimizes the *maximum absolute deviation* is

$$y = 0.125x^2 - 0.625x + 2.475.$$

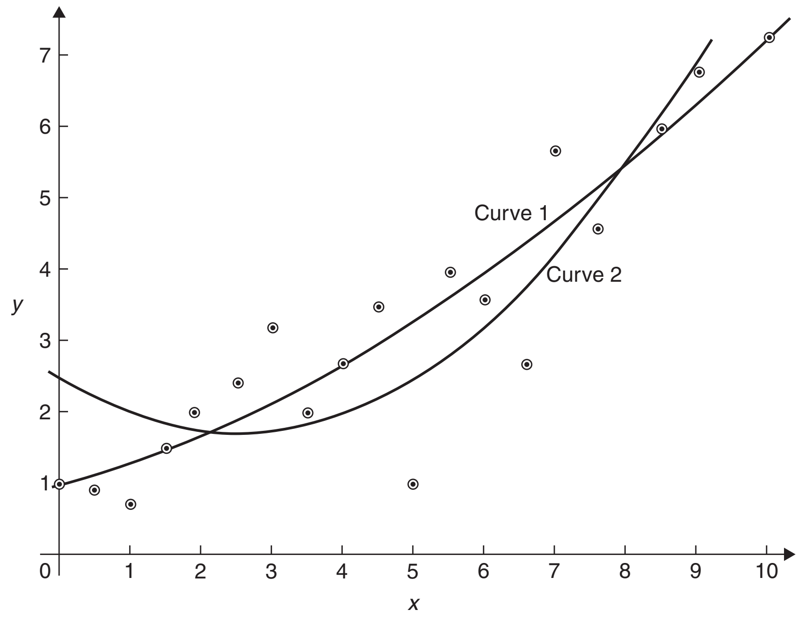


Figure 14.2

This is curve 2 shown in Figure 14.2. The maximum absolute deviation resulting from this curve is 1.475. (Points (0.0, 1.1), (3.0, 3.2), (5.0, 1.0) and (7.0, 5.7) all have this absolute deviation from the curve.)

A way of obtaining analytical solutions to these sorts of problems is described by Williams and Munford (1999).

14.12 Logical design

The optimal solution is shown in Figure 14.3. There are, of course, symmetric alternatives.

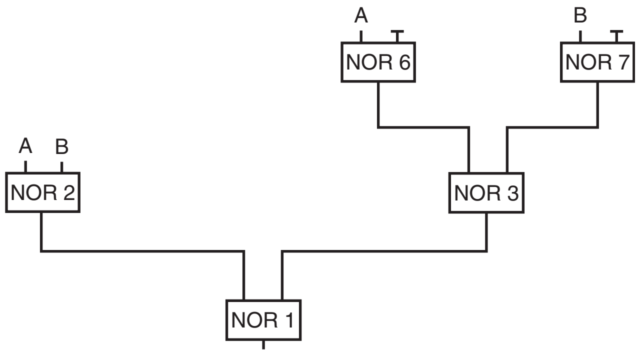


Figure 14.3

14.13 Market sharing

This model proves difficult to solve optimally although a feasible solution is comparatively easy to obtain. Using the original formulation with no extra constraints, a feasible solution was found after 21 nodes by minimizing the sum of percentage deviations. This solution gave a sum of percentage deviations of 8.49 with the maximum such deviation being 3.73% (the Region 3 OIL goal). The best solution obtained in this run was after 360 nodes with a sum of percentage deviations of 4.53, the maximum such deviation being 2.5% (the DELIVERY POINTS goal). After a total of 1995 nodes, the search was completed.

Minimizing the maximum percentage deviation produced a feasible solution after 37 nodes. This solution gave a maximum deviation of 2.5% (the OIL in region 3 goal). The sum of percentage deviations was 8.82. This was proved to be the optimal solution after 3158 nodes.

The second formulation was obtained by adding 228 ‘facets’ for the OIL goals in the three regions. The increased size of this model drastically reduced the number of nodes that could be explored in a given time. After 10 s with the objective of minimizing the sum of percentage deviations, 75 nodes had been explored with a minimum sum of percentage deviations of 11.28 obtained at node 61. The maximum deviation associated with this solution was 2.77% (the SPIRITS goal). When the objective was to minimize the maximum deviation, 36 nodes were explored with no feasible solution being found.

The most successful formulation was the third one, where six single 'tighter' constraints, logically equivalent to the six constraints implied by the OIL goals in the three regions, were added. With the objective of minimizing the sum of percentage deviations, a feasible solution was obtained after 35 nodes. The sum of percentage deviations was 8.83, and the maximum deviation was 2.5% (in the GROWTH prospects goals). This run was terminated after 394 nodes with no other feasible solutions being found.

With the objective of minimizing the maximum deviation, a feasible solution was found after 44 nodes with a maximum deviation of 3.15% (in the SPIRITS goal). The associated sum of percentage deviations was 12.14. A better solution was found at node 200 with a maximum deviation of 2.5% (in the GROWTH prospects goals). The associated sum of percentage deviations was 9.7. No better solutions were found after 303 nodes.

With hindsight, it is possible to observe that the minimax objective cannot be reduced below 2.5% because the slack variable in the GROWTH goal cannot be less than 0.2, given a right-hand side of 3.2. Therefore, the slack in this constraint was *fixed* at 0.2 and the surplus at 0. The third formulation of the problem was then rerun in order to minimize the sum of percentage deviations. At node 681, the optimal integer solution was found where the sum of percentage deviations was 7.806. The solution was proved optimal after 889 nodes. This optimal solution allocates the following retailers to D1:

M1, M2, M3, M4, M11, M16, M18, M21, M22.

All other retailers are to be assigned to division D2.

This problem has proved of considerable interest in view of its computational difficulty. A remodelling approach using 'basis reduction' is described by Aardal *et al*. (1999).

14.14 Opencast mining

The optimal solution is to extract the shaded blocks shown in Figure 14.4. This results in a profit of £17 500.

This solution was obtained in 28 iterations.

14.15 Tariff rates (power generation)

The following generators should be working in each period giving the following outputs:

Period 1	12 of type 1, output 10 200 MW
	3 of type 2, output 4 800 MW
Period 2	12 of type 1, output 16 000 MW
	8 of type 2, output 14 000 MW
Period 3	12 of type 1, output 11 000 MW
	8 of type 2, output 14 000 MW
Period 4	12 of type 1, output 21 250 MW
	9 of type 2, output 15 750 MW
	2 of type 3, output 3 000 MW
Period 5	12 of type 1, output 11 250 MW
	9 of type 2, output 15 750 MW

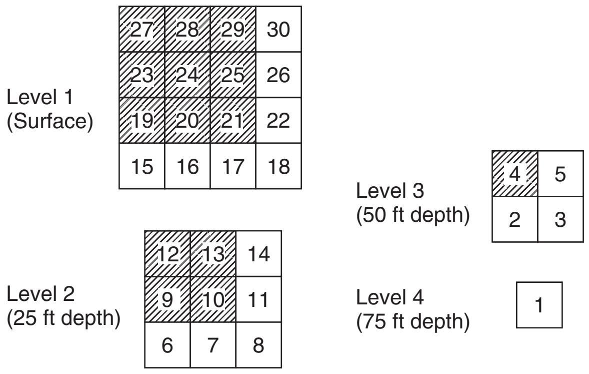


Figure 14.4

The total daily cost of this operating pattern is £988 540.

Deriving the marginal cost of production from a mixed integer programming model such as this encounters the difficulties discussed in Section 10.3. We could adopt the approach of fixing the integer variables at the optimal integer values and obtaining this economic information from the resulting linear programming model. The marginal costs then result from any changes within the optimal operating pattern, that is, without altering the numbers of different types of generator working in each period (although their levels of operation could vary).

The marginal costs of production per hour are then obtained from the shadow prices on the demand constraints (divided by the number of hours in the period). These give:

Period 1	£1.3 per megawatt hour
Period 2	£2 per megawatt hour
Period 3	£2 per megawatt hour
Period 4	£2 per megawatt hour
Period 5	£2 per megawatt hour

With this method of obtaining marginal valuations, there will clearly be no values associated with the reserve output guarantee constraints because all the variables have been fixed in these constraints.

As an alternative to using the above method of obtaining marginal valuations on the constraints, we could take the shadow prices corresponding to the continuous optimal solution. In this model, this solution (the continuous optimum) does not differ radically from the integer optimum. It gives the following operating pattern that is clearly unacceptable in practice because of the fractional number of generators working:

Period 1	12 of type 1,	output 10 200 MW
	2.75 of type 2,	output 4 800 MW
Period 2	12 of type 1,	output 15 200 MW
	8.46 of type 2,	output 14 800 MW
Period 3	12 of type 1,	output 10 200 MW
	8.46 of type 2,	output 14 800 MW
Period 4	12 of type 1,	output 21 250 MW
	9.6 of type 2,	output 16 800 MW
	1.3 of type 3,	output 1 950 MW
Period 5	12 of type 1,	output 10 200 MW
	9.6 of type 2,	output 16 800 MW

The resulting objective value (cost) is £985 164.

The shadow prices on the demand constraints imply the following marginal costs of production:

Period 1	£1.76 per megawatt hour
Period 2	£2 per megawatt hour
Period 3	£1.79 per megawatt hour
Period 4	£2 per megawatt hour
Period 5	£1.86 per megawatt hour

In this case, we can obtain a meaningful valuation for the 15% reserve output guarantee from the shadow prices on the appropriate constraints. The only non-zero valuation is on the constraint for period 4. This indicates that the cost of each guaranteed hour is £0.042. The range on the right-hand side coefficient of this constraint indicates that the 15% output guarantee can change between 2% and 52% with the marginal cost of each extra hour being £0.042.

14.16 Hydro power

The following thermal generators should be working in each period, giving the following outputs:

Period 1	12 of type 1,	output 10 565 MW
	3 of type 2,	output 5 250 MW
Period 2	12 of type 1,	output 14 250 MW
	9 of type 2,	output 15 750 MW
Period 3	12 of type 1,	output 10 200 MW
	9 of type 2,	output 15 750 MW
Period 4	12 of type 1,	output 21 350 MW
	9 of type 2,	output 15 750 MW
	1 of type 3,	output 1 500 MW
Period 5	12 of type 1,	output 10 200 MW
	9 of type 2,	output 15 750 MW

The only hydro generator to work is B in periods 4 and 5, producing an output of 1400 MW.

Pumping should take place in the following periods at the given levels:

Period 1	815 MW
Period 3	950 MW
Period 5	350 MW

Although it may seem paradoxical to both pump and run Hydro B in period 5, this is necessary to meet the requirement of the reservoir being at 16 m at the beginning of period 1, given that the Hydro can work only at a fixed level. It would be possible to use the model to cost this environmental requirement.

The height of the reservoir at the beginning of each period should be

Period 1	12 m
Period 2	17.63 m
Period 3	17.63 m
Period 4	19.53 m
Period 5	18.12 m

The cost of these operations is £986 630.

In solving this model, it is valuable to exploit the fact that the optimal objective value must be less than or equal to that reported in Section 14.15. This is for two reasons. First, the optimal solution in Section 14.15 using only thermal generators is a feasible solution to this model. Second, the 15% extra output guarantee can be met at no start-up cost using the hydro generators. When solving this model by the branch-and-bound method, the optimal objective value in Section 14.15 could be used as an 'objective cut-off' to prune the tree search.

Planning the use of hydro power by means of Stochastic Programming in order to model uncertainty is described by Archibald *et al*. (1999).

14.17 Three-dimensional noughts and crosses

The minimum number of lines of the same colour is four. There are many alternative solutions, one of which is given in Figure 14.5, where the top, middle and bottom sections of the cube are given. Cells with black balls are shaded.

This solution was obtained in 15 nodes. A total of 1367 nodes were needed to prove optimality.

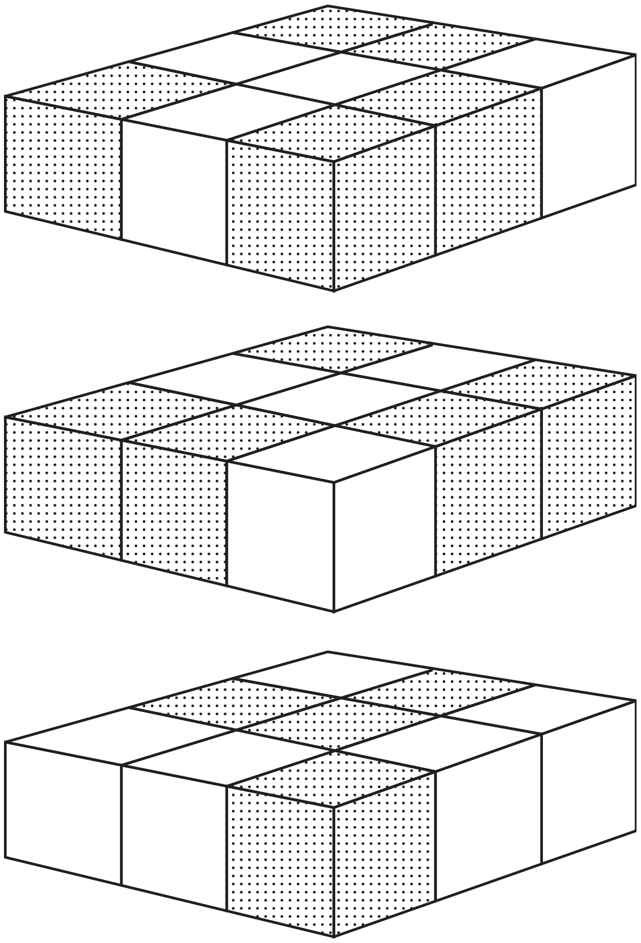


Figure 14.5

14.18 Optimizing a constraint

The ‘simplest’ version of this constraint (with minimum right-hand side coefficient) is

$$6x_1 + 9x_2 - 10x_3 + 12x_4 + 9x_5 - 13x_6 + 16x_7 + 14x_8 \leq 25.$$

This is also the equivalent constraint with the minimum sum of absolute values of the coefficients.

14.19 Distribution 1

The minimum cost distribution pattern is shown in Figure 14.6 (with quantities in thousands of tons).

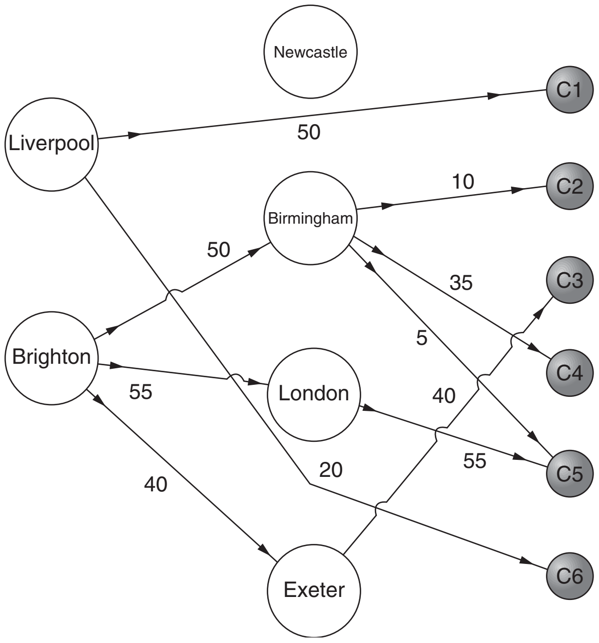


Figure 14.6

There is an alternative optimal solution in which the 40 000 ton from Brighton to Exeter comes from Liverpool instead.

This distribution pattern costs £198 500 per month.

Depot capacity is exhausted at Birmingham and Exeter. The value (in reducing distribution costs) of an extra ton per month capacity in these depots is £0.20 and £0.30, respectively.

This distribution pattern will remain the same as long as the unit distribution costs remain within certain ranges. These are given below (for routes which are to be used):

Route	Cost range
Liverpool to C1	$-\infty$ to 1.5
Liverpool to C6	$-\infty$ to 1.2
Brighton to Birmingham	$-\infty$ to 0.5
Brighton to London	0.3 to 0.8
Brighton to Exeter	$-\infty$ to 0.2
Birmingham to C2	$-\infty$ to 1.2
Birmingham to C4	$-\infty$ to 1.2
Birmingham to C5	0.3 to 0.7
London to C5	0.3 to 0.8
Exeter to C3	0 to 0.5

Depot capacities can be altered within certain limits. For the not fully utilized depots of Newcastle and London changing capacity within these limits has no effect on the optimal distribution pattern. For Birmingham and Exeter the effect on total cost will be £0.2 and £0.3 per ton per month within the limits. Outside certain limits the prediction of the effect requires resolving the problem. The limits are:

Depot	Capacity range
Birmingham	45 000–105 000 tons
Exeter	40 000–95 000 tons

N.B. All the above effects of changes are only valid if *one* thing is changed at a time within the permitted ranges. Clearly, the above solution does not satisfy the customer preferences for suppliers.

By minimizing the second objective, it is possible to reduce the number of goods sent by non-preferred suppliers to a customer to a minimum. This was done and revealed that it is impossible to satisfy all preferences. The best that could be done resulted in the distribution pattern shown in Figure 14.7, where customer C5 receives 10 000 tons from his non-preferred depot of London. This is the minimum cost such distribution pattern. (There are alternative patterns which also minimize the number of non-preferences but which cost more.) The minimum cost here is £246 000, showing that the extra cost of satisfying more customers preferences is £47 500.

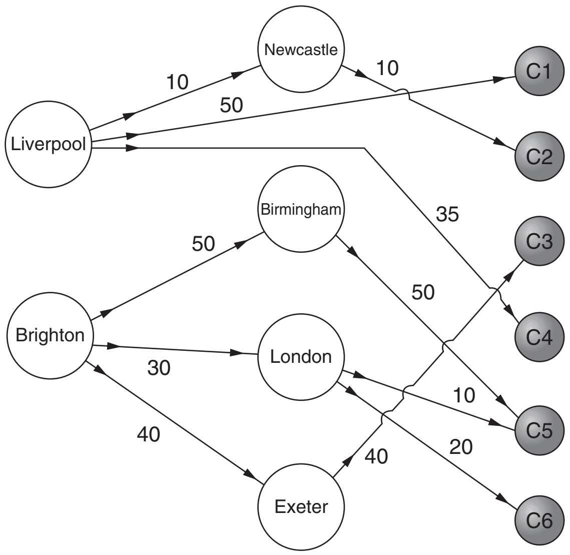


Figure 14.7

14.20 Depot location (distribution 2)

The minimum cost solution is to close down the Newcastle depot and open a depot in Northampton. The Birmingham depot should be expanded. The total monthly cost (taking account of the saving from closing down Newcastle) resulting from these changes and the new distribution pattern is £174 000. Figure 14.8 shows the new distribution pattern (with quantities in thousands of tons).

This solution was obtained in 40 iterations. The continuous optimal solution was integer. Therefore, no tree search was necessary.

14.21 Agricultural pricing

The optimal prices are:

Milk	£303 per ton
Butter	£667 per ton
Cheese 1	£900 per ton
Cheese 2	£1085 per ton

The resultant yearly revenue will be £1992 million. It is straightforward to calculate the yearly demands that will result from these prices. They are:

Milk	4 781 000 tons
Butter	384 000 tons
Cheese 1	250 000 tons
Cheese 2	57 000 tons

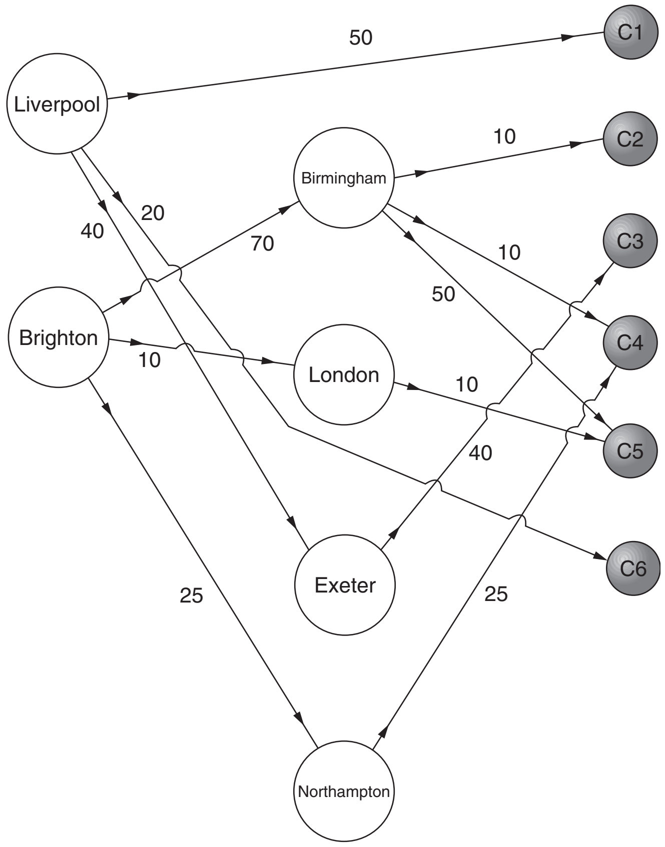


Figure 14.8

The economic cost of imposing a constraint on the price index can be obtained from the shadow price on the constraint. For this example, this shadow price in the optimal solution indicates that each £1 by which the new prices are allowed to increase the cost of past year's consumption would result in an increased revenue of £0.61.

14.22 Efficiency analysis

The efficient garages turn out to be 3 (Basingstoke), 6 (Newbury), 7 (Portsmouth), 8 (Alresford), 9 (Salisbury), 11 (Alton), 15 (Weymouth), 16

(Portland), 18 (Petersfield), 22 (Southampton), 23 (Bournemouth), 24 (Henley), 25 (Maidenhead), 26 (Fareham) and 27 (Romsey).

It should be observed that these garages may be efficient for different reasons. For example, Newbury has 12 times the staff of Basingstoke but only five times as much showroom space. It sells 10 times as many Alphas and 10.4 times as many Betas and makes nine times as much profit. This suggests that it makes more efficient use of showroom space but less of staff.

The other garages are deemed inefficient. They are listed in Table 14.8 in decreasing order of efficiency together with the multiples of the efficient garages that demonstrate them to be inefficient.

Table 14.8

Garage	Efficiency number	Multiples of efficient garages
19 Petworth	0.988	0.066(6) + 0.015(18) + 0.034(25) + 0.675(26)
21 Reading	0.982	1.269(3) + 0.544(15) + 1.199(16) + 2.86(24) + 1.37(25)
14 Bridport	0.971	0.033(3) + 0.470(16) + 0. 783(24) + 0.195(25)
2 Andover	0.917	0.857(15) + 0.215(25)
28 Ringwood	0.876	0.008(3) + 0.320(16) + 0.146(24)
5 Woking	0.867	0.952(8) + 0.021(11) + 0.009(22) + 0.148(25)
4 Poole	0.862	0.329(3) + 0.757(16) + 0.434(24) + 0.345(25)
12 Weybridge	0.854	0.797(15) + 0.145(25) + 0.018(26)
1 Winchester	0.840	0.005(7) + 0.416(8) + 0.403(9) + 0.333(15) + 0.096(16)
13 Dorchester	0.839	0.134(3) + 0.104(8) + 0.119(15) + 0.752(16) + 0.035(24) + 0.479(26)
20 Midhurst	0.829	0.059(9) + 0.066(15) + 0.472(16) + 0.043(18) + 0.009(25)
17 Chichester	0.824	0.058(3) + 0.097(8) + 0.335(15) + 0.166(16) + 0.236(24) + 0.154(26)
10 Guildford	0.814	0.425(3) + 0.150(7) + 0.623(8) + 0.192(15) + 0.168(16)

For example, the comparators to Petworth taken in the multiples given below use inputs of:

Staff	5.02
Showroom space	550 m^2
Category 1 population	2 (1000s)
Category 2 population	2 (1000s)
Alpha enquiries	7.35 (100s)
Beta enquiries	3.98 (100s)

to produce outputs of:

1.518 (1000s)	Alpha sales
0.568 (1000s)	Beta sales
1.568 (million pounds)	Profit

Clearly, this uses no more than the inputs used by Petworth but produces outputs at least 1.0119 times as great.

There is also a useful interpretation of the dual values on the input and output constraints. These can be regarded as *weightings* that the particular garage would like to place on its inputs and outputs so as to maximize this weighted ratio of outputs to inputs but keep the corresponding ratios for the other garages above 1 (i.e. not allow them to appear inefficient). In the case of Petworth, the dual values from solving the model turn out to be 0, 0.1557, 0.0618, 0.0158, 0 and 0 for the inputs and 0.1551, 0 and 0.495 for the outputs.

This gives a weighted ratio of

$$\frac{0.1551 \times 1.5 + 0.495 \times 1.55}{0.1557 \times 5.5 + 0.0618 \times 2 + 0.0158 \times 2} = 0.988.$$

This is clearly the efficiency number. Petworth weighs most heavily high outputs that bring it out in the best light and least heavily high inputs.

The *dual* formulation referred to in Sections 3.2 and 13.22 chooses the weights so as to maximize this ratio while not allowing the ratios for other garages (with these weights) to fall below 1.

14.23 Milk collection

The optimal solution is given in Figure 14.9 with dashes representing the first day's routes and dotted lines those of the second day. The total distance covered is 1229 miles.

In the formulation given in Section 13.23, this first solution produced subtours on day 1 around farms 2, 5 and 18; around 3, 16, 13, 4 and 19 and around 1, 8, 21, 9, 11, 7, 6 and 10 and on day 2, around farms 6, 7 and 20 and around all the other farms. This infeasible solution covers 1214 miles (there are alternative, equally good solutions). Subtour elimination constraints were then introduced to prevent the subtours (on both days to prevent them re-arising on the other day), resulting in another solution with no subtours on day 1 but subtours on day 2 around farms 1, 2, 17, 6, 7 and 10; around 4, 15, 3, 5, 14, 16 and 13 and around 8, 9 and 21. Subtour elimination constraints were introduced to prevent these subtours (on both days). This resulted in the optimal solution (no subtours) given in Figure 14.9.

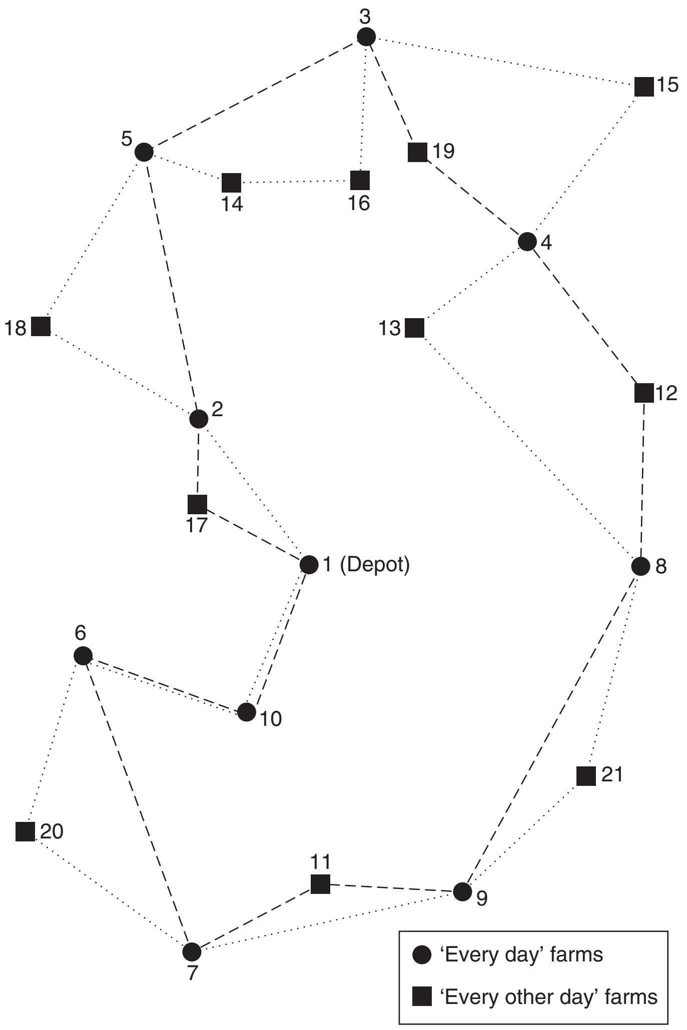


Figure 14.9

These stages required 34, 13 and 272 nodes, respectively. Out of interest, not introducing the 'disaggregated' constraints described in Section 13.23 resulted in the first stage taking 1707 nodes.

This problem is based on a larger one described by Butler, Williams and Yarrow (1997). That larger problem required a more sophisticated solution approach using generalizations of known results concerning the structure of the travelling salesman polytope.

14.24 Yield management

Numbers of seats have been rounded to nearest integers where necessary.

Period 1

Sell tickets at the following prices up to what is available:

First	£1200
Business	£900
Economy	£500

Set provisional prices for period 2:

First	£1150 if scenario 1 in period 1
	£1150 if scenario 2 in period 1
	£1300 if scenario 3 in period 1
Business	£1100 for all scenarios in period 1
Economy	£700 for all scenarios in period 1

Set provisional prices for period 3:

First	£1500 for all scenarios in periods 1 and 2
Business	£800 for all scenarios in periods 1 and 2
Economy	£480 for all scenarios in period 1 and scenarios 1 and 2 in period 2; £450 for all scenarios in period 1 and scenario 3 in period 2

(Provisionally) book three planes. Expected revenue is £169 544.

Period 2

Rerunning the model with the demand given (with hindsight) for the price levels decided for period 1 results in the following recommended decisions for period 2.

Sell tickets at the following prices up to what is available:

First	£1150
Business	£1100
Economy	£700

Set provisional prices for period 3:

First	£1500 for all scenarios in period 2
Business	£800 for all scenarios in period 2
Economy	£480 for scenarios 1 and 2 in period 2
	£450 for scenario 3 in period 2

(Provisionally) still book three planes. Expected total revenue is now £172 969.

Period 3

Rerunning the model with the known demands and price levels for periods 1 and 2 results in the following recommended decisions for period 3.

Sell tickets at the following prices up to what is available:

First	£1500
Business	£800
Economy	£480

(Provisionally) still book three planes. Expected total revenue is now £176 392.

Solution

The resultant solution at take-off (using demands during the final week) will therefore be:

Period 1

First Class	25 seats sold at £1200: Yield £30 000
Business Class	45 seats sold at £900: Yield £40 500
Economy Class	50 seats sold at £500: Yield £25 000

Period 2

First Class	50 seats sold at £1150: Yield £57 500
Business Class	45 seats sold at £1100: Yield £49 500
Economy Class	50 seats sold at £700: Yield £35 000

Period 3

First Class	40 seats sold at £1500: Yield £60 000
Business Class	25 seats sold at £800: Yield £20 000
Economy Class	36 seats sold at £480: Yield £17 280

Three planes will be needed. Total yield (subtracting the costs of the planes) will therefore be £184 780.

It will be necessary to reallocate four seats from Business to First Class and five seats from Economy to Business Class.

If the model is altered to maximize yield subject to expected demand (without recourse), the resultant solution (allowing for the given demands falling below those expected) is:

Period 1

First Class	24 seats sold at £1200: Yield £28 800
Business Class	39 seats sold at £900: Yield £35 100
Economy Class	50 seats sold at £500: Yield £25 000

Period 2

First Class	55 seats sold at £1150: Yield £63 250
Business Class	43 seats sold at £1100: Yield £47 300
Economy Class	49 seats sold at £700: Yield £34 300

Period 3

First Class	34 seats sold at £0.1500: Yield £51 000
Business Class	35 seats sold at £800: Yield £28 000
Economy Class	37 seats sold at £480: Yield £17 760

Three planes are needed. Total yield is £180 210.

This can clearly be improved by, in the last period, ‘topping-up’ to fill vacant capacity in the most beneficial way, that is, selling 39 Economy Class instead of 37, so filling the three planes.

This increases yield to £181 170, which is still considerably short of the yield produced by running the Stochastic Program with recourse.

14.25 Car rental

The numbers of cars in all aspects of the following solution have been rounded from the fractional answers that result from the linear programming model.

The company should own 624 cars and pursue the following policies that will result in a weekly profit of £122 398.

The estimated number of undamaged cars in each depot at the beginning of each day will be as follows (there will also be hired out cars not in any depot).

	Glasgow	Manchester	Birmingham	Plymouth
Monday	68	99	145	46
Tuesday	66	94	154	39
Wednesday	70	100	125	47
Thursday	69	115	117	44
Friday	71	102	126	44
Saturday	65	96	154	73

The estimated number of damaged cars in each depot at the beginning of each day will be as follows.

	Glasgow	Manchester	Birmingham	Plymouth
Monday	11	12	20	6
Tuesday	7	12	20	4
Wednesday	8	12	20	6
Thursday	9	12	20	5
Friday	11	12	20	5
Saturday	7	12	22	3

Of the undamaged cars, the following should be rented out each day, for the periods and destinations in the proportions given in the statement of the problem.

	Glasgow	Manchester	Birmingham	Plymouth
Monday	68	99	95	46
Tuesday	66	94	154	39
Wednesday	70	80	125	47
Thursday	69	115	111	44
Friday	71	102	70	0
Saturday	65	93	124	73

No transfers of undamaged cars should be made but the following transfers of damaged cars should be made (arriving the following day).

Glasgow to Manchester

Monday	Tuesday	Wednesday	Thursday	Friday	Saturday
3	2	3	2	3	2

Glasgow to Birmingham

Monday	Tuesday	Wednesday	Thursday	Friday	Saturday
5	5	3	4	9	5

Plymouth to Birmingham

Monday	Tuesday	Wednesday	Thursday	Friday	Saturday
5	3	6	5	5	3

The two repair depots of Manchester and Birmingham are fully occupied on all six days repairing 12 and 20 cars, respectively, each day. Repair capacity is clearly a limiting factor on the operation of the company. This is reflected by the high-shadow prices on the repair capacity constraints, which vary between £617 and £646 per car per day in both repairing depots. A plan to increase repair capacity forms the subject of problem 12.27. The solution above was obtained in 153 iterations.

14.26 Car rental 2

Only the Birmingham repair capacity should be increased, using both expansion options, to expand to 22 cars per day. Repair capacities in all depots are again fully used.

This allows the company to expand its fleet to 895 cars, resulting in a new weekly profit of £135 511. All demands still cannot be fully met.

This model solved in 5 nodes.

14.27 Lost baggage distribution

Two vans are needed. Solving the model, defined in Section 13.27 (a relaxation of the final model), leads to the ‘solution’ given in Figure 14.10. This required 2263 nodes to solve.

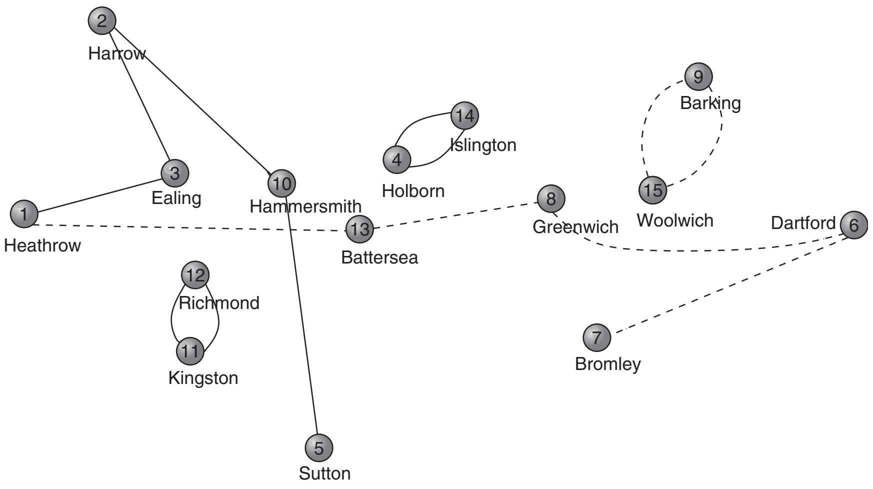


Figure 14.10

One van uses the bold lines and the other the dashed lines. Both take 120 min. Clearly, there are unacceptable subtours.

A total of 31 subtour elimination constraints needed to be appended, in 14 stages, in order to obtain a true solution with no subtours (there are a number of such solutions). This solution required 1296 nodes (a few seconds on a notepad computer). Fixing the number of vans needed at two and minimizing the maximum time needed by the vans took a further 561 nodes to solve (no additional subtour elimination constraints were needed). It produced the two routes illustrated in Figure 14.11, a bold line and a dashed line The times taken by the vans to reach their furthest locations are 99 and 100 min, respectively.

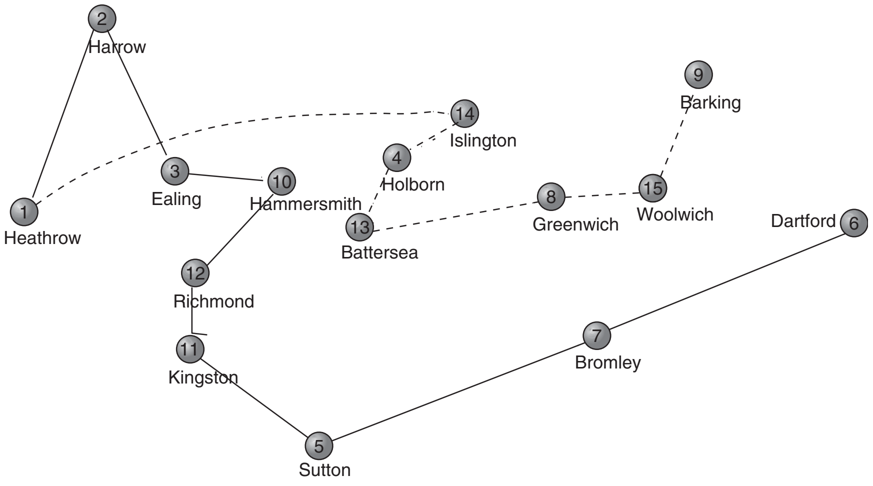


Figure 14.11

14.28 Protein folding

The model took 471 nodes to solve. Eight hydrophobic acids were matched (additional to those already contiguous) with folds between acids 3 and 4, 8 and 9, 22 and 23 and 35 and 36. The folded protein is illustrated in Figure 14.12 with the extra matches marked by dashed lines.

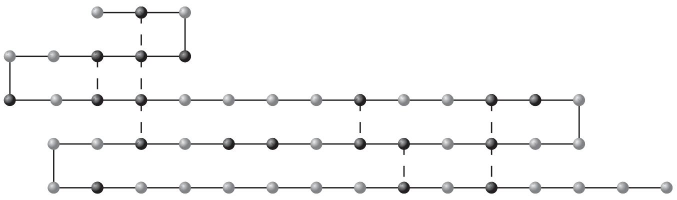


Figure 14.12

14.29 Protein comparison

This model took 253 branch-and-bound nodes to solve and resulted in the comparison illustrated in Figure 14.13 with five comparable edges demonstrating the lack of similarity between the two proteins.

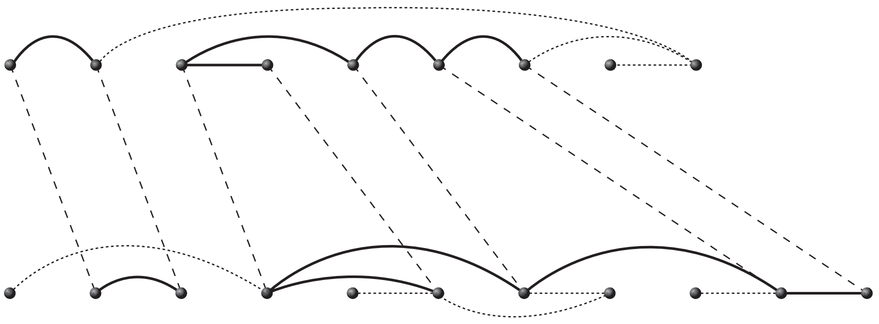


Figure 14.13

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