Hypothesis testing: The Central Limit Theorem at work!

To perform a hypothesis test, one must first have a structure to their test. Below is a 6-step guide to performing any hypothesis test:

Seven Steps to Conducting a Hypothesis Test

(1) Write what you know

(2) State the claim

(3) Write the Null and Alternative hypotheses (**H_0** and **H_1**)

(4) Draw a picture of the test

(5) Run the test (use critical regions or p-value method)

(6) Make the initial conclusion: Reject **H_0** *or* Fail to reject **H_0**

(7) Make the final conclusion: Reject the claim *or* Accept the claim

Running the test:

(i) Calculate the test statistic

(ii) Calculate the critical region: If **H_1** is ">" do a **right**-tailed test
If **H_1** is "<" do a **left**-tailed test
If **H_1** is "≠" do a **two**-tailed test

(iii) Draw Critical region and test statistic in diagram.
Then determine if the test statistic is in the critical region If it **is**, Reject **H_0**
If it **is not**, Fail to reject **H_0**

Test Type	**Test Statistic**	**Distribution**	
Proportion	$z = \frac{\hat{p}-p_0}{\sqrt{\frac{p_0 q_0}{n}}}$	Normal	
Mean (σ known)	$z = \frac{\hat{x}-\mu_0}{\frac{\sigma}{\sqrt{n}}}$	Normal	
Mean (σ unknown)	$t = \frac{\hat{x}-\mu_0}{\frac{s}{\sqrt{n}}}$	T	$\mathrm{df} = n-1$
Standard Deviation	$\chi^2 = \frac{(n-1)s^2}{\sigma_0^2}$	Chi-Squared	$\mathrm{df} = n-1$

How the test works

Recall the **Central Limit Theorem:**

> Consider a population of mean μ and standard deviation σ
> For a sample of size n, the value of the sample mean $\bar{x}$ follows a normal distribution with mean
> $\mu_{\bar{x}} = \mu$ and standard deviation $\sigma_{\bar{x}} = \frac{\sigma}{\sqrt{n}}$ (Note: This holds only if the population is normally distributed or if $n \geq 30$)

Using this, we determine if our assumption for the null hypothesis ($\mathbf{H_0}$) is reasonable or not.
If it is unlikely, by the **<u>Rare Event rule</u>**, our hypothesis is probably incorrect (i.e. reject $\mathbf{H_0}$).

In general, we use the following:

- If the test sample yields an <u>unlikely result</u>, it is probably incorrect (reject $\mathbf{H_0}$)
- If the test sample yields a <u>likely result</u>, it is probably correct (fail to reject $\mathbf{H_0}$)

For each parameter, the same technique is used, but with a different distribution:

Proportion	Standard Normal	(*as an approximation to the Binomial dist.*)
Mean (σ known)	Standard Normal	
Mean (σ unknown)	T	$(\mathrm{df} = n - 1)$
Standard Deviation	Chi-Squared	$(\mathrm{df} = n - 1)$

Example (testing a mean when σ is known)

Consider a population that is normally distributed with $\boldsymbol{\sigma} = 10$
We claim the population mean is $\boldsymbol{\mu}_0 = 100$
Use a sample of size $\boldsymbol{n} = 25$ and mean $\bar{\boldsymbol{x}} = 105.0$ to test the claim

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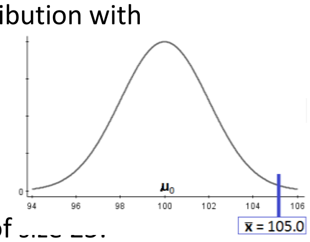
Assuming $\mathbf{H_0}$ is true (i.e. $\boldsymbol{\mu} = \boldsymbol{\mu}_0 = 100$), the sample mean will follow a distribution with

$$\boldsymbol{\mu}_{\bar{\boldsymbol{x}}} = \boldsymbol{\mu}_0 = 100 \quad \text{and} \quad \boldsymbol{\sigma}_{\bar{\boldsymbol{x}}} = \frac{\boldsymbol{\sigma}}{\sqrt{\boldsymbol{n}}} = \frac{10}{\sqrt{25}} = 2.0$$

If we plot the sample mean with the sampling distribution,
we get diagram on the right:



Note that the probability of getting 105.0 or higher is very low for a sample of size 25.
Indeed: $\mathbf{P}(\bar{\boldsymbol{x}} \geq 105.0) = 0.0062$ This is called the **<u>p-value</u>**

Since the probability of getting 105.6 or higher is so low, by the Rare Event Rule our assumption of $\boldsymbol{\mu} = \boldsymbol{\mu}_0 = 100$ is probably incorrect (reject $\mathbf{H_0}$)

For convenience, the above distribution is transformed into the standard normal distribution by using the ***z***-score of the sample mean (called the Test Statistic):

$$\boldsymbol{z} = \frac{\bar{\boldsymbol{x}} - \boldsymbol{\mu}_0}{\frac{\boldsymbol{\sigma}}{\sqrt{\boldsymbol{n}}}} = \frac{105.0 - 100}{\frac{10}{\sqrt{25}}} = 1.00$$

Then the p-value is calculated via the normal distribution: $\mathbf{P}(z \geq 1.00) = 0.0062$

Example 1: Testing a proportion

The XSORT method of gender selection is believed to **increases** the likelihood of birthing a girl.
14 couples used the XSORT method and resulted in the birth of 13 girls and 1 boy. Using a 0.05 significance level, test the claim that the XSORT method increases the birth rate of girls. (Assume the normal birthrate of girls is 0.5)

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(1) *Write what you know*

Original proportion: $\boldsymbol{p}_0 = 0.5$

Sample Size: $\boldsymbol{n} = 14$

Number of successes: $\boldsymbol{x} = 13$

Significance Level: $\boldsymbol{\alpha} = 0.05$

$\hat{\boldsymbol{p}} = \frac{x}{n} = \frac{13}{14} = 0.9286$

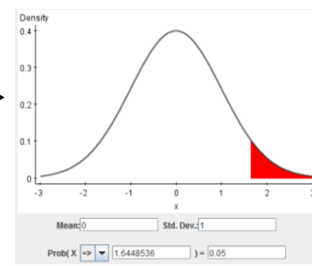
$\hat{\boldsymbol{q}} = 1 - \hat{p} = 0.0714$

(2) *State the Claim* The XSORT method increases the birth rate of girls: $(\boldsymbol{p} > \boldsymbol{p}_0 = 0.5)$

(3) *Write the Null and Alternative hypotheses* $\mathbf{H_0}: \boldsymbol{p} = 0.5$

$\mathbf{H_1}: \boldsymbol{p} > 0.5$

(4) *Draw a picture* (right tailed test)



(5) *Run the test*

(i) Calculate the test statistic: $z = \frac{\hat{\boldsymbol{p}} - \boldsymbol{p}_0}{\sqrt{\frac{\boldsymbol{p}_0 \boldsymbol{q}_0}{\boldsymbol{n}}}} = \frac{0.9286 - 0.5}{\sqrt{\frac{0.5 \cdot 0.5}{14}}} = 3.207$

(ii) Determine the critical region(s): $z_\alpha = z_{0.05} = 1.64$

Critical region: $(1.64, \infty)$

(iii) The test statistic DOES lie in the critical region

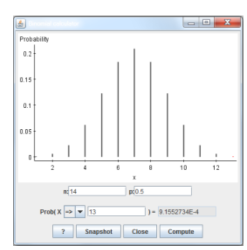
(6) *Make the initial conclusion*: Reject $\mathbf{H_0}$ (since the test statistic lies in the critical region)

(7) *Make the final conclusion*: **Accept the claim** (since the claim is equivalent to $\mathbf{H_1}$)

Conceptual meaning

Assuming $\mathbf{H_0}$ is true ($p = 0.5$), we use a binomial distribution (with $n = 14$) to find the probability of getting 13 or more girls: $\mathrm{P}(x \geq 13) = 0.00092$

By the **Rare Event Rule,** this sample indicates our hypothesis ($p = 0.5$) does not hold (i.e. reject $\mathbf{H_0}$).



Note: By the normal approximation of the Binomial Distribution (see Ch.6),
we can approximate the distribution with a normal distribution with mean np and standard deviation $\sqrt{npq}$.
This is why we use the normal distribution when testing a proportion.

Example 2: Testing a <u>mean (σ known)</u>

A sample of 54 bears has a mean weight of 237.9 lb. Assuming that ***σ*** is known to be 37.8 lb. use a 0.05 significance level to test the claim that the population mean of all such bear weights does not equal 250 lb.

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(1) *<u>Write what you know</u>*

Original mean: $\mu_0 = 250$

Sample Size: $n = 54$

Sample Mean: $\bar{x} = 237.9$

Population Standard Deviation: $\sigma = 37.8$

Significance Level: $\alpha = 0.05$

(2) *<u>State the Claim</u>*

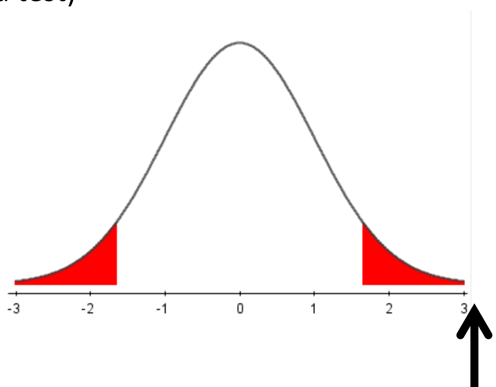
The mean weight of bears is not 250lb: i.e. $\mu \neq 250$

(3) *<u>Write the Null and Alternative hypotheses</u>*

$\mathbf{H_0}: \mu = 250$

$\mathbf{H_1}: \mu \neq 250$

(4) *<u>Draw a picture</u>* (two tailed test)



(5) *<u>Run the test</u>*

(i) Calculate the test statistic: $z = \frac{\hat{\mu}-\mu_0}{\frac{\sigma}{\sqrt{n}}} = \frac{-0.5}{\sqrt{\frac{0.5\cdot 0.5}{14}}} = 3.207$

(ii) Determine the critical region: $z_{\frac{\alpha}{2}} = z_{0.025} = 1.96$

Critical region: $(\infty, -1.96)$ and $(1.96, \infty)$

(iii) The test statistic DOES lie in the critical region

(6) *<u>Make an initial conclusion</u>*

Reject $\mathbf{H_0}$ (since the test statistic lies in the critical region)

(7) *<u>Make an initial conclusion</u>*

Accept the claim (since the claim is equivalent to $\mathbf{H_1}$)