Tutorial on Introduction to Sequential Monte Carlo methods

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Learning objectives

This tutorial is to introduce the Sequential Monte Carlo (SMC) method from a *practical* point of view. The main goal is that the participants can effectively use the method in their data analysis problem.

Materials

Lecture slides, code and data are available at
`https://github.com/VBayesLab/Tutorial-on-SMC`

Outline

Quick Introduction to Monte Carlo methods

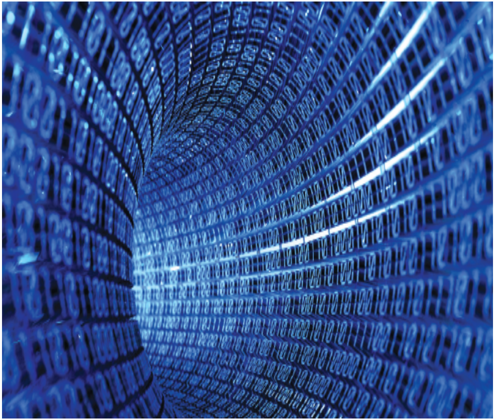
Motivating examples

Sequential Monte Carlo: Fixed Domain

Sequential Monte Carlo: Extended Domain

Monte Carlo methods

Monte Carlo simulation is at the heart of scientific computing[1]



[1] Picture credit: `https://sinews.siam.org/Details-Page/the-future-of-scientific-computation`

Monte Carlo methods

A basic problem in statistics is to compute an integral of the form

$$\mathcal{I} = \mathbb{E}[h(X)] = \int_{\mathcal{X}} h(x)\pi(x)dx$$

where $\pi(x)$ is a pdf on space $\mathcal{X}$, $X \sim \pi(x)$, and some function $h(x) : \mathcal{X} \mapsto \mathbb{R}$. Often, $\mathcal{X}$ is high dimensional.

E.g.

$$\mu_X = \mathbb{E}[X] = \int x\pi(x)dx, \quad \mathrm{Cov}(X) = \int (x - \mu_X)(x - \mu_X)^\top \pi(x)dx$$

$$\mathbb{P}(X \in A) = \int I_A(x)\pi(x)dx$$

where $I_A(x) = 1$ if and only if $x \in A$.

In most cases, we can't compute $\mathcal{I}$ analytically, have to use computers.

Monte Carlo problems

Monte Carlo methods are to deal with two main problems

P1 Generating samples from a probability distribution of interest with pdf $\pi(x)$.

P2 Estimating an integral of the form

$$\mathcal{I} = \int h(x)\pi(x)dx = \mathbb{E}_{X\sim\pi(x)}\big(h(X)\big)$$

for some function $h(x)$.

Problem 2 can be solved from Problem 1. Sometimes, it's more convenient and more efficient to solve Problem 2 directly.

Monte Carlo methods

Suppose that we are able to use a computer to generate

- i.i.d. samples $X_i \overset{iid}{\sim} \pi(x)$, $i = 1, \ldots, n$, or
- dependent, but ergodic Markov chain $\{X_i\}_{i \geq 1}$ with equilibrium distribution π.

Let

$$\widehat{\mathcal{I}}_n := \frac{1}{n} \sum_{i=1}^{n} h(X_i)$$

By LLN, $\widehat{\mathcal{I}}_n \xrightarrow{a.s.} \mathcal{I}$. For the iid case, $\mathbb{V}(\widehat{\mathcal{I}}_n) = \mathbb{V}(h(X))/n \longrightarrow 0$ as $n \to \infty$ regardless of the dimension of the integral.

Why MC methods work? Unlike numerical methods that spend all computational resources equally on the entire domain $\mathcal{X}$, Monte Carlo methods efficiently focus on regions with high π-density.

Monte Carlo methods

Example.

$$\mathcal{I} = \int_{\mathbb{R}} x^2 \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) dx = 1$$

Let try the following in Matlab

```
x = normrnd(0,1,n,1); Ihat = sum(x.*x)/n
```

n	100	1000	10000	1,000,000
$\widehat{\mathcal{I}}_n$	0.8098	0.9702	1.0057	1.0010
MC error	0.0569	0.0410	0.0312	0.0011

Note: *Reporting MC error is a "must" in any scientific reports that use MC.*

Monte Carlo methods

But wait... How can we use a computer to generate samples $X_i \sim \pi(x)$?

Note: there are two different uses of the notation $X \sim \pi(x)$. One means "distributed as" and the other means "sampled from".

Generating samples from a given probability distribution, or from a sequence of distributions, is the main focus of Monte Carlo methods.

This tutorial is about the Sequential Monte Carlo method, that generates samples from a sequence of distributions.

Outline

Example 1: Volatility modelling and forecasting

Let $y_{1:t} = \{y_1, ..., y_t\}$ be daily returns of a financial stock up to day t. Want to forecast the volatility $\sigma_{t+1}^2 = \mathbb{V}(y_{t+1}|y_{1:t})$.

GARCH model:

$$\begin{aligned} y_t &= \sigma_t \epsilon_t, \quad \epsilon_t \sim N(0,1) \\ \sigma_t^2 &= w + \alpha\sigma_{t-1}^2 + \beta y_{t-1}^2 \end{aligned}$$

The model parameter is $\theta = (w, \alpha, \beta)$. The posterior

$$\pi_t(\theta) = \frac{p(\theta)p(y_{1:t}|\theta)}{Z_t}, \; Z_t = \int p(\theta)p(y_{1:t}|\theta)d\theta$$

We want to sample from $\pi_t(\theta)$ sequentially. This provides a principled and convenient way to produce volatility forecast sequentially as data arrives

$$p(\sigma_{t+1}^2|y_{1:t}) = \int p(\sigma_{t+1}^2|\theta, y_t)\pi_t(\theta)d\theta, \; t = n+1, n+2, ...$$

We need some "long-enough" data $y_{1:n}$.

A bit about notations

In most cases, we only know the target density $\pi(x)$ up to a normalizing constant, i.e.,

$$\pi(x) = \frac{\gamma(x)}{Z}$$

where function $\gamma(x)$ is known but constant $Z = \int \gamma(x)dx$ is unknown.

Then, we often write

$$\pi(x) \propto \gamma(x)$$

to mean that we know $\pi(x)$ up to a constant.

E.g., the posterior distribution is often known up to a constant

$$\pi(\theta) = p(\theta|y) = \frac{1}{Z} \times \underbrace{p(\theta)}_{\text{prior}} \times \underbrace{p(y|\theta)}_{\text{likelihood}}, \text{ written as } \pi(\theta) \propto p(\theta)p(y|\theta)$$

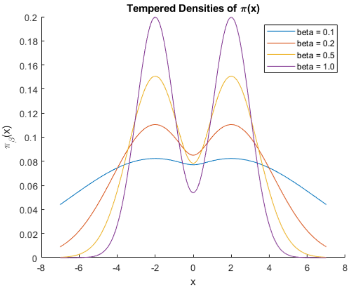
$Z = p(y) = \int p(\theta)p(y|\theta)d\theta$ is unknown, called marginal likelihood.

Example 2: Annealed importance sampling

- In many cases, it's challenging to sample from a target distribution $\pi(x)$: e.g., multimodal, non-standard shape
- A strategy is to design a sequence

$$\pi_t(x) \propto \pi(x)^{\beta_t}$$

  where $0 < \beta_1 < \cdots < \beta_T = 1$: sequence of temperatures.
- We first sample from π_1 (easy task as π_1 is very flat), then π_2, ..., until $\pi_T(x) = \pi(x)$.



Tempering annealed densities. If β_{t-1} is close to β_t, then π_{t-1} is close to π_t.

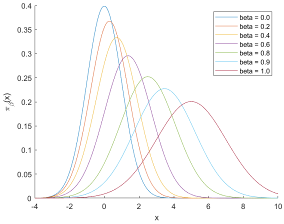
Example 3: Sequential sampling from bridged densities

- π_0: distribution that's easy to sample from; e.g., prior.
- $\pi(x)$: target distribution; e.g., the posteiror.
- Construct a sequence of "bridging" distributions from π_0 to π as follow

$$\pi_t(x) \propto \underbrace{\pi_0(x)}_{\text{easy}}{}^{1-\beta_t} \underbrace{\pi(x)}_{\text{difficult}}{}^{\beta_t},$$

$$0 = \beta_0 < \cdots < \beta_T = 1.$$

- We first sample from π_0 (easy to sampe from by definition), then π_1, ..., until $\pi_T(x) = \pi(x)$.



Bridging densities from $N(0,1)$ to $N(5,2)$.

In case $\pi(\theta)$ is a posterior $\pi_0(\theta) \propto p(\theta)p(y|\theta)$, with prior $\pi_0(\theta)$ and likelihood $p(y|\theta)$,

$$\pi_t(\theta) \propto \pi_0(\theta)p(y|\theta)^{\beta_t}$$

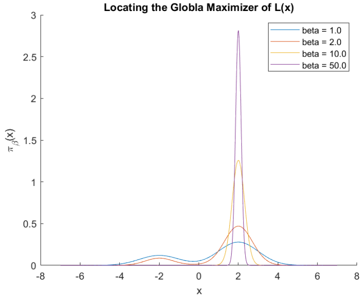
Often known as likelihood annealing method.

Example 4: Finding global maximizer

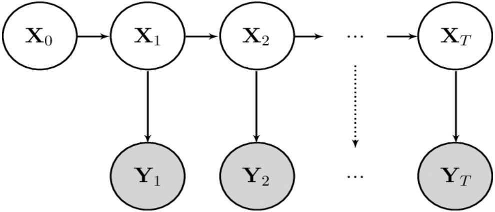
- $f(x)$: a non-negative objective function to maximize.
- construct a sequence of distributions as follow

$$\pi_t(x) \propto f(x)^{\beta_t}$$

  where $\beta_t \uparrow \infty$.
- for large β_t, $\pi_t(x)$ concentrates its mass at the global maximizer of $f(x)$.



Example 5: State-space models



$\{X_t\}_{t\geq 0}$: hidden/latent Markov process with

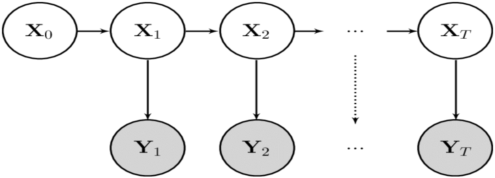
$$X_0 \sim \mu(x_0), \quad X_t|X_{t-1} = x_{t-1} \sim f(x_t|x_{t-1})$$

$\{Y_t\}_{t\geq 1}$: observed process, independent conditionally on $\{X_t\}_{t\geq 1}$:

$$Y_t|X_t = x_t \sim g(y_t|x_t)$$

Main task: Sample from $p(x_{1:t}|y_{1:t})$ for $t = 1, 2, \ldots$. Known as the particle filter problem.

Example 5: State-space models



There are abundant of state-space models, used in many scientific fields.

Stochastic volatility model: Let $\{Y_t\}_{t\geq 1}$ be financial asset returns

$$\begin{aligned} Y_t|X_t = x_t &\sim N\left(0, e^{x_t}\right) \\ X_t|X_{t-1} = x_{t-1} &\sim N(\mu + \phi x_{t-1}, \sigma^2). \end{aligned}$$

Here $\{X_t = \log \mathbb{V}(Y_t|X_t)\}_{t\geq 1}$ is the log-volatility process of interest. Financial risk management requires samples from X_1 (given y_1), then X_2 (given $y_{1:2}$), etc.

Two settings of SMC problems

Fixed domain: we want to approximate a sequence of distributions with the same domain

$$\pi_t(x), \quad x \in \mathcal{X}, \quad t = 1, 2, \ldots.$$

Examples 1-4 above belong to this setting.

Extended domain: we want to approximate the sequence of distributions with extended domain

$$\pi_t(x_{1:t}), \quad x_{1:t} \in \mathcal{X}^{\otimes t}, \quad t = 1, 2, \ldots.$$

Example 5, state-space models, belongs to this setting.

The two settings share many similarities in term of algorithmic design, but also have some substantial difference.

Outline

Sequential Monte Carlo: Fixed Domain

- Let $\{\pi_t\}_{t\geq 1}$ be a sequence of distributions defined on $\mathcal{X}$; each $\pi_t(x)$ is *known up to a normalizing constant*

$$\pi_t(x) = \underbrace{\frac{1}{Z_t}}_{\text{unknown}} \times \underbrace{\gamma_t(x)}_{\text{computable for every } x}$$

- Task: Sample from π_t and estimate Z_t *sequentially*: first sample from π_1 and estimate Z_1, then sample from π_2 and estimate Z_2, etc.

At each time t, we will use a set of weighted samples $\{W_t^{(i)}, X_t^{(i)}\}_{i=1}^M$, called particles, to approximate $\pi_t(x)$. That is, $\pi_t(x)$ is approximated by

$$\widehat{\pi}_t(x) = \sum_{i=1}^{M} W_t^{(i)} \delta_{X_t^{(i)}}(x)$$

where $0 \leq W_t^{(i)} \leq 1$, $\sum_i W_t^{(i)} = 1$. δ is the Dirac function.

SMC is based on Importance Sampling idea

Intuition: If the consecutive distributions π_t and π_{t-1} are "close" to each other for all t, then samples from π_{t-1} should be used to assist with sampling from π_t.

$$
\begin{aligned}
\int g(x)\pi_t(x)dx &= \int g(x)\gamma_t(x)dx \Big/ \int \gamma_t(x)dx \\
&= \int g(x)\frac{\gamma_t(x)}{\gamma_{t-1}(x)}\pi_{t-1}(x)dx \Big/ \int \frac{\gamma_t(x)}{\gamma_{t-1}(x)}\pi_{t-1}(x)dx \\
&= \int g(x)w_t(x)\pi_{t-1}(x)dx \Big/ \int w_t(x)\pi_{t-1}(x)dx \\
&= \int g(x)W_t(x)\pi_{t-1}(x)dx
\end{aligned}
$$

where $w_t(x) = \frac{\gamma_t(x)}{\gamma_{t-1}(x)}$ is unnormalized weight, and $W_t(x) = w_t(x) \Big/ \int w_t(z)\pi_{t-1}(z)dz$ is normalized weight.

Sequential Monte Carlo: Weighted Particles

Suppose at time $t-1$, we have a set of particles $\{X_{t-1}^{(i)}\}_{i=1}^{M}$ approximating $\pi_{t-1}(x)$.

As

$$\int g(x)\pi_t(x)dx = \int g(x)W_t(x)\pi_{t-1}(x)dx,$$

the weighted particles $\{W_t^{(i)}, X_{t-1}^{(i)}\}_{i=1}^{M}$ approximate $\pi_t(x)$ where

$$W_t^{(i)} = \frac{w_t\left(X_{t-1}^{(i)}\right)}{\sum_{i=1}^{M} w_t\left(X_{t-1}^{(i)}\right)}, \quad i = 1, ..., M$$

That is,

$$\widehat{\pi}_t(x) = \sum_{i=1}^{M} W_t^{(i)} \delta_{X_{t-1}^{(i)}}(x)$$

Sequential Monte Carlo: Resampling

The weighted particles $\{W_t^{(i)}, X_{t-1}^{(i)}\}_{i=1}^M$ approximate $\pi_t(x)$.

We now resample the particles $X_{t-1}^{(i)}$ with respect to their weights $W_t^{(i)}$ to get equally weighted particles:

- Each particle $X_{t-1}^{(i)}$ is copied $N_t^{(i)}$ times, with $\mathbb{E}(N_t^{(i)}) = MW_t^{(i)}$, $\sum_i N_t^{(i)} = M$.
- This is resampling *with* replacement.
- Many resampling methods can be used: multinomial resampling, stratified resampling, residual resampling.

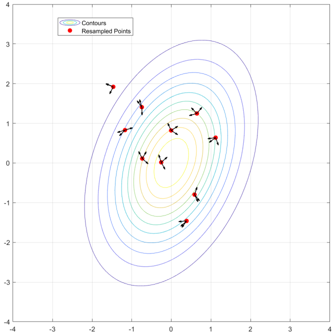
After resampling $\{W_t^{(i)}, X_{t-1}^{(i)}\}_{i=1}^M$, we obtain equally weighted particles $\{1/M, X_t^{(i)}\}_{i=1}^M$ approximating $\pi_t(x)$:

$$\widehat{\pi}_t(x) = \frac{1}{M}\sum_{i=1}^M \delta_{X_t^{(i)}}(x).$$

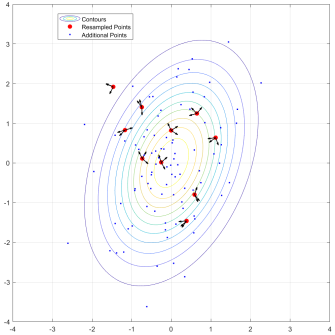
Sequential Monte Carlo: Depletion and Markov move

Resampling removes particles with low weights and replicates those with high weights. Might lead to only a few distinct particles - called depletion issue.

We need a Markov move step to "refresh" the particles, making them explore better the sample space.



Without Markov move



With Markov move

Sequential Monte Carlo: Depletion and Markov move

Markov move is often performed using the Metropolis-Hasting algorithm.

For each resampled particle $X_t^{(i)}$:

- Generate a proposal $X' \sim N(X_t^{(i)}, \Sigma_t)$, with Σ_t the sample covariance of the weighted particles $\{W_t^{(i)}, X_{t-1}^{(i)}\}_{i=1}^{M}$
- Set $X_t^{(i)} \leftarrow X'$ with the acceptance probability

$$\alpha = \min\left(1, \frac{\gamma_t(X')}{\gamma_t(X_t^{(i)})}\right),$$

otherwise keep $X_t^{(i)}$ unchanged.

$$\Sigma_t = \sum_i W_t^{(i)} X_{t-1}^{(i)} {X_{t-1}^{(i)}}^\top - \left(\sum_i W_t^{(i)} X_{t-1}^{(i)}\right)\left(\sum_i W_t^{(i)} X_{t-1}^{(i)}\right)^\top$$

Sequential Monte Carlo

In words, SMC moves a cloud of particles through the sequence of distributions $\{\pi_t(x)\}_{t=1}^{T}$ [2]. The cloud of particles at step t approximate $\pi_t(x)$.

The evolution of the particle cloud from one step to another consists of three steps: reweighting, resampling and moving.

[2] For simplicity, we assume that it is easy to sample from the initial distribution π_1

SMC for Fixed Domain: basic algorithm

Initialization: Generate M particles $X_1^{(i)} \sim \pi_1(x)$, $i = 1, ..., M$.

For $t = 2, ..., T$

- **Reweighting.** Compute normalized weights

$$w_t^{(i)} = \frac{\gamma_t(X_{t-1}^{(i)})}{\gamma_{t-1}(X_{t-1}^{(i)}))}, \quad W_t^{(i)} = w_t^{(i)} \Big/ \sum_j w_t^{(j)}$$

- **Resampling.** Resample weighted particles $\{W_t^{(i)}, X_{t-1}^{(i)}\}_{i=1}^M$ to get equally weighted particles $\{1/M, X_t^{(i)}\}_{i=1}^M$.
- **Markov move.** For $i = 1, ..., M$:
  1. Generate a proposal $X' \sim N(X_t^{(i)}, \Sigma_t)$
  2. Set $X_t^{(i)} \leftarrow X'$ with the acceptance probability

$$\alpha = \min\left(1, \frac{\gamma_t(X')}{\gamma_t(X_t^{(i)})}\right).$$

Note: for each particle, one often runs Markov step a few times, e.g., 5.

SMC for Fixed Domain: a bit on theory

Let $\{X_t^{(i)}\}_{i=1}^M$ be the set of particles approximating $\pi_t(x)$ at step t. The integral of interest $\mathbb{E}_{\pi_t}(g) = \int g(x)\pi_t(x)dx$ can be estimated by

$$\widehat{\mathbb{E}_{\pi_t}^{(M)}(g)} = \frac{1}{M}\sum_i g(X_t^{(i)})$$

It can be shown (Gilks and Berzuini, JRSSB 2001; Del Moral et al, JRSSB 2006) that

- $\widehat{\mathbb{E}_{\pi_t}^{(M)}(g)} \to \mathbb{E}_{\pi_t}(g)$ a.s. as $M \to \infty$
- $$\frac{\widehat{\mathbb{E}_{\pi_t}^{M}(g)} - \mathbb{E}_{\pi_t}(g)}{\sqrt{V_t(g)}} \Rightarrow N(0,1)$$

  $V_t(g) > 0$ has a complicated form (ignored here).

SMC for Fixed Domain: Estimating Z_t

SMC is well-known for its ability to estimate the normalizing constant Z_t.

$$\frac{Z_t}{Z_{t-1}} = \int w_t(x)\pi_{t-1}(x)dx$$

$$\widehat{\frac{Z_t}{Z_{t-1}}} = \frac{1}{M}\sum_i w_t\left(X_{t-1}^{(i)}\right)$$

If $Z_1 = 1$, then

$$Z_t = \frac{Z_2}{Z_1} \times \cdots \times \frac{Z_t}{Z_{t-1}}$$

$$\widehat{Z}_t = \prod_{s=1}^{t}\left(\frac{1}{M}\sum_i w_s\left(X_{s-1}^{(i)}\right)\right)$$

It can be shown that $\mathbb{E}(\widehat{Z}_t) = Z_t$.

Recap...

We have now covered the basic SMC algorithm.

Next, we will discuss in details some specific versions of SMC for various settings

- Likelihood Annealing SMC algorithm
- Data Annealing SMC algorithm

Likelihood Annealing SMC algorithm

Likelihood Annealing SMC is a special, but widely-used SMC algorithm: it moves a set of particles initially generated from the prior, to a set of particles approximating the posterior. It also provides an estimate of marginal likelihood - important for model comparison.

θ: model parameter

$p(\theta)$: prior distribution. Assume that we can sample from it and evaluate it.

$p(y|\theta)$: likelihood function.

Sequence of likelihood annealing distributions:

$$\pi_t(\theta) \propto p(\theta)p(y|\theta)^{a_t}$$

with annealing levels $0 = a_0 < a_1 < ... < a_T = 1$.

Note that: $\pi_0(\theta) = p(\theta)$ and $\pi_T(\theta) = p(\theta|y)$ the posterior.

Likelihood Annealing SMC algorithm

How to select T? In general we need a large T when the posterior is "weird" or high-dimensional. How to select a_t? Naive choice $a_t = t/T$ not always works well.

Guiding principle: selecting $\{a_t\}$ such that π_t is close enough to π_{t-1}. Specifically, we want the normalized importance weights

$$W_t(\theta) = w_t(\theta) \Big/ \int w_t(\theta)\pi_{t-1}(\theta)d\theta, \quad w_t(\theta) = p(y|\theta)^{a_t - a_{t-1}}$$

are of a high quality, i.e. having a small variance.

Let $\{\theta_{t-1}^{(i)}\}_{i=1}^M$ be M samples from $\pi_{t-1}(\theta)$. The quality of the weighted particles $\{W_t(\theta_{t-1}^{(i)}), \theta_{t-1}^{(i)}\}_{i=1}^M$, as an approximation of $\pi_t(\theta)$, is measured by Effective Sample Size

$$\mathrm{ESS}(a_t) = \frac{1}{\sum_{i=1}^M \left(W_t(\theta_{t-1}^{(i)})\right)^2}$$

- $0 < \mathrm{ESS} \leq M$. Higher ESS the better.

Likelihood Annealing SMC algorithm

Adaptive method for selecting the annealing levels $\{a_t\}$:

- Select an initial large T, e.g. $T = 10,000$.
- Let $\tilde{a}_i = (i/T)^3$, $i = 0, 1, ..., T$. Why cubic?
- Let $a_0 = \tilde{a}_0 = 0$.
- Select $a_1 = \min\{\tilde{a}_i : \tilde{a}_i > a_0 \text{ and } \mathrm{ESS}(\tilde{a}_i) < cM\}$, for some 0¡c¡1. That is, the next annealing level a_1 is the smallest $\tilde{a}_i > a_0$ such that ESS computed at $\tilde{a}_i$ is less than the threshold cM.
- c is a subjective choice. Common chocie $c = 0.8$.
- Select $a_2 = \min\{\tilde{a}_i : \tilde{a}_i > a_1 \text{ and } \mathrm{ESS}(\tilde{a}_i) < cM\}$. That is, the next annealing level a_2 is the smallest $\tilde{a}_i > a_1$ such that ESS computed at $\tilde{a}_i$ is less than the threshold cM.
- etc.

Likelihood Annealing SMC algorithm

Initialization: Sample $\theta_j \sim p(\theta)$ for $j = 1, ..., M$, $t \leftarrow 0, a_t \leftarrow 0$, log_llh $\leftarrow 0$.

While $a_t < 1$:

- $t \leftarrow t + 1$
- **Select** a_t **and reweighting:**
  - For each i such that $\tilde{a}_i > a_{t-1}$, compute

$$w_j = p(y|\theta_j)^{\tilde{a}_i - a_{t-1}}, \;\; W_j \propto w_j, \;\; \mathrm{ESS}_i = 1/\sum_{j=1}^{M} W_j^2$$

  - Increase i until $\mathrm{ESS}_i < cM$ for some $0 < c < 1$. Set $a_t \leftarrow \tilde{a}_i$.
- **Resampling** $\{W_j, \theta_j\}_{j=1}^M$ to obtain the new equally-weighted particles $\{\theta_j\}_{j=1}^M$.
- **Markov move**: For each $j = 1, ..., M$,
  1. Generate a proposal $\theta_j' \sim \mathcal{N}(\theta_j, \Sigma_t)$
  2. Set $\theta_j = \theta_j'$ with the probability $\min\left(1, \frac{p(y|\theta_j')^{a_t} p(\theta_j')}{p(y|\theta_j)^{a_t} p(\theta_j)}\right)$
- **Update log of marginal likelihood**: log_llh := log_llh + $\log\left(\sum_{j=1}^M w_j\right)$.

Example: Likelihood Annealing SMC for GARCH model

Let $\{y_t, t = 1, ...,\}$ be financial returns. Wish to model $\sigma_t^2 = \mathbb{V}(y_t | y_{1:t-1})$.

The GARCH model:

$$y_t = \sigma_t \epsilon_t, \quad \epsilon_t \sim N(0, 1)$$
$$\sigma_t^2 = w + \alpha \sigma_{t-1}^2 + \beta y_{t-1}^2, \quad t = 2, 3, ...$$

where $w > 0$, $\alpha > 0, \beta > 0$ and $\alpha + \beta < 1$. Let's parameterize $\alpha = \psi_1(1 - \psi_2)$ and $\beta = \psi_1 \psi_2$ with $0 < \psi_1, \psi_2 < 1$.

Use an inverse Gamma prior IG$(1, 1)$ for w and an uniform prior $U(0, 1)$ for ψ_1 and ψ_2.

Hence, the working model

$$y_t = \sigma_t \epsilon_t, \quad \epsilon_t \sim N(0, 1)$$
$$\sigma_t^2 = w + \psi_1(1 - \psi_2)\sigma_{t-1}^2 + \psi_1 \psi_2 y_{t-1}^2, \quad t = 2, 3, ...$$

The model parameters $\theta = (w, \psi_1, \psi_2)$.

Example: Likelihood Annealing SMC for GARCH model

Data: SP500 weekly indexes from Jan 1988 to Nov 2018, leading to $N = 1612$ returns.

Use the first $y_{1:n=1000}$ as training data.

We want to sample from the posterior $p(\theta|y_{1:n})$ and estimate the marginal likelihood

$$p(y_{1:n}) = \int p(\theta)p(y_{1:n}|\theta)d\theta$$

Code running!

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Example: Likelihood Annealing SMC for GARCH model

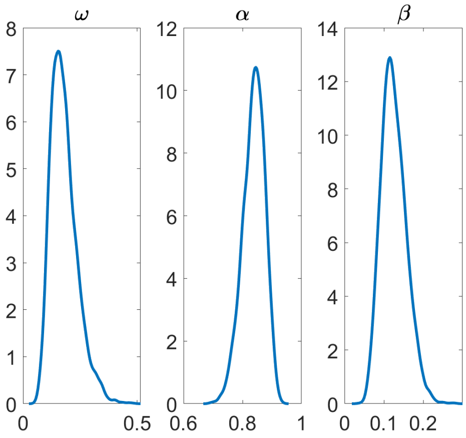


Figure: Posterior distributions of w, α and β. Estimated by Likelihood Annealing SMC algorithm.

Example: Likelihood Annealing SMC for LSTM-GARCH model

Nguyen et al. JAE 2024 propose the LSTM-GARCH model:

$$
\begin{aligned}
y_t &= \sigma_t \epsilon_t, \quad \epsilon_t \overset{iid}{\sim} t_\nu && t = 1, 2, ..., n \\
\sigma_t^2 &= \omega_t + \alpha y_{t-1}^2 + \beta \sigma_{t-1}^2 && t = 2, ..., n \\
\omega_t &= \beta_0 + \beta_1 h_t && t = 2, ..., n, h_1 = 0 \\
h_t &= g_t^o \times \text{sig}(C_t) && \text{LSTM cell output} \\
g_t^f &= \tanh(v_f x_t + w_f h_{t-1} + b_f) && \text{forget gate} \\
g_t^i &= \tanh(v_i x_t + w_i h_{t-1} + b_i) && \text{input gate} \\
x_t^d &= \tanh(v_d x_t + w_d h_{t-1} + b_d) && \text{data input} \\
g_t^o &= \text{sig}(v_o x_t + w_o h_{t-1} + b_o) && \text{output gate} \\
C_t &= g_t^f \times C_{t-1} + g_t^i \times x_t^d && \text{cell state}
\end{aligned}
$$

GARCH v.s. LSTM-GARCH model?

Model comparison and selection is an essential in statistical modelling!

- Likelihood Annealing SMC provides an efficient estimate of the marginal likelihood - key quantity for model comparison.
- For SP500 data:

	GARCH	LSTM-GARCH
log-llh estimate	-2077.95	-2056.10

Table: log of marginal likelihood estimate for GARCH and LSTM-GARCH models

Data Annealing SMC algorithm

Likelihood annealing SMC is suitable for in-sample analysis, as it approximates the posterior $p(\theta|y_{1:n})$ where $y_{1:n}$ denotes the training data.

For out-of-sample data-expanding forecasts where posterior of θ is updated once new data arrive, we use Data Annealing SMC: generating particles from

$$
\begin{aligned}
\pi_0(\theta) &= p(\theta|y_{1:n}) \\
\pi_1(\theta) &= p(\theta|y_{1:n+1}) \propto p(y_{1:n+1}|\theta)p(\theta) \propto \pi_0(\theta)p(y_{n+1}|\theta, y_{1:n}), \\
\ldots & \quad \ldots \\
\pi_t(\theta) &= p(\theta|y_{1:n+t}) \propto \pi_{t-1}(\theta)p(y_{n+t}|\theta, y_{1:n+t-1}), t = 2, 3, \ldots
\end{aligned}
$$

$$
\pi_t(\theta) \propto \underbrace{\pi_{t-1}(\theta)}_{\text{previous posterior}} \underbrace{p(y_{n+t}|\theta, y_{1:n+t-1})}_{\text{new information}}.
$$

Data Annealing SMC algorithm

Wish to approximate thre sequence of data-expanding distributions

$$\pi_t(\theta) \quad = \quad p(\theta|y_{1:n+t}) \propto \pi_{t-1}(\theta)p(y_{n+t}|\theta, y_{1:n+t-1}), t = 1, 2, ...$$

At time $t=0$, we already have a set of particles $\{W_j,\theta_j\}_{j=1}^M$ approximating $\pi_0(\theta)$ using likelihood annealing SMC.

At each time $t>0$, given a set of particles $\{W_j,\theta_j\}_{j=1}^M$ approximating $\pi_t(\theta)$:

- we can approximate the posterior predictive distribution of future data y_{n+t+1}

$$p(y_{n+t+1}|y_{1:n+t})=\int p(y_{n+t+1}|\theta,y_{1:n+t})\pi_t(\theta)d\theta$$

- when the data point y_{n+t+1} is available, we update π_{t+1} via

$$\pi_{t+1}(\theta)\propto\pi_t(\theta)p(y_{n+t+1}|\theta,y_{1:n+t}).$$

Fixed-data v.s. data-expanding forecast approaches

Fixed-data forecast approach computes the posterior predictive distribution of future data y_{n+t+1} as

$$p(y_{n+t+1}|y_{1:n}) = \int p(y_{n+t+1}|\theta, y_{1:n+t})\pi_0(\theta)d\theta.$$

Data-expanding forecast approach computes the posterior predictive distribution of future data y_{n+t+1} as

$$p(y_{n+t+1}|y_{1:n+t}) = \int p(y_{n+t+1}|\theta, y_{1:n+t})\pi_t(\theta)d\theta.$$

The latter takes into account the new information.

Data Annealing SMC algorithm

Given weighted particles $\{\theta_j, W_j\}_{j=1}^M$ approximating $\pi_0(\theta)$.
For $t = 0, 1, ...$

- **Forecasting:** Use the weighted particles $\{W_j, \theta_j\}_{j=1}^M$ for predicting y_{n+t+1}
- **Updating:** Given data y_{n+t+1}, approximate distribution π_{t+1}
  - compute weights $w_j = W_j p(y_{n+t+1}|y_{1:n+t}, \theta_j)$, $W_j \propto w_j, j = 1, ..., M$ and $\text{ESS} = \frac{1}{\sum_{j=1}^M (W_j)^2}$.
  - **if** ESS $< cM$ for some $0 < c < 1$, **then**
    - **Resampling** from $\{\theta_j, W_j\}_{j=1}^M$ to obtain the new equally-weighted particles $\{\theta_j, W_j = 1/N\}_{j=1}^M$.
    - **Markov move**: for each $j = 1, ..., M$
      - Generate a proposal $\theta_j' \sim N(\theta_j, \Sigma_t)$
      - Set $\theta_j = \theta_j'$ with the probability $\min\left(1, \frac{p(y_{1:n+t+1}|\theta_j')p(\theta_j')}{p(y_{1:n+t+1}|\theta_j)p(\theta_j)}\right)$.

Example: Data Annealing SMC for GARCH model

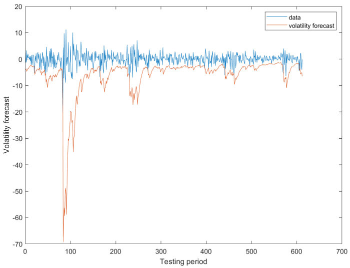
Data: SP500 weekly indexes from Jan 1988 to Nov 2018, leading to $N = 1612$ returns.

Use the first $y_{1:n=1000}$ as training data.

We want to construct one-step-ahead volatility forecast

$$\mathbb{V}(y_{n+t+1}|y_{1:n+1\theta}), \quad t = 0, 1, ...$$

Example: Data Annealing SMC for GARCH model



	PPS	Violate	Quantile Loss
Fixed-data forecast	2.2560	15	0.1120
Expanding-data forecast	2.1485	7	0.0965

Table: Forecast predictive metrics of fixed-data approach v.s. expanding-data approach.

Outline

Sequential Monte Carlo: Extended Domain

Not covered in this lecture!

Summary

We've covered the basic SMC algorithm and several specific versions of it.

We focused on the practical aspect of the method.

I hope this lecture equips data analysis practitioners with the tools and confidence to apply the method in practice.