2. Addition of angular momenta

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1 Uncertainty relations associated to the angular momentum

Exercise 2.1 Consider a system of angular momentum j in the eigenstate $|j, m\rangle$ of the operators $\hat{J}^2$ and $\hat{J}_z$.

(a) Find the average values $\left\langle \hat{\vec{J}} \right\rangle$, $\left\langle \hat{J}_x^2 \right\rangle$, $\left\langle \hat{J}_y^2 \right\rangle$ and $\left\langle \hat{J}_z^2 \right\rangle$.

(b) Find the dispersions ΔJ_x, ΔJ_y et ΔJ_z.

(c) Show that ΔJ_x et ΔJ_y are linked by a Heisenberg-like inequality.

2 Two spins 1/2 : brute force analysis

Exercise 2.2 We consider two spins 1/2 for which we define the total spin as $\hat{\vec{S}} = \hat{\vec{S}}_1 + \hat{\vec{S}}_2$. The product base of the total Hilbert space $\mathcal{E} = \mathcal{E}_{\text{spin}\,1} \otimes \mathcal{E}_{\text{spin}\,2}$ is $\{|\uparrow\uparrow\rangle, |\uparrow\downarrow\rangle, |\downarrow\uparrow\rangle, |\downarrow\downarrow\rangle\}$.

(a) Show that in this base the operators $\hat{S}^2$ and $\hat{S}_z$ are given by the following matrices :

$$\hat{S}^2 = \hbar^2 \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix} \qquad \hat{S}_z = \hbar \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{1}$$

(b) Determine the eigenvalues of $\hat{S}^2$ and the associated eigenvectors.

(c) What is the action of $\hat{S}_z$ on these states ? Conclude.

3 Two coupled spins in a magnetic field

Exercise 2.3 We consider two spins 1/2, α and β, in a magnetic field $\vec{B}$ along the z-axis, i.e. $\vec{B} = (0, 0, B)$. The magnetic moments of the two spins are coupled to each other so that the total Hamiltonian reads

$$\hat{H} = g\mu_{\mathrm{B}}\vec{B} \cdot \left(\hat{\vec{S}}_\alpha + \hat{\vec{S}}_\beta\right) + J\hat{\vec{S}}_\alpha \cdot \hat{\vec{S}}_\beta\,. \tag{2}$$

(a) Write this Hamiltonian in the product base $\{|\uparrow\uparrow\rangle, |\uparrow\downarrow\rangle, |\downarrow\uparrow\rangle, |\downarrow\downarrow\rangle\}$.

(b) Using the relation $2\hat{\vec{S}}_\alpha \cdot \hat{\vec{S}}_\beta = \hat{S}^2 - \hat{S}_\alpha^2 - \hat{S}_\beta^2$, show that

$$[\hat{H}, \hat{S}^2] = 0 \quad \text{and} \quad [\hat{H}, \hat{S}^2] = 0\,. \tag{3}$$

(c) Write the Hamiltonian in the base of states that is defined by the values of the total angular momentum. What are the eigenvalues ?