Selected Topic In Physics, 2024, 1–8
doi:
Original Article

Overview On Quantum Hall Effect

Vo Chau Duc Phuong[1,*]

[1]Condensed Matter and Statistical Physics, ICTP, Italy

*Corresponding author. vcdphuong@outlook.com

Abstract

about one line on the link between the classical and quantum hall effect. Some line on the effect of QHE on or from the topology or Berry phase of material

Key words: Overview, quantum Hall effect

Introduction

From classical Hall effect

The Hall effect happens when a conductor with a flow of current being placed in an external magnetic field $\mathbf{B}$. The Lorentz force of the magnetic field make the charge move to one side of the conductor. The equilibrium will be established when the charge density of both side create a strong enough electrical field $\mathbf{E}$. The relation between the current density and electric field can be described by Ohm's law (see Appendix A):

$$\mathbf{J} = \sigma \mathbf{E} \quad \rightleftharpoons \quad \mathbf{E} = \rho \mathbf{J}, \tag{1}$$

where the conductivity σ, in principle, is a second rank tensor, showing the affect on a direction from electric field of another direction. In the absent of the magnetic field or in the anisotropy materials, the off-diagonal element can be considering to be 0 without any further concern. In that case, the Ohm's law read:

$$\mathbf{J} = \left(\frac{n_e e^2 \tau_L}{m_e} \right) \mathbf{E}, \tag{2}$$

in which, n_e, e, τ_L, and m_e is density of electron, element charge, relaxation time (from scattering), and mass of electron, respectively.

Placing this system an external uniform magnetic field $\mathbf{B} = (0, 0, B)$ and using the convention of choosing x-axis parallel to the current direction. In this convention, Ohm's law reads:

$$\begin{pmatrix} E_x \\ E_y \end{pmatrix} = \begin{pmatrix} \rho_{xx} & \rho_{xy} \\ -\rho_{xy} & \rho_{yy} \end{pmatrix} \begin{pmatrix} J_x \\ J_y \end{pmatrix}.$$

The *resistivity* $(\rho)_{ij}$ define as the inverse of the *conductivity* $(\sigma)_{ij}$. From Drude model, we have explicitly:

$$\rho = \frac{1}{\sigma_{DC}} \begin{pmatrix} 1 & \omega_B \tau \\ -\omega_B \tau & 1 \end{pmatrix} \tag{3}$$

The off diagonal element

$$\rho_{xy} = \frac{\omega_B \tau}{\sigma_{DC}}$$

independence of the scattering time τ. Measuring the ratio between the V_y and current I_x

$$R_{xy} = \frac{V_y}{I_x} = \frac{E_y L}{J_x L} = \frac{E_y}{J_x}, \tag{4}$$

for so called *the resistance* R. Equation above correspond to,

$$R_{xy} = \frac{E_y}{J_x} = -\rho_{xy} = -\frac{\omega_B \tau}{\sigma_{DC}}. \tag{5}$$

Defining the Hall resistance as

$$R_H = -\frac{R_{xy}}{B} = \frac{\rho_{xy}}{B} = \frac{1}{ne}, \tag{6}$$

showing that the Hall resistance is a constant in the classical regime.
So far, we show that the classical resistance tensor have the form of depend on either the scattering time τ or the magnetic field B:

$$\rho = \begin{pmatrix} \frac{m}{ne^2\tau} & \frac{B}{ne} \\ -\frac{B}{ne} & \frac{m}{ne^2\tau} \end{pmatrix} \tag{7}$$

To the discover of the quantum Hall effect

Thing changes when we cooling down the system to extreme and increase the magnetic field ($\sim 1T$). As shown in (refKitzling 1980), at certain gate voltages, the conductivity go to 0 even when the density of carrier increase ref(Ando 1971) (corresponding to the strength and the concentration of the scatterer in the article). Further investigation showing that due to the governing of the quantum mechanics, there are two separate phenomenons, called *integer* and *fractional* quantum Hall effect. Hereby then, we will overview the basic concepts of these two effects and leave the more theoretically part in another section (see section 3 and section **??**).

Integer quantum Hall effect

From the precisely measurement of refKitzling 1980, which will brought him a Nobel award in 1980, as can be seen in figure (include Kitzling figure and integer qunatum hall effect figure).

The *longitude* and *transverse resistance* showing a very interesting properties in the low temperature and high magnetic field.

The *longitude resistance* ρ_{xx} go to 0 at certain gate voltages (B fixed in the figure ref figure kitzling) et vice verse (vary the field B while keep other parameters), showing that there's none resist for the current, or in another words: Superconducting.

Even more interesting is the Hall resistance, or *transverse resistance* ρ_{xy}. It jumps from plateau to plateau, significantly showing the quantization properties. In section 3, It will be shown to be obey[1]

$$\rho_{xy} = \frac{2\pi\hbar}{e^2}\frac{1}{\nu} \quad \nu \in \mathbb{Z}\backslash\{0\} \tag{8}$$

This properties shows that through measuring the Hall resistance, not only evaluate directly the Kitzling constant $2\pi\hbar/e^2$ but also a well-known *fine structure constant* α.

Fractional quantum Hall effect

Recently, we just discussed the results on the nearly pure samples. This "impurity" has another name is *disorder*. Interestingly, as we decrease the *disorder* of the sample, i.e. made it purer, better, the plateau in the data disappear, which will be covered in refInterger QHE.

As we increase the disorder even more, the integer peaks and plateaus become less prominent. Instead of that, according to refTsui 1982 data, there are other plateaus obey the same relation shows in (8), however with ν as a fractional number:

$$\nu \in \mathbb{Q}.$$

Not all fractions appear, some are proof to be more prominent than the others.

In general, while the integer quantum Hall effect can be explained by the free electron model, the interactions between the electrons have to be taken into accounted to explain the fractional quantum Hall effect.

Materials

It is worth to have some lines about the materials, in which we found and observed both the integer and fractional effects.The integer effect has been observed first time in Si MOSFET refKitzling 1980 and then later on in $GaAs - GaAsAl$ heterostructure[2]. More recently works also show these effect in Gaphene and other materials. Sometimes, the appear of quantum Hall effect in two material can be the same in spirit but differ in details.

The Integer Quantum Hall effect

Long Story Short

According to the data of experiments, we have the Hall's resistance in compare with the Drude model:

$$\rho_{xy} = \frac{2\pi\hbar}{e^2\nu} = \frac{B}{ne}. \tag{9}$$

[1] A little remind: the non-zero integer $\mathbb{Z}\backslash\{0\}$ has been used instead of the $\mathbb{N}^*$ for the skew symmetry of ρ

[2] both of these material have density of electron $n \sim 10^{11} - 10^{12}cm^{-2}$

Giving us the density of charge need to reach the resistivity corresponding with number ν:

$$n = \frac{eB}{2\pi\hbar}\nu = \frac{B}{\Phi_0}\nu, \tag{10}$$

where

$$\Phi_0 = \frac{2\pi\hbar}{e} = \frac{h}{e}. \tag{11}$$

Considering when the system completely fill the ν-th Landau level, there's no where for the electron to run inside that Landau's state. The electrons stay in the same place, cause no scattering, which also mean $\tau \to \infty \Rightarrow \rho_{xx} \to 0$ as the data show[3].

A particle inside the electromagnetic field according to Ehrenfest's theorem have the velocity

$$\begin{aligned} i\hbar\dot{\mathbf{x}} &= \left[\mathbf{x}, \frac{(\mathbf{p}-e\mathbf{A})^2}{2m}\right] = \left[\mathbf{x}, \frac{\mathbf{p}^2 - 2eA\mathbf{p} + (eA)^2}{2m}\right] \\ &= \frac{1}{2m}[\mathbf{x}, \mathbf{p}.\mathbf{p}] - \frac{1}{2m}2e\mathbf{A}[\mathbf{x}, \mathbf{p}] \\ \Rightarrow m\dot{\mathbf{x}} &= \mathbf{p} - e\mathbf{A} \end{aligned} \tag{12}$$

The total current of the system is:

$$I = -e\langle\dot{\mathbf{x}}\rangle = -\frac{e}{m}\sum_{\text{filled}}\langle\psi|\mathbf{p} - e\mathbf{A}|\psi\rangle \tag{13}$$

$$= -\frac{e}{m}\sum_{n}\sum_{k_y}\langle\psi_{n,k}(x)|\mathbf{p} - e\mathbf{A}|\psi_{n,k}(x)\rangle \tag{14}$$

If we chose the Landau gauge convention: $\mathbf{A} = (0, xB, 0)$, and place electric field $\mathbf{E}$ along the O_x, along the x direction, we have:

$$I_x = -\frac{e}{m}\sum_{k_y}\sum_{n=1}^{\nu}\langle\psi_{n,k}(x)|\frac{\hbar}{i}\partial_x|\psi_{n,k}(x)\rangle\,, \tag{15}$$

this one vanishes due to the orthogonal properties of the Hermitian function. Meanwhile, along the other direction:

$$\begin{aligned} I_y &= -\frac{e}{m}\sum_{n}^{\nu}\sum_{k_y}\langle\psi_{n,k_y}(x)|\frac{\hbar}{i}\partial_y - eBx|\psi_{n,k_y}(x)\rangle \\ &= -\frac{e}{m}\sum_{n,k_y}\langle\psi_{n,k_y}|\frac{\hbar}{i}\partial_y|\psi_{n,k_y}\rangle - eB\langle x\rangle\,. \end{aligned} \tag{16}$$

According to (45) for the harmonic oscillator, we have:

$$\begin{aligned} \langle x - x_0\rangle &= \left\langle x - \left(\frac{\hbar k_y}{eB} - \frac{mE}{eB^2}\right)\right\rangle = 0 \\ \Rightarrow \langle x\rangle &= \left\langle\frac{\hbar\partial_y}{ieB} - \frac{mE}{eB^2}\right\rangle. \end{aligned} \tag{17}$$

Substituting (17) into (16):

$$I_y = -\frac{e}{m}\frac{E}{B}\sum_{n,k_y}1 = -\sum_{k_y}e\nu\frac{E}{B}, \tag{18}$$

the sum over degeneracy k_y give us (42):

$$J_y = \frac{I_y}{L_xL_y} = \frac{eB}{2\pi\hbar}e\nu\frac{E}{B} = \frac{e^2\nu E}{2\pi\hbar} \tag{19}$$

[3] These condition happen due to our assumption that the thermal energy k_BT very small in compare with the gap caused by Landau levels.

Combining those results into matrix to get

$$\mathbf{E} = \begin{pmatrix} E \\ 0 \end{pmatrix}; \quad \mathbf{J} = \begin{pmatrix} 0 \\ \frac{e^2\nu}{2\pi\hbar}E. \end{pmatrix} \tag{20}$$

Compare with Ohm's law (1) to see that:

$$\sigma_{xx} = 0; \quad \sigma_{xy} = \frac{e^2\nu}{2\pi\hbar} \Rightarrow \rho_{xx} = 0; \quad \rho_{xy} = -\frac{2\pi\hbar}{e^2\nu}$$

Agree well with the experiments results. But there are several interesting properties that can't be captured in the discussion above.

Edge Modes

The edge mode can play an important role in the system. But before that, something should be defined: if the particles can only move *one way* along the line, then it is called *Chiral*. In one sample, if two sides restricted the particle in two opposite directions then it is opposite chirality on two sides.

Considering the system in with the edge characterized by

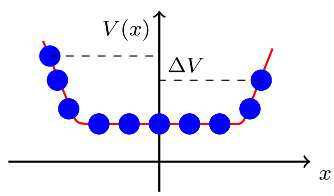


Fig. 1: Difference in Fermi energy at both sides Edge's potential $V(x)$

the potential $V(x)$, shown in Fig. 1. The group velocity can be obtained as

$$v_{k_y} = \frac{1}{\hbar}\frac{\partial E_{k_y}}{\partial k_y}$$

Using results from eq. (48) to get

$$v_{k_y} = -\frac{1}{eB}\frac{\partial V}{\partial x} \tag{21}$$

From this, we can calculate the current of system:

$$\begin{aligned} I_y =& -\frac{e}{2\pi}\int v_{k_y} dk_y = \frac{e}{2\pi B}\int \frac{\partial V}{\partial x}dx\frac{eB}{\hbar} \\ =& \frac{e^2}{2\pi\hbar}\Delta V = \frac{e^2}{2\pi\hbar}U_x \end{aligned} \tag{22}$$

The ratio give us the Hall conductivity of exact one Landau level

$$\sigma_{yx} = \frac{I_y}{U_x} = \frac{e^2}{2\pi\hbar} \tag{23}$$

As we can see that the overall I_y will be depended only from the difference between two side of the system's potential. No matter what kind of function of $V(x)$ is, if the Fermi level are higher than the middle of the function and $V(x)$ is smooth in that range, then the macro phenomenon will affected by the edge of the system, called *edge mode*.

Role of Disorder

The other aspect that weren't covered is the role of disorder in the existence of the plateaus. We will showing that, the impurity play a key roles in the existence of the continuous plateaus.

Quantizating the center of mass (X, Y) from (26)

$$\begin{aligned} x_0 \to X(t) &= x + \frac{1}{\omega_B}\dot{y} = x + \frac{\pi_y}{m\omega_B} \\ y_0 \to Y(t) &= y - \frac{\pi_x}{m\omega_B}, \end{aligned} \tag{24}$$

which some properties [4,5], which sometime can be called as the Heisenberg uncertainty of the oscillation. Putting it in the equation of motions:

$$\begin{aligned} i\hbar\frac{\mathrm{d}X}{\mathrm{d}t} =& [X, H + V] = [X, H] + [X, V] = [X, V] \\ =& [X, Y]\frac{\partial V}{\partial Y} = il_B^2\frac{\partial V}{\partial Y} \\ i\hbar\frac{\partial Y}{\partial t} =& -il_B^2\frac{\partial V}{\partial X}. \end{aligned}$$

These equations showing that the center of the circle will drift along the equipotentials.

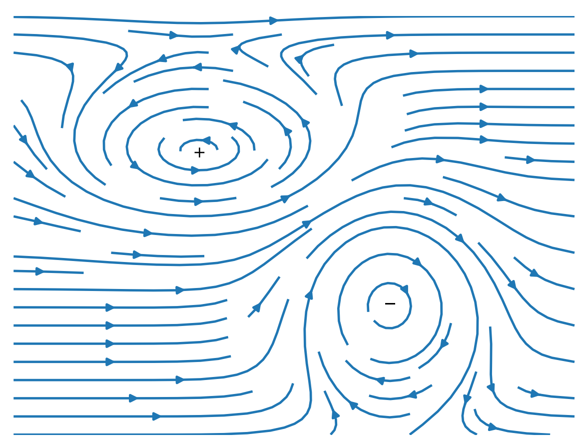


Fig. 2: The movement of particles around the minima and maxima

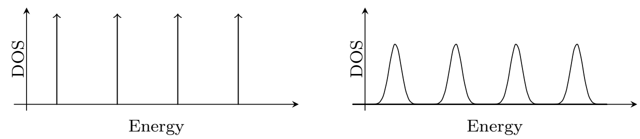
The sample that we measure on, will never be a perfect one. It will have some dust on the surface or have some improper cell that break the total symmetry. Overall, the *impurity* can be characterized as random peaks of the potential on the surface. Using the set of equations Eq. (25), we plot in Fig. 2 the flow of the center of the electron's circle when it flow through the sample.

The particle that can move freely through the sample can be illustrated as the line that can move from the left to the right continuously. The one that being trapped around the maxima (+) and minima (−).

These trapped particles will have the results in the spread out of the density of states (DOS), results in both edges of the peak about the Landau energy level as shown in Fig. . Not only that,

[4] $[X, Y] = \frac{i\hbar}{m\omega_B} = \frac{i\hbar}{eB} = il_B^2$

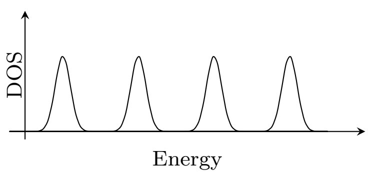
[5] $[X, H] = \left[x + \frac{\pi_y}{m\omega_B}, H\right] = 0 \leftarrow [\pi_x, \pi_y] = iq\hbar B$



(a) Without disorder (b) with disorder

Fig. 3: The energy spectrum for without disorder (a) and with it (b).

the disorder also play an important role in the existence of the plateau in the spectrum. As shown in (42), the number of particle in a Landau level depend proportional to the magnetic field flux $\Phi = BL_xL_y$. If in the initial, we fill the system up to n-th Landau level, and decrease the magnetic field B. Now, each Landau level can accommodate less particles,



Fractional Hall Effect

Conclusion

Some Conclusions here.

Particle in Magnetic field

Cyclotron Oscillation

Considering an electron moves in a magnetic field perpendicular to the moving plane, the equation of motion for the particle charge -e and mass m is

$$m\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = -e\mathbf{v}\times\mathbf{B} \tag{25}$$

Solution for this equation is the oscillation of the electron around a circle:

$$\mathbf{r}(t) : \begin{cases} x(t) & = x_0 - R\sin(\omega_B t + \phi) \\ y(t) & = y_0 + R\cos(\omega_B t + \phi) \end{cases}, \tag{26}$$

All these parameters are arbitrary, except the frequency ω_B, which is called cyclotron frequency:

$$\omega_B = \frac{eB}{m}, \tag{27}$$

which clearly depend proportional to magnetic field B.

Drude Model

Putting the system into the external electric field $\mathbf{E}$ along Ox and keep the magnetic field $\mathbf{B}$ perpendicular the Oxy plane, the equation of motion become:

$$m\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = -e\mathbf{E} - e\mathbf{v}\times\mathbf{B} - \frac{m\mathbf{v}}{\tau}, \tag{28}$$

in which the last term corresponding to the scattering of the electron characterized by the parameter τ, called scattering time. If the system archive the equilibrium, then

$$\mathbf{E} = \mathbf{v}\times\mathbf{B} - \frac{m\mathbf{v}}{\tau e}. \tag{29}$$

Substituting the current density J into (29) through

$$\mathbf{J} = -ne\mathbf{v}, \tag{30}$$

to have Ohm's law in the matrix form:

$$\begin{pmatrix} 1 & \omega_B\tau \\ -\omega_B\tau & 1 \end{pmatrix}\begin{pmatrix} J_x \\ J_y \end{pmatrix} = \frac{e^2n\tau}{m}\begin{pmatrix} E_x \\ E_y \end{pmatrix}, \tag{31}$$

or in the inverse form

$$\mathbf{J} = \sigma\mathbf{E}. \tag{32}$$

In which, the σ was called the *conductivity tensor*. Without the present of the magnetic field $\mathbf{B}$, (32) reduce to the DC conductivity form:

$$\mathbf{J} = \sigma_{DC}\mathbf{E} \quad \text{with} \quad \sigma_{DC} = \frac{ne^2\tau}{m},$$

For general case, we derive explicitly from (31):

$$\sigma = \frac{\sigma_{DC}}{1+\omega_B^2\tau^2}\begin{pmatrix} 1 & -\omega_B\tau \\ \omega_B\tau & 1 \end{pmatrix} \tag{33}$$

Landau Levels

Considering 2-D system confined within Oxy plane with the magnetic field being placed perpendicular to the system.

Landau Gauge

Fixing the gauge by choosing the Landau gauge ($\mathbf{A} = (0, Bx, 0), \phi = 0$, which is vector and scalar potential of the electromagnetic field, respectively), the Hamiltonian of the system reads:

$$\begin{aligned} H &= \frac{(\mathbf{p}+e\mathbf{A})^2}{2m} \\ &= \frac{p_x^2}{2m} + \frac{(p_y+eBx)^2}{2m} + \frac{p_z^2}{2m}. \end{aligned} \tag{34}$$

In this convention, there is no y, z in this Hamiltonian. Therefore, we have an Ansatz:

$$\psi(x,y,z) = e^{i(k_yy+k_zz)}\psi_x(x). \tag{35}$$

Substituting (35) and (34) into Schrödinger equation to have[6]:

$$\left(\frac{p_x^2}{2m} + \frac{(\hbar k_y+eBx)^2}{2m} + \frac{\hbar^2k_z^2}{2m}\right)\psi_x(x) = E_{k_y,k_z}\psi_x(x). \tag{36}$$

[6] In this case , we note that $[H, y] = [H, z] = 0$, leave the p_y as a constant

Excluding the free or extreme confined part of k_z depend on the system, (36) leaves us with:

$$\left(\frac{p_x^2}{2m}+\frac{m\omega_B^2}{2}\left(x+\frac{\hbar k_y}{eB}\right)^2\right)\psi_x(x)=E_{k_y}\psi_x(x). \tag{37}$$

Using the same convention for the harmonic oscillator, we introduce the ladder operators:

$$\hat{x}-k_y l_B^2=\sqrt{\frac{\hbar}{2m\omega_B}}\left(a^\dagger+a\right),\quad l_B^2=\frac{\hbar}{eB}, \tag{38}$$

$$\hat{p}=i\sqrt{\frac{\hbar m\omega_B}{2}}\left(a^\dagger-a\right). \tag{39}$$

Which allow us to rewrite the Hamiltonian in term of these ladder operators:

$$H=\hbar\omega_B\left(\frac{1}{2}+a^\dagger a\right)=\hbar\omega_B\left(\frac{1}{2}+n_x\right) \tag{40}$$

From here, we clearly seeing that the eigenvalue have discrete value of $\hbar\omega_B$, which in turn corresponding to the k_y.

Overall, the wave function now depend on two quantum numbers $n\in\mathbb{N}$ and $k_y\in\mathbb{R}$:

$$\psi_{n,k_y}\sim e^{ik_y y}H_n(x+kl_B^2)e^{-\frac{(x+kl_B)^2}{l_B^2}}$$

Considering the particle in the box with the length of Oy is L_y, the boundary condition give us:

$$k_y=\frac{N2\pi}{L_y},\quad N\in\mathbb{Z}, \tag{41}$$

give us a very large number of degeneracy states. Applying the boundary condition x direction, the particle have to stay inside the box:

$$0\le x\le L_x$$

At each degeneracy k_y, the particle will oscillate about the point $x=-k_y l_B^2$. Therefore, these is limit for k:

$$-\frac{L_x}{l_B^2}\le k_y\le 0.$$

Give us the total number of state inside the box for one filled Landau level:

$$\mathrm{N}=\frac{L_y}{2\pi}\int_{-\frac{L_x}{l_B^2}}^{0}dk=\frac{L_yL_x}{2\pi l_B^2}=\frac{L_xL_yeB}{2\pi\hbar}. \tag{42}$$

Since there is no difference in the density of states for the other level n, the density of states spectrum can be seen as multiple poles, called 'Dirac comb' (not Dirac cone, sound a like and have a little connection between both of it). A little remind about the density of state for an 2-D free electron gas is it a reverse Heavyside function at $E=E_F$, when applying the magnetic field, it concentrate into the pole with insane number in a small width that can be seen as Dirac delta function.

But in this case, clearly seeing that not only the symmetry in translation along x axis have been broke: $[x,H]\neq 0$, also with the rotation symmetry about the z axis also has been broke:

$$[\hat{H},\hat{L}_z]=[\hat{H},x\partial_y-y\partial_x]=-[H,y\partial_x]\neq 0 \tag{43}$$

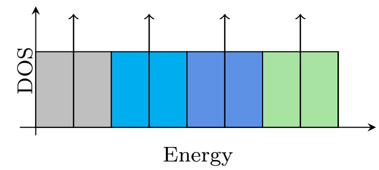


Fig. 4: Dirac comb with constant DOS of 2D Fermi gas in background

Including The Electric Field

If we including the electric field into the system along Ox, we have the Hamiltonian:

$$H=\frac{(p_x^2+(p_y-eBx)^2)}{2m}+eEx \tag{44}$$

With a little manipulate, we have:

$$\begin{aligned}H=&\frac{p_x^2}{2m}+\frac{\hbar^2k_y^2-2\hbar k_yeBx+(eBx)^2+2meEx}{2m}\\=&\frac{p_x^2}{2m}+(eB)^2\frac{x^2-2x(\hbar k_y/eB-mE/e^2B)}{2m}+\frac{\hbar^2k_y^2}{2m}\\=&\frac{p_x^2}{2m}+\frac{e^2B^2}{2m}\left(x-\frac{\hbar k_y}{eB}+\frac{mE}{eB^2}\right)^2\\&-eE\left(k_yl_B^2+\frac{eE}{m\omega_B^2}\right)+\frac{m}{2}\frac{E^2}{B^2}\end{aligned} \tag{45}$$

$$E_{n,k}=\hbar\omega_B\left(n+\frac{1}{2}\right)-eE\left(kl_B^2+\frac{eE}{m\omega_B^2}\right)+\frac{m}{2}\frac{E}{B} \tag{46}$$

The eigenstates are the same but with an extra shift $-\frac{mE}{eB^2}$.

An Useful Note

A worthy more generalized note need to talk about for later use. Considering the Hamiltonian have the form:

$$H=\frac{(p_x^2+(p_y+eBx)^2)}{2m}+V(x)$$

Expanding the potential $V(x)$ by Taylos's series around some point x_0, neglect the quadratic term to have:

$$H=\frac{(p_x^2+(p_y+eBx)^2)}{2m}+V_0(x_0)+\left.\frac{\partial V(x)}{\partial x}\right|_{x=x_0}(x-x_0).$$

Choosing the ground potential $V_0(x_0)$ and the origin of the reference frame to have

$$H=\frac{(p_x^2+(p_y-eBx)^2)}{2m}+\left.\frac{\partial V(x)}{\partial x}\right|_{x=x_0}x. \tag{47}$$

Doing the sane process as (45), we arrive with the eigenenergies:

$$E_{n,k,x}=\hbar\omega_B\left(n+\frac{1}{2}\right)-e\nabla_xV\left(kl_B^2+\frac{e\nabla_xV}{m\omega_B^2}\right)+\frac{m}{2}\frac{\nabla_xV}{B} \tag{48}$$

The k_y and x have the relation for the shift of wave function: $x=-kl_B^2+\frac{m\nabla_xV}{eB^2}$.

Symmetric Gauge

This lead us to an adaptation, choosing a symmetric gauge:

$$\mathbf{A} = -\frac{1}{2}\mathbf{r} \times \mathbf{B} = -\frac{yB}{2}\vec{i} + \frac{xB}{2}\vec{j}. \tag{49}$$

This convention broke translation symmetry in both x and y directions. On another hand, it reserve the rotation symmetry about the z-axis:

$$[H, x\partial_y - y\partial_x] = 0$$

in which,

$$H = \frac{(p_x - yeB/2)^2}{2m} + \frac{(p_y + xeB/2)^2}{2m}. \tag{50}$$

Due to this convention, the canonical momentum vector has to be defined:

$$\boldsymbol{\pi} = \mathbf{p} - q\mathbf{A} = \mathbf{p} + e\mathbf{A}, \tag{51}$$

$$[\pi_x, \pi_y] = iq\hbar\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) = iq\hbar B \tag{52}$$

$$a = \frac{1}{\sqrt{2\hbar B}}(\pi_x + i\pi_y), \tag{53}$$

In contract with the canonical momentum $\mathbf{p}$, the mechanical momentum $\boldsymbol{\pi}$ is the one represent the kinetic of the particle in the laboratory frame, therefore it is called "Kinetic momentum" sometime. Which give

$$[a, a^\dagger] = 1,$$

$$H = \frac{1}{2}\boldsymbol{\pi}\cdot\boldsymbol{\pi} = \hbar\omega_B(a^\dagger a + \frac{1}{2}).$$

Defining another operator:

$$\tilde{\boldsymbol{\pi}} = \mathbf{p} - e\mathbf{A}, \tag{54}$$

this choice is **not** *gauge invariant* (for more information, see appendix E). The commutator of the new operators obeys:

$$[\tilde{\boldsymbol{\pi}}_x, \tilde{\boldsymbol{\pi}}_y] = ie\hbar B. \tag{55}$$

The commutator this new operators and the mechanical momentum:

$$[\pi_x, \tilde{\pi}_x] = 2ie\hbar\frac{\partial A_x}{\partial x}, \quad [\pi_y, \tilde{\pi}_y] = 2ie\hbar\frac{\partial A_y}{\partial y} \tag{56}$$

$$[\pi_x, \tilde{\pi}_y] = [\pi_y, \tilde{\pi}_x] = ie\hbar\left(\frac{\partial A_x}{\partial y} + \frac{\partial A_y}{\partial x}\right) \tag{57}$$

By the choice of symmetric gauge 49, all the commutators above vanish. Allow define new ladder operators:

$$b = \frac{1}{\sqrt{2e\hbar B}}(\tilde{\pi}_x + i\tilde{\pi}_y); \quad b^\dagger = \frac{1}{\sqrt{2e\hbar B}}(\tilde{\pi}_x - i\tilde{\pi}_y) \tag{58}$$

These two operators obey

$$[b, b^\dagger] = 1$$

From this, we can construct the wave function as the simple harmonic for two set of operator:

$$a\,|0,0\rangle = b\,|0,0\rangle = 0$$

$$|n,m\rangle \equiv \frac{(a^\dagger)^n (b^\dagger)^m}{\sqrt{m!n!}}\,|0,0\rangle$$

Berry Phase and Some Relations

...

Abelian Berry Phase and Berry Connection

Considering a generalize Hamiltonian, expressed in term of degree of freedoms x^α, in example: position or spins, and parameters λ^i, in example, the spring constant in the harmonic oscillator problem.

$$H = H(x^\alpha, \lambda^i).$$

Now, gently varying the parameter in respect to the time with an assuming that in the initial time, the system is in the eigenstate $|\psi\rangle$: $|\psi\rangle \to |\psi(\lambda(t))\rangle$. The parameter will be vary so gentle that the system can't be excited to another eigenstate, we don't talk about degeneracy, at least at this point.

Finally, the parameter have to return to the same value as if when it start to move, making a closed trajectory in the parameter space. Since the starting point is the eigenstate, the final state have to be proportional to it initial state: $|\psi\rangle \to e^{i\gamma}\,|\psi\rangle$. Neglect the evolution term in respect with the time $exp(-iEt/\hbar)$, there is still another phase that contribution to the global phase, called *Berry phase*.

Calculate the Berry Phase

At time t, the Hamiltonian have solution $|\psi(t)\rangle$. Assuming that they can be expand in the basis $\{|n(t)\rangle\}$ that satisfied the time-independence Schrödinger equation:

$$H(t)\,|n(t)\rangle = E_n(t)\,|n(t)\rangle \tag{59}$$

From Schrödinger equation,

$$i\hbar\partial_t\,|\psi(t)\rangle = H(t)\,|\psi(t)\rangle$$

$$i\hbar\sum_n \partial_t c_n(t)\,|n(t)\rangle = \sum_n H(t)c_n(t)\,|n(t)\rangle$$

$$i\hbar\sum_n \dot{c}_n(t)\,|n(t)\rangle + c_n(t)|\dot{n(t)}\rangle = \sum_n c_n(t)E_n(t)\,|n(t)\rangle$$

Multiplying both sides with $\langle m(t)|$ and using the eigenstate orthogonal properties

$$i\hbar\dot{c}_m(t) + i\hbar\sum_n c_n(t)\left\langle m(t)\middle|\dot{n(t)}\right\rangle = c_m(t)E_m(t)$$

$$\dot{c}_m(t) + c_m(t)\left(\left\langle m(t)\middle|\dot{m(t)}\right\rangle + i\frac{E_m(t)}{\hbar}\right) = \sum_{n\neq m} c_n(t)\frac{\langle m(t)|\,\dot{H}\,|n(t)\rangle}{E_m(t) - E_n(t)}$$

Using the assuming that $\frac{\dot{H}}{E_m(t)-E_n(t)}$ too small and $E_m(t) - E_n(t) \neq 0$ with $m \neq n$ to neglect this term, then

$$\dot{c}_m(t) = -c_m(t)\left(i\frac{E_m(t)}{\hbar} + \left\langle m(t)\middle|\dot{m(t)}\right\rangle\right) \tag{60}$$

$$\Rightarrow c_m(t) = c_m(0)e^{-\int_0^t (i\frac{E_m(t')}{\hbar} + \langle m(t')|\dot{m(t')}\rangle)dt'}. \tag{61}$$

In here, we define the *Berry connection* (or Berry potential):

$$\mathcal{A}_i(\lambda) \equiv i\,\langle n(t)|\,\frac{\partial}{\partial\lambda^i}\,|n(t)\rangle \tag{62}$$

Substituting into (61) to get

$$c_m(t) = c_m(0)e^{-\int_0^t \frac{iE_m(t')}{\hbar}}e^{i\gamma}, \tag{63}$$

in which

$$\gamma = \oint_C \mathcal{A}_i d\lambda^i$$

Instead of depend on the varying time, *geometry phase* depend on the path of the parameter in the parameter space. The term

$$e^{i \oint_C \mathcal{A}_i d\lambda^i}$$

will be called *Berry phase.*

In the same spirit with the gauge transformation in electromagnetism, the physical information can be extract by calculating the *Berry curvature* of the connection,

$$\mathcal{F}_{ij}(\lambda) \equiv \frac{\partial \mathcal{A}_i}{\partial \lambda^j} - \frac{\partial \mathcal{A}_j}{\partial \lambda^i}. \tag{64}$$

Using Stokes's theorem for the *Berry phase*

$$e^{i\gamma} = exp\Big(- i \oint_C \mathcal{A}_i(\lambda) d\lambda^i \Big) = exp\Big(- i \int \mathcal{F}_{ij} dS^{ij} \Big)$$

With the S_{ij} is the surface which being closed by contour C[7].

Chern Number

Aharonov-Bohm effect

In two-split experiment for electron, we place a solenoid in between two split and the screen (include graphic on Aharonov-Bohm effect). This solenoid will be put in some kind of "concrete", which will eliminate all magnetic field B outside of it's wall. According to the Stoke's theorem for a circle with radius r bigger than the wall:

$$\Phi = \int \mathbf{B} d\mathbf{S} = \int_{in} (\nabla \times \mathbf{A}) dS = \oint_C \mathbf{A} dl \tag{65}$$

According to the gauge invariance, we can chose $\mathbf{A}$ freely. By choosing $\mathbf{A}$ that have angular symmetry, it will give us a specific magnetic flux that we will need later:

$$\Phi = A_\phi 2\pi r \Rightarrow A_\phi = \frac{\Phi}{2\pi r} \tag{66}$$

Long story short

The solenoid will stay somewhere between two split (not right in the middle, it will be too perfect) and near the split's wall so that the particle can not pass by the solenoid, it still effected by the potential vector $\mathbf{A}$ from that solenoid, causing phase difference for electron depend on whether they pass upper or below split. The results will be the shift in the spectrum on the screen. As we chance the $\mathbf{B}$ in the solenoid, $\mathbf{A}$ will also chance and causing the spectrum to chance (can be spread out, narrow in or moving left and right).

Long story long

Put a particle into a box, the box will stay near some magnetic field $\mathbf{B}$ shielded. The box has to be small enough that the vector potential $\mathbf{A}$ is constant within the box. Hamiltonian of the particle will be:

$$H = \frac{1}{2m}(\mathbf{p} + e\mathbf{A}(\mathbf{X}))^2 + V(x - X), \tag{67}$$

in which $\mathbf{X}$ is the center of the box. Choosing the gauge at the initial position so that $\mathbf{A}(\mathbf{X}_0) = 0$. At the initial, the particle will stay in the ground state:

$$\psi_0(x - X_0).$$

Slowly move the box to another position X, the wave function will be:

$$\psi(x - X) = exp\Big(- \frac{ie}{\hbar} \int_{\mathbf{X}_0}^{\mathbf{X}} \mathbf{A}(\mathbf{x}) d\mathbf{x} \Big) \psi(\mathbf{x} - \mathbf{X}_0)$$

If the box move along a closed path C, compare two wave function:

$$\psi(x - X_0) \to e^{i\gamma} \psi(\mathbf{x} - \mathbf{X}_0) = exp\Big(- \frac{ie}{\hbar} \oint_C \mathbf{A}(\mathbf{x}) d\mathbf{x} \Big) \psi(\mathbf{x} - \mathbf{X}_0)$$

Easily seeing from the generalize definition of *Berry phase* the relation for magnetic field:

$$\boldsymbol{\mathcal{A}}(\mathbf{X}) = \frac{e}{\hbar} \mathbf{A}(\mathbf{x} = \mathbf{X})$$

Generally, if a particle with charge q move around a region with magnetic flux Φ, it will pick up a Aharonov-Bohm phase:

$$e^{i \frac{q}{\hbar} \Phi} \tag{68}$$

Spectral Flow

Consider an electron move only in a circle around the localized solenoid. As we mentioned, the solenoid will be characterized by the magnetic flux Φ. It will be wise to choose the symmetry gauge (66) around the z-axis, the Hamiltonian of the system will have the form:

$$H = \frac{1}{2m}(p_\phi - qA_\phi)^2 = \frac{1}{2mr^2}\Big(\frac{\hbar}{i}\partial_\phi - \frac{q\Phi_0}{2\pi} \Big)^2. \tag{69}$$

The eigenvalues and eigenfunctions will be:

$$\psi_{n,r}(\phi) = \frac{1}{\sqrt{2\pi r}} e^{in\phi}, \tag{70}$$

$$E_n(\Phi) = \frac{\hbar^2}{2mr^2}\Big(n - \frac{q\Phi}{2\pi\hbar} \Big)^2 = \frac{\hbar^2}{2mr^2}\Big(n + \frac{\Phi}{\Phi_0} \Big)^2. \tag{71}$$

In here, one again, we meet the *quantum flux* $\Phi_0 = \frac{h}{q} = \frac{2\pi\hbar}{e}$. From (71), we seeing that in the case $\forall \frac{\Phi}{\Phi_0} \in$ Z, the energy spectrum will be the same. Otherwise, as it should be since there is nothing constrains the external parameter ϕ to be integer times of the quantum flux, the energy spectrum gets shifted.

$$\psi_n(\Phi) \xrightarrow{\Phi \to \Phi + \Phi_0} \psi_n(\Phi + \Phi_0) = \psi_{n+1}(\Phi)$$

This is an interesting result, the particle never experience the magnetic field B inside the solenoid but still being affected by it vector potential $\mathbf{A}$ and results in the shift of energy spectrum. As we increase the magnetic flux by one time quantum flux $\Phi \to \Phi + \Phi_0$, the energy of that particle will be shifted from $E_n(\Phi) \to E_{n+1}(\Phi)$, increasing the quantum number n to $n + 1$.

These results is an example of a brand of a wider situation called "spectral flow", whence under the chance of the external parameter, each individual state morphed into each others while spectrum chance from itself to itself.

[7] An interesting example can be found in the Tong's lecture note

Remind Some Old Things

Gauge Invariance

In quantum mechanics, the field's gauge transform will have the form:

$$\mathbf{A} \to \mathbf{A}' = \mathbf{A} + \nabla\Lambda$$
$$\psi \to \psi' = \psi e^{i\frac{q}{\hbar}\Lambda}$$

With any function $\Lambda = \Lambda(x, y, z)$, this transformation work. But in contract, any $\mathbf{F}(x, y, z)$ which is not a gradient of any function that transform $\mathbf{A} \to \mathbf{A}' = \mathbf{A} + \mathbf{F}(x, y, z)$, will not work. For example, $\Lambda = x^n y^m \,\forall n, m \in \mathbb{N}$, will work, but $\mathbf{F}(x, y, z) = (y, x, 0)$ will not.

The transformation of the wave function will cancel out the eigenvalue π of the observable, $\boldsymbol{\pi} = \mathbf{p} - q\mathbf{A}$. Doing the transform

$$\begin{aligned}\boldsymbol{\pi}\psi = \pi\psi \to \boldsymbol{\pi'}\psi' &= \left(\frac{\hbar}{i}\nabla - q\mathbf{A} - q\mathbf{A}\nabla\Lambda\right)\psi e^{i\Lambda/\hbar} \\ &= \left(\frac{\hbar}{i}\nabla\psi - q\mathbf{A}\psi\right)e^{i\Lambda/\hbar} = \pi\psi e^{i\Lambda/\hbar}.\end{aligned}$$

Vice versa, the opposite in sign is not, $\tilde{\boldsymbol{\pi}} = \mathbf{p} + q\mathbf{A}$ in the transform give

$$\begin{aligned}\tilde{\boldsymbol{\pi}}\psi = \tilde{\pi}\psi \to \tilde{\boldsymbol{\pi}}'\psi' &= \left(\frac{\hbar}{i}\nabla + q\mathbf{A} + q\mathbf{A}\nabla\Lambda\right)\psi e^{i\Lambda/\hbar} \\ &= \left(\frac{\hbar}{i}\psi\nabla + q\mathbf{A}\psi\right)e^{i\Lambda/\hbar} + 2q\mathbf{A}\psi\nabla\Lambda e^{i\Lambda/\hbar} \\ &= \tilde{\pi}\psi e^{i\Lambda/\hbar} + 2q\mathbf{A}\psi\nabla\Lambda e^{i\Lambda/\hbar},\end{aligned}$$

Definitely defend on the gauge choosing, or *variance*.

References