Introduction To Quantum Hall Effect

Vo Chau Duc Phuong

The Abdus Salam International Center for Theo retical Physics

February 28, 2025

Table of Contents

The Classical Hall Effect

The equilibrium of the Hall effect can be described using Ohm's law in convention:

$$\begin{pmatrix} E_x \\ E_y \end{pmatrix} = \begin{pmatrix} \rho_{xx} & \rho_{xy} \\ -\rho_{xy} & \rho_{yy} \end{pmatrix} \begin{pmatrix} J_x \\ 0 \end{pmatrix},$$

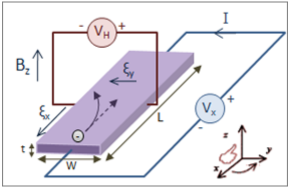


Figure: Classical Hall effect

in which:

$$\rho_{xy} = \frac{E_y}{J_x} = -\frac{B}{ne}, \qquad \rho_{xx} = \frac{E_x}{J_x} = \frac{m}{ne^2\tau}$$

Classically:

$$E_y \propto B \text{ and } E_x \text{ depend on scattering parameter } \tau$$

The Quantum Hall Effect

First introduced in 1980[1] and later on being investigated, The resistance in a MOSFET under a strong magnetic field shows interesting properties:

At certain point:

$$\rho_{xx} = 0.$$

$$\rho_{xy} \sim \frac{1}{\nu}, \quad \nu \in \mathbb{N}$$

Between these points:

$$\rho_{xy} = const.$$

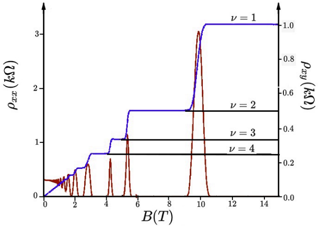


Figure: Quantum Hall Resistance
Taken from n.d.

[1] Klitzing, Dorda, and Pepper 1980.

Therefore, we will explain this effect as:

- Why does $\rho_{xx} \to 0$ is a peaks at some certain points and 0 otherwise?
- Why these plateux exist?

For a 2D electron gas in an external field field $\mathbf{B} = (0, 0, B)$ & $\mathbf{E} = (E, 0, 0)$. If we choose Landau gauge $\mathbf{A} = (0, Bx, 0)$, the eigenstates will be the Landau levels with the energy:

$$E_{\nu,k_y} = \hbar\omega_B\left(\nu + \frac{1}{2}\right) - eE\left(k_y l_B^2 + \frac{eE}{m\omega_B^2}\right) + \frac{m}{2}\frac{E}{B}, \tag{1}$$

in which

$$\omega_B = \frac{eB}{m}, \quad l_B = \frac{\hbar}{eB}$$

Recover the classical drift along $\mathbf{E} \times \mathbf{B}$ direction:

$$v_y = \frac{1}{\hbar}\frac{\partial E_{\nu,k_y}}{\partial k_y} = -\frac{E}{B} \tag{2}$$

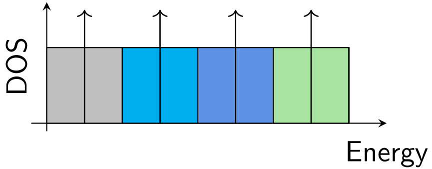


Figure: From constant DOS to Dirac comb

Conductivity

Each filled Landau levels have the degeneracy (in this convention, it's k_y) that when we take over the sum to get:

$$\mathbf{I} = -e\langle\dot{\mathbf{x}}\rangle = -e\sum_{n=1}^{\nu}\sum_{k_y}\left\langle\psi_{nk_y}\middle|\frac{\hbar}{i}\nabla - \mathbf{A}\middle|\psi_{nk_y}\right\rangle$$

$$\Rightarrow \qquad I_x = 0, \qquad I_y = -\sum_{k_y} e\nu\frac{E}{B} = \frac{e^2\nu E}{2\pi\hbar}$$

This result in:

$$\rho_{xx} = 0, \qquad \rho_{xy} = \frac{2\pi\hbar}{e^2\nu}, \tag{3}$$

in which ν is the total filled number of Landau levels.

But these conduction above not explained everything!

Revisit the calculation of (7) from (2) with a more generalize approach (Taylor expand $V(x)$ up to first order) give:

$$v_y = -\frac{1}{eB}\frac{\partial V(x)}{\partial x} \tag{4}$$

$$\begin{aligned} \sigma_{xy} = \frac{E_y}{I_x} &= \sum_\nu \frac{e}{EL_x}\int \frac{dk}{2\pi} v_y(x) \\ &= \sum_\nu \frac{e}{\cancel{EL_x}} \frac{\cancel{V(x_{max}) - V(x_{min})}}{2\pi\hbar} = \frac{\nu e^2}{2\pi\hbar} \end{aligned} \tag{5}$$

Invert:

$$\rho_{xy} \propto 1/\nu$$

$\Rightarrow$ As long as the $\partial_x V(x)$ smooth enough, only the difference of the edges create the quantize value!

But, why are the plateaus rounded rather than sharp, as seen in Fig. 2?
$\Rightarrow$ It turns out that disorder (impurities) play a crucial role!

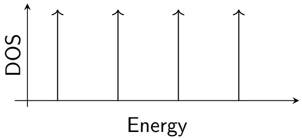
Disorder causes:

- Perturbation → Broader side (peaks!).
- Catch the localized state → Plateaus!

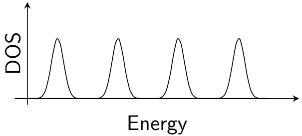
The Impurity act as a perturbation, causing the broad edge of the energy.

Sample too perfect? → flat spectrum

But:



(a) Without disorder



(b) with disorder

Too much disorder → not recognize the peaks!

From the calculation of the Landau levels (1), there's the degeneracy k_y in each Landau levels ν:

- Decrease B → more bands filled
- Total number of electrons: constant

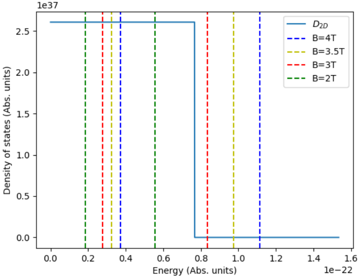


Figure: Illustrated number of filled levels when decrease B

From the calculation of the Landau levels (1), there's the degeneracy k_y in each Landau levels ν:

- Decrease B → more bands filled
- Total number of electrons: constant

But:

- Same filled levels ν: $N_e \propto B$.

Where do the electrons go?
They're still there!
(just not in the bands!)

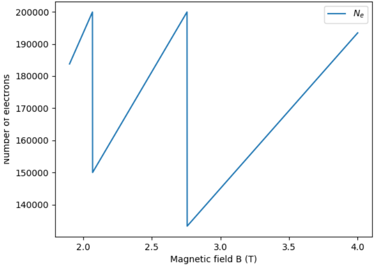


Figure: Illustrating number of electrons accommodated in Landau level when decreasing B

Disorder causes:

- Perturbation → Broader side (peaks!).
- Catch the localized state → Plateux!

The impurity → broad peaks (higher or lower energy than the center)→ localized by the impurity

Localized state don't contribute in conduction → plateux

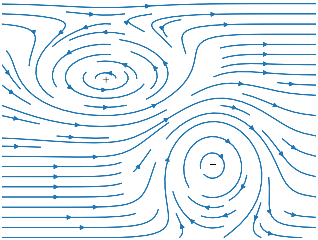


Figure: Movement of center of mass localized under impurity's maximum + or minimum −.

When decrease B but not filled the next level yet:

→ The electrons will populate the localized states!

The Quantum Hall Effect

So, let summary two main aspects:

- Edge states make sure quantized values.
- Impurity create the peaks and the plateux.

Partly filled levels?
→ Impurity create scattering inside level → longitude peaks.

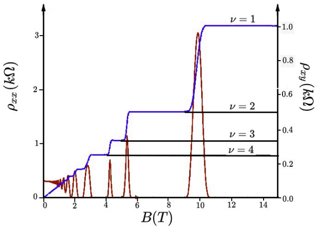


Figure: Quantum Hall Resistance Taken from n.d.

So, what do the edge states have to do with the topology?

These states enable a "highway current" that flows along the boundaries of the sample without backscattering, even in the presence of impurities.

Therefore, as long as the currents:

- not cut on the other (change the topology).
- stay non-localized, no back scattering.
- remain well-separated to prevent tunneling.

The system will exhibit quantized conductance and dissipationless transport.

Everything have been explained! Or not?

Beyond The Integer

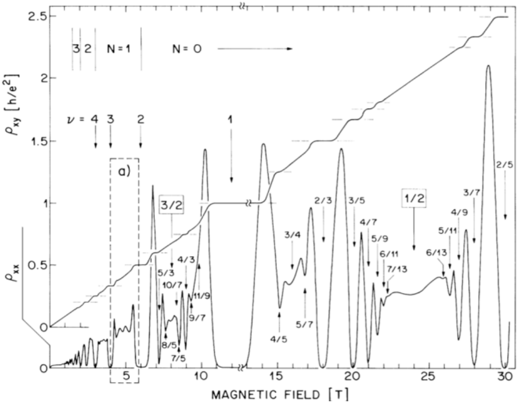


Figure: Fraction Hall Effect, from David Tong's lecture notes

Thank you for your listening.

Reference:

Advanced Quantum Mechanics II PHYS 40202 — oer.physics.manchester.ac.uk (n.d.). https://oer.physics.manchester.ac.uk/AQM2/Notes/Notes-4.4.html. [Accessed 15-02-2025].

Klitzing, K. v., G. Dorda, and M. Pepper (1980). “New Method for High-Accuracy Determination of the Fine-Structure Constant Based on Quantized Hall Resistance”. In: *Phys. Rev. Lett.* 45 (6), pp. 494–497.