Supplement For The Exam

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1 One Dimension chain of fermions

We have the Hamiltonian[1]:

$$H = \sum_{i=-\infty}^{\infty} \left(J_1 a_i^\dagger a_{i+1} + J_1 a_{i+1}^\dagger a_i + J_2 a_i a_{i+1} + J_2 a_{i+1}^\dagger a_i^\dagger - 2B a_i^\dagger a_i \right) \tag{1}$$

Doing the Fourier transformation to have:

$$a_r = \frac{1}{\sqrt{N}} \sum_k a_k e^{ikr}, \quad a_r^\dagger = \frac{1}{\sqrt{N}} \sum_k a_k^\dagger e^{-ikr} \tag{2}$$

Substituting into Hamiltonian to get:

$$\begin{aligned} H =& \frac{1}{N} \sum_{r,k,k'} \Big(J_1 e^{i(k-k')r} e^{ik'} a_k^\dagger a_{k+1} + J_1 e^{-ik(r+1)} e^{ik'r} a_k^\dagger a_k \\ & - 2B e^{i(k-k')r} a_{k'}^\dagger a_k + J_2 a_k^\dagger e^{-ik(r+1)} a_{k'}^\dagger e^{-ik'r} + J_2 a_k e^{ikr} e^{ik'(r+1)} a_{k'} \Big) \end{aligned}$$

Using the identity: $\sum_r e^{i(k-k')r} = N\delta_{k,k'}$ to omit the summation, the Hamiltonian been left with:

$$H = \sum_k (J_1 e^{ik} + J_1 e^{-ik} - 2B) a_k^\dagger a_k + J_2 \sum_{kk'} a_k^\dagger a_{k'}^\dagger e^{-ik} \delta_{-kk'} + \sum_{kk'} a_k a_{k'} \delta_{k-k'} e^{ik'} \tag{3}$$

Doing some manipulate to yield:

$$H = \sum_k 2(J_1 \cos(k) - B) a_k^\dagger a_k - iJ_2 \sum_k (a_k^\dagger a_{-k}^\dagger + a_k a_{-k}) \sin(k) \tag{4}$$

Invert the summation of the last term to get:

$$\begin{aligned} H =& \sum_k 2\varepsilon(k) a_k^\dagger a_k + i\Delta_k a_k^\dagger a_{-k}^\dagger - i\Delta_k a_{-k} a_k \\ =& \sum_k \varepsilon(k)(a_k^\dagger a_k + 1 - a_k a_k^\dagger) + i\Delta_k a_k^\dagger a_{-k}^\dagger - i\Delta_k a_{-k} a_k \end{aligned} \tag{5}$$

in which:

$$\varepsilon(k) = \sum_k (J_1 \cos(k) - B); \quad \Delta_k = J_2 \sin(k) \tag{6}$$

Or, can be written in the Bogoliubov-de-Gennes form (neglect the constant $\sum_k \varepsilon(k)/2$, since it give a constant):

$$H = \sum_k \begin{pmatrix} a_k^\dagger & a_{-k} \end{pmatrix} \begin{pmatrix} \varepsilon_k & -i\Delta_k \\ i\Delta_k & -\varepsilon_k \end{pmatrix} \begin{pmatrix} a_k \\ a_{-k}^\dagger \end{pmatrix} \tag{7}$$

To diagonalize this matrix, we find the Bogoliubov transformation:

$$\begin{pmatrix} c_k \\ c_{-k}^\dagger \end{pmatrix} = \begin{pmatrix} \bar{u} & -v \\ \bar{v} & u \end{pmatrix} \begin{pmatrix} \gamma_k \\ \gamma_{-k}^\dagger \end{pmatrix} \tag{8}$$

[1]This Hamiltonian, I suppose, is come from the XY model under the Jordan-Wigner transformation

Substituting into the BdG Hamiltonian to get:

$$H = \sum_k \begin{pmatrix} \gamma_k^\dagger & \gamma_{-k} \end{pmatrix} \begin{pmatrix} u & v \\ -\bar{v} & \bar{u} \end{pmatrix} \begin{pmatrix} \varepsilon & -i\Delta \\ i\Delta & -\varepsilon \end{pmatrix} \begin{pmatrix} \bar{u} & -v \\ \bar{v} & u \end{pmatrix} \begin{pmatrix} \gamma_k \\ \gamma_{-k}^\dagger \end{pmatrix} \tag{9}$$

Doing the multiplication:

$$H = \sum_k \begin{pmatrix} \gamma_k^\dagger & \gamma_{-k} \end{pmatrix} \begin{pmatrix} \varepsilon(|u|^2 - |v|^2) - i\Delta(u\bar{v} - \bar{u}v) & -2uv\varepsilon - i\Delta(u^2 + v^2) \\ -2\bar{u}\bar{v}\varepsilon + i\Delta(\bar{u}^2 + \bar{v}^2) & -(\varepsilon(|u|^2 - |v|^2) - i\Delta(u\bar{v} - \bar{u}v)) \end{pmatrix} \begin{pmatrix} \gamma_k \\ \gamma_{-k}^\dagger \end{pmatrix} \tag{10}$$

The off-diag terms have to be vanished, therefore, we can use the convention:

$$u_k = e^{i\phi_1}\cos(\theta/2), \quad v_k = e^{i\phi_2}\sin(\theta/2) \tag{11}$$

Or without loss of generality:

$$u_k = \cos(\theta/2), \quad v_k = i\sin(\theta/2) \tag{12}$$

so that the off-diag term become:

$$-\sin(\theta)\varepsilon - i\Delta\cos(\theta) = 0 \Rightarrow \tan(\theta) = \frac{\Delta}{\varepsilon} \tag{13}$$

Give us:

$$|u|^2 - |v|^2 = \cos(\theta) = \frac{\varepsilon}{\sqrt{\varepsilon^2 + \Delta^2}} \tag{14}$$

$$u\bar{v} - \bar{u}v = i(-\cos(\theta/2)\sin(\theta/2) - \cos(\theta/2)\sin(\theta/2)) = -i\sin(\theta) = \frac{-\Delta}{\sqrt{\varepsilon^2 + \Delta^2}} \tag{15}$$

$$\Rightarrow E(k) \equiv \varepsilon(|u|^2 - |v|^2) - i\Delta(u\bar{v} - \bar{u}v) = \sqrt{\varepsilon^2 + \Delta^2} = \sqrt{(J_1\cos(k) - B)^2 + J_2^2\sin^2(k)} \tag{16}$$

Or:

$$H = \sum_k \begin{pmatrix} \gamma_k^\dagger & \gamma_{-k} \end{pmatrix} \begin{pmatrix} E_k & 0 \\ 0 & -E_k \end{pmatrix} \begin{pmatrix} \gamma_k \\ \gamma_{-k}^\dagger \end{pmatrix} = \sum_k 2E_k \left(\gamma_k^\dagger \gamma_k - \frac{1}{2} \right) + \varepsilon(k) \tag{17}$$

So the spectrum of the quasiparticle is:

$$\epsilon(k) = 2E_k = 2\sqrt{\varepsilon^2 + \Delta^2} = 2\sqrt{(J_1\cos(k) - B)^2 + J_2^2\sin^2(k)} \tag{18}$$

In the limit we are considering: $B = 0, J_1 = J_2 = J$:

$$\epsilon(k) = 2J \quad \forall k \tag{19}$$

In this limit, the spectrum is a constant at every energy.

2 Correction to the other question:

2.1 Specific Heat

From the previous question, we have:

$$E = -\langle BM_z \rangle = -B\langle M_z \rangle = -mBL\left(\frac{mB}{T}\right) = -mB\left(\coth\left(\frac{mB}{T}\right) - \frac{T}{mB}\right) \tag{20}$$

Therefore:

$$C_v = \frac{\partial E}{\partial T} = 1 - \frac{m^2B^2}{T^2}\frac{1}{\sinh^2(mB/T)} \tag{21}$$

Plot it to get:

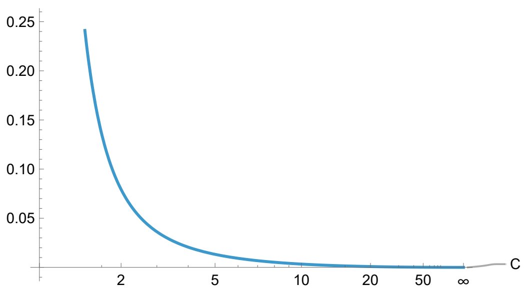
If we considering the quantum case, we have the expected value of S_z:

$$H = -g_s 2\mu_B \vec{B}\vec{S} = -\mu_B B S_z \tag{22}$$

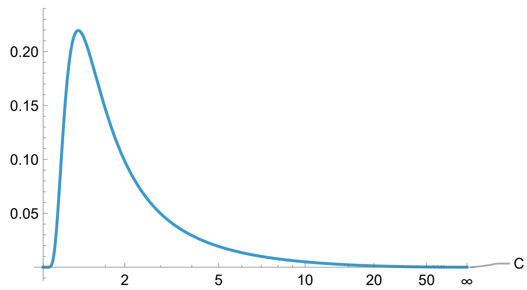
It give:

$$\langle S_z \rangle = \frac{1}{2}\tanh\left(\frac{\mu_B B}{T}\right), C_v = -\mu_B B\frac{\partial S_z}{\partial T} = -\partial_{(T/\mu_B B)}\tanh\left(\frac{\mu_B B}{T}\right) = \frac{1}{2T^2}\operatorname{sech}^2\left(1/T\right) \tag{23}$$

And plot it to get the Schottky anomaly, which have a peak at low T.



(a) Semi-classical case.



(b) Quantum case.