# Spin Dynamics

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**Abstract**

Working with the spin in the time-dependence magnetic field, I will briefly introduce the Bloch equation and solving it using the particular solution and using Green function for an equivalent approach, but much more generalize.

From last discussion, placing the magnetic moment inside the magnetic field give it a energy:

$$\mathcal{H} = -\vec{M} \cdot \vec{H}(t) = -g_J \mu_B S^\beta H^\beta \tag{1}$$

Considering the magnetic field has form: $\vec{H} = H_0 \vec{e}_z + H_\perp \vec{e}_\perp$. We are working in the quantum approach, so that:

$$[S^\alpha; S^\beta] = i\hbar \epsilon^{\alpha\beta\gamma} S^\gamma \tag{2}$$

Considering the equation of motions:

$$\begin{aligned}
d_t S^\alpha = \frac{i}{\hbar}[\mathcal{H}, S^\alpha] &= -\frac{i}{\hbar} g_J \mu_B H^\beta \left[S^\beta, S^\alpha\right] \\
&= -\frac{i}{\hbar} g_J \mu_B H^\beta i\hbar \epsilon^{\beta\alpha\gamma} S^\gamma = g_J \mu_B \epsilon^{\gamma\beta\alpha} S^\gamma H^\beta \\
d_t M^\alpha &= g_J \mu_B \epsilon^{\gamma\beta\alpha} M^\gamma H^\beta \\
\Rightarrow d_t \vec{M} &= g_J \mu_B \vec{M} \times \vec{H}
\end{aligned} \tag{3}$$

(3) is called Gyroscope equation. Since the total momentum give us the constrain on 3 axis, therefore this is a non-linear problem. We can define the oscillation in time of the magnetic field $H$ can be seen as the perturbation in time and perpendicular to the initial field, defining:

$$H_\perp = h(t)\vec{e}_x; \quad \max|h(t)| \ll H_0$$

Associating with it and mention that we are working in the quantum mechanics (where the $L_z$ is quantized), we should define the convention notation: $\delta m_z = M_0 - m_z$ A little of transformation give us:

$$d_t \vec{m} = \gamma \begin{vmatrix} i & j & k \\ m_x & m_y & M_0 - \delta m_z \\ h(t) & 0 & H_0 \end{vmatrix} \tag{4}$$

Since the field's oscillation is very small $h(t)$ and assuming the initial position of the vector is also along the $Oz$, we will neglect the term with order 2. Therefore, we arrive with

$$\begin{cases} d_t m_x &= m_y H_0 \gamma \\ d_t m_y &= \gamma(m_x H_0 - h(t) M_0) \\ d_t m_z &= 0 \end{cases}$$

The third equation immediate give us:

$$m_z = Const, \tag{5}$$

a.k.a it will rotating around the Z direction, its magnetic amplitude will draw a circle around the z axis on the sphere. A little a side here, since we are working in the spin conservation system, therefore:

$$\begin{aligned}
d_t M^2 &\propto [\mathcal{H}, \vec{M} \cdot \vec{M}] \propto [\vec{M} \cdot \vec{H}, \vec{M} \cdot \vec{M}] = [M^\alpha H^\alpha, M^\beta M^\beta] \\
&= [M^\alpha, M^\beta M^\beta] H^\alpha = (M^\beta [M^\alpha, M^\beta] + M^\beta [M^\alpha, M^\beta]) H^\alpha \\
&\propto 2 M^\beta \epsilon^{\alpha\beta\gamma} M^\gamma H^\alpha = 2\vec{M} \cdot (\vec{M} \times \vec{H}) = 0.
\end{aligned} \tag{6}$$

Therefore, the total magnetism is conserved, give us the convention basis by $(\theta, \phi)$. The other two equations in set of gyroscope equation are:

$$d_t m_x = \gamma m_y H_0 \tag{7}$$
$$d_t m_y = -\gamma(m_x H_0 - M_0 h(t)) \tag{8}$$

Defining the classical gyroscope frequency: $\omega_0 = \frac{|e|H_0}{mc} = \gamma H_0$. Take derivative in respect to time the first equation and substituting the second equation into it:

$$d_t^2 m_x = -\omega_0^2 m_x + M_0 \gamma h(t) \omega_0$$

Rearrange it a little bit,

$$d_t^2 m_x + \omega_0^2 m_x = M_0 \gamma h(t) \omega_0 \tag{9}$$

This is the same equation as the pendulum oscillate under the external oscillating force. Without the resonant $h(t) = h_0 \cos(\omega t); \omega \neq \omega_0$, the solution can be easily obtain by finding the particular solution: $x_p = C\cos(\omega t)$

$$C(\omega_0^2 - \omega^2) = \gamma h(t) m_0 \omega_0$$

$$C = \frac{\gamma h(t) m_0 \omega_0}{\omega_0^2 - \omega^2}. \tag{10}$$

Give us the general solution:

$$x_{non-resonant}(t) = A\cos(\omega_0 t) + B\sin(\omega_0 t) + \frac{\gamma h(t) m_0 \omega_0}{\omega_0^2 - \omega^2}\cos(\omega t) \tag{11}$$

But this is boring, let finding the resonant solutions, then: $F(t) = F_0 \cos(\omega_0 t)$. The homogeneous solutions will be the same:

$$x_h(t) = A\cos(\omega_0 t) + B\sin(\omega_0 t) \tag{12}$$

But since the $\cos(\omega_0 t)$ already is the homogeneous part, we will use this instead:

$$\begin{aligned} x_p(t) &= at\cos(\omega_0 t) + b\sin(\omega_0 t) \\ d_t^2 x_p(t) &= -2\omega_0 a \sin(\omega_0 t) + 2\omega_0 b\cos(\omega_0 t) - \omega_0^2 x_p(t) \\ F_0\cos(\omega_0 t) &= 2\omega_0 b\cos(\omega_0 t) - 2\omega_0 a\sin(\omega_0 t) \end{aligned}$$

Therefore, the general solution will be:

$$x(t) = \left(A - \frac{F_0 t}{2\omega_0}\right)\cos(\omega_0 t) + B\sin(\omega_0). \tag{13}$$

But this approach have a base problem, we have to carefully check the particular solution that whether it give the resonant or not, especially that the $F(t)$ can depend on more complex function than just a oscillation one. Therefore, we will need another approach that can, somewhat dueling with $F(t)$ on the RHS in general. The idea will be:

$$M_\alpha(\vec{r}, t) = \int d\vec{r} dt' \chi_{\alpha\beta}(\mathbf{r} - \mathbf{r}', t - t') H_\beta(t'), \tag{14}$$

in which $\chi_{\alpha\beta}$ is the susceptibility tensor. In simple language, it mean that the $\alpha$ component is and should be affected by every component of $H$ with some factor that showing the symmetry of the system, which also means that we can try to find a convention that satisfy:

$$\chi_{\alpha\beta} = \chi\delta_{\alpha\beta} \tag{15}$$

A little introduce, suppose we have an equation have the form:

$$\mathcal{L}(\{x^n\}, \{d_x^n\})y(x) = f(x) \tag{16}$$

Operationalize every linear operator in a single operator, and "secretly" sneak the delta definition into the RHS:

$$\mathcal{L}(\{x^n\}, \{d_x^n\})y(x) = \int dx' \delta(x' - x) f(x') \tag{17}$$

And since the operator is depend on $x$ but not $x'$, we will effect to find the invert operator $\mathcal{L}^{-1}$ so that:

$$y(x) = \mathcal{L}^{-1} f(x) = \int dx' \mathcal{L}^{-1} f(x') \delta(x - x') = \int dx' \mathcal{G}(x - x') F(x') dx' \tag{18}$$

And we arrive at the definition of the Green's function $\mathcal{G}(x-x')$:

$$\mathcal{L}\mathcal{G}(x-x') = \delta(x-x') \tag{19}$$

Okay, now let apply to our problem:

$$(\hat{d_t^2} + \omega_0^2)m_x = M_0\gamma h(t)\omega_0 \tag{20}$$

or, we have to find $\mathcal{G}(t-t')$ so that:

$$(\hat{d_t^2} + \omega_0^2)\mathcal{G}(t-t') = \delta(t-t') \tag{21}$$

Okay, again, I am lazy and $d_t(t-t') = d_t(t)$ so let define: $t - t' = \tau$, let assume $t - t'$, it mean that the past will effect the future, keep in mind this idea, we will return later. Take the Fourier transform for both side[1]:

$$\begin{aligned}(-\omega^2 + \omega_0^2)\mathcal{G}(\omega) &= 1 \\ \mathcal{G}(\omega) &= \frac{1}{\omega_0^2 - \omega^2}\end{aligned} \tag{22}$$

Let have a look at how we use the Green function in (18), the integral run from $-\infty \to \infty$, but the action of the field from the future have to have no effect on the reaction of the function at the present. Therefore, we will define a family of Green function $\mathcal{G}^R$, call retard Green function, so that:

$$\mathcal{G}^R : \begin{cases} \neq 0, & \text{if } t > t', \text{ or } \tau > 0 \\ = 0, & \text{if } t < t', \text{ or } \tau < 0 \end{cases} \tag{23}$$

Take the invert Fourier Transform to have:

$$\mathcal{G}^R(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega e^{-i\omega\tau}\mathcal{G}^R(\omega) \tag{24}$$

According to Jordan's lemma on the complex integral, we will need to take the lower semicircle contour ($-\infty \to \infty \to -i\infty \to -\infty$). To strengthen the definition of retard Green function, we know that the pole of $\mathcal{G}(\omega)$ is on the real axis, we will use $\omega \to \omega + i\delta$, which will be take the limit to 0 after the calculation:

$$\begin{aligned}\mathcal{G}^R(\tau > 0) &= \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega\tau}\mathcal{G}^R(\omega) = -\int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{i\omega\tau}\frac{1}{(\omega + i\delta)^2 - \omega_0^2} \\ &= \frac{\sin(\omega_0\tau)}{\omega_0}e^{-\delta\tau} \\ \Rightarrow \mathrm{G}^R(\tau) &= \frac{\sin(\omega_0\tau)}{\omega_0}e^{-\delta\tau}\theta(\tau)\end{aligned} \tag{25}$$

So, we have the general solution in the form:

$$m_x(t) = \int_{-\infty}^{\infty} \frac{\sin(\omega_0\tau)}{\omega_0}e^{-\delta\tau}\theta(t-t')(\gamma\omega_0 M_0)h(t')dt', \tag{26}$$

using the definition of the susceptibility:

$$\chi_{xx}^R(\tau) = \gamma\omega_0 M_0 \mathcal{G}^R(\tau) \tag{27}$$

and,

$$\chi_{xx}^R(\omega) = \gamma\omega_0 M_0\mathcal{G}^R(\omega) = -\gamma\omega_0 M_0 \frac{1}{(\omega + i\delta)^2 - \omega_0^2} = -\frac{\gamma}{2}M_0\left(\frac{1}{\omega - (\omega_0 - i\delta)} - \frac{1}{\omega + \omega_0 + i\delta}\right)$$

$$\frac{1}{x + i\delta} \xrightarrow{\delta \to 0} = P.V\left(\frac{1}{x}\right) - i\pi\delta(x) \tag{28}$$

$$\mathrm{Im}(\chi_{xx}^R(\omega)) = \frac{\gamma M_0 \pi}{2}(\delta(\omega - \omega_0) - \delta(\omega + \omega_0)) \tag{29}$$

$$\mathrm{Re}(\chi_{xx}^R(\omega)) = \frac{\gamma M_0}{2}\left(\frac{1}{\omega + \omega_0} - \frac{1}{\omega - \omega_0}\right) = -\frac{\gamma\omega_0 M_0}{\omega^2 - \omega_0^2} \tag{30}$$

[1]

$$\tilde{\delta}(\omega) = \int_{-\infty}^{\infty} \delta(t)e^{i\omega t}dt = 1, \quad \forall\omega$$

That's done for the idea case. Now, we will work with a phenomenon modification for the gyroscope equation. Introduce the parameter $T_1, T_2$ characterize the relaxation of the system to the stable state as

$$d_t m_z = -\gamma h m_y - \frac{m_z - M_0}{T_1} \tag{31}$$

$$d_t m_\alpha = \gamma(\vec{M} \times \vec{H})_\alpha - \frac{m_\alpha}{T_2} \tag{32}$$

With the same argument, neglecting the term $hm_y$ give us:

$$d_t(m_z - M_0) = -\frac{m_z - M_0}{T_1}$$
$$m_z - M_0 = Ce^{-t/T_1}$$

This equation give us the idea that from the point of view of the field, whatever the $m_z$ point into, it will eventually align with the $H_0$ along the $Oz$ axis when $t \to \infty$ is $T_1$ is finite. From the two last equations, we are doing the same procedure:

$$\begin{aligned}
d_t m_x &= \omega_0 m_y - \frac{m_x}{T_2} \\
d_t m_y &= -\omega_0 m_x - \frac{m_y}{T_2} + \gamma M_0 h(t) \\
d_t^2 m_x &= \omega_0 d_t m_y - \frac{d_t m_x}{T_2} = \omega_0 \left( -\omega_0 m_x - \frac{m_y}{T_2} + \gamma M_0 h(t) \right) - \frac{d_t m_x}{T_2} \\
&= -\omega_0^2 m_x - \frac{1}{T_2}\left( d_t m_x + \frac{m_x}{T_2} \right) + \gamma M_0 \omega_0 h(t) - \frac{1}{T_2} d_t m_x \\
\Rightarrow d_t^2 m_x + \frac{2}{T_2} d_t m_x &+ \left( \omega_0^2 + \frac{1}{T_2^2} \right) m_x = \gamma M_0 \omega_0 h(t)
\end{aligned}$$

So, we need to find $\mathcal{G}$ that:

$$\left( d_t^2 + \frac{2}{T_2} d_t + \left( \omega_0^2 + \frac{1}{T_2^2} \right) \right) \mathcal{G}(t - t') = \delta(t - t') \tag{33}$$

$$\left( \left( d_t^2 + \frac{2}{T_2} d_t + \frac{1}{T_2^2} \right) + \omega_0^2 \right) \mathcal{G}(t - t') = \delta(t - t') \tag{34}$$

$$\left( \left( d_t + \frac{1}{T_2} \right)^2 + \omega_0^2 \right) \mathcal{G}(t - t') = \delta(t - t') \tag{35}$$

$$\left( \left( -i\omega + \frac{1}{T_2} \right)^2 + \omega_0^2 \right) \mathcal{G}(\omega) = 1 \tag{36}$$

$$\mathcal{G}(\omega) = -\frac{1}{(\omega + i/T_2)^2 - \omega_0^2} \tag{37}$$

Therefore:

$$\mathcal{G}^R(\tau) = -\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{-i\omega\tau} d\omega}{(\omega + i/T_2) - \omega_0^2} \theta(\tau) = \frac{\sin(\omega_0 \tau) e^{-\tau/T_2}}{\omega_0} \tag{38}$$

Give us:

$$\chi_{xx}(\tau) = \mathcal{G}^R(\tau) e^{-\tau/T_2} \theta(\tau)(\gamma m_0 \omega_0) \tag{39}$$

$$\chi_{xx}(\omega) = -\gamma m_0 \omega_0 \frac{1}{\left( \omega + \frac{i}{T_2} \right)^2 - \omega_0^2} = -\frac{\gamma m_0}{2} \left( \frac{1}{\omega + \frac{i}{T_2} - \omega_0} - \frac{1}{\omega + \frac{i}{T_2} + \omega_0} \right) \tag{40}$$

$$\mathrm{Im}\{\chi_{xx}\} = \frac{\gamma m_0}{2T_2} \left( \frac{1}{(\omega - \omega_0)^2 + \frac{1}{T_2^2}} + \frac{1}{(\omega + \omega_0)^2 + \frac{1}{T_2^2}} \right) \xrightarrow{\omega \to 0} \frac{2 m_0 \omega \omega_0 \gamma}{T_2 \left( \omega_0^2 + T_2^{-2} \right)^2} \tag{41}$$

$$\mathrm{Re}[\chi_{xx}] = -\frac{m_0 \gamma}{2} \left( \frac{\omega - \omega_0}{(\omega - \omega_0)^2 + \frac{1}{T_2^2}} + \frac{\omega + \omega_0}{(\omega + \omega_0)^2 + \frac{1}{T_2^2}} \right) \xrightarrow{\omega \to 0} \frac{m_0 \omega_0 \gamma}{\omega_0^2 + T_2^{-2}} \tag{42}$$

And we will have the relation:

$$\mathrm{Im}\{\chi_{xx}\} = \frac{2\omega}{T_2 \left( \omega_0^2 + T_2^{-2} \right)} \mathrm{Re}\{\chi_{xx}\} \tag{43}$$

or:

$$\chi_{xx}(\omega) = \mathrm{Re}(\chi_{xx})\left(1 + i\frac{2\omega}{T_2\left(\omega_0^2 + T_2^{-2}\right)}\right) \tag{44}$$