Homework Of Superconductivity and Magnetism

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1 Superconductivity

- Considering the 2-D potential have the form:

$$U(r) = \begin{cases} -U_0, & r < a \\ 0, & r > a \end{cases} \tag{1}$$

Find the shallow energy level when $U_0 \ll h^2/ma^2$ for the case $M_z = 0$, where M_z is the projection of the orbital moment on the z-axis.
In the polar coordinate, the kinetic have the form:

$$K(r,\theta)\psi(r,\theta) = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\psi(r,\theta)}{\partial r}\right) + \frac{1}{r}\frac{\partial^2\psi(r,\theta)}{\partial\theta^2} \tag{2}$$

Since $M_z = 0$, we have the ansatz:

$$\psi(r,\theta) = \frac{1}{\sqrt{4\pi}}R(r) \tag{3}$$

Therefore, the second term of (2) vanished. The stationary Schrödinger equation have the form:

$$-\frac{\hbar^2}{2m}\left(\frac{1}{r}\frac{\partial R}{\partial r} + \frac{\partial^2 R}{\partial r^2}\right) + V(r)R = ER$$

Or in the form of Bessel equation:

$$r^2\frac{\partial^2 R}{\partial r^2} + r\frac{\partial R}{\partial r} + \frac{2m(E-V)}{\hbar^2}R = 0 \tag{4}$$

Inside the circle, we have the equation:

$$r^2\frac{\partial^2 R}{\partial r^2} + r\frac{\partial R}{\partial r} + \frac{2m(E+U_0)}{\hbar^2}R = 0 \tag{5}$$

Which have the general solution:

$$R(r) = C_1 J_0\left(\sqrt{\frac{2m(E+U_0)}{\hbar^2}}r\right) + C_2 Y_0\left(\sqrt{\frac{2m(E+U_0)}{\hbar^2}}r\right) \tag{6}$$

But since the radius have to be continuous at $r = 0$, where $Y_0(r)$ is not, therefore: $C_2 = 0$. Hence

$$R(r) = C_1 J_0\left(\sqrt{\frac{2m(E+U_0)}{\hbar^2}}r\right) \tag{7}$$

Outside the circle, we have:

$$r^2\frac{\partial^2 R}{\partial r^2} + r\frac{\partial R}{\partial r} - \frac{2m|E|}{\hbar^2}R = 0 \tag{8}$$

Which yields the Bessel modified function as the solution:

$$R(r) = C_3 I_0\left(\sqrt{\frac{2m|E|}{\hbar^2}}r\right) + C_2 K_0\left(\sqrt{\frac{2m|E|}{\hbar^2}}r\right) \tag{9}$$

The constrain condition now is $R(r)$ have to vanished at $r \to \infty$, which $I_0\left(\sqrt{\frac{2m|E|}{\hbar^2}}r\right)$ is not, therefore:

$$R(r) = C_2 K_0\left(\sqrt{\frac{2m|E|}{\hbar^2}}r\right) \tag{10}$$

At $r = a$, the function have to be continuous:

$$C_2 = C_1 \frac{J_0\left(\sqrt{\frac{2m(E+U_0)}{\hbar^2}}a\right)}{K_0\left(\sqrt{\frac{2m|E|}{\hbar^2}}a\right)} \tag{11}$$

And also their derivative, we using the properties of Bessel function:

$$\frac{\mathrm{d}}{\mathrm{d}x}J_0(x) = -J_1(x); \quad \frac{\mathrm{d}}{\mathrm{d}x}K_0(x) = -K_1(x) \tag{12}$$

to get:

$$\sqrt{\frac{E+U_0}{|E|}} C_1 J_1\left(\sqrt{\frac{2m(E+U_0)}{\hbar^2}}a\right) = C_2 K_1\left(\sqrt{\frac{2m|E|}{\hbar^2}}r\right) \tag{13}$$

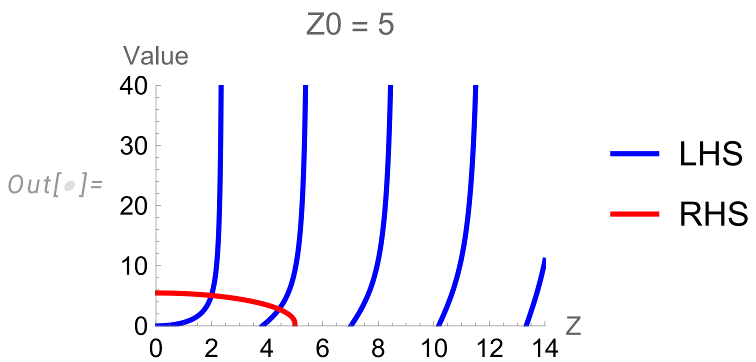
Or:

$$\sqrt{\frac{E+U_0}{|E|}} = \frac{K_1\left(\sqrt{\frac{2m|E|}{\hbar^2}}a\right)}{J_1\left(\sqrt{\frac{2m(E+U_0)}{\hbar^2}}a\right)} \frac{J_0\left(\sqrt{\frac{2m(E+U_0)}{\hbar^2}}a\right)}{K_0\left(\sqrt{\frac{2m|E|}{\hbar^2}}a\right)} \tag{14}$$

set $Z = \sqrt{2m(E+U_0)}a/\hbar$, $Z_0 = \sqrt{2mU_0}a/\hbar$, equation become:

$$\begin{aligned} \frac{Z}{\sqrt{Z^2 - Z_0^2}} &= \frac{K_1(\sqrt{Z_0^2 - Z^2})}{J_1(Z)} \frac{J_0(Z)}{K_0(\sqrt{Z_0^2 - Z^2})} \\ Z\frac{J_1(Z)}{J_0(Z)} &= \sqrt{Z^2 - Z_0^2}\frac{K_1(\sqrt{Z_0^2 - Z^2})}{K_0(\sqrt{Z_0^2 - Z^2})} \end{aligned} \tag{15}$$

Plot these two, we have: According to the LHS, we seeing that however the U_0 small, always exist a solution Z for



both side to match. At very shallow case: $U_0 \ll \frac{\hbar^2}{2ma^2}$ or $Z \ll 1$, we need to approximate the