Lab 1

Consider a partial differential equation

$$u_{xx} + u_{yy} = -1 - u - \lambda u^2. \tag{1}$$

with Dirichlet boundary conditions

$$\begin{aligned} u(0, y) &= 0, \\ u(1, y) &= 0, \\ u(x, 0) &= 0, \\ u(x, 1) &= 0. \end{aligned} \tag{2}$$

Your task is to study the behaviour of the following equation based on different parameter values.

1. (1 point) Derive functional for the equation (1).

Since the equation has second-order derivatives u_{xx} and u_{yy}, we assume that the corresponding functional contains a quadratic gradient term and an unknown function $G(u)$. We therefore seek the functional in the following form:

$$J[u] = \int_0^1 \int_0^1 \left[\frac{1}{2}\left(u_x^2 + u_y^2\right) + G(u)\right] dx\, dy$$

For a functional of the form $J[u] = \int F(u, u_x, u_y)\, dx\, dy$ the Euler-Lagrange equation is given by

$$\frac{\partial F}{\partial u} - \frac{d}{dx}\left(\frac{\partial F}{\partial u_x}\right) - \frac{d}{dy}\left(\frac{\partial F}{\partial u_y}\right) = 0$$

So, in our case, $F(u, u_x, u_y) = \frac{1}{2}\left(u_x^2 + u_y^2\right) + G(u)$. Let's compute the partial derivatives: $\frac{\partial F}{\partial u} = G'(u)$, $\frac{\partial F}{\partial u_x} = u_x$, $\frac{\partial F}{\partial u_y} = u_y$.

And substituting these expressions into the Euler-Lagrange equation: $G'(u) - u_{xx} - u_{yy} = 0 \longrightarrow G'(u) = u_{xx} + u_{yy}$

Comparing this equation with the given differential equation (1), we can notice that $G'(u) = -1 - u - \lambda u^2$.

Let's take the integral with respect to u: $G(u) = -u - \frac{1}{2}u^2 - \frac{\lambda}{3}u^3$. Note, that there is no constant since it does not affect the Euler-Lagrange equation.

And substituting the expression for $G(u)$ into the functional, we obtain

$$J[u] = \int_0^1 \int_0^1 \left[\frac{1}{2}\left(u_x^2 + u_y^2\right) - u - \frac{1}{2}u^2 - \frac{\lambda}{3}u^3\right] dx\, dy$$

2. (5 points) Apply numerical bifurcation analysis with iterative generalised Kantorovich method with only one term to construct bifurcation path. How many qualitatively different solutions are observed in this equation for different parameter values?

We assume a one-term separated approximation of the form $u(x,y) \approx h(x)\,g(y)$, where g is chosen to satisfy the boundary conditions in y: $g(0) = g(1) = 0$, like $g(y) = \sin(\pi y)$, which automatically enforces the boundary conditions in the y-direction. The remaining boundary conditions imply $h(0) = h(1) = 0$.

Substituting our approximations into (1) gives $h''(x)g(y) + h(x)g''(y) = -1 - h(x)g(y) - \lambda h(x)^2 g(y)^2$

To obtain an equation for $h(x)$, we do the following - multiply by $g(y)$ and integrate over $y \in (0,1)$.

Let's denote

$$A = \int_0^1 g^2(y)\,dy, \qquad B = \int_0^1 g(y)g''(y)\,dy, \qquad I_1 = \int_0^1 g(y)\,dy, \qquad I_3 = \int_0^1 g^3(y)\,dy.$$

Using these quantities we obtain the reduced nonlinear ODE

$$A\,h''(x) + B\,h(x) + A\,h(x) + I_1 + \lambda I_3\,h(x)^2 = 0, \qquad h(0) = h(1) = 0.$$

For the choice of $g(y)$, the integrals can be computed explicitly:

$$A = \int_0^1 \sin^2(\pi y)\,dy = \frac{1}{2}, \qquad B = \int_0^1 \sin(\pi y)(-\pi^2 \sin(\pi y))\,dy = -\frac{\pi^2}{2}, \qquad I_1 = \int_0^1 \sin(\pi y)\,dy = \frac{2}{\pi}, \qquad I_3 = \int_0^1 \sin^3(\pi y)\,dy = \frac{4}{3\pi}.$$

Substituting these values into our ODE and dividing by $A = 1/2$ gives the one-term Kantorovich model

$$h''(x) + (1-\pi^2)\,h(x) + \frac{4}{\pi} + \frac{8\lambda}{3\pi}\,h(x)^2 = 0, \qquad h(0) = h(1) = 0.$$

Finally, the approximate solution of the original problem is reconstructed as

$$u(x,y) \approx h(x)\sin(\pi y).$$

The obtained equation is a nonlinear two-point boundary value problem. To solve it numerically, we can discretise x and solve the resulting nonlinear algebraic system by an iterative method. Let's code it - we will apply Finite Differences and Newton's Methods (all code, explainations and step-by-step solutions can be found in .ipynb document).

Briefly, here is the answer for the last question:

Using the one-term Kantorovich approximation, three qualitatively different solution types are observed:

1. Main branch (small positive amplitude, $\max(h) \approx 0.08$): exists for all λ
2. Upper branch (large positive amplitude): exists for $\lambda > \lambda_2^* \approx 2.58$
3. Lower branch (negative amplitude): exists for $\lambda < \lambda_1^* \approx -2.58$

The near-symmetry of the critical values λ_1^* and λ_2^* is a numerical observation for the reduced Kantorovich model and does not follow from an exact symmetry of the original partial differential equation.

The bifurcation diagram is constructed by plotting the scalar quantity $\max_{x\in[0,1]} h(x)$ as a function of the parameter λ.

The number of solutions depends on the parameter range:

1. $\lambda < -2.58$: 2 solutions (main + lower)
2. $-2.58 < \lambda < 2.58$: 1 solution (main only)
3. $\lambda > 2.58$: 2 solutions (main + upper)

Two fold (saddle-node) bifurcations occur at $\lambda_1^* \approx -2.58$ and $\lambda_2^* \approx +2.58$.

$\boxed{\text{Implementation details and additional numerical results are provided in the accompanying Jupyter notebook.}}$

3. (2 points) Next apply iterative generalised Kantorovich method with two terms to construct bifurcation path. Does this affect the bifurcation picture of the equation?

The numerical and analytical evidence shows that the use of a two-term iterative generalized Kantorovich approximation does *NOT* alter the bifurcation structure of the problem obtained with the one-term approximation.

We consider the two-term approximation

$$u(x, y) \approx h_1(x)\sin(\pi y) + h_2(x)\sin(2\pi y),$$

which leads, after application of the iterative generalized Kantorovich method, to the coupled system of nonlinear ordinary differential equations

$$h_1'' + (1-\pi^2)h_1 + \frac{4}{\pi} + \lambda\left(\frac{8}{3\pi}h_1^2 + \frac{32}{15\pi}h_2^2\right) = 0, \quad h_2'' + (1-4\pi^2)h_2 + \frac{64\lambda}{15\pi}h_1 h_2 = 0,$$

subject to the boundary conditions

$$h_1(0) = h_1(1) = h_2(0) = h_2(1) = 0.$$

Second equation contains no constant forcing term, since $\int_0^1 \sin(2\pi y)\,dy = 0$ and as a consequence, the trivial solution $h_2(x) \equiv 0$ satisfies both the differential equations and boundary conditions for all values of the parameter λ and for any solution $h_1(x)$ of first equation.

Substituting $h_2 \equiv 0$ into the first equation reduces the two-term system exactly to the one-term Kantorovich equation.

To examine the possibility of a bifurcation from the trivial solution $h_2 \equiv 0$, we consider the linearized equation

$$h_2'' + \left[(1-4\pi^2) + \lambda\gamma_{21}h_1(x)\right]h_2 = 0,$$

where $\gamma_{21} = 64/(15\pi)$.

The constant coefficient $1 - 4\pi^2 \approx -38.5$ is strongly negative. Numerical results show that the coupling term $\lambda\gamma_{21}h_1(x)$ remains insufficient to compensate this negative shift in the relevant parameter range. Consequently, the associated linear operator remains negative definite and does not admit a zero eigenvalue.

Therefore, no secondary bifurcation associated with the excitation of the second mode occurs.

As a direct consequence:

1. the critical bifurcation values λ_1^* and λ_2^* are identical for the one-term and two-term Kantorovich approximations;
2. the number and qualitative structure of solution branches (main, upper, and lower) remain unchanged;
3. all *computed* solutions of the two-term system satisfy $h_2(x) \approx 0$ in the investigated parameter range.

Implementation details and additional numerical results are provided in the accompanying Jupyter notebook.

4. (2 points) Finally, apply PINN for constructing bifurcation path. Here you probably will need arclength continuation method. About this method you can read in the following article: `https://arxiv.org/pdf/2507.09782v1`

$\boxed{\text{Implementation details and additional numerical results are provided in the accompanying PINN directory.}}$