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# Mittag–Leffler stability of nabla discrete fractional-order dynamic systems

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**Abstract** In the present study, the definition of discrete Mittag–Leffler stability is derived to characterize convergence rule of the pseudostates for nabla discrete fractional-order dynamic systems. Applying the Lyapunov stability theory, some new criteria are proposed to determine asymptotic stability of the zero equilibrium. In addition, by applying fractional comparison principle, the results are extended from Caputo discrete fractional-order systems to Riemann–Liouville systems. Moreover, a useful inequality is proposed to further improve the availability of the presented methods. Finally, some meticulously designed simulations are provided to verify the correctness and practicability of the elaborated stability notions.

**Keywords** Nabla discrete fractional-order systems · Discrete Mittag–Leffler stability · Asymptotic stability · Fractional direct Lyapunov method

## 1 Introduction

It has been more than three hundred years since Newton and Leibniz founded classical calculus. On the basis of traditional calculus, fractional calculus was established when the order is extended from integer to fraction (arbitrary) [1]. Many plants exhibit fractional phenomena in practical engineering, and this property provides a possibility for the application of fractional calculus [2,3]. Given its unique and significant properties, such as long memory [4], nonlocality [5], and infinite dimensionality [6], etc, fractional calculus has been widely used in heat conduction [7], viscoelastic materials [8], thermal diffusion [9], capacitance [10], and so on. Especially in systems and control, a great deal of excellent research work has emerged based on fractional calculus [11,12]. In both theoretical perfection and engineering practice, fractional calculus has greatly promoted the development of related fields. In [13], the authors provide a state-of-the-art review to reveal the applications in real world of fractional calculus, which shows broad application prospects of this potential field.

Stability analysis is the most important index in the field of systems and control [14]. The classic Lyapunov method becomes an attractive strategy, which is very convenient and practical because it does not need to explicitly solve the differential equations. In continuous fractional-order nonlinear systems, many exciting research results for stability analysis have yielded. The Mittag–Leffler stability definition is proposed to describe the dynamics of the system, and the fractional direct Lyapunov method is introduced creatively in [15,16], which inspired many scholars and derived a series of pioneering work. Based on methodology of the frequency-distributed model, a Lyapunov approach has been presented to analyze the stabil-

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ity of fractional-order systems [17]. The stability of fractional-order nonlinear systems with additive disturbance and delayed fractional-order composite systems is investigated in [18] and [19], respectively. A new fractional differential inequality is proposed in [20] and can be used for finding Lyapunov functions conveniently. Furthermore, this critical method of the fractional direct Lyapunov method under vector case [21] is extended from scalar case, thereby improving the practicability. Besides, two novel inequalities on Riemann–Liouville derivative [22] and three generalized power law inequalities [23] are proposed to fully exploit Lyapunov direct method. For continuous fractional-order systems, other novel work on stability analysis can be found in [24–26].

In terms of discrete fractional-order nonlinear dynamic systems, related work on stability analysis is scattered along the literature and faces many challenges. Given the characteristics of the difference, it is impossible to describe the limit of the system states through the $\varepsilon - \delta$ language. Furthermore, the fractional difference is usually difficult to determine directly. The stability issue becomes quite difficult to address. In [27], a theorem is proved for discrete fractional direct Lyapunov method. Besides, the authors extend an inequality from the continuous case [20] and give a sufficient condition for stability of Caputo delta fractional difference equations. The stability criterion under the definition of Riemann–Liouville is also given in [28], yet, it should be mentioned that Refs. [27] and [28] use noncausal forward difference and the time series depends on the system order. The definition of discrete Mittag–Leffler stability given in [29] also adopts the forward difference and limits the initial instant to 0, which makes the conclusion less general. To compensate for the difficulty in obtaining fractional difference of Lyapunov functions, Wei et al. [30] proposed some useful inequalities under different definitions, which bring great convenience to the use of discrete fractional direct Lyapunov method. Nevertheless, most current studies of stability analysis concentrate on convergence in the steady state, while the convergence rule is rarely investigated. How to characterize the dynamics of the system is still an open problem. Thus far, for discrete fractional-order systems, systematic and complete framework for stability analysis has not been conducted.

Motivated by the above discussion, system stability can be characterized by an attenuation of the energy function at any rate. The specific contributions of this paper are concluded in the following.

i) The definition of discrete Mittag–Leffler stability is proposed to characterize the convergence rule of the pseudostates for nabla discrete fractional-order nonlinear dynamic systems. The relationship between the discrete Mittag–Leffler function and the dynamics of the system is established.
ii) This paper presents the extended Lyapunov direct method of Caputo or Riemann–Liouville discrete fractional-order nonlinear systems. Moreover, an inequality is proposed to make the presented methods more convenient to use. In this way, stability of such systems can be easily determined.
iii) The work exhibited in this paper adopts the backward difference, and the sampling time is independent on the order. Furthermore, the initial instant is no longer restricted to 0, which makes the presented results relatively general and further fills the gap produced by those existing approaches.

The remainder of this paper is structured as follows. Section 2 mainly introduces some necessary basic definitions and knowledge needed in this paper. In Sect. 3, the discrete Mittag–Leffler stability definition is proposed. Besides, the discrete fractional direct Lyapunov method and a useful inequality are given. Section 4 contains some meticulously designed examples to verify the correctness and practicability of the elaborated methods. In the end, Sect. 5 summarizes and forecasts the full paper.

## 2 Preliminaries

Some necessary definitions, facts, and the crucial discrete Mittag–Leffler function are introduced in this section.

**Definition 1** Assume $f : \mathbb{N}_{a+1-n} \to \mathbb{R}$. Then the $n$-th nabla backward difference is defined as [31]

$$\nabla^n f(k) \triangleq \sum_{j=0}^{n} (-1)^j \binom{n}{j} f(k-j), \tag{1}$$

where $k \in \mathbb{N}_{a+1}$, $\mathbb{N}_{a+1} \triangleq \{a+1, a+2, a+3, \ldots\}$, $a \in \mathbb{R}$, $n \in \mathbb{Z}_+$, $\binom{p}{q} \triangleq \frac{\Gamma(p+1)}{\Gamma(q+1)\Gamma(p-q+1)}$, and the Gamma function $\Gamma(x) \triangleq \int_0^{+\infty} \mathrm{e}^t t^{x-1} \mathrm{d}t$.

**Definition 2** Assume $f : \mathbb{N}_{a+1} \to \mathbb{R}$. Then the $\alpha$-th sum is defined as [31]

$$ {}_a^{\mathrm{R}}\nabla_k^{-\alpha} f(k) \triangleq \sum_{j=0}^{k-a-1} (-1)^j \binom{-\alpha}{j} f(k-j), \tag{2} $$

where $\alpha \in (n-1, n)$, $n \in \mathbb{Z}_+$, $a \in \mathbb{R}$ and $k \in \mathbb{N}_{a+1}$.

For simplicity, let $t^{\overline{r}} \triangleq \frac{\Gamma(t+r)}{\Gamma(t)}$, $t \in \mathbb{N}$, $r \in \mathbb{R}$. Then one can equivalently express the formula in (2) as

$$ \begin{aligned} {}_a^{\mathrm{R}}\nabla_k^{-\alpha} f(k) &= \sum_{j=0}^{k-a-1} \frac{(j+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)} f(k-j) \\ &= \sum_{j=a+1}^{k} \frac{(k-j+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)} f(j). \end{aligned} \tag{3} $$

**Definition 3** The nabla Caputo and Riemann–Liouville fractional differences (if they exist) of a function $f : \mathbb{N}_{a+1-n} \to \mathbb{R}$ are defined as [31]

$$ {}_a^{\mathrm{C}}\nabla_k^{\alpha} f(k) \triangleq {}_a^{\mathrm{R}}\nabla_k^{\alpha-n} \nabla^n f(k), \tag{4} $$

$$ {}_a^{\mathrm{R}}\nabla_k^{\alpha} f(k) \triangleq \nabla^n {}_a^{\mathrm{R}}\nabla_k^{\alpha-n} f(k), \tag{5} $$

respectively, where $\alpha \in (n-1, n)$, $n \in \mathbb{Z}_+$, $k \in \mathbb{N}_{a+1}$ and $a \in \mathbb{R}$.

Given the importance of Mittag–Leffler function in fractional calculus, the discrete Mittag–Leffler function given below becomes the key to describe the solutions of discrete fractional-order dynamic systems.

**Definition 4** The discrete Mittag–Leffler function is defined as [31]

$$ \mathcal{F}_{\alpha,\beta}(\lambda, k, a) \triangleq \sum_{i=0}^{+\infty} \lambda^i \frac{(k-a)^{\overline{i\alpha+\beta-1}}}{\Gamma(i\alpha+\beta)}, \tag{6} $$

where $\alpha > 0$, $\beta \in \mathbb{R}$, $k \in \mathbb{N}_a$ and $a \in \mathbb{R}$.

**Definition 5** Assume $f : \mathbb{N}_{a+1} \to \mathbb{R}$. Then the nabla Laplace transform ($\mathcal{N}$-transform) is [31]

$$ \mathcal{N}_a \{f(k)\} \triangleq \sum_{k=1}^{+\infty} (1-s)^{k-1} f(k+a). \tag{7} $$

More exactly, such a transform should be called the sampling-free nabla Laplace transform, since the sampling period is assumed to be 1.

Moreover, the $\mathcal{N}$-transform of discrete Mittag–Leffler function is [31]

$$ \mathcal{N}_a \left\{ \mathcal{F}_{\alpha,\beta}(\lambda, k, a) \right\} = \frac{s^{\alpha-\beta}}{s^\alpha - \lambda}, \tag{8} $$

for $|1-s| < 1$, $|s^\alpha| > |\lambda|$. The reference [32] also shows that

$$ \begin{aligned} \mathcal{N}_a \left\{ {}_a^{\mathrm{C}}\nabla_k^{\alpha} f(k) \right\} =& s^\alpha \mathcal{N}_a\{f(k)\} \\ & - \sum_{i=0}^{n-1} s^{\alpha-i-1} \nabla^i f(k) |_{k=a}, \end{aligned} \tag{9} $$

where $\alpha \in (n-1, n)$ and $n \in \mathbb{Z}_+$.

*Remark 1* The above definitions clarify the basic principles of nabla discrete fractional calculus, which lay a foundation for the subsequent work of this paper. It is worth noting that an inseparable relationship exists between discrete Mittag–Leffler function and solutions of discrete fractional-order systems. In the following, the $\mathcal{N}$-transform becomes an important tool to analyze this problem.

## 3 Main results

### 3.1 Nabla discrete fractional-order dynamic systems

We study the general nabla discrete fractional-order nonlinear dynamic systems, given by

$$ {}_a\nabla_k^{\alpha} x(k) = f(k, x), \tag{10} $$

where $\nabla$ signifies either Riemann–Liouville or Caputo fractional difference operator, $x(a)$ is initial condition, $0 < \alpha < 1$, $k \in \mathbb{N}_{a+1}$, $f : [a+1, \infty] \times \Omega \to \mathbb{R}^n$ is locally Lipschitz in $x$ and $\Omega \to \mathbb{R}^n$ is state space containing the $x_0 = 0$. The equilibrium point for system (10) is given in the following definition.

**Definition 6** For discrete fractional-order system (10), the constant $x_0$ is said to be an equilibrium point if and only if $f(k, x_0) = {}_a\nabla_k^{\alpha} x_0$, $k \in \mathbb{N}_{a+1}$.

*Remark 2* No loss of generality occurs when $x_0 = 0$ is considered as the equilibrium point in all of our claims. For example, assume that $\bar{x} \neq 0$ is the equilibrium point and take a variable substitution $y(k) = x(k) - \bar{x}$. Thus, the $\alpha$-th Caputo fractional difference of $y(k)$ can be obtained as

$$ \begin{aligned} {}_a^{\mathrm{C}}\nabla_k^{\alpha} y(k) &= {}_a^{\mathrm{C}}\nabla_k^{\alpha} (x(k) - \bar{x}) = f(k, x) \\ &= f(k, y + \bar{x}) \\ &= g(k, y). \end{aligned} $$

Hence, $g(k, 0) = 0$ and for the system with respect to the new variable $y$, the origin is an equilibrium point.

On the other hand, the $\alpha$-th Riemann–Liouville difference of $y(k)$ can be obtained as

$$\begin{aligned} {}_a^{\mathrm{R}}\nabla_k^{\alpha} y(k) &= {}_a^{\mathrm{R}}\nabla_k^{\alpha}(x(k)-\bar{x}) = f(k,x) - \bar{x}\tfrac{(k-a)^{\overline{-\alpha}}}{\Gamma(1-\alpha)} \\ &= f(k, y+\bar{x}) - \bar{x}\tfrac{(k-a)^{\overline{-\alpha}}}{\Gamma(1-\alpha)} \\ &= \bar{g}(k,y). \end{aligned}$$

Similarly, $\bar{g}(k,0)=0$ and for the system with respect to the new variable $y$, the origin is an equilibrium point. Consequently, considering origin as the equilibrium point is not a loss of generality under either Caputo or Riemann–Liouville definition.

Generally speaking, one can conclude that the solution to system (10) exists and is unique if $f(k,x)$ is locally Lipschitz in $x$ and locally bounded [33]. Specifically, Caputo discrete fractional nonautonomous system (10) and the Lipschitz condition have some internal relationships. These will lead to the discrete Mittag–Leffler stability definition.

**Lemma 1** $\left\|{}_a^{\mathrm{R}}\nabla_k^{-\alpha} f(k,x(k))\right\| \le {}_a^{\mathrm{R}}\nabla_k^{-\alpha}\|f(k,x(k))\|$ *holds for* $f(k,x)$ *in* (10)*, where* $\|\cdot\|$ *denotes an arbitrary norm and* $\alpha \ge 0$.

*Proof* Bearing in mind (3) and the properties of norm, we get

$$\begin{aligned} \left\|{}_a^{\mathrm{R}}\nabla_k^{-\alpha} f(k,x(k))\right\| &= \left\|\sum_{j=a}^{k}\tfrac{(k-j+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)} f(j,x(j))\right\| \\ &\le \sum_{j=a}^{k}\tfrac{(k-j+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)}\|f(j,x(j))\| \\ &= {}_a^{\mathrm{R}}\nabla_k^{-\alpha}\|f(k,x(k))\|, \end{aligned}$$

in which we use the fact that $\frac{(k-j+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)} \ge 0$ and complete the proof.

**Theorem 1** *In* (10)*, let* ${}_a\nabla_k^{\alpha} x(k) = {}_a^{\mathrm{C}}\nabla_k^{\alpha} f(k)$*. If* $f(\cdot)$ *is locally Lipschitz on* $x$ *with Lipschitz constant* $\lambda$ *and is locally bounded, then* $\|x(k)\| \le \|x(a)\|\mathcal{F}_{\alpha,1}(\lambda,k,a)$*, where* $\alpha \in (0,1)$.

*Proof* Applying the fractional sum operator ${}_a^{\mathrm{R}}\nabla_k^{-\alpha}$ from the left and right of (10), respectively, the following holds utilizing Lemma 1 and the Lipschitz condition

$$\begin{aligned} \big|\|x(k)\| - \|x(a)\|\big| &\le \|x(k)-x(a)\| \\ &= \|{}_a^{\mathrm{R}}\nabla_k^{-\alpha} f(k,x(k))\| \\ &\le {}_a^{\mathrm{R}}\nabla_k^{-\alpha}\|f(k,x(k))\| \\ &\le \lambda\,{}_a^{\mathrm{R}}\nabla_k^{-\alpha}\|x(k)\|. \end{aligned}$$

A nonnegative function $z(k)$ exists and further yields

$$\|x(k)\| - \|x(a)\| = \lambda\,{}_a^{\mathrm{R}}\nabla_k^{-\alpha}\|x(k)\| - z(k). \tag{11}$$

Moreover, taking the $\mathcal{N}$-transform to (11) gives

$$\|X(s)\| = \frac{\|x(a)\|s^{\alpha-1} - s^{\alpha}Z(s)}{s^{\alpha}-\lambda}, \tag{12}$$

where $\|X(s)\| = \mathcal{N}\{\|x(k)\|\}$. Thus, we have $\|x(k)\| = \|x(a)\|\mathcal{F}_{\alpha,1}(\lambda,k,a) - z(k) * \mathcal{F}_{\alpha,0}(\lambda,k,a)$ by taking the inverse $\mathcal{N}$-transform to (12), where $*$ denotes the nabla convolution operator, i.e.,
$x(k) * y(k) \triangleq \sum_{j=a+1}^{k} x(k-j+a+1)\,y(j)$.
Considering the $z(k) * \mathcal{F}_{\alpha,0}(\lambda,k,a) \ge 0$ due to the fact that $\mathcal{F}_{\alpha,0}(\lambda,k,a) \ge 0$, it then follows that $\|x(k)\| \le \|x(a)\|\mathcal{F}_{\alpha,1}(\lambda,k,a)$, which completes the proof.

### 3.2 Discrete Mittag–Leffler stability

Stability analysis is the most important index in systems and control. Lyapunov direct method plays a significant role in stability analysis which is known that if a Lyapunov candidate function exists, then the system is stable. Next, for discrete fractional-order nonlinear systems, we first define discrete Mittag–Leffler stability to successfully apply fractional direct Lyapunov method.

**Definition 7** (Definition of discrete Mittag–Leffler stability). System (10) is called Mittag–Leffler stable if

$$\|x(k)\| \le \left\{m(x(a))\mathcal{F}_{\alpha,\beta}(\lambda,k,a)\right\}^{b}, \tag{13}$$

where $b > 0$, $\alpha \in (0,1)$, $m(x)$ is locally Lipschitz on $x$ with Lipschitz constant $m_0$, $m(0)=0$ and $m(x) \ge 0$. If inequality (13) is satisfied for any initial state $x(a)$, then system (10) is globally Mittag–Leffler stable.

### 3.3 Lyapunov direct method

As a fractional extension of exponential stability, the proposed Mittag–Leffler stability can be used to distinguish asymptotic stability and analyze the convergence rule of the pseudostates for corresponding discrete fractional-order systems. In the present study, fractional direct Lyapunov method is extended from continuous case to discrete case and the conditions for achieving Mittag–Leffler stability are given.

**Theorem 2** *Let $V(k, x(k)) : [a+1, \infty) \times \mathbb{D} \to \mathbb{R}$ be locally bounded and locally Lipschitz on $x$, where $\mathbb{D} \subset \mathbb{R}^n$ is state space containing the origin. Assuming that*

$$\alpha_1 \|x(k)\|^b \leq V(k, x(k)) \leq \alpha_2 \|x(k)\|^{bc}, \tag{14}$$
$${}_a^{\mathrm{C}}\nabla_k^{\beta} V(k, x(k)) \leq -\alpha_3 \|x(k)\|^{bc}, \tag{15}$$

*where $k \in \mathbb{N}_{a+1}$, $\beta \in (0,1)$, $x \in \mathbb{D}$ and $\alpha_1, \alpha_2, \alpha_3, b, c > 0$, then the equilibrium point $x(k) = 0$ is Mittag–Leffler stable. Moreover, $x(k) = 0$ is globally Mittag–Leffler stable if the assumptions hold globally on $\mathbb{R}^n$.*

*Proof* It is known from (14) and (15) that

$${}_a^{\mathrm{C}}\nabla_k^{\beta} V(k, x(k)) \leq -\alpha_3 \alpha_2^{-1} V(k, x(k)).$$

A nonnegative function $z(k)$ exists and satisfies

$${}_a^{\mathrm{C}}\nabla_k^{\beta} V(k, x(k)) + z(k) = -\alpha_3 \alpha_2^{-1} V(k, x(k)). \tag{16}$$

Taking the $\mathcal{N}$-transform of (16) yields

$$s^{\beta} V_f(s) - V(a, x(a)) s^{\beta-1} + Z(s) = -\alpha_3 \alpha_2^{-1} V_f(s),$$

where $V_f(s) = \mathcal{N}\{V(k, x(k))\}$. Consequently, we have

$$V_f(s) = \frac{V(a, x(a)) s^{\beta-1} - Z(s)}{s^{\beta} + \alpha_3 \alpha_2^{-1}}.$$

If $x(a) = 0$, namely $V(a, x(a)) = 0$, then $x = 0$ is the solution of (10). If $x(a) \neq 0$, $V(a, x(a)) \geq 0$. Let $\lambda = -\alpha_3 \alpha_2^{-1}$. Based on the Lipschitz condition and inverse $\mathcal{N}$-transform, it is known that

$$\begin{aligned} V(k, x(k)) = & V(a, x(a)) \mathcal{F}_{\beta,1}(\lambda, k, a) \\ & - z(k) * \mathcal{F}_{\beta,\beta}(\lambda, k, a) \end{aligned}$$

is the unique solution of (16). Given $\mathcal{F}_{\beta,\beta}(\lambda, k, a)$ is positive, then

$$V(k, x(k)) \leq V(a, x(a)) \mathcal{F}_{\beta,1}(\lambda, k, a). \tag{17}$$

Substituting (17) into (14) gives

$$\|x(k)\| \leq \left[\frac{V(a,x(a))}{\alpha_1} \mathcal{F}_{\beta,1}(\lambda, k, a)\right]^{\frac{1}{b}}.$$

Let $m = \frac{V(a,x(a))}{\alpha_1} \geq 0$, where $m = 0$ holds if and only if $x(a) = 0$. Given that $V(k, x(k))$ is positive definite and monotonically decrescent, and $V(a, x(a)) = 0$ if and only if $x(a) = 0$, it is known that $m(0) = 0$ and $m$ is also Lipschitz. Consequently, system (10) is Mittag–Leffler stable which completes the proof.

Next, considering the connection between Caputo and Riemann–Liouville difference operators, Theorem 2 is further extended to Riemann–Liouville definition. The following lemma is essential.

**Lemma 2** ${}_a^{\mathrm{C}}\nabla_k^{\beta} M(k) \leq {}_a^{\mathrm{R}}\nabla_k^{\beta} M(k)$ *holds for* $\beta \in (0,1)$ *and* $M(a) \geq 0$.

*Proof* From the definitions of fractional operators given before, we have

$${}_a^{\mathrm{C}}\nabla_k^{\beta} M(k) = {}_a^{\mathrm{R}}\nabla_k^{\beta} M(k) - \tfrac{(k-a)^{\overline{-\beta}}}{\Gamma(1-\beta)} M(a),$$

where $\beta \in (0,1)$ and $M(a) \geq 0$. Considering that $\frac{(k-a)^{\overline{-\beta}}}{\Gamma(1-\beta)} \geq 0$, hence, ${}_a^{\mathrm{C}}\nabla_k^{\beta} M(k) \leq {}_a^{\mathrm{R}}\nabla_k^{\beta} M(k)$, which has completed the proof.

**Theorem 3** *In the case of Riemann–Liouville definition, let $V(k, x(k)) : [a+1, \infty) \times \mathbb{D} \to \mathbb{R}$ be locally bounded and locally Lipschitz on $x$, where $\mathbb{D} \subset \mathbb{R}^n$ is state space containing the origin. Assuming that*

$$\alpha_1 \|x(k)\|^b \leq V(k, x(k)) \leq \alpha_2 \|x(k)\|^{bc}, \tag{18}$$
$${}_a^{\mathrm{R}}\nabla_k^{\beta} V(k, x(k)) \leq -\alpha_3 \|x(k)\|^{bc}, \tag{19}$$

*where $k \in \mathbb{N}_{a+1}$, $\beta \in (0,1)$, $x \in \mathbb{D}$ and $\alpha_1, \alpha_2, \alpha_3, b, c > 0$, then one has $\|x(k)\| \leq \left[\frac{V(a,x(a))}{\alpha_1} \mathcal{F}_{\beta,1}(\lambda, k, a)\right]^{\frac{1}{b}}$ and the equilibrium point $x(k) = 0$ is Mittag–Leffler stable. Moreover, $x(k) = 0$ is globally Mittag–Leffler stable if the assumptions hold globally on $\mathbb{R}^n$.*

*Proof* Considering that $V(k, x(k)) \geq 0$ and Lemma 2, we have

$${}_a^{\mathrm{C}}\nabla_k^{\beta} V(k, x(k)) \leq {}_a^{\mathrm{R}}\nabla_k^{\beta} V(k, x(k)),$$

which implies ${}_a^{\mathrm{C}}\nabla_k^{\beta} V(k, x(k)) \leq -\alpha_3 \|x\|^{bc}$. The following proof is the same as Theorem 2.

The convergence rule of the pseudostates can be described in terms of Mittag–Leffler stability, implying asymptotic stability for discrete fractional-order systems. Next, using class–$\mathcal{K}$ functions, a new criterion is provided to distinguish asymptotic stability of system (10) by Lyapunov direct method. Preceding to our main results, the discrete fractional comparison principle and definition of discrete class–$\mathcal{K}$ function are given.

**Lemma 3** *Let $x(a) = y(a)$ and ${}_a^{\mathrm{C}}\nabla_k^{\beta} x(k) \geq {}_a^{\mathrm{C}}\nabla_k^{\beta} y(k)$ when $\beta \in (0,1)$. Then $x(k) \geq y(k)$.*

*Proof* Considering that ${}_a^{\mathrm{C}}\nabla_k^{\beta} x(k) \geq {}_a^{\mathrm{C}}\nabla_k^{\beta} y(k)$, a non-negative function $z(k)$ exists and satisfies

$$ {}_a^{\mathrm{C}}\nabla_k^{\beta} x(k) = z(k) + {}_a^{\mathrm{C}}\nabla_k^{\beta} y(k). \tag{20} $$

Taking the $\mathcal{N}$-transform of (20), one has $s^{\beta}X(s) - s^{\beta-1}x(a) = Z(s) + s^{\beta}Y(s) - s^{\beta-1}y(a)$. Hence, $X(s) = s^{-\beta}Z(s) + Y(s)$ on account of $x(a) = y(a)$. Applying the inverse $\mathcal{N}$-transform we can get $x(k) = {}_a^{\mathrm{R}}\nabla_k^{-\beta} z(k) + y(k)$. It is known from $z(k) \geq 0$ and (2) that $x(k) \geq y(k)$.

**Definition 8** (Definition of discrete class–$\mathcal{K}$ function [34]). A function $\gamma : [0, k) \to [0, \infty)$ belongs to the class–$\mathcal{K}$ if and only if it is strictly monotonically increasing and $\gamma(0) = 0$.

**Theorem 4** *For Caputo discrete fractional-order system (10), the origin is an equilibrium point. Suppose that a positive definite and monotonically decrescent Lyapunov function $V(k, x(k))$ exists and satisfies*

$$ \gamma_1(\|x(k)\|) \leq V(k, x(k)) \leq \gamma_2(\|x(k)\|) \tag{21} $$

*and*

$$ {}_a^{\mathrm{C}}\nabla_k^{\beta} V(k, x(k)) \leq -\gamma_3(\|x(k)\|), \tag{22} $$

*where $\beta \in (0, 1)$ and $\gamma_1, \gamma_2, \gamma_3$ are discrete class–$\mathcal{K}$ functions. Then system (10) is asymptotically stable.*

*Proof* By combining (21) and (22) the following holds

$$ {}_a^{\mathrm{C}}\nabla_k^{\beta} V(k, x(k)) \leq -\gamma_3\left(\gamma_2^{-1}(V(k, x(k)))\right). \tag{23} $$

Owing to $V(k, x(k))$ is positive definite and monotonically decrescent, we can conclude that $V(a, x(a)) \geq 0$ and $V(k, x(k)) \leq V(a, x(a))$ naturally.
**Case 1:** Assume that a constant $k_1 \geq a$ exists and satisfies $V(k_1, x(k_1)) = 0$, which means $x(k_1) = 0$. We can conclude that $x(k) = 0$ for $k \geq k_1$ and $x = 0$ is the equilibrium point.
**Case 2:** Suppose that a constant $\varepsilon > 0$ exists and $V(k, x(k)) \geq \varepsilon$ for $k \geq a$. Then

$$ 0 < \varepsilon \leq V(k, x(k)) \leq V(a, x(a)), \ k \geq a. \tag{24} $$

Substituting (24) into (23) yields

$$ \begin{aligned} -\gamma_3\left(\gamma_2^{-1}(V(k, x(k)))\right) &\leq -\gamma_3\left(\gamma_2^{-1}(\varepsilon)\right) \\ &= -\frac{\gamma_3\left(\gamma_2^{-1}(\varepsilon)\right)}{V(a, x(a))} V(a, x(a)) \\ &\leq -lV(k, x(k)), \end{aligned} $$

where $0 < l = \frac{\gamma_3\left(\gamma_2^{-1}(\varepsilon)\right)}{V(a, x(a))}$. Therefore,

$$ \begin{aligned} {}_a^{\mathrm{C}}\nabla_k^{\beta} V(k, x(k)) &\leq -\gamma_3\left(\gamma_2^{-1}(V(k, x(k)))\right) \\ &\leq -lV(k, x(k)). \end{aligned} $$

Consequently, $V(k, x(k)) \leq V(a, x(a))\mathcal{F}_{\beta,1}(-l, k, a)$ following the same proof of Theorem 2, which is contradicted with $V(k, x(k)) \geq \varepsilon$.

Combining the conclusions of **Case 1** and **Case 2**, one has $\lim_{k\to\infty} V(k, x(k)) = 0$. Hence, $\lim_{k\to\infty} x(k) = 0$ according to (21) which implies system (10) is asymptotically stable.

*Remark 3* Using class-$\mathcal{K}$ functions, Theorem 4 presents the discrete fractional direct Lyapunov method. Stability of Caputo or Riemann–Liouville discrete fractional-order systems is uniformly described employing the proposed approach. In addition, given that fractional difference is difficult to be obtained directly, analyzing the stability of such systems in combination with some useful inequalities is necessary in practice. The following theorem provides a useful and rather general inequality which makes the fractional direct Lyapunov method more applicable.

**Theorem 5** *If the Lyapunov function $V(k, x(k))$ is convex and differentiable with respect to $x$, then the following inequality holds for all $k \in \mathbb{N}_{a+1}$*
$$ {}_a^{\mathrm{C}}\nabla_k^{\alpha} V(k, x(k)) \leq \frac{\partial V(k,x(k))}{\partial x(k)} {}_a^{\mathrm{C}}\nabla_k^{\alpha} x(k), $$
*where $\alpha \in (0, 1)$, $x \in \mathbb{R}$, and $\frac{\partial V(k,x(k))}{\partial x(k)}$ is the partial differential of the Lyapunov function $V(k, x(k))$.*

*Proof* Recalling the definitions of fractional difference and sum given before, we have

$$ \begin{aligned} {}_a^{\mathrm{R}}\nabla_k^{\alpha} x(k) &= \nabla^{1} {}_a^{\mathrm{R}}\nabla_k^{\alpha-1} x(k) = \tfrac{(k-a)^{\overline{-\alpha}}}{\Gamma(1-\alpha)} x(k) \\ &\quad - \textstyle\sum_{j=a+1}^{k} \tfrac{(k-j+1)^{\overline{-\alpha-1}}}{\Gamma(-\alpha)} [x(k) - x(j)]. \end{aligned} $$

Then, according to the relationship between Caputo and Riemann–Liouville difference, we can obtain

$$ \begin{aligned} {}_a^{\mathrm{C}}\nabla_k^{\alpha} x(k) &= \tfrac{(k-a)^{\overline{-\alpha}}}{\Gamma(1-\alpha)} [x(k) - x(a)] \\ &\quad - \textstyle\sum_{j=a+1}^{k} \tfrac{(k-j+1)^{\overline{-\alpha-1}}}{\Gamma(-\alpha)} [x(k) - x(j)]. \end{aligned} $$

Similarly, ${}_{a}^{\mathrm{C}}\nabla_{k}^{\alpha} V\left(k, x\left(k\right)\right)$ can be obtained as

$$
\begin{aligned}
&{}_{a}^{\mathrm{C}}\nabla_{k}^{\alpha} V\left(k, x\left(k\right)\right) \\
&= \tfrac{(k-a)^{\overline{-\alpha}}}{\Gamma(1-\alpha)}\left[V\left(k, x\left(k\right)\right) - V\left(a, x\left(a\right)\right)\right] \\
&- \textstyle\sum_{j=a+1}^{k} \tfrac{(k-j+1)^{\overline{-\alpha-1}}}{\Gamma(-\alpha)}\left[V\left(k, x\left(k\right)\right) - V\left(j, x\left(j\right)\right)\right].
\end{aligned}
$$

We can further calculate that

$$
\begin{aligned}
&{}_{a}^{\mathrm{C}}\nabla_{k}^{\alpha} V\left(k, x\left(k\right)\right) - \tfrac{\partial V(k,x(k))}{\partial x(k)}\,{}_{a}^{\mathrm{C}}\nabla_{k}^{\alpha} x\left(k\right) \\
&= \tfrac{(k-a)^{\overline{-\alpha}}}{\Gamma(1-\alpha)} \\
&\left\{V\left(k, x\left(k\right)\right) - V\left(a, x\left(a\right)\right) - \tfrac{\partial V(k,x(k))}{\partial x(k)}\left[x\left(k\right) - x\left(a\right)\right]\right\} \\
&- \textstyle\sum_{j=a+1}^{k} \tfrac{(k-j+1)^{\overline{-\alpha-1}}}{\Gamma(-\alpha)} \\
&\left\{V\left(k, x\left(k\right)\right) - V\left(j, x\left(j\right)\right) - \tfrac{\partial V(k,x(k))}{\partial x(k)}\left[x\left(k\right) - x\left(j\right)\right]\right\}.
\end{aligned}
$$

Given that $V\left(k, x\left(k\right)\right)$ is convex and differentiable with respect to $x$, we have

$$
\begin{aligned}
&V\left(k, x\left(k\right)\right) - V\left(a, x\left(a\right)\right) \\
&\qquad - \tfrac{\partial V(k,x(k))}{\partial x(k)}\left[x\left(k\right) - x\left(a\right)\right] \le 0
\end{aligned}
$$

and

$$
\begin{aligned}
&V\left(k, x\left(k\right)\right) - V\left(j, x\left(j\right)\right) \\
&\qquad - \tfrac{\partial V(k,x(k))}{\partial x(k)}\left[x\left(k\right) - x\left(j\right)\right] \le 0.
\end{aligned}
$$

Consequently, it is known that

$$
{}_{a}^{\mathrm{C}}\nabla_{k}^{\alpha} V\left(k, x\left(k\right)\right) - \tfrac{\partial V(k,x(k))}{\partial x(k)}\,{}_{a}^{\mathrm{C}}\nabla_{k}^{\alpha} x\left(k\right) \le 0
$$

due to $\frac{(k-a)^{\overline{-\alpha}}}{\Gamma(1-\alpha)} > 0$ and $-\frac{(k-j+1)^{\overline{-\alpha-1}}}{\Gamma(-\alpha)} > 0$, which has completed the proof.

*Remark 4* Theorem 5 gives a useful inequality to further improve the practicality of the discrete fractional direct Lyapunov method. This inequality is also applicable to the vector case, i.e., $x \in \mathbb{R}^{n}$, and more general compared with those special cases proposed by Wei et al. in [30]. Specifically speaking, inequalities (16), (17), and (18) of Theorem 1 in [30] can be obtained directly using proposed Theorem 5. Next, let $y\left(k\right) = x^{m}\left(k\right)$, it is known from Theorem 5 that

$$
\begin{aligned}
{}_{a}^{\mathrm{C}}\nabla_{k}^{\alpha} x^{2m}\left(k\right) &= {}_{a}^{\mathrm{C}}\nabla_{k}^{\alpha} y^{2}\left(k\right) \le 2y\left(k\right){}_{a}^{\mathrm{C}}\nabla_{k}^{\alpha} y\left(k\right) \\
&= 2x^{m}\left(k\right){}_{a}^{\mathrm{C}}\nabla_{k}^{\alpha} x^{m}\left(k\right),
\end{aligned}
$$

which is consistent with inequality (14) of Theorem 1 in [30]. Similarly, for inequality (15), let $y\left(k\right) = x^{\frac{2m}{n}-1}\left(k\right)$, we can get the same result as follows

$$
\begin{aligned}
{}_{a}^{\mathrm{C}}\nabla_{k}^{\alpha} x^{\frac{2m}{n}}\left(k\right) &= {}_{a}^{\mathrm{C}}\nabla_{k}^{\alpha} y^{\frac{2m}{2m-n}}\left(k\right) \\
&\le \frac{2m}{2m-n} y^{\frac{n}{2m-n}}\,{}_{a}^{\mathrm{C}}\nabla_{k}^{\alpha} y\left(k\right) \\
&= \frac{2m}{2m-n} x\left(k\right){}_{a}^{\mathrm{C}}\nabla_{k}^{\alpha} x^{\frac{2m}{n}-1}\left(k\right).
\end{aligned}
$$

Thus, Theorem 1 in [30] can be concluded as a special case of Theorem 5 in this study. Moreover, presented Theorem 5 only requires the Lyapunov function to be convex and differentiable, thereby avoiding other very restrictive assumptions.

*Remark 5* By using the criteria proposed in this study, the stability of general discrete fractional-order systems can be determined, either under the Caputo or Riemann–Liouville definition. Then, the inequality in Theorem 5 makes the presented methods more convenient to use. At this point, the basic framework of analyzing the stability of discrete fractional-order systems by Lyapunov direct method is established. Compared with the previous work, the proposed methods adopt the backward difference instead of the noncausal forward difference and the time series is no longer dependent on the order of the system. Given the initial instant is no longer restricted to 0, the presented results are relatively general.

## 4 Illustrative examples

The correctness and practicability of the proposed methods are demonstrated by some detailed simulations in this section. To conveniently conclude this study, we use the inequalities mentioned in the literature [30].

*Example 1* The Caputo discrete fractional-order system is described by

$$
\begin{cases}
{}_{a}^{\mathrm{C}}\nabla_{k}^{\alpha} x_{1}\left(k\right) = -x_{1}\left(k\right) + x_{2}^{3}\left(k\right), \\
{}_{a}^{\mathrm{C}}\nabla_{k}^{\alpha} x_{2}\left(k\right) = -x_{1}^{3}\left(k\right) - x_{2}\left(k\right),
\end{cases} \tag{25}
$$

where $0 < \alpha < 1$. Let the Lyapunov candidate function be $V\left(k, x\left(k\right)\right) = x_{1}^{4}\left(k\right) + x_{2}^{4}\left(k\right)$. Utilizing the inequalities of Theorem 1 in [30], we calculate that

$$
\begin{aligned}
{}_{a}^{\mathrm{C}}\nabla_{k}^{\alpha} V\left(k, x\left(k\right)\right) =& {}_{a}^{\mathrm{C}}\nabla_{k}^{\alpha}\left(x_{1}^{4}\left(k\right) + x_{2}^{4}\left(k\right)\right) \\
\le& \, 4x_{1}^{3}\left(k\right){}_{a}^{\mathrm{C}}\nabla_{k}^{\alpha} x_{1}\left(k\right) \\
&+ 4x_{2}^{3}\left(k\right){}_{a}^{\mathrm{C}}\nabla_{k}^{\alpha} x_{2}\left(k\right) \\
=& -4x_{1}^{4}\left(k\right) - 4x_{2}^{4}\left(k\right) \\
=& -4V\left(k, x\left(k\right)\right).
\end{aligned}
$$

The same result can be obtained by using Theorem 5. Consequently, the system in (25) is Mittag–Leffler stable according to Theorem 2.

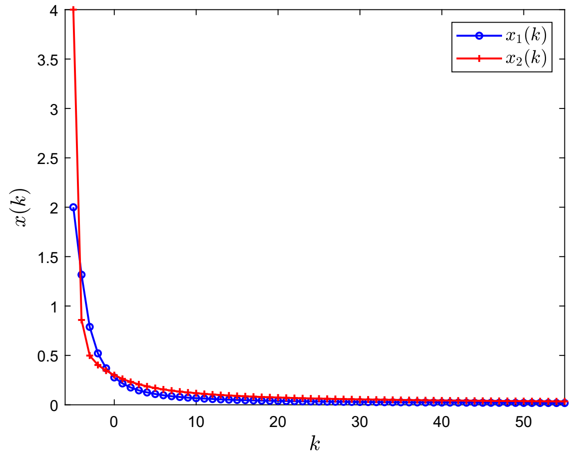


**Fig. 1** State response of the system in (25)

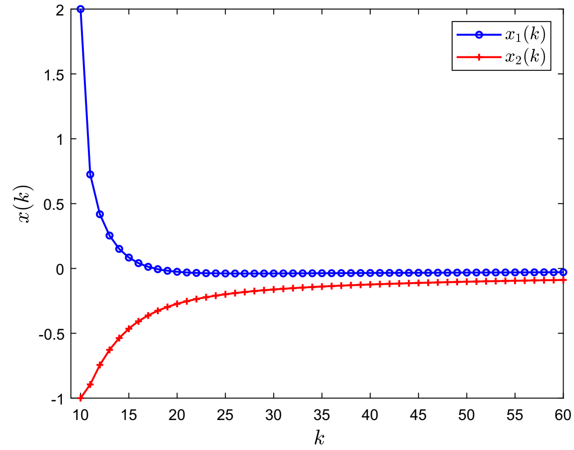


**Fig. 2** State response of the system in (26)

The parameters of system (25) are set as follows, $\alpha = 0.8, a = -5, x_1(-5) = 2, x_2(-5) = 4$. Clearly, Fig. 1 shows the asymptotic stability of system (25).

*Example 2* To fully illustrate the practicability of Theorem 4, we analyze the following Caputo discrete nonlinear system

$$\begin{cases} {}_a^{\mathrm{C}}\nabla_k^{\alpha} x_1(k) = -x_1^3(k) + x_2(k), \\ {}_a^{\mathrm{C}}\nabla_k^{\alpha} x_2(k) = -x_1(k) - x_2^{\frac{5}{3}}(k). \end{cases} \tag{26}$$

Now, let the Lyapunov candidate function be

$$V(k, x(k)) = x_1^2(k) + x_2^2(k).$$

Using the inequalities of Theorem 1 in [30], then

$$\begin{aligned} {}_a^{\mathrm{C}}\nabla_k^{\alpha} V(k, x(k)) =& {}_a^{\mathrm{C}}\nabla_k^{\alpha}\left(x_1^2(k) + x_2^2(k)\right) \\ \le& \, 2x_1(k)\, {}_a^{\mathrm{C}}\nabla_k^{\alpha} x_1(k) \\ &+ 2x_2(k)\, {}_a^{\mathrm{C}}\nabla_k^{\alpha} x_2(k) \\ =& -2x_1^4(k) - 2x_2^{\frac{8}{3}}(k) \\ \le& 0. \end{aligned}$$

The same conclusion can be generated by Theorem 5. Hence, system (26) is asymptotically stable by applying Theorem 4.

It is straightforward to see the evolution of the system states for $\alpha = 0.6$ when $a = 10, x_1(10) = 2$ and $x_2(10) = -1$ in Fig. 2. Similarly, we can get the asymptotic stability of system (26) which confirms the analytical analysis presented before.

*Example 3* To verify the correctness of Theorem 3 in Riemann–Liouville definition case, consider the following Riemann–Liouville discrete fractional-order nonlinear system

$${}_a^{\mathrm{R}}\nabla_k^{\alpha} x(k) = -x(k) - x^3(k). \tag{27}$$

$V(k, x(k)) = x^{\frac{4}{3}}(k)$ is the selected Lyapunov candidate function. Using the inequalities of Theorem 2 in [30], then

$$\begin{aligned} {}_a^{\mathrm{R}}\nabla_k^{\alpha} V(k, x(k)) &= {}_a^{\mathrm{R}}\nabla_k^{\alpha} x^{\frac{4}{3}}(k) \le \frac{4}{3} x^{\frac{1}{3}}(k)\, {}_a^{\mathrm{R}}\nabla_k^{\alpha} x(k) \\ &= -\frac{4}{3} x^{\frac{4}{3}}(k) - \frac{4}{3} x^{\frac{10}{3}}(k) \le -\frac{4}{3} x^{\frac{4}{3}}(k) \\ &= -\frac{4}{3} V(k, x(k)). \end{aligned}$$

Thus, system (27) is Mittag–Leffler stable applying Theorem 3.

Figure 3 presents the state response for $\alpha = 0.3, 0.6$, and $0.8$, respectively, when $a = 5$, which implies asymptotic stability of system (27) as anticipated from the analysis.

For the case of Riemann–Liouville definition, it is worth noting that the initial value, i.e., $x(a)$, is infinite. To describe the evolution of system state more intuitively, Fig. 4 shows the $(1 - \alpha)$-th fractional sum of corresponding state response, which is obtained by summing over a finite number of terms here.

*Example 4* We say that the inequality in Theorem 5 is more general than those existing results. To illustrate

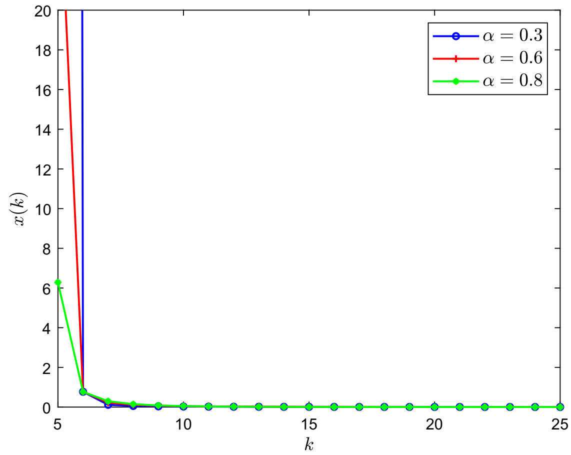


**Fig. 3** State response of the system in (27)

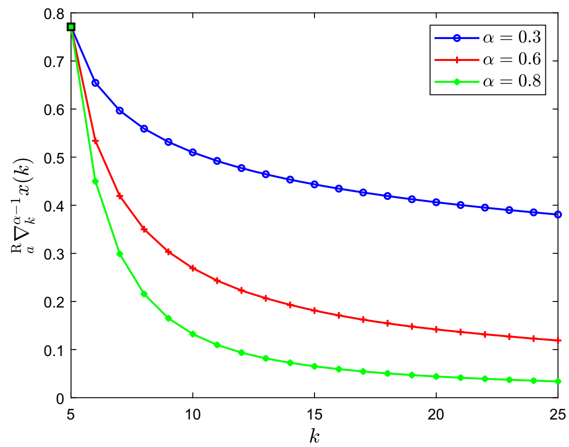


**Fig. 4** $(1-\alpha)$-th sum of state response

this, consider a Caputo discrete system described by

$$\begin{cases} {}_{a}^{\mathrm{C}}\nabla_k^{\alpha} x_1(k) = x_2(k), \\ {}_{a}^{\mathrm{C}}\nabla_k^{\alpha} x_2(k) = -\sin(x_1(k)) - x_2(k), \end{cases} \tag{28}$$

where $\alpha = 0.8$, $a = 8$, $x_1(8) = 2$, $x_2(8) = -1$ and $-\pi \le x_1(k) \le \pi$. Then, choose a Lyapunov candidate function

$$\begin{aligned} V(k, x(k)) =& \frac{1}{2}x_2^2(k) + \frac{1}{2}[x_1(k) + x_2(k)]^2 \\ &+ 2[1 - \cos(x_1(k))]. \end{aligned}$$

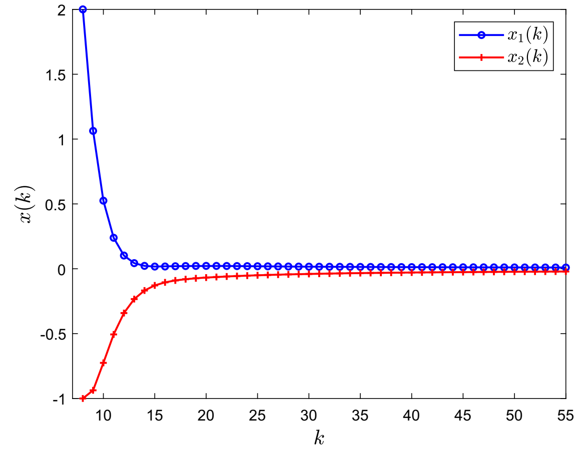


**Fig. 5** State response of the system in (28)

Taking the inequality in Theorem 5 into account, we arrive at

$$\begin{aligned} {}_{a}^{\mathrm{C}}\nabla_k^{\alpha} V(k, x(k)) \le& x_2(k)\, {}_{a}^{\mathrm{C}}\nabla_k^{\alpha} x_2(k) \\ &+ [x_1(k)+x_2(k)]\, {}_{a}^{\mathrm{C}}\nabla_k^{\alpha} [x_1(k)+x_2(k)] \\ &+ 2\sin(x_1(k))\, {}_{a}^{\mathrm{C}}\nabla_k^{\alpha} x_1(k) \\ =& -x_2^2(k) - x_1(k)\sin(x_1(k)) \\ \le& 0. \end{aligned}$$

Accordingly, one gets the asymptotic stability of system (28) applying Theorem 4. The simulation results depicted in Fig. 5 show the evolution of system states clearly, which further confirms our claims.

## 5 Conclusions

In this study, a new definition of discrete Mittag–Leffler stability is proposed, producing a novel stability description of discrete fractional-order systems. Besides, using Lyapunov direct method, some useful criteria for analyzing stability of nabla discrete fractional-order systems are derived. The presented methods are applicable to both Caputo and Riemann–Liouville definitions, and a useful inequality is given to further improve the practicality of the discrete fractional direct Lyapunov method. Moreover, through some illustrative examples, it is shown that correctness and effectiveness of the given methods are verified. In future work, we will study the stability of specific discrete fractional-order systems which have not yet been fully developed, for instance, variable-order systems, distributed-order systems, and systems with order $\alpha > 1$.

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**Compliance with ethical standards**

**Conflicts of interest** The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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