EE5103 Computer Control Systems: Homework #3 Solution

Q1

The controller is required to be in form of

$$R(q)u(k) = T(q)u_c(k) - S(q)y(k), \tag{1}$$

whose z transform is

$$U(z) = \frac{T(z)}{R(z)}U_c(z) - \frac{S(z)}{R(z)}Y(z). \tag{2}$$

Therefore, the block diagram for this control system is

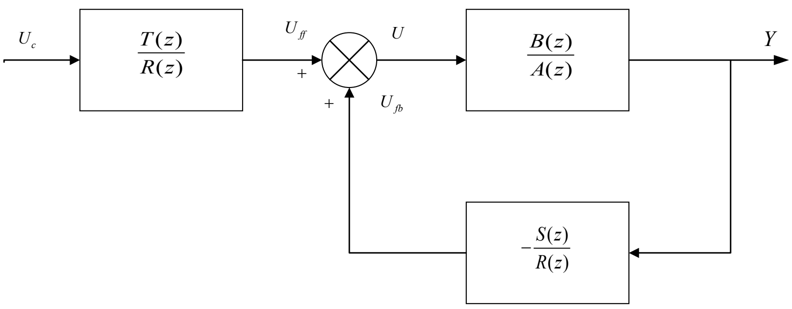

Figure 1 Block diagram of the closed-loop system

And we have

$$\frac{B(z)}{A(z)} = H(z) = \frac{z+0.9}{z^2 - 2.5z + 1}. \tag{3}$$

By designing the above two degree-of-freedom controller, we aim to obtain a closed-loop system with the following characteristic polynomial

$$A_m(z) = z^2 - 1.8z + 0.9. \tag{4}$$

To match the specified closed-loop denominator (poles), we mainly need to manipulate the feedback term $S(z)/R(z)$ since the closed-loop transfer function caused from the control law (2) is computed to be

$$\frac{Y(z)}{U_c(z)} = \frac{B(z)T(z)}{A(z)R(z)+B(z)S(z)} = \frac{B(z)T(z)}{A_{cl}(z)}. \tag{5}$$

Now the question is: what is the closed-loop polynomial $A_{cl}(z)$ in (5)? Since we are required to achieve $A_m(z)$ as the closed-loop denominator at the end, $A_{cl}(z)$ must originally be in some form of $A_{cl}(z) = A_o(z)A_m(z)$ and then reduced to $A_m(z)$ by cancelling $A_o(z)$. Now the remaining task is to determine $A_o(z)$. Generally, $A_o(z)$ may stem simply from polynomial order matching (when $A_{cl}(z)$ has a higher order than the required $A_m(z)$) or from additional requirements like zero cancellation, disturbance rejection etc. This is a basic principle for model matching of lecture 5, which applies to all the problems in this assignment.

a)

If the process zero is cancelled, that is, there is no $B(z)$ in the final form of (5), then it can be inferred that the closed-loop denominator before cancellation is

$$A_{cl}(z) = B(z)A_m(z) = (z+0.9)A_m(z) = (z+0.9)(z^2-1.8z+0.9) \tag{6}$$

Obviously, we have $\deg(R(z)) = \deg(A_{cl}(z)) - \deg(A(z)) = 1$. Therefore, the general form of $R(z) = r_0 z + r_1$. Additionally, since there is a factor $z+0.9$ in $A_{cl}(z)$, the $R(z)$ must be in this form $R(z) = r(z+0.9)$. Further, suppose $S(z) = s_o z + s_1$ due to $\deg(S(z)) \le \deg(R(z))$, then we have

$$\begin{aligned} A_{cl}(z) &= A(z)R(z) + B(z)S(z) \\ &= (z^2-2.5z+1)r(z+0.9) + (z+0.9)(s_o z + s_1) \\ &= (z+0.9)[rz^2 + (s_0 - 2.5r)z + r + s_1]. \end{aligned} \tag{7}$$

From (6) and (7), it can be inferred that

$$rz^2 + (s_0 - 2.5r)z + r + s_1 = z^2 - 1.8z + 0.9 \tag{8}$$

And it can be solved as $r = 1$ and $s_0 = 0.7$, $s_1 = -0.1$. Hence

$$\frac{S(z)}{R(z)} = \frac{0.7z - 0.1}{z+0.9} \tag{9}$$

- *Even you simply assume a general form $R(z) = r_0 z + r_1$, you can get the same result as above, though the computation may be a little more involved.*
- *If you notice that the leading coefficient is 1 in **Error! Reference source not found.**, you can directly infer that $r = 1$. Therefore, you can always work with the most general form, or you may continue by simplifying the general form through observing the specific structures. For the remaining questions, we usually simply choose the leading coefficient of R(z) to be 1 without further explanation since it is apparent that the leading coefficient of the desired $A_{cl}(z)$ is also 1 in the given questions.*

After we get $R(z)$ and $s(z)$, it is time now to choose $T(z)$. The requirement is that the steady state gain from $u_c(z)$ to $y(z)$ is one. For the closed-loop transfer function (5), the steady state gain can be computed by $Y(1)/U_c(1)$. We can assume a general form for $T(z) = t_0 z + t_1$ because of $\deg(T(z)) \le \deg(R(z))$. Then, we have

$$\frac{Y(1)}{U_c(1)} = \left.\frac{B(z)T(z)}{A_{cl}(z)}\right|_{z=1} = \frac{t_0 + t_1}{1 - 1.8 + 0.9} = 1, \tag{10}$$

which further gives

$$t_0 + t_1 = 0.1. \tag{11}$$

Thus $T(z)$ can have many solutions. The lowest order (0) one is

$$t_0 = 0, t_1 = 0.1 \Rightarrow T(z) = 0.1. \tag{12}$$

The whole controller is designed as follows:

$$\boxed{\begin{aligned} &R(z) = z + 0.9 \\ &S(z) = 0.7z - 0.1 \\ &T(z) = 0.1 \\ &U(z) = \frac{T(z)}{R(z)}U_c(z) - \frac{S(z)}{R(z)}Y(z) \end{aligned}} \tag{13}$$

In this case, from (5), the closed-loop transfer function is

$$G(z) = \frac{B_m(z)}{A_m(z)} = \frac{Y(z)}{U_c(z)} = \frac{0.1}{z^2 - 1.8z + 0.9}. \tag{14}$$

(Note: here you can choose other values for $T(z)$ once (10) is satisfied.)

b)

If the process zero is not cancelled, based on the above analysis, the order of the closed-loop characteristic polynomial is $\deg(A_{cl}(z)) = \deg(A(z)R(z) + B(z)S(z)) > \deg(A(z)) = 2$ since the order of $R(z)$ is at least one. Then, because the desired one is just $\deg(A_m(z)) = 2$, we have to include another auxiliary factor $A_o(z)$ such that the closed-loop characteristic polynomial can be matched as

$$A_{cl}(z) = A_o(z)A_m(z) = zA_m(z) = z(z^2 - 1.8z + 0.9) \tag{15}$$

Here we choose $A_o(z) = z$ because we want the lowest order one whose poles are all at the origin.

Now it can be inferred that $\deg(R(z)) = \deg(A_{cl}(z)) - \deg(A(z)) = 1$. Similarly, we shall assume $R(z) = r_0 z + r_1$, $S(z) = s_o z + s_1$. Then, we notice that the leading coefficient of both $A_{cl}(z)$ and $A(z)$ is 1, which indicates that $r_0 = 1$ since there exists $A(z)R(z) + B(z)S(z) = A_{cl}(z)$. Now we have $R(z) = z + r_1$.

Then, note that the additional $A_o(z)$ in (15) must be cancelled in the final form because the closed-loop should have the characteristic polynomial $A_m(z)$. This is the responsibility of $T(z)$, which is designed to be $T(z) = t_0 z$ due to two facts: (1) $T(z)$ should include the factor $A_o(z) = z$ and (2) $\deg(T(z)) \le \deg(R(z)) = 1 \Rightarrow \deg(T(z)) = 1$.

Finally, the closed-loop transfer function is

$$\begin{aligned} G(z) = \frac{Y(z)}{U_c(z)} &= \frac{B(z)T(z)}{A(z)R(z) + B(z)S(z)} = \frac{t_0 z(z+0.9)}{(z^2 - 2.5z + 1)(z + r_1) + (z + 0.9)(s_o z + s_1)} \\ &= \frac{t_0 z(z+0.9)}{z^3 + (r_1 + s_0 - 2.5)z^2 + (0.9\mathrm{s}_0 - 2.5\mathrm{r}_1 + \mathrm{s}_1 + 1)z + r_1 + 0.9 s_1} \end{aligned} \tag{16}$$

According to (15) and (16), we have

$$z^3 + (r_1 + s_0 - 2.5)z^2 + (0.9\mathrm{s}_0 - 2.5\mathrm{r}_1 + \mathrm{s}_1 + 1)z + r_1 + 0.9 s_1 = z(z^2 - 1.8z + 0.9) \tag{17}$$

Thus, the following equations are derived

$$\begin{cases} r_1 + s_0 - 2.5 = -1.8 \\ 0.9\mathrm{s}_0 - 2.5\mathrm{r}_1 + \mathrm{s}_1 + 1 = 0.9 \\ r_1 + 0.9 s_1 = 0 \end{cases} \tag{18}$$

whose solutions is

$$\begin{cases} r_1 = 0.1618 \\ s_0 = 0.5382 \\ s_1 = -0.1798 \end{cases} . \tag{19}$$

And then from (16) and (17) we get

$$G(z)=\frac{t_0(z+0.9)}{z^2-1.8z+0.9} \tag{20}$$

Besides, it is required that $G(1)=1$ for a unit steady-state gain:

$$G(1)=\frac{1.9t_0}{0.1}=1 \Rightarrow t_0=\frac{1}{19}\approx 0.0526 \tag{21}$$

After all, the controller is designed as follows

$$\boxed{\begin{aligned} &R(z)=z+0.1618\\ &S(z)=0.5382z-0.1798\\ &T(z)=\frac{1}{19}z\\ &U(z)=\frac{T(z)}{R(z)}U_c(z)-\frac{S(z)}{R(z)}Y(z)\end{aligned}} \tag{22}$$

And the closed-loop transfer function in this case is

$$G(z)=\frac{Y(z)}{U_c(z)}=\frac{z+0.9}{19(z^2-1.8z+0.9)}. \tag{23}$$

c)

After we finished the design, a Simulink model can be easily built according to Figure 1, which is exhibited in Figure 2. Run the simulation with a sampling period of 1ms.

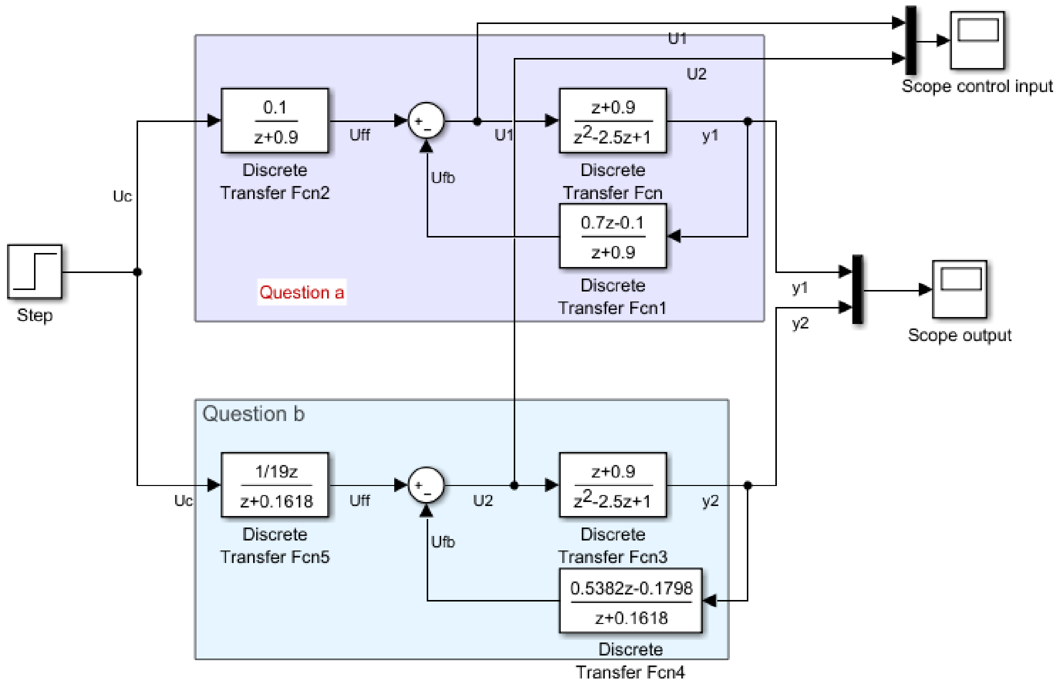

Figure 2 Simulink model

The closed-loop output and control input profiles are shown in Figure 3 and Figure 4 respectively.

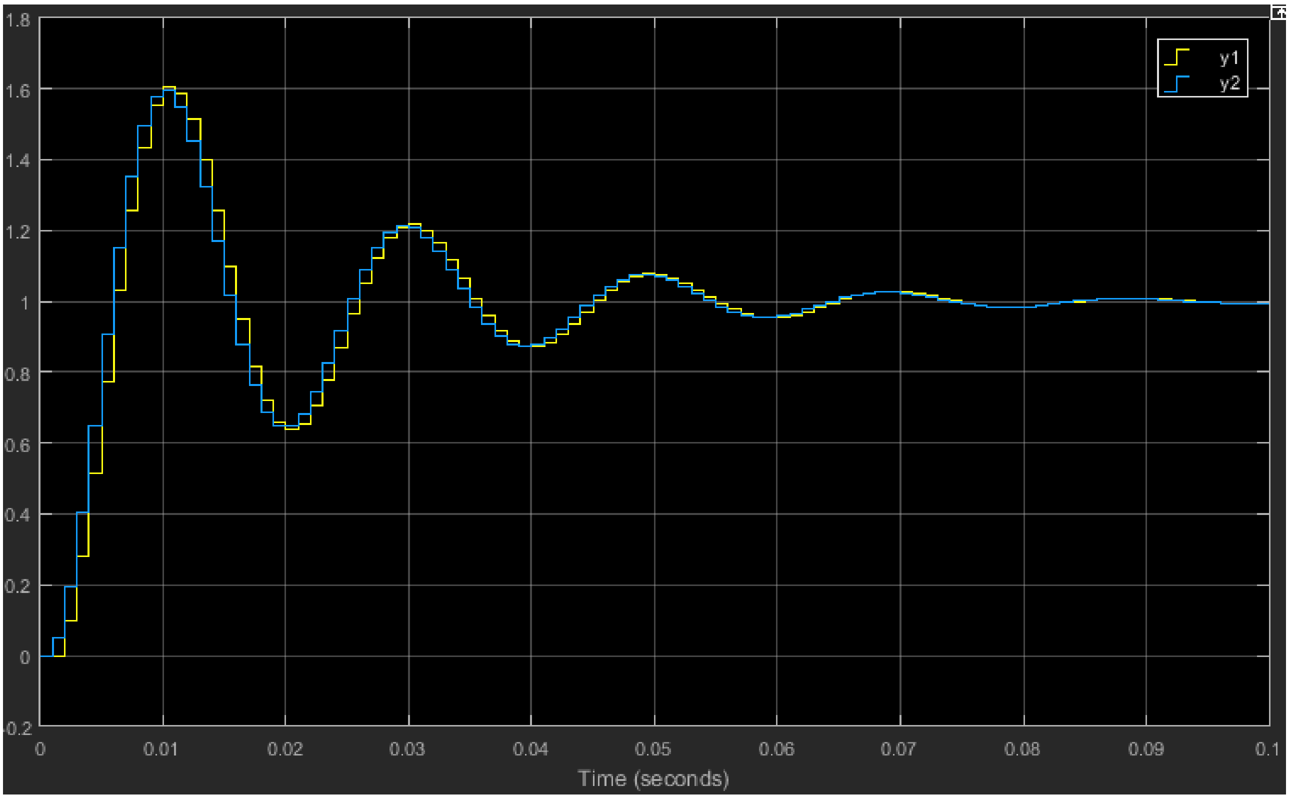

Figure 3 System output for the two cases

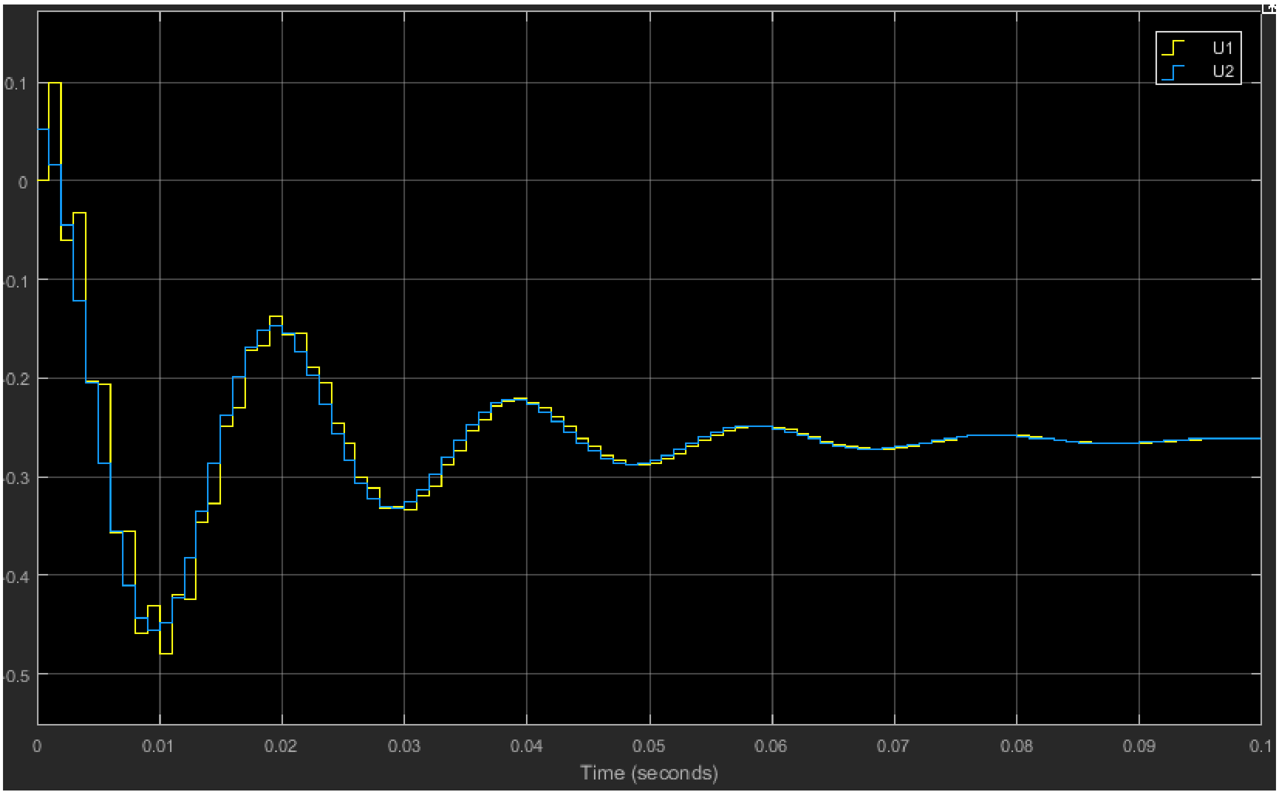

Figure 4 Control input signals for the two cases

As can be seen, the outputs in the two cases are quite similar. However, in Figure 4 it is noted that the control input signal for case 1 (with zero cancellation) is more seriously oscillating before arriving at the steady state. Besides, the control input for case 1 has a slightly larger magnitude. What is the reason for such behavior of the control input? Since we are studying the control input $u(k)$, we first write down the transfer function from the command signal $u_c(k)$ to the control input $u(k)$ as follows (for both the two cases)

$$\begin{cases} U(z) = \dfrac{T(z)}{R(z)}U_c(z) - \dfrac{S(z)}{R(z)}Y(z) \\ \dfrac{Y(z)}{U(z)} = \dfrac{B(z)}{A(z)} \end{cases} \Rightarrow H_u(z) = \frac{U(z)}{U_c(z)} = \frac{A(z)T(z)}{A(z)R(z)+B(z)S(z)} = \frac{A(z)T(z)}{A_{cl}(z)}. \quad (24)$$

By inserting $A_{cl}(z)$ from (6) and (15) into the above equation for case 1 and case 2 respectively, we get $H_u(z)$ for the two cases:

$$\begin{aligned} &\text{case 1} \quad H_u(z) = \frac{0.1(z^2-2.5z+1)}{(z+0.9)(z^2-1.8z+0.9)} \\ &\text{case 2} \quad H_u(z) = \frac{0.0526z(z^2-2.5z+1)}{z(z^2-1.8z+0.9)} = \frac{0.0526(z^2-2.5z+1)}{z^2-1.8z+0.9} \end{aligned} . \quad (25)$$

The poles of the above two transfer functions differ from each other, as follows

$$\begin{aligned} &\text{case 1 poles} \quad [-0.9, \quad 0.9+0.3i, \quad 0.9-0.3i] \\ &\text{case 2 poles} \quad [0.9+0.3i, \quad 0.9-0.3i] \end{aligned} \quad (26)$$

The additional pole of case 1 has a negative real part. We know that a pole with a negative real part in z transfer function would cause oscillation and this is the main reason for the above phenomenon we have observed.

In practice, we would favor a smoother control input with a smaller magnitude, which is easier to implement and more cost-friendly. Hence with the comparison, the second controller (without zero cancellation) is preferred.

Q2

a)

The structure of the closed-loop system is the same with the one in Figure 1 and is shown in Figure 5. The obtained closed-loop transfer function is

$$\frac{Y(z)}{U_c(z)} = \frac{B(z)T(z)}{A(z)R(z)+B(z)S(z)} = \frac{B(z)T(z)}{A_{cl}(z)}. \quad (27)$$

Comparing equation (27) with the reference model $A_m(z)$, we can see that the factor $B(z) = z - 0.8$ disappears in the reference model, which implies zero cancellation. Therefore, $R(z)$ must be comprised of $B(z)$ and possibly another factor.

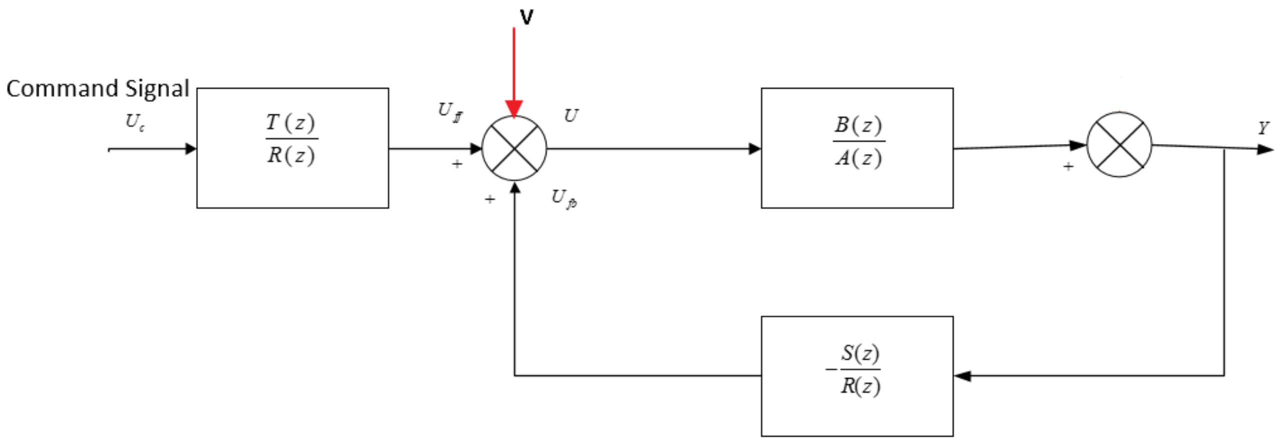

Figure 5 Diagram graph of the control system

Furthermore, here we should pay attention to another requirement on constant disturbance rejection. Assume the disturbance is $v(k)$. Then, from Figure 5 the transfer function from the disturbance v to the output y is resolved to be

$$G_v(z) = \frac{Y(z)}{V(z)} = \frac{B(z)R(z)}{A(z)R(z)+B(z)S(z)}. \tag{28}$$

To eliminate the disturbance effect at steady state, we require the steady state gain of $G_v(z)$ to be zero, that is, $G_v(1)=0$. This further request $R(z)$ to contain a factor $(z-1)$. Remember that $R(z)$ is also required to cancel the zero. In total, $R(z)$ should include both $(z-1)$ and $(z-0.8)$. Similarly, since there exists zero cancellation, the closed-loop polynomial is in form of $A_{cl}(z) = (z-0.8)A_m(z)A_0(z)$.

Now what is $A_0(z)$? We know that if we choose $R(z)=(z-1)(z-0.8)$, then the general form of $S(z)$ is $S(z)=s_0z^2+s_1z+s_2$. Since $A(z)R(z)$ has a degree of 4, to match the two denominator polynomials in **Error! Reference source not found.**, we may choose $A_0(z)=z$. Now we have

$$A(z)R(z)+B(z)S(z) = (z^2-4z+4)(z-1)(z-0.8)+(z-0.8)(s_0z^2+s_1z+s_2) \tag{29}$$

and

$$A_{cl}(z) = (z-0.8)A_m(z)A_o(z) = (z-0.8)z^3. \tag{30}$$

Equaling (29) to (30) yields

$$\begin{cases} s_0-5=0 \\ s_1+8=0 \\ s_2-4=0 \end{cases} \Rightarrow \begin{cases} s_0=5 \\ s_1=-8. \\ s_2=4 \end{cases} \tag{31}$$

Now the closed-loop transfer function is

$$\frac{Y(z)}{U_c(z)} = \frac{B(z)T(z)}{A(z)R(z)+B(z)S(z)} = \frac{T(z)}{z^3} \tag{32}$$

To eliminate the additional $A_0(z)$ and get $H_m(z)$, it is obvious that $T(z) = z$.

Therefore, the controller is

$$\boxed{\begin{cases} R(z) = (z-1)(z-0.8) = z^2 - 1.8z + 0.8 \\ S(z) = 5z^2 - 8z + 4 \\ T(z) = z \end{cases}} \tag{33}$$

Key points:

- $R(z)$ is structured both for zero cancellation and disturbance rejection;
- $S(z)$ works together with $R(z)$ to match the closed-loop denominator;
- $T(z)$ is used to match the closed-loop numerator.

b)

The closed-loop transfer function is

$$G(z) = \frac{Y(z)}{U_c(z)} = \frac{B(z)R(z)}{A(z)R(z)+B(z)S(z)} H_{ff}(z). \tag{34}$$

The closed-loop transfer function from the disturbance to the output is still

$$G_v(z) = \frac{Y(z)}{V(z)} = \frac{B(z)R(z)}{A(z)R(z)+B(z)S(z)}. \tag{35}$$

To reject constant disturbance, we need $R(z)$ involves a $(z-1)$ factor. If $R(z)$ is chosen to be of degree one, then we have $S(z) = s_0 z + s_1$. Since the degree of $A(z)R(z)$ is 3, to match the closed-loop polynomials there are 3 equations (the leading coefficient is fixed to be 1) but only 2 unknows s_0 and s_1, which usually has no solutions. Therefore, $R(z)$ should have a degree of at least 2. Supposing $R(z) = (z + r_1)(z - 1)$ and $S(z) = s_0 z^2 + s_1 z + s_2$, we can get

$$A(z)R(z)+B(z)S(z) = (z^2 - 4z + 4)(z + r_1)(z-1) + (z - 0.8)(s_0 z^2 + s_1 z + s_2). \tag{36}$$

To match $A_m(z) = z^2$, we need another term $A_0(z) = z^2$ and get

$$A_{cl}(z) = A_m(z)A_0(z) = z^4. \tag{37}$$

Comparing (36) and (37), we obtain

$$\begin{cases} r_1 + s_0 - 5 = 0 \\ s_1 - 0.8s_0 - 5r_1 + 8 = 0 \\ 8r_1 - 0.8s_1 + s_2 - 4 = 0 \\ -4r_1 - 0.8s_2 = 0 \end{cases} \Rightarrow \begin{cases} r_1 = -2.2222 \\ s_0 = 7.2222 \\ s_1 = -13.3333 \\ s_2 = 11.1111 \end{cases}. \tag{38}$$

Therefore, the feedback controller is

$$\boxed{\frac{S(z)}{R(z)} = \frac{7.2222z^2 - 13.3333z + 11.1111}{(z-2.2222)(z-1)}.} \tag{39}$$

Now we have $R(z)$ and $S(z)$. From (34) we further obtain

$$\frac{B(z)R(z)}{A(z)R(z)+B(z)S(z)} H_{ff}(z) = \frac{B(z)R(z)}{A_{cl}(z)} H_{ff}(z). \tag{40}$$

To match the closed-loop transfer function (40) with the reference model $H_m(z) = B_m(z)/A_m(z)$, we get

$$\frac{B(z)R(z)}{A_{cl}(z)} H_{ff}(z) = \frac{B_m(z)}{A_m(z)} \Rightarrow H_{ff}(z) = \frac{B_m(z)A_{cl}(z)}{A_m(z)B(z)R(z)} = \frac{B_m(z)A_0(z)}{B(z)R(z)}. \tag{41}$$

Inserting the relevant terms into (41) gives

$$\boxed{H_{ff}(z) = \frac{z^2}{(z-0.8)(z-2.2222)(z-1)}.} \tag{42}$$

Note:

1. For a second thought, you may find that if we keep the $R(z)$ and $S(z)$ in the solution to question a) unchanged and let $H_{ff}(z) = T(z)/R(z)$, we can also meet the requirement. That is to say, the control configuration in a) is exactly a special case of b). However, the $H_{ff}(z)$ structure in b) will give your more freedom since there is no requirement on its denominator form and it is a more general control configuration. For exercise and examination purpose, you should follow different procedures to finish the design in these two cases, although a) can be regarded as a special case of b). In this assignment, if you simply write $H_{ff}(z) = T(z)/R(z)$ by reusing the result obtained in a), you will only get 1 point.
2. In this case b), $R(z)$ is only responsible for disturbance rejection since $H_{ff}(z)$ doesn't appear in the transfer function from disturbance to output. All the other work will be done

by $H_{ff}(z)$ since it has more freedom than case a).

3. In the above, to make the computation easier, we have chosen a $R(z)$ of a lowest possible degree. Unless explicitly specified, you can of course choose a higher order of $R(z)$. Similarly, we choose a $A_0(z)$ whose poles are all at the origin to facilitate the calculation. You can choose other forms of $A_0(z)$ as long as its poles are stable. Overall, the only requirement is that the closed-loop transfer function should match the given reference model.

Q3

Step 1: design a reference model (4 points)

To satisfy the closed-loop performance requirements, we first propose a standard 2nd-order reference model like what we have done in Q3 of homework 2.

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}, \tag{43}$$

where ζ is the damping ration and ω_n is the natural frequency.

According the performance specifications, overshoot satisfies

$$M_p = e^{\frac{-\pi\zeta}{\sqrt{1-\zeta^2}}} < 0.1 \tag{44}$$

Then we have $0.5911 < \zeta < 1$, let $\zeta = 0.75$. (*You can choose any value inside the range.)*

The 1% settling time is required that

$$t_s \cong \frac{4.6}{\zeta\omega_n} < 10 \tag{45}$$

Thus, we have

$$\omega_n > \frac{4.6}{10\zeta} = 0.6133 \tag{46}$$

We choose $\omega_n = 1$. The reference model in the continuous-time is

$$H_m(s) = \frac{1}{s^2 + 1.5s + 1} \tag{47}$$

Step 2: find the open-loop transfer function of the vehicle (2 points)

Applying Laplace transform to the vehicle's differential equation gives

$$ms^2Y(s)+bsY(s)=U(s), \tag{48}$$

from which we can easily get the transfer function

$$H(s)=\frac{Y(s)}{U(s)}=\frac{1}{ms^2+bs}. \tag{49}$$

Inserting the numerical values into (49), we finally obtain

$$H(s)=\frac{1}{1000s^2+200s}. \tag{50}$$

Step 3: discretization (4 points)

The sampling period for our digital control system is 0.5s. By adopting the zero-order hold discretization method, we get the discrete time equivalence of (50) as

$$H(z)=\frac{0.0001209\,z+0.000117}{z^2-1.905z+0.9048}=\frac{B(z)}{A(z)}. \tag{51}$$

Similarly, the reference model (47) needs to be discretized. The result is

$$H_m(z)=\frac{0.09689\,z+0.07538}{z^2-1.3z+0.4724}=\frac{B_m(z)}{A_m(z)}. \tag{52}$$

Step 4: design the two-degree-of-freedom controller to match the reference model (10 points)

Now our task is to design a two-degree-of-freedom digital control to make the vehicle behave like the reference model, that is, the closed-loop transfer function should match the reference model $H_m(z)$. This step is quite like Q2 a). We only list the key procedures here. For details, please refer to the solution of Q2 a).

First notice that $H(z)$ has a different zero compared with $H_m(z)$. Therefore, the zero of the open-loop transfer function $H(z)=B(z)/A(z)$ must have been **cancelled**. Otherwise, the zero of $H(z)$ would remain in the closed-loop transfer function, which cannot match $H_m(z)$ exactly. Besides, the controller is required to resist constant disturbance. In total, we know that $R(z)$ must include $B(z)(z-1)$. For simplicity, we just take $R(z)=(z-1)B(z)$. Then, the general form for $S(z)$ is $S(z)=s_0z^2+s_1z+s_2$.The closed-loop characteristic polynomial before zero cancellation is $A_{cl}(z)=B(z)A_m(z)A_0(z)$. Our purpose is to make

$$A_{cl}(z)=A(z)R(z)+B(z)S(z). \tag{53}$$

By inserting the polynomials into (53), we get

$$A_m(z)A_0(z)=A(z)(z-1)+S(z). \tag{54}$$

Because both $A_m(z)$ and $A(z)$ are of degree 2, we can choose $A_0(z) = z$ to facilitate computing. Then, from (54) we further obtain

$$(z^2 - 1.3z + 0.4724)z = (z^2 - 1.905z + 0.9048)(z-1) + s_0 z^2 + s_1 z + s_2, \tag{55}$$

which yields

$$\begin{cases} s_0 - 2.905 = -1.3 \\ s_1 + 2.8098 = 0.4724 \\ s_2 - 0.9048 = 0 \end{cases} \Rightarrow \begin{cases} s_0 = 1.605 \\ s_1 = \text{-}2.3374. \\ s_3 = 0.9048 \end{cases} \tag{56}$$

The remaining task for now is to match the numerator of the reference model (52). The closed-loop system has a transfer function

$$G(z) = \frac{B(z)T(z)}{A(z)R(z) + B(z)S(z)} = \underbrace{\frac{B(z)T(z)}{B(z)A_m(z)A_0(z)} = \frac{T(z)}{A_m(z)A_0(z)}}_{\text{zero cancellation}}. \tag{57}$$

To make $G(z)$ equal to $H_m(z)$, we have

$$\frac{T(z)}{A_m(z)A_0(z)} = \frac{B_m(z)}{A_m(z)} \Rightarrow T(z) = A_0(z)B_m(z). \tag{58}$$

In summary, the two-degree-of-freedom digital controller is designed to be

$$\boxed{\begin{aligned} R(z) &= (z-1)(0.0001209\, z + 0.000117) \\ S(z) &= 1.605z^2 - 2.3374z + 0.9048 \\ T(z) &= z(0.09689\, z + 0.07538) \end{aligned}}. \tag{59}$$

To verify our controller, a Simulink model for the above control system can be designed. (This is NOT counted in your marks. Here we'd like to demonstrate how you can simulate a digital control system in Simulink to obtain a visual perception of the two-degree-of-freedom controller.)

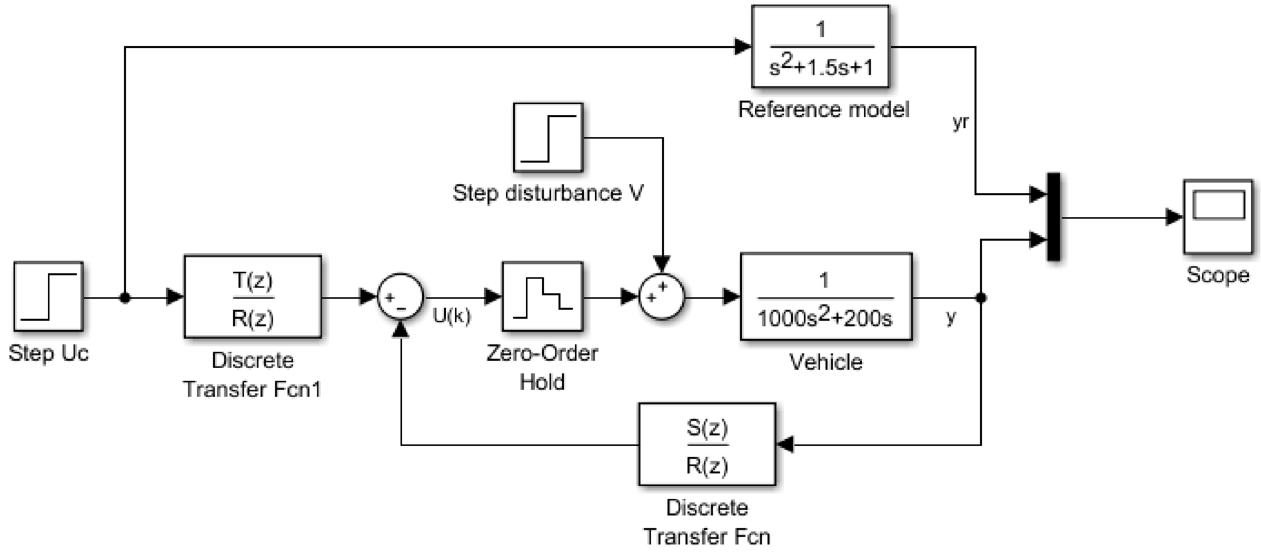

Figure 6 Simulink model for the digital control system

To build the above model, please refer to Figure 5. In discrete-time simulation using Simulink, remember to

- Set the sample time properly for all the discrete blocks, such the zero-order hold and the discrete transfer function.
- Place a zero-order hold before your plant (here the vehicle) to hold the discrete control signal $u(k)$ into a continuous one since all physical plants in reality work in continuous-time domain.
- As for sampling in a digital control system, the discrete transfer function block automatically does this. That is, you can feed a continuous-time signal, like y in the above, into a discrete block directly.

The step response of the above system with no disturbance is shown in Figure 7. As we can see, the actual output y of the plant (vehicle) can track the reference model output y_r quite closely. The overshoot is less than 10% and the setting time is less than 10s. Now you might appreciate the power of control system design: we can drive the plant as we want.

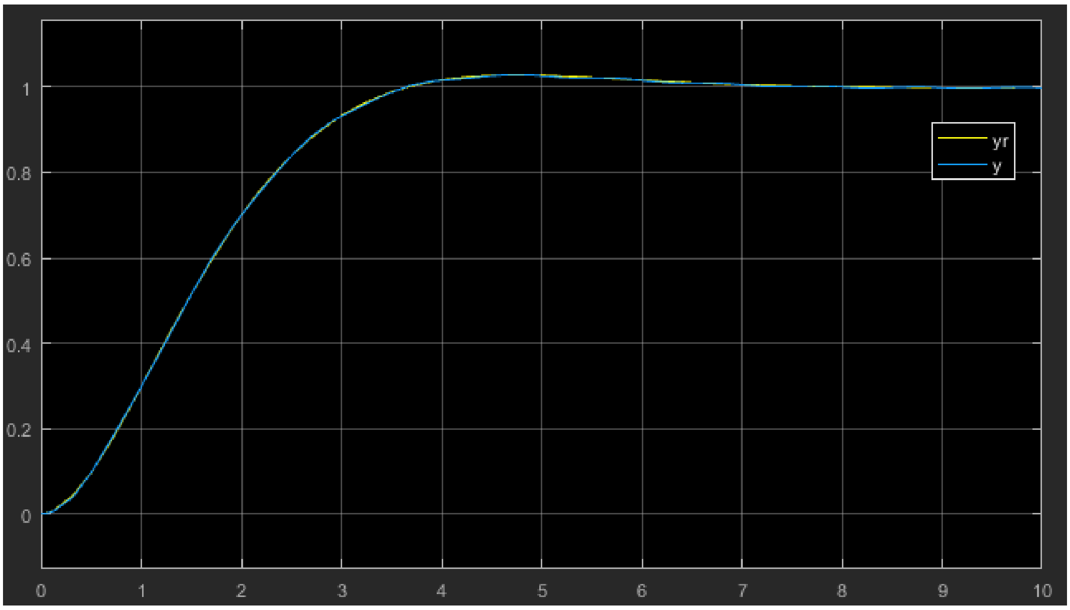

Figure 7 Output of the reference model y_r vs. output of the vehicle y

To test the disturbance rejection capacity of our digital control system, inject a constant disturbance $v = 1$ at time $t = 10$. The simulation result is given in Figure 8.

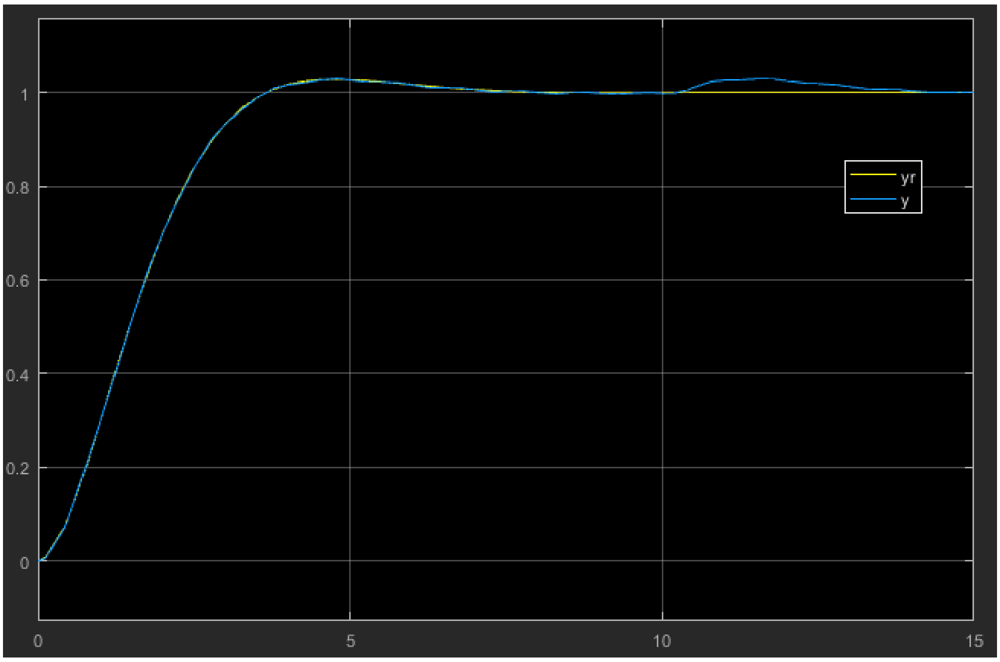

Figure 8 Disturbance rejection: a constant disturbance is injected at the plant input at time 10s.

Just like what we expect, the effect of the constant disturbance is eliminated at steady state. Therefore, our digital control system can reject constant disturbance as required. That's wonderful!

Q4

a)

The given input-output model is

$$y(k+1) = y(k) + \frac{cu(k-1)}{y^2(k-1)+1}. \tag{60}$$

From the input-output model we can get

$$cu(k-1) = \left(y(k+1) - y(k)\right)\left(y^2(k-1)+1\right), \tag{61}$$

which further leads to

$$u(k) = \frac{1}{c}\left(y(k+2) - y(k+1)\right)\left(y^2(k)+1\right). \tag{62}$$

In one-step-ahead controller, just make $y(k+2) = r(k+2)$ and replace other *future output* with their predicted values using current and past data available. That is, substitution of $y(k+1)$ by (60) into (62) gives

$$u(k) = \frac{1}{c}\left(r(k+2) - y(k) - \frac{cu(k-1)}{y^2(k-1)+1}\right)\left(y^2(k)+1\right). \tag{63}$$

b)

We need to assure that the plant output $y(k)$ and the controller output (aka plant input) $u(k)$ are both bounded for perfect tracking.

Since $y(k) = r(k)$ in perfect tracking, the plant output is bounded as long as the reference $r(k)$ is bounded.

From (62), in the case of perfect tracking, we have $y(k+2) = r(k+2)$ and $y(k+1) = r(k+1)$, the controller output is

$$\begin{aligned} u(k) &= \frac{1}{c}\left(y(k+2) - y(k+1)\right)\left(y^2(k)+1\right) \\ &= \frac{1}{c}\left(r(k+2) - r(k+1)\right)\left(r^2(k)+1\right). \end{aligned} \tag{64}$$

For any reference signal $r(k)$, as long as it is bounded, the right hand side of (64), i.e., $\left(r(k+2) - r(k+1)\right)\left(r^2(k)+1\right)$, is also bounded. Therefore, to make $u(k)$ bounded, the only requirement is $c \neq 0$.

In summary, to achieve perfect tracking the condition of c is $c \neq 0$.