Q_1:

a) $Y(s) = \frac{1}{s(Ts+1)} \cdot U(s) \overset{T=0.5}{\Rightarrow} 0.5\ddot{y}(t) + \dot{y}(t) = u(t)$

$\because x_1(t) = \dot{y}(t),\ x_2(t) = y(t) \Rightarrow \dot{x}_1(t) = \ddot{y}(t) = -2x_1(t) + 2u(t) \quad \dot{x}_2(t) = \dot{y}(t) = x_1(t)$

$\therefore \dot{x}(t) = \begin{pmatrix} -2 & 0 \\ 1 & 0 \end{pmatrix} x(t) + \begin{pmatrix} 2 \\ 0 \end{pmatrix} u(t), \quad y(t) = \begin{pmatrix} 0 & 1 \end{pmatrix} x(t)$

Convert into discrete-time model: $x(k+1) = \Phi x(k) + \Gamma u(k)$, $y(k) = c\,x(k)$

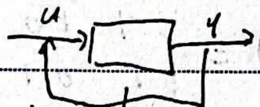
$\because e^{At} = L^{-1}(sI - A)^{-1} = L^{-1}\begin{pmatrix} s+2 & 0 \\ -1 & s \end{pmatrix}^{-1} = \begin{pmatrix} e^{-2t} & 0 \\ \frac{1}{2} - \frac{1}{2}e^{-2t} & u(t) \end{pmatrix}$

$\Rightarrow \Phi = e^{At}\Big|_{t=h} = \begin{pmatrix} e^{-2h} & 0 \\ \frac{1}{2} - \frac{1}{2}e^{-2h} & 1 \end{pmatrix}, \quad \Gamma = \int_0^h e^{At}dt \cdot B = \begin{pmatrix} \frac{1}{2} - \frac{1}{2}e^{-2h} & 0 \\ \frac{1}{4}e^{-2h} - \frac{1}{4} + \frac{1}{2}h & h \end{pmatrix}\begin{pmatrix} 2 \\ 0 \end{pmatrix}$

$= \begin{pmatrix} 1 - e^{-2h} \\ \frac{1}{2}e^{-2h} - \frac{1}{2} + h \end{pmatrix}$

$\Rightarrow x(k+1) = \begin{pmatrix} e^{-2h} & 0 \\ \frac{1}{2} - \frac{1}{2}e^{-2h} & 1 \end{pmatrix} x(k) + \begin{pmatrix} 1 - e^{-2h} \\ \frac{1}{2}e^{-2h} - \frac{1}{2} + h \end{pmatrix} u(k), \quad y(k) = \begin{pmatrix} 0 & 1 \end{pmatrix} x(k)$

b) To design the dead-beat feedback controller,



place poles at 0 $\Rightarrow A_m(\Phi) = \Phi^2 = \begin{pmatrix} e^{-4h} & 0 \\ \frac{1}{2}(1 - e^{-4h}) & 1 \end{pmatrix}$

$W_c = (\Gamma \quad \Phi\Gamma) = \begin{pmatrix} 1 - e^{-2h} & e^{-2h} - e^{-4h} \\ \frac{1}{2}e^{-2h} - \frac{1}{2} + h & \frac{1}{2}e^{-4h} - \frac{1}{2}e^{-2h} + h \end{pmatrix} \Rightarrow W_c^{-1} = \begin{pmatrix} \frac{\frac{1}{2}e^{-4h} - \frac{1}{2}e^{-2h} + h}{h \cdot (e^{-2h} - 1)^2} & \frac{e^{-2h}}{h \cdot (e^{-2h} - 1)} \\ \frac{-\frac{1}{2}e^{-2h} + \frac{1}{2} - h}{h \cdot (e^{-2h} - 1)^2} & \frac{-1}{h(e^{-2h} - 1)} \end{pmatrix}$

$\Rightarrow L = \begin{pmatrix} 0 & 1 \end{pmatrix} W_c^{-1} A_m(\Phi) = \begin{pmatrix} \frac{-h \cdot e^{-4h} - \frac{1}{2}e^{-2h} + \frac{1}{2}}{h(1 - e^{-2h})^2} & \frac{1}{h(1 - e^{-2h})} \end{pmatrix}$

Controller: $u(k) = -L\,x(k) = \begin{pmatrix} \frac{h \cdot e^{-4h} + \frac{1}{2}e^{-2h} - \frac{1}{2}}{h \cdot (1 - e^{-2h})^2} & \frac{1}{h(e^{-2h} - 1)} \end{pmatrix} x(k)$

c) Assume that the maximum value of $u(k)$ is at $k=0$.

$|u(0)| = |-L\,x(0)| = \left| \frac{2h \cdot e^{-4h} + 5 \cdot e^{-2h} - 5}{4h(1 - e^{-2h})^2} \right| < 1 \Rightarrow$ can't solve by hand.

Q_2:

a) $\because$ $x(k)$ and $v(k)$ can be measured $\Rightarrow z(k) = \begin{pmatrix} x(k) \\ v(k) \end{pmatrix}$ can be measured.

$$\Rightarrow \begin{pmatrix} x_1(k+1) \\ x_2(k+1) \\ v(k+1) \end{pmatrix} = \begin{pmatrix} 0.5 & 1 & 1 \\ 0.5 & 0.7 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1(k) \\ x_2(k) \\ v(k) \end{pmatrix} + \begin{pmatrix} 0.2 \\ 0.1 \\ 0 \end{pmatrix} u(k), \quad y(k) = (1\ 0\ 0) \begin{pmatrix} x_1(k) \\ x_2(k) \\ v(k) \end{pmatrix}$$

design $u(k) = -L_c \cdot z(k) = -L \cdot x(k) - L_w \cdot v(k)$

Only consider L_w $\Rightarrow$ choose L_w such that $\Phi_{xw} - \Gamma \cdot L_w = 0$

$\therefore \Phi_{xw} - \Gamma \cdot L_w = \begin{pmatrix} 1 \\ 0 \end{pmatrix} - \begin{pmatrix} 0.2 \\ 0.1 \end{pmatrix} L_w = \begin{pmatrix} 1 - 0.2 L_w \\ -0.1 \cdot L_w \end{pmatrix} \Rightarrow$ cannot make it as 0.

So we need to design L, let all the $(\Phi - \Gamma L)$ poles placed at $0 \Rightarrow A_m(z) = z^2$

$$L = (l_1\ \ l_2) \Rightarrow zI - (\Phi - \Gamma L) = \begin{pmatrix} z - 0.5 + 0.2 l_1 & 0.2 l_2 - 1 \\ 0.1 l_1 - 0.5 & z + 0.1 l_2 - 0.7 \end{pmatrix}$$

let $|zI - (\Phi - \Gamma L)| = A_m(z) \Rightarrow l_1 = \frac{45}{14} \quad l_2 = \frac{39}{7} \Rightarrow L = \left(\frac{45}{14}\ \ \frac{39}{7}\right)$
$= (3.2143\ \ 5.5714)$

Now analyze the T.F. from $v(k)$ to $y(k)$

$x(k+1) = (\Phi - \Gamma L) x(k) + (\Phi_{xw} - \Gamma \cdot L_w) v(k) \Rightarrow X(z) = (zI - \Phi + \Gamma L)^{-1} (\Phi_{xw} - \Gamma \cdot L_w) v(z)$

$Y(z) = C \cdot X(z) \Rightarrow H_w(z) = C(zI - \Phi + \Gamma L)^{-1} (\Phi_{xw} - \Gamma \cdot L_w)$

let $H_w(1) = 0 \Rightarrow l_w = \frac{75}{14} = 5.3571$

So the controller: $u(k) = (-3.2143\ \ -5.5714)\, x(k) - 5.3571\, v(k)$

b) $\because$ $x(k)$ can be measured, when $k = 0$, we have

$x_1(1) = 0.5 x_1(0) + x_2(0) + 0.2 u(0) + v(0)$
$x_2(1) = 0.5 x_1(0) + 0.7 x_2(0) + 0.1 u(0)$ $\Rightarrow$ we can get $u(0)$, $v(0)$

$\because$ $v(k)$ is constant $\Rightarrow$ then $x(k)$ and $v(k)$ can be measured, same as a).

we design the same dead-beat controller: $u(k) = (-3.2143\ \ -5.5714)\, x(k) - 5.3571\, v(k)$

c) $\because$ only $y(k)$ can be measured.

we assume $x(k)$ and $v(k)$ can be measured, design the feed-back controller.

$u(k) = (-3.2143\ \ -5.5714)\, x(k) - 5.3571\, v(k)$

Now design observer to estimate $x(k)$ and $v(k)$.

we have $z(k+1) = \begin{bmatrix} \Phi & \Phi_{xw} \\ 0 & \Phi_w \end{bmatrix} z(k) + \begin{bmatrix} \Gamma \\ 0 \end{bmatrix} u(k)$, $y(k) = [c \quad 0]\, z(k)$

$$\hat{z}(k+1) = \begin{bmatrix} \Phi & \Phi_{xw} \\ 0 & \Phi_w \end{bmatrix} \hat{z}(k) + \begin{bmatrix} \Gamma \\ 0 \end{bmatrix} u(k) + K\left(y(k) - \hat{y}(k)\right), \quad \hat{y}(k) = [c \quad 0]\, \hat{z}(k)$$

$$\Rightarrow z(k+1) - \hat{z}(k+1) = \begin{bmatrix} \Phi & \Phi_{xw} \\ 0 & \Phi_w \end{bmatrix} \left(z(k) - \hat{z}(k)\right) - K\left(y(k) - \hat{y}(k)\right), \quad y(k) - \hat{y}(k) = [c \quad 0]\left(z(k) - \hat{z}(k)\right)$$

let $e(k) = z(k) - \hat{z}(k) \Rightarrow e(k+1) = (\Phi - Kc)\, e(k)$

we design dead-beat observer, $A_o(z) = z^3$

$$K = (k_1 \quad k_2 \quad k_3)^T, \quad (zI - (\Phi - Kc)) = \begin{pmatrix} z+k_1-0.5 & -1 & -1 \\ k_2 - 0.5 & z - 0.7 & 0 \\ k_3 & 0 & z-1 \end{pmatrix}$$

$$\det|zI - (\Phi - Kc)| = z^3 \Rightarrow K = (2.2 \quad -0.6433 \quad 3.3333)^T$$

$\therefore$ controller : $u(k) = (-3.2143 \quad -5.5714)\, \hat{x}(k) - 5.3571\, \hat{v}(k)$

Q7:

a) overshoot : $M_p = e^{-\pi\xi/\sqrt{1-\xi^2}} = 5\% \Rightarrow \xi = 0.7$ damping ratio

natural frequency : $\omega_n = \frac{4.6}{t_s \cdot \xi} = \frac{4.6}{8 \times 0.7} = 0.82$

$$\therefore H(s) = \frac{\omega_n^2}{s^2 + 2\xi \cdot \omega_n \cdot s + \omega_n^2} = \frac{0.67}{s^2 + 1.15\xi + 0.67}$$

Convert into discrete time model, $H(z) = \frac{0.003223z + 0.003102}{z^2 - 1.885z + 0.8914}$

b) $800\ddot{y}(t) + 200\dot{y}(t) = u(t)$

$x_1(t) = y(t) \Rightarrow \dot{x}_1(t) = x_2(t)$

$x_2(t) = \dot{y}(t) \Rightarrow \dot{x}_2(t) = \ddot{y}(t) = -\frac{1}{4}x_2(t) + \frac{1}{800}u(t)$

$$\Rightarrow \dot{x}(t) = \begin{pmatrix} 0 & 1 \\ 0 & -\frac{1}{4} \end{pmatrix} x(t) + \begin{pmatrix} 0 \\ \frac{1}{800} \end{pmatrix} u(t)$$

$$y(t) = (1 \quad 0)\, x(t)$$

Convert into discrete time model: $x(k+1) = \Phi x(k) + \Gamma u(k)$, $y(k) = c\,x(k)$

$$e^{At} = L^{-1}(sI-A)^{-1} = L^{-1}\begin{pmatrix} s & -1 \\ 0 & s+\frac{1}{4} \end{pmatrix}^{-1} = L^{-1}\begin{pmatrix} \frac{1}{s} & 4\left(\frac{1}{s}-\frac{1}{s+\frac{1}{4}}\right) \\ 0 & \frac{1}{s+\frac{1}{4}} \end{pmatrix} = \begin{pmatrix} u(t) & 4u(t)-4\cdot e^{-\frac{1}{4}t} \\ 0 & e^{-\frac{1}{4}t} \end{pmatrix}$$

~~h=0.1~~

$$\Phi = e^{Ah} = \begin{pmatrix} 1 & 0.09876 \\ 0 & 0.9753 \end{pmatrix},\quad \Gamma = \int_0^{0.1} e^{At}\,dt \cdot B = \begin{pmatrix} 6.1982\times10^{-6} \\ 1.2345\times10^{-4} \end{pmatrix}$$

$$\therefore x(k+1) = \begin{pmatrix} 1 & 0.09876 \\ 0 & 0.9753 \end{pmatrix} x(k) + \begin{pmatrix} 6.1982\times10^{-6} \\ 1.2345\times10^{-4} \end{pmatrix} u(k),\quad y(k) = (1 \;\; 0)\,x(k)$$

c) $u_{fb}(k) = -L\,x(k) \Rightarrow x(k+1) = \Phi x(k) + \Gamma(-Lx(k) + u_{ff}(k)) = (\Phi - \Gamma L)x(k) + \Gamma \cdot u_{ff}(k)$

place the $(\Phi - \Gamma L)$ poles same as $A_m(z) = z^2 - 1.885z + 0.8914$ $\quad L = (l_1 \;\; l_2)$

$$\det|zI - (\Phi - \Gamma L)| = \begin{vmatrix} z-1+6.1982\times10^{-6}\cdot l_1 & 6.1982\times10^{-6}\cdot l_2 - 0.09876 \\ 1.2345\times10^{-4}\cdot l_1 & z-0.9753+1.2345\times10^{-4}\cdot l_2 \end{vmatrix}$$

$$\Rightarrow \begin{cases} 1.2345\times10^{-4}\cdot l_2 + 6.1982\times10^{-6}\cdot l_1 - 1.9753 = -1.885 \\ 6.1468\times10^{-6}\cdot l_1 - 1.2345\times10^{-4}\cdot l_2 = 0.8914 - 0.9753 \end{cases} \Rightarrow \begin{cases} l_1 = 518.43 \\ l_2 = 705.44 \end{cases}$$

$\therefore$ feedback controller: $u_{fb}(k) = (-518.43 \;\; -705.44)\,x(k)$

d) Let's derive the T.F. from $u_c(k)$ to $y(k)$ $\quad \because u_{ff}(k) = H_{ff} \cdot u_c(k)$

$\Rightarrow x(k+1) = (\Phi - \Gamma L)x(k) + \Gamma \cdot H_{ff} \cdot u_c(k)$

$\Rightarrow z \cdot X(z) = (\Phi - \Gamma L)X(z) + \Gamma \cdot H_{ff} \cdot U_c(z) \Rightarrow X(z) = (zI - \Phi + \Gamma L)^{-1}\Gamma \cdot H_{ff}\, U_c(z)$

$\Rightarrow Y(z) = C \cdot X(z) = C \cdot (zI - \Phi + \Gamma L)^{-1} \cdot \Gamma \cdot H_{ff} \cdot U_c(z)$

To place zero, $H_{ff} = \dfrac{B_m(z)}{B(z)}$

$$\because H_{cl}(z) = \frac{B(z)}{A_m(z)} \Rightarrow H_{cl}(z) = (1 \;\; 0)\begin{pmatrix} z-0.9968 & -0.0944 \\ 0.064 & z-0.8882 \end{pmatrix}^{-1}\begin{pmatrix} 6.1982\times10^{-6} \\ 1.2345\times10^{-4} \end{pmatrix}$$

$$= \frac{6.1982\times10^{-6}z + 6.1484\times10^{-6}}{z^2 - 1.885z + 0.8914}$$

$$\therefore H_{ff}(z) = \frac{0.003223z + 0.003102}{6.1982\times10^{-6}z + 6.1484\times10^{-6}}$$

e) $\because$ only $y(k)$ is measurable, we assume $x(k)$ is available,

design the feedback controller same as c), $u_{fb}(k) = (-518.43 \quad -705.44)\, x(k)$

Now design observer to estimate $x(k)$

we have $x(k+1) = \Phi x(k) + \Gamma u(k) \qquad y(k) = c\, x(k)$

$\hat{x}(k+1) = \Phi \hat{x}(k) + \Gamma u(k) + K(y(k) - \hat{y}(k)) \quad \hat{y}(k) = c\, \hat{x}(k)$

let $e(k) = x(k) - \hat{x}(k)$, $e(k+1) = (\Phi - Kc)\, e(k)$

we design dead-beat observer, $A_o(z) = z^2$

$$zI - (\Phi - Kc) = \begin{pmatrix} z-1+k_1 & -0.09876 \\ k_2 & z-0.9753 \end{pmatrix}$$

$$\det|zI - \Phi + Kc| = z^2 + (k_1 - 1 - 0.9753)z + (1-k_1)\cdot 0.9753 + 0.09876\cdot k_2 = z^2$$

$\Rightarrow k_1 = 1.9753 \quad k_2 = 9.6315 \quad$ observer: $K = (1.9753 \quad 9.6315)^T$

$\therefore$ controller: ~~u(k) =~~ $u_{fb}(k) = (-518.43 \quad -705.44)\, \hat{x}(k)$

$\therefore u(k) = u_{fb}(k) + u_{ff}(k) \Rightarrow$ It's possible to use two-degree-of-freedom controller