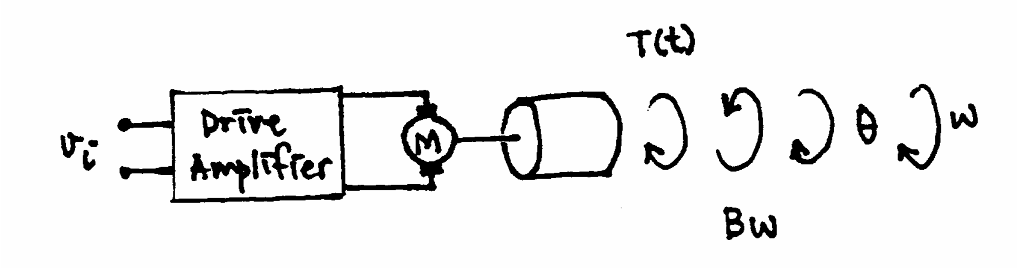
"State - Feedback"

An example of state feedback which you have met before

Permanent magnet D.C. motor



applied torque $T(t) = gv_i(t)$

(this is typically a good approximation since the electrical time constant is usually very small of the mechanical time constant)

Newton's 2nd law for rotational dynamics

$$J\frac{d}{dt}w = \sum torques$$ J: inertia of motor

$$= gv_i - Bw$$ & load about

$J\dot{\omega} = gv_i - B\omega$ rot^n axis

assuming the coupling shaft to the load is rigid.

Interested in the variables θ, w

$x_1 = \theta$

$x_2 = \dot{\theta} = w$

Further $\dot{\theta} = w$

Thus, together, we have

$F_{open-loop}$

$$\begin{bmatrix} \dot{\theta} \\ \dot{w} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -\frac{B}{J} \end{bmatrix} \begin{bmatrix} \theta \\ w \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{g}{J} \end{bmatrix} v_i$$

$\dot{x} = Fx + Gu$

[What are the poles of this open-loop system?]

Assume that we want to achieve a closed-loop with poles which are at the roots of

$$s^2 + 2s + 1 = 0$$

$w_n = 1$; & $\zeta = 1.0$

$s^2 + 2\zeta w_n s + w_n^2$

To achieve this, consider the feedback

$$v_i = -k_1\theta - k_2 w + 0$$

angular position & velocity feedback

Then closed-loop is

$$\begin{bmatrix} \dot{\theta} \\ \dot{w} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{g}{J}k_1 & -\frac{g}{J}k_2 - \frac{B}{J} \end{bmatrix} \begin{bmatrix} \theta \\ w \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{g}{J} \end{bmatrix} 0$$

(Ignore tracking of set-point for now.)

$\det\{sI - F_{closed-loop}\}$

$F_{closed-loop}$

[What are the poles of this closed-loop system?]

$\boxed{II}$ Pause & think!!!

Take Laplace Transform w.r.t. θ

or $\dot{w} = -\frac{g}{J}k_1\theta - (\frac{g}{J}k_2 + \frac{B}{J})w$

$\ddot{\theta}$ $\dot{\theta}$

i.e. $\left\{ s^2 + (\frac{g}{J}k_2 + \frac{B}{J})s + \frac{g}{J}k_1 \right\}\theta(s) = 0$

Closed-loop, with feedback gains k_1 & k_2

we want this to be

$s^2 + 2s + 1$

and values of the feedback gains k_1, k_2 can be calculated to achieve this.

Note: Given a matrix A, the characteristic polynomial $\alpha(s)$, of A, is defined as:

$$\alpha(s) = \det[sI - A]$$

Check this: This polynomial is the char polynomial of $F_{\text{closed-loop}}$

Summary:

Thus, given the open-loop system

$$\dot{x} = Fx + Gu$$

the feedback law

$$u = -kx = -\begin{bmatrix} k_1 & k_2 & \dots & k_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \quad + 0$$

can be used to change the characteristic polynomial of the system.

(Recall poles of system in state-space description.)

With feedback, system now is

$$= Fx + Gu$$

$$\begin{aligned} \dot{x} &= Fx - Gkx \quad + G0 \\ &= [F - Gk]x \quad + G0 \end{aligned}$$

Homogenous part determines poles of system which are at the roots of the characteristic equation

$$\det[sI - (F - Gk)] = 0$$

$F_{\text{closed-loop}}$

Design Philosophy

- Pick your favorite desired closed-loop pole locations, say, at $s = s_1, s_2, \ldots s_n$

  Not an easy matter.

  Pole locations has to be reasonable, subject to physical constraints like slewing rate of amplifiers, actuator saturation etc.

  (Will study methodology for this...)

- Then desired closed-loop characteristic eqn is

$$\alpha_c(s) = (s - s_1)(s - s_2)\ldots(s - s_n) = 0 \qquad \text{——} (1)$$

- For feedback

$$u = -kx$$

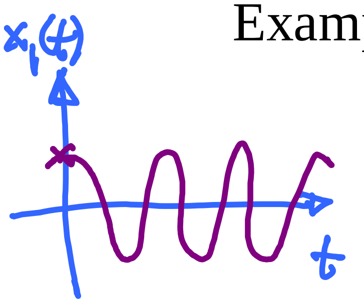
  obtained closed-loop char eqn is

$$\det[sI - (F - Gk)] = 0 \qquad \text{——} (2)$$

- Set (1) = (2) and equate coefficients to find k. Calculations usually tedious for high dimension system.

  In later sections, we will develop systematic methods to calculate k.

Example : Undamped Oscillator



Eigenvalues are $s = \pm j w_o$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -w_o^2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$\dot{x} = Fx + Gu$

Wish to relocate the poles so that both are at $s_{1,2} = -2w_o$

$$\alpha_c(s) = (s + 2w_o)^2 = s^2 + 4w_o s + 4w_o^2$$

i.e. desired c.l. characteristic polynomial

Control law

$$u = -\begin{bmatrix} k_1 & k_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$\therefore$ obtained closed-loop char ~~eqn~~ polynomial is

$$\det[sI - (F - Gk)]$$

$$= \det\left[\begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \left\{\begin{bmatrix} 0 & 1 \\ -w_o^2 & 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix}\begin{bmatrix} k_1 & k_2 \end{bmatrix}\right\}\right]$$

$$= \det\left[\begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -w_o^2 - k_1 & -k_2 \end{bmatrix}\right] = \det\begin{bmatrix} s & -1 \\ w_o^2 + k_1 & s + k_2 \end{bmatrix}$$

$$= s^2 + k_2 s + w_o^2 + k_1$$

Compare coeff to get

$$k_1 = 3w_o^2$$
$$k_2 = 4w_o$$

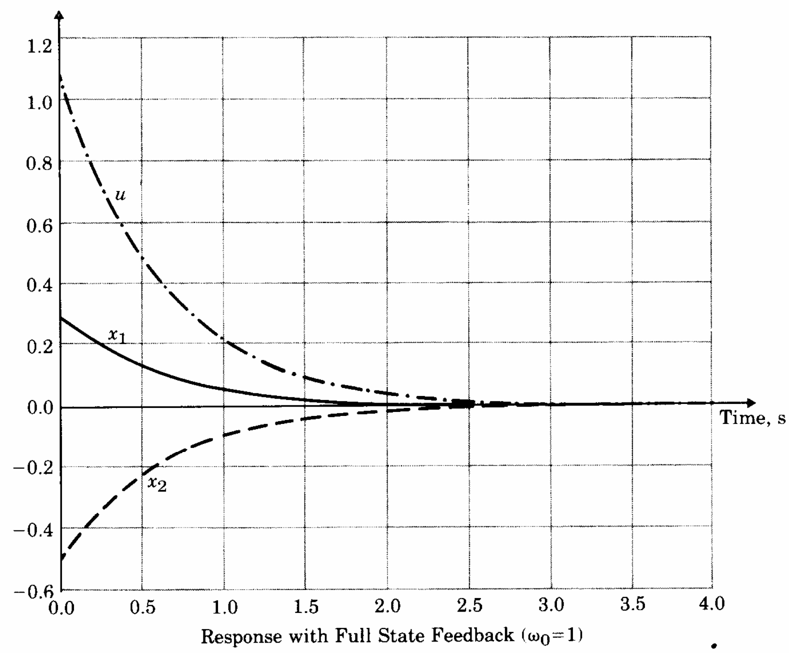
Response of closed-loop to i.c.

$$x_1 = 0.3 \quad , \quad x_2 = -0.5 \quad , \quad w_o = 1$$

FIGURE 6.14
Undamped oscillator response with full-state feedback ($\omega_0 = 1$).

Closed-loop poles at

$s = -2W_o$, and $-2W_o$



Two matters to be resolved

(i) Calculation of control gains by comparing coeff is tedious for n>3. Is there a simpler way ?

(ii) Can we always place the poles of the closed-loop system arbitrarily when using state feedback ?

(i) Calculation of control gains

- remember, state representations not unique
- are these representations where calculation of k is always easy ? answer is yes.

$\therefore$ use this approach:

Original rep^n	Rep^n where calculat^n of k is easy

$$\dot{x} = Fx + Gu \quad \xrightarrow{p=Tx} \quad \dot{p} = F'p + G'u$$

- Calculate the gain K' in the p system
- Then transform back to the x-system $\Delta\Delta$

[Compare with Method 1 !!]

A representation where calculation of K' is easy : "The Control Canonical Form".

Consider transfer function

$$H(s)=\frac{Y(s)}{U(s)}=\frac{b_1s^2+b_2s+b_3}{s^3+a_1s^2+a_2s+a_3}=\frac{b(s)}{a(s)}$$

or in d.e.

$$\dddot{y}+a_1\ddot{y}+a_2\dot{y}+a_3y=b_1\ddot{u}+b_2\dot{u}+b_3u$$

First introduce an auxiliary variable ξ where

$$\frac{\xi(s)}{U(s)}=\frac{1}{a(s)}$$

or $$\dddot{\xi}+a_1\ddot{\xi}+a_2\dot{\xi}+a_3\xi=u \qquad (6.52)$$

$$\frac{b(s)}{a(s)} = \frac{Y(s)}{U(s)} = \frac{Y(s)}{\xi(s)} \bullet \frac{\xi(s)}{U(s)} = \frac{Y(s)}{\xi(s)} \frac{1}{a(s)}$$

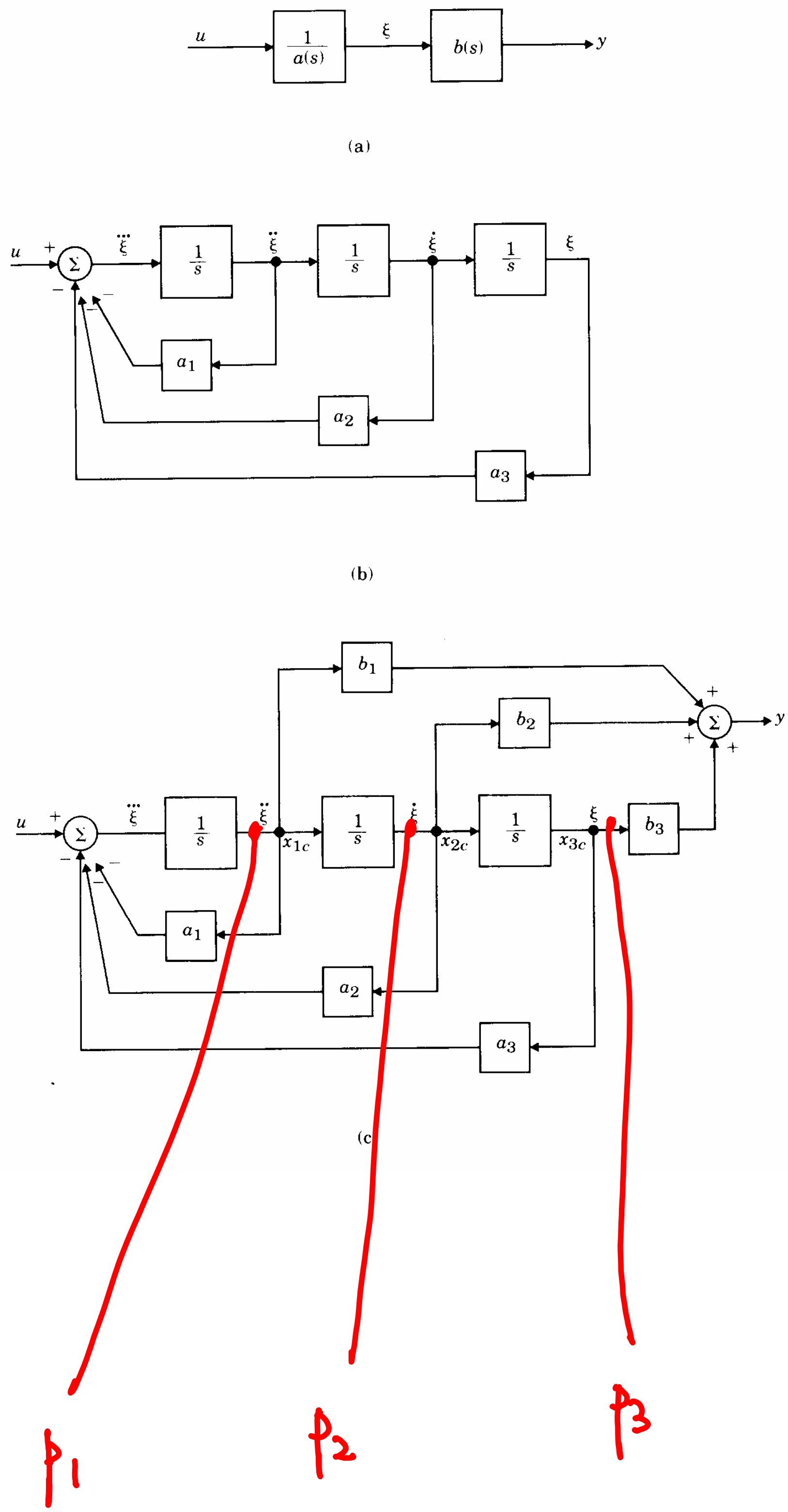
$$\therefore\ Y(s) = b(s)\xi(s)$$

or $$y = b_1\ddot{\xi} + b_2\dot{\xi} + b_3\xi \qquad (6.55)$$

(6.52) is easily patched up in an integrator patching diagram.

Then output (6.55) is easily picked out from outputs of integrators.

FIGURE 6.15
Derivation of control canonical form.



Represent output of integrator 1 as ρ_1
output of integrator 2 as ρ_2
output of integrator 3 as ρ_3

Then
$$\dot{\rho}_1 = \dddot{\xi} = -a_1\rho_1 - a_2\rho_2 - a_3\rho_3 + u$$
$$\dot{\rho}_2 = \rho_1$$
$$\dot{\rho}_3 = \rho_2$$

Check this

$\dot{\rho} = F_c\rho + G_c u$
$y = H_c\rho + J_c u$

coeff of $a(s)$

or
$$F_c = \begin{bmatrix} -a_1 & -a_2 & -a_3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \qquad G_c = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$I_{(n-1)\times(n-1)}$

$$H_c = \begin{bmatrix} b_1 & b_2 & b_3 \end{bmatrix} \qquad J_c = 0$$

The characteristic equation of this system is obtained simply from 1st row of F, and is

$$\det[sI - F_c] = s^3 + a_1 s^2 + a_2 s + a_3 = 0$$

$\frac{b(s)}{a(s)}$

This follows straightaway from our transfer function from which we started.

$G_c = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

Consider now state-feedback of the form

$$u = -K'\rho = -\begin{bmatrix} K_1' & K_2' & K_3' \end{bmatrix} \begin{bmatrix} \rho_1 \\ \rho_2 \\ \rho_3 \end{bmatrix}$$

Then

Check this

$$\begin{aligned} \dot{\rho} &= F_c \rho + G_c u \\ &= \left[F_c - G_c K' \right] \rho \end{aligned}$$

and

$I_{(n-1)\times(n-1)}$

$$F_c - G_c K' = \begin{bmatrix} -a_1 - K_1' & -a_2 - K_2' & -a_3 - K_3' \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Form of $F_c - G_c K'$ is exactly same as F_c

$\therefore$ we know that c.e. is $\det\{sI - [F_c - G_c K']\} =$

$$s^3 + (a_1 + K_1')s^2 + (a_2 + K_2')s + (a_3 + K_3') = 0$$

$\therefore$ if we desire a closed-loop c.e.

$$\alpha_c(s) = s^3 + \alpha_1 s^2 + \alpha_2 s + \alpha_3 = 0$$

Straightaway we have the desired gains

$$a_i + K_i' = \alpha_i \qquad \text{or} \qquad K_i' = -a_i + \alpha_i = \alpha_i - a_i$$

$$i = 1,2,3$$

(or in the general case $i = 1,2,\ldots n$)

<u>How to use this method</u>

- your system is $\dot{x} = Fx + Gu$

  method works for $\dot{\rho} = F_c\rho + G_c u$

  $\therefore$ must find the transformation T

$$\rho = Tx$$

where

$$\left.\begin{aligned} F_c &= TFT^{-1} \\ G_c &= TG \end{aligned}\right\} \text{——— (*)}$$

and $\quad u = -K'\rho = -K'Tx = -Kx$

where $\quad K = K'T$

T is found from the relations in (*)

- thus find T that satisfies (*) transform to p system

  find K' from $K_i' = -a_i + \alpha_i$

  then transform back with

  $$K = K'T$$

- Ackermann's formula does this automatically for us.

  If your desired closed-loop c.e. is

  $$\alpha_c(s) = s^n + \alpha_1 s^{n-1} + \ldots + \alpha_n = 0$$

  then Ackermann's formula says that your feedback law must be

  $$u = -Kx$$

  where

  $$K = \begin{bmatrix} \underbrace{0 \quad 0 \quad \ldots \quad 0}_{1 \times (n-1)} & \vdots & 1 \end{bmatrix} \zeta^{-1} \alpha_c(F)$$

$\dot{x} = Fx + Gu$

$y = Hx$

$$\zeta = \begin{bmatrix} G & FG & F^2G & \dots & F^{n-1}G \end{bmatrix}$$ controllability matrix

n is the order of the system

$$\alpha_c(F) = F^n + \alpha_1 F^{n-1} + \dots + \alpha_n I$$

where $\alpha_c(s) = s^n + \alpha_1 s^{n-1} + \dots + \alpha_n$ was the desired char poly.

Example : Undamped Oscillator

- previously, we compared coeff
- trivial in this case, but we wish to illustrate principle.
- Ackermann's formula easily incorporated in calculations using software packages e.g. MATLAB

$F = \begin{bmatrix} 0 & 1 \\ -w_0^2 & 0 \end{bmatrix}$

$G = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$\alpha_c(s) = (s+2w_0)^2 = s^2 + 4w_0 s + 4w_0^2$

$$\alpha_1 = 4w_o \quad ; \quad \alpha_2 = 4w_o^2$$

$$\alpha_c(F) = \dots = \begin{bmatrix} 3w_o^2 & 4w_o \\ -4w_o^3 & 3w_o^2 \end{bmatrix}$$

$$\zeta = \begin{bmatrix} G & FG \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad ; \quad \therefore \quad \zeta^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$K = \begin{bmatrix} K_1 & K_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 3w_o^2 & 4w_o \\ -4w_o^3 & 3w_o \end{bmatrix}$$

$$= \begin{bmatrix} 3w_o^2 & 4w_o \end{bmatrix} \qquad \text{as before}$$

Question (ii) : Practical considerations notwithstanding, can we place the closed-loop poles anywhere we like in all systems ?

Part of answer is given by Ackermann's formula
To place closed-loop poles at roots of

$$\alpha_c(s) = 0$$

$\alpha_c(s) = s^n + \alpha_1 s^{n-1} + \dots + \alpha_n$

the feedback law

$$u = -Kx = -[0 \quad \dots \quad 0 \quad 1]\zeta^{-1}\alpha_c(F)x$$

is needed.

Obviously, K cannot be calculated if the "controllability matrix"

$$\zeta = [G \quad FG \quad \dots \quad F^{n-1}G]$$

is singular

$\dot{x} = Fx + Gu$

$y = Hx$

A matrix M is singular iff $\det[M] = 0$

$\therefore$ for a linear time-invariant system

$$\dot{x} = Fx + Gu$$

the following statements are equivalent

(a) the closed-loop poles can be arbitrarily placed with the feedback $u = -kx$;

(b) ζ is non-singular ; $\zeta \triangleq [G \,\vdots\, FG \,\vdots\, \ldots F^{n-1}G]$

(c) the system is "controllable."

That is (a) $\Leftrightarrow$ (b) $\Leftrightarrow$ (c)

A Rough Physical Meaning for Controllability

$$\dot{x} = Fx + Gu$$

If the c.e. $\det[sI - F] = 0$ has no repeated roots, then it is possible to transform the representation to a diagonal form.

i.e. can find a T, with $p = Tx$ so that

$$\dot{p} = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix} p + \begin{bmatrix} g_1' \\ g_2' \\ \vdots \\ g_n' \end{bmatrix} u$$

If the system is controllable, then

all $g_i' \neq 0$

- thus every transformed state is "controlled" directly by the input

If system is not controllable, then some of the g_i will be zero. Thus, some of the components of p are not affected by the input u.

Warning :

While controllability can be checked by non-singularity of ζ, physical insight must also be used.

Exercise:

$$G(s) = \frac{(s+\alpha)}{(s+1)(s+2)}$$

- Method 1
- Method 2

Method 3

- $G(s) = \frac{(-1+\alpha)}{(s+1)(-1+2)} + \frac{(-2+\alpha)}{(s+2)(-2+1)}$

Example

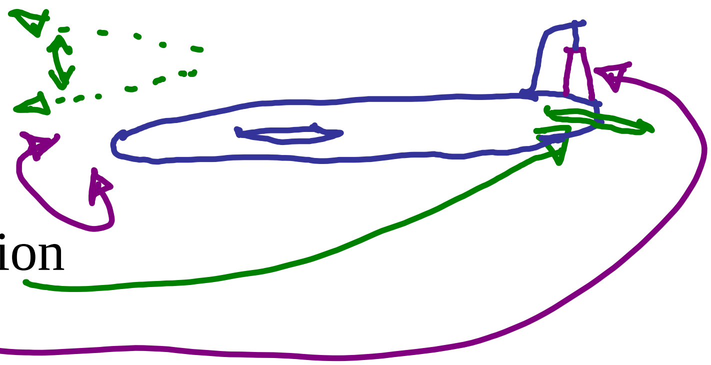
$$\begin{bmatrix} \dot{x}_p \\ \dot{x}_r \end{bmatrix} = \begin{bmatrix} F_p & \varepsilon \\ 0 & F_r \end{bmatrix} \begin{bmatrix} x_p \\ x_r \end{bmatrix} + \begin{bmatrix} G_p & 0 \\ 0 & G_r \end{bmatrix} \begin{bmatrix} \delta_e \\ \delta_a \end{bmatrix}$$

x_p : pitch motion

x_r : roll motion

δ_e : elevator section

δ_a : aileron



ε a very small number

ε represent weak coupling from x_r to x_p

mathematical test (used foolishly) would show that

$$\begin{bmatrix} \dot{x}_p \\ x_r \end{bmatrix} = \begin{bmatrix} F_p & \varepsilon \\ 0 & F_r \end{bmatrix} \begin{bmatrix} x_p \\ x_r \end{bmatrix} + \begin{bmatrix} 0 \\ G_r \end{bmatrix} \delta_a$$

is controllable !

Check controllability matrix. Look at its relative values!

But would be physically unreasonable to control x_p using δ_a directly.