Note Title 5/8/2013

1

$$\ddot{y} + a_1 \dot{y} + a_2 y = b u$$

Transfer function

$$\frac{Y(s)}{U(s)} = \frac{b}{s^2 + a_1 s + a_2}$$

For this simple example, let's assume that we identify the variables:

$$x_1(t) \triangleq y(t)$$

$$x_2(t) \triangleq \dot{y}(t)$$

With this identification, we have:

In matrix form, this is

$$\dot{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}} = \overset{F}{\begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix}} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \overset{G}{\begin{bmatrix} \quad \\ \quad \end{bmatrix}} u$$

$$y = \underset{H}{\begin{bmatrix} \quad & \quad \end{bmatrix}} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Simple non-linear d.e

$$\ddot{\theta} + \omega_0^2 \sin\theta = \frac{T_c}{ml^2}$$

Suppose we are interested in the variables:

$$x_1(t) \triangleq \omega_0 \theta(t)$$

$$x_2(t) \triangleq \dot{\theta}(t)$$

state-variable form!!!

Then, we have

$$\boxed{\begin{aligned} \dot{x}_1(t) &= \omega_0 \dot{\theta}(t) = \omega_0 x_2 \\ \dot{x}_2(t) &= \ddot{\theta}(t) = -\omega_0^2 \sin\left(\frac{x_1}{\omega_0}\right) + \frac{T_c}{ml^2} \end{aligned}}$$

For linearization and small deviations about the equilibrium

$$x_1 = 0 , \quad x_2 = 0$$

First, we note that for small deviation about the above, we have

$$\sin\left(\frac{x_1}{w_0}\right) \simeq \frac{x_1}{w_0}$$

[from $\sin\alpha \simeq \alpha$ for small α]

and this next gives the state-variable equations as:

$$\dot{x}_1 =$$

$$\dot{x}_2 =$$

$$=$$

leading to the linearised state-variable equations

$$\begin{bmatrix} \dot{x}_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} & \\ & \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} \\ \\ \end{bmatrix} u$$

Time-varying systems example

$$\ddot{y} + a_1(t)\,\dot{y} + a_2(t)\,y = b(t)\,u$$

Casting this in state-variable form is straightforward with, for example,

$$x_1(t) \triangleq y(t)$$

$$x_2(t) \triangleq \dot{y}(t)$$

$\vdots$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -a_2(t) & -a_1(t) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ b(t) \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

State-Variable Realizations

Method 1

$$\dddot{y} + a_1 \ddot{y} + a_2 \dot{y} + a_3 y \qquad — (1.1)$$

$$= b_1 \ddot{u} + b_2 \dot{u} + b_3 u$$

$$\frac{Y(s)}{U(s)} = \frac{b_1 s^2 + b_2 s + b_3}{s^3 + a_1 s^2 + a_2 s + a_3}$$

— (2.1)

Consider the intermediate variable $\xi(t)$

defined by:

$$\frac{\xi(s)}{U(s)} = \frac{1}{s^3 + a_1 s^2 + a_2 s + a_3} \quad \text{— (2.2)}$$

Then, in the time-domain, this is equivalent to:

— (3.1)

With (2.1) and (2.2), we must have

$$Y(s) = (b_1 s^2 + b_2 s + b_3)\,\xi(s)$$

In time-domain, this is

— (3.2)

With this, we now set up the patching diagram:

Identify each output of an integrator as a state variable

Then, note that we now have:

$$\dot{x}_1 =$$

$$\dot{x}_2 =$$

$$\dot{x}_3 =$$

$$y =$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} \\ \\ \\ \end{bmatrix} u \quad \text{— (4.1a)}$$

$$y = \begin{bmatrix} & & \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad \text{— (4.1b)}$$

(4.1a) and (4.1b) constitute a state-variable representation of the d.e. (1.1)

$$\dddot{y} + a_1 \ddot{y} + a_2 \dot{y} + a_3 y \qquad \text{—— (1.1)}$$
$$= b_1 \ddot{u} + b_2 \dot{u} + b_3 u$$

Think of how the above pattern automatically generalises to:

- 5th order
- ⋮
- nth order

$$y^{(5)} + a_1 y^{(4)} + a_2 y^{(3)} + a_3 \ddot{y} + a_4 \dot{y} + a_5 y = b_1 u^{(4)} + b_2 u^{(3)} + b_3 \ddot{u} + b_4 \dot{u} + b_5 u$$