EE5110 Segment 3

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Q1

a) Calculus of variations

The input signal is :

$$u = -\frac{a}{2}y + \frac{\dot{y}}{2}$$

The function $J(y, u)$ changes to:

$$J(y) = \int_0^{\infty} [y^2 + (-\frac{a}{2}y + \frac{\dot{y}}{2})^2]dt$$
$$= \int_0^{\infty} [(1 + \frac{a^2}{4})y^2 + \frac{1}{4}\dot{y}^2]dt - \frac{a}{2}\int_0^{\infty} y\dot{y}dt$$

$\because$

$$\int_0^{\infty} y\dot{y}dt = y^2|_0^{\infty} - \int_0^{\infty} y\dot{y}dt$$
$$\int_0^{\infty} y\dot{y}dt = \frac{1}{2}[y^2(\infty) - y^2(0)]$$

$\therefore$

$$J(y) = \int_0^{\infty} [(\frac{a^2}{4} + 1)y^2 + \frac{1}{4}\dot{y}^2]dt - \frac{a}{4}y^2(\infty) + \frac{a}{4}c^2$$

Let $z(t)$ denote any function of t with the property that $J(z)$ exists. Take ε to be a scalar parameter.

$$J(y_0 + \varepsilon z) = \int_0^{\infty} [(\frac{a^2}{4} + 1)(y_0 + \varepsilon z)^2 + \frac{1}{4}(\dot{y}_0 + \varepsilon \dot{z})^2]dt - \frac{a}{4}y^2(\infty) + \frac{a}{4}c^2$$

The $J(y_0 + \varepsilon z)$ must have an absolute minimum at $\varepsilon = 0$

$$\frac{d}{d\varepsilon}J(y_0 + \varepsilon z)|_{\varepsilon=0} = 0$$

$$J(y_0 + \varepsilon z) = \int_0^{\infty} [(\frac{a^2}{4} + 1)y_0^2 + \frac{1}{4}\dot{y}_0^2]dt + 2\varepsilon \int_0^{\infty} [(\frac{a^2}{4} + 1)(y_0 z) + \frac{a^2}{4}\dot{y}_0 \dot{z}]dt$$
$$+ \varepsilon^2 \int_0^{\infty} [(\frac{a^2}{4} + 1)z^2 + \frac{a^2}{4}\dot{z}^2]dt - \frac{a^2}{4}y^2(\infty) + \frac{a^2}{4}c^2$$

We see then that the variational condition derived is :

$$\int_0^{\infty} [(\frac{a^2}{4} + 1)y_0^2 + \frac{1}{4}\dot{y}_0^2]dt = 0$$

$$2\varepsilon \int_0^{\infty} [(\frac{a^2}{4} + 1)(y_0 z) + \frac{a^2}{4}\dot{y}_0 \dot{z}]dt = 0$$

$$int_0^{\infty} [(\frac{a^2}{4} + 1)z^2 + \frac{a^2}{4}\dot{z}^2]dt - \frac{a^2}{4}y^2(\infty) + \frac{a^2}{4}c^2 = 0$$

Since $y_0 + \varepsilon\ z$ is an admissible function satisfies the initial condition :

$$y_0(0) + \varepsilon z(0) = c$$

We see that $z(0) = 0$.

Since the left − hand side must be zero for all admissible z, we suspect that

$$(\frac{a^2}{4} + 1)y_0 - \frac{1}{4}\ddot{y}_0 = 0$$

First, we use T to replace ∞, $\dot{y}_0(T) = 0$. And we obtain no condition on $\dot{y}_0(0)$. We can get :

$$y_0(0) = c,\ \dot{y}_o(T) = 0$$

The general solution of the differential equation is :

$$y = c_1 e^{\sqrt{4+a^2}\cdot t} + c_2 e^{-\sqrt{4+a^2}\cdot t}$$

Using the boundary conditions, we have the two equations to determine the coefficients c_1 and c_2.

$$c = c_1 + c_2$$
$$0 = c_1 e^{\sqrt{4+a^2}\cdot T} - c_2 e^{-\sqrt{4+a^2}\cdot T}$$

Solving, we obtain the expression :

$$y_o(t) = c(\frac{e^{\sqrt{4+a^2}(t-T)} + e^{-\sqrt{4+a^2}(t-T)}}{e^{-\sqrt{4+a^2}\cdot T} + e^{\sqrt{4+a^2}\cdot T}}) = c\frac{cosh(\sqrt{4+a^2}(t-T))}{cosh(\sqrt{4+a^2}\cdot T)}$$

Let $T \to \infty$, *We have*

$$y_o(t) = c(\frac{e^{\sqrt{4+a^2}(t-T)} + e^{-\sqrt{4+a^2}(t-T)}}{e^{-\sqrt{4+a^2}\cdot T} + e^{\sqrt{4+a^2}\cdot T}}) = c(\frac{e^{\sqrt{4+a^2}(t-2T)} + e^{-\sqrt{4+a^2}\cdot t}}{e^{-2\sqrt{4+a^2}\cdot T} + 1}) \to ce^{-\sqrt{4+a^2}\cdot t}$$

$$\dot{y}_o(t) = c\sqrt{4+a^2}\cdot(\frac{e^{\sqrt{4+a^2}(t-T)} - e^{-\sqrt{4+a^2}(t-T)}}{e^{-\sqrt{4+a^2}\cdot T} + e^{\sqrt{4+a^2}\cdot T}}) = c\sqrt{4+a^2}\cdot(\frac{e^{\sqrt{4+a^2}(t-2T)} - e^{-\sqrt{4+a^2}\cdot t}}{e^{-2\sqrt{4+a^2}\cdot T} + 1}) \to -c\sqrt{4+a^2}\cdot e^{-\sqrt{4+a^2}\cdot t}$$

$$\dot{y}_o = -\sqrt{4+a^2}\cdot y_o(t)$$

So we have the control laws

b) Dynamic programming

Optimal Value function :

$$V(c,T) = \min_y J(y)$$

$$J(y) = \int_0^\Delta + \int_\Delta^T = (c^2+u^2)\Delta + V(c+(ac+2u)\Delta, T-\Delta) + O(\Delta^2)$$

We can use Taylor series to relate $V(c+(ac+2u)\Delta, T-\Delta)$ *with* $V(c,T)$, $J(y)$ *will change to*

$$V(c,T) = \min_u[(c^2+u^2)\Delta + V(c,T) + \frac{\partial V}{\partial c}(ac+2u)\Delta - \frac{\partial V}{\partial T}\Delta + O(\Delta^2)]$$

Ignoring the higher order terms of Δ, *we have*

$$\frac{\partial V}{\partial T} = \min_u[(c^2+u^2) + \frac{\partial V}{\partial c}(ac+2u)]$$

When $T \to \infty$, $V(c,T)$ *becomes* $V(c)$,

$$V(c) = \min_u[(c^2+u^2)\Delta + V(c+(ac+2u)\Delta)] + O(\Delta^2)$$
$$0 = \min_u[(c^2+u^2) + \dot{V}(c)(ac+2u)]$$

Take the derivative respect to u gives $2u + 2\dot{V}(c)a = 0$, *so*

$$u = -\dot{V}(c)$$
$$0 = (c^2 + (\dot{V}(c))^2) + \dot{V}(c)(ac - 2\dot{V}(c))$$
$$\dot{V}^2(c) - ac\dot{V}(c) + c^2 = 0$$

So we have two possibilities, with the condition $V(0) = 0$, *we can obtain two possible solutions :*

$$V(c) = \frac{a+\sqrt{a^2+4}}{2}$$
$$V(c) = \frac{a-\sqrt{a^2+4}}{2}$$

Since $V(c) \geqslant 0$, *we see that* $V(c) = \frac{a+\sqrt{a^2+4}}{2}$, *the optimal value can be easily obtained as*

$$u = -\frac{a+\sqrt{a^2+4}}{2}c$$

Since $y(0) = c$, *so we have* $u(0) = \frac{a+\sqrt{a^2+4}}{2}\ y(0)$. *At any time t, we will have the control law :*

$$u(t) = -\frac{a+\sqrt{a^2+4}}{2}y(t)$$

The results from two method are same

Q2

We write the optimal value function as

$$V_N(c) = \min_{u_n} J_N(y,u)$$

After $u(0)$ *is chosen, the new state of the system is* $y(1) = 2c + au(0)$, *The cost function takes the form*

$$c^2 + u^2(0) + \sum_{n=1}^{N}(y^2(n) + u^2(n))$$

$The\ long\ term\ cost\ can\ be\ expressed\ as\ optimal\ value\ starting\ from\ 2c + au(0)\ withe\ N-1\ steps\ left$

$$\sum_{n=1}^{N}(y^2(n) + u^2(n)) = V_{N-1}(ac + 2u(0))$$

$Then$

$$V_N(c) = \min_{u(0)}[c^2 + u^2(0) + V_{N-1}(ac + 2u(0))]$$

$For\ the\ continuous\ case\ we\ have V(c,T) = c^2 r(T)$

$It\ is\ reasonable\ to\ guess\ that$

$$V_N(c) = c^2 r_N$$

$$c^2 r_N = \min_{u(0)}[c^2 + u^2(0) + (ac + 2u(0))^2 r_{N-1}]$$

$The\ value\ of\ u(0)\ that\ minimizes\ is\ readily\ obtained\ by\ differentiation$

$$2u(0) + 2a(2a + au(0))r_{N-1} = 0$$

$$u(0) = -\frac{2acr_{N-1}}{1 + 4r_{N-1}}$$

$Using\ this\ value,\ we\ obtain\ the\ recurrence\ relation$

$$r_N = 1 + \frac{a^2 r_{N-1}}{1 + 4r_{N-1}}$$

$At\ each\ time\ t = k,\ the\ input\ control\ is$

$$u(k) = -\frac{2acr_{N-k-1}y(k)}{1 + 4r_{N-k-1}}$$

$When\ N \to \infty,\ let\ r = \lim_{N\to\infty} r_N,\ then\ r\ is\ the\ positive\ root\ of\ the\ quadratic\ equation$

$$r = \frac{(a^2 + 3) + \sqrt{a^4 + 6a^2 + 25}}{8}$$

$The\ control\ signal\ will\ change\ to:$

$$\lim_{N\to\infty} u(0) = -\frac{2acr}{1 + 4r}$$

$$\therefore\ for\ the\ inftyite\ time\ process,\ the\ optimal\ feedback\ controller\ is:$$

Q3

$Assume\ that\ the\ lifeguard\ will\ run\ to\ (a,0),\ and\ then\ swim\ to\ the\ swimmer.\ The\ parameter\ we\ can\ get\ from\ question:\ v$
$/s.\ The\ optimal\ function\ can\ be\ expressed\ as:$

$$T = \min_a[(\frac{\sqrt{a^2+10^2}}{v_1}) + (\frac{\sqrt{(20-a)^2+(-10)^2}}{v_2})]$$

$Take\ the\ derivative\ respect\ to\ a,\ we\ get$

$$\frac{29a^2}{100} - 10a + 129 = 0$$

$$a = \frac{10}{0.58} = 17.24$$

$So,\ the\ shortest\ time\ path\ is\ that\ lifeguard\ run\ to\ (18.823, 0)\ and\ then\ swim\ to\ the\ swimmer.\ The\ shortest\ time\ is:$

$$t_{min} = \frac{\sqrt{a^2+10^2}}{v_1} + \frac{\sqrt{(20-a)^2+(-20)^2}}{v_2} = 9.17s$$

In [1]:
```
import numpy as np
np.sqrt(17.24**2+100)/5+np.sqrt((20-17.24)**2+100)/2
```

Out[1]: 9.173008873571147

Q4

Method1: Programming view

$This\ question\ we\ can\ refer\ to\ Dijkstra,\ Floyd\ method.\ It\ has\ the\ same\ principle\ with\ optimal\ solution\ method.$
$The\ basic\ logicis:$
$Based\ on\ the\ algorism,\ suppose\ we\ get\ the\ optimal\ result\ of\ set\ S,\ S\ includes\ the\ places\ we\ traveled.$
$We\ have\ n\ place,\ suppose\ we\ traveled\ k\ places,\ k<n\ and\ belongs\ to\ S,\ the S(k)\ should\ also\ be\ optimal\ result.$

$We\ can\ put\ all\ attractions\ and\ hotel\ in\ the\ x\ y\ plane\ and\ sort\ them\ by\ x\ coordinate\ from\ small\ to\ large\ p_0, p_1, p_2, \ldots, p_n. p_0$

$In\ order\ to\ get\ the\ shortest\ circle,\ we\ set\ dist(i,j) = \sqrt{(x_i - xj)^2 + (y_i - y_j)^2}\ as\ the\ distance\ between\ place\ i\ and\ j.$

$step1: Set\ S\ contains\ only\ the\ source\ point.\ S = p_0,\ and\ the\ distance\ of\ p_0\ is\ 0. U\ contains\ vertices\ other\ than\ places,\ i.$
$and\ we\ can\ calculate\ all\ the\ distance\ between\ two\ places.$

$step2$
$:\ Select\ a\ vertex\ p_k\ from\ U\ with\ the\ smallest\ distance\ from\ p_0\ and\ add\ p_k\ to\ S\ (the\ selected\ distance\ is\ the\ shortest\ path$

$step3: Take\ k\ as\ the\ newly\ considered\ intermediate\ point\ and\ modify\ the\ distance\ of\ each\ vertex\ in\ U;$
$If\ the\ distance\ from\ the\ source\ point\ p_0\ to\ the\ vertex\ p_u\ (through\ the\ vertex\ p_k)\ is\ shorter\ than\ the\ original\ distance$
$(without\ passing\ through\ the\ vertex\ p_k),\ the\ distance\ value\ of\ the\ vertex\ p_u\ is\ modified,$
$and\ the\ modified\ distance\ value\ is\ the\ distance\ of\ the\ vertex\ p_k\ plus\ the\ distance\ between\ two\ places.$

$step4: Repeat\ the\ step2\ and\ step\ 3\ until\ all\ the\ places\ are\ in\ S\ set.$

Method2: Mathmatic view

$First,\ we\ can\ put\ all\ attractions\ and\ hotel\ in\ the\ x\ y\ plane\ and\ sort\ them\ by\ x\ coordinate\ from\ small\ to\ large\ p_0, p_1, p_2, \ldots$

$Assume\ that\ V_{i,j}\ (i \leqslant j)\ is\ the\ shortest\ closed\ curve\ which\ contain\ p_0, p_1, \ldots, p_n.$

$This\ path\ goes\ from\ p_i\ to\ p_0,\ and\ then\ goes\ from\ p_0\ right\ to\ p_j. So,\ V_{n,n}\ is\ what\ we\ want\ in\ this\ topic.$

$Assume\ that\ the\ length\ of\ V_{i,j}\ is\ l(i,j),\ the\ distance\ between\ p_i\ and\ p_j\ is\ dist(i,j) = \sqrt{(x_i - xj)^2 + (y_i - y_j)^2}$

$In\ the\ path\ V_{i,j},\ p_i\ is\ in\ the\ path\ p_i \to p_1,\ p_j\ is\ in\ the\ path\ p_1 \to p_j.\ Now,\ let's\ talk\ about\ the\ position\ of\ p_{j-1}$

(1) $i < j-1$

$Because\ p_{j-1}\ is\ on\ the\ right\ side\ of\ p_i,\ so\ p_{j-1}\ is\ in\ the\ path\ p_1 \to p_j.$
$Besides,\ p_{j-1}\ is\ the\ rightmost\ point\ except\ p_j,\ so\ it\ connect\ to\ p_j\ directly.\ We\ can\ get$

$$l(i,j) = l(i,j-1) + dist(j-1,j)$$

(2) $i = j-1$

$In\ this\ case,\ p_{j-1}\ is\ p_i,\ so\ p_{j-1}\ is\ in\ the\ path\ p_i \to p_1.\ Any\ point\ from\ p_1, p_2, \ldots, p_{j-2}\ can\ connect\ to\ p+j.$

$Assume\ that\ point\ is\ p_k (1 \leqslant k \leqslant j-2).\ We\ need\ to\ chose\ an\ appropriate\ point\ p_k\ so\ that\ we\ can\ get\ the\ shortest\ l(i,j)$

$$l(i,j) = \min_{1 \leqslant k \leqslant j-2} [l(k,j-1) + dist(k,j)]$$

(3) $i = j$

$This\ only\ happens\ when\ i = j = n.\ In\ this\ case,\ p_{n-1}\ connect\ to\ p_n,\ we\ can\ get:$

$$l(n,n) = l(n-1,n) + dist(n-1,n)$$

$In\ conclusion\ the\ optimal\ function\ is:$

$$l(i,j) = \begin{cases} l(i,j-1) + dist(j-1,j), & i < j-1 \\ \min_{1 \leqslant k \leqslant j-2} [l(k,j-1) + dist(k,j)], & i = j-1 \\ l(n-1,n) + dist(n-1,n), & i = j = n \end{cases}$$

$This\ function\ is\ what\ we\ want.$