- choose your $\alpha_c(s)$.
  calculate control gain K ;

- if all states can be measured, skip this step.
  if states need to be estimated
  choose your $\alpha_e(s)$;

- use the loop structure shown.
  use a package like Matrix$_x$ or MATLAB,
  find the transfer function from $r^1 \mapsto y$.

- for the t.f. $r^1 \mapsto y$, find the gain at $s = j\omega = 0$.
  Then use $r^1 = K_s r$ where K_s is a scaling gain so that

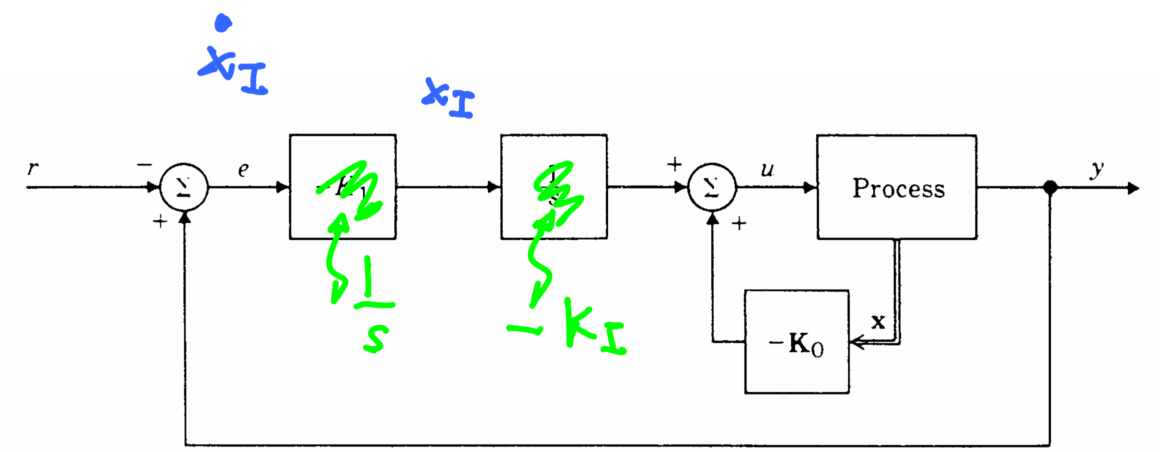
$$K_s G_{r^1 \mapsto y}(s) \Big|_{s=j\omega=0} = 1$$

Integral Control

- important because in every system, there is possibility of disturbance

- state feedback considered thus far will not reject a persistent disturbance

- inclusion of integral control also automatically takes care of steady state reference signal tracking.



Proper way to introduce integral action:

"state augmentation"

Also consider

① $\dot{x} = Fx + Gu + G_1 w$

a constant disturbance

Plant

$$\dot{x} = Fx + Gu$$

$$y = Hx$$

Augment (i.e. add in) an extra state

$$\dot{x}_I = y = Hx$$

$\dot{x}_I = y - r$

② Suppose we also need $[y(t) - r] \to 0$

What then?

Then, augmented system becomes

$$\begin{bmatrix} \dot{x} \\ \dot{x}_I \end{bmatrix} = \begin{bmatrix} F & 0 \\ H & 0 \end{bmatrix} \begin{bmatrix} x \\ x_I \end{bmatrix} + \begin{bmatrix} G \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} H & 0 \end{bmatrix} \begin{bmatrix} x \\ x_I \end{bmatrix}$$

or

$$\left.\begin{aligned} \dot{\bar{x}} &= \bar{F}\bar{x} + \bar{G}u \\ y &= \bar{H}\bar{x} \end{aligned}\right\}$$

Now that this is in the standard state-space form, we can apply all the methods we talked out.

Example:

Plant $$\frac{Y(s)}{U(s)} = \frac{1}{s+3}$$

- wish to have integral control with closed-loop poles at $\omega_n = 5, \zeta = 0.5$

  i.e. $\alpha_c(s) = s^2 + 5s + 25$

- augmented system

$$\begin{bmatrix} \dot{x}_I \\ \dot{x} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -3 \end{bmatrix} \begin{bmatrix} x_I \\ x \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u + \begin{bmatrix} -1 \\ 0 \end{bmatrix} r$$

(should not bother you by now whether

$\bar{x} = [x_I \quad x]^T$ or $\bar{x} = [x \quad x_I]^T$!!

You should be able to keep track of the states at this stage!!)

- calculate your control gains K from

$$\det[sI - \begin{bmatrix} 0 & 1 \\ 0 & -3 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} K] = s^3 + 5s + 25$$

This gives

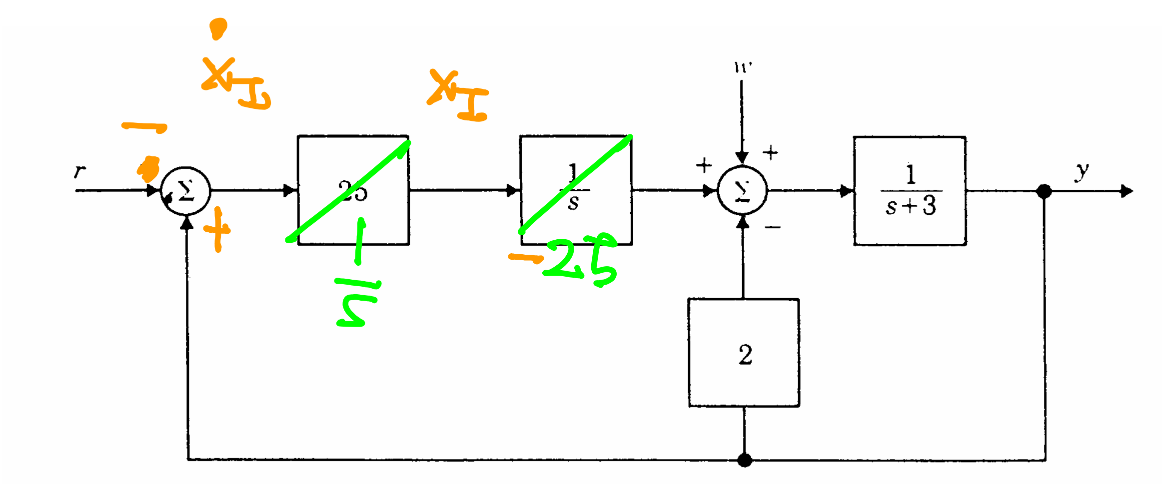
$$K = [25 \quad 2]$$

and $\qquad u = -K\begin{bmatrix} x_I \\ x \end{bmatrix} \qquad$ in this case.

Integral control example Block diagram:

- zero steady state error to step r

- zero steady state error to constant (persistent) disturbance w.



An Appendix on Solution of State Equations

Not for exams

System $\dot{x} = Fx + Gu$

$y = Hx$

Consider first $u(t) \equiv 0$, with $x(0) = x_o$

Define a matrix exponential function by the power series

$$e^{Ft} = (I + Ft + \frac{F^2 t^2}{2!} + \ldots)$$

$$= \sum_{K=0}^{\infty} \frac{F^K t^K}{K!} \qquad \text{with} \qquad F^o = I$$

K!=1

$$\dot{x} = Fx; \qquad x(0) = x_o$$

Assume solution is given by

$$x(t) = e^{Ft} x_o$$

Check if this is correct

First note

$$\frac{d}{dt}e^{Ft} = \frac{d}{dt}\left[\sum_{K=0}^{\infty}\frac{F^K t^{K-1}}{K!}\right]$$

$$= \sum_{K=1}^{\infty}\frac{F^K t^{K-1}}{(K-1)!}$$

$$= F\sum_{j=0}^{\infty}\frac{F^j t^j}{j!} = Fe^{Ft}$$

Now let us see if solution correct.

- at $t = 0$,

$$e^{Ft}x_o = e^{F.0}x_o = Ix_o = x_o$$

using def^n of e^{Ft} with $t = 0$

$\therefore$ fits at $t = 0$

- Check if $\dot{x} = Fx$

L.H.S. $= \dot{x} = \frac{d}{dt}(x(t)) = \frac{d}{dt}(e^{Ft}x_o)$

$= Fe^{Ft}x_o$ from above

$= Fx$ = R.H.S.

$\therefore$ fits state-space d.e. as well.

$\therefore$ solution to

$$\dot{x} = Fx; \quad x(0) = x_o$$

is $\quad x(t) = e^{Ft} x_o$

$\Delta\Delta$

What about case with non-zero $u(t)$?

$$\dot{x} = Fx + Gu; \qquad x(0) = x_o$$

Solution is in fact

$$x(t) = e^{Ft} x_o + \int_0^t e^{F(t-\tau)} Gu(\tau) d\tau$$

Compare this with your "A"-level; JC; or high school mathematics!

$\dot{x} = fx + gu$

$x(0) = x_0$

Solution confirmed in same way:

- at $t = 0$

$t = 0, \qquad e^{Ft} x_o = x_o \qquad$ (from previous)

$t = 0, \qquad \int_0^t e^{F(t-\tau)} Gu(\tau) d\tau = \int_0^0 [\qquad] d\tau = 0$

$\therefore$ fits at $t = 0$

$x(t) = e^{ft} x_0 + \int_0^t e^{f(t-\tau)} g\, u(\tau) d\tau$

- Check if $\dot{x} = Fx + Gu$

L.H.S. $= \dot{x} = \dfrac{d}{dt}x(t)$

$$= \frac{d}{dt}\left[e^{Ft}x_o + \int_0^t e^{F(t-\tau)}Gu(\tau)d\tau \right]$$

remember that $\dfrac{d}{dt}\int_0^t \alpha(t,\tau)d\tau$

$$= \alpha(t,\tau)\,\Big|_{\tau=t} + \int_0^t \frac{d}{dt}\alpha(t,\tau)d\tau$$

$$\therefore \; \frac{d}{dt}\left[e^{Ft}x_o + \int_0^t e^{F(t-\tau)}Gu(\tau)d\tau \right]$$

$$= Fe^{Ft}x_o + e^{F(t-\tau)}Gu(t)\Big|_{\tau=t} + \int_0^t Fe^{F(t-\tau)}Gu(\tau)d\tau$$

$$= F\left[e^{Ft}x_o + \int_0^t e^{F(t-\tau)}Gu(\tau)d\tau \right] + Gu(t)$$

$$= Fx(t) + Gu(t) \qquad = R.H.S.$$

$\therefore$ state-space equation is satisfied

**

Thus for

$$\dot{x} = Fx + Gu\,; \quad x(0) = x_o$$

$$y = Hx$$

Solution is

$$x(t) = e^{Ft}x_o + \int_0^t e^{F(t-\tau)}Gu(\tau)d\tau$$

and

$$y(t) = Hx(t)$$

$$= \underbrace{He^{Ft}x_o}_{\text{zero input response}} + \underbrace{\int_0^t He^{F(t-\tau)}Gu(\tau)d\tau}_{\text{zero state response}}$$

Note that when we considered impulse response,
We said, let

$$\frac{Y(s)}{U(s)} = G(s)$$

then $y(t) = \int_0^t g(t-\tau)u(\tau)d\tau = g(t) \underset{\text{convolution}}{*} u(t)$

where $g(t)$ is impulse response.

From above, impulse response $= g(t) = He^{Ft}G$

$\Delta\Delta$

Ways to calculate e^{Ft} (Franklin & Powell pp. 410)

- important previously
- not so important nowadays with advent of inexpensive computing facilities, and "canned" control system software

1. Matrix Exponential Series

use the def^n directly

$$e^{Ft} = I + Ft + \frac{F^2t^2}{2!} + \frac{F^3t^3}{3!} + \ldots$$

- can only be used for F with all negative e-values
- truncate when high-order terms sufficiently small

2. Inverse Laplace Transform

$$e^{Ft} = L^{-1}\left\{(sI - F)^{-1}\right\}$$

- invert the matrix $(sI - F)$;
- find the inverse Laplace transform for each element $(sI - F)^{-1}{}_{ij}$

3. Diagonalization of System Matrix F

- can only be used if F is diagonalizable

Optional further investigation: Generalizes to the "Jordan" matrix for repeated eigenvalues

find transformation T so that

$$T\Lambda = FT \ \text{ or } \ \Lambda = T^{-1}FT$$

$$\left[t_1 \; \vdots \; t_2 \; \vdots \; \dots \; \vdots \; t_n\right]\begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix} = F\left[t_1 \; \vdots \; t_2 \; \vdots \; \dots \; \vdots \; t_n\right]$$

thus columns of T are in fact eigenvectors of F, and λ_i are corresponding eigenvalues

(use a CAD package, if possible, to find T)

then

$$e^{Ft} = T\begin{bmatrix} e^{\lambda_1 t} & & & 0 \\ & e^{\lambda_2 t} & & \\ & & \ddots & \\ 0 & & & e^{\lambda_n t} \end{bmatrix}T^{-1}$$

Not for exams!

Digital Control: State-Space Methods

Recall that there are two main design philosophies in digital control

(a) Discrete-time implementation of continuous-time controllers

(b) Direct design of digital controllers

(a) Discrete-Time Implementations of Continuous-Time Controllers

- Straightforward
  use rules like
  - Forward Rectangular Rule
  - Backward Rectangular Rule
  - Bilinear Transformation
  - Matched Pole-Zero Approximation

- sampling interval, h, must be chosen carefully
  typically $\frac{1}{10}$ of smallest time constant in system

Each of these methods have advantages & disadvantages.

If h chosen as suggested (i.e. $\frac{1}{10}$ of smallest time constant in system), then all the above methods can be used in the discrete-time implementation of the continuous-time controller.

Example using Forward Rectangular Rule (also called Delta operator).

- Some notation first.

  use the notation $p = \frac{d}{dt}$, the differential operator, where:

  i.e. $$px(t) = \frac{d}{dt}x(t)$$

  and introduce the forward-shift operator, q, where:

  $$qx(kh) = x(\overline{k+1}h),\ k = 0,1,2,\ldots$$

  note that p and q are time-domain operators corresponding frequency domain variable are

p	$\rightarrow$	s
q	$\rightarrow$	z
time domain		frequency (or complex variable) domain

forward rectangular rule:

substitute $\dfrac{z-1}{h}$ for s

equivalently, in time domain

substitute $\dfrac{q-1}{h}x(kh)$ for $\dfrac{d}{dt}x(t)$

at $t = kh$

i.e. use $\dfrac{x(\overline{k+1}\,h)-x(kh)}{h}$ for $\dfrac{d}{dt}x(t)$

Discrete-time implementation of controller in Franklin & Powell pp361 example

Plant is $G(s) = \dfrac{1}{s^2}$

i.e. $\dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$

$y = \begin{bmatrix} 1 & 0 \end{bmatrix} x$

control roots to be placed at

$s = (-1 \pm j1)/\sqrt{2}$ $\quad (\omega_n = 1, \zeta = 0.7)$

this gives $K = \begin{bmatrix} 1 & \sqrt{2} \end{bmatrix}$

Estimator error roots are at

$$s = -2.5 \pm j4.3 \qquad (\omega_n = 5, \zeta = 0.5)$$

this gives $\quad L = \begin{bmatrix} 5 \\ 25 \end{bmatrix}$

Thus the continuous-time controller is

$$\dot{\hat{x}} = (F - GK - LH)\hat{x} + Ly$$

$$= \left\{ \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & \sqrt{2} \end{bmatrix} - \begin{bmatrix} 5 \\ 25 \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix} \right\} \hat{x}$$

$$+ \begin{bmatrix} 5 \\ 25 \end{bmatrix} y$$

$$\dot{\hat{x}} = \begin{bmatrix} -5 & 1 \\ -26 & -\sqrt{2} \end{bmatrix} \hat{x} + \begin{bmatrix} 5 \\ 25 \end{bmatrix} y$$

$$u = -\begin{bmatrix} 1 & \sqrt{2} \end{bmatrix} \hat{x}$$

The time constants correspond to roots $\quad \tau \simeq \frac{1}{\omega_n}$

at $s = 0$ (plant)

$s = (-1 \pm j1)/\sqrt{2} \qquad (\omega_n = 1, \zeta = 0.7)$ (control) $\tau_c \simeq 1$

$s = -2.5 \pm j4.3 \qquad (\omega_n = 5, \zeta = 0.5)$ (estimation) $\tau_e \simeq \frac{1}{5}$

$\therefore$ choose h to be $\frac{1}{10}$ of smallest time constant

i.e. $h = \frac{1}{10}\left(\frac{1}{5}\right) = 0.02$ seconds

$$\dot{\hat{x}} = \begin{bmatrix} -5 & 1 \\ -26 & -\sqrt{2} \end{bmatrix} \hat{x} + \begin{bmatrix} 5 \\ 25 \end{bmatrix} y$$

discrete-time implementation:

$$\dot{\hat{x}}(t) \cong \frac{q-1}{h}\hat{x}(kh) = \frac{\hat{x}(\overline{k+1}h) - \hat{x}(kh)}{h}$$

$$u = -\begin{bmatrix} 1 & \sqrt{2} \end{bmatrix} \hat{x}$$

$$\therefore \frac{q-1}{h}\hat{x}(kh) = \begin{bmatrix} -5 & 1 \\ -26 & -\sqrt{2} \end{bmatrix}\hat{x}(kh) + \begin{bmatrix} 5 \\ 25 \end{bmatrix} y(kh)$$

$h = 0.02$

i.e. $\hat{x}(\overline{k+1}\{0.02\})$

$$= \begin{bmatrix} 1 - 5\times 0.02 & 0.02 \\ -26\times 0.02 & 1 - 0.02\times\sqrt{2} \end{bmatrix}\hat{x}(0.02k)$$

$$+ \begin{bmatrix} 5\times 0.02 \\ 25\times 0.02 \end{bmatrix} y(0.02k) \quad \#$$

$k = 0, 1, 2, 3, \ldots$

$$u(0.02k) = -\begin{bmatrix} 1 & \sqrt{2} \end{bmatrix}\hat{x}(0.02k)$$

#

- thus design is carried out in continuous-time
- controller is implemented digitally in discrete-time, with $\hat{x}$ and u updated every $h = 0.02$ seconds. ΔΔ

(b) Direct Design of Digital (State-Space) Controllers

This requires that we first obtain a discrete-time state-space representation of the plant. (Franklin & Powell pp 562~567)

Obtaining Discrete-Time State Representation

Plant $\dot{x} = Fx + Gu$; $x(0) = x_o$

$y = Hx$

Use sampling interval h

Recall that solution to state-equation is

$$x(t) = e^{Ft}x_o + \int_0^t e^{F(t-\tau)}Gu(\tau)d\tau$$

- assume a zero-order hold

  then $u(\tau) = u(kh) \qquad kh \le \tau < (k+1)h$; $k = 0, 1, 2, 3, \ldots$

or, more generally:

$$x(t) = e^{F(t-t_0)}x(t_0) + \int_{t_0}^{t} e^{F(t-\tau)}Gu(\tau)d\tau$$

then, using the solution to the state-equation, x(kh) may be related to x(kh+h) as

$t = \overline{k+1}\,h$

$t_0 = kh$

$$x(kh+h) = e^{Fh}\underline{x(kh)} + \int_{kh}^{kh+h} e^{F(kh+h-\tau)}\,d\tau\,G\underline{u(kh)}$$
$$y(kh) = H\underline{x(kh)}$$

and the discrete-time state-equation takes the form:

$$x(k+1) = \Phi x(k) + \Gamma u(k)$$
$$y(k) = Hx(k) \qquad k = 0,1,2,\ldots$$

cf $\dot{x} = Fx + Gu$

$y = Hx$

where the sampling interval h is assumed i.e. x(k) means x(kh) [for simplicity of notation!]

and $\Phi = e^{Fh}$

$$\Gamma = \int_{kh}^{kh+h} e^{F(kh+h-\tau)}\,d\tau\,G$$
$$= \int_0^h e^{F\eta}\,d\eta\,G$$

for the subst^n of variables $\eta = kh + h - \tau$

Check this!

Thus, the discrete-time state-variable theory proceeds in exactly the same way as the continuous-time case, using the discrete-time state-variable description:

$$\boxed{\begin{aligned} x(k+1) &= \Phi x(k) + \Gamma u(k) \\ y(k) &= Hx(k) \end{aligned}}$$

compare with

$\dot{x} = Fx + Gu$

$y = Hx$

Poles of discrete-time s.s. system: "a pole is a value of discrete-time frequency such that the system equation has a non-trivial solution when the forcing input is zero."

i.e. without a forcing input, the system responds as

$$x(k) = \alpha^k x_o \qquad \text{for i.c.} \qquad x(0) = x_o$$

then $\qquad x(k+1) = \Phi x(k) + \Gamma u(k)$

$u \equiv 0$

i.e. $\qquad \alpha^{k+1} x_o = \Phi \alpha^k x_o$

$$\alpha^k \{\alpha I - \Phi\} x_o = 0$$

where α is a pole of the system in the z-plane

For a non-trivial solution $(x_o \neq 0)$

must have $\det[\alpha I - \Phi] = 0$

i.e. $\boxed{\text{poles, z, are the roots of } \det[zI - \Phi] = 0}$

(cf. c.t. s.s. poles are roots of $\det[sI - F] = 0$)

State Feedback

consider control

$$u(k) = -Kx(k)$$

then overall system is

$$x(k+1) = \Phi x(k) + \Gamma u(k)$$
$$= [\Phi - \Gamma K] x(k)$$

Compare:

$\dot{x} = Fx + Gu$

$= (F - GK)x$

for $u = -Kx$

$\therefore$ the poles of the system are arbitrarily assignable if $\zeta(\Phi, \Gamma)$ has full rank.

- Exact correspondence carries over from continuous-time

Compare:

$\dot{\hat{x}} = F\hat{x} + Gu + L(y - H\hat{x})$

State-Estimation

Consider full-order estimator

$$\hat{x}(k+1) = \Phi\hat{x}(k) + \Gamma u(k) + L[y(k) - H\hat{x}(k)]$$

and

$\dot{\tilde{x}} = (F - LH)\tilde{x}$

State-error dynamics is, $\tilde{x} = \hat{x} - x$,

$$\tilde{x}(k+1) = [\Phi - LH]\tilde{x}(k)$$

and arbitrary estimator dynamics $\alpha_e(z)$ can be specified if $\mathrm{O}(H, \Phi)$ is full-rank.

Thus, once the sampled state-space equation in discrete-time directly is obtained, the controller can be designed directly in discrete-time using state-feedback & state-estimation.