Result:

Given a system

$$\dot{x} = Fx + Gu$$
$$y = Hx$$

an observer (state-estimator)

$$\dot{\hat{x}} = F\hat{x} + Gu + L(y - H\hat{x})$$

can be constructed with the property that

$$(x - \hat{x}) \to 0 \qquad \text{exponentially}$$

iff the observability matrix

$$O(H,F) = \begin{bmatrix} H \\ HF \\ HF^2 \\ \vdots \\ HF^{n-1} \end{bmatrix}$$

has full rank. $\Delta\Delta$

Procedure to find L

- Decide on the desired eigenvalues for your estimator error dynamics. Say, they are roots of

$$\alpha_0(s) = s^n + \alpha_1 s^{n-1} + \alpha_2 s^{n-2} + \ldots + \alpha_n = 0$$

- These will be eigenvalues of $(F - LH)$, or equivalently, eigenvalues of $F^T - H^T L^T$.

state feedback

$F - GK$

- Ackermann's formula then provides

$$L^T = [0 \quad \ldots \quad 0 \quad 1]\zeta^{-1}(F^T, H^T)\alpha_o(F^T)$$

Book Example 6.3 (Undamped Oscillator)

Undamped Oscillator

- same state-space system as example from page 26 of Lecture notes.
- here, we want to construct estimator to estimate $\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -\omega_o^2 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} x$$

- open loop estimator cannot be used for state-estimation error will not decrease!

- use closed-loop state estimator with both estimator poles at $-10\omega_o$ (five times faster than controller poles.)

$$\alpha_e(s) = (s + 10\omega_o)^2 = s^2 + 20\omega_o s + 100\omega_o^2$$

$\alpha_o(s)$

$$\det[sI - (F - LH)] = s^2 + l_1 s + l_2 + \omega_o^2$$

Recall that in page 26, we chose $\alpha_c(s) = (s + 2\omega_o)^2$

thus we need

$$L = \begin{bmatrix} l_1 \\ l_2 \end{bmatrix} = \begin{bmatrix} 20\omega_o \\ 99w_o^2 \end{bmatrix}$$

$K = [k_1 \; k_2] = [3\omega_o^2 \;\; 4\omega_o^2]$

- combine the state-estimator with feedback law for u

$$\dot{\hat{x}} = F\hat{x} + Gu + L(y - H\hat{x})$$

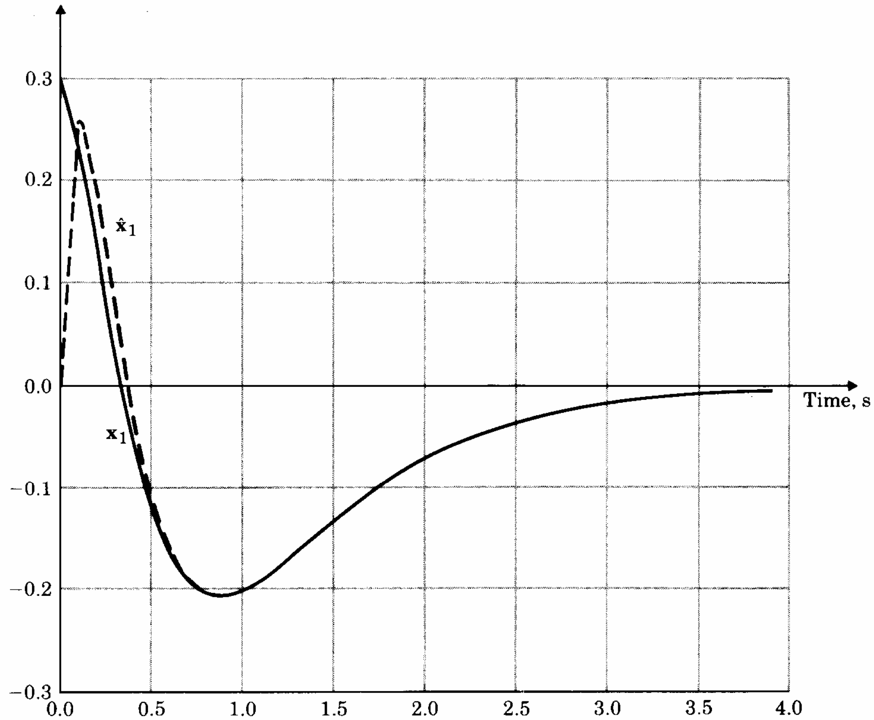
$$u = -K\hat{x}$$

$\dot{x} = Fx + Gu$

$y = Hx$

$u = -Kx$

FIGURE 6.23
Undamped-oscillator initial-condition response of y and $\hat{y}$.



$\omega_0 = 1$

FIGURE 6.24
Undamped-oscillator initial-condition response of $\dot{y}$ and $\hat{\dot{y}}$.

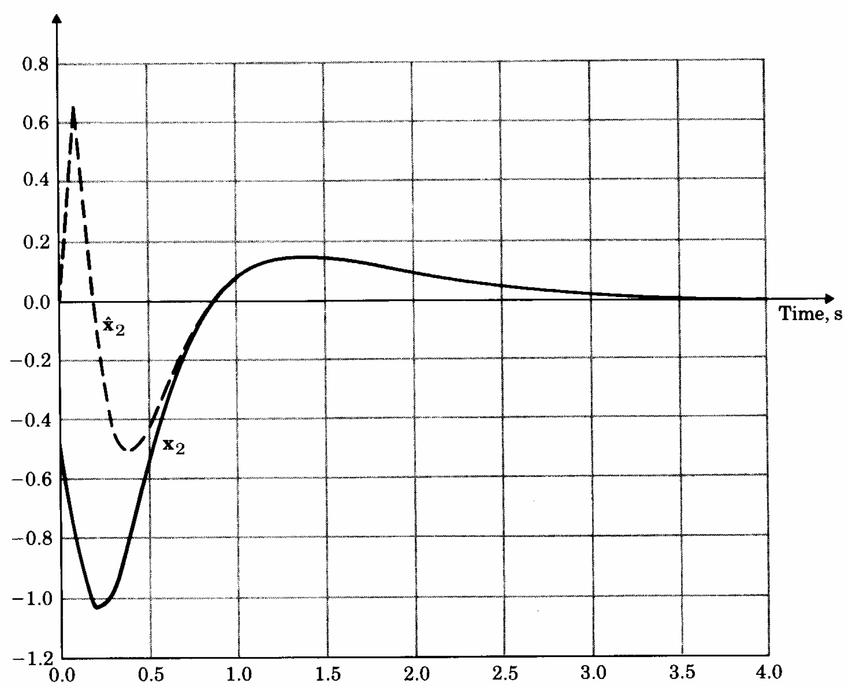
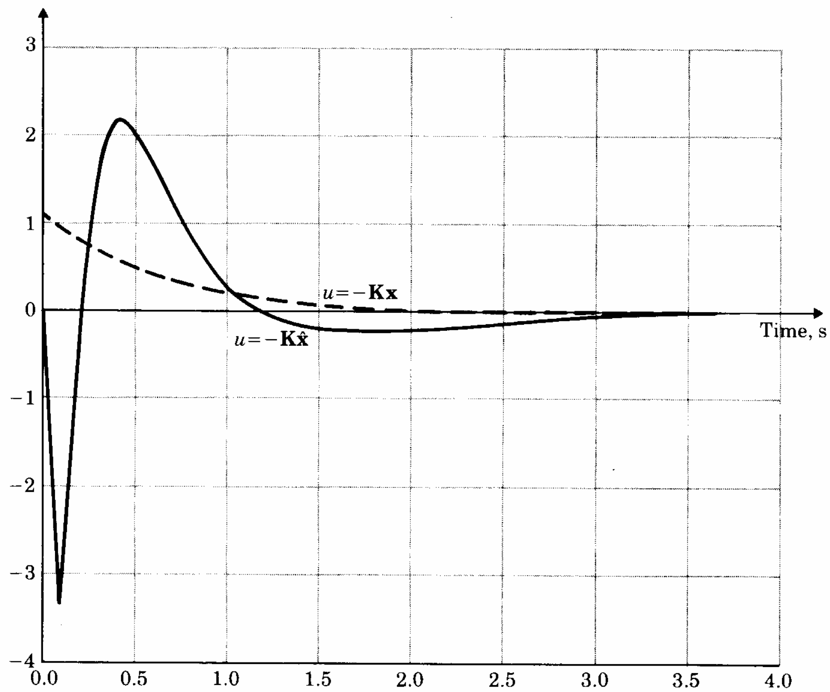


FIGURE 6.25
Comparison of control efforts for state and estimated state feedback.



$\boxed{w_0 = 1}$

Reduced Order Estimators (Friedland, pp 276~279)

- In many systems, we can measure quite a few of the states, and only need to re-construct the reminder

Thus, let the system be

$$\dot{x} = Fx + Gu \quad ; \quad y = Hx$$

$$\left. y_m = \begin{bmatrix} H_1 & \vdots & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ \cdots \\ x_2 \end{bmatrix} \quad \right\} \text{ dimension } n_1$$

where y_m is an n_1- dimension measurement vector
H_1 is $n_1 \times n_1$, and non-singular

- Note that here, I am allowing y_m to be n_1-dimensional. This is reasonable because several mesurements may be available. I also distinguish the measurements y_m from the output $y = Hx$, which is still restricted to one-dimension, i.e., SISO system.
- An example for H_1 might be $I_{n_1 \times n_1}$. Thus $y_m = x_1$ and the first n_1 states are measured directly.

$\dot{x} = Fx + Gu$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{array}{c c} & \begin{array}{cc} n_1 & n_2 \end{array} \\ \begin{array}{c} n_1 \\ n_2 \end{array} & \left[\begin{array}{c|c} F_{11} & F_{12} \\ \hline F_{21} & F_{22} \end{array}\right] \end{array} \begin{bmatrix} x_1 \\ \hline x_2 \end{bmatrix} + \begin{bmatrix} G_1 \\ \hline G_2 \end{bmatrix} u$$

$$y_m = \begin{array}{c c} & \begin{array}{cc} n_1 & n_2 \end{array} \\ n_1 & \left[\begin{array}{c|c} H_1 & 0 \end{array}\right] \end{array} \begin{bmatrix} x_1 \\ \hline x_2 \end{bmatrix}$$

Obviously x_1 need not be estimated.

It is recovered as

$$x_1 = H_1^{-1} y_m$$

H_1 is $n_1 \times n_1$
& non-singular

$$\dot{x}_2 = F_{21}x_1 + F_{22}x_2 + G_2 u$$

$$= F_{22}x_2 + \underbrace{F_{21}H_1^{-1}y_m + G_2 u}_{measurable}$$

Consider the form

$$\hat{x}_2 = L y_m + z;$$

L is $n_2 \times n_1$, & to be chosen.

to estimate x_2, with z generated by

$$\dot{z} = \bar{F}z + \bar{G}y_m + \bar{H}u$$

z clearly must have same dimension as x_2

i.e. dimension $n_2 = n - n_1$

$\therefore$ $\bar{F}$ is $n_2 \times n_2$
$\bar{G}$ is $n_2 \times n_1$
$\bar{H}$ is $n_2 \times 1$

$\bar{F}, \bar{G}, \bar{H}$ to be appropriately chosen in what follows

$\hat{x}_2 = L y_m + z$

$\dot{z} = \bar{F} z + \bar{G} y_m + \bar{H} u$

Look at state estimation error

$$\tilde{x}_2 = x_2 - \hat{x}_2$$

Then

$$\dot{\tilde{x}}_2 = \dot{x}_2 - \dot{\hat{x}}_2$$
$$= \{F_{22}x_2 + F_{21}x_1 + G_2 u\} - \{L\dot{y}_m + \dot{z}\}$$

But
$$\dot{y}_m = H_1 \dot{x}_1$$
$$= H_1\{F_{11}x_1 + F_{12}x_2 + G_1 u\}$$

Further
$$\dot{z} = \bar{F}z + \bar{G}y_m + \bar{H}u$$

Express everything in the $\tilde{x}_2$ dynamical equation in terms of $\tilde{x}_2, x_1, x_2,$ and u

$$\therefore \quad z = \hat{x}_2 - Ly_m$$
$$= (x_2 - \tilde{x}_2) - LH_1x_1$$

$\dot{\hat{z}} = \bar{F}z + \bar{G}y_m + \bar{H}u$

$$\dot{z} = \bar{F}(x_2 - \tilde{x}_2) - \bar{F}LH_1x_1 + \bar{G}H_1x_1 + \bar{H}u$$

Then making the appropriate substitution gives

$$\dot{\tilde{x}}_2 = \{F_{22}x_2 + F_{21}x_1 + G_2u\}$$
$$-\{LH_1F_{11}x_1 + LH_1F_{12}x_2 + LH_1G_1u\}$$
$$-\{\bar{F}x_2 - \bar{F}\tilde{x}_2 + (\bar{G}H_1 - \bar{F}LH_1)x_1 + \bar{H}u\}$$

$L\dot{y}_m$

$\dot{z}$

$\hat{x}_2 \triangleq Ly_m + z$

i.e.

$$\dot{\tilde{x}}_2 = \bar{F}\tilde{x}_2 + (F_{21} - LH_1F_{11} - \bar{G}H_1 + \bar{F}LH_1)x_1$$
$$+(F_{22} - LH_1F_{12} - \bar{F})x_2$$
$$+(G_2 - LH_1G_1 - \bar{H})u$$

i.e. $\dot{\tilde{x}}_2 = \bar{F}\tilde{x}_2$

make all these "nulls"! (i.e. zero)

Construct the observer by choosing

$$\bar{F} = F_{22} - LH_1F_{12}$$
$$\bar{G} = (F_{21} - LH_1F_{11} + \bar{F}LH_1)H_1^{-1}$$
$$\bar{H} = G_2 - LH_1G_1$$

$\bar{F} = F' - LH'$

$\hat{x}_2 = Ly_m + z$

$\dot{z} = \bar{F}z + \bar{G}y_m + \bar{H}u$

Then observer dynamics become

$$\dot{\tilde{x}}_2 = \bar{F}\tilde{x}_2 \quad = (F_{22} - L[H_1 F_{12}])\tilde{x}_2$$

and if $\bar{F}$ have eigenvalues all in the l.h.p. ,

then

$$\tilde{x}_2 \to 0 \quad \text{exponentially.}$$

What are conditions for this?

$$\bar{F} = F_{22} - L(H_1 F_{12})$$

F_{22}, H_1, F_{12} are fixed. L is a gain vector to be chosen.

This is in fact very much like full order estimator case, assigning the eigenvalues of $(F - LH)$.

<u>Fact</u>

Let $\quad y_m = [H_1 \;\vdots\; 0]\begin{bmatrix} x_1 \\ \hdashline x_2 \end{bmatrix} = H_m x \quad ; \quad y = Hx$

Then if the full state observability matrix $O(H_m, F)$ is of full rank,

L can always be chosen so that eigenvalues of $\bar{F} = F_{22} - L(H_1 F_{12})$ are in l.h.p.

Reduced-Order Estimator (Observer) Summary

For the system

$$
\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{array}{c|c} & \begin{array}{cc} n_1 & n_2 \end{array} \\ \begin{array}{c} n_1 \\ n_2 \end{array} & \left[\begin{array}{c|c} F_{11} & F_{12} \\ \hline F_{21} & F_{22} \end{array}\right] \end{array} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} G_1 \\ G_2 \end{bmatrix} u
$$

; $y = Hx$

$$
y_m = \begin{array}{cc} & \begin{array}{cc} n_1 & n_2 \end{array} \\ n_1 & \left[\begin{array}{c|c} H_1 & 0 \end{array}\right] \end{array} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}
$$

- x_1 is recovered directly as $x_1 = H_1^{-1} y_m$
- x_2 is estimated by

$\tilde{x}_2 = x_2 - \hat{x}_2$

$\dot{\tilde{x}}_2 = \bar{F}\tilde{x}_2$

$$
\boxed{\begin{aligned} &\hat{x}_2 = Ly_m + z \\ \text{with}\quad &\dot{z} = \bar{F}z + \bar{G}y_m + \bar{H}u \end{aligned}}
$$

where
$$
\begin{aligned}
\bar{F} &= F_{22} - LH_1F_{12} \\
\bar{G} &= (F_{21} - LH_1F_{11} + \bar{F}LH_1)H_1^{-1} \\
\bar{H} &= G_2 - LH_1G_1
\end{aligned}
$$

and L is chosen to ensure that eigenvalues of $\bar{F}$ are all in the l.h.p.

Example: (Franklin & Powell, pp 354~355)

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\omega_o^2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$y_m = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Thus, we can only measure the position, and we want to estimate the velocity x_2 only.

Want to place roots of reduced-order estimator-error characteristic equation at $-10\omega_o$.

This characteristic eqn as we have seen is from

$$\dot{\tilde{x}}_2 = \overline{F}\tilde{x}_2; \quad \text{or} \;\; \det\left[sI - \overline{F}\right] = 0$$

$$\overline{F} = F_{22} - LH_1F_{12}$$

$$= 0 - L(1)(1) = -L$$

$\therefore$ require

$$\det\left[sI - \overline{F}\right] = (s - (-L)) = s + 10\omega_o = 0$$

i.e. $L = 10\omega_o$

$$\bar{G} = (F_{21} - LH_1F_{11} + \bar{F}LH_1)H_1^{-1}$$
$$= (-\omega_o^2 - 10\omega_o \cdot 0 + (-10\omega_o)10\omega_o \cdot 1)1^{-1}$$
$$= -101\omega_o^2$$

$$\bar{H} = 1 - (10\omega_o)(1)(0) = 1$$

Thus x_2 is estimated by

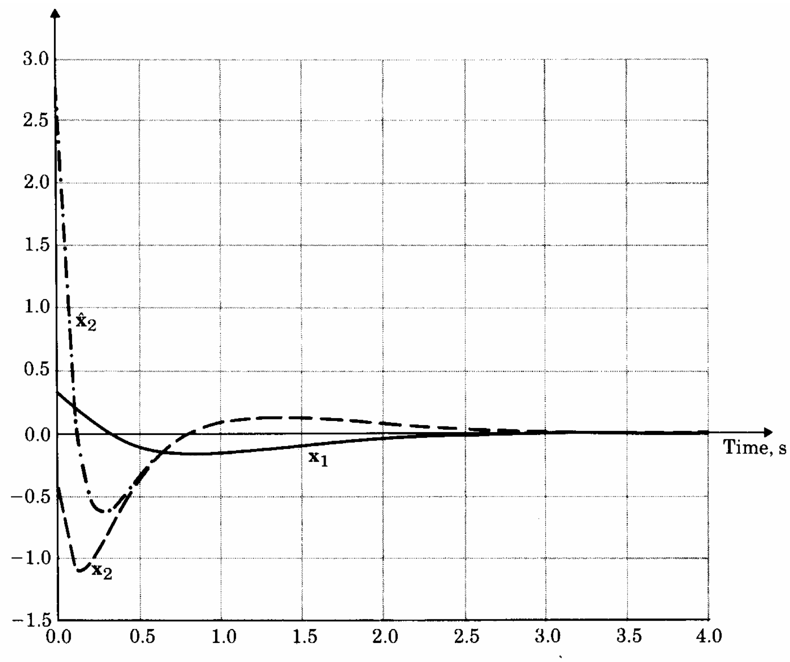
$$\hat{x}_2 = 10\omega_o y_m + z$$
$$\dot{z} = -10\omega_o z - 101\omega_o^2 y_m + u$$

You may note that this is the same estimator as obtained in the book. The difference is that the treatment is more general here, and will handle cases where we have multiple measurements through y_m.

Read
- Estimator Pole Selection
  (Franklin & Powell) pp 356~358

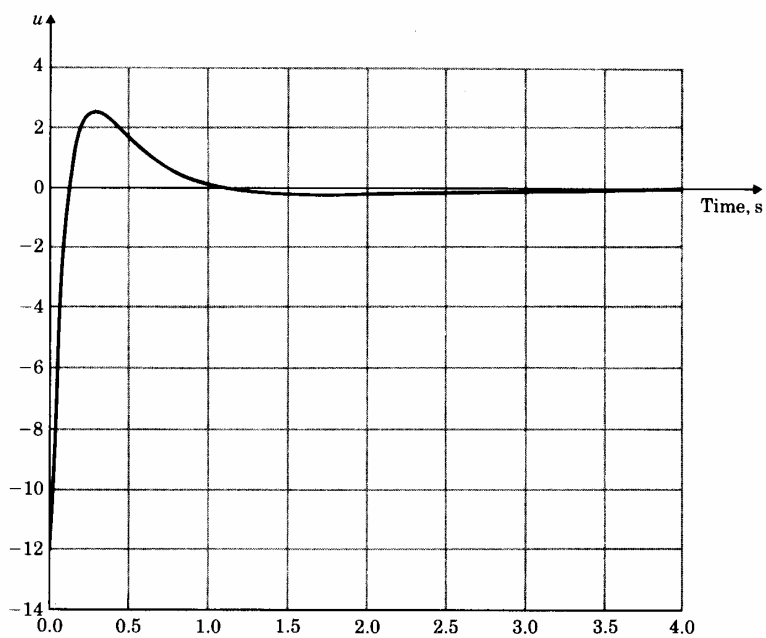
- Further examples on observers
  (Friedland pp 279~284)

FIGURE 6.28
Initial-condition response of reduced-order controller.



$\omega_0 = 1$

FIGURE 6.29
Control effort for reduced-order controller.



$u = -k_1 x_1 - k_2 \hat{x}_2$

Estimator Pole Selection

- rule of thumb: choose estimator poles 2 to 6 times faster than closed-loop poles from control

  — faster the desired response, generally larger the corresponding fb signal

  control law: $u = -K\hat{x}$ (fb signal)

  fast $\alpha_c(s)$, large u

  actuator may saturate

  estimator: fb signal to estimator is Hx

  $$L(y - H\hat{x})$$

  fast $\alpha_e(s)$, $\rightarrow$ large $L(y - H\hat{x})$ signal

  this is all right as this signal is only a voltage in analog implementation or digital word in discrete-implementation

- large sensor noise, choose slower estimator poles

- read also SRL corresponding to choice of $\alpha_e(s)$

Exercise

For the restricted case where only the output $y \in IR^1$ (i.e. a one-dimensional real number) can be measured, show that the restricted-case reduced-order estimator given by simply applying

$$y_m = y$$

in the development given here.

Compensator Design:

Combined Control Law and Estimator

considered

- state feedback
- state estimation

states not directly measurable, we said, use

$$u = -K\hat{x}$$

Would that work?

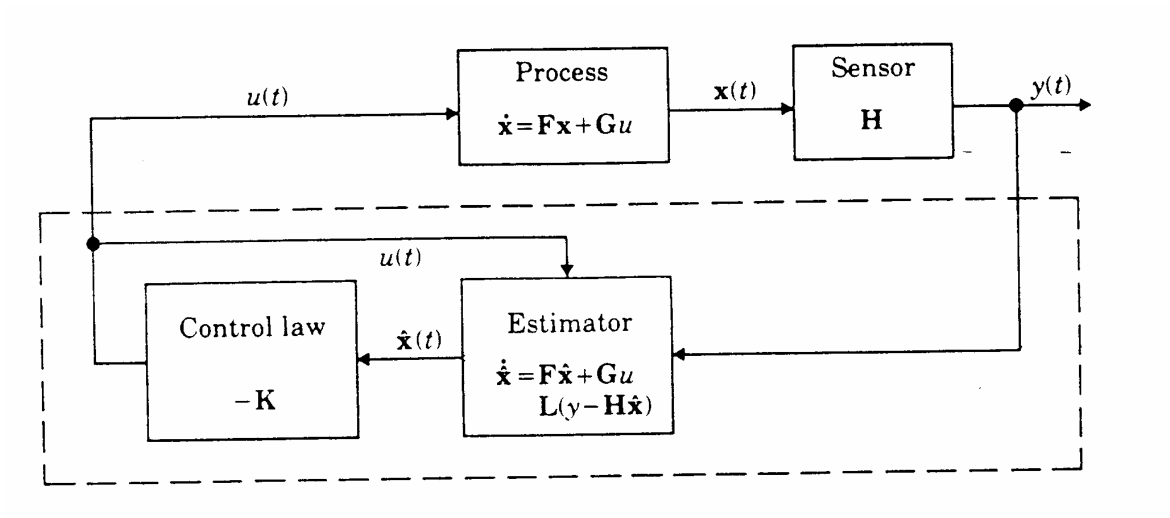
To answer this question, entire structure of estimator & "control" must be considered together.

Plant:

$$\left.\begin{aligned} \dot{x} &= Fx + Gu \\ y &= Hx \end{aligned}\right\}$$

Complete Controller:

$$\left.\begin{aligned} \dot{\hat{x}} &= F\hat{x} + Gu + L(y - H\hat{x}) \\ u &= -K\hat{x} \end{aligned}\right\}$$


Process
$\dot{\mathbf{x}} = \mathbf{F}\mathbf{x} + \mathbf{G}u$
$u(t)$
$\mathbf{x}(t)$
Sensor
$\mathbf{H}$
$y(t)$
$u(t)$
Control law
$-\mathbf{K}$
$\hat{\mathbf{x}}(t)$
Estimator
$\dot{\hat{\mathbf{x}}} = \mathbf{F}\hat{\mathbf{x}} + \mathbf{G}u$
$\mathbf{L}(y - \mathbf{H}\hat{\mathbf{x}})$

Must write state equations for

plant + complete controller

Easier to analyze in terms of the state estimation errors $\tilde{x} = x - \hat{x}$

$$\dot{x} \quad \cancel{\tilde{x}} = Fx + Gu = Fx + G(-K\hat{x})$$

$$\dot{\tilde{x}} = (F - LH)\tilde{x}$$

$$= Fx - GK(x - \tilde{x})$$

$$= (F - GK)x + GK\tilde{x}$$

$\therefore$ complete state of system is $\begin{bmatrix} x \\ \tilde{x} \end{bmatrix}$

Remark: An equivalent complete state is $\begin{bmatrix} x \\ \hat{x} \end{bmatrix}$;

but this is related by

$$\begin{bmatrix} x \\ \hat{x} \end{bmatrix} = \left[\begin{array}{c|c} I & 0 \\ \hline I & -I \end{array}\right] \begin{bmatrix} x \\ \tilde{x} \end{bmatrix} = T \begin{bmatrix} x \\ \tilde{x} \end{bmatrix}$$

$$\tilde{x} \triangleq x - \hat{x}$$

i.e. $\hat{x} = x - \tilde{x}$

It is easier to analyze the $\begin{bmatrix} x \\ \tilde{x} \end{bmatrix}$ system; and we should always work in the easier system.

$\Delta\Delta$

Complete state-equation for plant + controller is

$\therefore$

$$\underbrace{\begin{bmatrix} \dot{x} \\ \dot{\tilde{x}} \end{bmatrix}}_{\dot{\bar{x}}} = \underbrace{\begin{bmatrix} F - GK & GK \\ 0 & F - LH \end{bmatrix}}_{\bar{F}} \underbrace{\begin{bmatrix} x \\ \tilde{x} \end{bmatrix}}_{\bar{x}}$$

Characteristic equation of closed-loop is

$$\det \begin{bmatrix} sI - (F - GK) & -GK \\ 0 & sI - (F - LH) \end{bmatrix} = 0$$

and $\because$ it is block triangular, equiv to

$$\det[sI - (F - GK)] \det[sI - (F - LH)] = 0$$

or $$\alpha_c(s)\alpha_e(s) = 0$$

"Poles of combined system is union of control roots and estimator roots."

Thus, allows us to design control law and estimator independently. "Separation Principle". (Kalman)

Note: Similar development possible for reduced-order estimator/observer