

$$\ddot{y} + a_1 \dot{y} + a_2 y = b u$$

Transfer function

$$\frac{Y(s)}{U(s)} = \frac{b}{s^2 + a_1 s + a_2}$$


For this simple example, let's assume that we identify the variables:


$$x_1(t) \triangleq y(t)$$

$$x_2(t) \triangleq \dot{y}(t)$$

With this identification, we have:

In matrix form, this is

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} \quad \\ \quad \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} \quad \\ \quad \end{bmatrix} u$$


$$y = \begin{bmatrix} \quad \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$


Simple non-linear d.e

$$\ddot{\theta} + \omega_0^2 \sin \theta = \frac{T_c}{ml^2}$$

Suppose we are interested
in the variables:

$$x_1(t) \triangleq \omega_0 \theta(t)$$

$$x_2(t) \triangleq \dot{\theta}(t)$$

state-
variable
form!!!

Then, we have

$$\dot{x}_1(t) = \omega_0 \dot{\theta}(t) = \omega_0 x_2$$

$$\dot{x}_2(t) = \ddot{\theta}(t) = -\omega_0^2 \sin\left(\frac{x_1}{\omega_0}\right) + \frac{T_c}{ml^2}$$

For linearization and small
deviations about the equilibrium

$$x_1 = 0, \quad x_2 = 0$$

Proof, we note that for small
deviation about the above, we
have

$$\sin\left(\frac{x_1}{w_0}\right) \approx \frac{x_1}{w_0}$$

$$\left[\text{for } \sin \alpha \approx \alpha \text{ for small } \alpha \right]$$

and this next gives the state-
variable equations as:

$$\dot{x}_1 =$$

$$\dot{x}_2 =$$

$$=$$

leading to the linearised
state-variable equations

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} \quad \\ \quad \end{bmatrix} u$$

Time-varying systems example

$$\ddot{y} + a_1(t) \dot{y} + a_2(t) y = b(t) u$$

Cast this in state-variable form
is straightforward with, for
example,

$$x_1(t) \triangleq y(t)$$

$$x_2(t) \triangleq \dot{y}(t)$$

⋮

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -a_2(t) & -a_1(t) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$+ \begin{bmatrix} 0 \\ b(t) \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

State-Variable Realizations

Method 1

$$y^{(3)} + a_1 y'' + a_2 y' + a_3 y = (1.1)$$

$$= b_1 u^{(3)} + b_2 u'' + b_3 u'$$

$$\frac{Y(s)}{U(s)} = \frac{b_1 s^2 + b_2 s + b_3}{s^3 + a_1 s^2 + a_2 s + a_3}$$

— (2.1)

Consider the intermediate
variable $z(t)$

defined by:

$$\frac{\sum(s)}{u(s)} = \frac{1}{s^3 + a_1 s^2 + a_2 s + a_3}$$

Then, in the three-domain, — (2.2)
this is equivalent to:

— (3.1)

With (2.1) and (2.2), we must

have
$$Y(s) = (b_1 s^2 + b_2 s + b_3) \sum(s)$$

In three-domain, this is

— (3.2)

With this, we now set up the patchy diagram =

Identify each output of an
integrator as a state-variable

Then, note that we now have:

$$\dot{x}_1 =$$

$$\dot{x}_2 =$$

$$\dot{x}_3 =$$

$$y =$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} =$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} \\ \\ \end{bmatrix} u$$

— (4.1a)

$$y =$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

— (4.1b)

(4.1a) and (4.1b) constitute a state-variable representation of the d.e. (1.1)

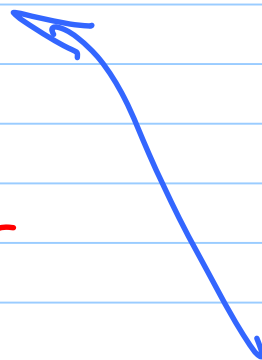
$$\begin{aligned} \overset{\infty}{y} + a_1 \overset{\infty}{y} + a_2 \overset{0}{y} + a_3 \overset{0}{y} & \quad \text{--- (1.1)} \\ & = b_1 \overset{\infty}{u} + b_2 \overset{0}{u} + b_3 u \end{aligned}$$

Think of how the above pattern automatically generalises to:

• k^{th} order

...

• n^{th} order



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$$y + a_1 y + a_2 y + a_3 y + a_4 y$$

$$+ a_5 y =$$

$$b_1 u + b_2 u + b_3 u + b_4 u + b_5 u$$