Relating State-Variable Design to Transform Method

$\dot{x} = Fx + Gu$
$y = Hx$

State Variable Controller (including estimator)

$$\dot{\hat{x}} = (F - GK - LH)\hat{x} + Ly$$
$$u = -K\hat{x}$$

$\dot{\hat{x}} = F\hat{x} + Gu + L(y - H\hat{x})$

This means that the transfer function from y to u $(y \mapsto u)$ is

$$G_c(s) = \frac{U(s)}{Y(s)} = -K[sI - (F - GK - LH)]^{-1} L$$

Similar development possible for reduced-order estimator:

State equations

$$\left.\begin{aligned} \dot{x}_c &= A_r x_c + B_r y \\ u &= C_r x_c + D_r y \end{aligned}\right\}$$

Franklin & Powell pp361

- work thru the given expr^{ns} for yourself

and transfer function $y \to u$

$$G_{cr}(s) = \frac{U(s)}{Y(s)} = C_r(sI - A_r)^{-1} B_r + D_r$$

- We will calculate $G_c(s)$ and $G_{cr}(s)$ for a few examples

- Will note that compensators designed using state-variable methods are very similar to compensators designed using transform methods. (True only because we are restricting ourselves to being able to measure u and y. We then estimate all n states in full-order estimator, or (n-1) states in the restricted reduced-order estimator treated in the book.)

- I want to point out, in addition, that if we can measure all the states, or more than one relevant measurements are allowed as in $y_m \notin IR^1$, then state-feedback is much more versatile and robust than simple output feedback.
  Remember this in your working environment.

Example: Double integrator plant

$$G(s) = \frac{1}{s^2}$$

$$\Rightarrow \qquad \dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} x$$

$\alpha_c(s)$

- place control roots at $s = (-1 \pm j1)/\sqrt{2}$ $(\omega_n = 1, \zeta = 0.7)$

$$\alpha_c(s) = s^2 + \sqrt{2}s + 1$$

then $\quad K = \begin{bmatrix} 1 & \sqrt{2} \end{bmatrix}$

$\alpha_e(s)$

- place estimator error roots at $\omega_n = 5, \zeta = 0.5$

$$\alpha_e(s) = s^2 + 5s + 25$$

then $\quad L = \begin{bmatrix} 5 \\ 25 \end{bmatrix}$

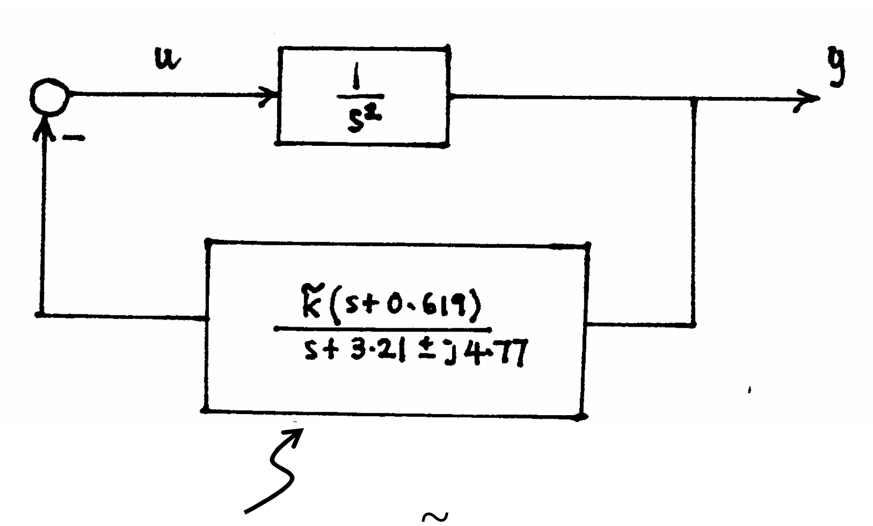
- compensator transfer function

$$G_c(s) = -K[sI - (F - GK - LH)]^{-1} L$$
$$= \frac{-40.4(s + 0.619)}{s + 3.21 \pm j4.77}$$

- comparison with transform methods:
  - looks very much like lead compensator
  - zero on the real axis to the right of its poles
  - however has two complex poles, instead of one
  - zero provides "derivative-like" feedback, two poles provide some smoothing of sensor noise

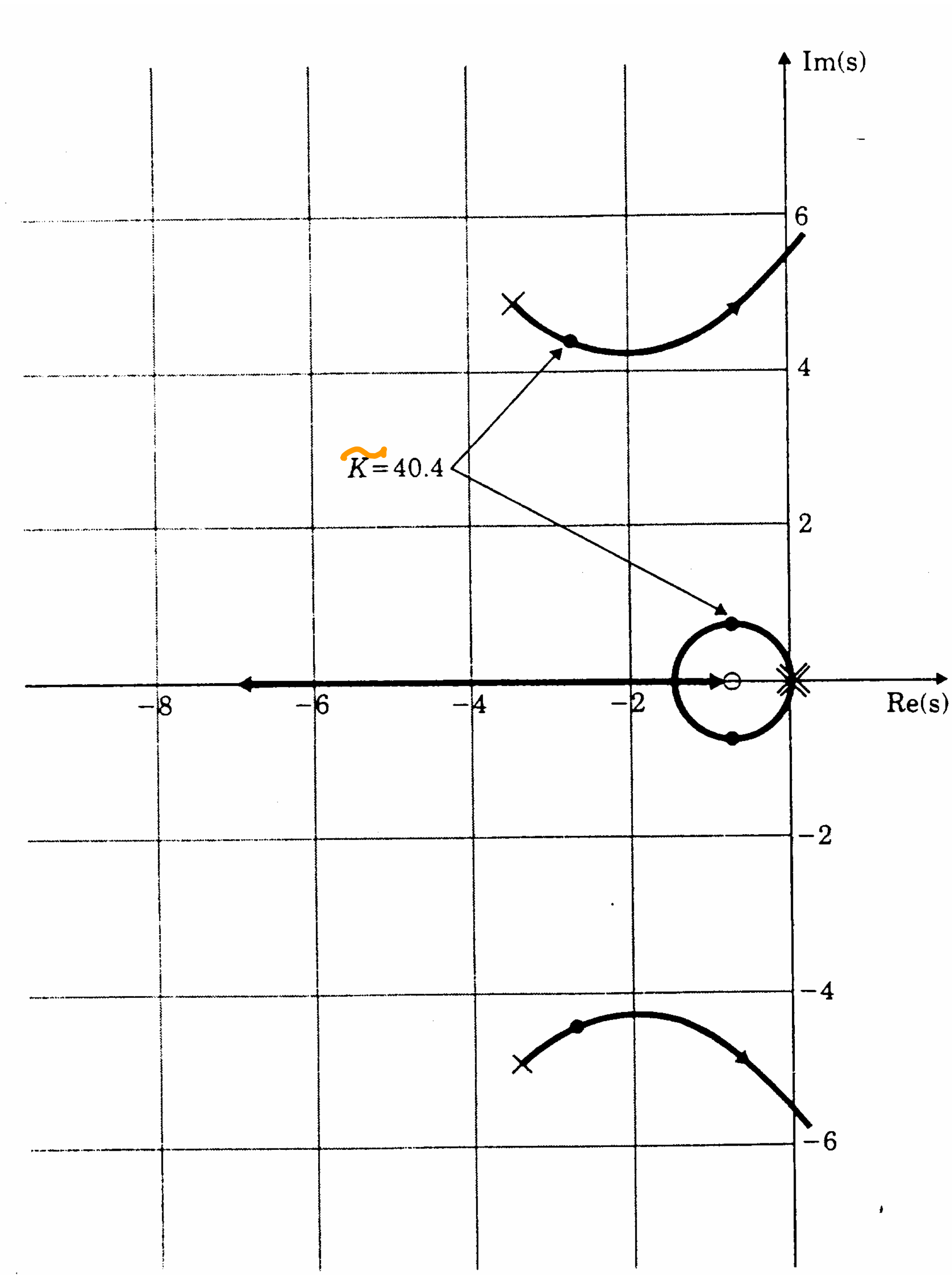
- analysis of this compensator using transform techniques

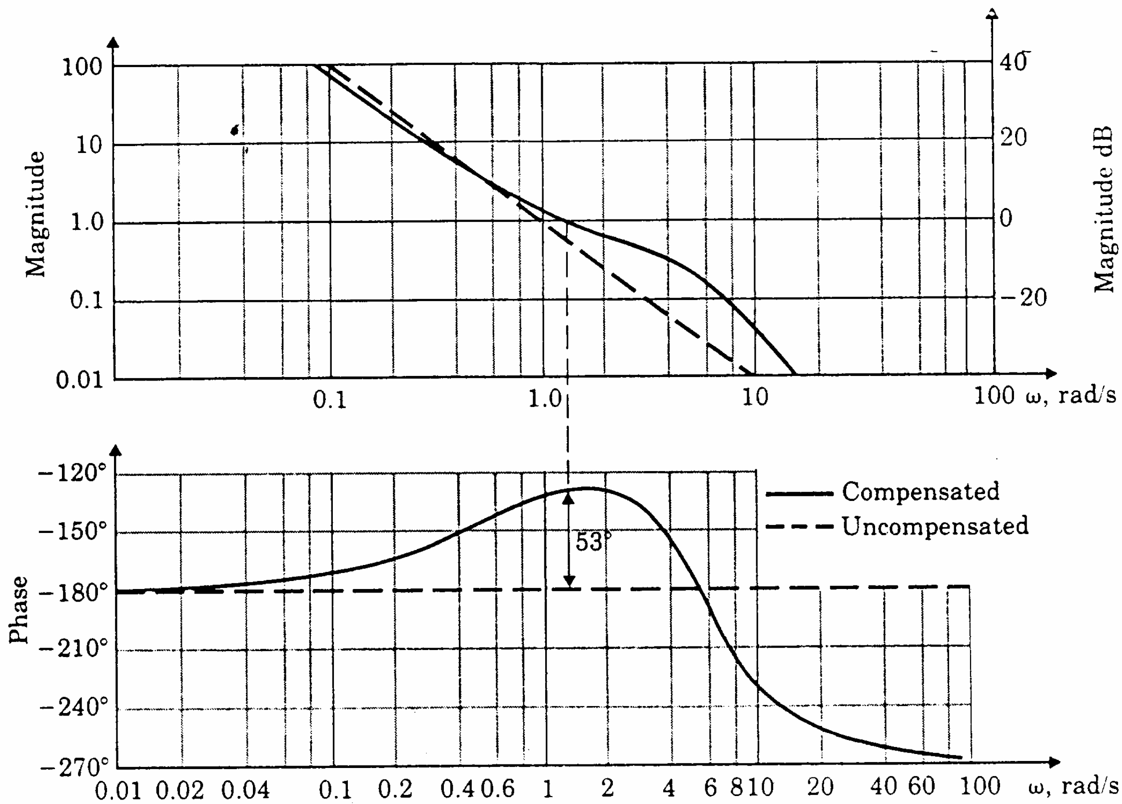
  Loop is in fact



$G_c(s)$, with $\tilde{K} = 40.4$

  - root locus point of view
  - Bode plot point of view


Im(s)
Re(s)
$\tilde{K}$=40.4
6
4
2
−2
−4
−6
−8
−6
−4
−2



100
10
1.0
0.1
0.01
Magnitude
40
20
0
−20
Magnitude dB
0.1
1.0
10
100 ω, rad/s
−120°
−150°
−180°
−210°
−240°
−270°
Phase
53°
Compensated
Uncompensated
0.01 0.02 0.04 0.1 0.2 0.4 0.6 1 2 4 6 8 10 20 40 60 100 ω, rad/s

- Double Integrator Plant, Reduced-Order Estimator

$\alpha_{er}(s) = (s+5)$

  - put estimator pole at –5

    thus $L = 5$

$x_1 = \tilde{H}_1^{-1} y_m$

$\hat{x}_2 = L y_m + z$

$\dot{z} = \bar{F} z + \bar{G} y_m + \bar{H} u$

  - compensator equations

$$\dot{x}_c = -6.41 x_c - 33.1 y$$

$$u = -1.41 x_c - 8.07 y$$

  - compensator t.f.

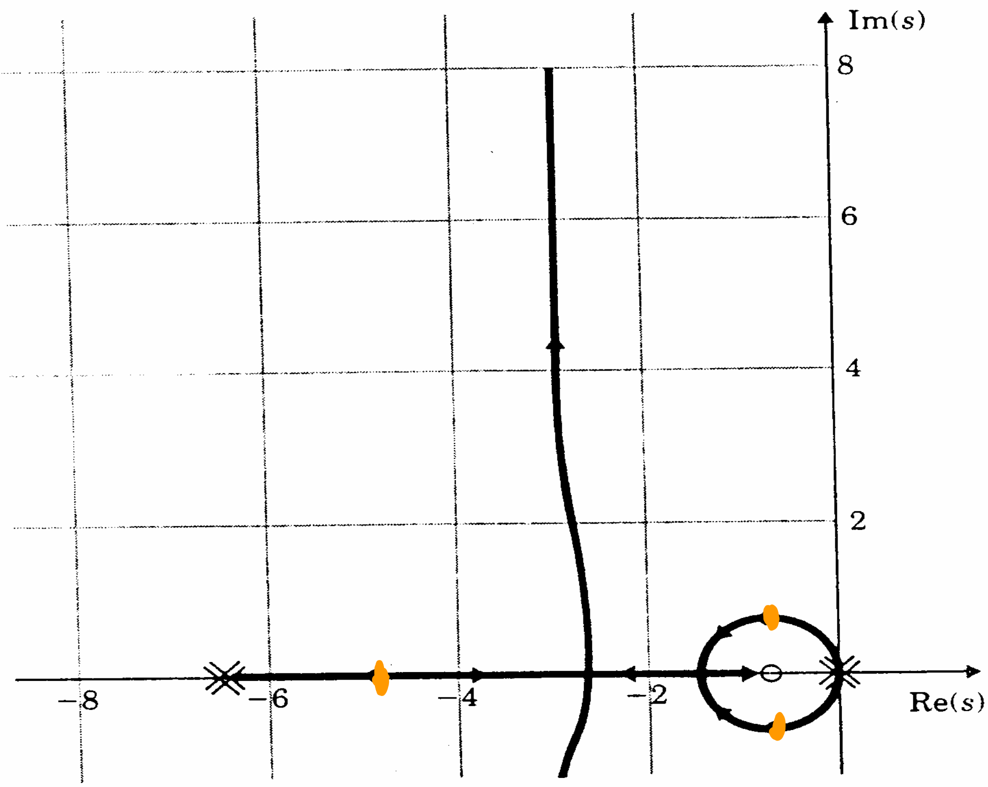
$\tilde{K}$

$$G_{cr}(s) = \frac{-8.07(s+0.62)}{s+6.4}$$

$G(s) = \frac{1}{s^2}$

This is in fact a lead network

- root-locus
- Bode plot



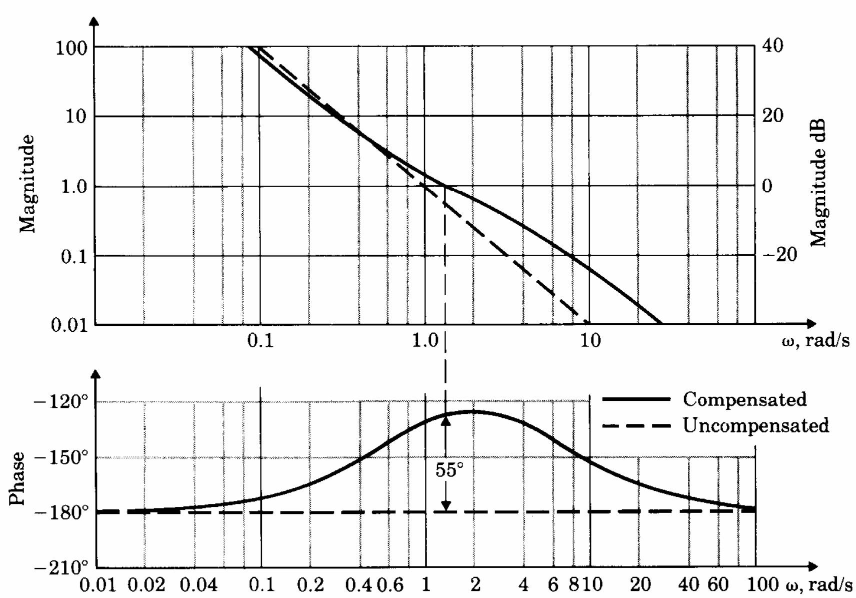


FIGURE 6.35
Frequency response of the $1/s^2$ example with a reduced-order estimator.

Another example

- consider ITAE design vs SRL design

Plant

$$G(s) = \frac{10}{s(s+2)(s+8)}$$

State Representation (in “observer canonical form”)

$$F = \begin{bmatrix} -10 & 1 & 0 \\ -16 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}; \qquad G = \begin{bmatrix} 0 \\ 0 \\ 10 \end{bmatrix}$$

$$H = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}; \qquad J = 0$$

- choose state feedback poles using ITAE criterion at $w_o = 2$
  - use Table 6.1(a), replace s with $\frac{s}{2}$, and expand out the expression.

    This gives

$$\alpha_{c(s)} = (s+1.42)(s+1.04 \pm j2.14)$$
$$= s^3 + 3.5s^2 + 8.6s + 8$$

by comparing coeff, or using Ackermann's formula, find that

$$K = \begin{bmatrix} -46.4 & 5.76 & -0.65 \end{bmatrix}$$

- choose full-order estimator error dynamics poles at 3rd order ITAE location $\omega_o = 6$.

  Computing $\alpha_e(s) = (s+4.25)(s+3.13 \pm j6.41)$
  $$= s^3 + 10.5s^2 + 77.4s + 216$$

  direct coeff comparison, (simple because H,F is in observer form)

  $$\det[sI - (F - LH)] = \alpha_e(s)$$

  gives

  $$L = \begin{bmatrix} 0.5 \\ 61.4 \\ 216 \end{bmatrix}$$

  $y \to u$

- Compensator transfer function $\cancel{u \to y}$

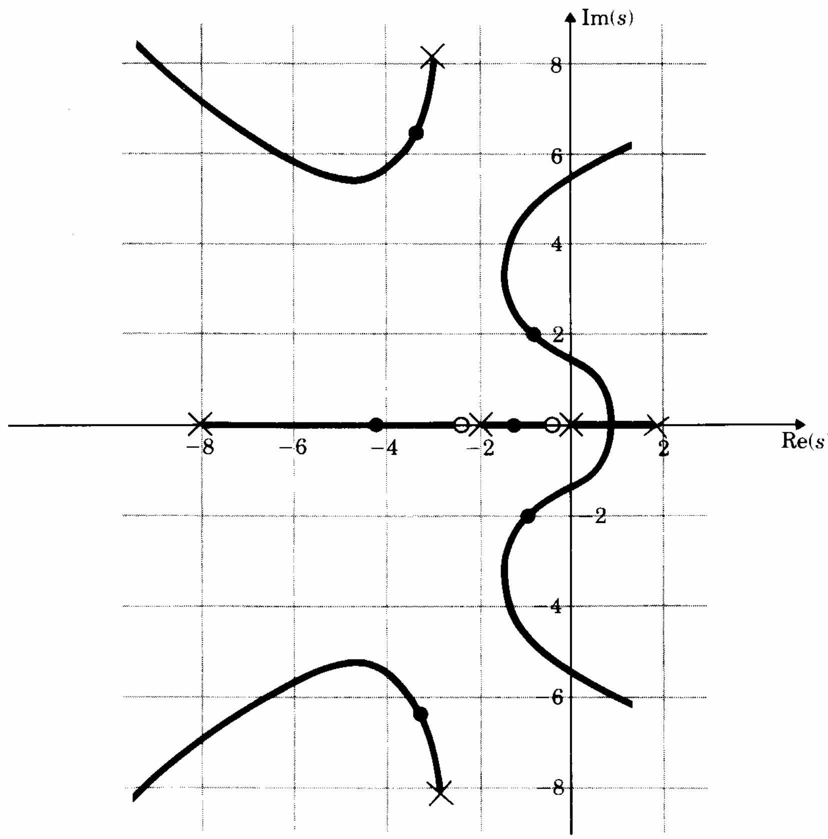
$$G_c(s) = -K[sI - (F - GK - LH)]^{-1} L$$

$$= -190 \frac{(s+0.432)(s+2.10)}{(s-1.88)(s+2.94 \pm j8.32)}$$

$\tilde{K}$

$$G(s) = \frac{10}{s(s+2)(s+8)}$$

FIGURE 6.37
Root locus for ITAE pole assignment.



- root locus viewpoint
  - closed-loop poles are at specified locations $\alpha_c(s), \alpha_e(s)$
  - however, compensator itself, viewed as a unit, is unstable. Has an unstable root at $s = +1.88$

    Thus, overall closed-loop is stable, but compensator as an individual unit, is unstable.

    Not desirable, because compensator unit cannot be bench-tested. (Or, at least, it is difficult to do so.)

    It is not that compensators are not allowed to be unstable. In many cases, you may have no choice, and you have to live with it. However, in this case, this situation is not necessary, and we will see later that a better choice of $\alpha_c(s)$ corrects the situation.
  - main problem here is that such a $G_c(s)$ leads to a conditionally stable system.

$$G(s) = \frac{40}{s(s+2)(s+8)}$$

"Restricted" Reduced-Order Estimator

- use same ITAE design for fb poles
- estimator poles: use ITAE design with $\omega_o = 6$;

$$\alpha_e(s) = s^2 + 8.4s + 36$$

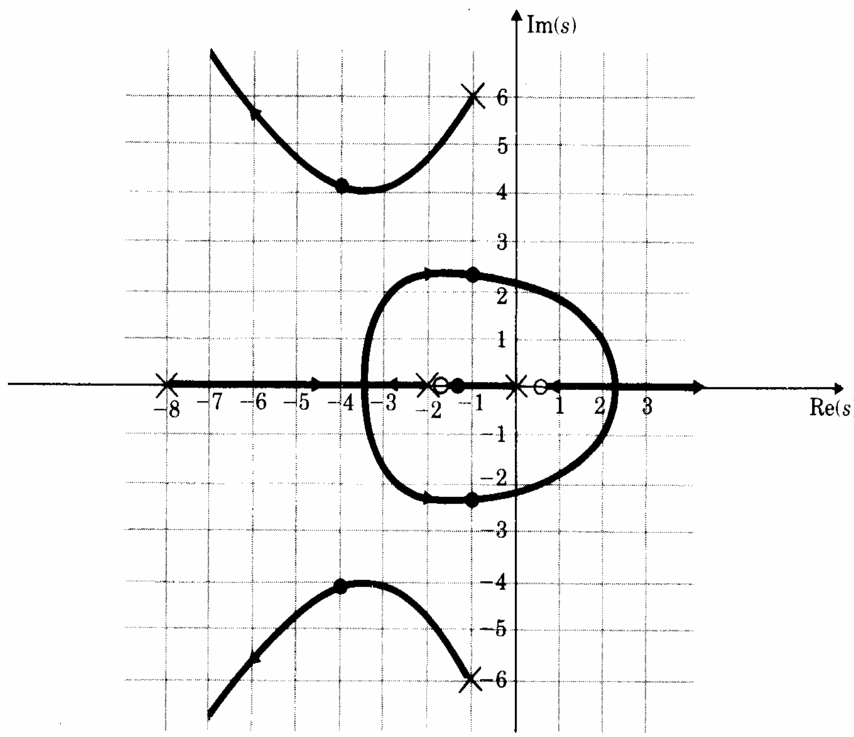
this gives $L = \begin{bmatrix} 8.4 \\ 36 \end{bmatrix}$

- compensator transfer function

$$G_{cr}(s) = 21.416\frac{(s-0.718)(s+1.87)}{s+0.95 \pm j6.17}$$

- root locus point of view
  - this is a little better than if we estimate all the states. (Aside: also reinforces my comment that the more measurements are used, the "better" it gets.)
  - The zero in the r.h.p., though, is undesirable. (Called non-minimum phase zero.) (Less undesirable than r.h.p. pole.)

FIGURE 6.38
Root locus for an ITAE reduced-order controller.



$G(s) = \frac{10}{s(s+2)(s+8)}$

Same Example using an Optimal Design

$\dot{x} = Fx + Gu + G_1 w$

$y = Hx + v$

$z = H_1 x$

consider SRL approach

assume $G_1 = G$; $H_1 = H$

SRL for state fb:

$1 + \rho\, G_0(-s) G_0(s) = 0$

$G_0(s) \triangleq H_1 \{sI - F\}^{-1} G$

then SRL is the same for controller & estimator.

$J_{SRL} = \int_0^\infty \rho z^2 + u^2 \, d\tau$ for state fb

(the locus is the same, but you pick different desired pole locations. Picking fb poles gives $\alpha_c(s)$, & ρ; picking estimator poles gives $\alpha_e(s)$, & q.)

$J_{SRL} = \int_0^\infty q e_1^2 + w^2 \, d\tau$

$e_1 \triangleq \{y - H\hat{x}\}$

for state est

SRL for estimator/observer

$1 + q\, \bar{G}_0(-s) \bar{G}_0(s) = 0$

$\bar{G}_0(s) \triangleq H \{sI - F\}^{-1} G_1$

Pick factors of $\alpha_c(s)$ as $-2 \pm j1.56,\ -8.04$

$\alpha_e(s)$ as $-4 \pm j4.9\ ,\ -9.169$

this leads to

$$K = \begin{bmatrix} -0.275 & 0.218 & 0.204 \end{bmatrix}$$

$$L = \begin{bmatrix} 7.17 \\ 97.5 \\ 368 \end{bmatrix}$$

** [N.B.: SRL for state est not in exams!!]

$$G_0(s) = H_1\{sI - F\}^{-1} G$$

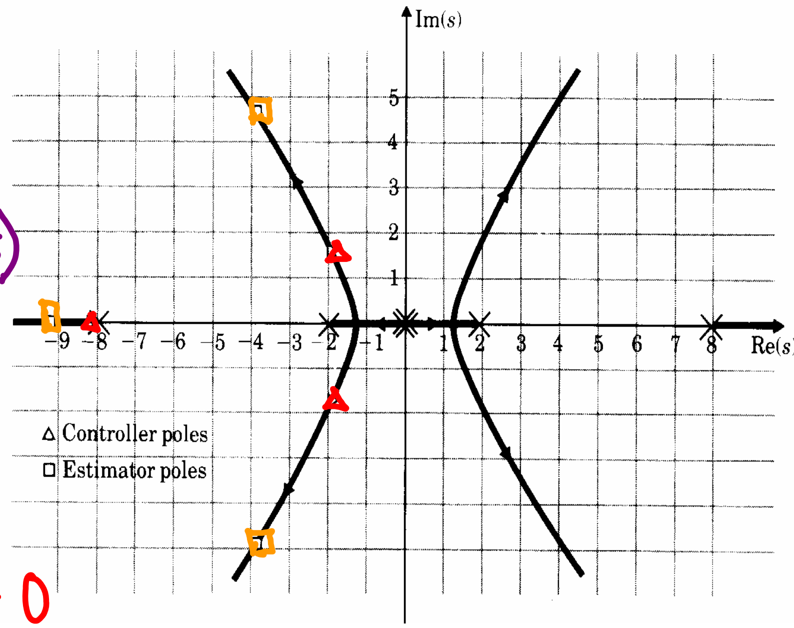


FIGURE 6.39
Symmetric root locus.

Here

$$G_0(s) = \bar{G}_0(s) = G(s) = \frac{10}{s(s+2)(s+8)}$$

$$1 + \rho\, G_0(-s)\, G_0(s) = 0$$

$$1 + q\, \bar{G}_0(-s)\, \bar{G}_0(s) = 0$$

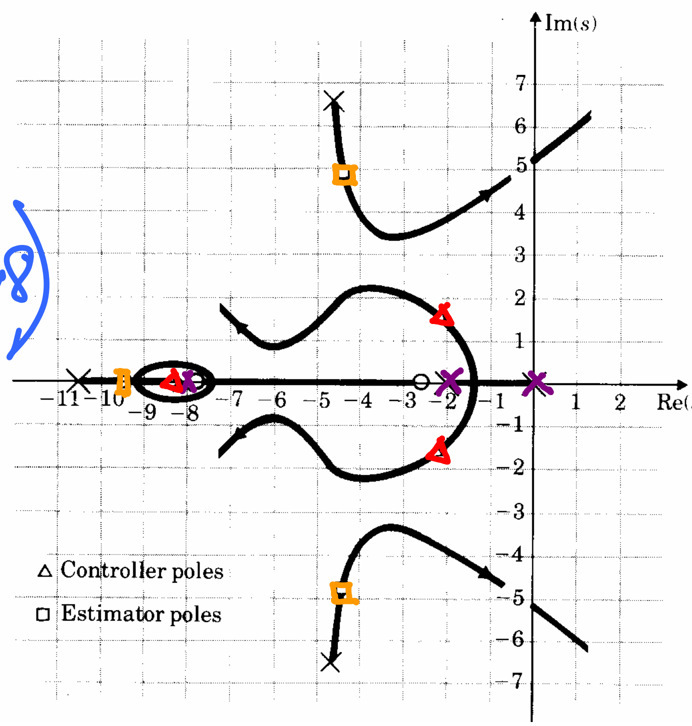


FIGURE 6.40
Root locus for optimum pole assignment.

$$G_0(-s) = \frac{10}{[-s](-s+2)(-s+8)}$$

Compensator has t.f.:

$$G_c(s) = \frac{-94.5(s+7.98)(s+2.52)}{(s+4.28 \pm j6.42)(s+10.6)}$$

$\tilde{K}$

$$G(s) = \frac{10}{s(s+2)(s+8)}$$

Resulting compensator t.f. $y \to u$

$$G_c(s) = \frac{-94.5(s+7.98)(s+2.52)}{(s+4.28 \pm j6.42)(s+10.6)}$$

Observations:

- magnitude of gains in K are smaller than in ITAE design. Since $u = -K\hat{x}$, this roughly means that magnitudes of u req'd are smaller.

- root-locus point of view :

  compensator is stable, minimum phase.

  improved design because SRL design results in root at $s = -8$ almost unchanged (recall earlier discussion on movement of poles, and req'd control effort.)

  best use of control effort is to shift roots at $s = 0, s = -2$, and leave $s = -8$ root almost untouched.

Results from SRL design gives us better insights into pole selection problem.

Thus, ITAE design can be used, but must be used wisely.

<u>Re-design using ITAE</u> (with consideration of only moving poles that are a problem.)

- Do not move the pole at $s = -8$

  (Earlier, we move everything to around a natural freq $\omega_o = 2 rads^{-1}$. In the process $s = -8$ was moved to $s = -1.4$.)

  Thus, specify two poles by ITAE with $\omega_o = 2$. Leave 3rd pole at $s = -8$.

i.e. $\alpha_c(s) = (s + 1.41 \pm j1.41)(s + 8)$

2nd order ITAE with $\omega_o = 2$

this gives $K = \begin{bmatrix} -4.69 & 2.34 & 0.828 \end{bmatrix}$

compare with $K = \begin{bmatrix} -46.4 & 5.76 & -0.65 \end{bmatrix}$

from "naive" straightfwd ITAE

- Remember estimator poles are also poles of the closed loop. $\therefore$ same consideration applies.

Choose $\alpha_e(s) = (s+1.24 \pm j4.24)(s+8)$

2nd order ITAE with $\omega_o = 6$

then

$$L = \begin{bmatrix} 6.48 \\ 87.9 \\ 288 \end{bmatrix}$$

compare with earlier: $L = \begin{bmatrix} 7.17 \\ 97.5 \\ 368 \end{bmatrix}$

Overall compensator is

$y \mapsto u$ $$G_c(s) = \frac{-414(s+2.78)(s+8)}{(s+4.13 \pm j5.29)(s+9.05)}$$

- Stable, minimum phase

- pole selection must be done with due consideration of open-loop dynamics.

Introduction of Reference Input

(Section 6.7 in Franklin & Powell) – earlier Eds
Section 7.8 & 7.9 – 4th Ed

- Several methods mentioned there. However, all not very satisfactory.

- A much better way to include the reference input is to incorporate integral control into the state fb method. This will be discussed in detail.

Approach #1 (Set-Point) – [Scaling Gain]

Simplest way to introduce reference input without using integral control: