Friedland pp 229-231

Bass-Gura Formula for State Feedback Gain

$$u = -\tilde{K}^T x \quad = -Kx \qquad \dot{x} = Fx + Gu$$

$$\tilde{K}^T = [K_1 \quad K_2 \quad \dots \quad K_n] \qquad y = Hx$$

Open loop c.e. $\quad s^n + a_1 s^{n-1} + \dots + a_n = 0$

Desired closed-loop c.e. $\quad s^n + \alpha_1 s^{n-1} + \dots + \alpha_n = 0$

$$\tilde{K} = \begin{bmatrix} K_1 \\ K_2 \\ \vdots \\ K_n \end{bmatrix} = \left[(\zeta W)^T\right]^{-1} \begin{bmatrix} \alpha_1 - a_1 \\ \alpha_2 - a_2 \\ \vdots \\ \alpha_n - a_n \end{bmatrix}$$

ζ is controllability matrix !!!

$$W = \begin{bmatrix} 1 & a_1 & \dots & a_{n-1} \\ 0 & 1 & \dots & a_{n-2} \\ \vdots & & \ddots & \\ 0 & \dots & 0 & 1 \end{bmatrix} \qquad \zeta = [G \,\vdots\, FG \,\vdots\, \dots F^{n-1}G]$$

• Exercise : use Bass-Gura's formula in the oscillator example

Transformations between realizations

$$\dot{x} = F_1 x + G_1 u \quad \xrightarrow[p=Tx]{} \quad \dot{p} = F_2 p + G_2 u$$

then
$$F_2 = TF_1T^{-1}$$
$$G_2 = TG_1$$

$$\zeta_2 = \begin{bmatrix} G_2 & F_2G_2 & \dots & F_2^{n-1}G_2 \end{bmatrix}$$
$$= T\begin{bmatrix} G_1 & F_1G_1 & \dots & F_1^{n-1}G_1 \end{bmatrix}$$

i.e. $\zeta_2 = T\zeta_1$

$\Rightarrow \quad T = \zeta_2\zeta_1^{-1}$

Think of Bass-Gura formula; & relationship to "Control Canonical Form."

- Obviously, this can only be used if you have the original (F_1, G_1) system and you want to transform to a "canonical" form (F_2, G_2) system.
- Exercise : Verify this for the oscillator example.

Selection of Pole Locations

- Previously saw that if

$$\dot{x} = Fx + Gu$$

is controllable, we can place the closed-loop poles at any desired location by feedback

$$u = -Kx$$

You will be able to explore this in CA3!

- As to what is reasonable choice, practical considerations will come in.

  - the farther the closed-loop poles are placed from open-loop poles, the larger the control signal u is going to be.

  - open-loop zeros attract nearby poles.
    $\therefore$ choice of closed-loop poles far away from *o.l.* zeros will require large control gains (applicable when output feedback is used.)

∴ best to only consider a design that corrects undesirable aspects of response only (e.g. stabilize an unstable system without requiring a high bandwidth.)

We need some systematic way to pick desired closed-loop poles. $\alpha_c(s) = s^n + \alpha_1 s^{n-1} + \dots + \alpha_n$

1 • use prototype pole-locations, understanding their time-responses & frequency response

[CA1 Expt 1 will explore]

2 • use an optimal criterion to pick the poles

3 • use an optimal criterion, calculate the control gains, and look at the frequency response.

[CA1 Expt 1 ditto]

Reading: Read also from F&P book, "7.4.1 Dominant Second-Order Poles" method, to choose $\alpha_c(s)$

form $\alpha_c(s)$ from these

(1) Prototype Pole-locations

Step [a] • find the tabulated pole locations

Step [b] • use Ackermann's formula to calculate the gains.

See the prototype time-response plots

(2) Symmetric Root Locus Method

Consider the system

$$\dot{x} = Fx + Gu \quad ; \quad y = Hx$$

Pick a linear combination of the states that are important to you.

For example if $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad ; \quad n = 3$

and if errors in x_1 only is important, consider

$$z = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = H_1 x$$

and so on.

TABLE 6.1
Prototype Response Poles

(a) ITAE transfer functions	$\cancel{k}$ n	**Pole locations for ω_0 = 1 rad / s**†
	1	$s + 1$
	2	$s + 0.7071 \pm j0.7071$‡
	3	$(s + 0.7081)(s + 0.5210 \pm j1.068)$
	4	$(s + 0.4240 \pm j1.2630)(s + 0.6260 \pm j0.4141)$
	5	$(s + 0.8955)(s + 0.3764 \pm j1.2920)(s + 0.5758 \pm j0.5339)$
	6	$(s + 0.3099 \pm j1.2634)(s + 0.5805 \pm j0.7828)(s + 0.7346 \pm j0.2873)$
(b) Bessel transfer functions	$\cancel{k}$ n	**Pole locations for ω_0 = 1 rad / s**†
	1	$s + 1$
	2	$s + 0.8660 \pm j0.5000)$‡
	3	$(s + 0.9420)(s + 0.7455 \pm j0.7112)$
	4	$(s + 0.6573 \pm j0.8302)(s + 0.9047 \pm j0.2711)$
	5	$(s + 0.9264)(s + 0.5906 \pm j0.9072)(s + 0.8516 \pm j0.4427)$
	6	$(s + 0.5385 \pm j0.9617)(s + 0.7998 \pm j0.5622)(s + 0.9093 \pm j0.1856)$

†Pole locations for other values of ω_0 can be obtained by substituting s/ω_0 for s everywhere.
‡The factors $(s + a + jb)(s + a - jb)$ are written as $(s + a \pm jb)$ to conserve space.

$$J_{ITAE} = \int_0^\infty t\left|(y-r)\right|dt$$

Consider system of order n.
Say, $n = 2$, for example.

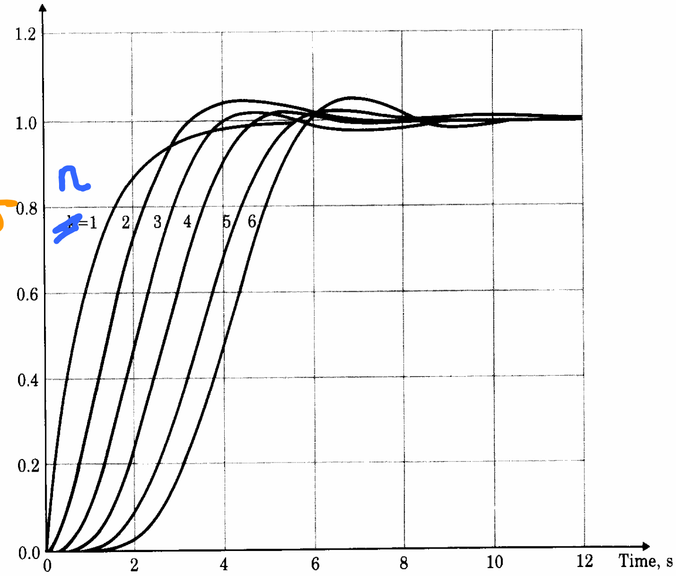
If choice is made to choose an ITAE response of bandwidth $w_0 = 3$ rad s^{-1},

then char equation is

$$\left[\frac{s}{w_0} + 0.707 + j0.707\right]\left[\frac{s}{w_0} + 0.707 - j0.707\right] = 0$$

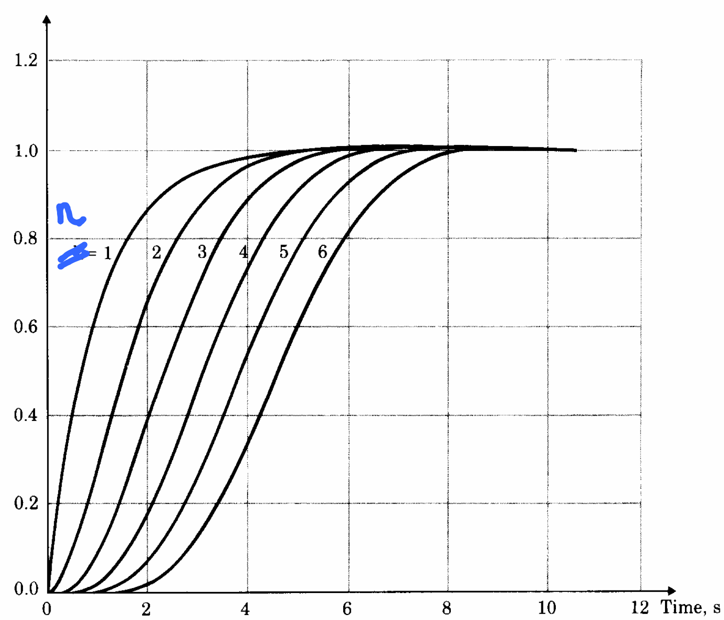
FIGURE 6.16
Step response of ITAE prototypes for $\omega_0 =$ 1 rad/s.

$$J_{ITAE} = \int_0^{\infty} t\,|y-r|\,dt$$



Section 6.4 / Selection of Pole Locations for Good Design

FIGURE 6.17
Step responses of the Bessel prototype systems for $\omega_0 =$ 1 rad/s.



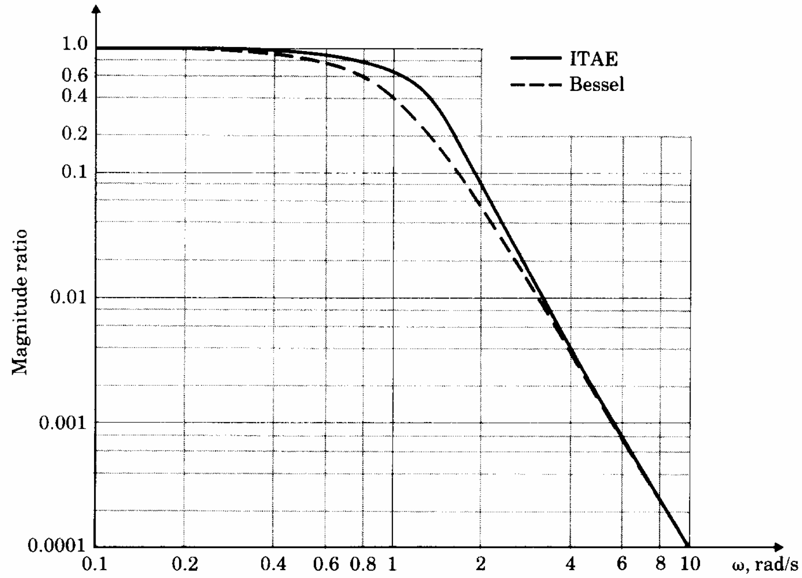


FIGURE 6.18
Frequency response magnitude of ITAE and Bessel prototype systems.

Then find the closed-loop pole locations that minimize

weighting $\rho > 0$

$$\underset{\text{SRL}}{J} = \int_0^\infty \left[\rho z^2(t) + u^2(t)\right] dt$$

Fact 1 : For a given ρ, the control that minimizes J is given by

$$u = -K_\rho x$$

Fact 2 : For a given ρ, the resulting closed-loop poles that minimizes J are given by the stable roots of the Symmetric Root Locus (SRL)

$\dot{x} = Fx + Gu$

$y = Hx$

$z = H_1 x$

$$\boxed{1 + \rho G_0(-s) G_0(s) = 0}$$

$$1 + \rho \frac{n(-s)}{d(-s)} \frac{n(s)}{d(s)} = 0$$

where $G_0(s)$ is given by

$$G_0(s) = \frac{Z(s)}{U(s)} = H_1(sI - F)^{-1} G = \frac{n(s)}{d(s)}$$

$$\frac{Y(s)}{U(s)} = H\{sI - F\}^{-1} G$$

$1 + \rho G_0(-s) G_0(s) = 0$

$\rho > 0$

Note that the SRL is symmetric about the imaginary axis, i.e.

whenever $s = s_0$ is a root of the SRL for a particular ρ, $s = -s_0$ is also a root.

$\therefore$ for every unstable root, there exists a stable root.

Order of SRL: 2n, i.e. $\forall \rho$, there are 2n roots.

How to use the SRL :

- plot the SRL, ρ being the root-locus parameter
- decide on the weighting on state error (linear combination) $z(t)$ vs. weighting on $u(t)$. This chooses the value of ρ.
- From the SRL, (assuming n=3), there will be $2n = 6$ roots for that value of ρ. Pick the $n = 3$ stable ones, and these are your desired closed-loop poles. $\alpha_c(s)$
- Calculate K from *Ackermann*'s formula needed to place the closed-loop poles at locations chosen.

$1 + \lambda \frac{r(s)}{q(s)} = 0 \qquad \lambda > 0$

Example : Inverted Pendulum

$J_{SRL} = \int_0^\infty \rho z^2 + u^2 \, dt$

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ a^2 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ -1 \end{bmatrix} u$$

Select output to be minimized as

$$z = \begin{bmatrix} 1 & 1 \end{bmatrix} x$$

then $\frac{Z(s)}{U(s)} = G_0(s) = -\frac{s+1}{s^2 - a^2}$

$G_0(-s)$

$1 + \rho \left[-\frac{(-s)+1}{(-s)^2 - a^2} \right] \cdot \left[-\frac{s+1}{s^2 - a^2} \right] = 0$

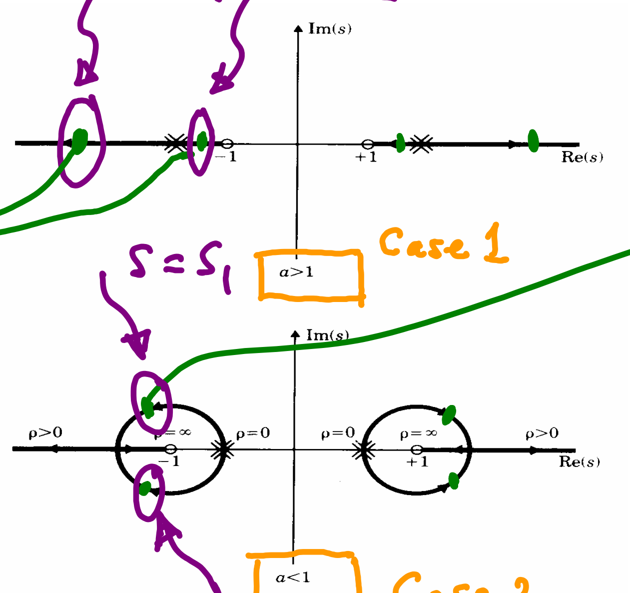
$G_0(s)$

- look at the SRL
- pick the pair of closed-loop poles (stable) that are desirable $n=2$ λ
- form $\alpha_c(s)$, and calculate K using Ackermann's formula

$\alpha_c(s) = (s-s_1)(s-s_2)$

$s = s_1 \quad s = s_2$

FIGURE 6.20
Symmetric root locus for Example 6.2.



$\rho_1 = 5$

$\alpha_c(s) = (s-s_1)(s-s_2)$

(3) Full LQR design method, with consideration of frequency response

- see "Control System Design", Friedland, pp. 343 ~ 345
- also Expt ~~2~~ 1

Method requires use of a control system design package like Matrix_x or MATLAB

Consider system

$$\dot{x} = Fx + Gu$$

Find a control to minimize

$$V = \int_0^\infty \left(x^T Q x + u^T R u \right) dt$$

Compare this with the SRL criterion

Again the control that achieves this has the structure

$$u = -Kx$$

and $\quad K = f_{lqr}(F, G, Q, R)$

Automatically calculated in software packages.

Improvement over SRL method because it allows you complete control over how to weigh each of the states, not just linear combinations.

Example :

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

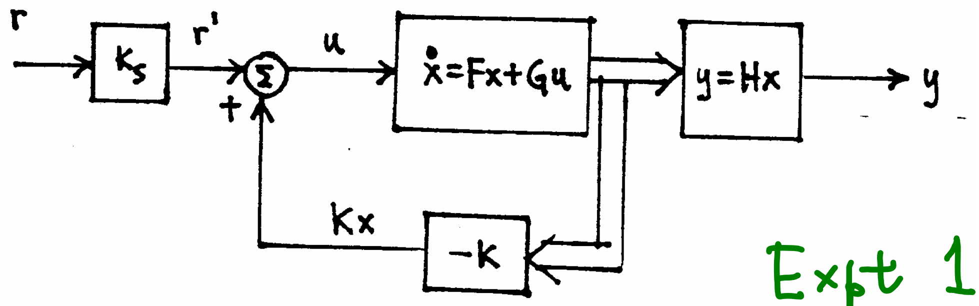
$$Q = \begin{bmatrix} 10 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 10 \end{bmatrix}$$

- means that errors in x_1 and x_3 are 10 times as important as errors in x_2.
- not possible in SRL method.

Interactively find weightings Q and R to meet frequency response specifications.

Being a method based on minimizing both “size” of x and u, it ensures that control effort will not be excessive. (Also true for SRL, but not for prototype design.)

Example



Exp't 1 will allow you to explore this.

Note: For next lecture, we will cover material from Page 102 ff first.

Using MATLAB

(a) Choose Q, R

Use MATLAB to find

$$K = f_{lqr}(F, G, Q, R)$$

MATLAB function is "LQR"

(b) Use "FEEDBACK" to form the closed loop

(c) Use "BODE" to check the frequency response between r' and y

(d) If it meets bandwidth requirements, calculate K_s to meet steady-state requirements, and you are done. Otherwise, go back to (a) and proceed again.

Reading: Read also from F&P book, "7.4.1 Dominant Second-Order Poles" method, to choose $\alpha_c(s)$

Estimator Design

In state-feedback, we assumed that we used the feedback law

$$u = -Kx$$

But not all states may be measurable.
It turns out that it is all right to use an estimate $\hat{x}$ of the state x as long as

$$\hat{x} - x \to 0 \qquad \text{exponentially}$$

(To be proved later.)

How to generate $\hat{x}$

Full Order Estimators

- Open Loop Estimator

System $\quad \dot{x} = Fx + Gu \quad ; \quad x(0)$

Estimator $\quad \dot{\hat{x}} = F\hat{x} + Gu \quad ; \quad \hat{x}(0)$

Would this work in general ?

Example : $n = 1$

System $\dot{x} = fx + u \quad ; \quad x(0) = 1$

Estimator $\dot{\hat{x}} = f\hat{x} + u \quad ; \quad \hat{x}(0) = 0$

since we do not know the state x

Consider the error

$$\tilde{x} = x - \hat{x}$$

then $\dot{\tilde{x}} = \dot{x} - \dot{\hat{x}} = fx - f\hat{x}$

$$= f\tilde{x} \quad ; \quad \tilde{x}(0) = x(0) - \hat{x}(0) = 1$$

i.e. $\boxed{\dot{\tilde{x}} = f\tilde{x} \quad ; \quad \tilde{x}(0) = 1}$

also look at this from viewpoint of s-v. equations

and $\tilde{x}(t) = x(t) - \hat{x}(t) = e^{ft}\tilde{x}(0)$

Thus $\tilde{x}(t) \to 0$ only if $f < 0$

$\tilde{x}(t)$ grows exponentially if $f > 0$

$\therefore$ open loop estimator

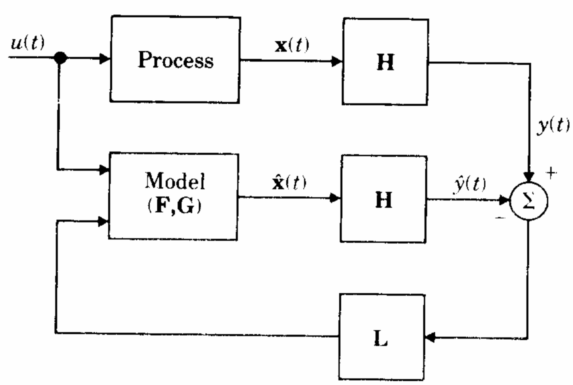
(a) cannot be used for systems with any open-loop unstable pole ;

(b) even if all open-loop poles are stable, estimation error $\tilde{x}$ may ~~be~~ not decrease fast enough if there are slow poles in the system

The only practical estimators are typically

- Closed-Loop Estimators

FIGURE 6.22
Closed-loop estimator.



"Closed-Loop Estimator/Observer"

$\therefore$ given the system

$$\dot{x} = Fx + Gu$$
$$y = Hx \qquad (6.75a)$$

and only u and y can be measured; to estimate the states, use

$$\dot{\hat{x}} = F\hat{x} + Gu + L(y - H\hat{x}) \qquad (6.75b)$$

where

$$L = \begin{bmatrix} l_1 \\ l_2 \\ \vdots \\ l_n \end{bmatrix}$$

is an estimator gain chosen to achieve satisfactory dynamics of the error $\tilde{x} = x - \hat{x}$

From (6.75a) and (6.75b), error dynamics is now given by

$$\dot{\tilde{x}} = (F - LH)\tilde{x}$$

and the poles of the error system are at

$$\det[sI - (F - LH)] = 0$$

Fast poles means faster decay of the error.

$\dot{x} = fx + u$

Example : $n = 1$ (continued)

$x(0) = x_0$

Assume $f = 1$, and $y = 2x$

$\tilde{x} \triangleq x - \hat{x}$

Now use $\dot{\hat{x}} = \hat{x} + u + l(y - 2\hat{x})$

then error dynamics is

$$\dot{\tilde{x}} = (1 - l.2)\tilde{x}$$

alternatively, if desired

choosing $l = 2$, say, this gives

$$\dot{\tilde{x}} = -3\tilde{x}$$

$\alpha_e(s) = s+3$

or $$\tilde{x}(t) = e^{-3t}\tilde{x}(0)$$

(Obviously, this example is rather constrained; but the point should be clear.)

$\dot{\tilde{x}} = (F - LH)\tilde{x}$

Therefore

- L should be chosen so that eigenvalues of $F - LH$ are stable, and sufficiently fast
- the closed-loop observer is simply a dynamic system implemented using analog components, on a digital computer.

Can L always be chosen so that the eigenvalues of the error system are assigned at arbitrary stable locations ?

To answer this question, we go back to our state-feedback results.

Recall that for

$$\dot{x} = Fx + Gu$$
$$u = -Kx$$

leading to

$$\dot{x} = (F - GK)x \qquad \text{——— (1)}$$

- The poles of the system (1) are the eigenvalues of $(F - GK)$
- The poles of (1) can be assigned at the roots of $\alpha_c(s) = s^n + \alpha_1 s^{n-1} + \ldots + \alpha_n = 0$ as long as the pair (F, G) is controllable, or, equivalently, $\zeta = [G, FG, F^2G, \ldots F^{n-1}G]$

  is of full rank.
  Ackermann's formula gives the exact value for K. ΔΔ

State-fb problem

Observer Problem :

$$\dot{\tilde{x}} = (F - LH)\tilde{x}$$

$\dot{x} = (F - GK)x$

$\tilde{x}(t) \triangleq x(t) - \hat{x}(t)$

We wish to find $L = \begin{bmatrix} l_1 \\ l_2 \\ \vdots \\ l_3 \end{bmatrix}$ so that we can assign the eigenvalues of $(F - LH)$ at the location we choose.

Borrow mathematical conditions from state-fb problem.

- first, note that a square matrix A, and its transpose A^T has the same eigenvalues.
- $\therefore$ to desire a set of eigenvalues for $(F - LH)$ is the same as desiring it for
$$(F - LH)^T = F^T - H^T L^T$$
- $(F - GK)$ can have eigenvalues assigned by K if (F, G) pair is controllable.
Thus, $(F^T - H^T L^T)$ can have ditto by L^T if (F^T, H^T) is controllable

$\zeta_c(F, G) = [G \;\; FG \;\; \ldots \;\; F^{n-1}G]$

- $\therefore\ \zeta(F^T, H^T)$
  $= \left[H^T, F^T H^T, (F^T)^2 H^T, \ldots (F^T)^{n-1} H^T\right]$

  must be of full rank.

$\zeta(F^T, H^T)$ is of full rank if its transpose

$\zeta^T(F^T, H^T)$ is of full rank

$$\zeta^T(F^T, H^T)$$
$$= \left[H^T, (F^T)H^T, (F^T)^2 H^T, \ldots (F^T)^{n-1} H^T\right]^T$$

$$= \begin{bmatrix} H \\ HF \\ HF^2 \\ \vdots \\ HF^{n-1} \end{bmatrix}$$

$= O(H, F)$ the observability Matrix given in book

i.e. $O(H, F)$ must be full rank in order to be able to calculate L to assign the eigenvalues of $\dot{\tilde{x}} = (F - LH)\tilde{x}$