1

State Space Design

Introduction

- "State space"

  i.e. the set, or space, containing the states.

Idea came from state-variable description of differential equations — a re-writing of a high-order d.e. into first order form.

E.g.

a simple example

$$\ddot{y} + a_1\dot{y} + a_2 y = bu$$

can be written as

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -a_2 x_1 - a_1 x_2 + bu$$

The d.e. written in this form can be easily solved on the computer. (using numerical integration.) Control system design using the state-variable description is called state-space design.

- Control System Design

state-variable methods	—	modern	—	state space
root locus & frequency response	—	classical	—	transform

- Design objective

  same in both methods e.g. tracking, regulation

  in situations where state-variables can be measured, state-space design methods are more powerful.

- Advantages of state-space design becomes obvious in the relative simplicity with which multi-input multi-output systems can be treated.

Also state-variable representation provides information on internal representation not available from only input-output data.

System Description

Motion of any finite dimensional dynamic system can be expressed as a set of first-order d.e.

Example: Newton's law for a single mass M moving in one dimension under an applied force F

$$M\ddot{x} = F \qquad \text{(second order d.e.)}$$

Define one state variable x_1 as position $x_1 = x$ the other state as velocity $x_2 = \dot{x} = \dot{x}_1$. Then in state-variable form

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = \frac{F}{M}$$

Input

$u = F$

And if we are interested in the output

$$y = x_1$$

These can be cast into the matrix form

$$\dot{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \overbrace{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}^{F} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \overbrace{\begin{bmatrix} 0 \\ \frac{1}{M} \end{bmatrix}}^{G} u \qquad u = F$$

$$y = \underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}_{H} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

The general compact notation is

$$\boxed{\begin{aligned} \dot{x} &= Fx + Gu + G_1 w \\ y &= Hx + Ju + n_1 \end{aligned}}$$

additional disturbances incorporated easily

for linear systems.

See also two-mass example in book.

Mostly, we consider SISO systems. Thus, unless otherwise stated

$$\dot{x} = Fx + G\overset{1x1}{u}$$

$$\underset{1x1}{y} = H\underset{nx1}{x} + Ju$$

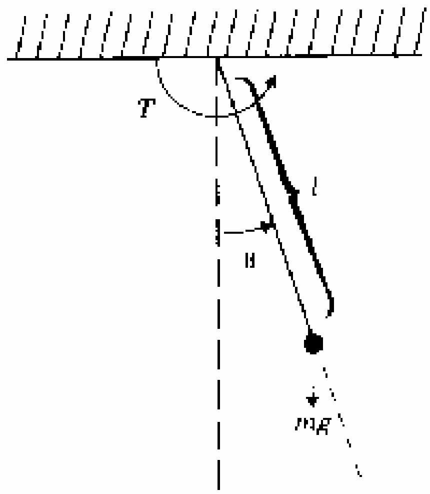
$\underset{nxn}{F}$	$\underset{nx1}{G}$
$\underset{1xn}{H}$	$\underset{1x1}{J}$

where n is the dimension of the state vector.

Example : Motion of simple pendulum

$$\ddot{\theta} + w_o^2 \sin\theta = \frac{T_c}{ml^2}$$

FIGURE 6.4
A simple pendulum.



Choose the state-variables

$$x_1 = w_o\theta$$
$$x_2 = \dot{\theta}$$

Then

$$\dot{x}_1 = w_o x_2$$
$$\dot{x}_2 = -w_0^2 \sin\left(\frac{x_1}{w_o}\right) + \frac{T_c}{ml^2}$$

Show this!

Input $u = T_c$

Equations are non-linear.
They are in state-variable form.

In state-variable form, equilibrium easily checked by having

looks at situation with $u = 0$

$$\dot{x}_1 = 0\ ; \quad \dot{x}_2 = 0$$

For no applied torque T_c, possible equilibrium states are

$x_1 \triangleq w_o \theta$

$x_2 \triangleq \dot{\theta}$

$$x_1^* = 0 \quad \text{or} \quad \pi w_o$$
$$x_2^* = 0$$

since

$\sin \alpha \simeq \alpha$ for small α

Consider small deviations about the down position. Linearization about $x_1^* = x_2^* = 0$ for small deviations yield

$$\dot{x}_1 = w_o x_2$$

$$\dot{x}_2 = -w_o x_1 + \frac{T_c}{ml^2}$$

Thus

$$F = \begin{bmatrix} & \\ & \end{bmatrix}; \qquad G = \begin{bmatrix} \\ \end{bmatrix}$$

$$H = [\qquad]\ ; \qquad J =$$

Some reasons for using state-variable form

- to study more general models

  state-variable can easily include non-linear as well as time-varying systems.

  We have already seen the non-linear state variable equation example in dynamics of pendulum for large angular motions.

  Time-varying systems

  $$\ddot{y} + a_1(t)\dot{y} + a_2(t)y = b(t)u$$

  Transfer function description not possible state variable description straightforward.

- introduce ideas of geometry into d.e.

  plot of position vs velocity is phase plane, this shows motion of system in two-dimension. State-space generalizes this idea to higher dimensions. Visualize solutions of d.e. as trajectory in state-space of appropriate dimension.

- to connect internal and external descriptions

  state-variables show the internal dynamics of the system.

  Through the matrix representation, a simple relationship can be established to the external input-output description.

  Transfer-function description focuses on input-output representation only.

Relation of states to analog-module patching

State variable description of system has a direct correspondence to analog-module patching diagrams

Consider dynamical system described by d.e.

$$\dddot{y} + a_1\ddot{y} + a_2\dot{y} + a_3 y = b_1\ddot{u} + b_2\dot{u} + b_3 u$$

Transfer function is

$$\frac{Y(S)}{U(S)} = \frac{b_1 s^2 + b_2 s + b_3}{s^3 + a_1 s^2 + a_2 s + a_3}$$

Consider analog-module patch-up of above system.

- assume fundamental building block is an integrator unit.

(remember most analog-module integrators in practice have a negative sign!

Method 1 :

Use an operator notation $p \equiv \dfrac{d}{dt}$

i.e. $px(t)$ is equivalent to $\dfrac{dx(t)}{dt}$

then d.e. is

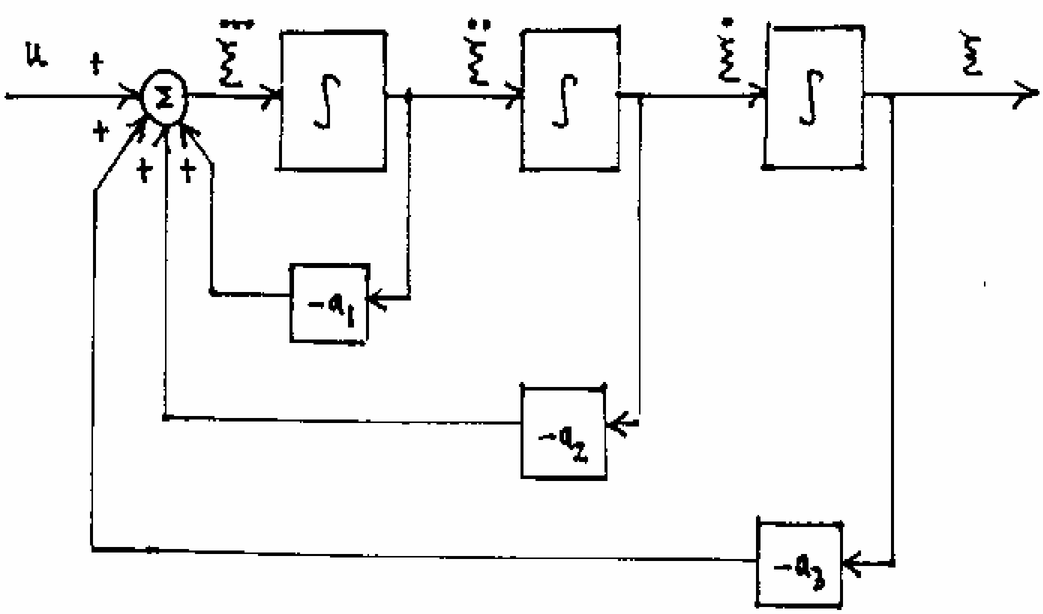
$$\left(p^3 + a_1 p^2 + a_2 p + a_3\right)y = \left(b_1 p^2 + b_2 p + b_3\right)u \qquad \text{—— (1-1)}$$

Re-write as

$$(p^3 + a_1 p^2 + a_2 p + a_3)\xi = u \qquad \text{—— (1-2)}$$

$$\therefore \qquad y = (b_1 p^2 + b_2 p + b_3)\xi \qquad \text{—— (1-3)}$$

This is easily patched up as



Identify the states as

$$x_1 = \xi \ , \qquad x_2 = \dot{\xi} = p\xi \ , \qquad x_3 = \ddot{\xi} = p^2\xi$$

then the state-variable description of the system is

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} \quad & \quad & \quad \\ & & \\ & & \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} \quad \\ \\ \end{bmatrix} u$$

$$y = (b_1 p^2 + b_2 p + b_3)\xi$$

$$= \begin{bmatrix} b_3 & b_2 & b_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Method 2 :

$$\frac{Y(s)}{U(s)} = \frac{b_1 s^2 + b_2 s + b_3}{s^3 + a_1 s^2 + a_2 s + a_3}$$

$$\dddot{y} + a_1\ddot{y} + a_2\dot{y} + a_3 y = b_1\ddot{u} + b_2\dot{u} + b_3 u$$

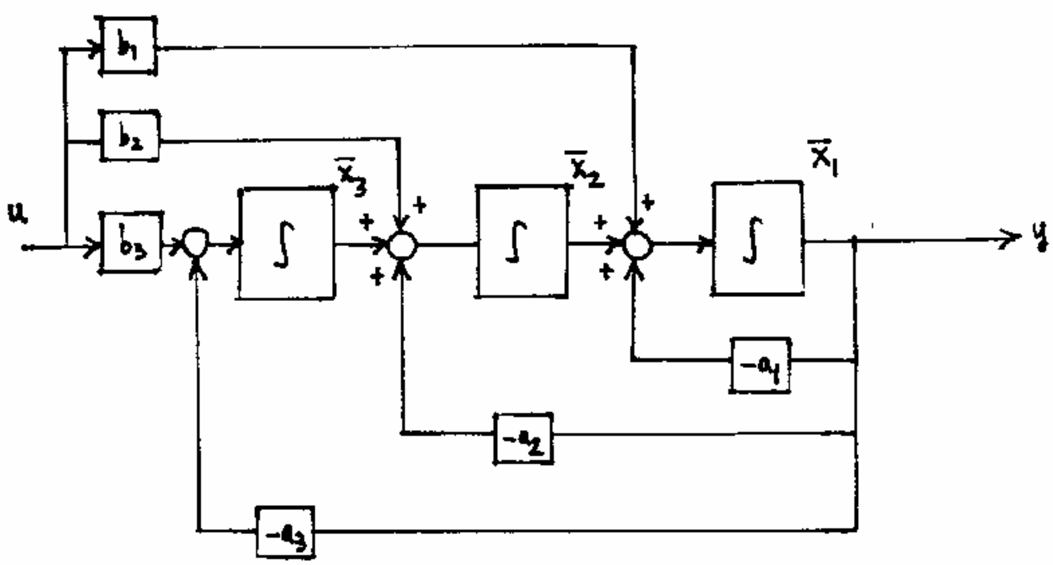
- collect together terms with equal number of differentiation operations, and re-write as

$$\dddot{y} = (-a_1\ddot{y} + b_1\ddot{u}) + (-a_2\dot{y} + b_2\dot{u}) + (-a_3 y + b_3 u)$$

- Integrate thru n number of times. (n = 3 here)

$$y = \int(-a_1 y + b_1 u) + \int\int(-a_2 y + b_2 u) + \int\int\int(-a_3 y + b_3 u)$$

- this is easily patched-up as



• mark outputs of integrators as states

Show this

$$\therefore \quad \begin{aligned} \dot{\bar{x}}_1 &= -a_1\bar{x}_1 + \bar{x}_2 + 0\bar{x}_3 + b_1 u \\ \dot{\bar{x}}_2 &= -a_2\bar{x}_1 + 0\bar{x}_2 + \bar{x}_3 + b_2 u \\ \dot{\bar{x}}_3 &= -a_3\bar{x}_1 + 0\bar{x}_2 + 0\bar{x}_3 + b_3 u \end{aligned}$$

and $\quad y = 1\bar{x}_1 + 0\bar{x}_2 + 0\bar{x}_3$

State Transformations

As we have seen, the choice of state variables to describe a given system is not unique. [Example = Method 1 & Method 2] However, any two different choices for the state representation will be related by a non-singular linear transformation.

Consider for example spacecraft example (in book)

$$\underbrace{\begin{bmatrix} \dot{q} \\ \dot{\theta} \\ \dot{\theta}_m \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 2 & -2 \end{bmatrix}}_{F} \begin{bmatrix} q \\ \theta \\ \theta_m \end{bmatrix} + \underbrace{\begin{bmatrix} 0.9 \\ 0 \\ 0 \end{bmatrix}}_{G} M_c + \underbrace{\begin{bmatrix} 0.9 \\ 0 \\ 0 \end{bmatrix}}_{G_1} M_d$$

$$y = \underbrace{\begin{bmatrix} 0 & 0 & 1 \end{bmatrix}}_{H} \begin{bmatrix} q \\ \theta \\ \theta_m \end{bmatrix}$$

A different choice of state representation could be simply a re-ordering

$$p = \begin{bmatrix} \theta_m \\ \theta \\ q \end{bmatrix}$$

compared with

$$x = \begin{bmatrix} q \\ \theta \\ \theta_m \end{bmatrix}$$

Then p & x are related by

the non-singular transformation

$$p = Tx \quad ; \quad T = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

and the new state representation is

Check this

$$\dot{p} = F'p + G'M_c + G_1'M_d$$
$$y = H'p$$

$$F' = \begin{bmatrix} -2 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} ; G' = G_1' = \begin{bmatrix} 0 \\ 0 \\ 0.9 \end{bmatrix} ; H' = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$$

For more complicated transformations, new and old representation matrices are related as

$$F' = TFT^{-1} \; ; \; G' = TG \; ; \; G_1' = TG_1 \; ; \; H' = HT^{-1}$$

show this!

Reasons for seeking alternative representations :

- Try to find a representation useful for a particular analysis. E.g. to calculate suitable controller gains, the "controller canonical form" is useful.

  Alternatively, to examine the poles of the system, a diagonal form for F' (if possible) is desirable.

... and other reasons...

System Transfer Functions

State representation has a simple relation to transfer functions

$$\dot{x} = Fx + Gu$$
$$y = Hx + Ju$$

Remember transfer function analysis assumes zero i.c.

Take Laplace transforms

$$sX(s) - x(0) = FX(s) + GU(s)$$
$$Y(s) = HX(s) + JU)(s)$$

Drop i.c.

$$Y(s) = H(sI - F)^{-1} GU(s) + JU(s)$$

Thus the input-output description (transfer function) is given by

$$\mathbf{G}(s) = \frac{Y(s)}{U(s)} = H(sI - F)^{-1} G + J$$

Poles, Zeros and Eigenvalues

Poles : If for some particular initial condition x_0, the system responds as $\overset{x_0}{\cancel{K_i}}e^{\lambda_i t}$ with no forcing function, then $s = \lambda_i$ is a pole of the transfer function **G**(s). ##

Thus for $u \equiv 0$

$$\dot{x} = Fx$$

Assume $x(t) = e^{\lambda_i t} x_0$ for i.c. $x(0) = x_0$

Then

$$\text{L.H.S.} = \dot{x} = \lambda_i e^{\lambda_i t} x_0$$

$$\text{R.H.S.} = Fx = Fe^{\lambda_i t} x_0$$

or
$$\boxed{(\lambda_i I - F)x_0 = 0}$$

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} a \\ -a \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

For a non-zero response, $x_0 \neq 0$, and this happens iff

$$\boxed{\det[\lambda_i I - F] = 0}$$

##

i.e. poles of **G**(s) are eigenvalues of F.

Zeros

A zero is a value of frequency s such that ~~is~~ if the input is

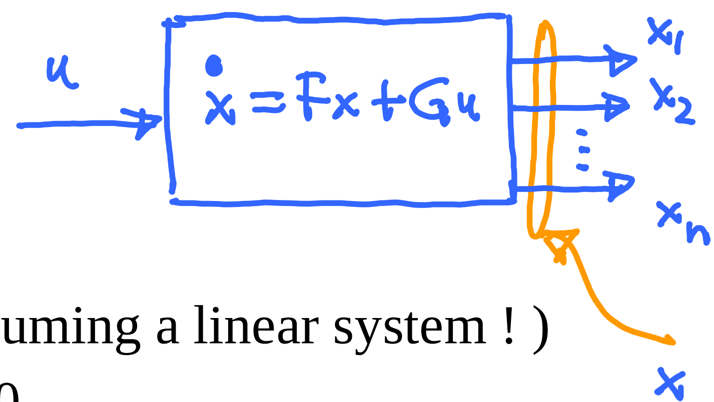
$$u = u_0 e^{st}$$

the output is identically zero (with zero i.c.).

Thus if the system is relaxed (zero i.c.), an input of $u_0 e^{st}$ produces zero output.

Thus for the state-space system

$$u = u_0 e^{st}$$



produces states

$$x = x_0 e^{st}$$

(Remember, we are assuming a linear system !)

and output $\qquad y \equiv 0$

Thus

$$\dot{x} = s e^{st} x_0 = F e^{st} x_0 + G u_0 e^{st}$$

or

$$[sI - F, -G]\begin{bmatrix} x_0 \\ \hdashline u_0 \end{bmatrix} = 0$$

and

$$y = Hx + Ju = He^{st}x_0 + Je^{st}u_0 \equiv 0$$

Combining, we have

$$\begin{bmatrix} sI - F & -G \\ H & J \end{bmatrix} \begin{bmatrix} x_0 \\ u_0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Thus, a zero of the state-space system is a value of s where the above equation has a non-trivial solution for $\begin{bmatrix} x_0 \\ u_0 \end{bmatrix}$. For SISO system, check that above matrix is square, and non-trivial solution to above corresponds to

$$\det\begin{bmatrix} sI - F & -G \\ H & J \end{bmatrix} = 0$$

##

$$G(s) = \frac{(s+1)}{(s+2)(s+3)} \xrightarrow[\text{Method 1 or Method 2}]{} \left.\begin{aligned} \dot{x} &= Fx + Gu \\ y &= Hx + Ju \end{aligned}\right\} \text{Calculate poles \& zeros}$$

Controller Design

State-space design method

- breaks naturally into two parts
- design feedback gain assuming all states measurable —— controller design
- for those states not measurable, design a state estimator —— estimator design
- combine the estimator & controller

Practical note:

In a real-world problem, design your system so that you can measure as many states as possible. Only estimate those states that you absolutely cannot measure. Feedback using measured states is much more robust to variations in system parameters.