NATIONAL UNIVERSITY OF SINGAPORE

EXAMINATION FOR
(Semester I: 2021/2022)

EE4302 – ADVANCED CONTROL SYSTEMS

November/December 2021 – Time Allowed: 2 Hours

INSTRUCTIONS TO CANDIDATES:

1. Please write your student number only. Do not write your name.
2. This question paper contains **FOUR** (4) questions and comprises **Eleven** (11) pages.
3. Answer **ALL** questions.
4. Note that the Questions do not carry equal marks.
5. This is an **OPEN BOOK** examination.
6. Relevant data are provided at the end of this examination paper.
7. Graphics/Programmable calculators are not allowed.

Q1 Consider the system given by the transfer function

$$\frac{Y(s)}{U(s)} = \frac{s+\beta}{(s+1)(s+2)(s+3)}$$

For this system, using partial fraction expansion (or any other preferred methodology), write next the system in the form

$$\frac{Y(s)}{U(s)} = \frac{g_1}{s+1} + \frac{g_2}{s+2} + \frac{g_3}{s+3}$$

showing clearly the exact expressions for g_1, g_2 and g_3. Further, also write the system as

$$\begin{aligned} \frac{X_1(s)}{U(s)} &= \frac{g_1}{s+1} \\ \frac{X_2(s)}{U(s)} &= \frac{g_2}{s+2} \\ \frac{X_3(s)}{U(s)} &= \frac{g_3}{s+3} \end{aligned}$$

Using the above, show clearly (with all necessary supporting diagrams and descriptions) the resulting matrix form of the state-variable system with the time-domain signals $x_1(t)$, $x_2(t)$ and $x_3(t)$ as the state-variables.

[10 marks]

With this expansion above, provide all necessary analysis (and appropriate detailed descriptions) to show that if any one of the coefficients g_1, g_2 or g_3 should take on the zero value, then certainly the controllability property of the system is now lost.

[7 marks]

Q2 Consider an oscillator system (in the open-loop) given by

$$\begin{aligned} \dot{x}_1(t) &= x_2(t) \\ \dot{x}_2(t) &= -w_0^2\ x_1(t) + u(t) \\ y(t) &= x_1(t) \end{aligned}$$

where w_0 is a positive-valued constant. Here $y(t)$ is the measured output of the system to be controlled, and $r(t)$ is a set-point command signal which will be applied to the closed-loop system.

It is desired to use the state-feedback method (with scaling gain)

$$u(t) = -k_1 x_1(t) - k_2 x_2(t) + k_s r(t)$$

to attain to a stable (and sufficiently fast) closed-loop with both closed-loop poles at $-5w_0$, and also with 0 dB steady-state gain.

Using Ackermann's formula (the formula may be found in the Data Sheet at the end of this Examination script), calculate the required values of k_1 and k_2 to achieve this. Show clearly all the steps in your calculation.

Next, using the Bass-Gura's formula (the formula may also be found in the Data Sheet at the end of this Examination script), likewise calculate the required values of k_1 and k_2 to achieve this. Again, show clearly all the steps in your calculation.

[10 marks]

Finally, for the case with $w_0 = 1$, calculate the required value of the scaling gain k_s to attain the specified 0 dB steady-state gain in the closed-loop. Show clearly all the steps in your calculation.

[8 marks]

Q3 The Plant and Nonlinearity in Figure Q3 are given as

$$\frac{Y(s)}{U(s)} = \frac{3}{s(2s+1)}$$

$$u(t) = \begin{cases} e(t) & \text{for} \quad -1 \leq e(t) \leq 1 \\ 1 & \text{for} \quad e(t) > 1 \\ -1 & \text{for} \quad e(t) < -1 \end{cases}$$

respectively.

a) Determine the isocline equation for the states defined as $x_1 = e$ and $x_2 = \dot{e}$.

[10 marks]

b) On a phase-plane of $-3 \leq x_1 \leq 3$ and $-3 \leq x_2 \leq 3$, sketch the isoclines for slopes of $\alpha = -1$, 0, 1, and ∞.

[10 marks]

c) Using the isoclines, sketch the trajectory of the states. Start from $x_1 = -\frac{1}{3}$, $x_2 = -1$ and end when $x_2 = 1.5$.

[10 marks]

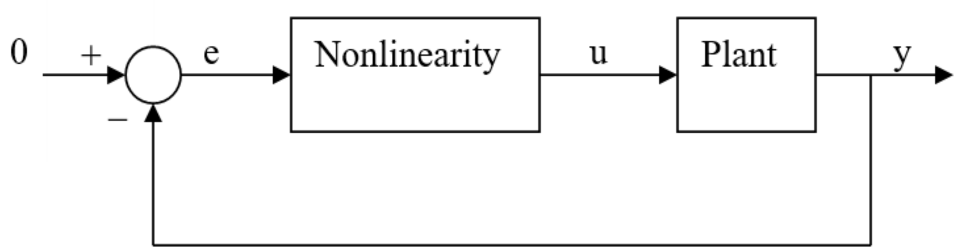

Figure Q3

Q4 Consider the nonlinear process

$$
\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -x_1^2 - x_1 x_2 + u \\
y &= x_1
\end{aligned}
$$

where u and y are the input and output respectively. The states are given by x_1 and x_2.

a) For $u = \bar{u}$, find the equilibrium points $\bar{x}_1$ $\bar{x}_2$ in terms of $\bar{u}$.

[5 marks]

b) Using the phase portrait in Figure Q4a, sketch the graph of x_1 versus t.

[15 marks]

c) In Figure Q4b, estimate x_2 at $x_1 = 10.05$, 10.1, 10.15, and 10.2. Using the estimates, sketch the phase portrait of x_2 versus x_1.

[15 marks]

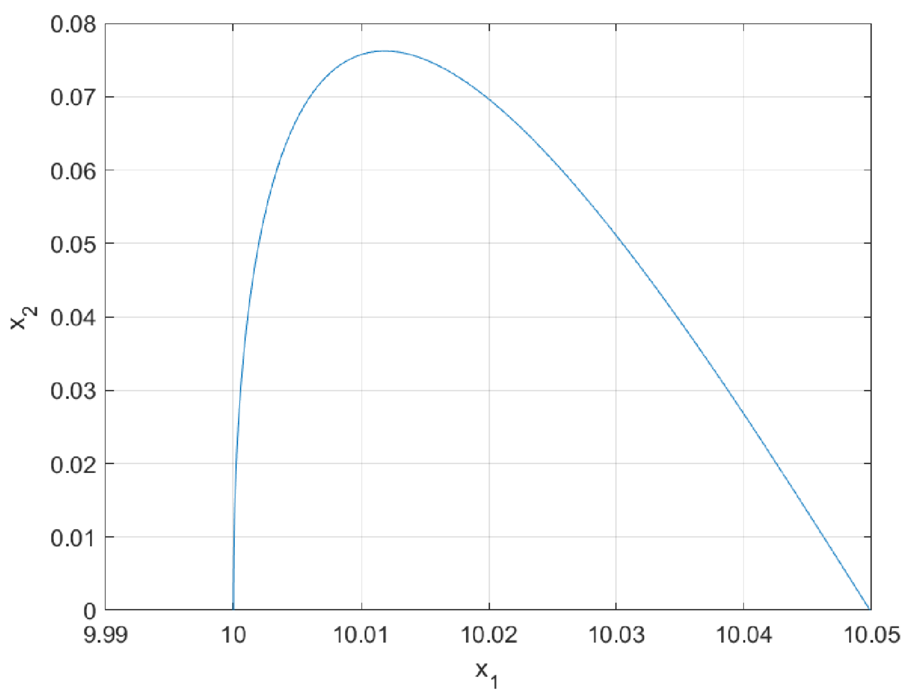

Figure Q4a

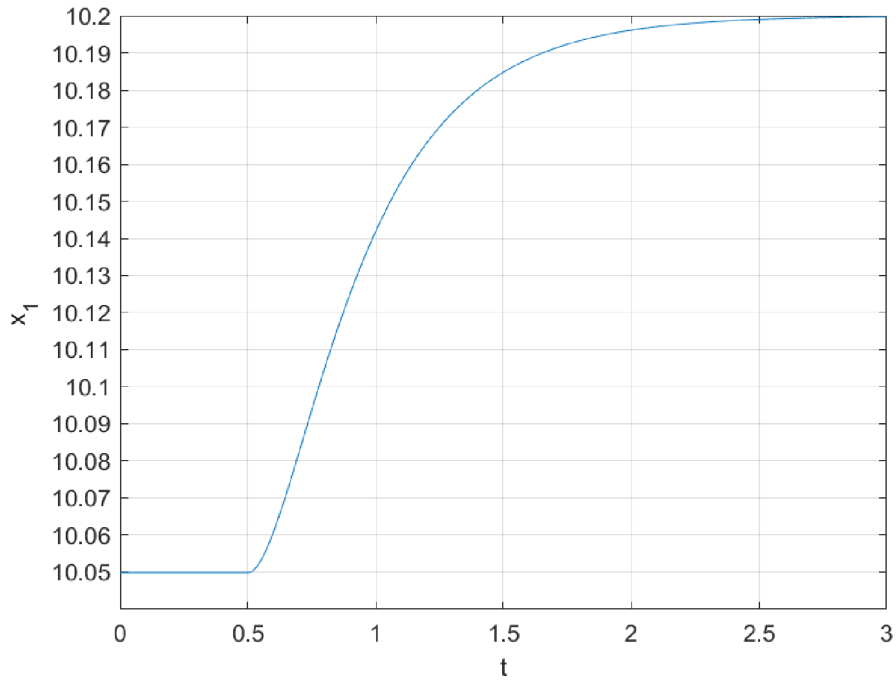

Figure Q4b

END OF QUESTIONS

DATA SHEET:

1. For the matrices $\mathbf{A} \in \mathbf{R}^{n \times n}$, $\mathbf{C} \in \mathbf{R}^{1 \times n}$, and $\mathbf{L} \in \mathbf{R}^{n}$, the eigenvalues of the matrix $(\mathbf{A} - \mathbf{LC})$ can be arbitrarily assigned by a suitable choice of $\mathbf{L}$ as long as

$$O(\mathbf{A}, \mathbf{C}) = \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \\ \vdots \\ \mathbf{CA}^{(n-1)} \end{bmatrix}$$

is non-singular.

2. For the linear system

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{Fx} + \mathbf{G}u \\ y &= \mathbf{Hx} \end{aligned}$$

where $\mathbf{x} \in \mathbf{R}^n$ and $u, y \in \mathbf{R}^1$, the controllability matrix of the system is given by

$$C(\mathbf{F}, \mathbf{G}) = \begin{bmatrix} \mathbf{G} & \mathbf{FG} & ... & \mathbf{F}^{(n-1)}\mathbf{G} \end{bmatrix}$$

If the characteristic polynomial of $\mathbf{F}$ is given by

$$\alpha(s) = s^n + a_1 s^{n-1} + a_2 s^{n-2} + ... + a_n$$

then the state-feedback $u = -\mathbf{Kx}$ which yields the closed-loop characteristic polynomial

$$\alpha_c(s) = s^n + \alpha_1 s^{n-1} + \alpha_2 s^{n-2} + ... + \alpha_n$$

can be calculated using the Bass-Gura's formula

$$\mathbf{K} = \begin{bmatrix} \alpha_1 - a_1 & \alpha_2 - a_2 & ... & \alpha_n - a_n \end{bmatrix} \{C(\mathbf{F}, \mathbf{G})W\}^{-1}$$

where

$$W = \begin{bmatrix} 1 & a_1 & a_2 & ... & a_{n-1} \\ 0 & 1 & a_1 & ... & a_{n-2} \\ \vdots & \vdots & \ddots & ... & \vdots \\ 0 & 0 & 0 & ... & 1 \end{bmatrix}$$

Equivalently and alternatively, it can also be calculated using the Ackermann's formula

$$\mathbf{K} = \begin{bmatrix} 0 & 0 & \ldots & 0 & 1 \end{bmatrix} \{C(\mathbf{F}, \mathbf{G})\}^{-1} \alpha_c(\mathbf{F})$$

where

$$\alpha_c(\mathbf{F}) = \mathbf{F}^n + \alpha_1 \mathbf{F}^{n-1} + \alpha_2 \mathbf{F}^{n-2} + \ldots + \alpha_n \mathbf{I}$$

3. For the system

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3 \\ &\vdots \\ \dot{x}_n &= -a_n x_1 - a_{n-1} x_2 - \ldots - a_1 x_n + b_0 u \\ y &= x_1 \end{aligned}$$

the characteristic polynomial is

$$\alpha_o(s) = s^n + a_1 s^{n-1} + \ldots + a_n$$

and the transfer function is

$$\frac{Y(s)}{U(s)} = \frac{b_0}{s^n + a_1 s^{n-1} + \ldots + a_n}$$

4. For the triple

$$\begin{aligned} A_m &= \begin{bmatrix} 0 & 1 & 0 \\ a_1 & a_2 & a_3 \\ 1 & 0 & 0 \end{bmatrix} \\ b_m &= \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} \\ c_m &= \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \end{aligned}$$

the equivalent transfer function is

$$c_m^\top [sI - A_m]^{-1} b_m = \frac{-a_3}{s^3 - a_2 s^2 - a_1 s - a_3}$$

5. In the development of a reduced-order observer, note that a state-variable system with state vector $\mathbf{x}$ of order n, where the first n_1 state-variables, in a vector $\mathbf{x}_1$ are

essentially measurable, can be written as:

$$\begin{aligned}
\dot{\mathbf{x}}_1 &= \mathbf{F}_{11}\mathbf{x}_1 + \mathbf{F}_{12}\mathbf{x}_2 + \mathbf{G}_1 u \\
\dot{\mathbf{x}}_2 &= \mathbf{F}_{21}\mathbf{x}_1 + \mathbf{F}_{22}\mathbf{x}_2 + \mathbf{G}_2 u
\end{aligned}$$

with the remaining n_2 state-variables, in a vector $\mathbf{x}_2$ to be estimated (or observed). Here, $\mathbf{F}_{11}$, $\mathbf{F}_{12}$, $\mathbf{F}_{21}$ and $\mathbf{F}_{22}$ are all known system matrices (obtained by calibration data, for example), and the measurement is typically given by

$$\mathbf{y}_m = \mathbf{H}_1\mathbf{x}_1$$

where $\mathbf{H}_1$ is also a known $(n_1 \times n_1)$ system matrix. Under these circumstances, a suitable form for the estimator for $\mathbf{x}_2(t)$ is

$$\begin{aligned}
\hat{\mathbf{x}}_2 &= \mathbf{L}\mathbf{y}_m + \mathbf{z} \\
\dot{\mathbf{z}} &= \bar{\mathbf{F}}\mathbf{z} + \bar{\mathbf{G}}\mathbf{y}_m + \bar{\mathbf{H}}u
\end{aligned}$$

where a suitable set of matrices defining the reduced-order observer are given by:

$$\begin{aligned}
\bar{\mathbf{F}} &= \mathbf{F}_{22} - \mathbf{L}\mathbf{H}_1\mathbf{F}_{12} \\
\bar{\mathbf{G}} &= \left\{\mathbf{F}_{21} - \mathbf{L}\mathbf{H}_1\mathbf{F}_{11} + \bar{\mathbf{F}}\mathbf{L}\mathbf{H}_1\right\}\mathbf{H}_1^{-1} \\
\bar{\mathbf{H}} &= \mathbf{G}_2 - \mathbf{L}\mathbf{H}_1\mathbf{G}_1
\end{aligned}$$

6. Prototype Response Tables

	k	Pole Locations for $\omega_0 = 1\ rad/s^a$
ITAE	1	$s + 1$
	2	$s + 0.7071 \pm 0.7071j^b$
	3	$(s + 0.7081)(s + 0.5210 \pm 1.068j)$
	4	$(s + 0.4240 \pm 1.2630j)(s + 0.6260 \pm 0.4141j)$
	5	$(s + 0.8955)(s + 0.3764 \pm 1.2920j)(s + 0.5758 \pm 0.5339j)$
Bessel	1	$s + 1$
	2	$s + 0.8660 \pm 0.5000j^b$
	3	$(s + 0.9420)(s + 0.7455 \pm 0.7112j)$
	4	$(s + 0.6573 \pm 0.8302j)(s + 0.9047 \pm 0.2711j)$
	5	$(s + 0.9264)(s + 0.5906 \pm 0.9072j)(s + 0.8516 \pm 0.4427j)$

[a] Pole locations for other values of ω_0 can be obtained by substituting s/ω_0 for s.
[b] The factors $(s + a + bj)(s + a - bj)$ are written as $(s + a \pm bj)$ to conserve space.

Laplace Transform Table

Laplace Transform, F(s)	Time Function, f(t)
1	$\delta(t)$ (unit impulse)
$\frac{1}{s}$	$u(t)$ (unit step)
$\frac{1}{s^2}$	t
$\frac{1}{s^n}$	$\frac{t^{n-1}}{(n-1)!}$ ($n =$ positive integer)
$\frac{1}{s+a}$	e^{-at}
$\frac{a}{s(s+a)}$	$1 - e^{-at}$
$\frac{1}{(s+a)^2}$	te^{-at}
$\frac{a^2}{s(s+a)^2}$	$1 - e^{-at}(1 + at)$
$\frac{1}{(s+a)^n}$	$\frac{1}{(n-1)!}t^{n-1}e^{-at}$ ($n =$ positive integer)
$\frac{1}{(s+a)(s+b)}$	$\frac{e^{-at}-e^{-bt}}{b-a}$
$\frac{1}{s(s+a)(s+b)}$	$\frac{1}{ab}\left[1 + \frac{1}{a-b}(be^{-at} - ae^{-bt})\right]$
$\frac{\omega}{s^2+\omega^2}$	$\sin \omega t$
$\frac{\omega^2}{s(s^2+\omega^2)}$	$1 - \cos \omega t$
$\frac{s}{s^2+\omega^2}$	$\cos \omega t$
$\frac{\omega}{(s+a)^2+\omega^2}$	$e^{-at} \sin \omega t$
$\frac{s+a}{(s+a)^2+\omega^2}$	$e^{-at} \cos \omega t$
$\frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$	$\frac{\omega_n}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t} \sin \omega_n \sqrt{1-\zeta^2}t$
$\frac{s}{s^2+2\zeta\omega_n s+\omega_n^2}$	$-\frac{1}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t} \sin(\omega_n \sqrt{1-\zeta^2}t - \phi)$ $\phi = \tan^{-1} \frac{\sqrt{1-\zeta^2}}{\zeta}$
$\frac{\omega_n^2}{s(s^2+2\zeta\omega_n s+\omega_n^2)}$	$1 - \frac{1}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t} \sin(\omega_n \sqrt{1-\zeta^2}t + \phi)$ $\phi = \tan^{-1} \frac{\sqrt{1-\zeta^2}}{\zeta}$

END OF PAPER