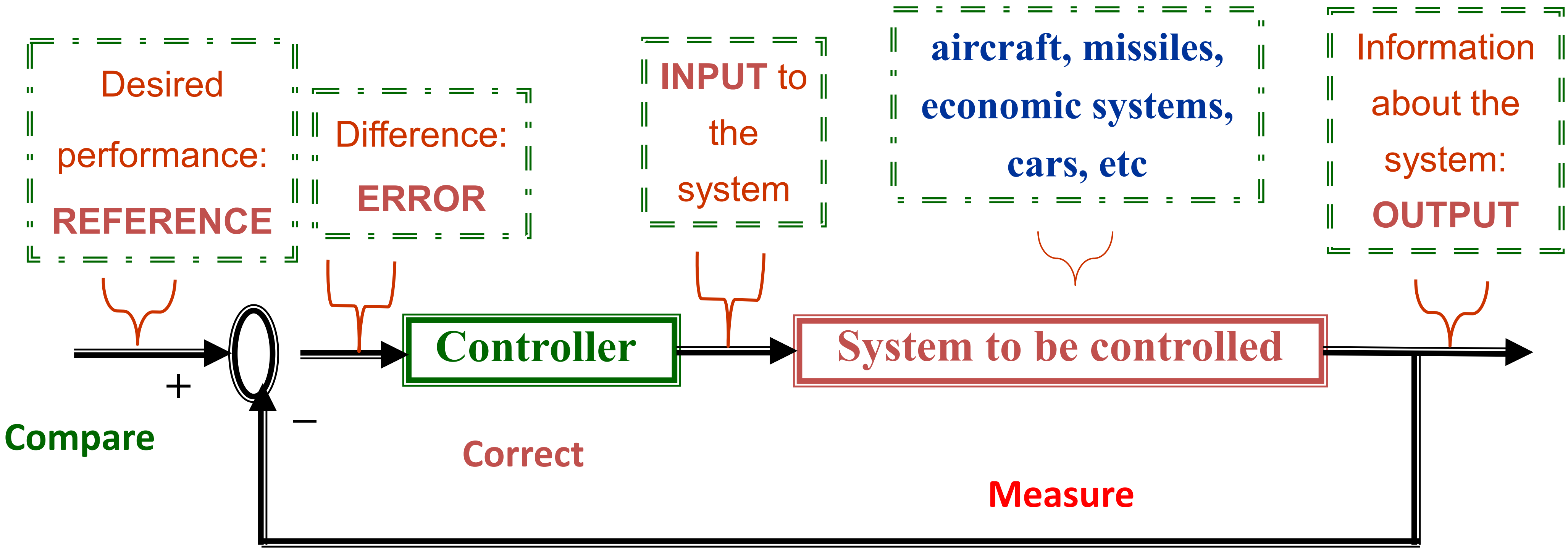
EE5101/ME5401: Linear Systems: Part II
Optimal Control

Xiang Cheng

Associate Professor

Department of Electrical & Computer Engineering
The National University of Singapore

Phone: 65166210 Office: Block E4-08-07
Email: elexc@nus.edu.sg



•**Feedback: Measure —Compare –-Correct**

Objective: To make the system OUTPUT and the desired REFERENCE as close as possible, i.e., to make the ERROR as small as possible.

Chapter 8   Quadratic Optimal Control

In Chapter Seven, we introduced the first type of controller, pole placement controller.

How to design the desired poles?

From the performance specification, build a reference model, which can meet all the requirements. The poles of the reference model are the desired poles for the closed loop.

What is the key to obtain the solution to pole placement ?

The controllable canonical form! The solution to SISO system is unique. But the solution to multiple-input system is not unique as there are multiple ways to design the desired controllable canonical form with the same poles.

How many reference models can you build to meet the time domain performance specifications?

There are many reference models to meet the requirements. Among all the models, i.e., which model should we select to design the controller? Can we find out the best one?

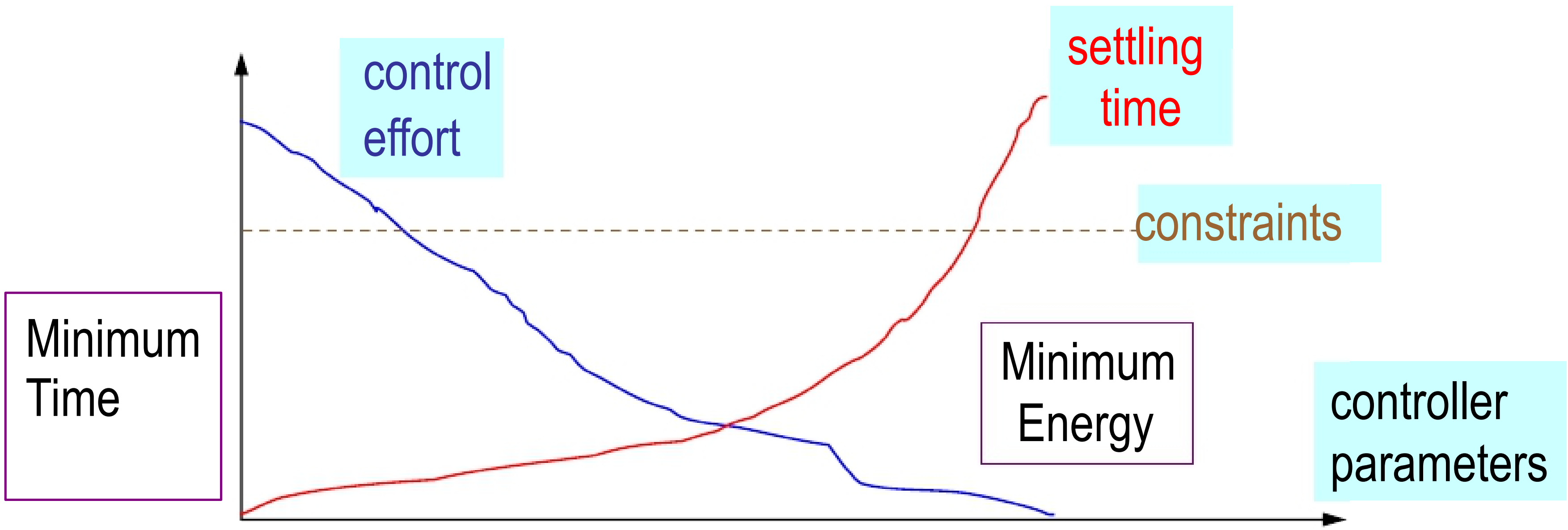
The question of course depends upon the criterion for “BEST”.

Optimization under constraints

Control specifications often have multiple objectives which conflict, e.g.

- reduce the steady state error $\Rightarrow$ increase gain $\Rightarrow$ higher energy cost

Optimization is to achieve the best trade-off among all the desired objectives.
It is to make balance between the speed (settling time) and cost.



Optimal Control Problem: Given a plant to be controlled and the specifications to be met by control system, the specifications are first cast into a specific index or cost function, and the control is sought to minimize the cost function.

Examples include

- Minimum time
- Minimum energy
- Linear Quadratic Regulator (LQR)-minimum mixed error/energy

We focus on LQR.

What is LQR?

The system is described by the standard *linear* state space model:

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t), \mathbf{x}(0) \neq 0, \qquad (1)$$
$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t),$$

The objective is to bring the state from *non-zero initial value to zero (equilibrium point)*. This is the *regulation* problem.

The problem is cast into **the following** *quadratic* **cost function,**

$$J = \frac{1}{2}\int_0^\infty \left(\mathbf{x}^T\mathbf{Q}\mathbf{x} + \mathbf{u}^T\mathbf{R}\mathbf{u}\right)dt. \qquad (2)$$

The LQR optimal control is to find the control law that minimizes (2). The optimal control law turns out to be in the form of *linear* state feedback:

$$\mathbf{u} = -\mathbf{K}\mathbf{x}, \qquad (3)$$

Is this controller in the same mathematical form as that for pole placement?

Yes. But the computation for K will be totally different!

Of course, you may ask why the optimal control has to be in this form. Is it possible for a nonlinear controller to achieve a better performance?

Later we are going to prove that the optimal solution has to be in this form!

What is Q and R?

$$J = \frac{1}{2}\int_0^{\infty}\left(\mathbf{x}^T\mathbf{Q}\mathbf{x} + \mathbf{u}^T\mathbf{R}\mathbf{u}\right)dt.$$

The matrices **Q** and **R** appear most often in diagonal form. Though there is no inherent restriction to such a form.

In addition, it is assumed (will be explained shortly) that ***Q*** should be semi-positive definite, i.e.,

$$Q = \begin{bmatrix} q_1 & & & 0 \\ & q_2 & & \\ & & \ddots & \\ 0 & & & q_n \end{bmatrix}, \quad q_i \geq 0,\ i = 1, 2, \ldots, n,$$

whereas ***R*** be positive definite, i.e.,

$$R = \begin{bmatrix} r_1 & & & 0 \\ & r_2 & & \\ & & \ddots & \\ 0 & & & r_m \end{bmatrix}, \quad r_i > 0,\ i = 1, 2, \ldots, m.$$

Meaning of Q and R?

$$x^T Q x = q_1 x_1^2 + q_2 x_2^2 + \cdots + q_n x_n^2,$$

- q_i are relative weightings among x_i.
- *If* q_1 is bigger than q_2, say $10x_1^2 + x_2^2$

there is higher penalty/price on error x_1 than x_2,

and control will try to make smaller x_1^2 than x_2^2, vice versa.

The same can be said on $u^T R u = r_1 u_1^2 + r_2 u_2^2 + \cdots + r_m u_m^2$.

- r_i are relative weightings among u_i. If one control input is more expensive to implement than another, then put a higher weight for it.

For later use, let $Q = H^T H$. It is easy to see

$$Q = \begin{bmatrix} q_1 & & & 0 \\ & q_2 & & \\ & & \ddots & \\ 0 & & & q_n \end{bmatrix}, \quad H = \begin{bmatrix} \sqrt{q_1} & & & 0 \\ & \sqrt{q_2} & & \\ & & \ddots & \\ 0 & & & \sqrt{q_n} \end{bmatrix}$$

Why is LQR useful?

A scalar illustration

For the plant:

$$\dot{x} = x + u, \quad x(0) \neq 0, \tag{4}$$

we want to regulate the state to $x = 0$, with simple state feedback control: u=-kx.

Is the open loop stable?

No. That is why we need the feedback control to make x→0.

The closed loop is

$$\dot{x} = (1-k)x$$

What is the condition on the gain k to make x→0?

Any k>1 can do it! To pick the "BEST" k, we need more requirement.

If we are mainly concerned with the speed, then the larger k, the faster the response!

However, large gain means higher amount of control efforts!

Assume that we do not wish to apply any more control effort than is necessary. For example, we might wish to use as little power as possible. Thus we should keep u as small as possible!

Therefore, to keep a balance between speed and cost,

we want to minimize the following index

$$J = \frac{1}{2}\int_0^\infty \left(qx^2 + ru^2\right)dt, \tag{5}$$

- The factor of 1/2 is introduced for numerical convenience.
- The weighting factors q and r express the relative importance of keeping x and u near zero.
- If we place more importance on x , then we select q to be large relative to r. In this case, the state x will converge to 0 faster, but the control effort will be bigger, and energy cost higher.
- If we care more about the energy cost rather than the response speed, then we should set higher r.
- Although we are interested in minimizing J, the actual value of J is usually not of interest. This also means that we can set either q or r to unity for convenience because it is their relative weight that is important.
- This step also reduces the number of weighting factors to be selected.

It turns out that feedback-control law that minimizes J is a linear state feedback law (which will be proven later),

$$\mathbf{u} = -K\mathbf{x}, \tag{6}$$

substitute (6) into (5) with $r = 1$. This results in

$$J = \frac{1}{2}\int_0^\infty \left(qx^2 + u^2\right)dt = \frac{1}{2}\left(q + K^2\right)\int_0^\infty x^2 dt. \tag{7}$$

We want to find out the "BEST" K to minimize this cost function! But how?

From calculus, we know that if we want to find out the minimal of a function f(x), what should we do?

$$\frac{df(\mathrm{x})}{dx} = 0$$

For this example, we need to compute

$$\frac{dJ}{dK} = 0$$

$$J = \frac{1}{2}\int_0^{\infty}(qx^2 + u^2)dt = \frac{1}{2}(q + K^2)\int_0^{\infty} x^2 dt. \tag{7}$$

$$\frac{dJ}{dK} = K\int_0^{\infty} x^2 dt = 0$$

$$K = 0$$

Then u=-Kx=0, is this correct?

What is wrong with above derivation?

x(t) also depends upon K! we need to find out the expression of x(t)!

This is the most difficult part for LQR. But for the scalar case, we have a simple solution.

Plug in the control law, we have the closed loop system

$$\dot{x} = x - Kx = -(K-1)x, \tag{8}$$

and its solution for constant K is

$$x(t) = x(0)e^{-(K-1)t}.$$

In this case, substituting $x(t)$ into (7) gives

$$J = \frac{1}{2}(q + K^2)x^2(0)\int_0^{\infty} e^{-2(K-1)t}dt = \frac{(q+K^2)}{4(K-1)}x^2(0).$$

Now we can compute $\frac{dJ}{dK} = 0$

We will have $K^2 - 2K - q = 0.$

Its roots are $K_1 = 1 + \sqrt{1+q}, \quad K_2 = 1 - \sqrt{1+q}.$

Which one should we pick?

To assure the system is stable, we need k>1.

One root, K_1, will satisfy this condition.

The optimal solution:

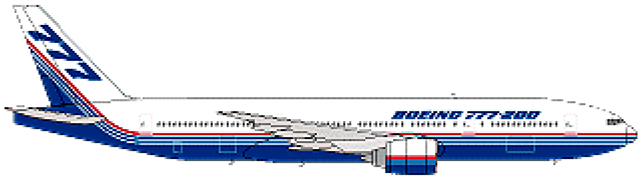
$$u = -Kx$$

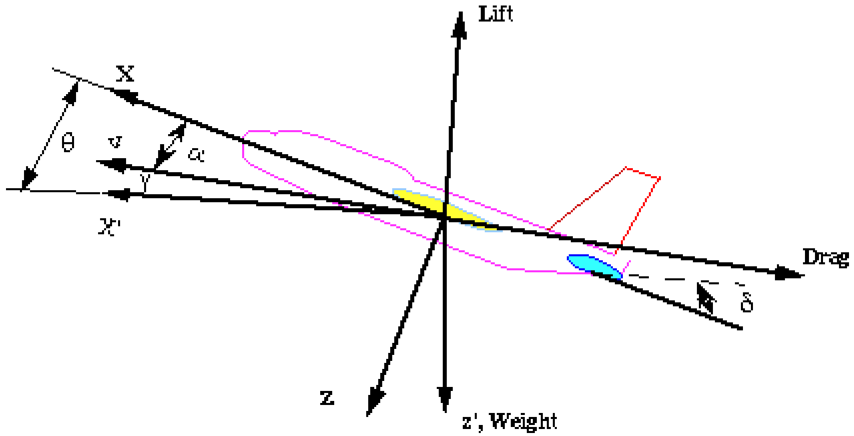
$$K = 1 + \sqrt{1+q}$$

Finally, we note that the design problem has been transformed into one of selecting a value for q.

- The larger q is, the larger will be the gain K, and the faster will $x(t)$ approach zero.
- However, the peak magnitude of u will be larger (with higher energy cost).
- The parameter q is selected to achieve a compromise between these effects. We will indicate a general procedure for doing this.

- **An Industrial Motivation: A pitch controller design for an aircraft**





α – Angle of attack

q – Pitch rate

θ – Pitch angle

δ_e – Elevator deflection angle

Model: The basic coordinate axes and forces acting on an aircraft are shown in the figure above. After some simplifications, the longitudinal equations of motion of an aircraft can be written as:

$$\dot{\alpha} = -0.313\alpha + 56.7q + 0.232\delta_e,$$

$$\dot{q} = -0.0139\alpha - 0.426q + 0.0203\delta_e,$$

$$\dot{\theta} = 56.7q.$$

Its state-space model is

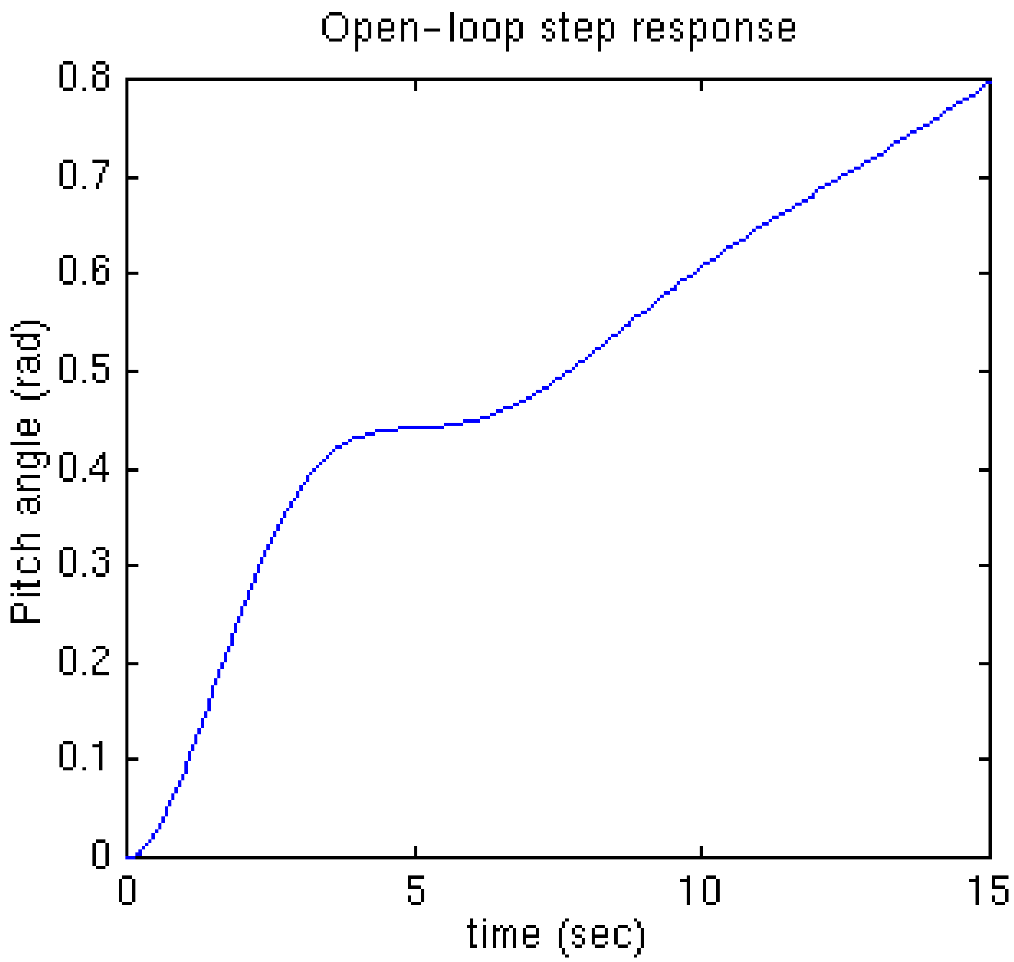
$$\begin{bmatrix} \dot{\alpha} \\ \dot{q} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} -0.313 & 56.7 & 0 \\ -0.0139 & -0.426 & 0 \\ 0 & 56.7 & 0 \end{bmatrix} \begin{bmatrix} \alpha \\ q \\ \theta \end{bmatrix} + \begin{bmatrix} 0.232 \\ 0.0203 \\ 0 \end{bmatrix} \delta_e,$$

$$y = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \alpha \\ q \\ \theta \end{bmatrix}.$$

Control design requirements

- Rise time: Less than 2 seconds;
- Overshoot: Less than 10%;
- 2% settling time: Less than 10 seconds.

Its open-loop response to a step input $\delta_e = 0.2$ rad (11°) is shown in the following figure.



Is the open loop stable?

From the plot, we see that the open-loop response does not satisfy the design criteria at all. In fact the open-loop is unstable.

Therefore, feedback control is needed to change the property of the system!
We will come back to this design example after we derive the general solution.

Is LQR problem solvable?

We need some assumptions before we can develop a procedure for computing **K**.

Consider a state space plant:

$$\begin{aligned} \dot{x}_1 &= x_1, \\ \dot{x}_2 &= x_2 + u. \end{aligned} \tag{10}$$

Let the performance index be

$$J = \frac{1}{2}\int_0^\infty \left(x_1^2 + u^2\right) dt. \tag{11}$$

Does the solution exist? And why?

It is easy to see from (10) that no minimum for J will exist because the state variable x_1 is uncontrollable and of an unstable mode. Thus the response of x_1 will be $x_1(t) = x_1(0)e^t$ regardless of what u and x_2 do. Therefore $x_1(t) \to \infty$ and no matter what $u(t)$ does.

$$J = \frac{1}{2}\int_0^\infty [x_1^2(0)e^{2t} + u^2]dt \to \infty.$$

Any control system of practical usefulness must be stable.
Any unstable plant must be stabilized by feedback control.

We can conclude that no minimum exists for the index J in this problem because of the following:

[1] The state variable x_1 is uncontrollable.

[2] The uncontrollable state variable x_1 is also unstable.

We make the following assumption:

Assumption 1. The system, (A, B), is controllable.

The choice of the performance index is also important. Consider another simple example:

$$\dot{x} = x + u,$$

$$J = \frac{1}{2}\int_0^\infty u^2 dt.$$

What is the input u(t) to minimize this index?

$u(t) = 0!$

What about x(t) then?

X(t) → *infinity* because the system is unstable!

To make sure that system is stable, we need the performance index to include some state variables!

Assumption 2. The pair (**A**, **H**) is completely observable, where **H** is any matrix such that $\mathbf{H}^T\mathbf{H} = \mathbf{Q}$.

Assumption 2 says that *all* the state variables will be "observed" by the performance index. This is necessary to assure the stability. We need this condition to prove the stability later.

The algorithm to be developed later is capable of stabilizing any mode that is controllable (Assumption 1) and that appears in the performance index (Assumption 2).

But will the control signal remain finite and hence implementable?

Assumption 3. The weighting matrices $\boldsymbol{Q}$ *and* $\boldsymbol{R}$ *are symmetric. The matrix* $\boldsymbol{R}$ *is positive definite, while the matrix* $\boldsymbol{Q}$ *is semi-positive definite.*

Assumption 3 is required for a minimum of J to exist with finite control.

For example, if one takes

$$J = \frac{1}{2}\int_0^\infty \left(x^2 - u^2\right)dt, R < 0,$$

$$J = \frac{1}{2}\int_0^\infty \left(-x^2 + u^2\right)dt, Q < 0.$$

J may be made as negative as one wishes (without a minimum) when u or $x \to \infty$. These situations are avoided by requiring Q and R to be positive or semi-positive definite, which implies $J \geq 0$.

But why do we require R>0?

Consider an example:

$$J = \frac{1}{2}\int_0^\infty x^2 dt,$$

$$\dot{x} = -x + u,$$

has a positive Q=1 and R=0, which is semi-positive.

What would happen in this case?

It is unacceptable because it results in $K -> \infty$

which implies that the input is infinity. *u(t)* must be finite due to physical limitations.

In summary, Assumptions 1 and 2 guarantee that the optimal control system is stable, while Assumption 3 ensures that the optimal control signal *u(t)* is bounded for physical realization.

§8.2 General Solution

It is time to find out the general solution.

The LQR problem consists of the system:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u},$$

$$\mathbf{y} = \mathbf{C}\mathbf{x},$$

and the index:

$$J = \frac{1}{2}\int_0^\infty \left(\mathbf{x}^T\mathbf{Q}\mathbf{x} + \mathbf{u}^T\mathbf{R}\mathbf{u}\right)dt.$$

We need to find out the state feedback control law:

$$\mathbf{u} = -K\mathbf{x},$$

such that the cost function J is minimized.

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u},$$

$$\mathbf{y} = \mathbf{C}\mathbf{x},$$

$$J = \frac{1}{2}\int_0^\infty \left(\mathbf{x}^T\mathbf{Q}\mathbf{x} + \mathbf{u}^T\mathbf{R}\mathbf{u}\right)dt.$$

Let's first follow the way we obtained the solution for the scalar example.
Plug in the control law,

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} = (A - BK)x$$

We can easily obtain the solution for x(t), $x(t) = e^{(A-BK)t}x(0)$

Then let's try to evaluate the cost

$$J = \frac{1}{2}\int_0^\infty \left(\mathbf{x}^T\mathbf{Q}\mathbf{x} + \mathbf{u}^T\mathbf{R}\mathbf{u}\right)dt.$$

$$= \frac{1}{2}\int_0^\infty \left(\mathbf{x}^T\mathbf{Q}\mathbf{x} + \mathbf{x}^T K^T \mathbf{R}K\mathbf{x}\right)dt$$

$$= \frac{1}{2}\int_0^\infty \mathbf{x}^T\left(\mathbf{Q} + K^T\mathbf{R}K\right)\mathbf{x}dt$$

$$= \frac{1}{2}\int_0^\infty \mathbf{x}(0)^T e^{(A-BK)^T t}\left(\mathbf{Q} + K^T\mathbf{R}K\right)e^{(A-BK)t}\mathbf{x}(0)dt$$

But we do not know how to compute the integral ! So this is the dead end!

So the solution to LQR is not as simple as the pole placement problem!
This time the controllable canonical form cannot play the magic trick for us.

Let's try to get some clue by considering the scalar case again:

$$\dot{x} = ax + bu$$

and the index:

$$J = \frac{1}{2}\int_0^\infty (qx^2 + ru^2)dt.$$

Let's try the Lyapunov method this time. Let's define a Lyapunov function

$$V(x) = px^2$$

where p>0, will be designed later.

Let's compute the change of the Lyapunov function with time:

$$\begin{aligned}\frac{dV(x)}{dt} &= 2px\dot{x}\\ &= 2px(ax+bu)\\ &= 2pax^2 + 2pbxu\end{aligned}$$

What is the condition on the change rate of the Lyapunov function to assure the stability?

$$\frac{dV(x)}{dt} \le 0$$

We need to find out p and u to make this happen.

What is the next step?

We need to find out a way to relate the Lyapunov function V with the cost function J!

If we can relate dV(x)/dt with $qx^2 + ru^2$, then we can connect the cost function with the Lyapunov function .

$$\frac{dV(x)}{dt} = 2pax^2 + 2pbxu \quad \Longleftrightarrow \quad qx^2 + ru^2$$

How to connect these two obviously different objects?

Let's force them together (a non-trivial idea!)

$$\frac{dV(x)}{dt} = 2pax^2 + 2pbxu$$
$$= 2pax^2 + 2pbxu + ru^2 + qx^2 - qx^2 - ru^2$$

Now we can analyze the expression of $\frac{dV(x)}{dt}$

The object is to design both p and u such that $\frac{dV(x)}{dt} \leq 0$

We notice that $\frac{dV(x)}{dt}$ contains negative terms $-qx^2 - ru^2$

We need to deal with the other terms

$$2pax^2 + 2pbxu + ru^2 + qx^2$$

$$\frac{dV(x)}{dt} = 2pax^2 + 2pbxu$$
$$= 2pax^2 + 2pbxu + ru^2 + qx^2 - qx^2 - ru^2$$

Can we design p and u such that

$$2pax^2 + 2pbxu + ru^2 + qx^2 = 0?$$

If this can be done, then we can make $\frac{dV(x)}{dt} \leq 0$

In order to deal with the cross term $2pbxu$ whose sign is not easy to determine, the commonly used trick is to complete the square.

$$2pax^2 + 2pbxu + ru^2 + qx^2$$
$$= \frac{1}{r}[\, 2pbrxu + r^2u^2] + 2pax^2 + qx^2$$
$$= \frac{1}{r}[\, (pbx)^2 + 2pbrxu + r^2u^2] - \frac{(pbx)^2}{r} + 2pax^2 + qx^2$$
$$= \frac{(pbx + ru)^2}{r} - \frac{p^2b^2}{r}x^2 + 2pax^2 + qx^2$$
$$= (2pa + q - \frac{p^2b^2}{r})x^2 + \frac{(pbx + ru)^2}{r}$$

$$\frac{dV(x)}{dt} = (2pa + q - \frac{p^2 b^2}{r})x^2 + \frac{(pbx + ru)^2}{r} - qx^2 - ru^2$$

We have not put any condition on p except that p>0.

In order to make $\frac{dV(x)}{dt} \le 0$

Let's first try to make the first term on the RHS to be zero. Therefore, let p satisfy

$$2pa + q - \frac{p^2 b^2}{r} = 0$$

We can easily verify that one root is positive and the other root is negative, and we just pick the positive root for p.

Then we have

$$\frac{dV(x)}{dt} = \frac{(pbx + ru)^2}{r} - qx^2 - ru^2$$

$$\frac{dV(x)}{dt} = \frac{(pbx+ru)^2}{r} - qx^2 - ru^2$$

Now we can integrate both sides over t from 0 to infinity, and have

$$\int_0^\infty \frac{dV(x)}{dt}dt = \int_0^\infty \frac{(pbx+ru)^2}{r}dt - \int_0^\infty (qx^2 + ru^2)dt$$

$$V(x,t)\Big|_{t=0}^{t=\infty} = \int_0^\infty \frac{(pbx+ru)^2}{r}dt - 2J$$

$$px^2(\infty) - px^2(0) = \int_0^\infty \frac{(pbx+ru)^2}{r}dt - 2J$$

Assume the system is stable, we have $x(\infty) = 0.$

Finally we have $J = \frac{1}{2}px^2(0) + \frac{1}{2}\int_0^\infty \frac{(pbx+ru)^2}{r}dt \geq \frac{1}{2}px^2(0)$

Finally we can design the controller.

How to choose u to make J reach the minimum?

$$pbx + ru = 0 \to u = \frac{-pb}{r}x = -kx$$

$$k = \frac{pb}{r}$$

So Lyapunov method will be the way to go!

Break

State-of-the-art control systems

Future Robots (2: 16m to 30m)

Now we are ready to extend this idea to deal with the general case:
The LQR problem consists of the system:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u},$$

$$\mathbf{y} = \mathbf{C}\mathbf{x},$$

and the index:

$$J = \frac{1}{2}\int_0^\infty \left(\mathbf{x}^T\mathbf{Q}\mathbf{x} + \mathbf{u}^T\mathbf{R}\mathbf{u}\right)dt.$$

We will follow the Lyapunov method again. In the scalar case, we have

$$V(x) = px^2$$

<u>What would it look like in the general case now?</u>

$$V(x) = x^T P x$$

where P is a positive definite matrix, P>0.

$$V(x) = x^T P x$$

P>0 is not enough, we will find out an equation for P such that

$$\frac{dV(x)}{dt} \leq 0$$

In the following, we need to compute

$$\frac{dV(x)}{dt}$$

$$\frac{dV(x)}{dt} = \frac{d}{dt}(x^T P x) = \dot{x}^T P x + x^T P \dot{x}$$

Using $\dot{x} = Ax + Bu$

$$\frac{dV(x)}{dt} = (x^T A^T + u^T B^T) P x + x^T P (Ax + Bu)$$

$$= x^T (A^T P + PA) x + u^T B^T P x + x^T P B u$$

$$= x^T (A^T P + PA) x + 2x^T P B u$$

Just like the scalar case, we force it to marry with the cost function such that

$$\frac{dV(x)}{dt} = x^T (A^T P + PA) x + 2x^T P B u + x^T Q x + u^T R u - x^T Q x - u^T R u$$
$$= x^T (A^T P + PA + Q) x + 2x^T P B u + u^T R u - x^T Q x - u^T R u$$

How did we deal with the cross term $2pbxu$ in the scalar case?

Complete the square!

$$\frac{dV(x)}{dt} = x^T(A^TP + PA + Q)x + 2x^TPBu + u^TRu - x^TQx - u^TRu$$

Complete the square for scalar case:

$$axu + bu^2 = c(\alpha x + \beta u)^2 + \gamma\,\mathrm{x}^2$$

This time we need to complete the square for the matrix quadratic form!

We want to put the cross terms in the following form:

$$2x^TPBu + u^TRu = (\Phi x + \Gamma u)^T\Omega(\Phi x + \Gamma u) + x^TDx$$

How to find out Ω, Φ , Γ $and\ D$?

$$(\Phi x + \Gamma u)^T\Omega(\Phi x + \Gamma u) + x^TDx = (x^T\ \Phi^T + u^T\ \Gamma^T)\Omega(\Phi x + \Gamma u) + x^TDx$$

$$= x^T\ \Phi^T\ \Omega\Phi x + x^T\ \Phi^T\ \Omega\,\Gamma u + u^T\ \Gamma^T\Omega\Phi x + u^T\ \Gamma^T\Omega\,\Gamma u + x^TDx$$

$$= x^T\ \Phi^T\ \Omega\Phi x + 2\,x^T\ \Phi^T\ \Omega\,\Gamma u + u^T\ \Gamma^T\Omega\,\Gamma u + x^TDx$$

Clearly we can see , $\Gamma^T\Omega\,\Gamma = R$

$\Phi^T\ \Omega\,\Gamma = PB$

One simple way is to set $\Omega\,\Gamma$=I , then

$\Gamma^T = R$ $\Rightarrow$ $\Gamma = R^T = R$

$\Phi^T = PB$ $\Rightarrow$ $\Phi = B^TP$

$\Omega\,\Gamma$=I $\Rightarrow$ $\Omega = R^{-1}$

How to complete the square for the quadratic form?

$$2x^T PBu + u^T Ru = (\Phi x + \Gamma u)^T \Omega (\Phi x + \Gamma u) + x^T Dx$$

$$(\Phi x + \Gamma u)^T \Omega (\Phi x + \Gamma u) + x^T Dx = x^T \Phi^T \Omega \Phi x + 2x^T \Phi^T \Omega \Gamma u + u^T \Gamma^T \Omega \Gamma u + x^T Dx$$

$$= 2 x^T \Phi^T \Omega \Gamma u + u^T \Gamma^T \Omega \Gamma u + x^T (\Phi^T \Omega \Phi + D) x$$

We have already determined

$$\Omega = R^{-1}$$

$$\Gamma = R$$

$$\Phi = B^T P$$

What about D?

$$\Phi^T \Omega \Phi + D = 0 \quad \Rightarrow \quad D = -\Phi^T \Omega \Phi = -PBR^{-1} B^T P$$

So finally, we have completed the square for the matrix quadratic form as follows:

$$2x^T PBu + u^T Ru = (\Phi x + \Gamma u)^T \Omega (\Phi x + \Gamma u) + x^T Dx$$

$$= (B^T Px + Ru)^T R^{-1} (B^T Px + Ru) - x^T PBR^{-1} B^T Px$$

Now let’s put it back to the equation for $\dfrac{dV(x)}{dt}$

$$\frac{dV(x)}{dt} = x^T(A^TP + PA + Q)x + 2x^TPBu + u^TRu - x^TQx - u^TRu$$

Completing the square for the quadratic form!

$$2x^TPBu + u^TRu = (B^TPx + Ru)^TR^{-1}(B^TPx + Ru) - x^TPBR^{-1}B^TPx$$

So overall we have

$$\frac{dV(\mathrm{x})}{dt} = x^T(A^TP + PA + Q - PBR^{-1}B^TP)x + (B^TPx + Ru)^TR^{-1}(B^TPx + Ru) - x^TQx - u^TRu$$

$$\frac{dV(\mathrm{x})}{dt} = x^T(A^TP + PA + Q - PBR^{-1}B^TP)x + (B^TPx + Ru)^TR^{-1}(B^TPx + Ru) - x^TQx - u^TRu$$

In order to make the first term on the RHS to be zero, we choose P such that

$$\mathbf{A}^T\mathbf{P} + \mathbf{P}\mathbf{A} + \mathbf{Q} = \mathbf{P}\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^T\mathbf{P}.$$

This is the so called *Algebraic Riccati Equation (ARE)*,

which corresponds to the equation in scalar case:

$$2pa + q - \frac{p^2b^2}{r} = 0$$

ARE is the key to the solution of optimal control problem!

After getting rid of the first term on the RHS, we have

$$\frac{dV(\mathrm{x})}{dt} = (B^TPx + Ru)^TR^{-1}(B^TPx + Ru) - x^TQx - u^TRu$$

$$\frac{dV(\mathrm{x})}{dt} = (B^T Px + Ru)^T R^{-1}(B^T Px + Ru) - x^T Qx - u^T Ru$$

Then, we integrate both sides. The LHS becomes

$$\int_0^\infty \frac{d}{dt}\left(\mathbf{x}^T\mathbf{P}\mathbf{x}\right)dt = \mathbf{x}^T(\infty)\mathbf{P}\mathbf{x}(\infty) - \mathbf{x}^T(0)\mathbf{P}\mathbf{x}(0)$$
$$= -\mathbf{x}^T(0)\mathbf{P}\mathbf{x}(0),$$

because $\mathbf{x}(\infty) \to \mathbf{0}$ from the assumed stability.

The RHS is given by

$$\int_0^\infty \left(\mathbf{x}^T\mathbf{P}\mathbf{B} + \mathbf{u}^T\mathbf{R}\right)\mathbf{R}^{-1}\left(\mathbf{B}^T\mathbf{P}\mathbf{x} + \mathbf{R}\mathbf{u}\right)dt$$
$$-\int_0^\infty \left(\mathbf{x}^T\mathbf{Q}\mathbf{x} + \mathbf{u}^T\mathbf{R}\mathbf{u}\right)dt$$

$$J = \frac{1}{2}\mathbf{x}^T(0)\mathbf{P}\mathbf{x}(0) + \frac{1}{2}\int_0^{\infty}(\mathbf{x}^T\mathbf{P}\mathbf{B} + \mathbf{u}^T\mathbf{R})\mathbf{R}^{-1}(\mathbf{B}^T\mathbf{P}\mathbf{x} + \mathbf{R}\mathbf{u})dt\,. \tag{13}$$

But $\mathbf{x}(0)$ and $\mathbf{P}$ are independent of the control input $\mathbf{u}$ to be chosen. Thus J can be minimized by considering only the integrand in (13). This is nonnegative and therefore has a minimum of zero.

Is it possible to make the integrand zero?

This minimum occurs at

$$\mathbf{B}^T\mathbf{P}\mathbf{x} + \mathbf{R}\mathbf{u} = \mathbf{0},$$

and this implies that

$$\mathbf{u} = -\mathbf{R}^{-1}\mathbf{B}^T\mathbf{P}\mathbf{x} = -\mathbf{K}\mathbf{x}.$$

Is the solution unique?

The solution is unique! It means that any other type of controller will have higher cost!

This method is beautiful as the state feedback control law is a natural result from this LQR rather than an assumption!

Theorem 1. *Consider the system:*

$$\dot{x} = Ax + Bu, \quad x(0) = x_0 \neq 0, \tag{14}$$

with the cost index:

$$J = \frac{1}{2}\int_0^\infty \left(x^T Qx + u^T Ru\right)dt, \tag{15}$$

where R is a positive definite and Q is semi-positive definite with $Q = H^T H$. *If (A,B) is controllable and (A,H) observable, the optimal control minimizing* (15) *is given by*

$$u(t) = -R^{-1}B^T Px(t), \tag{16}$$

where P is the symmetric positive definite solution of the algebraic Matrix Riccati equation:

$$A^T P + PA + Q - PBR^{-1}B^T P = 0. \tag{17}$$

But did we ever prove that the *positive definite solution* always exists for ARE? If it exists, is it unique?

Yes. It can be proven.
However, this is not trivial at all, which is beyond the scope of this course!

Example 1. Consider the system:

$$\dot{x} = x + u,$$

with the cost function:

$$J = \frac{1}{2}\int_0^\infty (x^2 + u^2)dt.$$

The ARE is

$$A^T P + PA + Q - PBR^{-1}B^T P = 0.$$

$$2P - P^2 + 1 = 0,$$

$$P = 1 \pm \sqrt{2}.$$

Which one to choose?

We choose the positive definite solution, namely,

$$P = 1 + \sqrt{2}.$$

Since $R = 1$ and $B = 1$, the optimal control is

$$u = -R^{-1}BPx = -(1 + \sqrt{2})x.$$

Example 2. Find the optimal control with the cost function:

$$J = \frac{1}{2}\int_0^{\infty}\left[x^T\begin{pmatrix} q_1 & 0 \\ 0 & q_2 \end{pmatrix}x + ru^2\right]dt, \tag{18}$$

for the system:

$$\dot{x} = \begin{pmatrix} 0 & 1 \\ -\alpha_0 & -\alpha_1 \end{pmatrix}x + \begin{pmatrix} 0 \\ 1 \end{pmatrix}u, \quad x(0) = x_0. \tag{19}$$

Let

$$P = \begin{pmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{pmatrix}.$$

The ARE $\qquad A^T P + PA + Q - PBR^{-1}B^T P = 0.$

$$\begin{pmatrix} 0 & -\alpha_0 \\ 1 & -\alpha_1 \end{pmatrix}\begin{pmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{pmatrix} + \begin{pmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -\alpha_0 & -\alpha_1 \end{pmatrix} + \begin{pmatrix} q_1 & 0 \\ 0 & q_2 \end{pmatrix}$$

$$- r^{-1}\begin{pmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \end{pmatrix}\begin{pmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{pmatrix} = 0. \tag{20}$$

$$-2\alpha_0 p_{12} + q_1 - p_{12}^2 r^{-1} = 0,$$

$$p_{11} - \alpha_1 p_{12} - \alpha_0 p_{22} - p_{12} p_{22} r^{-1} = 0,$$

$$2p_{12} - 2\alpha_1 p_{22} + q_2 - p_{22}^2 r^{-1} = 0,$$

There are multiple solutions

$$p_{12} = -r\alpha_0 \pm r\sqrt{\alpha_0^2 + q_1/r},$$

$$p_{22} = -r\alpha_1 \pm r\left[\alpha_1^2 + q_2/r + 2p_{12}/r\right]^{\frac{1}{2}},$$

$$\begin{aligned} p_{11} &= \alpha_1 p_{12} + \alpha_0 p_{22} + p_{12} p_{22} r^{-1} \\ &= (\alpha_1 r + p_{22})(\alpha_0 r + p_{12}) r^{-1} - \alpha_1 \alpha_0 r. \end{aligned}$$

Which one to choose?

Only one of them is positive definite, P>0!

The positive definite P is

$$p_{11} = r\left\{\alpha_1^2 + q_2/r + 2\left[-\alpha_0 + \sqrt{\alpha_0^2 + q_1/r}\right]\right\}^{\frac{1}{2}}\sqrt{\alpha_0^2 + q_1/r} - \alpha_0\alpha_1 r,$$

$$p_{12} = -r\alpha_0 + r\sqrt{\alpha_0^2 + q_1/r},$$

$$p_{22} = -r\alpha_1 + r\left[\alpha_1^2 + q_2/r + 2\left(-\alpha_0 + \sqrt{\alpha_0^2 + q_1/r}\right)\right]^{\frac{1}{2}}.$$

The optimal control is given by

$$u = -r^{-1}\begin{pmatrix}0 & 1\end{pmatrix}Px$$

$$= -\left[-\alpha_0 + \sqrt{\alpha_0^2 + q_1/r},\quad -\alpha_1 + \left[\alpha_1^2 + q_2/r + 2\left(-\alpha_0 + \sqrt{\alpha_0^2 + q_1/r}\right)\right]^{1/2}\right]x. \quad (21)$$

What is the most time consuming part for obtaining the solution?

It is not trivial to solve ARE!

How to select Q and R?

In pole placement, we design the reference model to meet the performance specifications, which provide the desired poles for the algorithm .

In LQR, do we need to provide desired poles directly?

No. We do not know where the desired poles are. Instead, we design the weighting matrices Q and R. Q and R have to be designed carefully to strike a balance between response speed and energy efficiency. General speaking, the bigger Q implies faster speed, while bigger R means less energy cost.

The selection of the weighting matrices Q and R is usually made on the basis of experience together with simulations of the results for different trial values. The following guidelines have emerged.

- Usually **Q** and **R** are selected to be diagonal so that specific state and control variables are penalized individually with higher weightings if their response is undesirable. To this end, the state variables and manipulated variables should represent sets of variables that are easily identifiable physically, rather than a set of transformed variables such as the model variables. This enables the designer to visualize the effects of the trial values for **Q** and **R**.

- The larger the elements of **Q** are, the larger are the elements of the gain matrix **K**, and the faster the state variables approach zero. On the other hand, the larger the elements of **R**, the smaller the elements of **K** which leads to slower response but smaller energy cost.

§8.3 Stability of the Optimal Control System

For the system

$$\dot{x} = Ax + Bu,$$

with the optimal control:

$$u = -Kx, \tag{22}$$

and

$$K = R^{-1}B^T P, \tag{23}$$

the closed-loop system is

$$\dot{x} = (A - BK)x, \qquad x(0) = x_0. \tag{24}$$

P satisfies the ARE equation:

$$A^T P + PA + Q - PBR^{-1}B^T P = 0.$$

In obtaining the optimal control law, we assumed that the system is stable such that $\mathbf{x}(\infty) \rightarrow \mathbf{0}.$ Now let's try to prove this.

Theorem 2. *If the triple* (A,B,H) *is controllable and observable, then the closed-loop system using the optimal control* (16) *which minimizes the quadratic criterion function* (15) *is asymptotically stable.*

Proof: Define the Lyapunov function:

$$V(x,t) = x^T P x > 0, \quad \text{if } x \neq 0,$$

Previously, we have already shown that

$$\frac{dV(\mathrm{x})}{dt} = x^T(A^T P + PA + Q - PBR^{-1}B^T P)x + (B^T Px + Ru)^T R^{-1}(B^T Px + Ru) - x^T Qx - u^T Ru$$

We choose P to satisfy the ARE

$$A^T P + PA + Q - PBR^{-1}B^T P = 0$$

We design the optimal controller such that

$$B^T Px + Ru = 0 \quad \Longrightarrow \quad u = -Kx, K = R^{-1}B^T P,$$

Overall, we have

$$\frac{dV(\mathrm{x})}{dt} = -x^T Qx - u^T Ru \leq 0$$

$$\frac{dV(\mathrm{x})}{dt} = -x^T Q x - u^T R u \le 0$$

This inequality only assures that $\dot{V} \le 0.$

Is this enough to guarantee asymptotical stability, i.e. $x(t) \to 0\,?$

No! we need to show that $\dot{V} < 0$ for all nonzero x. How to do that?

We need to take a closer look at $\dot{V}$

$$u = -Kx$$

$$\frac{dV(\mathrm{x})}{dt} = -(x^T Q x + (Kx)^T R (Kx))$$

When we discussed the meaning of Q and R, we mentioned that Q can be expressed as

$$Q = H^T H$$

$$\frac{dV(\mathrm{x})}{dt} = -\left(x^T H^T H x + (Kx)^T R (Kx)\right) = -\left((Hx)^T (Hx) + (Kx)^T R (Kx)\right)$$

$$\frac{dV(\mathrm{x})}{dt} = -\left((Hx)^T(Hx) + (Kx)^T R(Kx)\right)$$

We need to show that $\dot{V} < 0$ for all nonzero x.

Assume that $\dot{V}(x,t) = 0$ for some $x(t) \neq 0$,

then we have $Kx = 0$ and $Hx = 0$.

$$\dot{x} = Ax - BKx = Ax \quad \text{and} \quad x = e^{At}x_0.$$

Then $$Hx = He^{At}x_0 = 0$$

But we know that (A,H) is assumed to be observable.

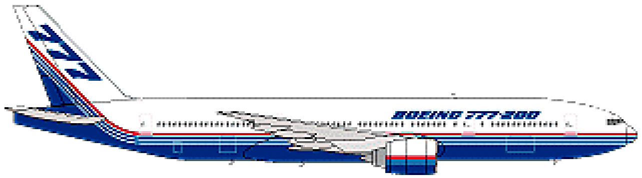
If Hx=0, then x must be 0 (due to observability)!

Therefore we have a contradiction!

Then, the closed loop must be asymptotically stable. The proof is complete.

The technique we used to prove -→ proof by contradiction → One of the most important tricks in math!

An Industrial Application: A pitch controller design for an aircraft revisited



Model:

$$\begin{bmatrix} \dot{\alpha} \\ \dot{q} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} -0.313 & 56.7 & 0 \\ -0.0139 & -0.426 & 0 \\ 0 & 56.7 & 0 \end{bmatrix} \begin{bmatrix} \alpha \\ q \\ \theta \end{bmatrix} + \begin{bmatrix} 0.232 \\ 0.0203 \\ 0 \end{bmatrix} \delta_e,$$

$$y = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \alpha \\ q \\ \theta \end{bmatrix}.$$

Design requirements

- Rise time: Less than 2 seconds;
- Overshoot: Less than 10%;
- 2% settling time: Less than 10 seconds.

Solution:

Let $R = 1$, $Q = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 50 \end{bmatrix}$. The optimal control is

$$u(t) = -R^{-1}B^T Px(t),$$

where

$$K = R^{-1}B^T P = [-0.6435 \quad 169.6950 \quad 7.0711].$$

The step response of the closed-loop system is shown in Figure 3. The rise time, overshoot, and settling time look satisfactory.

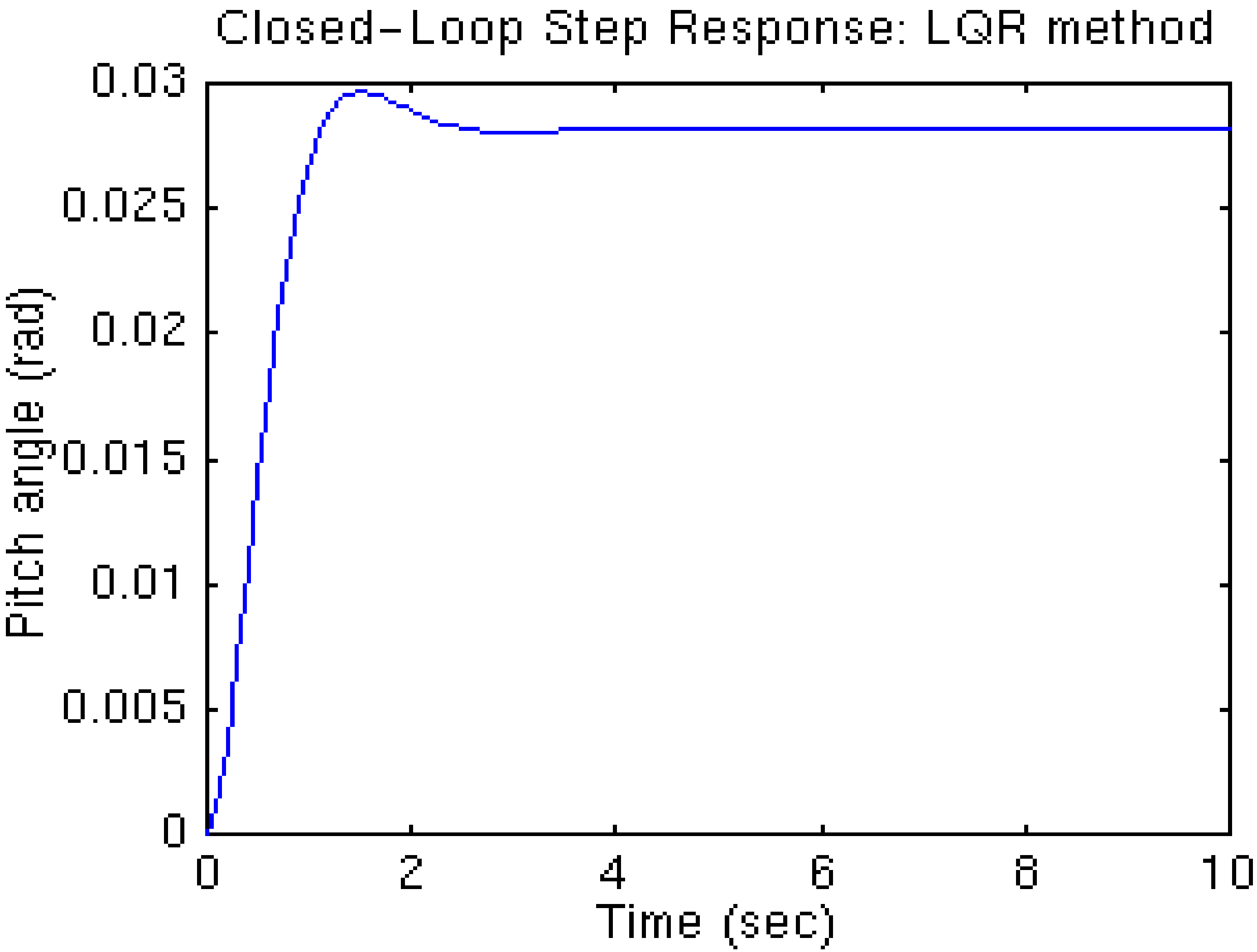


Figure 3 Optimal control for aircraft.

§8.4. How to Solve ARE?

The most time-consuming part of LQR is solving the ARE as it involves a set of nonlinear equations, as shown in example 2.

To find the positive definite solution of the Riccati equation:

$$PA + A^T P - PBR^{-1}B^T P + Q = 0,$$

people have developed the following systematic way of eigenvalue-eigenvector based algorithm.

Step 1: Form the $2n \times 2n$ matrix:

$$\Gamma = \begin{pmatrix} A & -BR^{-1}B^T \\ -Q & -A^T \end{pmatrix},$$

and find its n stable eigenvalues.

Step 2**:** Let the eigenvector of Γ corresponding to **stable** λ_i , $i = 1,\ldots, n,$ be

$$\begin{pmatrix} \nu_i \\ \mu_i \end{pmatrix}, i = 1, 2, \cdots, n.$$

where ν_i, μ_i are n-dimensional vectors.

Then it can be shown (detailed in the lecture notes) that

$$P\nu_i = \mu_i,\ i = 1, 2, \cdots, n,$$

Therefore we have

$$P\begin{bmatrix} \nu_1 & \nu_2 & \cdots & \nu_n \end{bmatrix} = \begin{bmatrix} \mu_1 & \mu_2 & \cdots & \mu_n \end{bmatrix}$$

Step 3: P is given by

$$P = [\mu_1,\ldots,\mu_n][\nu_1,\ldots,\nu_n]^{-1}.$$

Example 5. Consider

$$\dot{x} = -3x + u,$$

$$J = \frac{1}{2}\int_0^\infty \left(x^2 + u^2\right)dt.$$

Clearly, $A = -3$, $B = 1$, $Q = R = 1$.

Solution (i). The Riccati equation is

$$-6P - P^2 + 1 = 0,$$

$$P = -3 \pm \sqrt{10}.$$

The positive definite P of $\sqrt{10} - 3$ = 0.1622 gives

$$u = -Kx = -0.162x$$

Solution (ii) (Eigenvectors).

Form
$$\Gamma = \begin{bmatrix} A & -BR^{-1}B^T \\ -Q & -A^T \end{bmatrix} = \begin{bmatrix} -3 & -1 \\ -1 & 3 \end{bmatrix}$$

Its eigenvalues are $\pm\sqrt{10}$ and the eigenvector corresponding to $-\sqrt{10}$ is

$$\begin{bmatrix} \nu \\ \mu \end{bmatrix} = \begin{bmatrix} 1 \\ \sqrt{10} - 3 \end{bmatrix}$$

giving $P = \mu\nu^{-1} = 0.1622$, the same result.

Linear Quadratic Regulator (LQR)

- Reasons
- Solvability
- Methods

In particular, we showed that

- The conditions for solving LQR
- Two methods for solving ARE
- The stability proof of optimal control system.

What is the key technique to derive the ARE?

Use Lyapunov method and complete the square for matrix quadratic form!

Q & A…

THANK YOU !