EE5103 Lecture Two
Computer Control and Discrete-Time Systems

Xiang Cheng

Associate Professor

Department of Electrical & Computer Engineering
The National University of Singapore

Phone: 65166210 Office: Block E4-08-07
Email: elexc@nus.edu.sg

- We discussed lots of basic concepts last time.
- Systems
- Input and output
- Control
- Feedback
- Superposition
- Laplace Transform
- Transfer function
- Poles and zeros
- Stability
- State

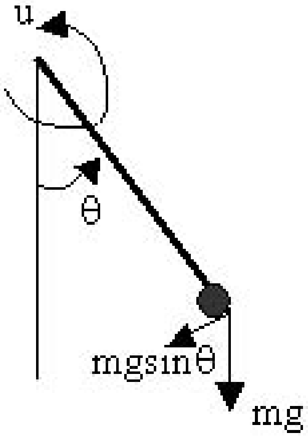
- If I want you to choose ONLY ONE that you think is the most important concept for control theory, what would you say?

Feedback

Feedback is the key to change the behavior of the system!

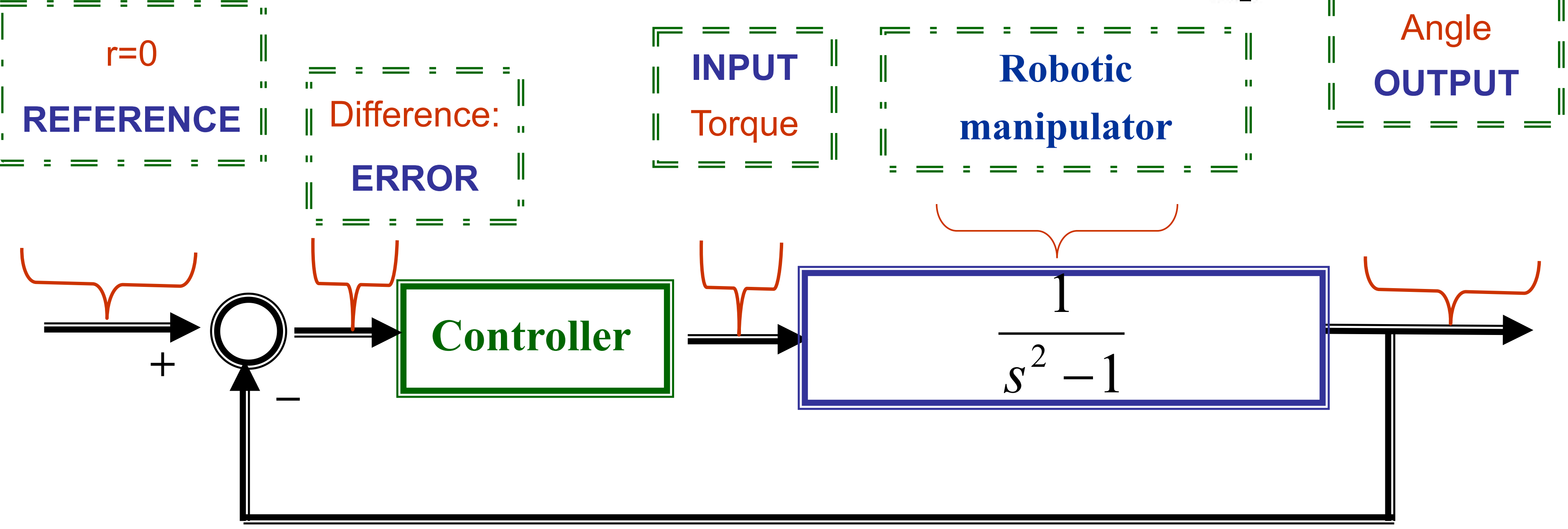
Control of the robotic manipulator

$$\ddot{\theta}(t)+\sin\theta(t)=u(t)$$



For small $y=\theta-\pi$

$$\ddot{y}(t)-y(t)=u(t)$$



r=0

REFERENCE

Difference:

ERROR

INPUT

Torque

Robotic manipulator

Angle

OUTPUT

+

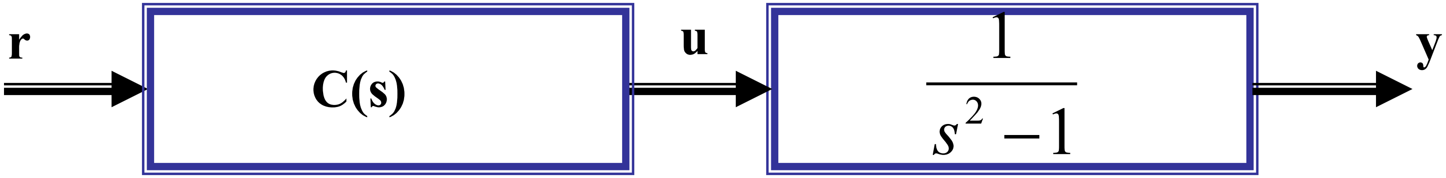
−

Controller

$$\frac{1}{s^2-1}$$

Objective: To bring the system OUTPUT to zero (upright position) for any small nonzero initial conditions.

Open-loop control



The transfer function from r to y is $C(s)\frac{1}{s^2-1}$

Is it possible to choose the controller C(s) such that the whole system is stable?

No.

We tried $C(s) = s-1 \quad \Longrightarrow \quad \frac{s-1}{s^2-1} \neq \frac{1}{s+1}$

$$Y(s) = \underbrace{H(s)U(s)}_{zero-state-response} + \underbrace{Y_o(s)}_{zero-input-response}$$

Last time, we showed that the zero-input-responses (u=0) are different!

How about the zero-state-responses? Are they the same or different?

$$\frac{s-1}{s^2-1}U(s) \Leftrightarrow \frac{1}{s+1}U(s)$$

The same!

Those common factors can always cancel out if they are only signals instead of T.F.

$$Y(s) = \underbrace{H(s)U(s)}_{zero-state-response} + \underbrace{Y_o(s)}_{zero-input-response}$$

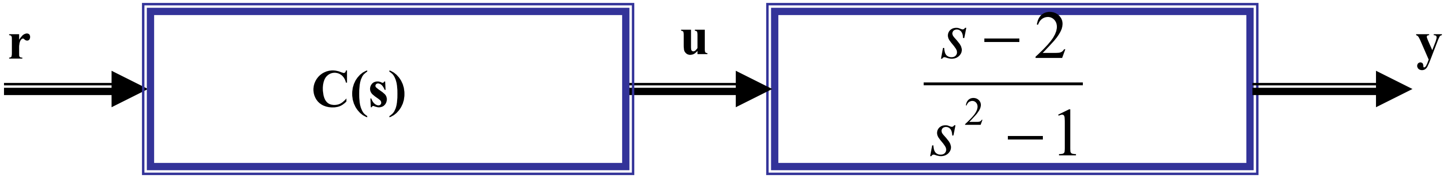
<u>If the system is stable, what is the steady state value of the zero-input-response ?</u>

Zero. So it is possible to cancel out the poles and zeros if the whole system is stable.

$$\frac{s+1}{(s+1)(s+2)} \approx \frac{1}{s+2}$$

If the whole system is stable, then the common poles and zeros can cancel out each other such that the reduced T.F. is good approximation of the original one.

Let’s take a look at another example.



Let $$C(s) = \frac{1}{s-2}$$

The transfer function from r to y is $$\frac{s-2}{(s-2)(s^2-1)}$$

Should we cancel out this common factor (s-2)?

Let’s write down the differential equation relating y and u

$$Y(s) = \frac{s-2}{s^2-1}U(s) \Longrightarrow (s^2-1)Y(s) = (\mathrm{s}-2)\,\mathrm{U(s)} \Longrightarrow \ddot{y} - y = \dot{u} - 2u$$

Let’s write down the equation relating r and u

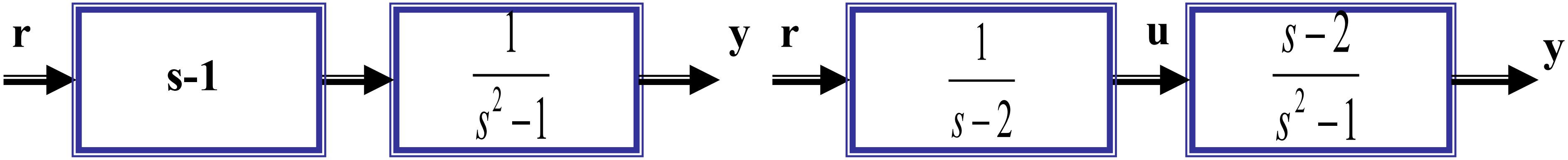
$$U(s) = \frac{1}{s-2}R(s) \Longrightarrow (\mathrm{s}-2)\,\mathrm{U(s)} = \mathrm{R(s)} \Longrightarrow \dot{u} - 2u = r$$

What is the differential equation relating r and y? $$\ddot{y} - y = r$$

What is the transfer function from r to y? $$\frac{1}{(s^2-1)}$$

We need to cancel out this common factor (s-2) in order to get the correct answer!

Let’s compare the two examples carefully.



The transfer function from r to y is

$$\frac{s-1}{(s^2-1)}$$

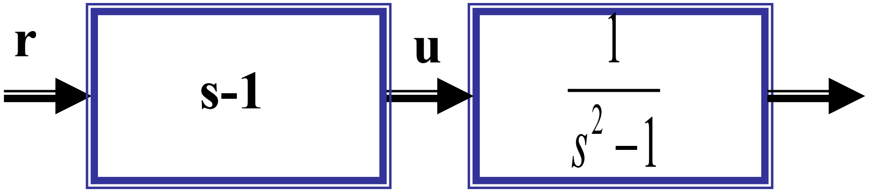
$$\frac{1}{(s^2-1)}$$

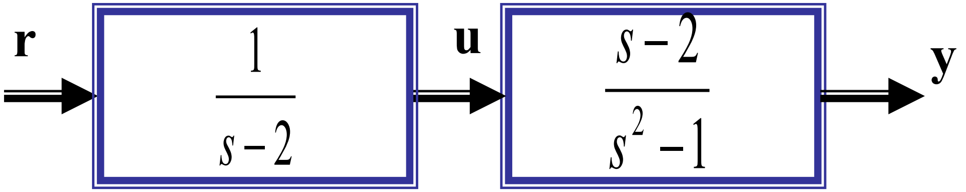
This common factor (s-1) cannot cancel out.

This common factor (s-2) has to cancel out.

What is the difference between these two common factors? And when should we cancel them out?

The answer lies in the different ways of these factors to be associated with the signals. Can you figure it out?





The transfer function from r to y is

$$\frac{s-1}{(s^2-1)}$$

$$\mathrm{U(s)} = (s-1)\,\mathrm{R}(s)$$

The first (s-1) is operating on r.

$$(s^2-1)Y(s) = (s+1)(s-1)Y(s) = \mathrm{U(s)}$$

The second (s-1) is operating on y.

Overall, we have

$$(s^2-1)Y(s) = (s-1)\,\mathrm{R}(s)$$

This common factor (s-1) cannot cancel out.

$$\frac{1}{(s^2-1)}$$

$$(\mathrm{s}-2)\,\mathrm{U(s)} = \mathrm{R(s)}$$

The first (s-2) is operating on u.

$$(s^2-1)Y(s) = (\mathrm{s}-2)\,\mathrm{U(s)}$$

The second (s-2) is also operating on u.

Overall, we have

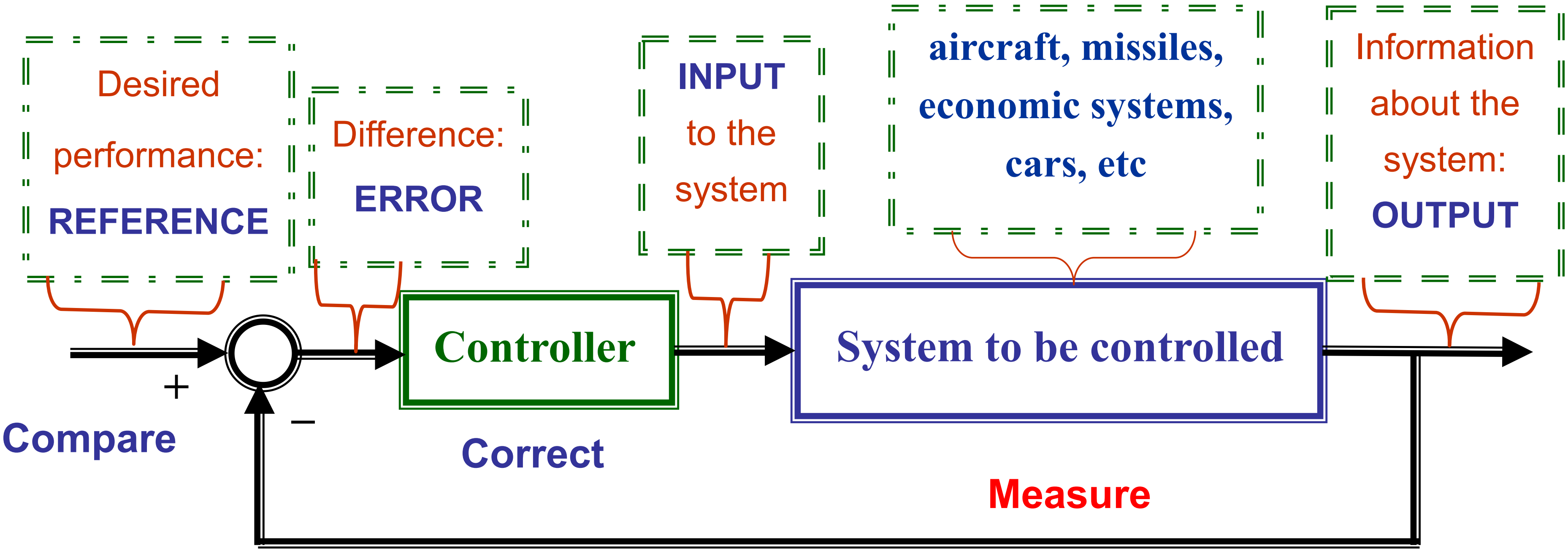
$$(s^2-1)Y(s) = \mathrm{R}(s)$$

This common factor (s-2) has to cancel out.

If the common factors are operating on the same variable, they should cancel out.

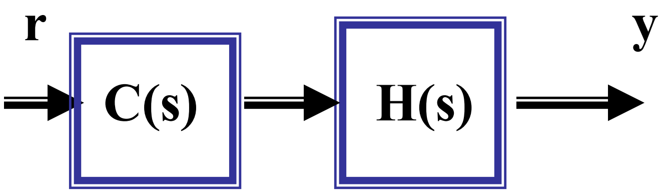
If the common factors are operating on different variables, they should remain.

What is a feedback control system?

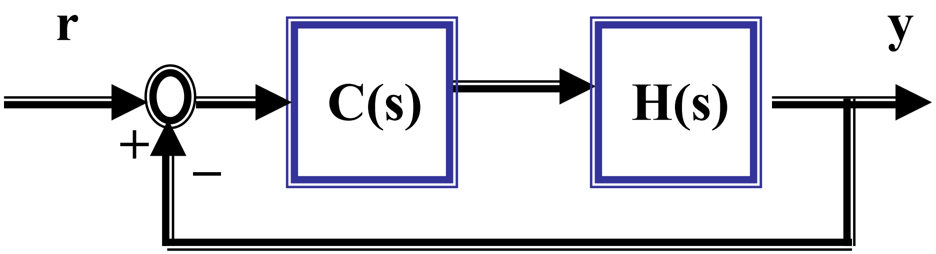


- Trio: Measure —Compare –-Correct

- Fundamental Concept of Control: **Feedback**



$$\frac{Y(s)}{R(s)} = C(s)H(s)$$



$$\frac{Y(s)}{R(s)} = \frac{C(s)H(s)}{1+C(s)H(s)}$$

- There are so many wonderful properties about the feedback controller.
- **What is the most important one?**

- If the system is unstable, is it possible to use open-loop control to make it stable?

- No. Only feedback controller can change the dynamic property of the whole system! Therefore, an unstable system can be stabilized by feedback!
- But it is a double-edged sword! It can also destabilize the system!
- Whenever a feedback controller is used, the dynamic property of the system has been changed!
- Therefore, you have to design your feedback controller carefully because it can make the performance better, and also possibly worse.
- However, there is one obvious advantage for open loop controller from the point of view of implementation, what is that?

It is much easier to implement!

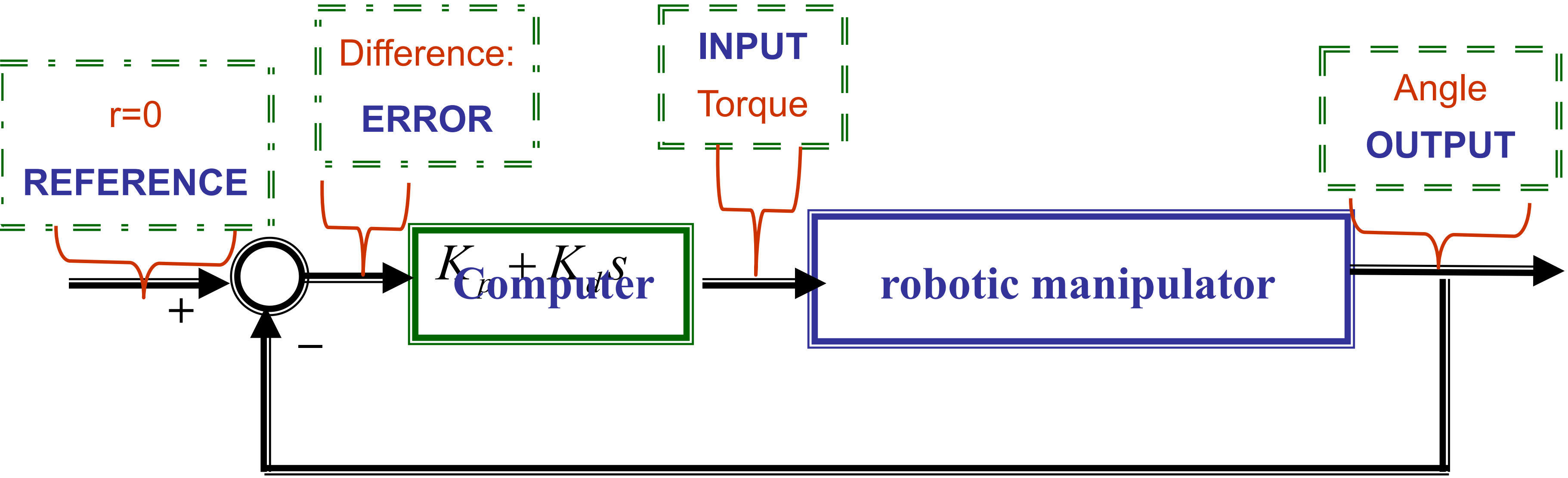
Drawing the feedback link in one second, which may cost the engineers one year!

What is a Computer Control System?

The controller is implemented by a computer instead of an analog circuit!

How to convert an analog controller into computer control system?

Can we simply replace the analog circuit controller with a computer?



What is the input to the computer? The error signal.

Is it continuous time or discrete-time? It is continuous-time.

Can you feed a continuous-time signal directly into a computer?

No. Computer can only deal with a sequence of digits.

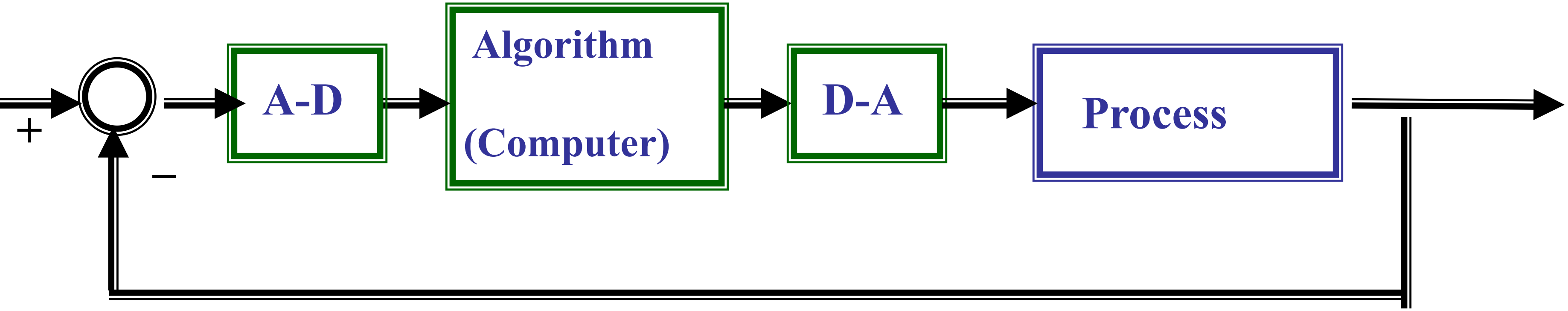
What is the output from the computer program?

A sequence of numbers---discrete-time signal

What is the input to the robotic manipulator?

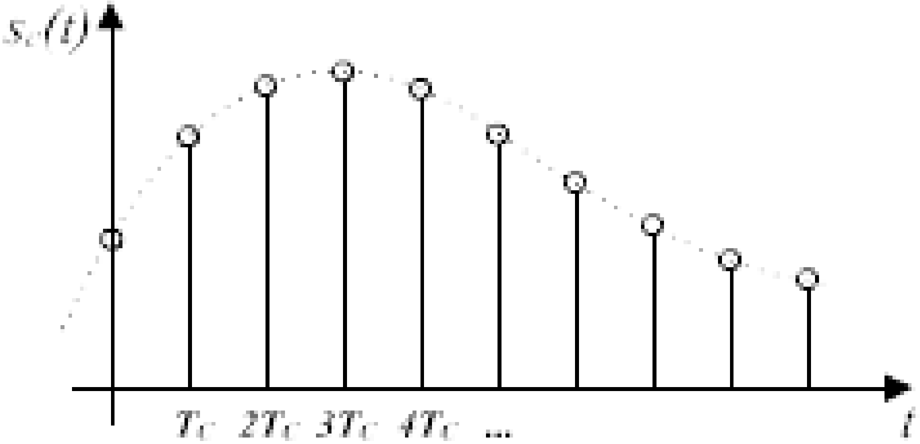
The torque is supposed to be continuous-time signal!

So there is another mismatch!



How to do the A-D?

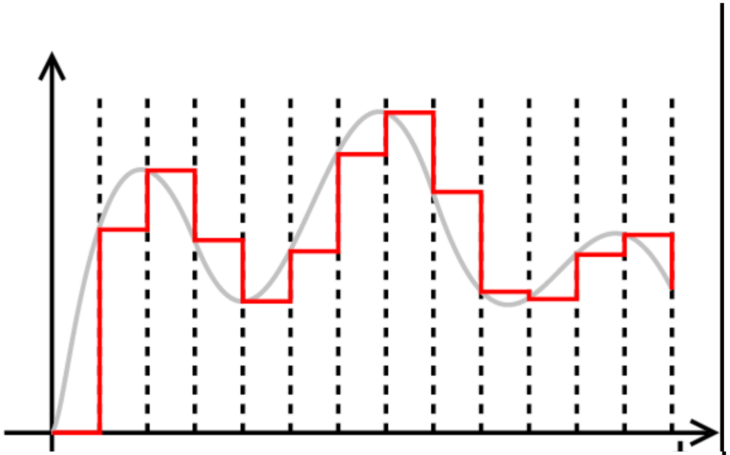
Sampling!



- The conversion is done at the sampling instants.
- The computer interprets the converted signal, $y(t_k)$, as a sequence of numbers.
- t_k is called the sampling instants, and the time between successive samplings is called the sampling period and is denoted as h.
- The sampling period h is normally a constant -→ uniform sampling

How to do the D-A? What’s the natural way of converting the discrete-time signal into continuous-time signal?

Zero-order Hold

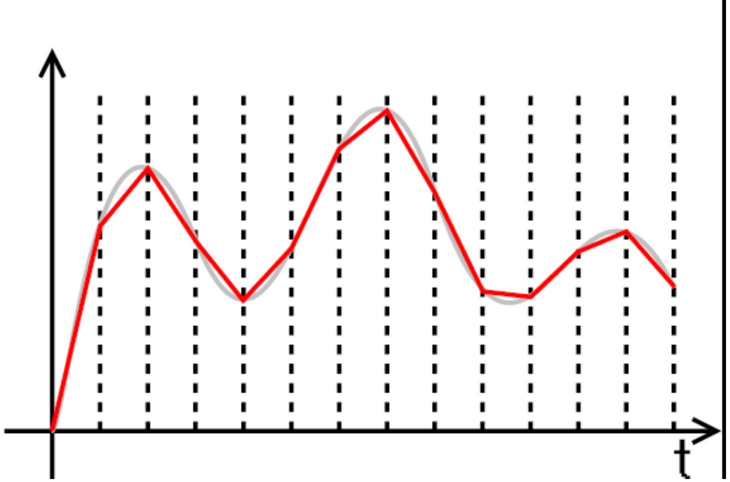


Is the resulting analog signal continuous or dis-continuous?

It is dis-continuous.

Can we make it piece-wise continuous?

First-order Hold



Why don’t we use First-order Hold to convert the control signal from discrete-time to continuous time?

For feedback control system, the control signal depends upon the measured output.

At present, can we measure the output at next step?

No.

Next we are going to derive the equations for sampled system. But first we need to know the solution for the continuous-time system.

How did we solve the differential equation? What is the magic tool?

We did it last time using Laplace Transform. Let's do it again.

$$\dot{x} = Ax + bu$$

$$y = cx$$

Since we need a precise solution, we need to consider the initial conditions.

$$sX(s) - x(0) = AX(s) + bU(s) \quad \Longrightarrow \quad \begin{aligned} &sX(s) - AX(s) = x(0) + bU(s) \\ &X(s) = (sI - A)^{-1}x(0) + (sI - A)^{-1}bU(s) \end{aligned}$$

$$Y(s) = cX(s) = c(sI - A)^{-1}x(0) + c(sI - A)^{-1}bU(s)$$

Let's find out the solution in time-domain!

$$(sI - A)^{-1} \quad \Longrightarrow \quad e^{At}$$

$$x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)}bu(\tau)d\tau$$

$$y = cx = ce^{At}x(0) + c\int_0^t e^{A(t-\tau)}bu(\tau)d\tau$$

Zero-input response

Zero-state-response

Now we are going to use the solution in the continuous-time to derive the state space model of the sampled system.

$$\dot{x} = Ax + bu \qquad x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)}bu(\tau)d\tau$$

$$y = cx \qquad y = cx = ce^{At}x(0) + c\int_0^t e^{A(t-\tau)}bu(\tau)d\tau$$

Assume that the state at time t_k is known, how to calculate the future state at time t, x(t), where t is in the interval $[t_k, t_{k+1}]$?

$$x(t) = e^{A(t-t_k)}x(t_k) + \int_{t_k}^t e^{A(t-\tau)}bu(\tau)d\tau$$

Is u(t) a constant in the whole interval $[t_k, t_{k+1})$? Yes. Zero-order-hold.

$$x(t) = e^{A(t-t_k)}x(t_k) + (\int_{t_k}^t e^{A(t-\tau)}d\tau)bu(t_k)$$

Next, let’s try to simplify the expression of the integral.

By change of variable $v = t - \tau$

$$\int_{t_k}^t e^{A(t-\tau)}d\tau = -\int_{t-t_k}^0 e^{Av}dv = \int_0^{t-t_k} e^{Av}dv$$

$$x(t) = e^{A(t-t_k)}x(t_k) + (\int_0^{t-t_k} e^{Av}dv)bu(t_k)$$

Now we are ready to compute the state at t_{k+1}, putting $t = t_{k+1}$

$$x(t_{k+1}) = e^{A(t_{k+1}-t_k)}x(t_k) + (\int_0^{t_{k+1}-t_k} e^{Av}dv)bu(t_k)$$

For periodic sampling with period h, we have $t_k = kh$ and $t_{k+1} - t_k = h$

$$x(kh+h) = e^{Ah}x(kh) + (\int_0^h e^{Av}dv)bu(kh) = \Phi x(kh) + \Gamma u(kh)$$

Choose the sampling period as the time unit

State-space Model

$$\dot{x} = Ax + bu$$
$$y = cx$$

Zero-order hold $\Longrightarrow$

$$x(k+1) = \Phi x(k) + \Gamma u(k)$$
$$y(k) = cx(k)$$

$$\Phi = e^{Ah}$$

$$\Gamma = (\int_0^h e^{Av}dv)b$$

How to calculate e^{At} ? $(sI - A)^{-1}$ $\Longrightarrow$ e^{At}

State-space Model

$$x(k+1) = \Phi x(k) + \Gamma u(k)$$
$$y(k) = cx(k)$$

Given x(k) and u(k), how to predict the future values of x(k)?

To get x(k+1) is simple:

$$x(k+1) = \Phi x(k) + \Gamma u(k)$$

How about x(k+2), …?

$$\begin{aligned} x(k+2) &= \Phi x(k+1) + \Gamma u(k+1) \\ &= \Phi(\Phi x(k) + \Gamma u(k)) + \Gamma u(k+1) \\ &= \Phi^2 x(k) + \Phi\Gamma u(k) + \Gamma u(k+1) \end{aligned}$$

How about x(k+n), …?

$$x(k+n) = \Phi x(k+n-1) + \Gamma u(k+n-1) = \underbrace{\Phi^n x(k)}_{\boxed{\text{zero-input response}}} + \underbrace{\Phi^{n-1}\Gamma u(k) + \Phi^{n-2}\Gamma u(k+1) + \cdots + \Gamma u(k+n-1)}_{\boxed{\text{zero-state response}}}$$

Given x(k) and input signal, can you predict the whole future of the system?

Yes. Therefore, it is verified that this is indeed the state space model.

- Input-Output Models

For continuous-time, we use differential equations to relate input and output.

Can we use differential equation to describe discrete-time signal?

No. For discrete-time, we use difference equation to relate the input and output.

$$y(k+1) = a_1 y(k) + a_2 y(k-1) + b_0 u(k) + b_1 u(k-1)$$

For continuous-time, we use differentiation operator. For discrete-time, we use

- **Shift-Operator** **---to simplify the representation**

forward-shift operator q $\qquad q\ f(k) = f(k+1)$

backward-shift operator or the delay operator q^{-1} $\qquad q^{-1} f(k) = f(k-1)$

- Operator calculus gives compact descriptions of systems

$$y(k+1) = a_1 y(k) + a_2 y(k-1) + b_0 u(k) + b_1 u(k-1)$$

$$y(k+1) - a_1 y(k) - a_2 y(k-1) = b_0 u(k) + b_1 u(k-1)$$

- Use of the shift operator gives

$$qy(k) - a_1 y(k) - a_2 q^{-1} y(k) = b_0 u(k) + b_1 q^{-1} u(k)$$

$$(q - a_1 - a_2 q^{-1}) y(k) = (b_0 + b_1 q^{-1}) u(k)$$

$$(q - a_1 - a_2 q^{-1}) y(k) = (b_0 + b_1 q^{-1}) u(k)$$

How to get rid of q^{-1} ?

$$q(q - a_1 - a_2 q^{-1}) y(k) = q(b_0 + b_1 q^{-1}) u(k)$$

$$(q^2 - a_1 q - a_2) y(k) = (b_0 q + b_1 \ ) u(k)$$

$$A(q) y(k) = B(q) u(k)$$

In terms of q

$$A(q) = q^2 - a_1 q - a_2$$

$$B(q) = b_0 q + b_1$$

This is a more compact representation than the original difference equation:

$$y(k+1) = a_1 y(k) + a_2 y(k-1) + b_0 u(k) + b_1 u(k-1)$$

How to solve difference equation?

$$y(k+1) = a_1 y(k) + a_2 y(k-1) + b_0 u(k) + b_1 u(k-1)$$

Given input u(k), how to find out output y(k) from this equation?

In continuous-time, what is the magic tool to solve differential equation?

We rely on Laplace Transform.

$$F(s) = \int_0^\infty f(t)e^{-st}dt$$

Can we directly apply Laplace transform to discrete-time signal?

No.

What is the fundamental basis for Laplace Transform? Why is Laplace Transform so powerful for linear system?

Superposition Principle

Is the superposition principle still true for discrete-time linear system?

Yes.

Break

- *10 Amazing AI Robots*

Fundamental property of linear system----Superposition Principle

Why is superposition principle so important?

Suppose you try to figure out the response of the system to a complicated input u.

$$u(t) \longrightarrow \boxed{\qquad} \longrightarrow ???$$

Let input u be resolved into a set of **basis functions**

$$u(t) = a_1\phi_1(t) + a_2\phi_2(t) + ...$$

$$\phi_1(t) \longrightarrow \boxed{\qquad} \longrightarrow y_1(t)$$

$$\phi_2(t) \longrightarrow \boxed{\qquad} \longrightarrow y_2(t)$$

if the system is linear

$$a_1\phi_1(t) + a_2\phi_2(t) + \cdots \longrightarrow \boxed{\qquad} \longrightarrow a_1y_1(t) + a_2y_2(t) + \cdots$$

The trick of superposition principle is to choose proper basis functions whose response can be easily obtained.

What did we choose for the continuous time system?

$$e^{st} \longrightarrow \boxed{} \longrightarrow H(s)e^{st}$$

In the discrete-time, what basis function should we use?

We can still try the exponential function just replacing t with t=kh!

$$e^{st} \xrightarrow{t \to kh} e^{skh} \xrightarrow{e^{sh} \to a} a^k$$

Then we can make a wild guess that

$$a^k \longrightarrow \boxed{} \longrightarrow H(a)a^k$$

Does it really work?

We need to verify whether they satisfy the equation or not.

$$y(k+1) = a_1 y(k) + a_2 y(k-1) + b_0 u(k) + b_1 u(k-1)$$

$$y(k+1) - a_1 y(k) - a_2 y(k-1) = b_0 u(k) + b_1 u(k-1)$$

$$u(k) = a^k, y(k) = H(a)a^k$$

$$H(a)a^{k+1} - a_1 H(a)a^k - a_2 H(a)a^{k-1} = b_0 a^k + b_1 a^{k-1} \Longrightarrow H(a) = \frac{b_0 + b_1 a^{-1}}{a - a_1 - a_2 a^{-1}}$$

So it works again!

Can we decompose any discrete-time signal f(k) by the exponentials a^k ?

$$f(k) = \frac{1}{2\pi i}\oint F(z) z^{k-1} dz \qquad z^k \rightarrow a^k$$

What is F(z)?

$$F(z) = \sum_{k=0}^{\infty} f(k) z^{-k} = f(0) + f(1) z^{-1} + f(2) z^{-2} + \cdots + f(k) z^{-k} + \cdots$$

F(z) is the Z- transform of f(k)

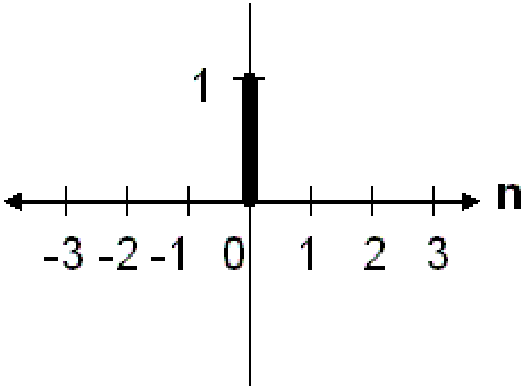
The equivalent of Laplace Transform in continuous-time.

- The z-Transform

$$F(z)=\sum_{k=0}^{\infty} f(k)z^{-k} = f(0)+f(1)z^{-1}+f(2)z^{-2}+\cdots+f(k)z^{-k}+\cdots$$

where z is a complex variable, just like in F(s), where s is complex variable.

- Examples. **What is z-transform of unit impulse signal $\delta(k)$?**



Just apply the definition we have,

$$F(z)=\sum_{k=0}^{\infty} f(k)z^{-k} = f(0)=1$$

- The z-Transform

$$F(z)=\sum_{k=0}^{\infty} f(k)z^{-k}=f(0)+f(1)z^{-1}+f(2)z^{-2}+\cdots+f(k)z^{-k}+\cdots$$

- What is Laplace transform of e^{at} ?

$$e^{at}\rightarrow\frac{1}{s-a}$$

What is z-transform of a^{k} ?

$$F(z)=\sum_{k=0}^{\infty} a^{k}z^{-k}=\sum_{k=0}^{\infty}(\frac{a}{z})^{k}=1+\frac{a}{z}+(\frac{a}{z})^{2}+\cdots$$

What is the little trick to calculate this type of summation (Geometric series)?

$$\frac{a}{z}F(z)=\frac{a}{z}+(\frac{a}{z})^{2}+\cdots$$

$$F(z)-\frac{a}{z}F(z)=\left(1-\frac{a}{z}\right)F(z)=1-\lim_{n\rightarrow\infty}\left(\frac{a}{z}\right)^{n}$$

$$\Longrightarrow \quad F(z)=\frac{1}{1-\frac{a}{z}}=\frac{z}{z-a}$$

What is the most important property of Laplace transform?

The relationship between y(t) and y’(t) in s-domain!

$$L(y(t)) = Y(s)$$

$$L(y'(t)) = sY(s) - y(0)$$

$$\frac{d}{dt} \Leftrightarrow s$$

Similarly, the most important property of z-transform is about the relationship between x(k) and x(k+1) in the z-domain

$$X(z) = z\{x(k)\} = \sum_{k=0}^{\infty} x(k) z^{-k} = x(0) + x(1)z^{-1} + x(2)z^{-2} + \cdots$$

$$z\{x(k+1)\} = \sum_{k=0}^{\infty} x(k+1) z^{-k} = x(1) + x(2)z^{-1} + x(3)z^{-2} + \cdots$$

How do we relate these two?

$$zX(z) = zx(0) + x(1) + x(2)z^{-1} + x(3)z^{-2} + \cdots$$

$$zX(z) = zx(0) + Z\{x(k+1)\} \quad \Longrightarrow \quad Z\{x(k+1)\} = zX(z) - zx(0)$$

X(0)=0 $\quad \Longrightarrow \quad z\{x(k+1)\} = zX(z)$

Compare with q {x(k)}=x(k+1)

$$q(forward - time - shift) \Leftrightarrow z$$

$$q^{-1}(backward - time - shift) \Leftrightarrow z^{-1}$$

Now we are ready to use z-transform to solve difference equations; for instance,

$$y(k+1) = a_1 y(k) + a_2 y(k-1) + b_0 u(k) + b_1 u(k-1)$$

$$y(k+2) - a_1 y(k+1) - a_2 y(k) = b_0 u(k+1) + b_1 u(k)$$

If the z-transform of both sides is taken, assuming zero initial conditions

$$z^2 Y(z) - a_1 z Y(z) - a_2 Y(z) = b_0 z U(z) + b_1 U(z)$$

It is very easy to get the solution of Y(z) in the z-domain.

$$Y(z) = \frac{b_0 z + b_1}{z^2 - a_1 z - a_2} U(z) = H(z) U(z)$$

What is the solution in the time-domain?

How to get y(k) in time-domain? How to do inverse z-Transform?

$$y(k) = \frac{1}{2\pi i}\oint Y(z) z^{k-1} dz$$

This is the hard way!

What is the easy way?

Partial Fraction Expansion:

$$Y(z) = \frac{z+1}{(z-0.5)(z-2)} = (\frac{-1}{z-0.5} + \frac{2}{z-2})$$

Use the z-Transform table : $\frac{z}{z-a} \to a^k$

There is a mismatch! There is one little trick to make it work!

$Use \frac{Y(z)}{z} instead!$

$$\frac{Y(z)}{z} = \frac{z+1}{z(z-0.5)(z-2)} = (\frac{1}{z} - \frac{2}{z-0.5} + \frac{1}{z-2})$$

$$y(k) = \delta(k) - 2(0.5)^k + 2^k, k \geq 0$$

There is another way to get the answer using Partial Fraction Expansion directly:

$$Y(z) = \frac{z+1}{(z-0.5)(z-2)} = (\frac{-1}{z-0.5} + \frac{2}{z-2}) \qquad \frac{z}{z-a} \to a^k$$

In order to use the z-Transform table , let's rewrite

$$Y(z) = z^{-1}\left(\frac{-z}{z-0.5} + \frac{2z}{z-2}\right) = z^{-1}V(z)$$

$$V(z) = \frac{-z}{z-0.5} + \frac{2z}{z-2}$$

Using the z-transform table, we get

$$v(k) = -(0.5)^k + 2 \times (2)^k, k \geq 0$$

What is y(k) then?

$$q^{-1}(backward - time - shift) \Leftrightarrow z^{-1}$$

$$y(k) = q^{-1}v(k) = v(k-1) = -(0.5)^{k-1} + 2 \times (2)^{k-1}, k \geq 1$$

What is y(0)? Let's try to find it out from the definition of z-transform

$$Y(z) = y(0) + y(1)z^{-1} + y(2)z^{-2} + \cdots$$

y(0)=0.

Let's apply z-transform to the state-space equation

$$x(k+1) = \Phi x(k) + \Gamma u(k)$$

$$y(k) = cx(k)$$

$$z(X(z) - x(0)) = \Phi X(z) + \Gamma U(z)$$

$$zX(z) - \Phi X(z) = zx(0) + \Gamma U(z)$$

$$(zI - \Phi)X(z) = zx(0) + \Gamma U(z)$$

$$X(z) = (zI - \Phi)^{-1} zx(0) + (zI - \Phi)^{-1} \Gamma U(z)$$

What is the output?

$$Y(z) = cX(z) = c(zI - \Phi)^{-1} zx(0) + c(zI - \Phi)^{-1} \Gamma U(z)$$

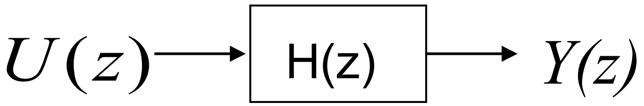
- If the initial conditions are zero, x(0)=0

$$Y(z) = c(zI - \Phi)^{-1} \Gamma U(z) = H(z)U(z)$$

- **What is the transfer function?** $H(z) = c(zI - \Phi)^{-1}\Gamma$

$$U(z) \longrightarrow \boxed{\text{H(z)}} \longrightarrow Y(z)$$

Transfer function: H(z) -----Transfer the input into output



<u>Since Y(z)=H(z)U(z), does the same input always produce the same output?</u>

Y(z)=H(z)U(z) only if the initial conditions are all zero.

If intial conditions are not zero, there are extra terms in the solution!.

$$Y(z)=cX(z)=c(zI-\Phi)^{-1}zx(0)+c(zI-\Phi)^{-1}\Gamma U(z)$$

zero-input response

zero-state response

Transfer function:

A representation of the system which is equivalent to difference equation.

$$q(forward - shift - operator) \Leftrightarrow z$$

Given a transfer function, how to write down the difference equation?

$$\frac{Y(z)}{U(z)} = \frac{z+2}{z^2-1} \qquad z^2 Y(z) - Y(z) = zU(z) + 2U(z)$$

$$q \Leftrightarrow z \qquad y(k+2) - y(k) = u(k+1) + 2u(k)$$

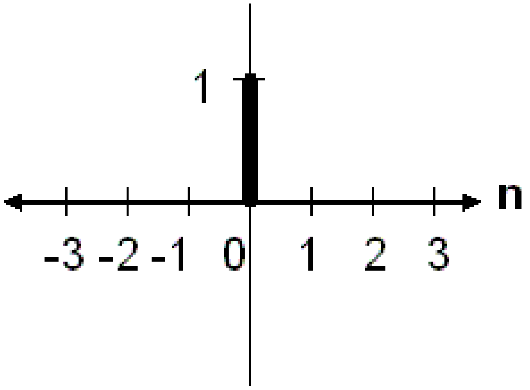
Given a difference equation, how to write down the transfer function?

$$y(k+1) = y(k) + y(k-1) + u(k-2) \qquad y(k+1) - y(k) - y(k-1) = u(k-2)$$

$$qy(k) - y(k) - q^{-1}y(k) = q^{-2}u(k) \quad q \Leftrightarrow z \quad zY(z) - Y(z) - z^{-1}Y(z) = z^{-2}U(z)$$

$$Y(z) = \frac{z^{-2}}{z-1-z^{-1}}U(z) \quad \Longrightarrow \quad Y(z) = \frac{1}{z^3 - z^2 - z}U(z)$$

What is the unit impulse response?



$$u(k) = \delta(k)$$

$$U(z) = 1, Y(z) = H(z)$$

Transfer function is the unit impulse response in z-domain!

- **What is the response to exponential a^k?**

$$a^k \longrightarrow \boxed{} \longrightarrow H(a)a^k$$

- The special case: a=1

$$1 \longrightarrow \boxed{} \longrightarrow H(1)$$

The static gain (the steady-state gain, or DC gain)------ H(1)

$$C \longrightarrow \boxed{} \longrightarrow H(1)C$$

What is steady-state gain in continuous time?

$$e^{st} \longrightarrow \boxed{} \longrightarrow H(s)e^{st} \quad \xRightarrow{s=0} \quad 1 \longrightarrow \boxed{} \longrightarrow H(0)$$

$$U(z) \longrightarrow \boxed{\text{H(z)}} \longrightarrow Y(z)$$

$$H(z) = c(zI - \Phi)^{-1}\Gamma = \frac{Q(z)}{P(z)}$$

- What are the poles? $P(z) = 0$

- What are the zeros? $Q(z) = 0$

What is P(z)?

$$(zI - \Phi)^{-1} = \frac{\begin{bmatrix} * & * \\ * & * \end{bmatrix}}{\det\{(zI - \Phi)\}} \quad \Longrightarrow \quad P(z) = \det\{(zI - \Phi)\}$$

How to get the eigenvalues of the matrix Φ ?

$$\det\{(\lambda I - \Phi)\} = 0$$

The eigenvalues of Φ $\Longrightarrow$ **The poles of the system**

The relationship between poles and stability:

Partial Fraction expansion of Transfer function (Impulse response)

$$H(s)=\frac{1}{s^2-1}=\frac{1}{2}\left(\frac{1}{s-1}-\frac{1}{s+1}\right)$$

If λ is the pole, then

$\frac{1}{s-\lambda}$ must be one component in H(s)

$e^{\lambda t}$ is component of the impulse response

$\lambda=\sigma+j\omega$ $e^{\sigma t}$ --stability

$e^{\lambda t}=e^{\sigma t}e^{j\omega t}$ $e^{j\omega t}$ ---oscillation

$\sigma<0$ $e^{\sigma t}\to 0$ Stable

$\sigma>0$ $e^{\sigma t}\to\infty$ Unstable

$\sigma=0$ $e^{\sigma t}=1$ Marginally Stable

$$H(z)=\frac{z+1}{(z-0.5)(z-2)}$$

If λ is the pole, then

$\frac{1}{z-\lambda}$ must be one component in H(z)

λ^k is component of the impulse response

$\lambda=\rho e^{j\theta}$ ρ^k ---stability

$\lambda^k=\rho^k e^{j\theta k}$ $e^{j\theta k}$ ---oscillation

$\rho<1$ $\rho^k\to 0$ Stable

$\rho>1$ $\rho^k\to\infty$ Unstable

$\rho=1$ $\rho^k=1$ Marginally Stable

Multiplicity of the poles

Partial Fraction expansion of Transfer function (Impulse Response)

$$H(s) = \frac{3s+1}{(s-1)(s+1)^2} = \frac{1}{s-1} - \frac{1}{s+1} + \frac{1}{(s+1)^2}$$

If λ is the multiple pole, then

$\frac{1}{(s-\lambda)^2}$ must be one component in H(s)

$te^{\lambda t}$ is component of the impulse response

$\lambda = \sigma + j\omega$ $te^{\sigma t}$ ---stability

$te^{\lambda t} = te^{\sigma t}e^{j\omega t}$ $e^{j\omega t}$ ---oscillation

$\sigma < 0$ $te^{\sigma t} \rightarrow 0$ Stable

$\sigma > 0$ $te^{\sigma t} \rightarrow \infty$ Unstable

$\sigma = 0$ $te^{\sigma t} = t \rightarrow \infty$ Unstable

$\frac{1}{(z-\lambda)^2}$ must be one component in H(z)

$k\lambda^k$ is component of the impulse response

$\lambda = \rho e^{j\theta}$ $k\rho^k$ ---stability

$k\lambda^k = k\rho^k e^{j\theta k}$ $e^{j\theta k}$ ---oscillation

$\rho < 1$ $k\rho^k \rightarrow 0$ Stable

$\rho > 1$ $k\rho^k \rightarrow \infty$ Unstable

$\rho = 1$ $k\rho^k = k \rightarrow \infty$ Unstable

Stability criterion

Continuous-time

$$\lambda = \sigma + j\omega$$

$\sigma < 0$ Stable

$\sigma > 0$ Unstable

$\sigma = 0$

Single

Marginal Stable

Multiple

Unstable

Discrete-time

$$\lambda = \rho e^{j\theta}$$

$\rho < 1$ Stable

$\rho > 1$ Unstable

$\rho = 1$

Single

Marginal Stable

Multiple

Unstable

For control system design, “marginal stable” is not considered as “stable”.

Now we know that the poles decide the stability. What about zeros?

- Meaning of zeros

$$H(s) = \frac{Q(s)}{P(s)} = \frac{a_1}{s - p_1} + \frac{a_2}{s - p_2} + \cdots \frac{a_n}{s - p_n}$$

The zeros decide the weights of the components in the impulse response.

There is another important meaning for zeros.

$$a^k \longrightarrow \boxed{\quad} \longrightarrow H(a)a^k$$

What is the output if a is the zero of the system, ie. H(a)=0?

Zero!

Signal blocking property

Example:

H(1)=0

$$1 \longrightarrow \boxed{\quad} \longrightarrow \quad ??? \quad 0$$

It means that the effect of the input can be completely rejected!

When do we want this to happen?

We will use this property for disturbance rejection later!

Zeros are related to an important concept in control system design:

- **Inverse of a system**

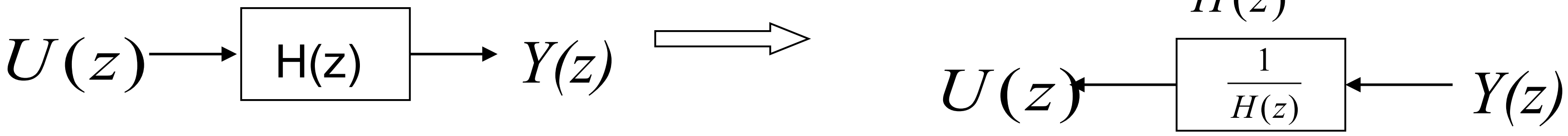
If the transfer function of a system is $H(z)=\dfrac{Q(z)}{P(z)}$

The inverse of a system is defined as $\dfrac{1}{H(z)}=\dfrac{P(z)}{Q(z)}$

The inverse of a system can be imagined as the original system with inverse direction of the signal flow: the input becomes output while the output becomes input!

$$Y(z)=H(z)U(z)$$

$$U(z)=\frac{1}{H(z)}Y(z)$$



- The inverse of a system may not have any physical meaning because cause and effect cannot be simply reversed in real world.
- It is just a mathematical concept which is useful for control system design.

- **Inverse of a system**

If the transfer function of a system is $H(z) = \dfrac{Q(z)}{P(z)}$

The inverse of a system is defined as $\dfrac{1}{H(z)} = \dfrac{P(z)}{Q(z)}$

Systems with stable and unstable inverses

- System with stable inverse (1/H(z) is stable): All the zeros are stable.

System with unstable inverse (1/H(z) is unstable):
The system contains at least one unstable zero.

- **If a system has unstable zeros, does it imply that the system is unstable ?**

No. It only implies that the inverse of the system is unstable.

This concept is important for controller design, we will discuss it in Chapter 4.

What happens to the poles of the system after sampling?

$$\begin{aligned} \dot{x} &= Ax + bu \\ y &= cx \end{aligned} \quad \xrightarrow{\text{Zero-order hold}} \quad \begin{aligned} x(k+1) &= \Phi x(k) + \Gamma u(k) \\ y(k) &= cx(k) \end{aligned}$$

$$\Phi = e^{Ah}$$

What are the eigenvectors and eigenvalues of A?

$$Av = \lambda v$$

Is v also the eigenvector of Φ ?

$$\Phi = e^{Ah} = I + Ah + \ldots + A^k \frac{h^k}{k!} + \ldots$$

$$\Phi v = e^{Ah} v = v + Ahv + \ldots + A^k \frac{h^k}{k!} v + \ldots \qquad Av = \lambda v, A^k v = \lambda^k v$$

$$\Phi v = v + h\lambda v + \ldots + \frac{h^k}{k!} \lambda^k v + \ldots = (1 + h\lambda + \ldots + \frac{(h\lambda)^k}{k!} + \ldots)v$$

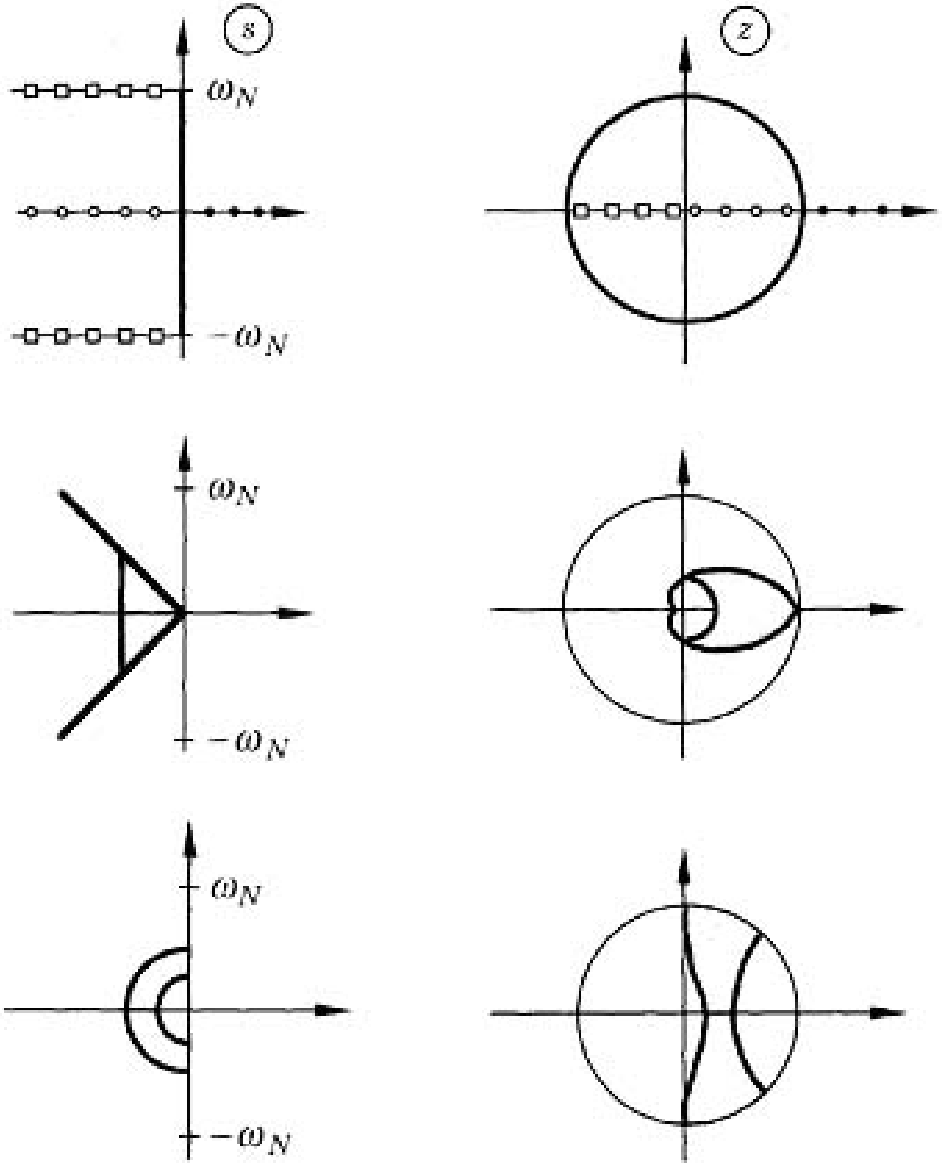
If $Av = \lambda v$ Then $\Phi v = e^{h\lambda} v$

$$\lambda \Rightarrow e^{h\lambda}$$

$$\lambda \Rightarrow e^{h\lambda}$$

$$\lambda = \sigma + j\omega \Rightarrow e^{h\lambda} = e^{h(\sigma + j\omega)} = e^{h\sigma}e^{jh\omega}$$

- This is the mapping from the continuous-time poles to the discrete-time poles.



- If p is the stable pole, is it stable in the discrete-time?

Yes.

- If p is the unstable pole, is it unstable in the discrete time?

Yes.

The map of the poles: $\lambda \Rightarrow e^{h\lambda}$

- **How about zeros? Can we use the same map to get zeros?**
- Example Second-order system

$$\frac{1}{s^2}$$

Using Table 2.1 , we can get the discrete transfer function

$$\frac{h^2(z+1)}{2(z-1)^2}$$

What is the zero for the sampled system?

$z = -1$.

- **Are there any zeros in the original continuous-time system?**

  - Zeros are introduced by sampling!

Is it possible to derive a simple formula to map the zeros of the system to zeros of the sampled system under zero-order-hold?

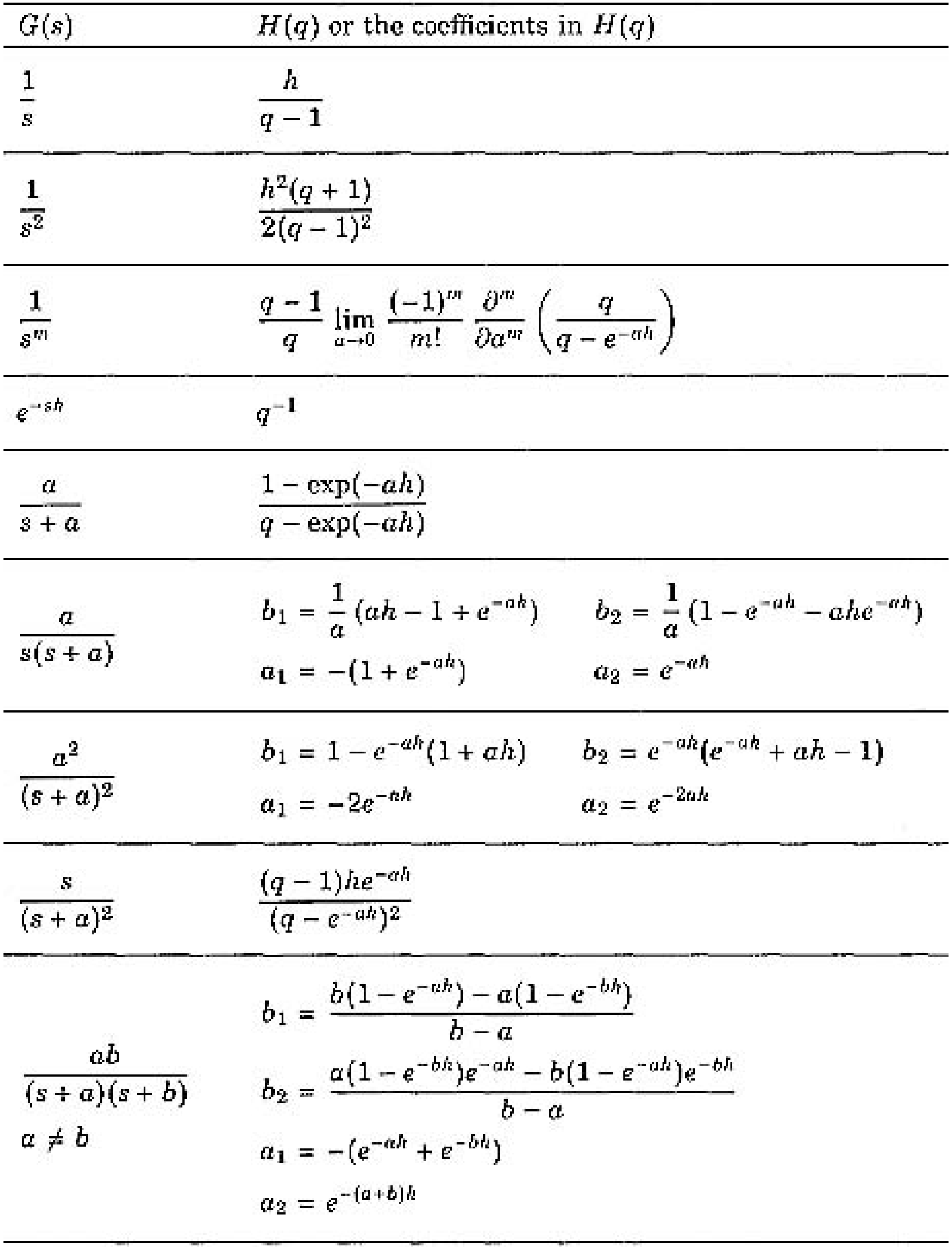
Impossible! You cannot have a formula to produce something from NOTHING.

Q & A…

THANK YOU !

$$H(z)=\frac{b_1 z+b_2}{z^2+a_1 z+a_2}$$

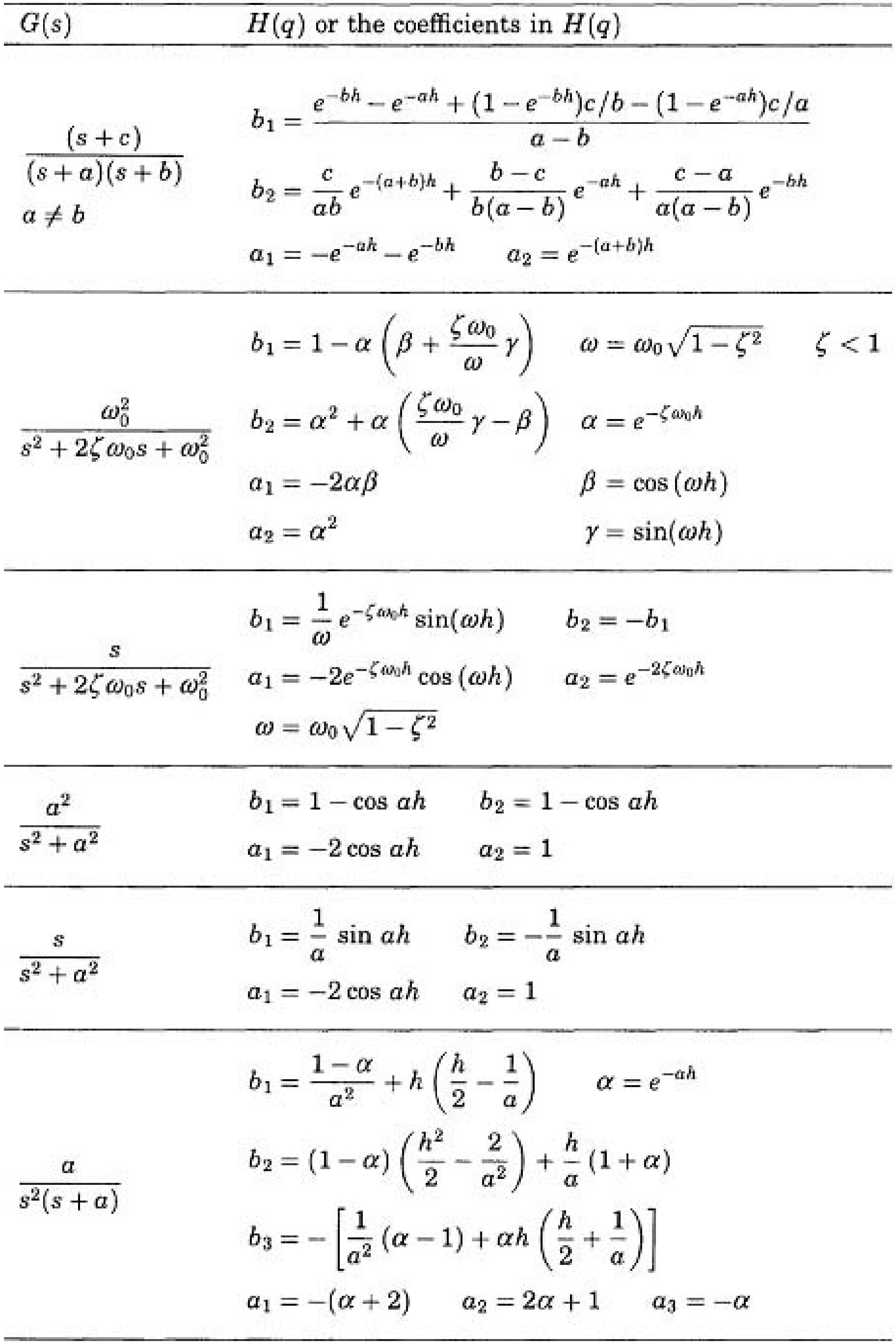
return



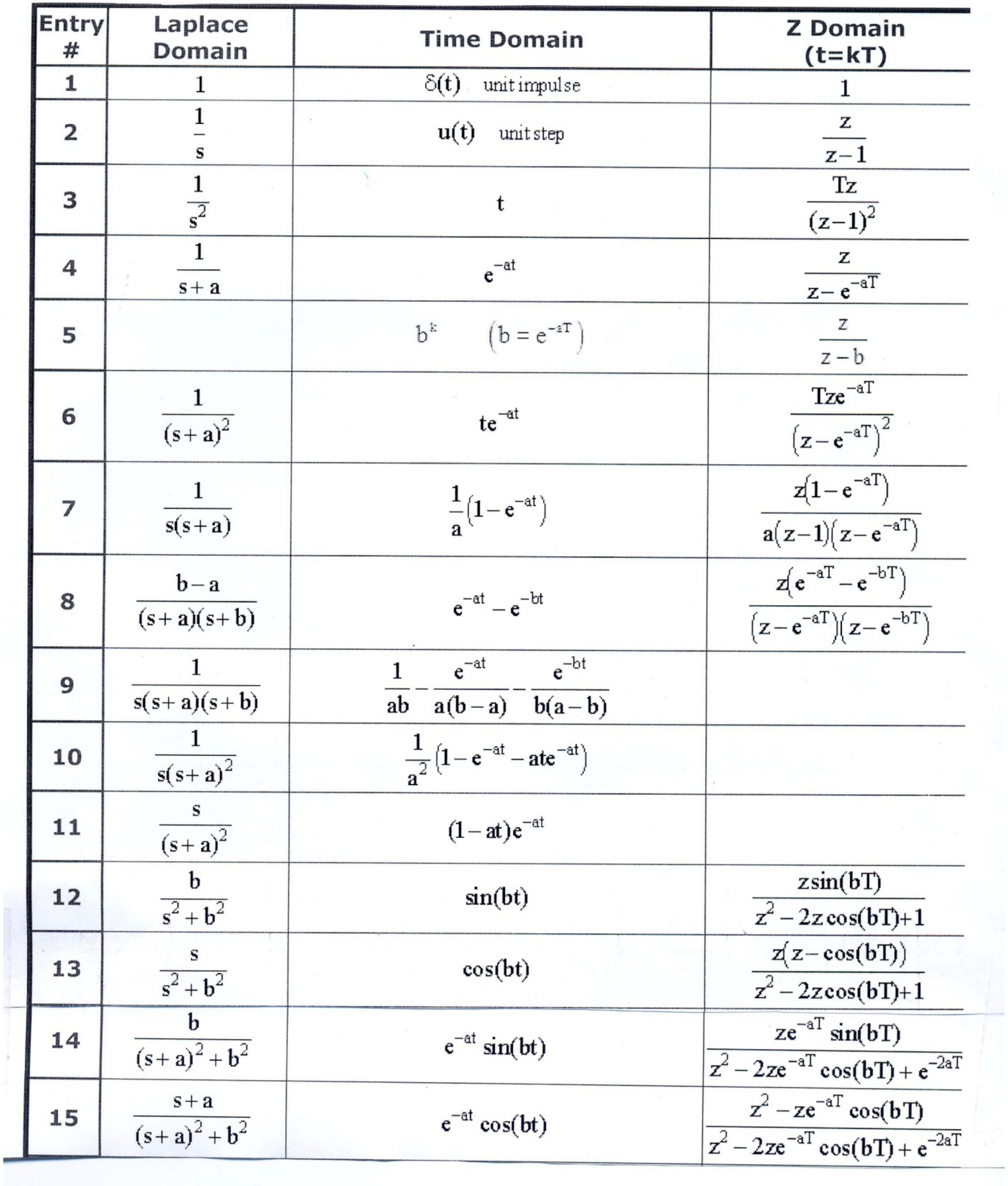
$G(s)$	$H(q)$ or the coefficients in $H(q)$
$\frac{1}{s}$	$\frac{h}{q-1}$
$\frac{1}{s^2}$	$\frac{h^2(q+1)}{2(q-1)^2}$
$\frac{1}{s^m}$	$\frac{q-1}{q}\lim_{a\to 0}\frac{(-1)^m}{m!}\frac{\partial^m}{\partial a^m}\left(\frac{q}{q-e^{-ah}}\right)$
e^{-sh}	q^{-1}
$\frac{a}{s+a}$	$\frac{1-\exp(-ah)}{q-\exp(-ah)}$
$\frac{a}{s(s+a)}$	$b_1=\frac{1}{a}(ah-1+e^{-ah})$ $\quad b_2=\frac{1}{a}(1-e^{-ah}-ahe^{-ah})$ $a_1=-(1+e^{-ah})$ $\quad a_2=e^{-ah}$
$\frac{a^2}{(s+a)^2}$	$b_1=1-e^{-ah}(1+ah)$ $\quad b_2=e^{-ah}(e^{-ah}+ah-1)$ $a_1=-2e^{-ah}$ $\quad a_2=e^{-2ah}$
$\frac{s}{(s+a)^2}$	$\frac{(q-1)he^{-ah}}{(q-e^{-ah})^2}$
$\frac{ab}{(s+a)(s+b)}$ $a\neq b$	$b_1=\frac{b(1-e^{-ah})-a(1-e^{-bh})}{b-a}$ $b_2=\frac{a(1-e^{-bh})e^{-ah}-b(1-e^{-ah})e^{-bh}}{b-a}$ $a_1=-(e^{-ah}+e^{-bh})$ $a_2=e^{-(a+b)h}$

$$H(z) = \frac{b_1 z + b_2}{z^2 + a_1 z + a_2}$$

return



$G(s)$	$H(q)$ or the coefficients in $H(q)$
$\frac{(s+c)}{(s+a)(s+b)}$ $a \neq b$	$b_1 = \frac{e^{-bh} - e^{-ah} + (1-e^{-bh})c/b - (1-e^{-ah})c/a}{a-b}$ $b_2 = \frac{c}{ab} e^{-(a+b)h} + \frac{b-c}{b(a-b)} e^{-ah} + \frac{c-a}{a(a-b)} e^{-bh}$ $a_1 = -e^{-ah} - e^{-bh} \qquad a_2 = e^{-(a+b)h}$
$\frac{\omega_0^2}{s^2 + 2\zeta\omega_0 s + \omega_0^2}$	$b_1 = 1 - \alpha\left(\beta + \frac{\zeta\omega_0}{\omega}\gamma\right) \qquad \omega = \omega_0\sqrt{1-\zeta^2} \qquad \zeta < 1$ $b_2 = \alpha^2 + \alpha\left(\frac{\zeta\omega_0}{\omega}\gamma - \beta\right) \qquad \alpha = e^{-\zeta\omega_0 h}$ $a_1 = -2\alpha\beta \qquad \beta = \cos(\omega h)$ $a_2 = \alpha^2 \qquad \gamma = \sin(\omega h)$
$\frac{s}{s^2 + 2\zeta\omega_0 s + \omega_0^2}$	$b_1 = \frac{1}{\omega} e^{-\zeta\omega_0 h} \sin(\omega h) \qquad b_2 = -b_1$ $a_1 = -2e^{-\zeta\omega_0 h}\cos(\omega h) \qquad a_2 = e^{-2\zeta\omega_0 h}$ $\omega = \omega_0\sqrt{1-\zeta^2}$
$\frac{a^2}{s^2 + a^2}$	$b_1 = 1 - \cos ah \qquad b_2 = 1 - \cos ah$ $a_1 = -2\cos ah \qquad a_2 = 1$
$\frac{s}{s^2 + a^2}$	$b_1 = \frac{1}{a}\sin ah \qquad b_2 = -\frac{1}{a}\sin ah$ $a_1 = -2\cos ah \qquad a_2 = 1$
$\frac{a}{s^2(s+a)}$	$b_1 = \frac{1-\alpha}{a^2} + h\left(\frac{h}{2} - \frac{1}{a}\right) \qquad \alpha = e^{-ah}$ $b_2 = (1-\alpha)\left(\frac{h^2}{2} - \frac{2}{a^2}\right) + \frac{h}{a}(1+\alpha)$ $b_3 = -\left[\frac{1}{a^2}(\alpha - 1) + \alpha h\left(\frac{h}{2} + \frac{1}{a}\right)\right]$ $a_1 = -(\alpha + 2) \qquad a_2 = 2\alpha + 1 \qquad a_3 = -\alpha$



Entry #	Laplace Domain	Time Domain	Z Domain (t=kT)
1	1	$\delta(t)$ unit impulse	1
2	$\frac{1}{s}$	$u(t)$ unit step	$\frac{z}{z-1}$
3	$\frac{1}{s^2}$	t	$\frac{Tz}{(z-1)^2}$
4	$\frac{1}{s+a}$	e^{-at}	$\frac{z}{z-e^{-aT}}$
5		$b^k \quad (b = e^{-aT})$	$\frac{z}{z-b}$
6	$\frac{1}{(s+a)^2}$	te^{-at}	$\frac{Tze^{-aT}}{\left(z-e^{-aT}\right)^2}$
7	$\frac{1}{s(s+a)}$	$\frac{1}{a}\left(1-e^{-at}\right)$	$\frac{z\left(1-e^{-aT}\right)}{a(z-1)\left(z-e^{-aT}\right)}$
8	$\frac{b-a}{(s+a)(s+b)}$	$e^{-at}-e^{-bt}$	$\frac{z\left(e^{-aT}-e^{-bT}\right)}{\left(z-e^{-aT}\right)\left(z-e^{-bT}\right)}$
9	$\frac{1}{s(s+a)(s+b)}$	$\frac{1}{ab}-\frac{e^{-at}}{a(b-a)}-\frac{e^{-bt}}{b(a-b)}$	
10	$\frac{1}{s(s+a)^2}$	$\frac{1}{a^2}\left(1-e^{-at}-ate^{-at}\right)$	
11	$\frac{s}{(s+a)^2}$	$(1-at)e^{-at}$	
12	$\frac{b}{s^2+b^2}$	$\sin(bt)$	$\frac{z\sin(bT)}{z^2-2z\cos(bT)+1}$
13	$\frac{s}{s^2+b^2}$	$\cos(bt)$	$\frac{z(z-\cos(bT))}{z^2-2z\cos(bT)+1}$
14	$\frac{b}{(s+a)^2+b^2}$	$e^{-at}\sin(bt)$	$\frac{ze^{-aT}\sin(bT)}{z^2-2ze^{-aT}\cos(bT)+e^{-2aT}}$
15	$\frac{s+a}{(s+a)^2+b^2}$	$e^{-at}\cos(bt)$	$\frac{z^2-ze^{-aT}\cos(bT)}{z^2-2ze^{-aT}\cos(bT)+e^{-2aT}}$

return

return