MU FEI A0260032Y HW1 EE5103

Q1

a) $u(t) = \frac{di(t)}{dt} + i + \int i\,dt$, $y(t) = \int i\,dt \Rightarrow \frac{dy}{dt} = i \Rightarrow \frac{d^2y}{dt^2} = \frac{di}{dt}$

$\Rightarrow \ddot{y}(t) + \dot{y}(t) + y = u$ Assume the initial conditions are zeros,

$\Rightarrow s^2 Y(s) + sY(s) + Y(s) = U(s) \Rightarrow \frac{Y(s)}{U(s)} = \frac{1}{s^2+s+1}$

b) $\dot{x}_1(t) = \dot{e}_0 = x_2(t)$

$\dot{x}_2(t) + x_2(t) + x_1(t) = u(t) \Rightarrow \dot{x}_2(t) = -x_1(t) - x_2(t) + u(t)$

From $\cancel{\Phi}$ $\dot{x} = Ax + Bu$ $y = Cx$

$\Rightarrow \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \qquad y = \begin{bmatrix} 1 & 0 \end{bmatrix} \cancel{\Phi} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$\Rightarrow \dot{x} = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \qquad y = \begin{bmatrix} 1 & 0 \end{bmatrix} x$

c) From $x(k+1) = \Phi x(k) + \Gamma u(k)$ $y(k) = Cx(k)$

$\because \Phi = e^{A} = L^{-1}[sI - A]^{-1}\big|_{t=1}$

$\because sI - A = \begin{pmatrix} s & -1 \\ 1 & s+1 \end{pmatrix} \Rightarrow (sI - A)^{-1} = \begin{pmatrix} \frac{s+1}{s^2+s+1} & \frac{1}{s^2+s+1} \\ \frac{-1}{s^2+s+1} & \frac{s}{s^2+s+1} \end{pmatrix}$

$$e^{At} \Rightarrow L^{-1}(sI-A)^{-1} = L^{-1}\begin{pmatrix} \frac{s+\frac{1}{2}+\frac{1}{\sqrt{3}}\cdot\left(\frac{\sqrt{3}}{2}\right)}{(s+\frac{1}{2})^2+(\frac{\sqrt{3}}{2})^2} & \frac{\frac{2}{\sqrt{3}}\cdot\left(\frac{\sqrt{3}}{2}\right)}{(s+\frac{1}{2})^2+(\frac{\sqrt{3}}{2})^2} \\ \frac{-\frac{2}{\sqrt{3}}\cdot\left(\frac{\sqrt{3}}{2}\right)}{(s+\frac{1}{2})^2+(\frac{\sqrt{3}}{2})^2} & \frac{s+\frac{1}{2}-\frac{1}{\sqrt{3}}\cdot\left(\frac{\sqrt{3}}{2}\right)}{(s+\frac{1}{2})^2+(\frac{\sqrt{3}}{2})^2} \end{pmatrix}$$

$$= \begin{pmatrix} e^{-\frac{1}{2}t}\cdot\cos(\frac{\sqrt{3}}{2}t) + \frac{1}{\sqrt{3}}\cdot e^{-\frac{1}{2}t}\cdot\sin(\frac{\sqrt{3}}{2}t) & \frac{2}{\sqrt{3}}e^{-\frac{1}{2}t}\cdot\sin(\frac{\sqrt{3}}{2}t) \\ -\frac{2}{\sqrt{3}}e^{-\frac{1}{2}t}\cdot\sin(\frac{\sqrt{3}}{2}t) & e^{-\frac{1}{2}t}\cdot\cos(\frac{\sqrt{3}}{2}t) - \frac{1}{\sqrt{3}}\cdot e^{-\frac{1}{2}t}\cdot\sin(\frac{\sqrt{3}}{2}t) \end{pmatrix}$$

$\therefore \Phi = \begin{pmatrix} 0.6597 & 0.5335 \\ -0.5335 & 0.1262 \end{pmatrix} \qquad \Gamma = \int_0^1 e^{Av}dv \cdot B = \begin{pmatrix} 0.8738 & 0.3403 \\ -0.3403 & 0.5335 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0.3403 \\ 0.5335 \end{pmatrix}$

$\therefore$ $\cancel{x(k+1) =}$ $C = \begin{bmatrix} 1 & 0 \end{bmatrix}$

d) Based on c), $z\cdot\left(X(z)-x(0)\right) = \Phi X(z) + \Gamma U(z)$ $Y(z) = CX(z)$

$$\Rightarrow Y(z) = CX(z) = C\left[(zI-\Phi)^{-1} z\cdot x(0) + (zI-\Phi)^{-1}\Gamma U(z)\right]$$

Assume $x(0)=0 \Rightarrow Y(z) = C\cdot(zI-\Phi)^{-1}\Gamma U(z)$

$$H(z) = C\cdot(zI-\Phi)^{-1}\Gamma = (1\ \ 0)\cdot\frac{1}{(z-0.6597)(z-0.1262)+0.5335^2}\begin{pmatrix} z-0.1262 & 0.5335 \\ -0.5335 & z-0.6597\end{pmatrix}\begin{pmatrix}0.3403\\0.5335\end{pmatrix}$$

$$= \frac{0.3403z+0.2417}{z^2-0.7859z+0.3679}$$

$$\Rightarrow y(k+2) - 0.7859y(k+1) + 0.3679y(k) = 0.3403u(k+1) + 0.2417u(k)$$

$$\Rightarrow y(k+1) = 0.7859y(k) - 0.3679y(k-1) + 0.3403u(k) + 0.2417u(k-1),\ k\geq 1$$

e) $\because u(k)=1|_{k\geq 0} \Rightarrow U(z) = \frac{z}{z-1}$ $x(0) = \begin{pmatrix}x_1(0)\\x_2(0)\end{pmatrix} = \begin{pmatrix}y(0)\\\dot{y}(0)\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix}$

$$\therefore Y(z) = C\cdot(zI-\Phi)^{-1}\cdot\left[z\cdot x(0) + \Gamma U(z)\right]$$

$$= (1\ \ 0)\cdot\frac{1}{z^2-0.7859z+0.3679}\begin{pmatrix} z-0.1262 & 0.5335 \\ -0.5335 & z-0.6597\end{pmatrix}\cdot\begin{pmatrix} z+0.3403\cdot\frac{z}{z-1} \\ 0.5335\cdot\frac{z}{z-1}\end{pmatrix}$$

$$= \frac{z^3-0.7860z^2+0.3679z}{z^3-1.7859z^2+1.1538z-0.3679} = \frac{z}{z-1}\cdot\frac{(z^2-0.7859z+0.3679)}{(z^2-0.7859z+0.3679)} = \frac{z}{z-1}$$

$$\cancel{\text{then } \frac{Y(z)}{z} = \frac{z^2+0.2141z-0.6321}{z^3-1.7859z^2+1.1538z-0.3679} = \frac{a_2}{z-a_1} + \frac{b_2}{z-b_1} + \frac{c_2}{z-c_1}}$$

$$\cancel{y(k) = a_2\cdot a_1^k + b_2\cdot b_1^k + c_2\cdot c_1^k}$$

$$y(k) = u(k),\ k\geq 0.$$

Q_2.

a) $G(s) = \frac{1}{s(s-1)} \Rightarrow$ no zeros ; $s=0, s=1$ are poles

$\because s=1>0$, it's not stable. because $G(s)=\frac{1}{s-1}-\frac{1}{s} \Rightarrow G(t)=e^{t}-u(t)$

Goes to infinity.

$H(s)=\frac{1}{G(s)} = s(s-1) \Rightarrow$ no poles $\Rightarrow$ it's stable.

because for inverse system, $\frac{Y(s)}{X(s)} = s(s-1) \Rightarrow y(t)=\frac{d^2x(t)}{dt^2}-\frac{dx(t)}{dt}$

for any physical $x(t)$, ~~$y(t)$~~ the output $y(t)$ is finite.

b) $G(s)=\frac{1}{s-1}-\frac{1}{s} \Rightarrow G(z)=\frac{z}{z-e^T}-\frac{z}{z-1}=\frac{(e^T-1)z}{(z-e^T)(z-1)}$

poles : $z=e^T, z=1$ ~~beca~~ $\because T>0$, $e^T>1$, it's not stable.

so we choose any sampling period "h" and the system is unstable.

c) $H(z)=\frac{1}{G(z)}=\frac{(z-e^T)(z-1)}{z\cdot(e^T-1)}$ poles: $z=0$. it's stable.

so we choose any "h" and it's stable.

Q_3:

a) $\Phi = \begin{pmatrix} 0.1 & 0.2 \\ 0.1 & 0.2 \end{pmatrix} \Rightarrow \lambda I-\Phi = \begin{pmatrix} \lambda-0.1 & -0.2 \\ -0.1 & \lambda-0.2 \end{pmatrix}$

let $\det(\lambda I-\Phi)=0 \Rightarrow \lambda_1=0.3, \lambda_2=0 \Rightarrow$ poles: $z_1=0.3, z_2=0 \Rightarrow$ stable.

$W_c = [\Gamma \ \ \Phi\Gamma] = \begin{bmatrix} 1 & 0.1 \\ 0 & 0.1 \end{bmatrix} \Rightarrow$ full rank $\Rightarrow$ controllable

$W_o = \begin{bmatrix} C \\ C\Phi \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0.1 & 0.2 \end{bmatrix} \Rightarrow$ full rank $\Rightarrow$ obserable.

b) we can write $x_1(k+1)=0.1x_1(k)+0.2x_2(k)+u(k)$ ①

$x_2(k+1)=0.1x_1(k)+0.2x_2(k)$ ②

$y(k)=x_1(k)$ ③

$x_1(k+2)=0.1x_1(k+1)+0.2x_2(k+1)+u(k+1)$ ④

$5\times$④ $-$ ①, then $\Rightarrow 5y(k+2)-y(k+1)=0.5y(k+1)+5u(k+1)-u(k)$

$\Rightarrow 5z^2\cdot Y(z)-1.5zY(z)=5z\cdot U(z)-U(z)$

$$\frac{Y(z)}{U(z)}=\frac{5z-1}{5z^2-1.5z}$$

c) From b), $5y(k+2) - 1.5y(k+1) = 5K u_c(k+1) - 5K\cdot y(k+1) - K\cdot u_c(k) + K\cdot y(k)$

$$5z^2\cdot Y(z) - 1.5z\cdot Y(z) + 5Kz\cdot Y(z) - K\cdot Y(z) = 5Kz\cdot U_c(z) - K\cdot U_c(z)$$

$$\Rightarrow \frac{Y(z)}{U_c(z)} = \frac{5Kz - K}{5z^2 + (5K-1.5)z - K}$$

d)

a_0	a_1	a_2
5	$5K-1.5$	$-K$
$5-\frac{K^2}{5}$	$(1+\frac{K}{5})(5K-1.5)$	0
$5-\frac{K^2}{5}-\frac{(1+\frac{K}{5})^2(5K-1.5)^2}{5-\frac{K^2}{5}}$	0	0

$$\Rightarrow \begin{cases} 5-\frac{K^2}{5} > 0 \\ 5-\frac{K^2}{5}-\frac{(1+\frac{K}{5})^2(5K-1.5)^2}{5-\frac{K^2}{5}} > 0 \end{cases}$$

$$\Rightarrow \begin{cases} |K| < 5 \\ (0.8K^2+4.7K+3.5)(-1.2K^2-4.7K+6.5) > 0 \end{cases} \Rightarrow -0.875 < K < 1.083$$

e) $u_c(k) = 1|_{k\geq 0} \Rightarrow U(z) = \frac{z}{z-1}$

$$Y(z) = \frac{z}{z-1}\cdot\frac{5Kz-K}{5z^2+(5K-1.5)z-K} \Rightarrow U(z)-Y(z) = \frac{z}{z-1}\cdot\frac{5z^2-1.5z}{5z^2+(5K-1.5)z-K}$$

$$\therefore \frac{U(z)-Y(z)}{z} = \frac{1}{z-1}\cdot\frac{5z^2-1.5z}{5z^2+(5K-1.5)z-K}$$

$$= \frac{3.5}{4K+3.5}\cdot\frac{1}{z-1} + \left(\frac{20K}{4K+3.5}z - \frac{3.5K}{4K+3.5}\right)\cdot\frac{1}{5z^2+(5K-1.5)z-K}$$

$$= \frac{3.5}{4K+3.5}\cdot\frac{1}{z-1} + \frac{b_2}{z-b_1} + \frac{c_2}{z-c_1}$$

$$\Rightarrow [u_c - y](k) = \frac{3.5}{4K+3.5}\cdot u(k) + b_2\cdot b_1^k + c_2\cdot c_1^k$$

Define $E(z) = U(z) - Y(z)$

$$\lim_{t\to\infty} E(t) = \lim_{z\to 1}(z-1)\cdot E(z) = \frac{z\cdot(5z^2-1.5z)}{5z^2+(5K-1.5)z-K} = \frac{3.5}{4K+3.5}$$

Q4.

a) $z \cdot Y(z) = -Y(z) + z^{-1} Y(z) + z^{-2} Y(z) + U(z) + 2U(z) \cdot z^{-1} + U(z) \cdot z^{-2}$

$$\frac{Y(z)}{U(z)} = \frac{1+2z^{-1}+z^{-2}}{z+1-z^{-1}-z^{-2}} = \frac{(z+1)^2}{(z+1)^2(z-1)}$$

poles : $z_1 = -1$, $z_2 = -1$, $z_3 = 1$; $\because$ multiple poles on $|z|=1$, It's not stable.

Inverse poles : $z_1 = -1$, $z_2 = -1$, $\because$ multiple poles on $|z|=1$, it's not stable.

b) Define $x_1(k) = y(k)$, $x_2(k) = y(k-1) + y(k-2) + 2u(k-1) + u(k-2)$

$x_3(k) = y(k-1) + u(k-1)$

$\Rightarrow$ $x_1(k+1) = -x_1(k) + x_2(k) + u(k)$

$x_2(k+1) = x_1(k) + x_3(k) + 2u(k)$

$x_3(k+1) = x_1(k) + u(k)$

$$\therefore \; x(k+1) = \begin{pmatrix} -1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} x(k) + \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} u(k), \quad y(k) = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} x(k)$$

$$W_c = [\Gamma \;\; \Phi\Gamma \;\; \Phi^2\Gamma] = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 1 & 1 & 1 \end{pmatrix} \Rightarrow \text{not full rank} \Rightarrow \text{not controlable}$$

$$W_o = \begin{bmatrix} C \\ C\Phi \\ C\Phi^2 \end{bmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & -1 & 1 \end{pmatrix} \Rightarrow \text{full rank} \Rightarrow \cancel{\text{oberv}} \text{ observable}$$