Chapter 9 Servo Control

§9.1 Introduction

What has been discussed? What is missing?

State feedback may be designed by pole placement, LQR and decoupling, all of which are related to transient performance. One notes

$$\text{Control Specifications}\begin{cases} \textit{Transient} \quad \textit{Accuracy}: \\ \quad e(t) = r(t) - y(t) \quad 0 \le t < \infty \\ \textit{Steady} \quad \textit{State} \quad \textit{Accuracy}: \\ \quad e(\infty) \end{cases}$$

How to achieve $e(\infty) = 0$

in face of disturbance w(t) and set-point change r(t)?

Asymptotic Tracking We wish $y(t) = r(t)$, $t \geq 0$, but this is impossible. What can be achieved is that for $r(t) \neq 0$ and $w(t) = 0$, there holds

$$\lim_{t\to\infty} e(t) = \lim_{t\to\infty}(r(t) - y(t)) = 0 .$$

This is called the asymptotic tracking.

Asymptotic Regulation (Disturbance Rejection) For $r(t) = 0$, $w(t) \neq 0$, if there holds

$$\lim_{t\to\infty} e(t) = \lim_{t\to\infty}(r(t) - y(t)) = \lim_{t\to\infty}(-y(t)) = 0 ,$$

it is called the asymptotic regulation.

If both asymptotic tracking and regulation are required, it is called the servo control problem.

How did we do in the classical SISO servo control with step inputs?

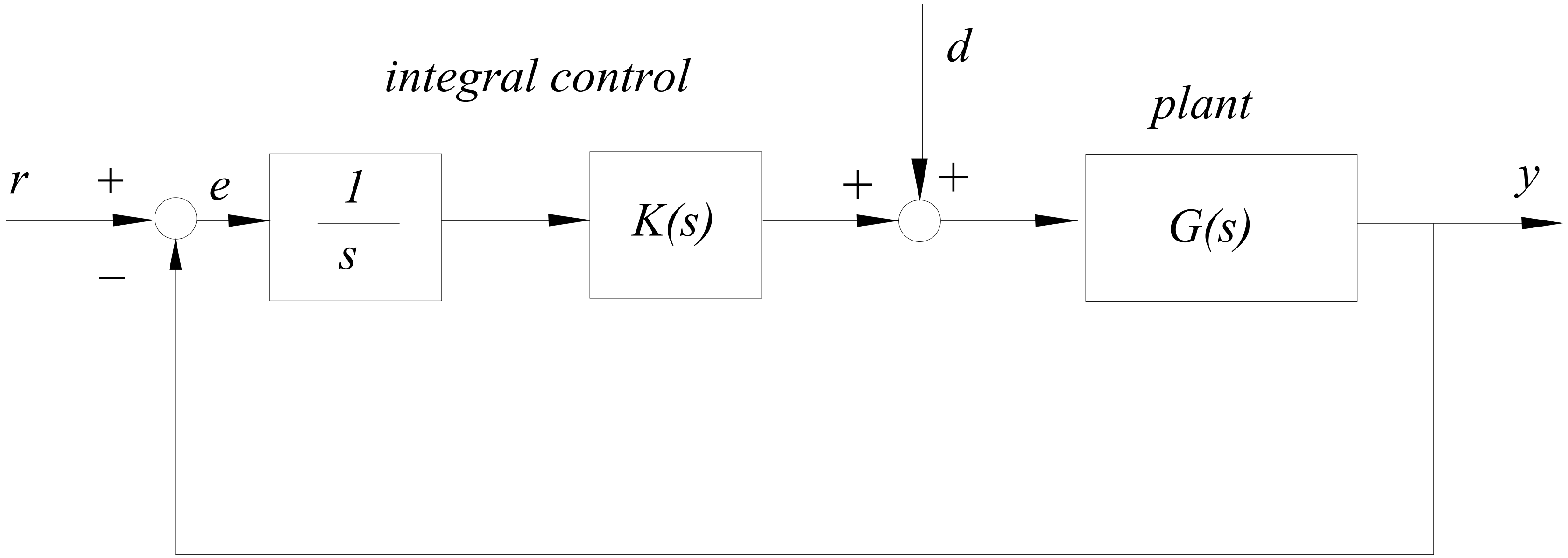

Figure 1 Classical integral control for SISO system.

An integral control can achieve a zero steady-state error in response to step inputs, as long as the closed-loop system is stable. Two key elements to achieve zero steady-state error in this case are

- an integrator: being a right servo mechanism for step inputs
- stabilization: enabling the servo mechanism to function in end.

The objective of this chapter is to extend this notion to

- General type of inputs
- Multivariable systems

It is possible only if we can have

- a right servo mechanism for any given input type;
- stabilization technique for a general plant.

That is,

Design = Servo mechanism + Stabilization!

We present

- Output feedback
- State feedback

Industrial Motivation: Antenna Servo Control

Process:

Model: It is desired to control the elevation of an antenna designed to track a satellite. Let the antenna and drive parts have a moment of inertia J and damping B. The equation of motion is given by

$$J\ddot{\theta} + B\dot{\theta} = T_c + T_d,$$

where θ is the angle, T_c is the net torque from the drive motor and T_d is the disturbance torque due to wind. By defining

$$a = \frac{B}{J},\ u = \frac{T_c}{B},\ d = \frac{T_d}{B},$$

the equation reduces to

$$\theta(s) = \frac{1}{s(s/a+1)}u(s) + \frac{1}{s(s/a+1)}d(s),\ \text{a=0.1.}$$

Control: The design objective is to achieve servo control when both the reference and disturbance are step signals.

§9.2 Polynomial Approach to General SISO Servo Problem

Consider the unity output feedback system.

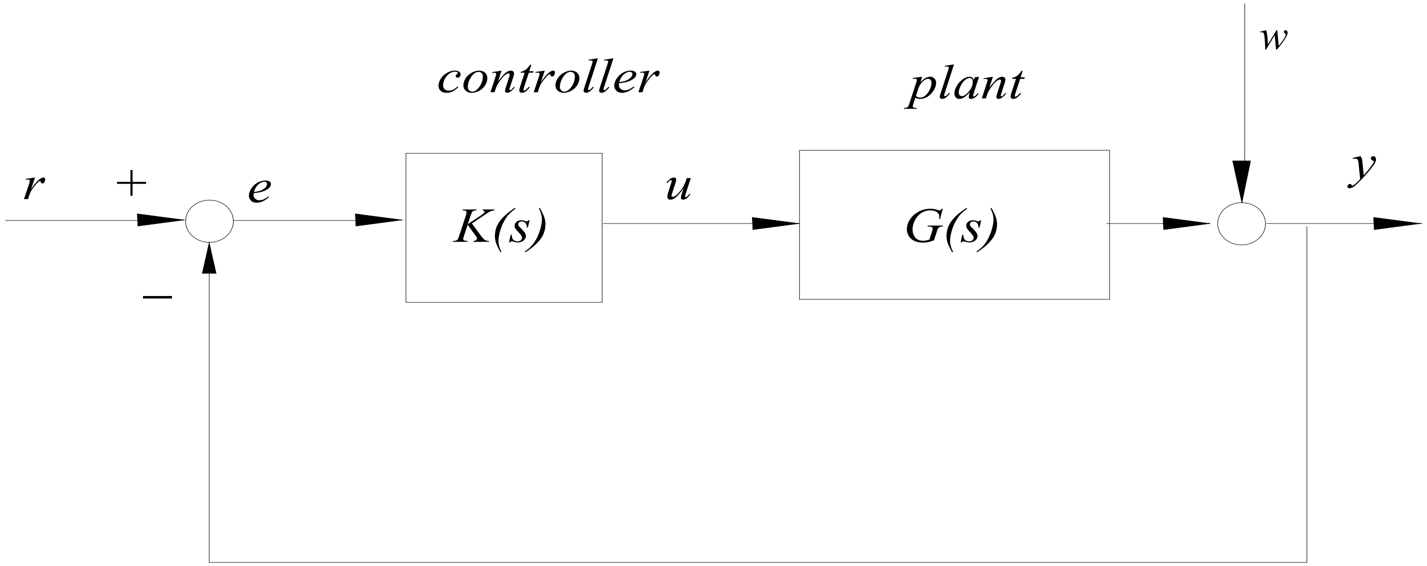

Figure 2 Unity output feedback system.

Objective: $e(\infty)=0$ ***in face of r and w***

Let

$$G(s) = \frac{N(s)}{D(s)} \text{ and } K(s) = \frac{B(s)}{A(s)}.$$

The closed-loop transfer function between y and r is

$$H = \frac{GK}{1+GK} = \frac{NB}{DA+NB}.$$

The feedback system is stable if and only if all the roots of its characteristic polynomial,

$$[D(s)A(s) + N(s)B(s)],$$

have negative real parts.

Pole Placement

Example 1. Let

$$G(s) = \frac{1}{s(s+2)}.$$

If $K(s)$ is a gain of k, then

$$H(s) = \frac{kG(s)}{1+kG(s)} = \frac{k}{s^2+2s+k}.$$

Its two poles cannot be arbitrarily assigned by choosing a value for k. For example, Let

$$s^2+2s+k = (s+2)(s+3) = s^2+5s+6.$$

Clearly, there is no k to meet the equation.

What to do? Increase controller order!

Next let

$$K(s) = \frac{B_0 + B_1 s}{A_0 + A_1 s},$$

with $A_1 \neq 0$. Then

$$H(s) = \frac{G(s)K(s)}{1 + G(s)K(s)}$$

$$= \frac{B_0 + B_1 s}{s(s+2)(A_1 s + A_0) + B_1 s + B_0}$$

$$= \frac{B_1 s + B_0}{A_1 s^3 + (2A_1 + A_0)s^2 + (2A_0 + B_1)s + B_0}.$$

The goal is to make the denominator of $H(s)$ equal to

$$P_c(s) = s^3 + F_2 s^2 + F_1 s + F_0,$$

where F_i are entirely arbitrary to have any desired roots. This leads to

$$A_1 s^3 + (2A_1 + A_0)s^2 + (2A_0 + B_1)s + B_0$$
$$= s^3 + F_2 s^2 + F_1 s + F_0$$

giving

$$A_1 = 1, \quad 2A_1 + A_0 = F_2,$$
$$2A_0 + B_1 = F_1, \quad B_0 = F_0,$$

or

$$A_1 = 1, \quad A_0 = F_2 - 2A_1,$$
$$B_1 = F_1 - 2F_2 + 4A_1, \quad B_0 = F_0.$$

A solution always exists for arbitrary $P_c(s)$. For example, if we assign the three poles of $H(s)$ as *-2* and $-2 \pm 2j$, then $P_c(s)$ becomes

$$\begin{aligned}P_c(s) &= s^3 + F_2 s^2 + F_1 s + F_0 \\ &= (s+2)(s+2+2j)(s+2-2j) \\ &= s^3 + 6s^2 + 16s + 16.\end{aligned}$$

For this set of poles, we have

$$A_1 = 1, \quad A_0 = 6 - 2 = 4,$$

$$B_1 = 16 - 2 \cdot 6 + 4 = 8, \quad B_0 = 16,$$

and

$$K(s) = \frac{8s + 16}{s + 4}.$$

This compensator will place the poles of $H(s)$ at *-2* and $-2 \pm 2j$. One may verify

$$H(s) = \frac{G(s)K(s)}{1+G(s)K(s)} = \frac{8s+16}{s^3+6s^2+16s+16}.$$

Indeed, $H(s)$ has poles at -2 and $-2 \pm 2j$.

What is the lesson learnt here?

Higher order of controller

>>> more free parameters

>>> better chance for pole placement!

But further Questions arise:

- Is pole placement always possible with a sufficiently high-order controller?
- What is the controller order?

We now look at the general case and establish the condition for achieving pole placement.

Let

$$G(s)=\frac{N(s)}{D(s)},\quad K(s)=\frac{B(s)}{A(s)},\quad H(s)=\frac{N_c(s)}{P_c(s)}$$

and $\deg N(s)\le \deg D(s)=n$. It follows that

$$H(s)=\frac{G(s)K(s)}{1+G(s)K(s)}=\frac{\dfrac{N(s)}{D(s)}\dfrac{B(s)}{A(s)}}{1+\dfrac{N(s)}{D(s)}\dfrac{B(s)}{A(s)}}$$

or

$$H(s)=\frac{N(s)B(s)}{D(s)A(s)+N(s)B(s)}. \tag{1}$$

The pole-placement problem is equivalent to solving

$$D(s)A(s) + N(s)B(s) = P_c(s), \tag{2}$$

where $D(s)$ and $N(s)$ are known, the roots of $P_c(s)$ are the desired poles of the overall system to achieve, and $A(s)$ and $B(s)$ are unknown polynomials to be determined.

Q1: Are there solutions?

Q2: What are the solution controller and its order?

Q3: Are solutions realizable (proper)?

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Coprimeness of two polynomials

Definition:

$\alpha(s)$ and $\beta(s)$ are coprime if they have no common factors, or roots.

Examples:

- $p_1 = (s+1)(s+2)$ and $p_2 = (s+3)(s+4)$ are coprime since they have no common factors or roots.

- $p_1 = (s+1)(s+2)$ and $p_2 = (s+2)(s+3)$ are NOT coprime since they have a common factor $(s+2)$.

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For Q1, if $D(s)$ and $N(s)$ both contain the factor $(s-2)$ or $D(s)=(s-2)\overline{D}(s)$ and $N(s)=(s-2)\overline{N}(s)$, then (2) becomes

$$
\begin{aligned}
&D(s)A(s)+N(s)B(s)\\
&=(s-2)[\overline{D}(s)A(s)+\overline{N}(s)B(s)]\\
&=P_c(s)
\end{aligned}
$$

$P_c(s)$ must contain the same common factor $(s-2)$. Thus, if $N(s)$ and $D(s)$ have a common factor, then not every root of $P_c(s)$ can be arbitrarily assigned. Therefore we assume from now on that $D(s)$ and $N(s)$ are coprime.

Coprimeness: necessary condition for Q1, later also sufficient

For Q2, Let

$$D(s) := D_0 + D_1 s + D_2 s^2 + \cdots + D_n s^n, \quad D_n \neq 0 \tag{3a}$$

$$N(s) := N_0 + N_1 s + N_2 s^2 + \cdots + N_n s^n, \tag{3b}$$

$$A(s) := A_0 + A_1 s + A_2 s^2 + \cdots + A_m s^m, \tag{4a}$$

$$B(s) := B_0 + B_1 s + B_2 s^2 + \cdots + B_m s^m, \tag{4b}$$

where D_i, N_i, A_i, B_i are all real numbers. Because $\deg P_c(s) = n + m$,

$$P_c(s) := F_0 + F_1 s + F_2 s^2 + \cdots + F_{n+m} s^{n+m}, \tag{5}$$

The substitution of these into (2) yields

$$
\begin{aligned}
&(D_0 + D_1 s + \cdots + D_n s^n)(A_0 + A_1 s + \cdots + A_m s^m) \\
&+(N_0 + N_1 s + \cdots + N_n s^n)(B_0 + B_1 s + \cdots + B_m s^m) \\
&= F_0 + F_1 s + \cdots + F_{n+m} s^{n+m},
\end{aligned}
$$

or

$$
\begin{aligned}
&(D_0 A_0 + N_0 B_0) + (D_1 A_0 + N_1 B_0 + D_0 A_1 + N_0 B_1)s \\
&+\cdots + (D_n A_m + N_n B_m)s^{n+m} = F_0 + F_1 s + \cdots + F_{n+m} s^{n+m}.
\end{aligned}
$$

or

$$
\begin{aligned}
D_0 A_0 + N_0 B_0 &= F_0\,, \\
D_1 A_0 + N_1 B_0 + D_0 A_1 + N_0 B_1 &= F_1\,, \\
&\vdots \\
D_n A_m + N_n B_m &= F_{n+m}\,.
\end{aligned}
$$

There are a total of $n+m+1$ equations, which can be re-written as

$$\underbrace{\begin{bmatrix} D_0 & N_0 & 0 & 0 & & 0 & 0 \\ D_1 & N_1 & D_0 & N_0 & & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & & 0 & 0 \\ D_n & N_n & D_{n-1} & N_{n-1} & \cdots & D_0 & N_0 \\ 0 & 0 & D_n & N_n & & D_1 & N_1 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & 0 & & D_n & N_n \end{bmatrix}}_{S_m} \begin{bmatrix} A_0 \\ B_0 \\ A_1 \\ B_1 \\ \vdots \\ A_m \\ B_m \end{bmatrix} = \begin{bmatrix} F_0 \\ F_1 \\ F_2 \\ \vdots \\ F_{m+n} \end{bmatrix}, \quad (6)$$

where the matrix S_m has

- *(m+n+1)* rows, being the number of equations; and
- *2(m+1)* columns, being number of unknowns

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Linear algebraic equation system

The equation system:

$$\Gamma\alpha = \beta$$

has a solution for α iff

$$Rank(\Gamma) = Rank\begin{bmatrix}\Gamma & \beta\end{bmatrix}$$

In particular, if β is arbitrary, the above becomes

$$Rank(\Gamma) = Rank\begin{bmatrix}\Gamma & \beta\end{bmatrix} = \text{full row rank}$$
$$= \text{No. of rows in } \Gamma$$

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Equation (6) has a solution for any $P_c(s)$ if and only if the matrix S_m has a **full row rank**. A necessary condition is that the number of rows or equations is smaller than or equal to the number of columns or unknowns:

$$n+m+1 \leq 2(m+1), \quad \text{or} \quad n-1 \leq m. \tag{7}$$

With $m \geq n-1$, it can be shown that the matrix S_m has a full row rank if and only if $D(s)$ and $N(s)$ are **coprime** or have no common factors. In this case, if $m=n-1$, the matrix S_m is a square matrix of order $2n$, and for every $P_c(s)$, the solution of (6) is unique. If $m \geq n$, then (6) has more unknowns than equations and solutions of (6) are not unique.

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Properness of Rational Functions

- **A *rational function*** is in form of $G(s) = \dfrac{\beta(s)}{\alpha(s)}$, where α, β are both polynomials.
- A rational function $G(s) = \dfrac{\beta(s)}{\alpha(s)}$ is called **proper** if $\lim_{s\to\infty} G(s) < \infty$, or $\deg(\alpha) \geq \deg(\beta)$. For example, $G_1(s) = \dfrac{s+2}{s+1}$ is proper while $G_2(s) = s$ is not proper
- A rational function $G(s)$ is called **strictly proper** if $\lim_{s\to\infty} G(s) = 0$, or $\deg(\alpha) > \deg(\beta)$. For example, $G_1(s) = \dfrac{s+2}{s+1}$ is proper but not strictly proper. $G_3(s) = \dfrac{1}{s+1}$ is strictly proper.

Realizability of Controllers. $K(s)$ is physically realizable iff $K(s)$ is proper. Thus, an improper function is not realizable. This is best illustrated in the discrete time domain.

- Consider an improper controller:

$$K(z) = z$$

$$U(z) = K(z)E(z) = zE(z)$$

$$u(k) = e(k+1)$$

so that the present output $u(k)$ of such a controller makes use of the future input $e(k+1)$ which is not available yet at the present. It is thus not possible to implement such a controller in real world.

- With a similar argument, the present output of a proper controller will make use of the present and past inputs, so available and realizable.

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For Q3 on the realizability of the compensator. If the plant $G(s)$ is strictly **proper**, then $N_n = 0$ and $D_n \neq 0$. The last equation of (6) becomes

$$A_m D_n + B_m N_n = A_m D_n = F_{n+m},$$

which implies

$$A_m = \frac{F_{n+m}}{D_n}, \qquad D_n \neq 0.$$

Thus if $F_{n+m} \neq 0$, then $A_m \neq 0$ and the compensator $K(s) = B(s)/A(s)$ is proper.

Theorem 1 *Consider the unity-feedback system shown in Figure 2 with a proper plant transfer function* $G(s) = N(s)/D(s)$ *with* $\deg D(s) = n$, *and* $N(s)$ *and* $D(s)$ *coprime. If* $m \geq n-1$, *then for any polynomial* $P_c(s)$ *of degree* $(n+m)$, *a proper compensator* $K(s) = B(s)/A(s)$ *of order m exists to achieve arbitrary pole placement. If* $m = n-1$, *the compensator is unique. If* $m \geq n$, *the compensators are not unique.*

Has the theorem given answers to three Qs raised before? Yes:

- Q1: Are there solutions? *Yes if* $N(s)$ *and* $D(s)$ *are coprime.*

- Q2: What are the solution controller and its order? *It should be at least the plant order minus one:* $m \geq n-1$.

- Q3: Are there realizable (proper) solutions? *Yes if the plant is strictly proper.*

Example 2. Let

$$G(s)=\frac{N(s)}{D(s)}=\frac{s-2}{(s+1)(s-1)}=\frac{\overset{N_0}{-2}+\overset{N_1}{s}+\overset{N_2}{0\cdot s^2}}{\underset{D_0}{-1}+\underset{D_1}{0\cdot s}+\underset{D_2}{s^2}}. \tag{8}$$

$D(s)$ and $N(s)$ have no common factor and $n=2$. Let a compensator of order $m=n-1=1$, be

$$K(s)=\frac{B_0+B_1 s}{A_0+A_1 s}.$$

We arbitrarily choose the three poles as -3, $-2\pm j$, so that

$$\begin{aligned} P_c(s) &:= (s+3)(s+2-j)(s+2+j) \\ &= \underset{F_0}{15}+\underset{F_1}{17s}+\underset{F_2}{7s^2}+\underset{F_3}{s^3}. \end{aligned}$$

Equation (6) becomes

$$\begin{bmatrix} D_0 & N_0 & 0 & 0 \\ D_1 & N_1 & D_0 & N_0 \\ D_2 & N_2 & D_1 & N_1 \\ 0 & 0 & D_2 & N_2 \end{bmatrix} \rightarrow \begin{bmatrix} -1 & -2 & 0 & 0 \\ 0 & 1 & -1 & -2 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} A_0 \\ B_0 \\ A_1 \\ B_1 \end{bmatrix} = \begin{bmatrix} 15 \\ 17 \\ 7 \\ 1 \end{bmatrix} \leftarrow \begin{bmatrix} F_0 \\ F_1 \\ F_2 \\ F_3 \end{bmatrix} \quad (9)$$

Its solution is

$$A_0 = \frac{79}{3}, \quad A_1 = 1, \quad B_0 = -\frac{62}{3}, \quad B_1 = -\frac{58}{3}.$$

It implies that

$$K(s) = \frac{-\frac{62}{3} - \frac{58}{3}s}{\frac{79}{3} + s} = \frac{-(58s + 62)}{3s + 79}$$

$$= \frac{-(19.3s + 20.7)}{s + 26.3} \qquad (10)$$

and the resulting overall system is

$$H(s) = \frac{-(58s + 62)(s - 2)}{3(s^3 + 7s^2 + 17s + 15)}.$$

Note that Theorem 1 is for pole placement, that is, we want to place closed-loop poles arbitrarily. But if we simply want to stabilize the plant, that is, to make

$$D(s)A(s)+N(s)B(s)=P_c(s) \tag{2}$$

stable, we can allow $D(s)$ and $N(s)$ have some stable common roots but no unstable common roots. One sees that

- if $D(s)$ and $N(s)$ both contain the factor $(s-2)$ or $D(s)=(s-2)\overline{D}(s)$ and $N(s)=(s-2)\overline{N}(s)$, then (2) becomes

$$\begin{aligned}&D(s)A(s)+N(s)B(s)\\&=(s-2)[\overline{D}(s)A(s)+\overline{N}(s)B(s)]\\&=P_c(s)\end{aligned}$$

  $P_c(s)$ must contain the same unstable common factor $(s-2)$ and thus always unstable;
- if $D(s)$ and $N(s)$ both contain the factor $(s+2)$ or $D(s)=(s+2)\overline{D}(s)$ and $N(s)=(s+2)\overline{N}(s)$, then (2) becomes

$$
\begin{aligned}
& D(s)A(s)+N(s)B(s) \\
&= (s+2)[\bar{D}(s)A(s)+\bar{N}(s)B(s)] \\
&= P_c(s)
\end{aligned}
$$

$P_c(s)$ must contain the same common stable factor $(s+2)$ but it can still be stable as long as $[A(s)\bar{D}(s)+B(s)\bar{N}(s)]$ is made stable by proper choice of A(s) and B(s), which is always possible if $\bar{D}(s)$ and $\bar{N}(s)$ are coprime.

Corollary 1 *Consider the unity-feedback system shown in Figure 2 with a proper plant transfer function* $G(s)=N(s)/D(s)$ *with* $\deg D(s)=n$ *and* $N(s)$ *and* $D(s)$ *having no common unstable root. Then, a proper compensator* $K(s)=B(s)/A(s)$ *exists to stabilize the plant.*

Have zero steady-state error to step input? No

Note from Example 2 that $H(0) = 124/45$. If we apply a unit-step reference input, the output will approach $124/45 = 2.76$. Thus the output of this overall system will not track asymptotically the step reference input.

Where are we now?

Recall that servo control design consists of

- servo mechanism determination and
- overall system stabilization.

We have just developed a solution for stabilization. We now need to determine a suitable servo mechanism to enable asymptotic tracking and disturbance rejection.

Servo mechanism determination

Suppose that the feedback system is stable. Let the reference and disturbance be $R(s) = \boldsymbol{L}[r(t)] = N_r(s)/D_r(s)$ and $W(s) = \boldsymbol{L}[w(t)] = N_w(s)/D_w(s)$, respectively. It is noted that the parts of $r(t)$ and $w(t)$ which **go to zero** as $t \to \infty$ have no effect on $y(t)$ as $t \to \infty$; Hence, assume that some roots of $D_r(s)$ and $D_w(s)$ have zero or positive real parts. Take the least common denominator of the unstable modes of $R(s)$ and $W(s)$, and assign it as a **polynomial** $Q(s)$ of degree q.

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Why the parts of $r(t)$ and $w(t)$ which go to zero as $t \to \infty$ have no effect on $y(t)$ as $t \to \infty$?

Let

$$R(s) = R^{+}(s) + R^{-}(s), R(s) = \frac{2}{(s+1)(s-1)} = \underbrace{\frac{1}{(s-1)}}_{\text{unstable}} - \underbrace{\frac{1}{(s+1)}}_{\text{stable}}$$

Then

$$Y(s) = H(s)R(s) = H(s)R^{+}(s) + H(s)R^{-}(s)$$

Let

$$Y^{-} := HR^{-} \qquad H, R^{-} \text{ are both stable}$$

Then, the final value theorem is applicable:

$$y^{-}(\infty) = \lim_{s\to 0} sH(s)R^{-}(s) = \lim_{s\to 0} s \cdot \lim_{s\to 0} H(s)R^{-}(s) = 0 \cdot a = 0$$

$$a < \infty.$$

Finding Q

Eg.1

$$R(s)=\frac{1}{s}\ ,\ W(s)=\frac{1}{s(s+1)}\ ,\quad \Rightarrow \quad Q=s$$

Eg.2

$$R(s)=\frac{1}{s^2}\ ,\ W(s)=\frac{1}{s}\ ,\quad \Rightarrow \quad Q=s^2$$

Eg.3 $r(t)=1+2t$:

$$R(s)=\frac{1}{s}+\frac{2}{s^2}=\frac{s+2}{s^2}\ ,\ W(s)=\frac{1}{s}\quad ,\quad \Rightarrow \quad Q=s^2$$

Eg.4 $r = 1(t),\ w(t) = \sin(\omega t):$

$$R(s) = \frac{1}{s},\ W(s) = \frac{\omega}{s^2 + \omega^2},\quad \Rightarrow\quad Q(s) = s\left(s^2 + \omega^2\right)$$

Eg.5 $r = 0,\ w(t) = \sin(\omega t):$

$$R(s) = 0 = 0/1,\ W(s) = \frac{\omega}{s^2 + \omega^2},\quad \Rightarrow\quad Q(s) = s^2 + \omega^2$$

Eg.6 $r = \cos(\omega t),\ w(t) = 0:$

$$R(s) = \frac{s}{s^2 + \omega^2},\quad \Rightarrow\quad Q(s) = s^2 + \omega^2$$

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Motivated from the special cases:

- Step input, $\frac{1}{s}$, $\quad Q(s)=s$, $\quad \Rightarrow \frac{1}{s}$ or $\frac{1}{Q(s)}$
- Ramp input, $\frac{1}{s^2}$, $\quad Q(s)=s^2$, $\quad \Rightarrow \frac{1}{s^2}$ or $\frac{1}{Q(s)}$
- $\sin(\omega t)$, $\frac{1}{s^2+\omega^2}$, $\quad Q(s)=s^2+\omega^2$, $\Rightarrow \frac{1}{s^2+\omega^2}$ or $\frac{1}{Q(s)}$

we assign the servo mechanism for general inputs as

$$\frac{1}{Q}.$$

We cascade it with a given proper plant

$$G(s)=\frac{N(s)}{D(s)},$$

to form a generalized plant:

$$G(s)\big/Q(s)=\frac{N(s)}{Q(s)D(s)}.$$

The generalized plant is strictly proper as long as the plant is proper and $Q(s)$ is a polynomial of degree 1 or above. It has to be stabilized, in order for the servo mechanism to work and eliminate the steady state error. It follows from Corollary 1 that the generalized plant can be stabilized if $N(s)$ and $D(s)Q(s)$ have no common unstable root. ***This requires that no root of*** $Q(s)$ ***is a zero of the plant*** $G(s)$. For example. Let

$$G(s) = \frac{s}{s+1},\ Q(s) = s,$$

then

$$N(s) = s, \text{ and } Q(s) = s \text{ are not coprime.}$$

Under coprimeness of polynomials $N(s)$ and $D(s)Q(s)$, a compensator $\widetilde{K}(s) = \dfrac{B(s)}{A(s)}$ of degree $l = n + q - 1$ will achieve pole-placement by considering $N/(DQ)$ of order $n + q$ as the plant.

Practically, specify a desired stable $P_c(s)$ of degree $n + q + l$ with $l \geq n + q - 1$ and solve the polynomial equation:

$$D(s)Q(s)A(s) + N(s)B(s) = P_c(s) \, , \qquad (12)$$

to get compensator $\widetilde{K}(s) = \dfrac{B(s)}{A(s)}$.

Theorem 2 *Consider the unity-feedback system shown in Figure 2 with a proper plant transfer function* $G(s) = N(s)/D(s)$ *with* $N(s)$ *and* $D(s)$ *coprime. Let* $Q(s)$ *be the least common denominator of the unstable modes of the set point* $R(s)$ *and disturbance* $W(s)$. *If N(s) and D(s)Q(s) have no common unstable root, then there is a proper compensator* $\tilde{K}(s)$ *which stabilizes the generalized plant,* $N/(DQ)$, *and the overall controller* $K = \tilde{K}/Q$ *is proper and achieves asymptotic tracking and regulation robustly.*

Proof . The stabilization part has been shown before. We now prove zero steady state error. Let $y_r(t)$ denote the output excited by $r(t)$ when $w(t) = 0$. Then

$$
\begin{aligned}
E_r(s) &= R(s) - Y_r(s) \\
&= R(s) - \frac{G(s)K(s)}{1+G(s)K(s)}R(s) \\
&= \frac{R(s)}{1+\frac{N(s)}{D(s)Q(s)}\frac{B(s)}{A(s)}} \\
&= \frac{D(s)A(s)}{D(s)Q(s)A(s)+N(s)B(s)} \cdot Q(s)\frac{N_r(s)}{D_r(s)}
\end{aligned}
$$

(13)

Since all unstable roots of $D_r(s)$ are canceled by $Q(s)$, all the poles of $E_r(s)$ are from $P_c(s) = A(s)D(s)Q(s) + B(s)N(s)$ and have negative real parts, and we have $e_r(t) \to 0$ as $t \to \infty$.

Similarly, let $y_w(t)$ be the output excited by $w(t)$ when $r(t) \equiv 0$. Then,

$$
\begin{aligned}
E_w(s) = R(s) - Y_w(s) &= 0 - \frac{1}{1+G(s)K(s)} W(s) \\
&= -\frac{A(s)D(s)Q(s)}{A(s)D(s)Q(s)+B(s)N(s)} W(s) \\
&= -\frac{A(s)D(s)}{A(s)D(s)Q(s)+B(s)N(s)} \cdot Q(s) \frac{N_w(s)}{D_w(s)}.
\end{aligned}
\tag{14}
$$

Since all unstable roots of $D_w(s)$ are canceled by $Q(s)$, all the poles of $E_w(s)$ have negative real parts, and we have $e_w(t) \to 0$ as $t \to \infty$. Thus, the disturbance rejection is also guaranteed.

It can readily be seen that the zero steady state error remains true under any perturbations in G and $\tilde{K}$ as long as the perturbations do not cause instability of the closed-loop system. (However, a perturbation of $Q(s)$ is not permitted). The proof is complete. $\triangleleft$

$1/Q(s)$ is inside the loop, referred to as the internal model principle. Any design based on this principle is deemed robust. For step inputs, $Q(s)=s$ and $[1/Q(s)]G(s)\widetilde{K}(s)$ is of type 1 transfer function, coinciding with the classical integral control theory.

The location of the internal model $\dfrac{1}{Q(s)}$ is not critical in the SISO case as long as $\dfrac{1}{Q}$ does not appear in the forward paths from r to e or from w to y. But if $\dfrac{1}{Q}$ appears, say from w to y, Q will not exist in the numerator of the transfer function from w to y to cancel unstable modes of $w(t)$.

Servo control design procedure:

(i) Obtain plant coprime polynomial fraction as $G = N(s)/D(s)$.

(ii) Determine Q from the given types of disturbance and reference.

(iii) Design $\tilde{K}$ to stabilize the generalized plant, $N/(DQ)$.

(iv) Form the servo controller as

$$K = \tilde{K}\frac{1}{Q}.$$

Example 3 Let

$$G(s) = \frac{s+3}{s-1} := \frac{N(s)}{D(s)}.$$

The reference is a sinusoid signal $r(t) = \sin(t)$, and the disturbance is of step type. Then we have $Q = s(s^2+1)$ coprime with N(s) and

$$\tilde{G} = \frac{1}{Q}G(s) = \frac{s+3}{s^4 - s^3 + s^2 - s},$$

of order *4*. A controller $\tilde{K}$ of order 3 can be determined from

$$DQA + NB = P_c. \tag{15}$$

If $P_c(s)$ is chosen as

$$\begin{aligned} P_c(s) &= (s+3)^6 (s+1) \\ &= s^7 + 19s^6 + 153s^5 + 675s^4 + 1755s^3 + 2673s^2 + 2187s + 729, \end{aligned}$$

then (15) is solved to get the solution as

$$A(s) = A_0 + A_1 s + A_2 s^2 + A_3 s^3 = 363+172\mathrm{s}+20\mathrm{s}^2+s^3$$

$$B(s) = B_0 + B_1 s + B_2 s^2 + B_3 s^3 = 243+769\mathrm{s}+571s^2+465s^3.$$

Thus $\tilde{K}$ is

$$\tilde{K} = \frac{465s^3 + 571s^2 + 769s + 243}{\mathrm{s}^3+20\mathrm{s}^2+172\mathrm{s}+363}. \tag{16}$$

The complete servo controller is

$$K(s) = \frac{1}{s\left(s^2+1\right)}\tilde{K}.$$

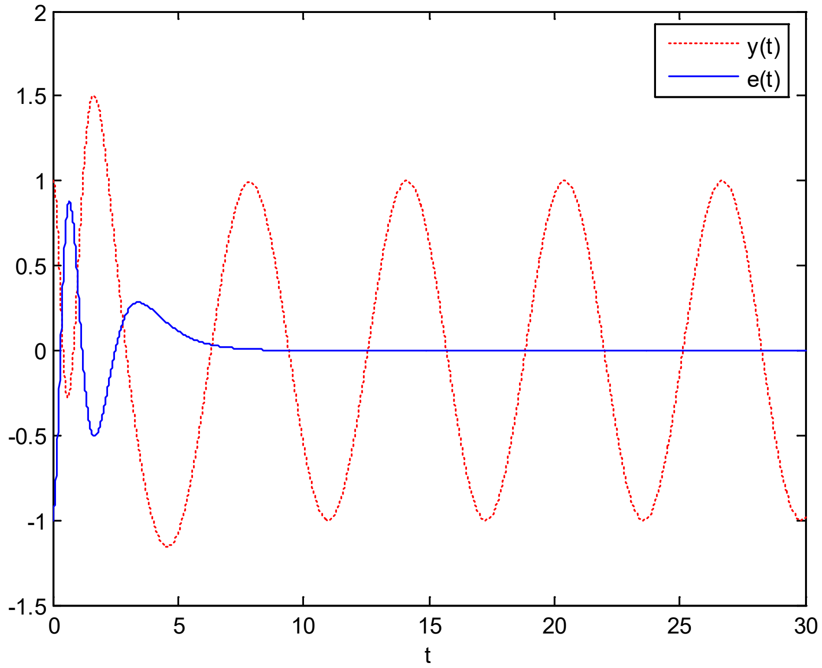


Figure 3 System response.

Industrial application: Antenna servo control re-visited

Model: The system is described by

$$\theta(s) = \frac{1}{s(s/a+1)} u(s) + \frac{1}{s(s/a+1)} d(s).$$

Design: For $a = 0.1$, $R(s) = 1/s$, $d(s) = 1/s$, we have $W(s) = \dfrac{1}{s^2(10s+1)}$

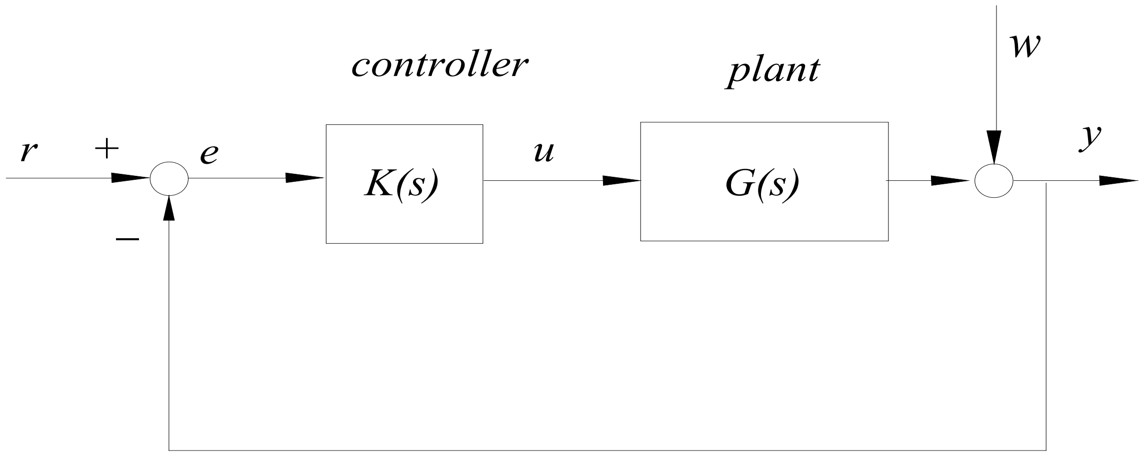


Figure 2 Unity output feedback system.

It follows that $Q(s) = s^2$ is co-prime with N(s), and

$$\tilde{G}(s) = \frac{1}{Q(s)} G(s) = \frac{1}{s^2} \frac{1}{s(10s+1)} = \frac{1}{s^3(10s+1)}$$

of order 4. A controller $\tilde{K}$ of order 3 can be determined from

$$DQA + NB = P_c$$

If $P_c(s)$ is chosen to be

$$P_c(s) = (s+2)^6 (s+1)$$
$$= s^7 + 13s^6 + 72s^5 + 220s^4 + 400s^3 + 432s^2 + 256s + 64,$$

then the solution is

$$A(s) = A_0 + A_1 s + A_2 s^2 + A_3 s^3 = 21.29 + 7.07\mathrm{s} + 1.29\mathrm{s}^2 + 0.1 s^3$$

$$B(s) = B_0 + B_1 s + B_2 s^2 + B_3 s^3 = 64 + 256\mathrm{s} + 432 s^2 + 378 s^3.$$

Thus $\tilde{K}$ is

$$\tilde{K} = \frac{378 s^3 + 432 s^2 + 256\mathrm{s} + 64}{0.1 s^3 + 1.29\mathrm{s}^2 + 7.07\mathrm{s} + 21.29},.$$

And the complete controller is

$$\begin{aligned} K(s) &= \frac{1}{s^2}\tilde{K} \\ &= \frac{378 s^3 + 432 s^2 + 256\mathrm{s} + 64}{\mathrm{s}^2\left(0.1 s^3 + 1.29\mathrm{s}^2 + 7.07\mathrm{s} + 21.29\right)}. \end{aligned}$$

The closed-loop response is given in Figure 4 and has a large overshoot.

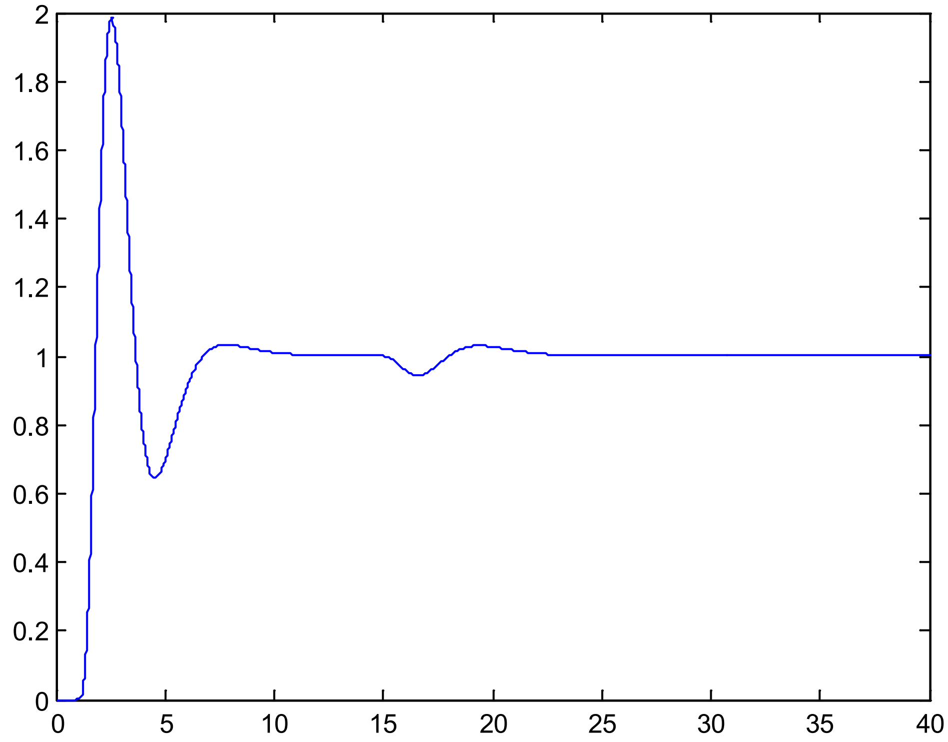


Figure 4 System step response (Design 1)

A simpler design could be achieved as follows.

- Servo mechanism. Since $Q(s) = s^2$, the open-loop GK should have the factor $1/s^2$. Noting that G(s) already has $1/s$, thus the controller only needs an integrator $1/s$, and can be given by

$$K(s) = \frac{\tilde{K}(s)}{s}$$

as long as $\tilde{K}$ stabilizes

$$\tilde{G}(s) = \frac{1}{s^2(10s+1)}.$$

- Stabilizer. Stable pole-zero cancellation may be used to simplify design and reduce controller order. Let $\tilde{K}(s)$ cancel the stable pole of G(s) so that

$$\tilde{K}(s) = (10s+1)(k_1 s + k_2)$$

and the open-loop becomes

$$GK = \tilde{G}\tilde{K} = \frac{k_1 s + k_2}{s^2}$$

giving the characteristic equation of the closed-loop as

$$P_c(s) = s^2 + k_1 s + k_2.$$

Choose $k_1 = k_2 = 4$ to assign two closed-loop poles both at -2. This leads to the following servo controller,

$$K(s) = \frac{(10s+1)(4s+4)}{s} = \frac{40s^2 + 44s + 4}{s}$$

which is of PID type.

The response from this new design is displayed in Figure 5 and has little overshoot.

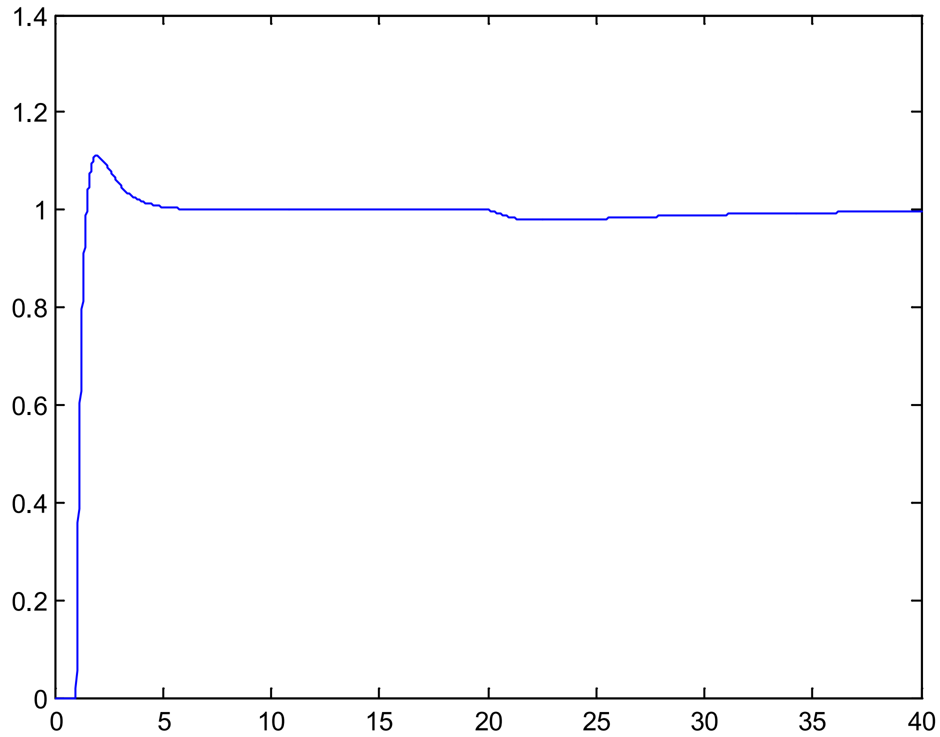


Figure 5 System step response (Design 2)

Remark on realizability of PID controller. It should be noted that PID in its ideal/theoretical form,

$$K_p(1+\frac{K_I}{s}+K_D s),$$

is not proper, thus not realizable. For implementation, so-called industrial D control is used to approximate the derivative term by

$$K_D s \approx \frac{K_D s}{1+(K_D/N)s}$$

where N is usually 3 to 20. Then, it becomes proper and realizable.

§9.3 Multivariable Integral Control

This section is to extend integral control to the MIMO case.

Consider an $m \times m$ plant:

$$\dot{x} = Ax + Bu + B_w w, \tag{17}$$

$$y = Cx, \tag{18}$$

The error is defined as

$$e = r - y. \tag{19}$$

Suppose that the disturbance w and reference r are **both of step type**. To achieve zero steady state error, like SISO integral control, we introduce one integrator to each channel:

$$v(t) = \int_0^t e\left(\mathrm{t}\right) d\tau .$$

Then, it follows that

$$\dot{v}\left(t\right) = e\left(t\right) = r - y\left(t\right) = r - Cx\left(t\right). \tag{20}$$

Equations (17), (18) and (20) form an augmented system:

$$\underset{\dot{\bar{x}}}{\begin{pmatrix} \dot{x} \\ \dot{v} \end{pmatrix}} = \underset{\bar{A}}{\begin{pmatrix} A & O \\ -C & O \end{pmatrix}} \underset{\bar{x}}{\begin{pmatrix} x \\ v \end{pmatrix}} + \underset{\bar{B}}{\begin{pmatrix} B \\ O \end{pmatrix}} u + \underset{\bar{B}_w}{\begin{pmatrix} B_w \\ O \end{pmatrix}} w + \underset{\bar{B}_r}{\begin{pmatrix} 0 \\ I \end{pmatrix}} r . \tag{21}$$

$$y = \begin{bmatrix} C & 0 \end{bmatrix} \begin{bmatrix} x \\ v \end{bmatrix} = \bar{C}\bar{x} \tag{22}$$

The controllability matrix is

$$
\begin{aligned}
Q_c &= \begin{pmatrix} B & AB & A^2B & \cdots \\ 0 & -CB & -CAB & \cdots \end{pmatrix} \\
&= \begin{pmatrix} A & B \\ -C & 0 \end{pmatrix} \begin{pmatrix} 0 & B & AB & \cdots \\ I & 0 & 0 & \cdots \end{pmatrix}.
\end{aligned}
\tag{23}
$$

The augmented system is controllable if and only if

(i) the plant is controllable and

(ii)

$$
rank \begin{pmatrix} A & B \\ -C & 0 \end{pmatrix} = n + m.
$$

or equivalently,

$$
rank \begin{pmatrix} A & B \\ C & 0 \end{pmatrix} = n + m. \tag{24}
$$

which implies that the plant does not have any zero at the origin.

If the augmented system is controllable, it can be stabilized by the state feedback control law:

$$u = -K\bar{x} = -\begin{bmatrix} K_1 & K_2 \end{bmatrix} \begin{bmatrix} x \\ v \end{bmatrix}$$

or

$$u(t) = -\underset{\substack{\uparrow \\ P}}{K_1 x} - \underset{\substack{\uparrow \\ I}}{K_2 \int_0^t e \, d\tau}, \tag{26}$$

which contains integral control. The feedback gain K may be determined by pole placement procedure or LQR.

The resultant feedback system is

$$\begin{pmatrix} \dot{x} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} A - BK_1 & -BK_2 \\ -C & O \end{pmatrix} \begin{pmatrix} x \\ v \end{pmatrix} + \begin{pmatrix} B_w \\ O \end{pmatrix} w + \begin{pmatrix} 0 \\ I \end{pmatrix} r. \tag{27}$$

$$\dot{\bar{x}} = \bar{A}_c \bar{x} + \bar{B}_w w + \bar{B}_r r$$

$$y = \begin{bmatrix} C & 0 \end{bmatrix} \begin{bmatrix} x \\ v \end{bmatrix} \tag{28}$$

$$y = \bar{C} \quad \bar{x}.$$

Once the feedback system is stable, each signal in the system in response to step inputs will be constant in the steady state, and so is $v(t)$. It follows that

$$e(t) = \dot{v}(t) = 0.$$

Thus, zero steady state error has been achieved.

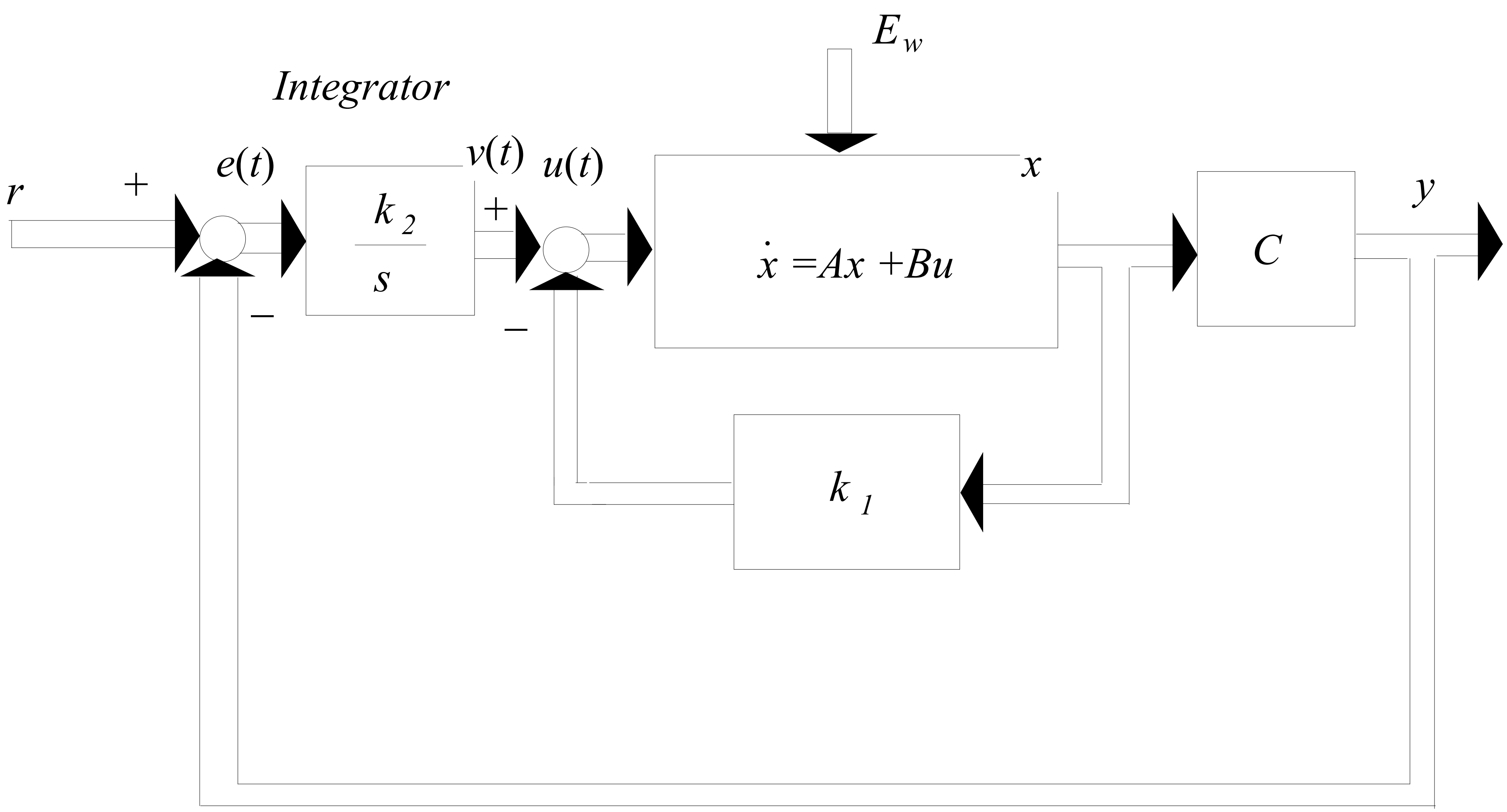


Figure 6 MIMO integral control system.

Example 4. Design an integral controller for

$$\dot{x} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} x + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u + \begin{pmatrix} 1 \\ 1 \end{pmatrix} w, \quad x(0) = 0,$$

$$y = [1 \quad 0]x.$$

One may readily check that it is controllable and meets (24).

The augmented system is

$$\begin{pmatrix} \dot{x} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ v \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} u + \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} w + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} r.$$

$$y = [1 \quad 0 \quad 0] \begin{bmatrix} x \\ v \end{bmatrix}.$$

The optimal control minimizing

$$J = \int_0^\infty (\|y\|_Q^2 + \|u\|_R^2)dt, \tag{29}$$

yields the feedback gain:

$$(K_1, K_2) = R^{-1}\left[B^T, 0^T\right]P, \tag{30}$$

where P is the positive definite solution of

$$\begin{aligned} &\begin{pmatrix} A & 0 \\ -C & 0 \end{pmatrix}^T P + P\begin{pmatrix} A & 0 \\ -C & 0 \end{pmatrix} + \begin{pmatrix} C^T \\ 0 \end{pmatrix} Q[C \quad 0] \\ &-P\begin{pmatrix} B \\ 0 \end{pmatrix} R^{-1}\left[B^T \quad 0^T\right]P = 0 \end{aligned} \tag{31}$$

For $Q=1, R=1$, the solution is

$$u = -[2, 2]\mathbf{x} + \int_0^t (r - y)d\tau.$$

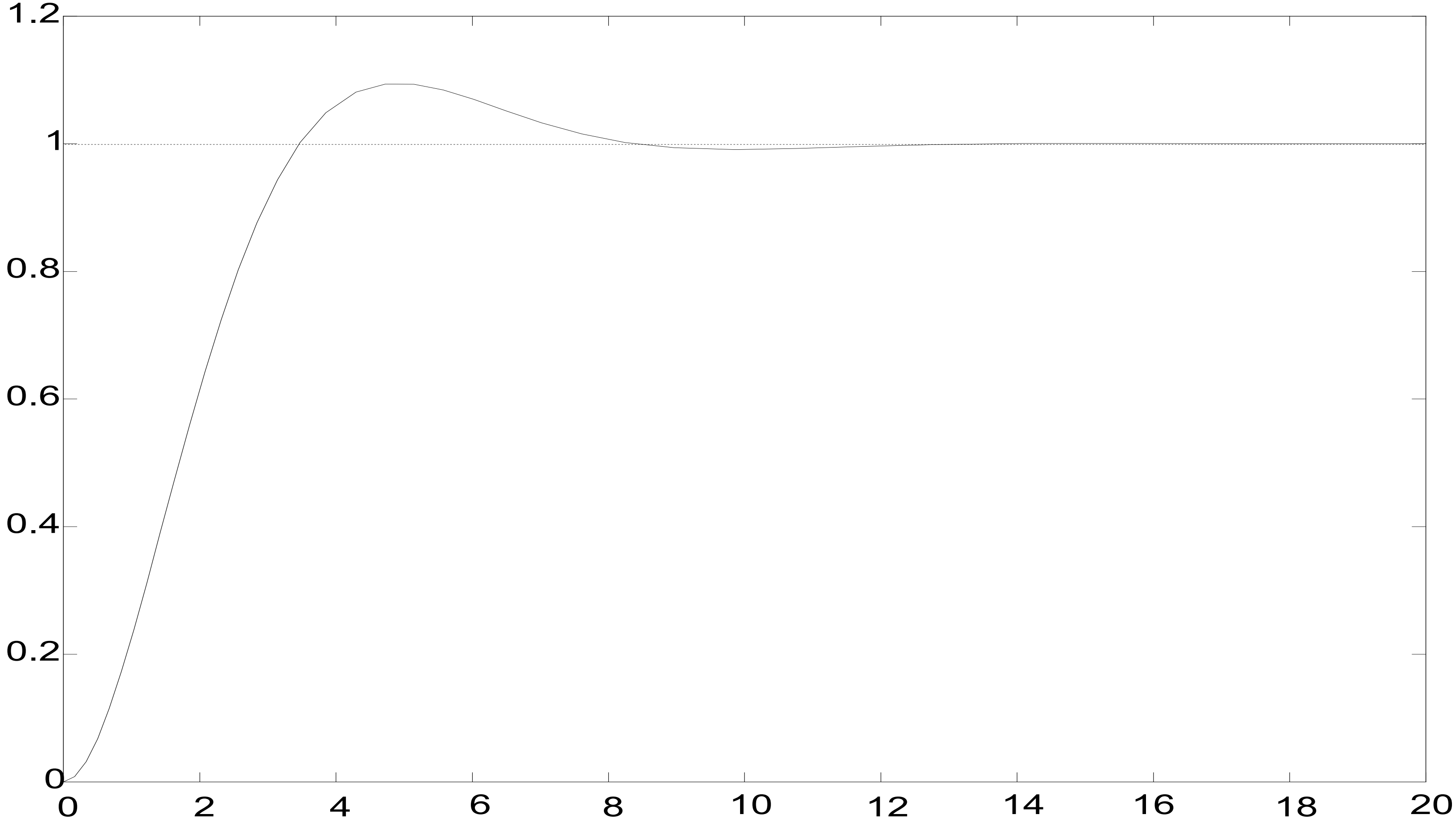


Figure 7 System step response for Example 4.

Example 5. Design an integral controller for

$$\dot{x} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} x + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} u + \begin{pmatrix} 1 \\ 0 \end{pmatrix} w, \quad x(0) = 0,$$

$$y = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} x.$$

One may readily check that it is controllable and meets (24). The augmented system is

$$\begin{pmatrix} \dot{x} \\ \dot{v} \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}}_{\bar{A}} \underbrace{\begin{pmatrix} x \\ v \end{pmatrix}}_{\bar{x}} + \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}}_{\bar{B}} u + \underbrace{\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}}_{\bar{B}_w} w + \underbrace{\begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}}_{\bar{B}_r} r.$$

$$y = \underbrace{\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}}_{\bar{C}} \begin{pmatrix} x \\ v \end{pmatrix}.$$

It can be checked that this augmented system is controllable. To stabilize the closed-loop augmented system, we can use pole placement or LQR method. Take the LQR optimal control for an example.

The optimal control minimizing

$$J = \int_0^\infty \left(\bar{x}^T Q \bar{x} + u^T R u \right) dt$$

yields the feedback gain for state feedback $u = -Kx$

$$K = R^{-1} \bar{B}^T P,$$

where P is the positive definite solution of the following Riccati equation

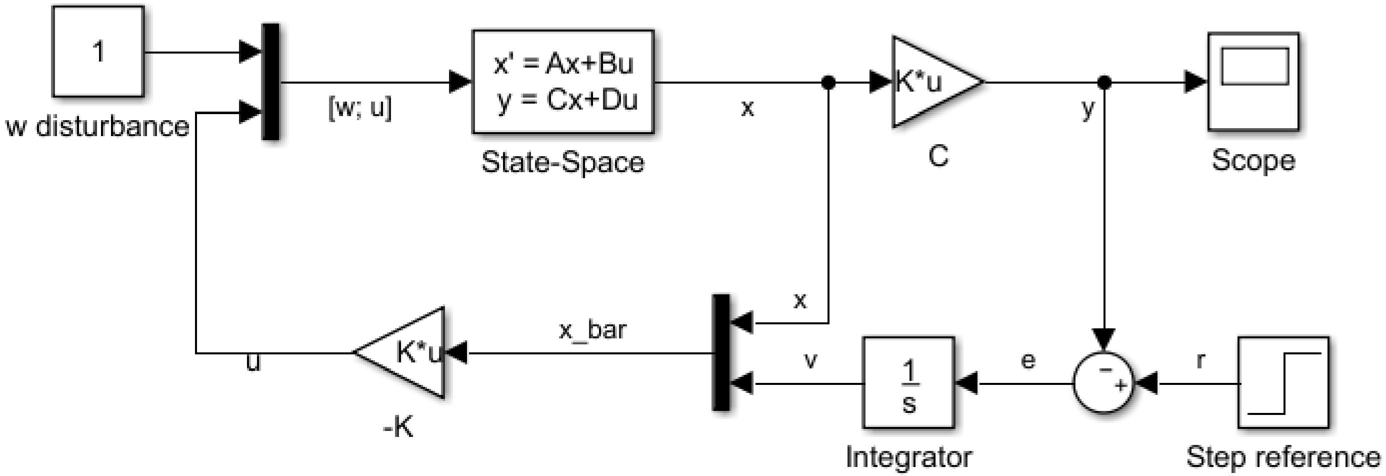
$$\bar{A}^T P + P\bar{A} + Q - P\bar{B}\bar{B}^T P = 0$$

For $Q = I, R = I$, it meets the requirement that $\left(\bar{A},\ Q^{0.5}\right)$ is observable for LQR design. Then, the solution can be solved as

$$K = \begin{bmatrix} 1.6464 & 0.4633 & -0.9626 & 0.2709 \\ 0.4633 & 1.9071 & -0.2709 & -0.9626 \end{bmatrix}.$$

(Hint: use "care" function of MATLAB to solve this high-dimensional Riccati equation.)

A Simulink model can be built to verify the integral control.



The result response of y to track the unit setting point $r = [1, 1]^T$ is

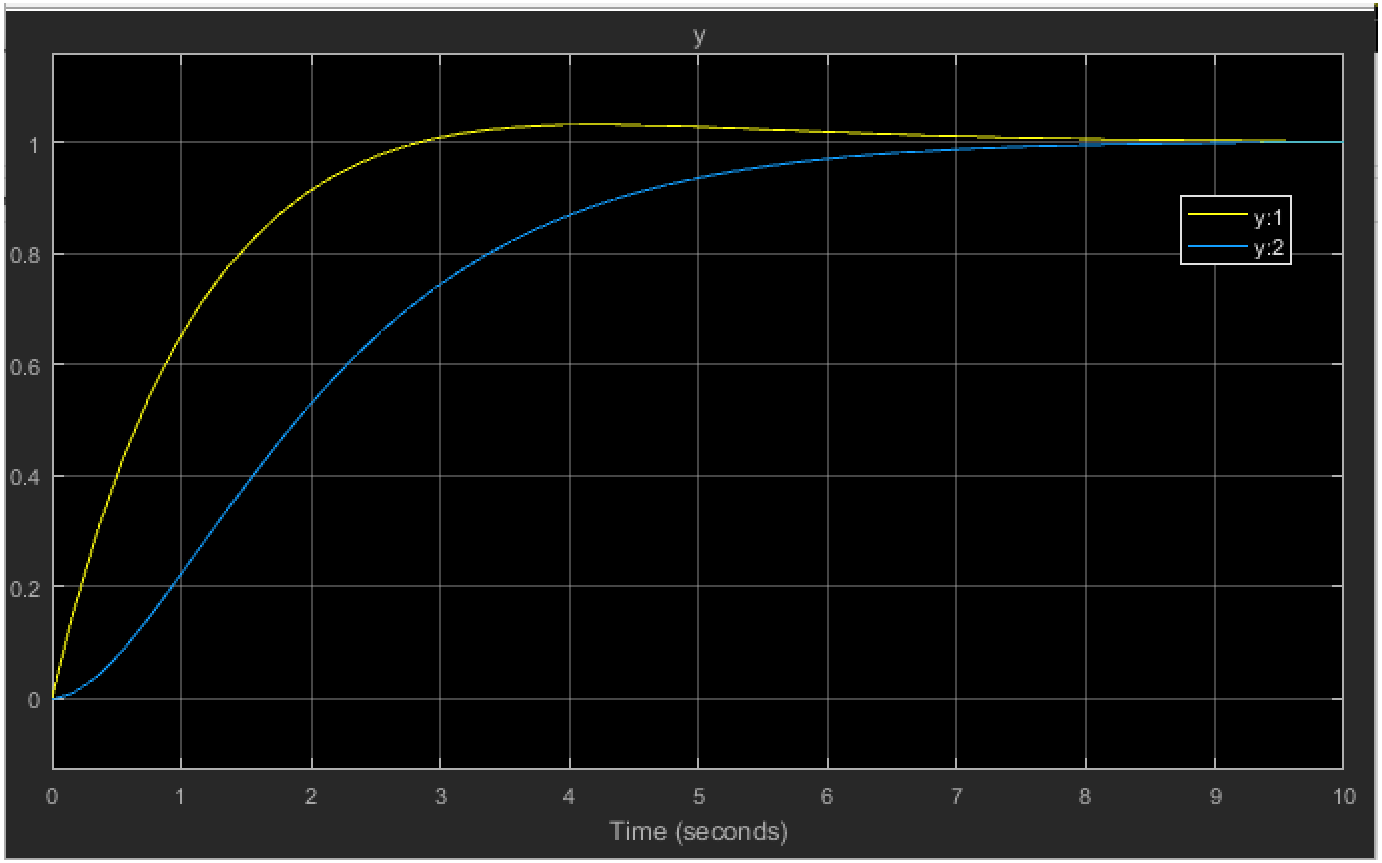


Figure 8 System step response for Example 5.