EE5103 Lecture Five
Pole-Placement Problem
Input-output Model Approach

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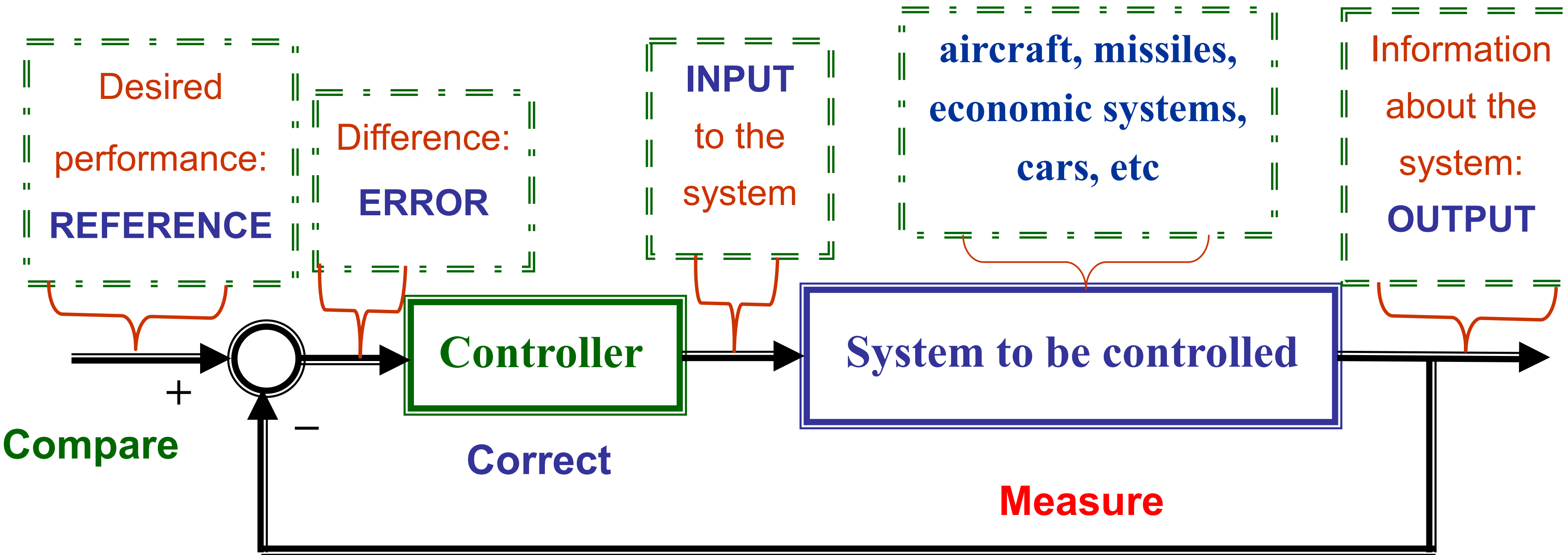
•What is a feedback control system?



•**Feedback: Measure —Compare –-Correct**

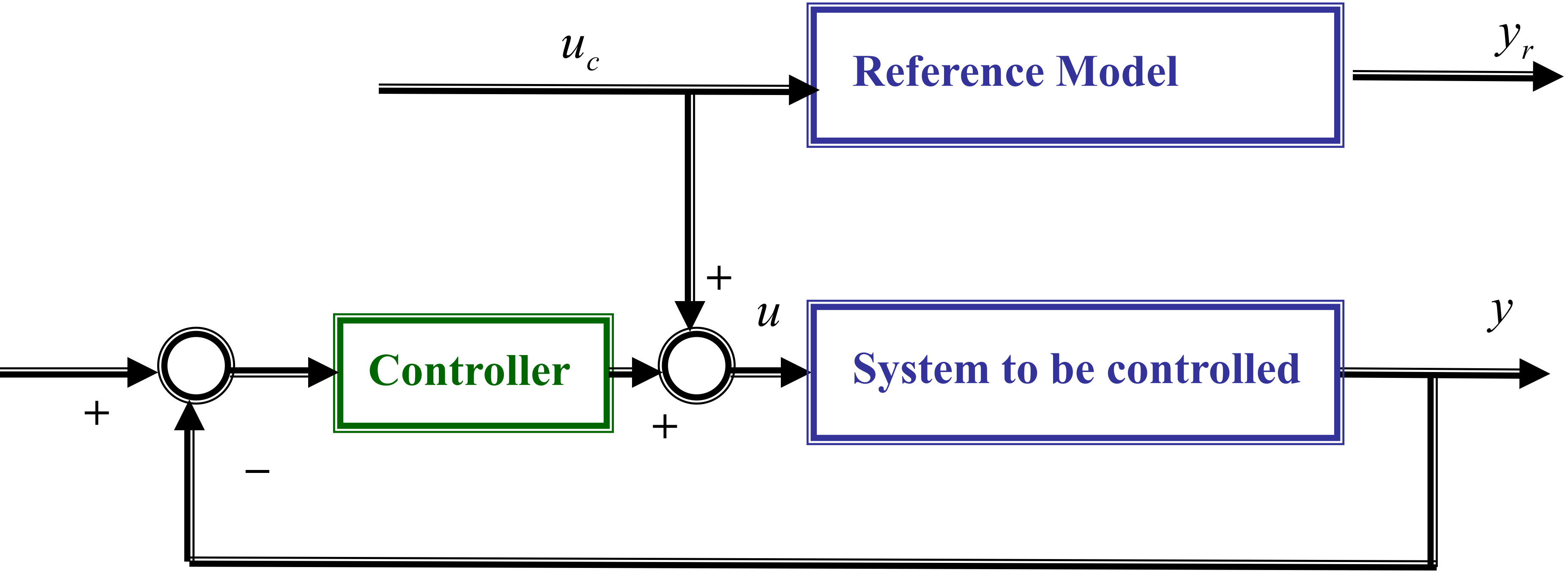
Objective: To make the system OUTPUT and the desired REFERENCE as close as possible, i.e., to make the ERROR as small as possible.

•Can we make the output follow arbitrary bounded signal? Certainly not.

How to specify the reference signal, or the desired output?

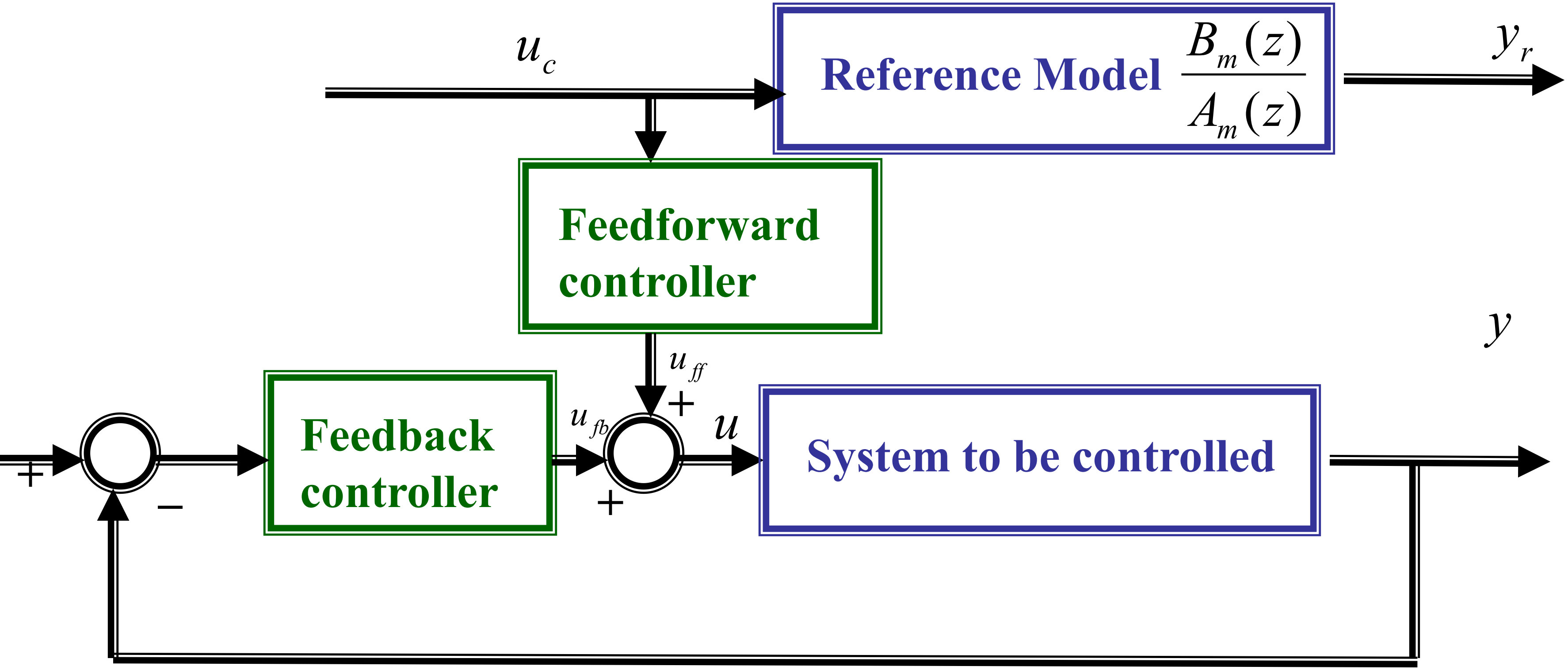
•The reference signal is specified by a well-behaved reference model.

•**Model-reference control:**



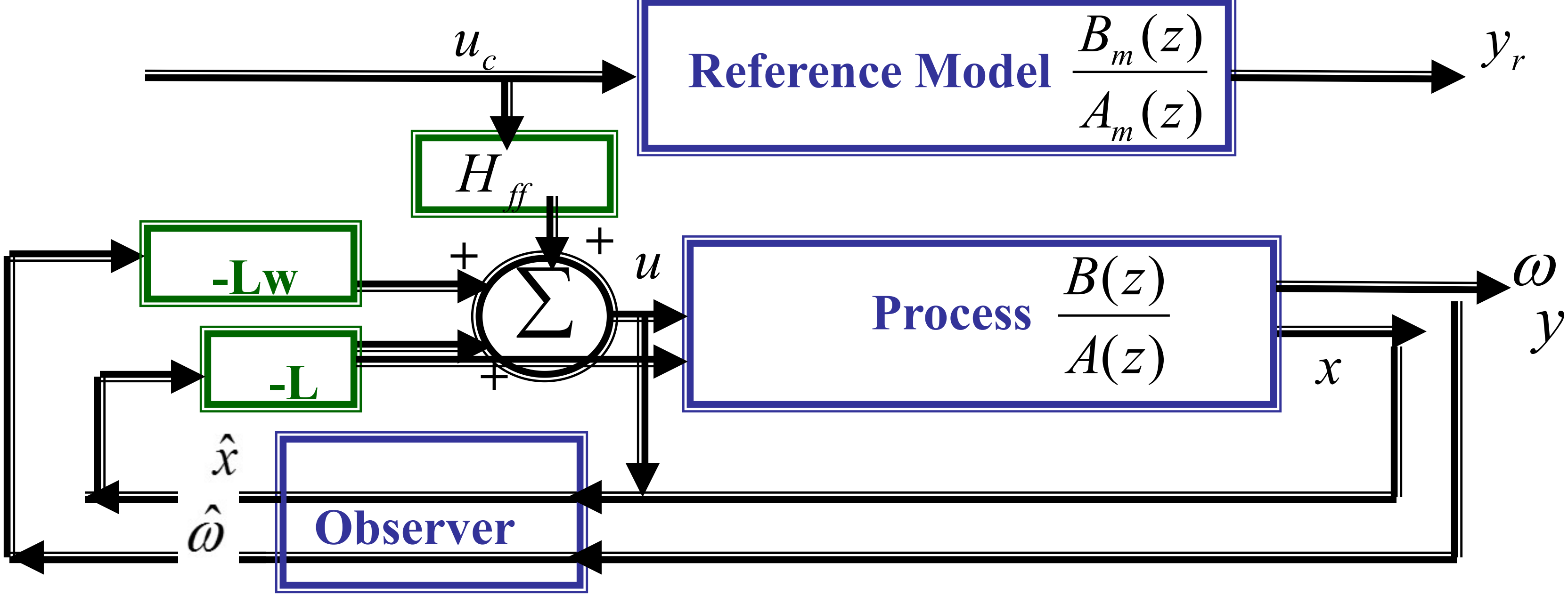
Objective: To make the closed loop model follow the reference model as close as possible.

Two-Degree-of-Freedom Controller



$$u = u_{fb} + u_{ff}$$

Pole placement	Zero placement
Match $A_m(z)$	**Match** $B_m(z)$



How to choose L? Pole Placement for the controller: $|zI-(\Phi-\Gamma L)|=A_m(z)$

How to choose Lw? Problem Dependent

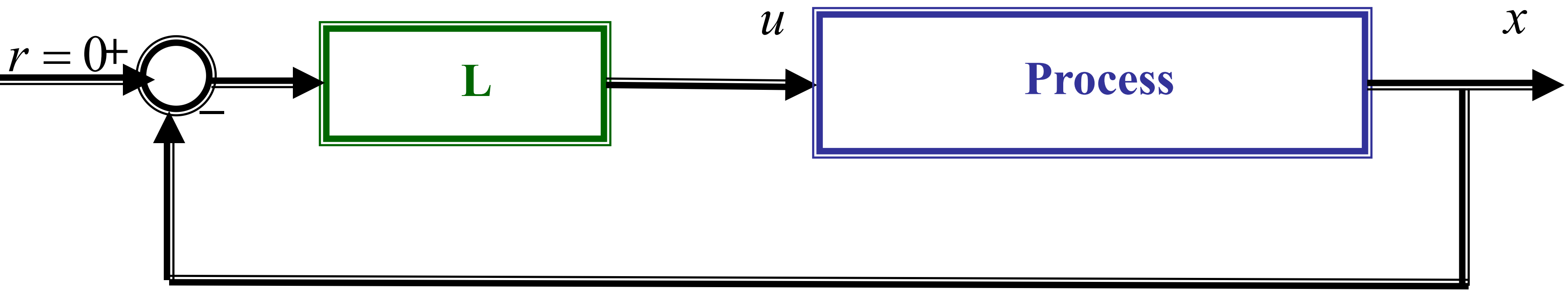
How to design the observer? Pole Placement for observer: $|zI-(\Phi-Kc)|=A_o(z)$

How to choose the Feedforward T.F.? $H_{ff}=\dfrac{B_m(z)}{B(z)}$

Zero Placement

<u>Separation Property!</u>

Step One: Proportional Control --- State Feedback Control



It is just a simple extension of the P control you have learned in the past:

$$u(k) = -Lx(k)$$

Closed Loop System:

$$x(k+1) = \Phi x(k) + \Gamma u(k) = (\Phi - \Gamma L)x(k)$$

If the system is controllable, the poles can be placed anywhere:

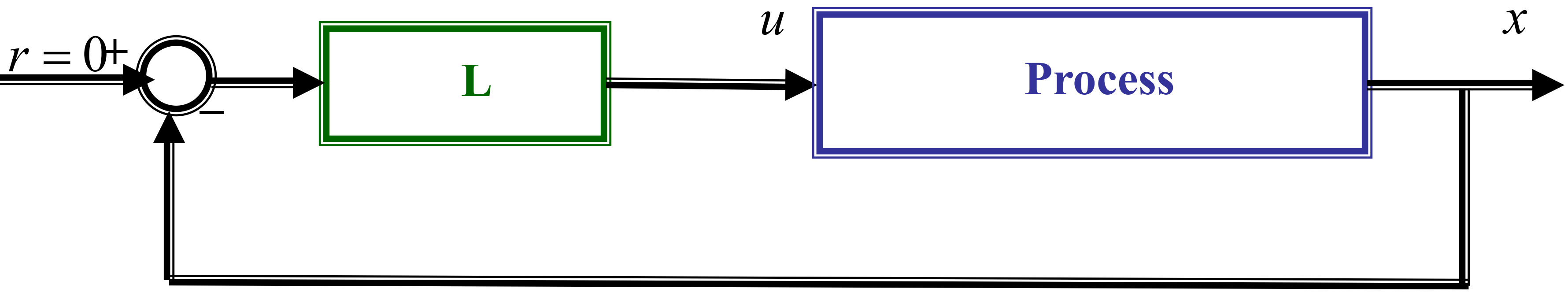
- Ackermann's formula $\quad L = \begin{bmatrix} 0 & \cdots & 0 & 1 \end{bmatrix} W_c^{-1} A_m(\Phi)$

Can you solve the problem without using Ackermann's formula?

Use direct comparison

$$|zI - (\Phi - \Gamma L)| \Longrightarrow A_m(z)$$

Step One: Proportional Control --- State Feedback Control



A very simple controller: $u(k) = -Lx(k)$

Closed Loop System: $x(k+1) = \Phi x(k) + \Gamma u(k) = (\Phi - \Gamma L)x(k)$

If the system is uncontrollable, can we still place the poles anywhere?

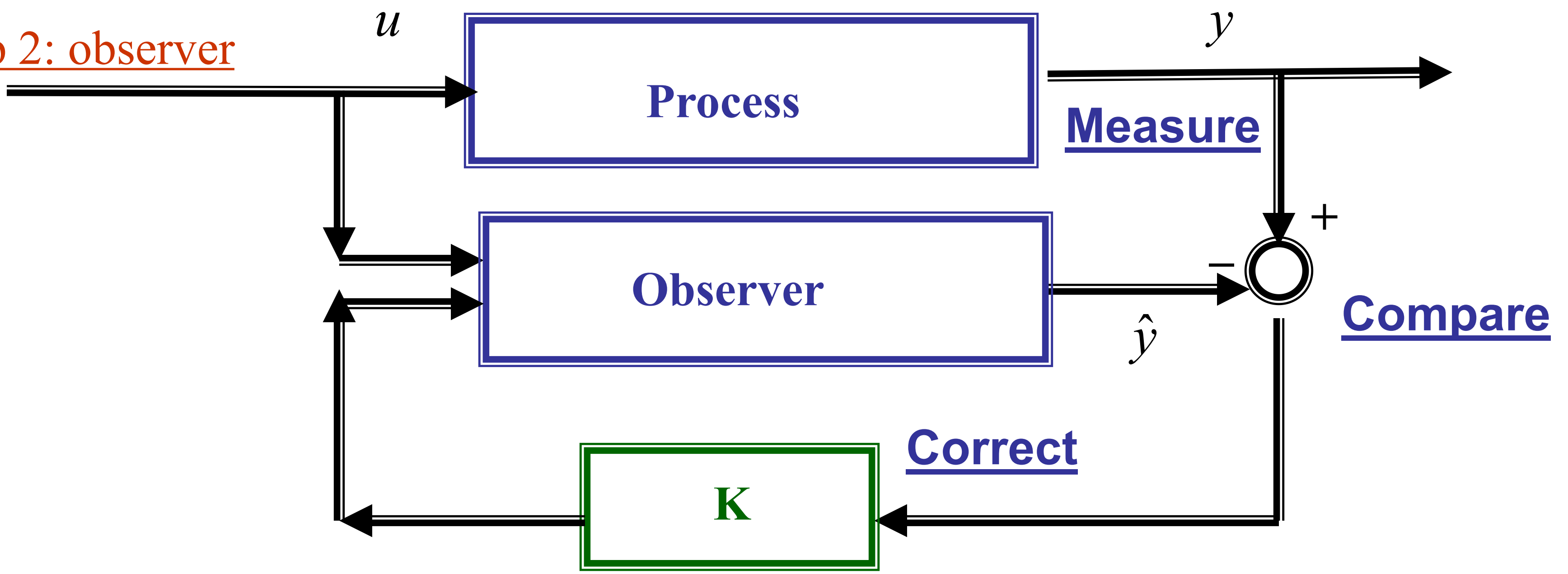
Some of the poles cannot be changed!

- If the uncontrollable poles are stable, then we can still control the system to some degree.

- Stabilizable.

- If the uncontrollable poles are unstable, then we have to think out of the box and find other more effective control inputs to make the system controllable!

Step 2: observer



$$x(k+1) = \Phi x(k) + \Gamma u(k)$$
$$y(k) = cx(k)$$

$$\hat{x}(k+1) = \Phi \hat{x}(k) + \Gamma u(k) + K(y(k) - \hat{y}(k))$$
$$\hat{y}(k) = c\hat{x}(k)$$

$$x(k+1) - \hat{x}(k+1) = \Phi(x(k) - \hat{x}(k)) - K(y(k) - \hat{y}(k))$$

$$e(k+1) = (\Phi - Kc)e(k)$$

How to choose K?

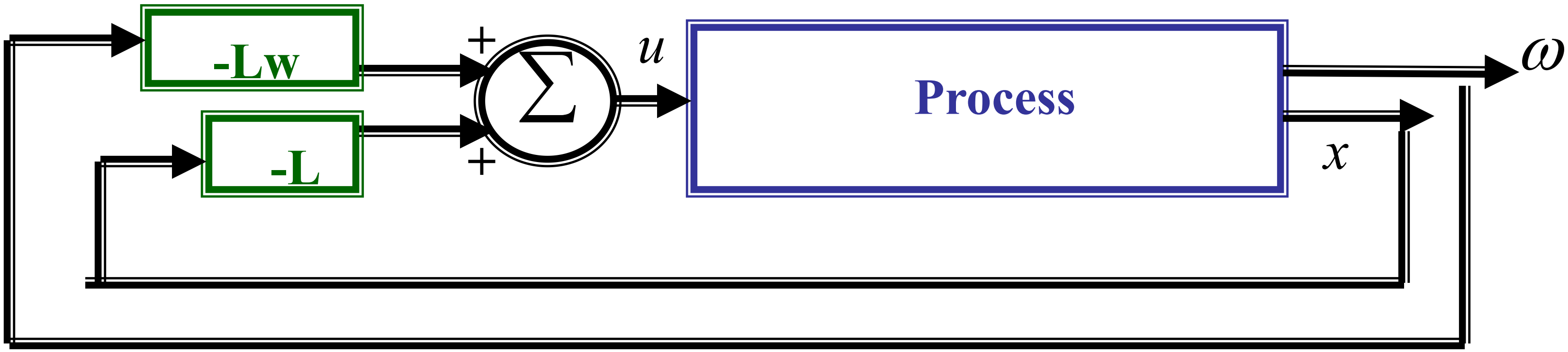
- Ackermann's formula $K = A_o(\Phi)(W_o)^{-1}[0 \quad \cdots \quad 0 \quad 1]^T$

Another way is to do direct comparison

$$|zI - (\Phi - Kc)| \Longrightarrow A_o(z)$$

Step 3: Output-feedback controller $u(k) = -L\hat{x}(k)$

Disturbance Rejection: **What is the trick?** Treat disturbance as another state variable!



$$u(k) = -L_c z(k) = -Lx(k) - L_\omega \omega(k)$$

$$\begin{bmatrix} x(k+1) \\ \omega(k+1) \end{bmatrix} = \begin{bmatrix} \Phi & \Phi_{x\omega} \\ 0 & \Phi_\omega \end{bmatrix} \begin{bmatrix} x(k) \\ \omega(k) \end{bmatrix} - \begin{bmatrix} \Gamma \\ 0 \end{bmatrix} \begin{bmatrix} L & L_\omega \end{bmatrix} \begin{bmatrix} x(k) \\ \omega(k) \end{bmatrix} = \begin{bmatrix} \Phi - \Gamma L & \Phi_{x\omega} - \Gamma L_\omega \\ 0 & \Phi_\omega \end{bmatrix} \begin{bmatrix} x(k) \\ \omega(k) \end{bmatrix}$$

$$y(k) = \begin{bmatrix} c & 0 \end{bmatrix} z(k)$$

- How to choose Lw? Try to make $\Phi_{x\omega} - \Gamma L_\omega = 0$

What if it is impossible to make $\Phi_{x\omega} - \Gamma L_\omega = 0$?

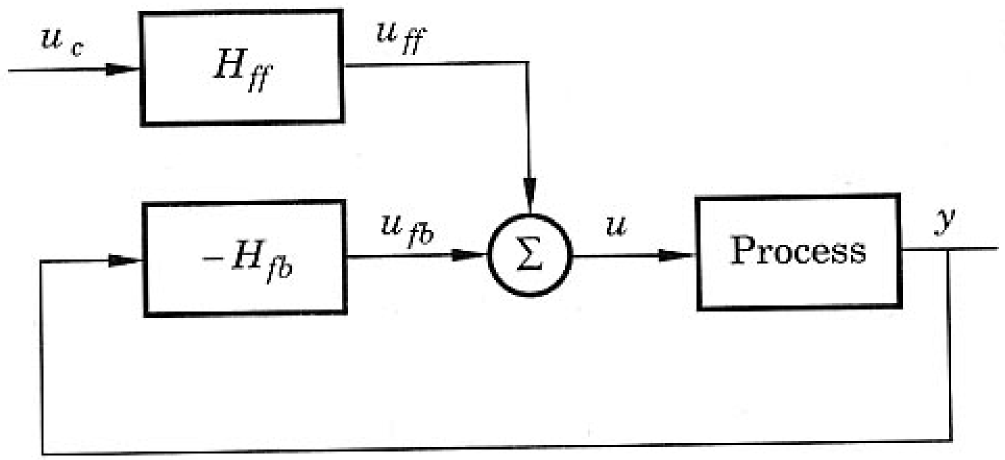
You need to get the TF from the disturbance to the output.

$$\omega(z) \longrightarrow \boxed{H_\omega(z)} \longrightarrow Y(z)$$

If the disturbance is constant, is it possible to to make its output ZERO?

Make the steady-state gain $H_\omega(1) = 0$

Tracking Problem: how to match the reference model?



$$\frac{Y(z)}{U_c(z)} = H_{ff}(z)\frac{B(z)}{A_m(z)} \qquad \rightarrow \frac{B_m(z)}{A_m(z)}$$

•<u>How to design the feed-forward controller?</u>

$$H_{ff}(z) = \frac{B_m(z)}{B(z)}$$

•<u>Under what conditions is perfect tracking attainable? How to make sure that the inputs and outputs are bounded?</u>

$$\frac{Y(z)}{U_c(z)} = \frac{B_m(z)}{A_m(z)}$$

$$\frac{U(z)}{U_c(z)} = \frac{A(z)B_m(z)}{B(z)A_m(z)}$$

B(z) is stable!

<u>Do we need the condition of stable inverse B(z) for pole-placement or stabilization?</u>

No. We need it only for perfect tracking (zero-placement).

One big problem: what if the state-space model is not available?

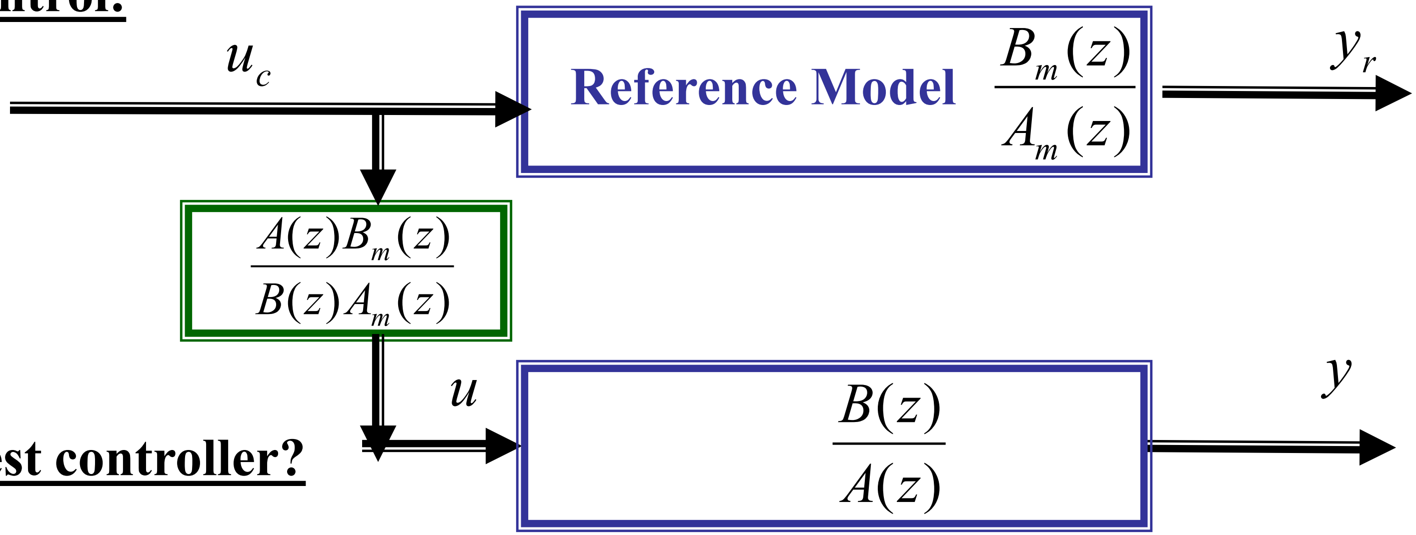
We can either find the realization of the input-output model
and use the state-space model approach.

Or we just work with the input-output model directly.

There are certain advantages in using the input output model directly!

•So we need a new approach to deal with input-output model directly.

Open Loop Control:



What is the simplest controller?

Let's derive the T.F. from command signal u_c to output y

$$Y(z) = \frac{B(z)}{A(z)} \frac{A(z)B_m(z)}{B(z)A_m(z)} U_c(z)$$

Should we cancel out A(z)? No. One A(z) is operating on u_c, the other A is on y.

Should we cancel out B(z)? Yes. Both of them are operating on the input u.

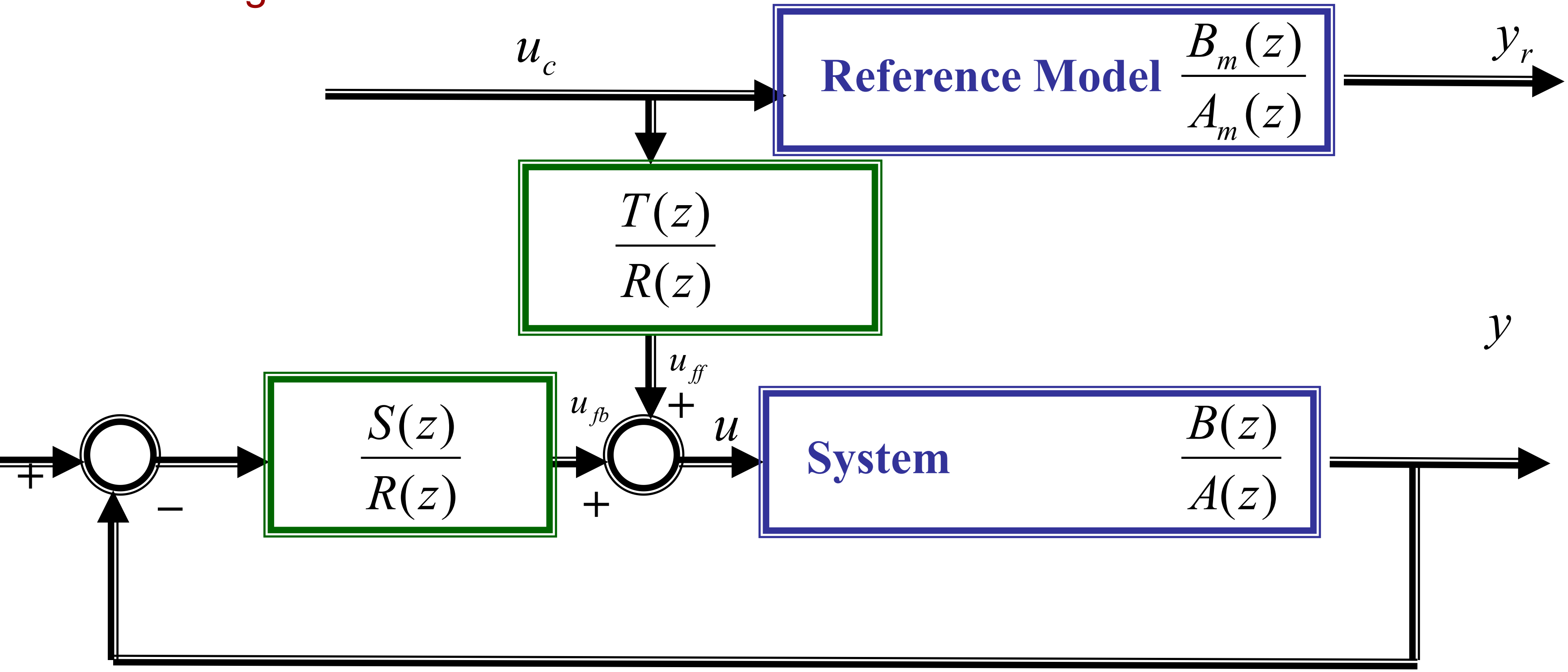
$$Y(z) = \frac{B_m(z)A(z)}{A_m(z)A(z)} U_c(z) \qquad U(z) = \frac{A(z)B_m(z)}{B(z)A_m(z)} U_c(z)$$

What is the condition to assure the output is bounded? A(z) is stable

What is the condition to assure the input is bounded? B(z) is stable

Both the system and its inverse have to be stable!

Two-Degree-of-Freedom Controller



What's the closed loop transfer function from command signal Uc to y?

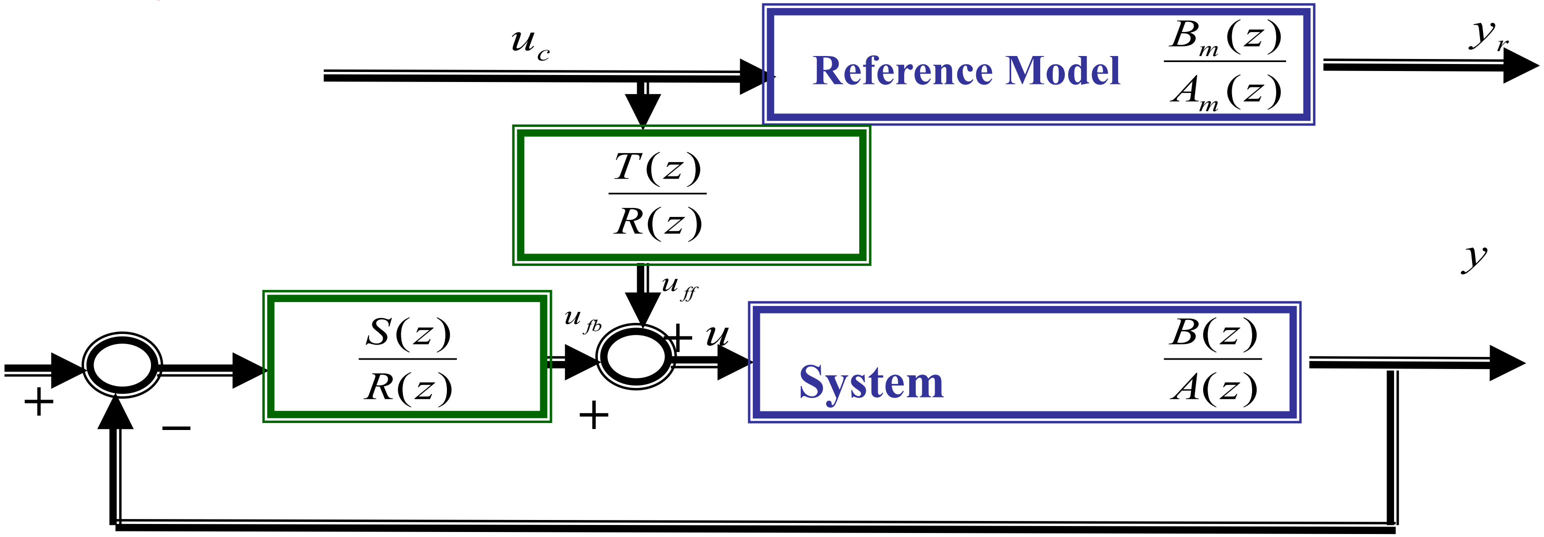
Let's first get the T.F. from u_{ff} to y:

What is the feed-forward T.F.? $\frac{B(z)}{A(z)}$

What is the open-loop T.F.? $\frac{B(z)}{A(z)}(-1)\frac{S(z)}{R(z)}(-1) = \frac{B(z)}{A(z)}\frac{S(z)}{R(z)}$

$$\frac{Y(z)}{U_{ff}(z)} = \frac{\frac{B(z)}{A(z)}}{1+\frac{B(z)}{A(z)}\frac{S(z)}{R(z)}} \Longrightarrow \frac{Y(z)}{U_{ff}(z)} = \frac{B(z)R(z)}{A(z)R(z)+B(z)S(z)}$$

Two-Degree-of-Freedom Controller



$$\frac{Y(z)}{U_{ff}(z)} = \frac{B(z)R(z)}{A(z)R(z)+B(z)S(z)} \qquad \frac{U_{ff}(z)}{U_c(z)} = \frac{T(z)}{R(z)}$$

$$\frac{Y(z)}{U_c(z)} = \frac{Y(z)}{U_{ff}(z)}\frac{U_{ff}(z)}{U_c(z)} = \frac{B(z)R(z)}{A(z)R(z)+B(z)S(z)}\frac{T(z)}{R(z)}$$

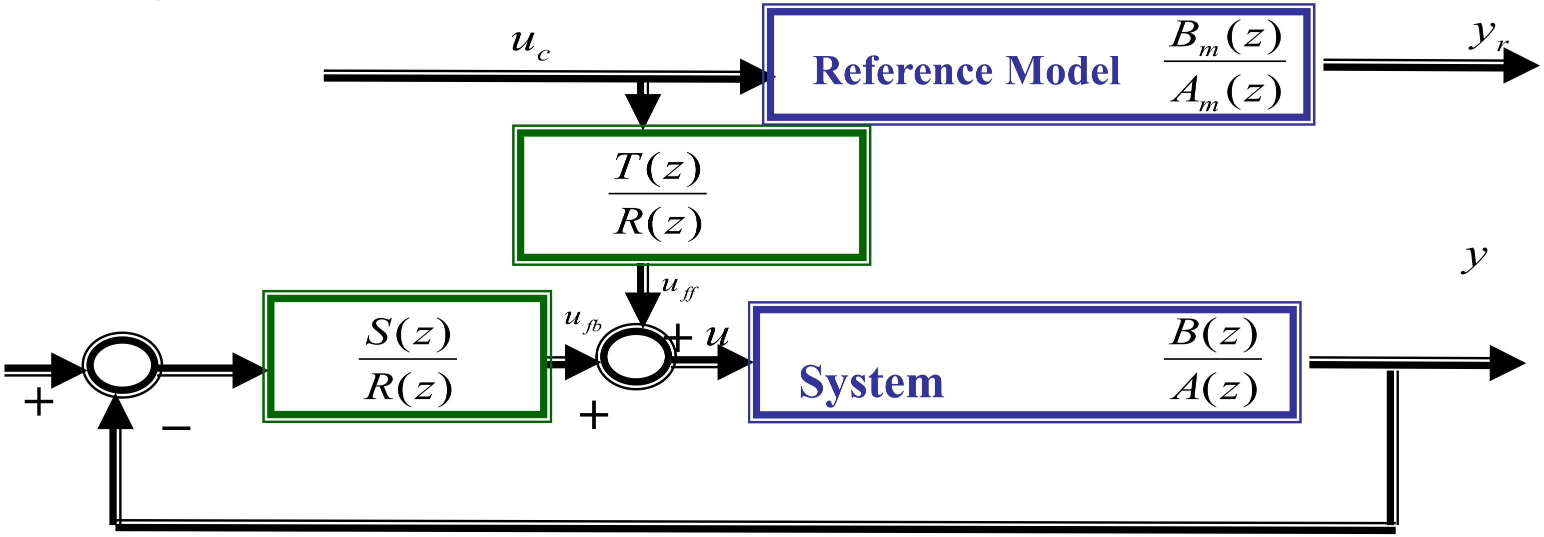
<u>Should we cancel out R(z)?</u>

Yes. Both of them are operating on u_{ff}.

So finally we have the following closed loop transfer function:

$$\frac{Y(z)}{U_c(z)} = \frac{T(z)B(z)}{A(z)R(z)+B(z)S(z)}$$

Two-Degree-of-Freedom Controller



$$\frac{Y(z)}{U_c(z)} = \frac{T(z)B(z)}{A(z)R(z)+B(z)S(z)}$$

How to design the feedback controller $\frac{S(z)}{R(z)}$? Match the poles or zeros?

Choose R(z) and S(z) such that

$$A_{cl}(z) = A(z)R(z) + B(z)S(z) = A_m(z)A_o(z)$$

Why not directly match $A_m(z)$?

The order of $A_m(z)$ may be lower than the closed loop.

$$\frac{Y(z)}{U_c(z)} = \frac{T(z)B(z)}{A(z)R(z)+B(z)S(z)} = \frac{T(z)B(z)}{A_m(z)A_o(z)} \Longrightarrow \frac{B_m(z)}{A_m(z)}$$

How to design the feedforward controller $\frac{T(z)}{R(z)}$? R(z) is already fixed!

- How to choose polynomial T(z)?

Compare it to the reference model, one simple way to choose T(z) is

$$T(z) = t_o A_o(z)$$

So the closed loop system becomes: $\frac{Y(z)}{U_c(z)} = \frac{T(z)B(z)}{A_m(z)A_o(z)} = \frac{t_o B(z)}{A_m(z)}$

At least the poles match those of the reference model!

What is the steady state gain of $\frac{t_o B(z)}{A_m(z)}$? $\frac{t_o B(1)}{A_m(1)}$

How to choose t_o to make the static gain unity?

$$\frac{t_o B(1)}{A_m(1)} = 1 \quad \rightarrow \quad t_o = \frac{A_m(1)}{B(1)}$$

$$\frac{Y(z)}{U_c(z)} = \frac{T(z)B(z)}{A(z)R(z)+B(z)S(z)} = \frac{T(z)B(z)}{A_m(z)A_o(z)} \Longrightarrow \frac{B_m(z)}{A_m(z)}$$

How to design the feedforward controller $\frac{T(z)}{R(z)}$?

I just showed the simple design: $T(z) = t_o A_o(z)$

So the closed loop system becomes: $\frac{Y(z)}{U_c(z)} = \frac{T(z)B(z)}{A_m(z)A_o(z)} = \frac{t_o B(z)}{A_m(z)}$

Obviously the simple design does not match the zeros.

- Is perfect tracking attainable? Can we match the zeros?

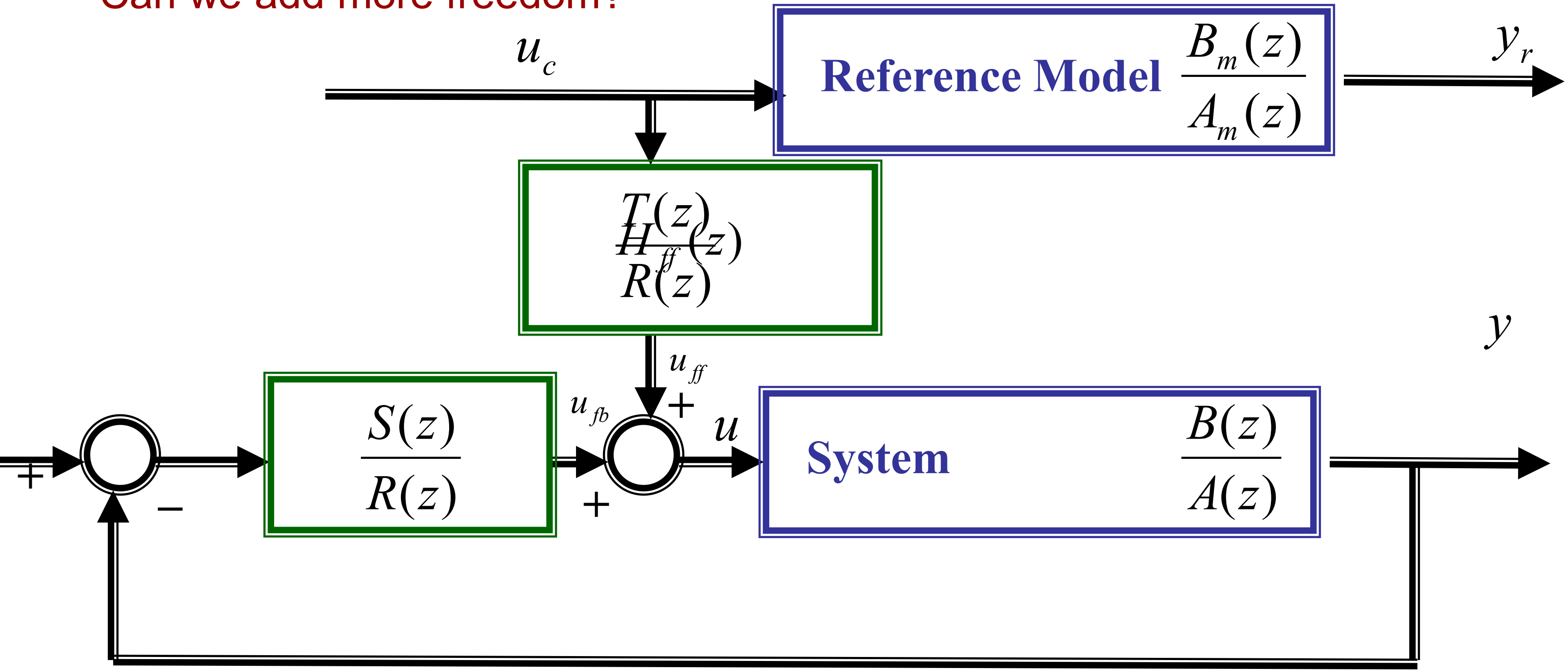
Let's try

$$T(z) = A_o(z)B_m(z) \Longrightarrow \frac{Y(z)}{U_c(z)} = \frac{B_m(z)B(z)}{A_m(z)}$$

- Is it possible to get rid of $B(z)$?

Yes. Under certain conditions! You are going to learn how to do it later today.

Can we add more freedom?



What's the closed loop transfer function from command signal Uc to y?

$$\frac{Y(z)}{U_{ff}(z)} = \frac{B(z)R(z)}{A(z)R(z)+B(z)S(z)} \qquad \frac{U_{ff}(z)}{U_c(z)} = H_{ff}(z) \Longrightarrow \frac{Y(z)}{U_c(z)} = H_{ff}(z)\frac{B(z)R(z)}{A(z)R(z)+B(z)S(z)}$$

Choose R(z) and S(z) such that

$$A_{cl}(z) = A(z)R(z)+B(z)S(z) = A_m(z)A_o(z)$$

$$\frac{Y(z)}{U_c(z)} = H_{ff}(z)\frac{B(z)R(z)}{A_m(z)A_o(z)}$$

Design $H_{ff}(z)$ to match $\frac{B_m(z)}{A_m(z)}$

$$H_{ff}(z)\frac{B(z)R(z)}{A_m(z)A_o(z)} = \frac{B_m(z)}{A_m(z)} \Longrightarrow H_{ff} = \frac{A_o(z)B_m(z)}{B(z)R(z)}$$

- This is the overall picture of the design of the two degree of freedom controller.

  - Is it simpler than the state space approach?
  - Do we need to design an observer?

We do not need an observer because we do need to estimate the state variables!

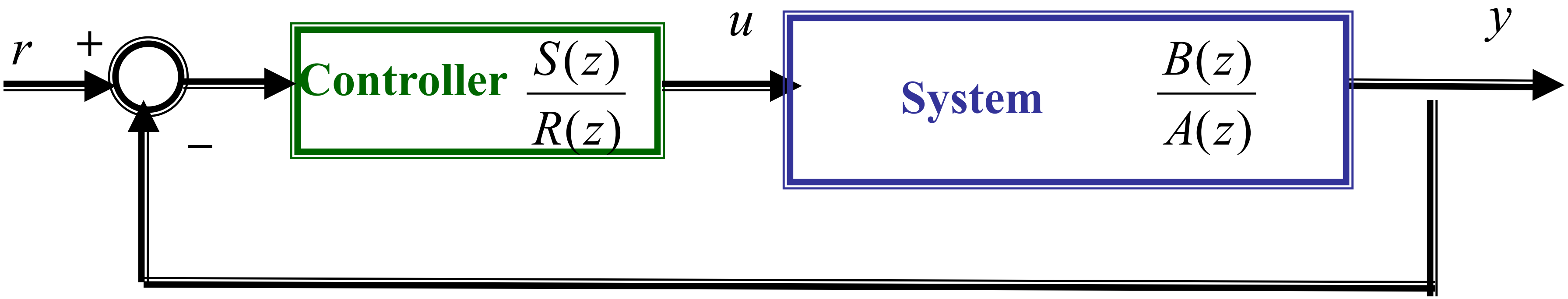
But how to solve the equation: $A_{cl}(z) = A(z)R(z) + B(z)S(z)$?

Diophantine Equation

Break

Self-Driving Car

As in the state-space approach, let's try to design the feedback controller first.



$$\frac{Y}{R}=\frac{\frac{S(z)}{R(z)}\frac{B(z)}{A(z)}}{1+\frac{S(z)}{R(z)}\frac{B(z)}{A(z)}}=\frac{S(z)B(z)}{R(z)A(z)+S(z)B(z)}$$

$$R(z)A(z)+S(z)B(z)\rightarrow A_{cl}(z)=A_m(z)A_o(z)$$

Example: the double integrator

$$Y(z)=\frac{\frac{h^2}{2}(z+1)}{(z-1)^2}U(z)$$

$$A(z)y=B(z)u$$

$$A(z)=(z-1)^2$$

$$B(z)=\frac{h^2}{2}(z+1)$$

Let's try the simplest controller, the proportional controller, $R=1, S=s_0$

$$RA+SB=(z-1)^2+\frac{h^2}{2}s_0(z+1)=z^2+(\frac{h^2}{2}s_0-2)z+\frac{h^2}{2}s_0+1 \Longrightarrow A_{cl}=z^2+p_1z+p_2$$

<u>Can we match them?</u>

Two equations with one design parameter s_0----mission impossible!

So we have to give more freedom to the controller!

Then try the first-order controller, $R = z + r_1$

$S = s_0 z + s_1$

Compare it to the P controller, how many extra control parameters? Two more!

Increase the order of the controller by ONE $\Longrightarrow$ Design parameters jump by TWO

$$AR + BS = (z-1)^2 (z + r_1) + \frac{h^2}{2}(z+1)(s_0 z + s_1)$$

$$= z^3 + (r_1 + \frac{h^2}{2}s_0 - 2)z^2 + (1 - 2r_1 + \frac{h^2}{2}(s_0 + s_1))z + r_1 + s_1\frac{h^2}{2}$$

•If the desired C.P. is $A_{cl} = z^3 + p_1 z^2 + p_2 z + p_3$

Match the coefficients: •How many equations? •How many design parameters?

So we can easily compute the design parameters by solving the three equations.

•What if the desired C.P. of the reference model is $A_m = z^2 + p_1 z + p_2$?

•Can we match A_m direcly? No!

Let's introduce A_o such that $A_{cl} = A_m A_o$

One simple choice: $A_o = z$ $\Longrightarrow$ $A_{cl} = z^3 + p_1 z^2 + p_2 z$

Can we match A_{cl} now? Yes!

What is the lesson? The controller must have sufficient degree of freedom!

Now let's consider the following question:

Given polynomials A(z) and B(z), and a third polynomial , $A_{cl}(z)$

can we find R(z) and S(z) such that

$$A(z)R(z) + B(z)S(z) = A_{cl}(z) \Longrightarrow$$ Diophantine equation

- Let's consider second order system

$$A(z) = a_0 z^2 + a_1 z + a_2$$
$$B(z) = b_0 z^2 + b_1 z + b_2$$
$$A_{cl}(z) = p_0 z^3 + p_1 z^2 + p_2 z + p_3$$

- Design a first order controller

$$R(z) = r_0 z + r_1$$
$$S(z) = s_0 z + s_1$$

- From $A(z)R(z) + B(z)S(z) = A_{cl}(z)$

compare the coefficients

$$
\begin{array}{lll}
z^3: & p_0 & a_0 r_0 + b_0 s_0 \\
z^2: & p_1 & a_1 r_0 + a_0 r_1 + b_1 s_0 + b_0 s_1 \\
z: & p_2 & a_2 r_0 + a_1 r_1 + b_2 s_0 + b_1 s_1 \\
1: & p_3 & a_2 r_1 + b_2 s_1
\end{array}
$$

- How many equations?

Four

- How many design parameters?

Four

Rewrite it in a compact matrix form,

$$\begin{bmatrix} a_0 & 0 & b_0 & 0 \\ a_1 & a_0 & b_1 & b_0 \\ a_2 & a_1 & b_2 & b_1 \\ 0 & a_2 & 0 & b_2 \end{bmatrix} \begin{bmatrix} r_0 \\ r_1 \\ s_0 \\ s_1 \end{bmatrix} = \begin{bmatrix} p_0 \\ p_1 \\ p_2 \\ p_3 \end{bmatrix}$$

- Sylvester matrix

- What is the condition on the Sylvester matrix for getting the solution?

The solution exists as long as the Sylvester matrix is non-singular!

- What is the condition for non-singularity of Sylvester matrix?

Let's take a look at a simple example:

$$\frac{Y(z)}{U(z)} = \frac{(z+1)}{(z+2)(z+1)}$$

$$A(z) = (z+2)(z+1)$$
$$B(z) = z+1$$

$$A(z)R(z) + B(z)S(z) = (z+2)(z+1)R(z) + (z+1)S(z)$$
$$= (z+1)[(z+2)R(z) + S(z)]$$

Is it possible to choose R and S such that AR+BS matches any A_{cl}?

Can you design R(z) and S(z) such that

$$(z+1)[(z+2)R(z) + S(z)] = z^3$$

It is impossible! Unless A_{cl} contains the same factor (z+1).

So one necessary condition to assure Sylvester matrix is nonsingular:
A(z) and B(z) should not have any common factors.

Sylvester matrix is 2nx2n matrix:

$$M_s = \underbrace{\begin{bmatrix} a_0 & 0 & \cdots & 0 & b_0 & 0 & \cdots & 0 \\ a_1 & a_0 & \ddots & \vdots & b_1 & b_0 & \ddots & \vdots \\ \vdots & a_1 & \ddots & 0 & \vdots & b_1 & \ddots & 0 \\ \vdots & \vdots & \ddots & a_0 & \vdots & \vdots & \ddots & b_0 \\ a_n & \vdots & \vdots & a_1 & b_n & \vdots & \vdots & b_1 \\ 0 & \ddots & \vdots & \vdots & 0 & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & a_n & 0 & \cdots & 0 & b_n \end{bmatrix}}_{<------n------><------n------>}$$

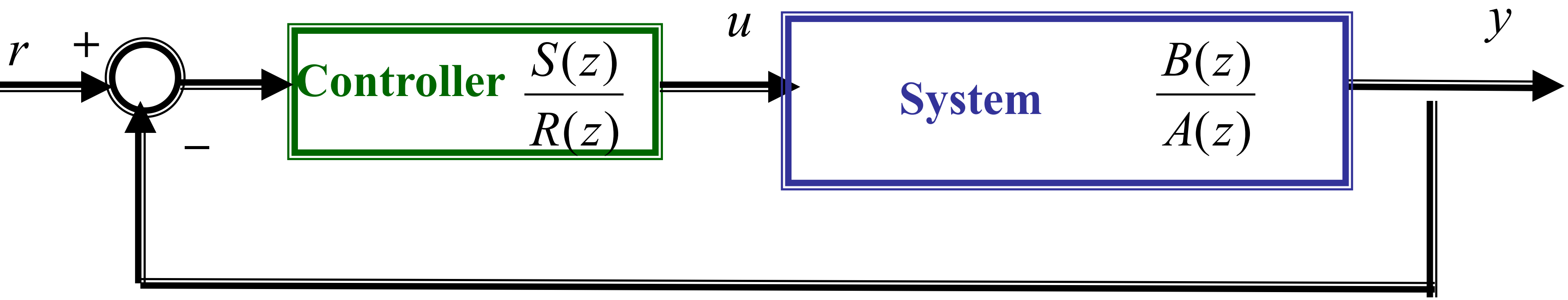
$$A(z) = a_0 z^n + a_1 z^{n-1} + \cdots + a_n$$

$$B(z) = b_0 z^n + b_1 z^{n-1} + \cdots + b_n$$

- <u>Sylvester Theorem:</u>

<u>The Sylvester matrix is nonsingular if and only if the two polynomials A(z) and B(z) have no common factors.</u>

The basic solution for pole placement:



Assumptions: the order of the system is n: Deg A=n, Deg B <= n

The order of controller is n-1

$$A(z) = a_0 z^n + a_1 z^{n-1} + \cdots + a_n \qquad R(z) = r_0 z^{n-1} + r_1 z^{n-2} + \cdots + r_{n-1}$$

$$B(z) = b_0 z^n + b_1 z^{n-1} + \cdots + b_n \qquad S(z) = s_0 z^{n-1} + s_1 z^{n-2} + \cdots + s_{n-1}$$

How many design parameters? 2n

The order of the closed loop: 2n-1 $\quad A_{cl}(z) = AR + BS = p_0 z^{2n-1} + p_1 z^{2n-2} + \cdots + p_{2n-1}$

Then if A and B have no common factors, R and S can be uniquely determined from

$$A(z)R(z) + B(z)S(z) = A_{cl}(z) \quad \Longrightarrow \quad \begin{bmatrix} r_0 \\ \vdots \\ r_{n-1} \\ s_0 \\ \vdots \\ s_{n-1} \end{bmatrix} = M_s^{-1} \begin{bmatrix} p_0 \\ p_1 \\ \vdots \\ \vdots \\ \vdots \\ p_{2n-1} \end{bmatrix}$$

Why is the order of the controller n-1? Can we use lower-order controller?

Assume the order of the controller is m, how many design parameters?

$$R(z) = r_0 z^m + r_1 z^{m-1} + \cdots + r_m$$
$$S(z) = s_0 z^m + s_1 z^{m-1} + \cdots + s_m$$

$\Longrightarrow$ 2(m+1)

What is the order of the closed loop AR+SB?

m+n

How many equations can you get by matching AR+SB with $A_{cl}(z)$?

m+n+1

The number of design parameters should be at least the number of the matching eqns.

$$2(m+1) \geq m+n+1 \quad \Longrightarrow \quad m \geq n-1$$

What would happen if the order of the controller is larger than n-1?

The number of design parameters exceeds the number of equations!

How many solutions?

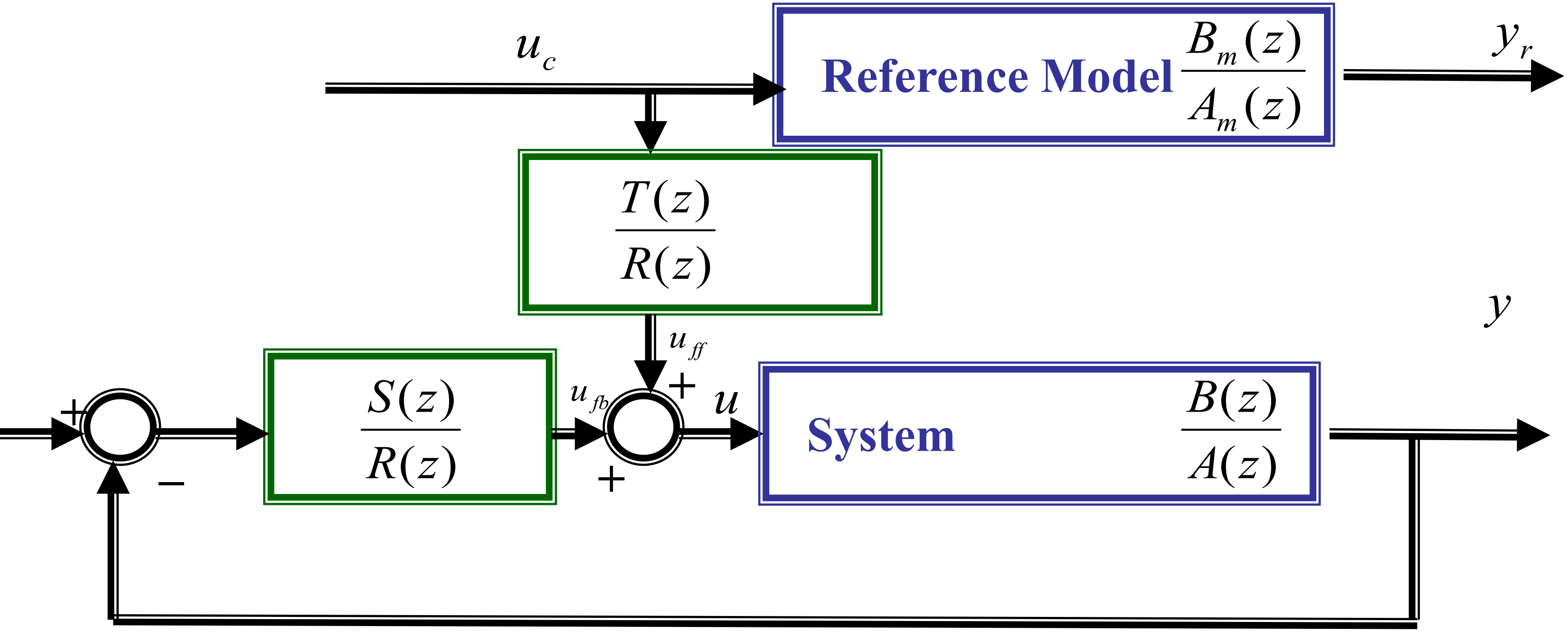
Infinity. The extra freedom can be used to satisfy other design specifications.

What are the other considerations:

1. Reduce the order of the whole system.
2. Disturbance rejection

Reduce the order----the motivation:

It is preferable sometimes to keep the order of the closed loop system as low as possible such that the design of controller is simpler and the dynamics of the system is also simpler.



What's the closed loop transfer function from command signal Uc to y?

$$\frac{Y(z)}{U_{ff}(z)} = \frac{B(z)R(z)}{A(z)R(z)+B(z)S(z)}$$

$$\frac{U_{ff}(z)}{U_c(z)} = \frac{T(z)}{R(z)}$$

$$\Longrightarrow \quad \frac{Y(z)}{U_c(z)} = \frac{T(z)B(z)}{A(z)R(z)+B(z)S(z)} = \frac{T(z)B(z)}{A_{cl}(z)}$$

$$\frac{Y(z)}{U_c(z)} = \frac{T(z)B(z)}{A_{cl}(z)}$$

How to reduce the order?

Reduce the order by pole-zero cancellation!

If B(z) is stable, can we cancel out B(z) by designing $A_{cl}(z)$ properly?

$$A_{cl}(z) = \overline{A}_{cl}(z)B(z) \Longrightarrow \frac{Y(z)}{U_c(z)} = \frac{T(z)B(z)}{\overline{A}_{cl}(z)B(z)} = \frac{T(z)}{\overline{A}_{cl}(z)}$$

If B(z) is stable, we can ger rid of B(z) by pole zero cancellation!

If B(z) contains both stable and unstable factors: $B = B^+ B^-$

$$A_{cl}(z) = \overline{A}_{cl}(z)B^+(z) \Longrightarrow \frac{Y(z)}{U_c(z)} = \frac{T(z)B(z)}{\overline{A}_{cl}(z)B^+(z)} = \frac{T(z)B^-(z)}{\overline{A}_{cl}(z)}$$

Can we cancel out the unstable factor B^-?

No!

$$\frac{Y(z)}{U_c(z)} = \frac{T(z)B(z)}{A_{cl}(z)}$$

Reduce the order by pole-zero cancellation!

If A(z) has stable poles $A = A^+A^-$ can we cancel out the stable factor A^+ ?

Let $A_{cl}(z) = \overline{A}_{cl}(z)A^+(z)$

Where to find the factors to cancel A^+ ?

$$T(z) = \overline{T}(z)A^+(z)$$

- Can we cancel out the stable factors of A(z) and B(z) at the same time?

Let $\begin{cases} A = A^+A^- \\ B = B^+B^- \end{cases} \Longrightarrow A_{cl}(z) = A^+B^+\overline{A}_{cl}$ $\quad T(z) = \overline{T}(z)A^+(z)$

$$\frac{B(z)T(z)}{A_{cl}(z)} = \frac{B^+B^-A^+\overline{T}}{A^+B^+\overline{A}_{cl}} = \frac{B^-\overline{T}}{\overline{A}_{cl}}$$

But how does this pole-zero cancellation affect the controller design, R(z) and S(z)?

•The corresponding Diophantine equation

$$A(z)R(z)+B(z)S(z)=A_{cl}(z)$$

becomes

$$A^{+}A^{-}R+B^{+}B^{-}S=A^{+}B^{+}\overline{A}_{cl}$$

Both terms on the left side have to contain $A^{+}(z)B^{+}(z)$

How to make the first term $A^{+}(z)A^{-}(z)R(z)$ contain $A^{+}(z)B^{+}(z)$?

Where to find $B^{+}(z)$? $\Longrightarrow$ $R=B^{+}\overline{R}$

How to make the second term $B^{+}(z)B^{-}(z)S(z)$ contain $A^{+}(z)B^{+}(z)$?

Where to find $A^{+}(z)$? $\Longrightarrow$ $S=A^{+}\overline{S}$

•How to design $\overline{R}$ and $\overline{S}$?

$$A^{+}A^{-}R+B^{+}B^{-}S=A^{+}B^{+}\overline{A}_{cl} \Longrightarrow A^{-}\overline{R}+B^{-}\overline{S}=\overline{A}_{cl}$$

Is the order of this Diophantine equation lower than the original one?

•Yes

So the controller design is simplified!

Example: $H(z)=\dfrac{(z-b)}{(z-1)(z-a)}$ |b|<1 and a>1 Reference model: $H_m(z)=\dfrac{z(1+p_1+p_2)}{z^2+p_1z+p_2}$

Z=b is a stable zero, which may be cancelled out.

Let's try the first order controller first. $R(z)=(z-b), S(z)=s_0z+s_1$

The closed loop:

$$R(z)A(z)+S(z)B(z)=(z-b)(z-1)(z-a)+(s_0z+s_1)(z-b)=(z-b)(z^2+(s_0-a-1)z+a+s_1)$$

The desired closed loop: $A_{cl}(z)=(z-b)(z^2+p_1z+p_2)$

Match the polynomials:

$$\begin{aligned}(s_0-a-1)&=p_1\\ a+s_1&=p_2\end{aligned} \quad\Longrightarrow\quad \begin{aligned}s_0&=p_1+a+1\\ s_1&=p_2-a\end{aligned}$$

The closed loop T.F. from the command signal to the output:

$$\frac{Y(z)}{U_c(z)}=\frac{(z-b)T(z)}{(z-b)(z^2+p_1z+p_2)}\approx\frac{T(z)}{(z^2+p_1z+p_2)}$$

How to choose T(z) to match the reference model?

$$T(z)=z(1+p_1+p_2)$$

The resulting controller is

$$U(z) = -\frac{S(z)}{R(z)}Y(z) + \frac{\mathrm{T}(z)}{R(z)}\mathrm{U}_c(z)$$

Put it in the time-domain, we have

$$R(q)u(k) = T(q)u_c(k) - S(q)y(k)$$

$$(q-b)u(k) = q(1+p_1+p_2)u_c(k) - (s_0 q + s_1)y(k)$$

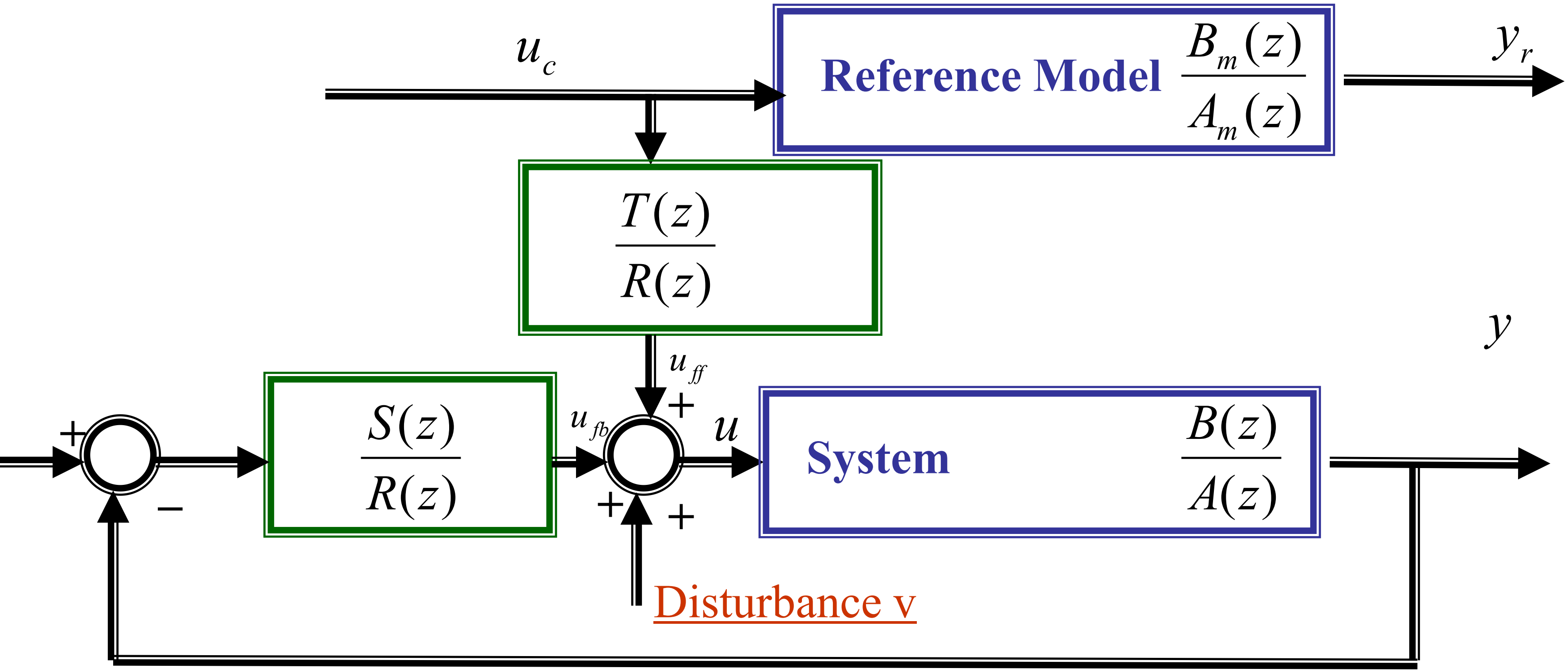
$$u(k+1) - bu(k) = (1+p_1+p_2)u_c(k+1) - s_0 y(k+1) - s_1 y(k)$$

$$u(k) = bu(k-1) + (1+p_1+p_2)u_c(k) - s_0 y(k) - s_1 y(k-1)$$

How to implement this by programming?

Do it by a loop!

Disturbance Rejection Problem



The simplest case: $A(z)Y(z) = B(z)(U(z)+V(z))$

Let's find the transfer function from disturbance v to output y for the closed loop:

What's the feed-forward transfer function from v to y? $\frac{B(z)}{A(z)}$

What's the open-loop transfer function from v to y? $\frac{B(z)}{A(z)}(-1)\frac{S(z)}{R(z)}(-1) = \frac{B(z)}{A(z)}\frac{S(z)}{R(z)}$

$$\frac{Y(z)}{V(z)} = \frac{\frac{B(z)}{A(z)}}{1+\frac{B(z)}{A(z)}\frac{S(z)}{R(z)}} = \frac{B(z)R(z)}{A(z)R(z)+B(z)S(z)} = \frac{B(z)R(z)}{A_{cl}(z)}$$

So overall, the output is affected by both the disturbance and the command signal

$$Y = \frac{BRV}{(RA+BS)} + \frac{BTU_c}{(RA+BS)}$$

Let's consider the simple case that the disturbance is an unknown constant.
Then the corresponding steady state output subjected to this constant input is simply the disturbance constant multiplied by the DC gain.

How do we compute the DC gain of a discrete-time transfer function G(z) ?

$$G(z)\,|_{z=1} = G(1)$$

What is the DC gain from the disturbance to the output?

$$\left.\frac{B(z)R(z)}{A_{cl}(z)}\right|_{z=1} = \frac{B(1)R(1)}{A_{cl}(1)}$$

How to design R(z) such that the DC gain is 0, i.e. R(1)=0?

$$R = (z-1)R'(z)$$

It means that there is an integrator in the controller!

What about other types of disturbances?

Although the disturbance is unknown to us, we can reject its influence as long as we know the key characteristics of the disturbance.

For instance, is it a constant, or is it periodic? If it is periodic, how many frequencies are there? If we have those information, we can easily reject its effect on the system at steady state.

Please refer to my lecture notes for more details.

Example: $y(k+1)=3y(k)+u(k)+v(k)$ $\Longrightarrow$ A(q)y(k)=B(q)(u(k)+v(k))

where v(k) is a constant disturbance:

How to design R(z)? $R=(z-1)R'(z)$

What is A(z) and B(z)?

A(z)=z-3 B(z)=1

The simplest controller is the first order controller

$$R(z)=(z-1)$$
$$S(z)=s_0z+s_1$$

The closed loop: $AR+BS=(z-3)(z-1)+s_0z+s_1=z^2+(s_0-4)z+s_1+3$

If the desired C.P. is $A_{cl}=z^2+p_1z+p_2$

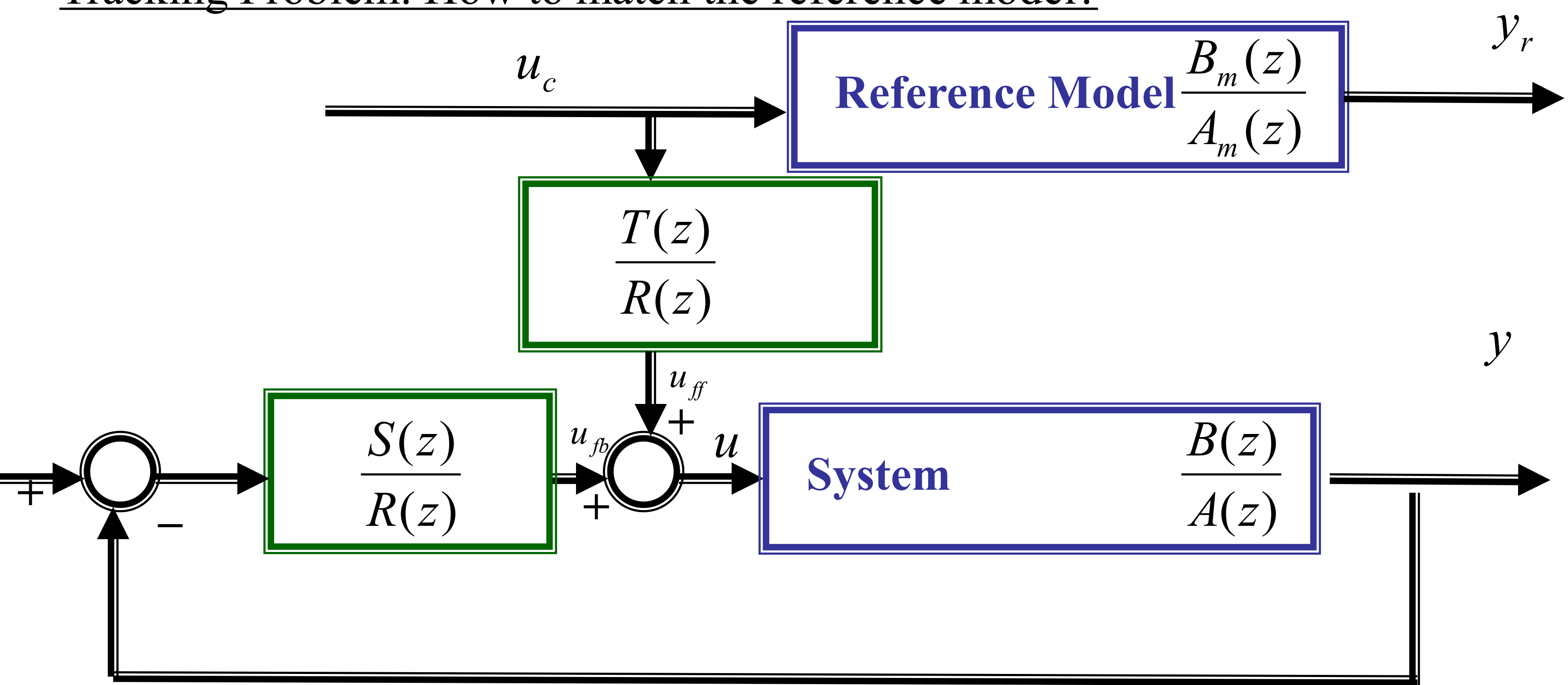
Can you match them? Of course!

$$\frac{Y(z)}{U_c(z)}=\frac{BT}{(RA+BS)}=\frac{T}{z^2+p_1z+p_2}$$

How to choose T?

To satisfy other requirements like zero placement.

Tracking Problem: How to match the reference model?



What's the closed loop transfer function from command signal Uc to y?

$$\frac{Y(z)}{U_{ff}(z)} = \frac{B(z)R(z)}{A(z)R(z)+B(z)S(z)} \qquad \frac{U_{ff}(z)}{U_c(z)} = \frac{T(z)}{R(z)} \Longrightarrow \frac{Y(z)}{U_c(z)} = \frac{T(z)B(z)}{A(z)R(z)+B(z)S(z)}$$

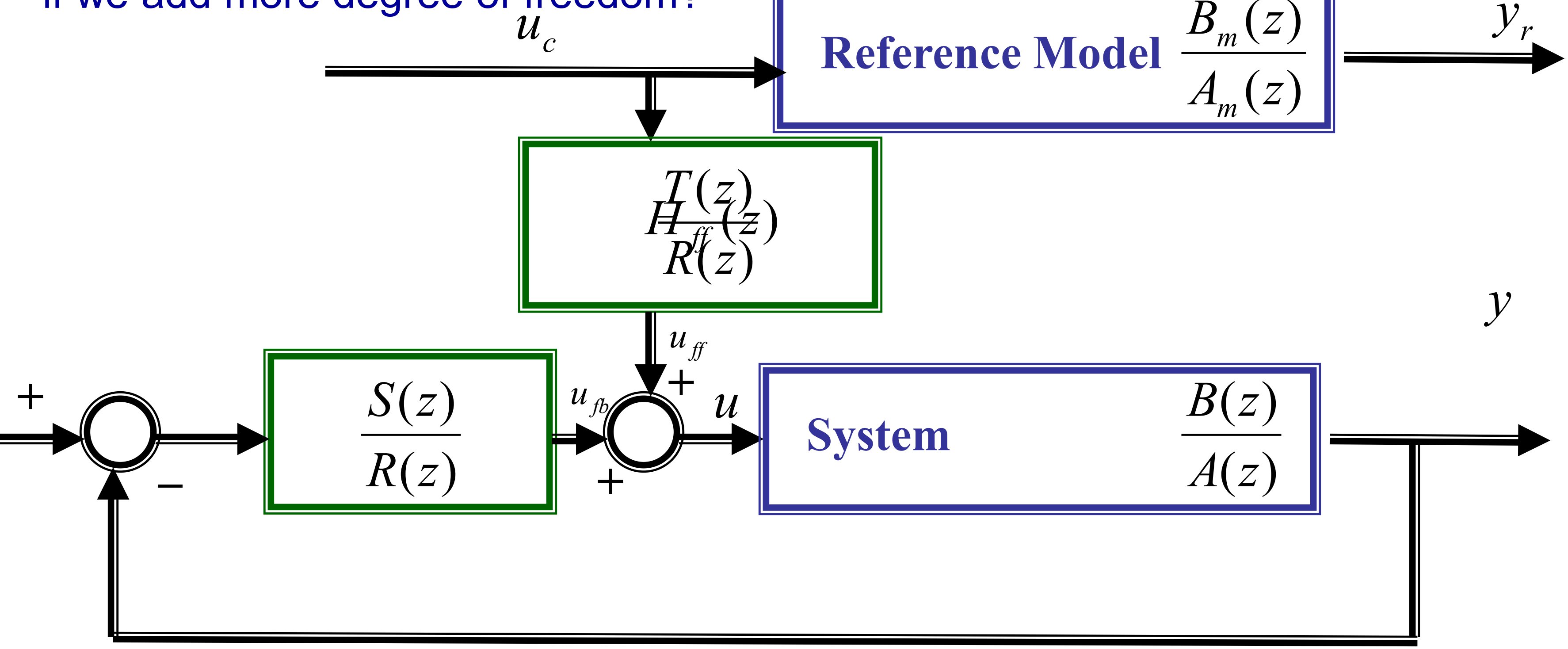
Can we cancel out B(z)? Yes if B(z) is stable! And let R(z) contain B(z)

Choose R(z) and S(z) such that $A_{cl}(z) = A(z)R(z)+B(z)S(z) = A_m(z)A_o(z)B(z)$

How to choose T(z) to match the reference model? $T(z) = B_m(z)A_o(z)$

$$\frac{Y(z)}{U_c(z)} = \frac{T(z)B(z)}{A(z)R(z)+B(z)S(z)} = \frac{B_m(z)A_o(z)B(z)}{A_m(z)A_o(z)B(z)} = \frac{B_m(z)}{A_m(z)}$$

Is the condition of stable inverse essential? Do we need this condition if we add more degree of freedom?



<u>What's the closed loop transfer function from command signal Uc to y?</u>

$$\frac{Y(z)}{U_{ff}(z)} = \frac{B(z)R(z)}{A(z)R(z)+B(z)S(z)} \qquad \frac{U_{ff}(z)}{U_c(z)} = H_{ff}(z) \Longrightarrow \frac{Y(z)}{U_c(z)} = H_{ff}(z)\frac{B(z)R(z)}{A(z)R(z)+B(z)S(z)}$$

Choose R(z) and S(z) such that

$$A_{cl}(z) = A(z)R(z)+B(z)S(z) = A_m(z)A_o(z)$$

$$\frac{Y(z)}{U_c(z)} = H_{ff}(z)\frac{B(z)R(z)}{A_m(z)A_o(z)}$$

Design $H_{ff}(z)$ to match $\frac{B_m(z)}{A_m(z)}$

$$H_{ff}(z)\frac{B(z)R(z)}{A_m(z)A_o(z)} = \frac{B_m(z)}{A_m(z)} \Longrightarrow H_{ff} = \frac{A_o(z)B_m(z)}{B(z)R(z)}$$

As in the state-space approach, we have to check whether all the signals are bounded.

What is the transfer function from the command signal to the output now?

$$\frac{Y(z)}{U_c(z)} = H_{ff}(z)\frac{B(z)R(z)}{A_m(z)A_o(z)} \xrightarrow{H_{ff} = \frac{A_o(z)B_m(z)}{B(z)R(z)}} \frac{Y(z)}{U_c(z)} = \frac{B_m(z)}{A_m(z)}$$

Are the outputs bounded? Yes!

Are the inputs bounded?

What is the transfer function from the command signal to the input ?

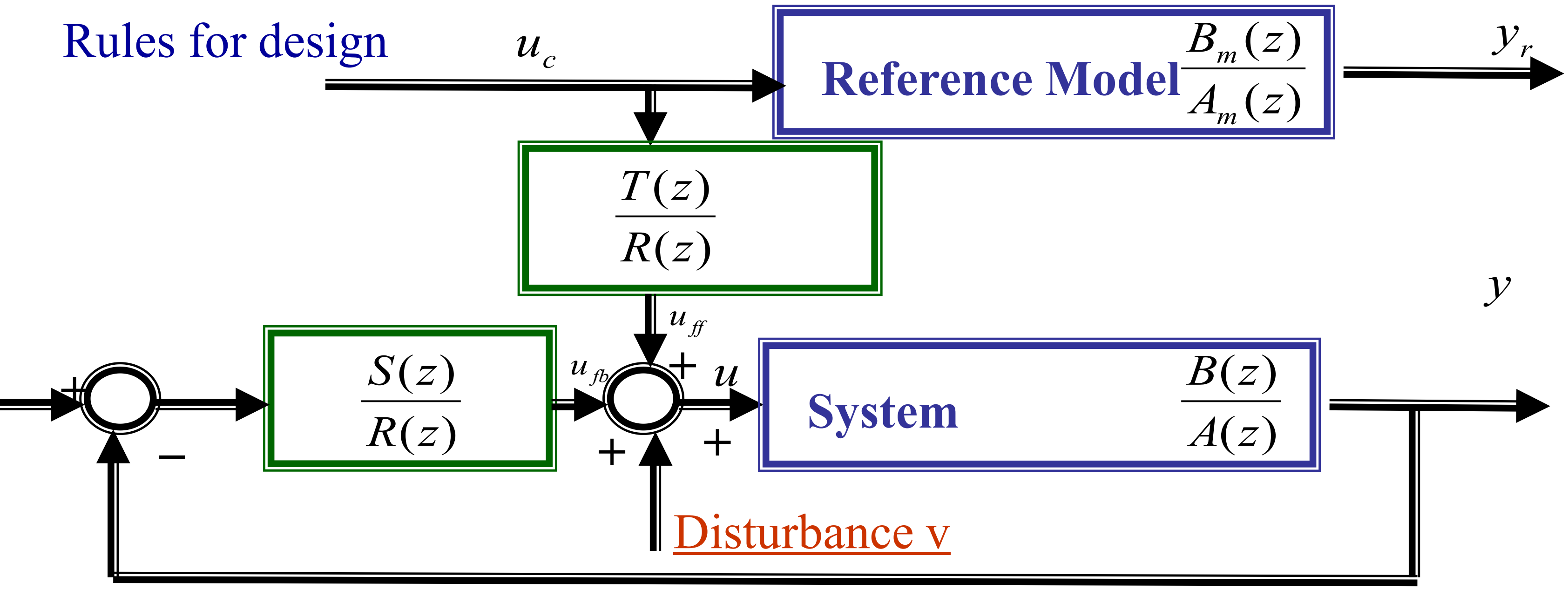
$$\frac{Y}{U_c} = \frac{B_m}{A_m} \xrightarrow{\frac{Y}{U} = \frac{B}{A}} \frac{U(z)}{U_c(z)} = \frac{A(z)\mathrm{B}_m(z)}{A_m(z)B(z)}$$

What is the condition to guarantee the boundedness of the input?

B(z) has to be stable!

Stable inverse condition is always needed for perfect tracking!

Rules for design



Occam's razor -- the simpler, the better.

Separation Property: Design feedback controller first, then build feedforward controller.

Step One: Figure out the design requirements on R(z).

- If need to cancel out zeros: R(z) must contain the zero polynomial B(z)
- Disturbance rejection: R must contain (z-1) (integrator) for constant disturbance.
- Causality conditions: Deg (R) >= Deg (S)

Step Two: Design R and S by solving the Diophantine equation $AR + BS = A_{cl}$

Step Three: Choose T or H_{ff} at the final stage, to satisfy other design requirements.

The order of controller can be increased in order to meet other design requirements.

Stable inverse condition is required for perfect tracking (perfect match of the reference model)

Design Example: ***speed control system***

Consider the vehicle, which has a weight $m = 1000$ kg. Assuming the average friction coefficient $b = 100$, design a speed control system such that the vehicle can reach 100 km/h from 0 km/h in 8 s with an overshoot less 5%. Assuming the sampling period $T = 0.6$ seconds.

The differential equation for the dynamics of the vehicle:

$$m\ddot{x} + b\dot{x} = u$$

In terms of speed, we have $m\dot{y} + by = u$

The T.F. for the continuous-time process is then

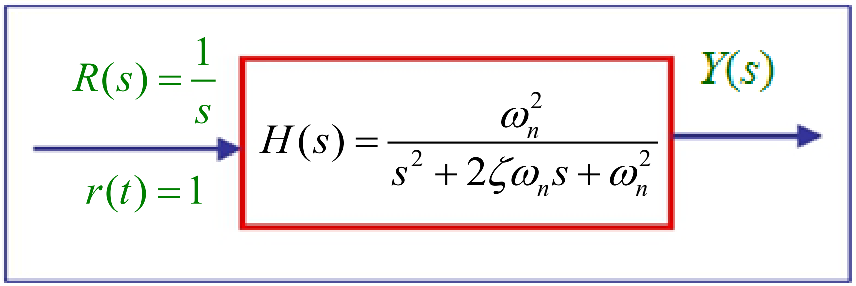
$$G(s) = \frac{1}{1000s + 100}$$

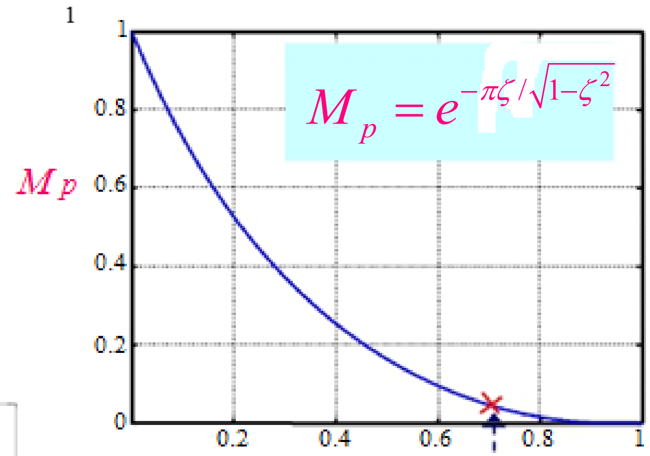
We can get the discrete-time T.F. from the continuous time T.F.

$$G(z) = \frac{0.00058}{z - 0.942}$$

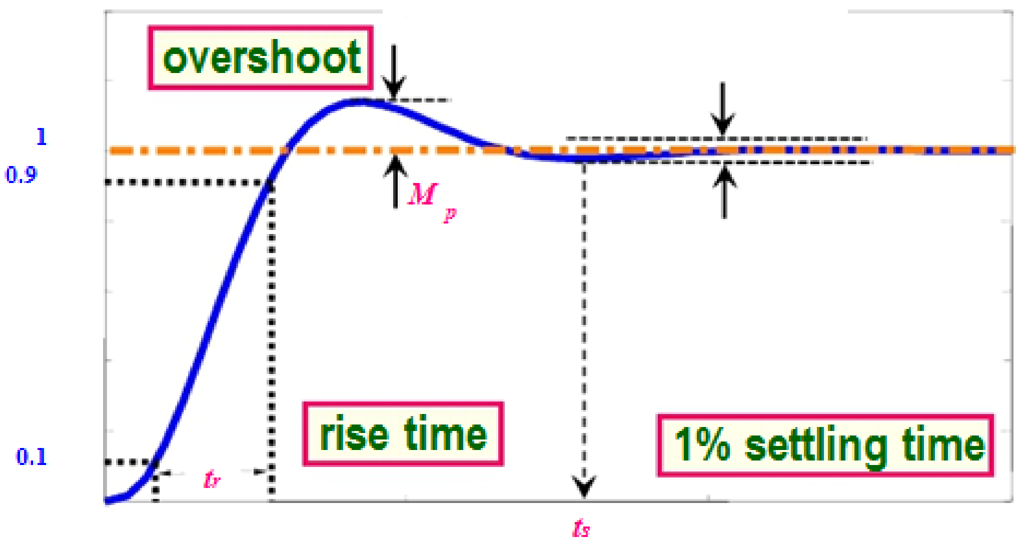
Get the continuous-time reference model:

First derive ξ and ω_n from the design specifications:





$$\zeta \geq 0.6901 \quad \Rightarrow \quad \zeta = 0.7$$



$$t_s \cong \frac{4.6}{\zeta\omega_n} \Rightarrow \omega_n \cong \frac{4.6}{t_s\zeta}$$

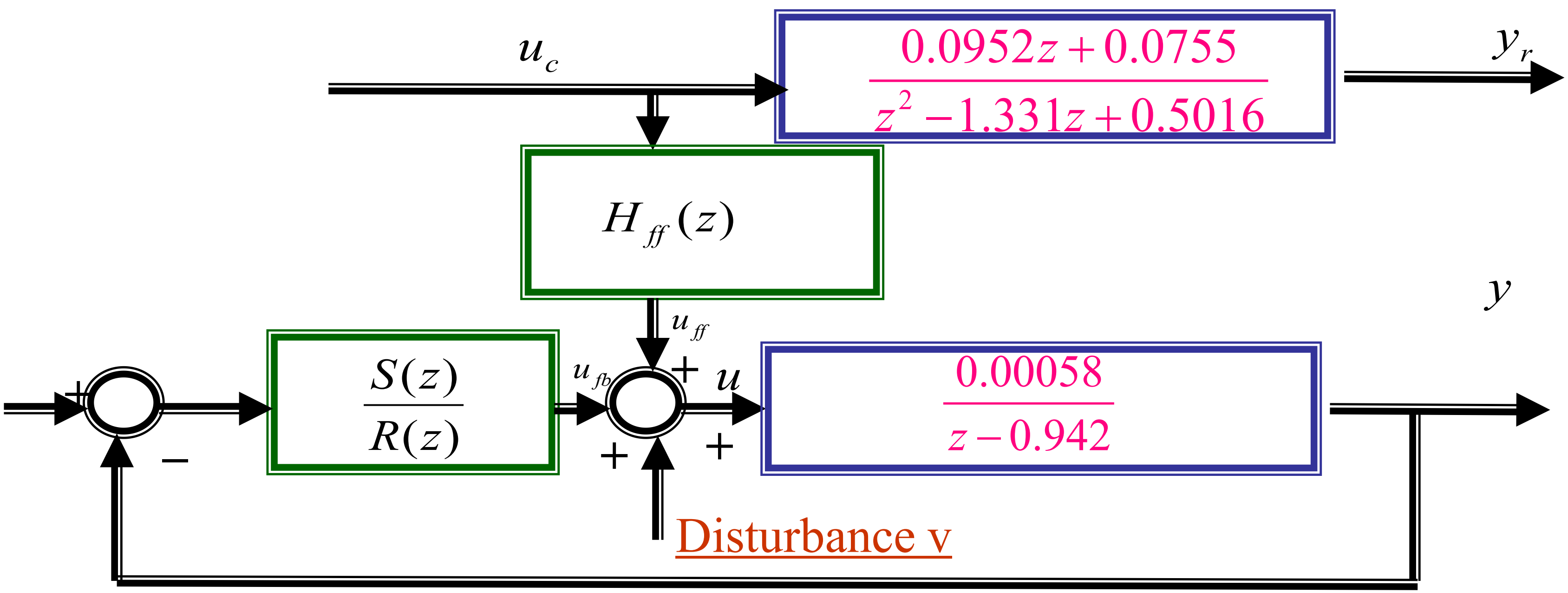
$$\Rightarrow \omega_n = \frac{4.6}{8\times 0.7} = 0.82$$

Desired CLTF $\quad \Rightarrow \quad H_d(s) = \dfrac{0.67}{s^2+1.15s+0.67}$

Convert the desired reference model from continuous time to discrete-time

You can use the conversion table 2.1 or the MATLAB command c2d.

$$H_d(z) = \frac{0.0952z + 0.0755}{z^2 - 1.331z + 0.5016}$$



We want to reject the constant disturbance, how to design R(z)?

$$R = (z-1)R'(z)$$

The simpler, the better. So first try $R = z - 1$ and $S = s_0 z + s_1$, it follows that

$$AR + BS = (z-0.942)(z-1) + 0.00058(s_0 z + s_1)$$
$$= z^2 + (0.00058 s_0 - 1.942)z + 0.00058 s_1 + 0.942$$

Compare with the $A_m(z)$

$$z^2 - 1.331z + 0.5016$$

Match the coefficients, we have

$$(0.00058 s_0 - 1.942) = -1.331 \qquad 0.00058 s_1 + 0.942 = 0.5016$$

$$\Rightarrow \qquad s_0 = 1053.4 \qquad s_1 = -759.3$$

We have

$$R = z - 1, \quad S = 1053.4z - 759.3$$

$$\frac{Y(z)}{U_c(z)} = H_{ff}(z)\frac{B(z)R(z)}{A(z)R(z)+B(z)S(z)}$$

$$\frac{Y(z)}{U_c(z)} = H_{ff}(z)\frac{B(z)R(z)}{A_m(z)}$$

Design $H_{ff}(z)$ to match $\frac{B_m(z)}{A_m(z)}$

$$H_{ff}(z)\frac{B(z)R(z)}{A_m(z)} = \frac{B_m(z)}{A_m(z)} \Longrightarrow H_{ff} = \frac{B_m(z)}{B(z)R(z)}$$

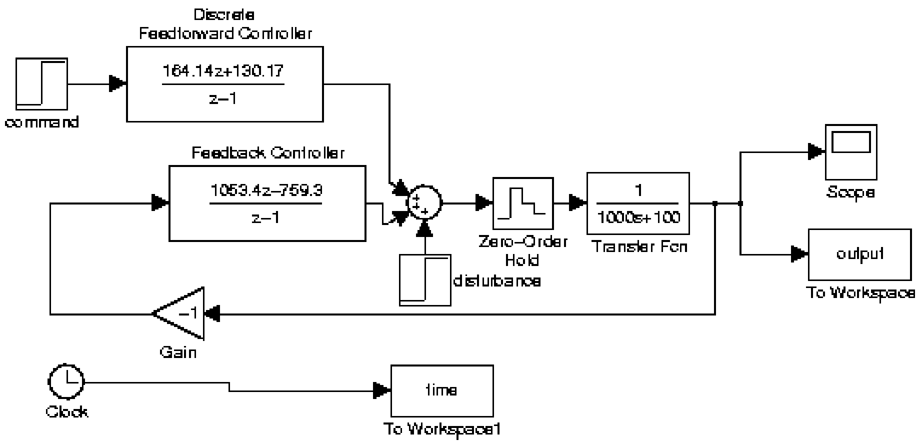
For feed-forward controller, we have

$$H_{ff}(z) = \frac{B_m}{BR} = \frac{0.0952z+0.0755}{0.00058(z-1)} = \frac{164.14z+130.17}{(z-1)}$$

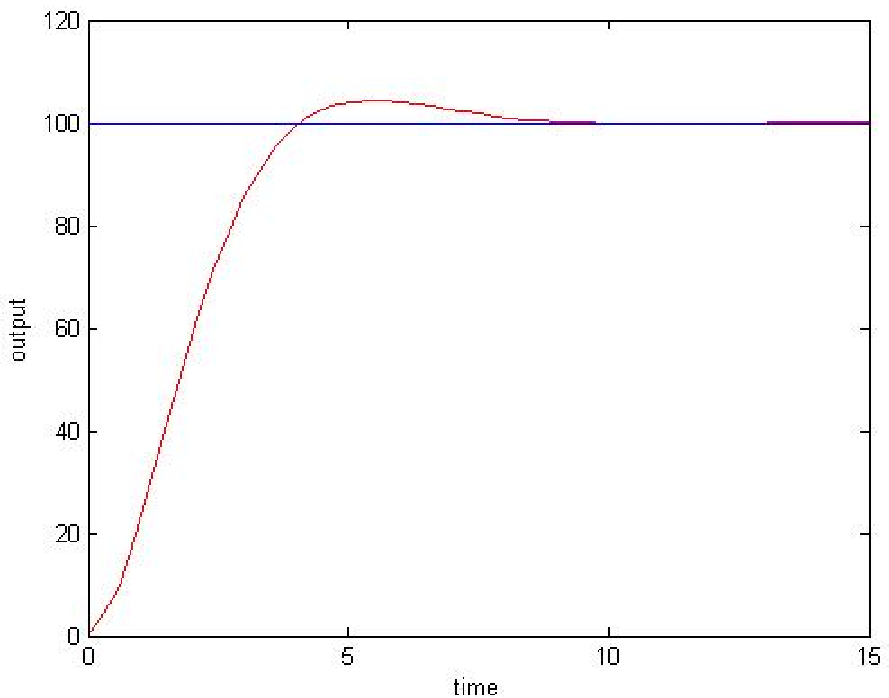
Therefore, the two-degree of freedom controller has the form of,

$$U(z) = -\frac{1053.4z-759.3}{z-1}Y(z)+\frac{164.14z+130.17}{z-1}U_c(z)$$

Now simulate the digital two-degree-of-freedom control system response with actual plant



Does it meet all the performance requirements?



Yes.

The overshoot is less than 5%.

The settling time is around 8s.

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Q & A…

THANK YOU !