Note Title 1/20/2011

Summary

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$$\dot{x} = Fx + Gu$$

$$y = Hx + Ju$$

assume system is "controllable"

i.e., $\mathcal{C} = [G \;\; FG \;\; F^2G \;\dots\; F^{n-1}G]$ is non-singular

Open-loop system has poles given by:

$$\underbrace{a(s) = \alpha_{ol}(s) = \overbrace{\det\{sI - F\}}^{\text{open-loop characteristic polynomial}}}_{} \underbrace{= 0}_{}$$

open-loop characteristic polynomial

$$\alpha_{ol}(s) = s^n + a_1 s^{n-1} + a_2 s^{n-2} + \dots + a_n$$

open-loop characteristic equation

Open-loop system is not satisfactory for ~~precision~~ control because of, for example,

* no feedback to ensure tracking errors $= 0$

* open-loop system dynamics are not satisfactory (e.g. too slow; unstable etc)

Taking the approach of state-feedback, i.e.

$$u = -k_1 x_1 - k_2 x_2 - \ldots - k_n x_n$$

$$= -\underbrace{K}_{1\times n}\,\underbrace{x}_{n\times 1} \qquad = -\tilde{K}^T x$$

$K = [k_1\ k_2\ \ldots\ k_n]$

$$\tilde{K} = \begin{bmatrix} k_1 \\ k_2 \\ \vdots \\ k_n \end{bmatrix}$$

With state-feedback, system now becomes:

$$\dot{x} = Fx + Gu$$

$$\dot{x} = Fx + G\{-Kx\}$$

$$\dot{x} = \{F - GK\}x$$

$$y = Hx + Ju$$

System, with state-feedback, above has (closed-loop) poles given by:

$$\underbrace{\alpha_c(s) = \det\{sI - [F - Gk]\} = 0}$$

closed-loop char polynomial

closed-loop char equation

The above is what we have obtained through state-feedback

In terms of desired closed-loop performance, we would specify a desired closed-loop characteristic polynomial:

$$\alpha_c(s) = s^n + \alpha_1 s^{n-1} + \alpha_2 s^{n-2} + \dots + \alpha_n$$

Recall
* motor example $\alpha_c(s) = s^2 + 2s + 1$
* oscillator example $\alpha_c(s) = (s + 2w_0)^2$

How to properly specify a desired $\alpha_c(s)$ was studied earlier today !!!

The necessary K (state-feedback) gain to yield the desired $\alpha_c(s)$ is given by either

Ackermann's formula

$$K = [0 \; 0 \; \cdots \; \underbrace{0}_{(n-1)} \; \vdots \; 1] \, \mathcal{C}^{-1} \alpha_c(F)$$

where $\mathcal{C} \triangleq [G \; \vdots \; FG \; \vdots \; F^2 G \; \vdots \; \cdots \; \vdots \; F^{n-1} G]$

or

Bass-Gura formula

$$\tilde{K} = \begin{bmatrix} k_1 \\ k_2 \\ \vdots \\ k_n \end{bmatrix} = \left[(\mathcal{C} W)^T \right]^{-1} \begin{bmatrix} \alpha_1 - a_1 \\ \alpha_2 - a_2 \\ \vdots \\ \alpha_n - a_n \end{bmatrix}$$

$$\mathcal{C} = \begin{bmatrix} G & FG & F^2G & \cdots & F^{n-1}G \end{bmatrix}$$

$$W = \begin{bmatrix} 1 & a_1 & a_2 & \cdots & a_{n-1} \\ 0 & 1 & a_1 & \cdots & a_{n-2} \\ \vdots & & \ddots & \ddots & \\ 0 & \cdots & & 0 & 1 \end{bmatrix}$$

$n \times n$

Transformations between realizations

$$\dot{x} = F_1 x + G_1 u \xrightarrow{p = Tx} \dot{p} = F_2 p + G_2 u$$

$$\mathcal{C}_2 = \begin{bmatrix} G_2 & F_2G_2 & \cdots & F_2^{n-1}G_2 \end{bmatrix}$$

$$F_2 = TF_1T^{-1}$$

$$G_2 = TG_1$$

$$G_2 = TG_1$$

$$F_2G_2 = \left\{TF_1T^{-1}\right\}\left\{TG_1\right\} = TF_1G_1$$

$$F_2^2 G_2 = \{T F_1 T^{-1}\}\{T F_1 T^{-1}\}\{T G_1\}$$

$$= T F_1^2 G_1$$

i.e.

$$\mathcal{C}_2 = \left[G_2 \;\; F_2 G_2 \;\; F_2^2 G_2 \; \ldots \; F_2^{n-1} G_2 \right]$$

$$= T \left[G_1 \;\; F_1 G_1 \;\; F_1^2 G_1 \ldots F_1^r G_1 \ldots F_1^{n-1} G_1 \right]$$

from pattern above

$\mathcal{C}_1$

i.e. $\mathcal{C}_2 = T\,\mathcal{C}_1$

i.e. $T = \mathcal{C}_2 \mathcal{C}_1^{-1}$

Selection of Pole Locations

i.e. how to choose

$$\alpha_c(s) = s^n + \alpha_1 s^{n-1} + \alpha_2 s^{n-2} + \ldots + \alpha_n$$

① Approach #1 : Prototype Response Tables

e.g. ITAE tables

$$J_{ITAE} = \int_0^\infty t\,|y-r|\,dt$$

② Approach #2 : Symmetric Root Locus (SRL)

$$\dot{x} = Fx + Gu; \quad y = Hx$$

Define a signal important to you ....

... called z where

$$z = H_1 x$$

SRL method gives choice of $\alpha_c(s)$ which minimises

$$J_{SRL} = \int_0^\infty (\rho z^2 + u^2)\, dt$$

where $\rho > 0$

Will need to draw the <u>Symmetric Root</u> Locus

$$\boxed{1 + \rho\, G_0(-s)\, G_0(s) = 0} \quad \rho > 0$$

where

$$G_0(s) \triangleq \frac{Z(s)}{U(s)} = H_1 \{sI - F\}^{-1} G$$

Recall EE2010 root Locus

$$1 + K G(s) = 1 + k \frac{b(s)}{a(s)} = 0$$

Approach #3 : Full optimization

Linear Quadratic Regulator approach

$$\dot{x} = Fx + Gu$$

$$y = Hx$$

This approach looks at minimizing

$$J_{LQR} = \int_0^{\infty} \left\{ x^T Q x + r u^2 \right\} dt$$

Q: a positive-definite matrix

r: a positive number

For example, with $Q = \begin{bmatrix} q_1 & 0 & 0 \\ 0 & q_2 & 0 \\ 0 & 0 & q_3 \end{bmatrix}$, i.e., $n = 3$

$q_1, q_2, q_3 > 0$;

here,

$$J_{LQR} = \int_0^\infty \left\{ q_1 x_1^2 + q_2 x_2^2 + q_3 x_3^2 + r u^2 \right\} dt$$

For next Wednesday's class, we will "leap-frog" to Page 102 of the Lecture Notes, on topic "Introduction of Reference Signal (Methods)"

And then, we will all be ready for Experiment 1 !!!