EE5101/ME5401: Linear Systems: Part II

Decoupling Control

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What is a coupled system?

Consider a boiler shown below.

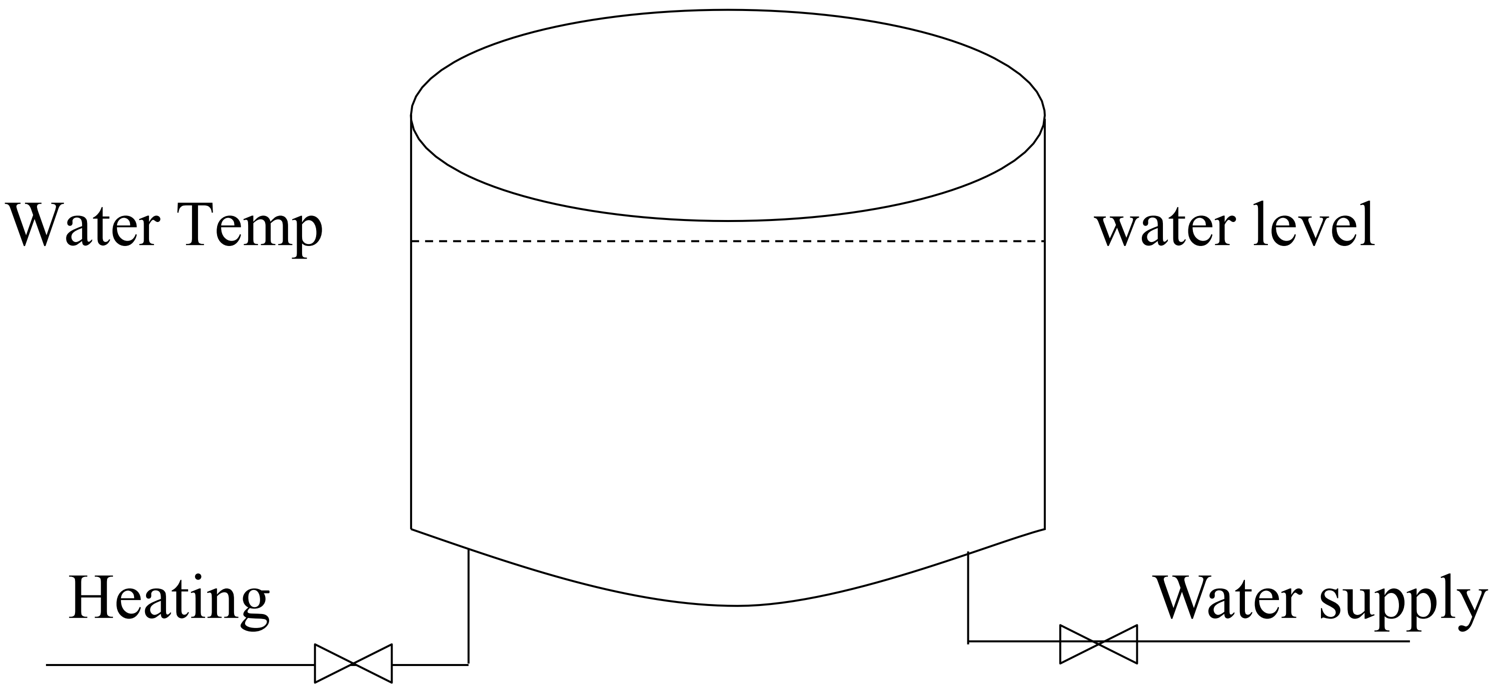


Figure 1 Boiler system.

Does heating affect water temperature? Of course. It is the main factor.

Does heating affect water level? Yes. A little bit.

The same is true for the other input: water supply, which affects both outputs.

This is a typical example of coupling. The interactions between loops may cause difficulty in operating it.

Let's look at this problem from the transfer functions. Consider a MIMO system:

$$\dot{x} = Ax + Bu,$$

$$y = Cx,$$

How to obtain the solution? What is the powerful tool to solve linear ODE?

Let's apply Laplace Transform

$$sX(s) - x(0) = AX(s) + BU(s) \Longrightarrow \begin{array}{l} sX(s) - AX(s) = x(0) + BU(s) \\ X(s) = (sI - A)^{-1}x(0) + (sI - A)^{-1}BU(s) \end{array}$$

$$Y(s) = CX(s) = C(sI - A)^{-1}x(0) + C(sI - A)^{-1}BU(s)$$

Zero-input response

Zero-state-response

If we only consider the response due to the input, we have the transfer function matrix.

But do keep in mind that the outputs are affected by both the initial condition and the inputs.

Consider an MIMO system:

$$\begin{aligned} \dot{x} &= Ax + Bu, \\ y &= Cx, \end{aligned} \tag{1}$$

The transfer function matrix is

$$Y(s) = C(sI - A)^{-1} BU(s) = G(s)U(s), \tag{2}$$

where G is supposed to be an *m by m* *square* matrix in general .

$$Y_1(s) = g_{11}(s)U_1(s) + g_{12}(s)U_2(s) + \cdots + g_{1m}(s)U_m(s) \ ,$$

$$Y_2(s) = g_{21}(s)U_1(s) + g_{22}(s)U_2(s) + \cdots + g_{2m}(s)U_m(s) \ ,$$

$$\vdots$$

$$Y_m(s) = g_{m1}(s)U_1(s) + g_{m2}(s)U_2(s) + \cdots + g_{mm}(s)U_m(s) \ . \tag{3}$$

If $g_{ij} \neq 0$, for some $i \neq j$, the system is called coupled since U_j affects not only Y_j but some Y_i, $i \neq j$.

What is a decoupled system?

If $g_{ii} \neq 0$ and $g_{ij} = 0$ for all $i \neq j$, the system is called decoupled. A decoupled system has a diagonal and non-singular *G(s)*:

$$
\begin{aligned}
y_1(s) &= g_{11}(s)u_1(s), \\
y_2(s) &= g_{22}(s)u_2(s), \\
&\vdots \\
y_m(s) &= g_{mm}(s)u_m(s).
\end{aligned}
\tag{4}
$$

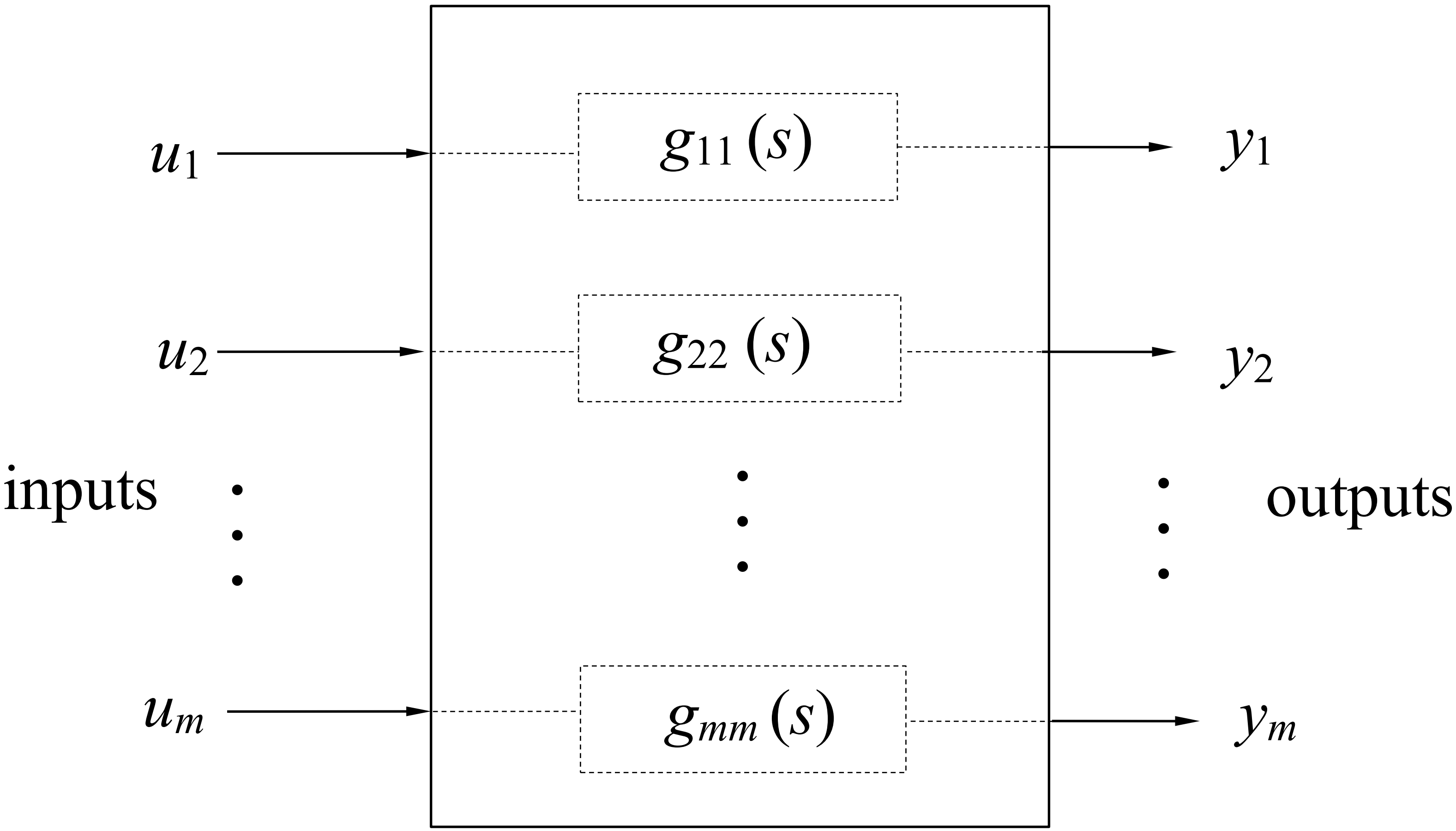


Figure 2   Decoupled system.

Which type of system is easier to control?

The decoupled system is easy to handle as we can treat it as many SISO systems and design the controller for each one using SISO method!

When we want to change one output, we only need to tune one input only!

An Industrial Motivation: distillation column control

Process:

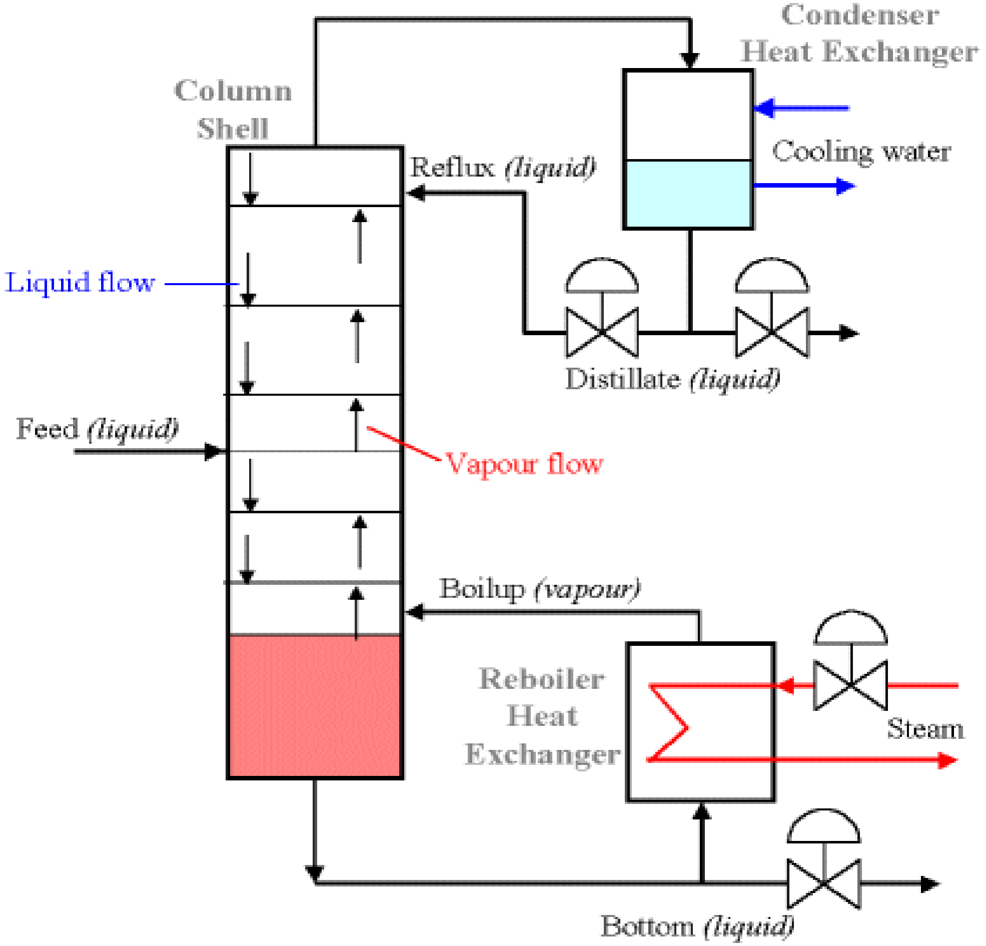


Figure 3 Distillation column.

Model: Wood and Berry (1973) transfer function model of a methanol-water distillation column is given by

$$\begin{bmatrix} X_D(s) \\ X_B(s) \end{bmatrix} = \begin{bmatrix} \dfrac{12.8}{16.7s+1}e^{-s} & \dfrac{-18.9}{21s+1}e^{-3s} \\ \dfrac{6.6}{10.9s+1}e^{-7s} & \dfrac{-19.4}{14.4s+1}e^{-3s} \end{bmatrix} \begin{bmatrix} R(s) \\ F(s) \end{bmatrix}$$

where $x_D(t)$ is the mole fraction of methanol in the top; $x_B(t)$ is the mole fraction of methanol in the bottom; $r(t)$ is the reflux flow rate; $f(t)$ is the steam flow rate.

Control requirements: decoupling, zero steady state error to step inputs and good dynamic performance.

What is decoupling?

- Decoupling control is to decouple a coupled plant such that feedback system is decoupled.

Why is decoupling useful?

- Decoupling is usually required in practice for easy operations.

How to decouple?

Two Schemes:

1) State Feedback -- with the state-space model
2) Output Feedback -- with the transfer function matrix

Let's start with some simple case, and then try to figure out the solution for the general case.

Illustrative Example

$$\dot{x} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} x + \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} u,$$

$$y = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} x = x.$$

Let's compute the transfer function matrix:

$$G(s) = C(sI - A)^{-1} B = \begin{bmatrix} s & -1 \\ -1 & s-1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

$$= \frac{1}{s^2 - s - 1} \begin{bmatrix} s-1 & 1 \\ 1 & s \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

$$= \frac{1}{s^2 - s - 1} \begin{bmatrix} s-1 & s \\ 1 & s+1 \end{bmatrix}$$

$$= \begin{bmatrix} \dfrac{s-1}{s^2 - s - 1} & \dfrac{s}{s^2 - s - 1} \\ \dfrac{1}{s^2 - s - 1} & \dfrac{s+1}{s^2 - s - 1} \end{bmatrix}$$

Is it a coupled or decoupled system? It is coupled!

$$\dot{x} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} x + \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} u,$$

$$y = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} x = x.$$

Let’s try to design state feedback control:

$$u = -Kx + Fr$$

Such that the transfer function matrix from R to Y is decoupled as simple as the integrator:

$$H(s) = \begin{bmatrix} \frac{1}{s} & 0 \\ 0 & \frac{1}{s} \end{bmatrix}$$

If the final TF is $$H(s)=\begin{bmatrix}\frac{1}{s} & 0 \\ 0 & \frac{1}{s}\end{bmatrix}$$

The corresponding state space equation is

$$\dot{x}=r$$

Is that really possible?

Let's design K first. We need to make all the state variables disappear on the right hand side.

$$\dot{\mathrm{x}}_1=x_2+u_1+u_2$$

$$\dot{\mathrm{x}}_2=x_1+x_2+u_2$$

To make the RHS zero, we just let

$$x_2+u_1+u_2=0$$

$$x_1+x_2+u_2=0$$

Is this possible by choosing proper values of the inputs?

We easily have

$$u_1=x_1$$

$$u_2=-x_1-x_2$$

$$u = -Kx + Fr$$

How about F? we know that with the state feedback we already have

$$\dot{x} = (A - BK)x + BFr$$

$$= \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \mathrm{Fr},$$

So we just need to design F such that

$$BF = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \mathrm{F} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\mathrm{F} = B^{-1} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$$

So B must be nonsingular!

In other words, the inputs are independently affecting the outputs.

Let's summarize what we have learned so far, when y=x:

$$\dot{x} = Ax + Bu$$

$$u = -Kx + Fr$$

$$\dot{x} = (A - BK)x + BFr$$

To completely decouple the system to integrators:

$$\dot{x} = r$$

We have

$$A - BK = 0$$

$$BF = I$$

Therefore, we design

$$A = BK \Rightarrow K = B^{-1}A$$

$$F = B^{-1}$$

We have to design both K and F to solve the decoupling problem!

For both of them, we need B to be nonsingular. Is that really necessary?

Let's change B to a singular matrix

$$\dot{x} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} x + \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} u,$$

$$y = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} x = x.$$

Let's try to design purely state feedback control first:

$$u = -Kx$$

$$\dot{x}_1 = x_2 + u_1 + u_2$$

$$\dot{x}_2 = x_1 + x_2 + u_1 + u_2$$

To make the RHS zero, we let

$$x_2 + u_1 + u_2 = 0$$

$$x_1 + x_2 + u_1 + u_2 = 0$$

Is it possible?

It is impossible now!

If B is not full rank, then inputs are not affecting the system independently!

§10.2 Decoupling by State Feedback

Consider the system

$$\dot{x} = Ax + Bu, \qquad x(0) = x_0,$$
$$y = Cx, \tag{5}$$

The open loop transfer function matrix: $G(s) = C(sI - A)^{-1} B$ (6)

Let the state feedback controller

$$u = -Kx + Fr. \tag{7}$$

The closed loop system is

$$\dot{x} = (A - BK)x + BFr,$$
$$y = Cx, \tag{8}$$

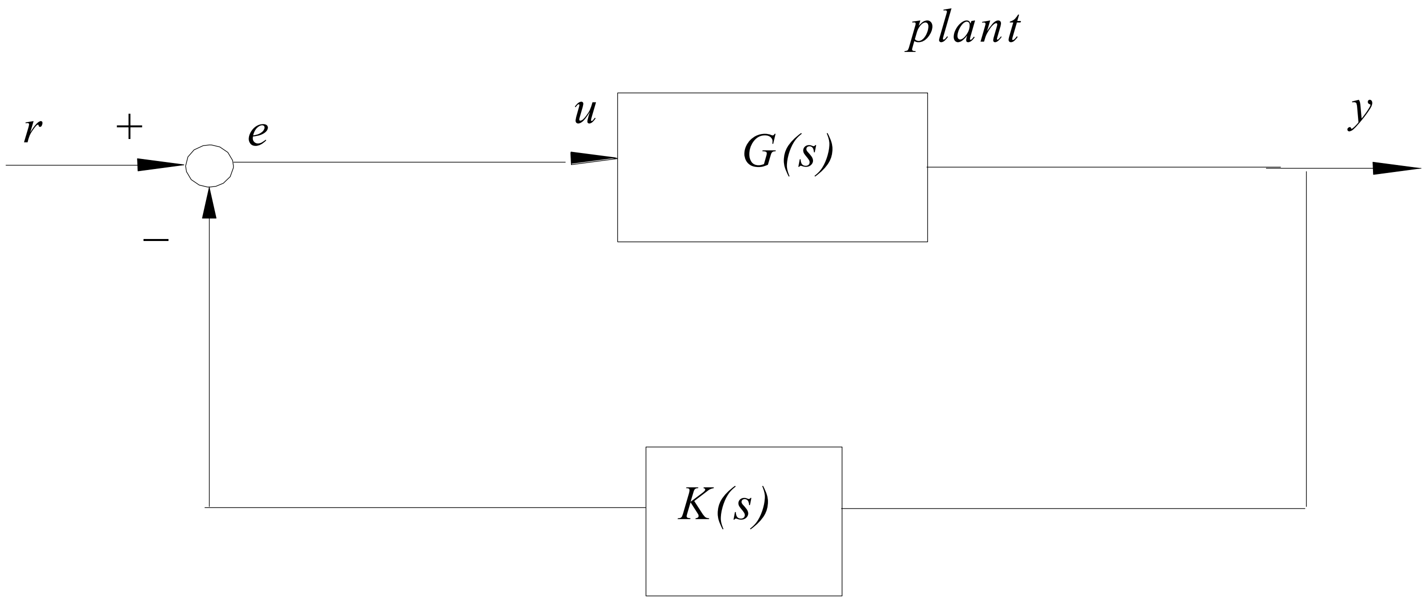
and the transfer function matrix of the feedback system is

$$H(s) = C(sI - A + BK)^{-1} BF. \tag{9}$$

The objective is to make H(s) a diagonal matrix (decoupled matrix).

First, let’s try to find out how the open loop G(s) is related to the closed loop H(s).

The open loop and closed loop TF of SISO system



$$Y(s) = G(s)U(s)$$

$$U(s) = R(s) - K(s)\,Y(s)$$

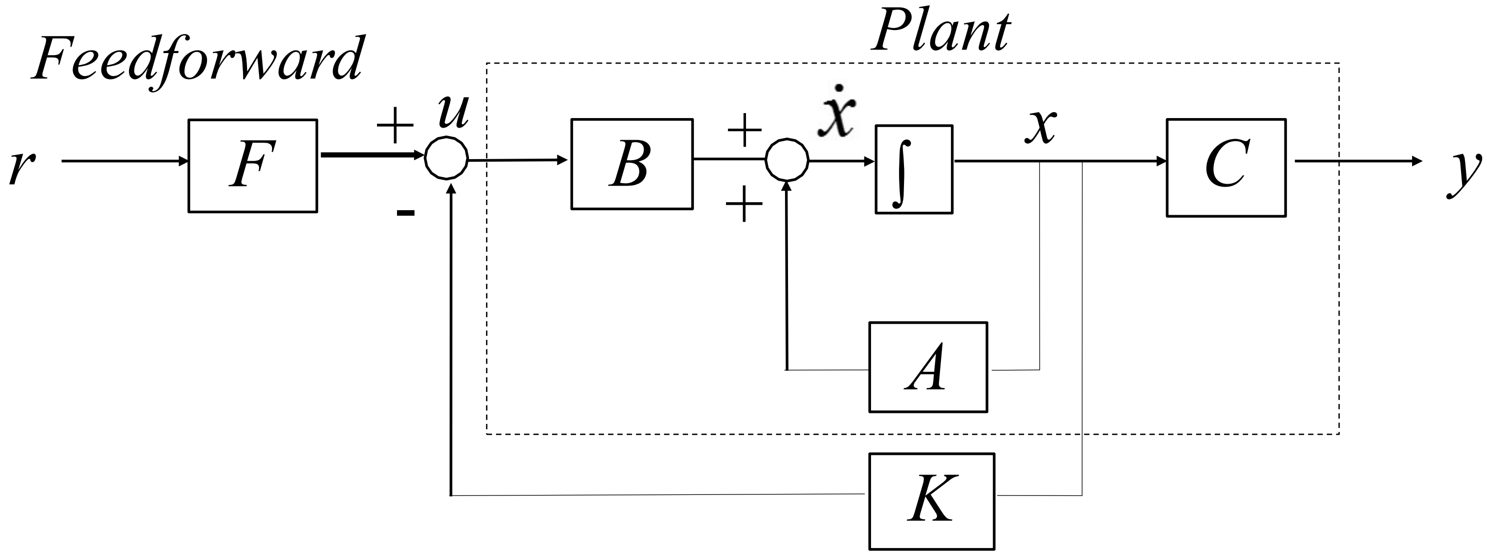
$$Y(s) = G(s)(R(s) - K(s)\,Y(s))$$

Closed loop TF $$H(s) = \frac{Y(s)}{R(s)} = \frac{G(s)}{1 + K(s)G(s)}$$

Feedforward TF

Open loop TF

Can we get similar relation for MIMO?



The open loop: $$Y(s) = G(s)U(s)$$

$$u = -Kx + Fr \quad \Rightarrow \quad U(s) = -KX(s) + FR(s)$$

$$\dot{x} = \left(A - BK\right)x + BFr, \quad \Rightarrow \quad X(s) = (sI - (A - BK))^{-1} BFR(s)$$

$$\begin{aligned} Y(s) &= G(s)U(s) \\ &= G(s)(FR - KX) \\ &= G(s)(FR - K(sI - (A - BK))^{-1} BFR) \\ &= G(s)(I - K(sI - (A - BK))^{-1} B)FR \end{aligned}$$

The closed loop transfer function matrix

$$H(s) = G(s)[I - K\left(sI - A + BK\right)^{-1} B]F$$

Does it look like the formula for the SISO system? No, this is not what we look for.

Proposition 1 The closed loop TF H(s) is related to the open loop TF G(s) by

$$H(s) = G(s)[I - K(sI - A + BK)^{-1} B]F$$
$$= G(s)\left[I + K(sI - A)^{-1} B\right]^{-1} F. \qquad (11)$$

Proof: we already showed the first equation. Just need to show that

$$I - K(sI - A + BK)^{-1} B = \left[I + K(sI - A)^{-1} B\right]^{-1}$$

If you encounter problem like this in linear algebra, what is the "SOP" to prove?

We just need to show

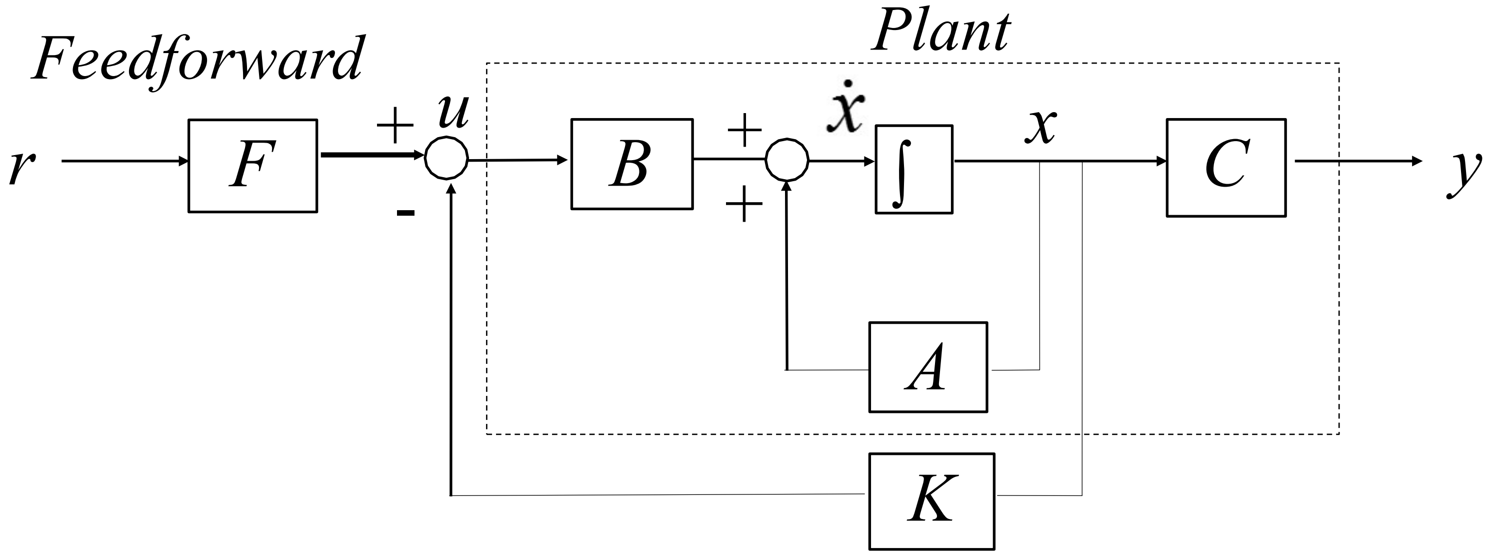
$$\left[I - K\left(sI - A + BK\right)^{-1} B\right]\left[I + K\left(sI - A\right)^{-1} B\right] = I$$

$$LHS = I - K\left(sI - A + BK\right)^{-1} B + K(sI - A)^{-1} B - K\left(sI - A + BK\right)^{-1} BK(sI - A)^{-1} B$$

$$= I - K\left(sI - A + BK\right)^{-1} \{([sI - A + BK] - [BK])\left(sI - A\right)^{-1}\}B$$

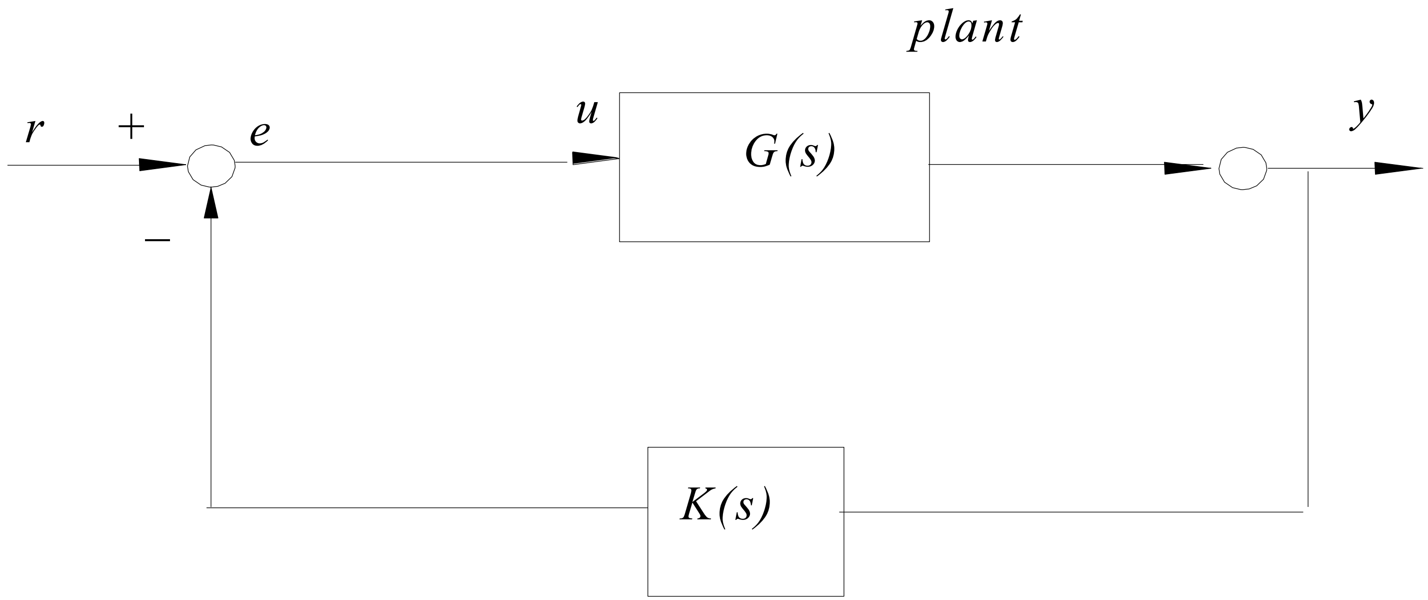
$$+K(sI - A)^{-1} B - K(sI - A + BK)^{-1} BK(sI - A)^{-1} B$$

$$= I = RHS$$



MIMO

$$H(s) = G(s)[I + K(sI - A)^{-1}B]^{-1}F$$



SISO

$$H(s) = \frac{G(s)}{1 + K(s)G(s)}$$

They are very similar to each other!

The closed loop transfer function matrix

$$H(s) = G(s)\left[I + K(sI - A)^{-1} B\right]^{-1} F.$$

where $G(s) = C(sI - A)^{-1} B.$

If we can make H(s) diagonal, is H(s) nonsingular?

Yes.

If H(s) is nonsingular, how about G(s)?

G(s) has to be nonsingular too!

This is the key condition for decoupling!

In the illustrative example, we assume y=x, which means C=I and we end up with the condition that B has to be nonsingular to guarantee that G(s) is nonsingular.

First of all, do we require C and B fully ranked to assure that G(s) is nonsingular?

Yes, but that is not enough to assure that G(s) is nonsingular!

In the following, we will develop a condition about non-singularity of the transfer function matrix which involves A, B and C!

In the following, we will develop a condition to assure that

$$G(s) = C(sI - A)^{-1} B$$

is nonsingular.

Let's still use the previous example to show how we check whether the transfer function matrix G(s) is non-singular or not.

$$\dot{x} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} x + \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} u,$$

$$y = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} x = x.$$

$$G(s) = C(\mathrm{s}I - A)^{-1} B = \frac{1}{s^2 - s - 1} \begin{bmatrix} s-1 & s \\ 1 & s+1 \end{bmatrix} = \begin{bmatrix} \dfrac{s-1}{s^2-s-1} & \dfrac{s}{s^2-s-1} \\ \dfrac{1}{s^2-s-1} & \dfrac{s+1}{s^2-s-1} \end{bmatrix}$$

$$\det\{G(s)\} = \det\{\begin{bmatrix} \dfrac{s-1}{s^2-s-1} & \dfrac{s}{s^2-s-1} \\ \dfrac{1}{s^2-s-1} & \dfrac{s+1}{s^2-s-1} \end{bmatrix}\} = \frac{1}{s^2 - s - 1} \neq 0$$

But getting the transfer function is time-consuming!

Is there a simple way to check without using the transfer function directly?

Let's manipulate the transfer function matrix a little bit.

$$G(s)=\begin{bmatrix}\dfrac{s-1}{s^2-s-1} & \dfrac{s}{s^2-s-1}\\ \dfrac{1}{s^2-s-1} & \dfrac{s+1}{s^2-s-1}\end{bmatrix}=\begin{bmatrix}\dfrac{1}{s} & 0\\ 0 & \dfrac{1}{s}\end{bmatrix}\begin{bmatrix}\dfrac{s^2-s}{s^2-s-1} & \dfrac{s^2}{s^2-s-1}\\ \dfrac{s}{s^2-s-1} & \dfrac{s^2+s}{s^2-s-1}\end{bmatrix}$$

Now the problem becomes to check the rank of

$$\begin{bmatrix}\dfrac{s^2-s}{s^2-s-1} & \dfrac{s^2}{s^2-s-1}\\ \dfrac{s}{s^2-s-1} & \dfrac{s^2+s}{s^2-s-1}\end{bmatrix}$$

Please note that the matrix has to be full rank for almost any value of s. So let s→infinity, we have

$$\begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix}$$

The rank of this matrix is easy to check now!

If we look at these numbers carefully, we can find out that they are the leading coefficients of the numerators of the transfer function matrix.

It turns out that we only need to compute this matrix instead of the transfer function matrix.

Since only leading coefficients of the numerator are needed, we can compute them using A,B, and C, without the complete information on the transfer function matrix G(s).

Let’s develop a systematic way to compute the leading coefficients of the numerator of the TF.

We need to introduce one important concept in transfer function:

Relative Degree

The difference between the **degree** of the denominator (number of poles) and **degree** of the numerator (number of zeros) is the **relative degree** of the **transfer function**.

$$H(s) = \frac{B(s)}{A(s)} = \frac{b_m s^m + b_{m-1} s^{m-1} + \cdots + b_0}{s^n + a_{n-1} s^{n-1} + \cdots + a_0}$$

$$\sigma = n - m$$

The transfer function is **proper** if $n \geq m$

b_m is the leading coefficient of the numerator.

Long Division

$$H(s) = \frac{b_m s^m \left(1 + \frac{b_{m-1}}{b_m} s^{-1} + \cdots + \frac{b_0}{b_m} s^{-m}\right)}{s^n \left(1 + a_{n-1} s^{-1} + \cdots + a_0 s^{-n}\right)}$$

$$= b_m s^{-(n-m)} (1 + \alpha_1 s^{-1} + \alpha_2 s^{-2} + \cdots)$$

$$= b_m s^{-\sigma} (1 + \alpha_1 s^{-1} + \alpha_2 s^{-2} + \cdots)$$

Any transfer function can be written as ONE polynomial (with infinite # of terms) of s^{-1} with the leading term as $b_m s^{-\sigma}$.

We will show that we can do similar thing with transfer function matrix!

<u>Break</u>

State-of-the-art control systems

<u>Future Robots (4: 45:60)</u>

In the following, we will try to express the transfer function matrix

$$G(s) = C(sI - A)^{-1} B$$

using polynomials of s^{-1}.

C is a mxn matrix. Partition C into m row vectors.

$$C = \begin{bmatrix} c_1^T \\ c_2^T \\ \vdots \\ c_m^T \end{bmatrix}$$

$$G(s) = \begin{bmatrix} c_1^T \\ c_2^T \\ \vdots \\ c_m^T \end{bmatrix} (sI - A)^{-1} B = \begin{bmatrix} c_1^T (sI - A)^{-1} B \\ c_2^T (sI - A)^{-1} B \\ \vdots \\ c_m^T (sI - A)^{-1} B \end{bmatrix} = \begin{bmatrix} g_1^T (s) \\ g_2^T (s) \\ \vdots \\ g_m^T (s) \end{bmatrix}$$

Where to find the polynomial of s^{-1}?

We have to find out a way to express

$$(sI - A)^{-1}$$

by polynomial of s^{-1}!

Is it possible to express

$$(sI - A)^{-1}$$

using polynomial of s^{-1}?

How to tackle this kind of problem? When you have no clue, then start with the simplest case, where A is not a matrix, but a scalar!

Can you express $(1-x)^{-1} = \dfrac{1}{1-x}$ by polynomial of x?

Taylor Series:

$$(1-x)^{-1} = \frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots \qquad |x| < 1$$

$$(y-x)^{-1} = y^{-1}(1 - \frac{x}{y})^{-1} = y^{-1}\left\{1 + \frac{x}{y} + (\frac{x}{y})^2 + \cdots\right\} \qquad \left|\frac{x}{y}\right| < 1$$

Similarly for a matrix A, one has

$$(sI-A)^{-1} = s^{-1}\left[I-\frac{A}{s}\right]^{-1} = s^{-1}\left[I+As^{-1}+A^2s^{-2}+\cdots\right]$$

$$= s^{-1}I + As^{-2} + A^2s^{-3} + \cdots \qquad \left\|\frac{A}{s}\right\| < 1$$

This polynomial expression is very useful for theoretical analysis!

It converts the operation of "division/inverse" into "summation"!

Another easy way to show this is to use the definition of the matrix exponential:

$$e^{At} = I + At + \ldots + A^k\frac{t^k}{k!} + \ldots$$

Taking Laplace Transform on both sides will lead to the polynomial expression.

The ith row of the transfer function matrix $G(s)$ can be expanded in polynomials of s^{-1} as

$$g_i(s)^T = c_i^T (sI - A)^{-1} B = c_i^T B s^{-1} + c_i^T AB s^{-2} + \cdots$$

It is similar to the scalar case: $H(s) = b_m s^{-\sigma}(1 + \alpha_1 s^{-1} + \alpha_2 s^{-2} + \cdots)$

Now let's check the relative degree:

If $c_i^T B \neq 0$ what is the relative degree and the leading coefficients?

$$c_i^T B \neq 0 \Rightarrow \sigma = 1$$

If $c_i^T B = 0, c_i^T AB \neq 0$ what is the relative degree?

$$c_i^T B = 0, c_i^T AB \neq 0 \Rightarrow \sigma = 2$$

So we can find out a SOP to get the relative degree and the leading coefficients using {A,B,C} only.

$$Define, \sigma_i,\ i = 1, 2, \cdots, m, as\ an\ integer\ by$$

$$\sigma_i = \begin{cases} \min\left(j \middle| c_i^T A^{j-1} B \neq 0^T,\ j = 1,\ 2,\ \cdots,\ n\right); \\ \\ n, \quad if \quad c_i^T A^{j-1} B = 0^T,\ j = 1,\ 2,\ \cdots,\ n. \end{cases} \tag{13}$$

$$g_i(s)^T = c_i^T (sI - A)^{-1} B = c_i^T B s^{-1} + c_i^T A B s^{-2} + \cdots$$

Calculation of Relative Degree σ_i

Let $i = 1$:

start with $j = 1, \ c_1^T A^{j-1} B = c_1^T A^0 B = c_1^T B \begin{cases} \neq 0 \rightarrow \sigma_1 = 1; \text{ go to } i = 2. \\ = 0, \rightarrow j = 2 \end{cases}$

$$j = 2, \quad c_1^T A^{j-1} B = c_1^T A B \begin{cases} \neq 0 \rightarrow \sigma_1 = 2, \; go \text{ to } i = 2 \\ = 0 \rightarrow j = 3, \ldots.. \end{cases}$$

Let $i = 2$:

$\vdots$

Let $i = m$:

BTW, in the calculation process, if at one step, all of the vectors,

$$c_i^T A^{j-1} B = 0 \qquad for \qquad j = 1, 2, \cdots n$$

It implies that the whole row is ZERO! Then it is guaranteed that G(s) is singular, and decoupling is impossible!

Example 1 Let

$$\dot{x} = \begin{pmatrix} 0 & 1 \\ -1 & -2 \end{pmatrix} x + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} u,$$

$$y = \begin{pmatrix} 1 & 0.5 \\ 1 & 1 \end{pmatrix} x.$$

One proceeds as follows:

$i = 1 \qquad c_1^T = \begin{bmatrix} 1 & 0.5 \end{bmatrix}$

$$c_1^T B = \begin{bmatrix} 1 & 0.5 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0.5 \end{bmatrix} \neq 0 \rightarrow \sigma_1 = 1$$

$i = 2 \qquad c_2^T = \begin{bmatrix} 1 & 1 \end{bmatrix}$

$$c_2^T B = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \end{bmatrix} \neq 0 \rightarrow \sigma_2 = 1$$

Example 2 Let

$$\dot{x} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix} x + \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 0 & -1 \end{pmatrix} u,$$

$$y = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} x.$$

One proceeds as follows:

$i = 1$ $\quad c_1^T = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix},$

$$c_1^T B = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \end{bmatrix};$$

$$c_1^T AB = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \end{bmatrix} \neq \mathbf{0} \rightarrow \sigma_1 = 2;$$

$i = 2 \qquad c_2^T = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}$

$$c_2^T B = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \end{bmatrix} \neq 0 \rightarrow \sigma_2 = 1.$$

The *i*th row of the transfer function $G(s)$ can be expanded in polynomials of s^{-1} as

$$g_i(s)^T = c_i^T(sI - A)^{-1}B = c_i^T B s^{-1} + c_i^T A B s^{-2} + \cdots$$

Note from definition of σ_i that

$$c_i^T B = c_i^T A B = \cdots = c_i^T A^{\sigma_i - 2} B = 0,\ c_i^T A^{\sigma_i - 1} B \neq 0$$

One sees

$$\begin{aligned}
g_i^T(s) &= c_i^T (sI - A)^{-1} B = c_i^T B s^{-1} + c_i^T A B s^{-2} + c_i^T A^2 B s^{-3} + \cdots \\
&= c_i^T A^{\sigma_i - 1} B s^{-\sigma_i} + c_i^T A^{\sigma_i} B s^{-(\sigma_i + 1)} + \cdots \\
&= s^{-\sigma_i}\left[c_i^T A^{\sigma_i - 1} B + c_i^T A^{\sigma_i} B s^{-1} + \cdots\right]
\end{aligned}$$

$$\begin{aligned}
c_i^T A^{\sigma_i} B s^{-1} + \cdots &= c_i^T A^{\sigma_i}\left(s^{-1} I + A s^{-2} + A^2 s^{-3} + \cdots\right) B \\
&= c_i^T A^{\sigma_i}(sI - A)^{-1} B
\end{aligned}$$

$$g_i^T(s) = s^{-\sigma_i}\left[c_i^T A^{\sigma_i - 1}B + c_i^T A^{\sigma_i}\left(sI - A\right)^{-1} B\right]$$

$$G\left(s\right) = \begin{bmatrix} g_1^T(s) \\ g_2^T(s) \\ \vdots \\ g_m^T(s) \end{bmatrix} = \begin{bmatrix} s^{-\sigma_1}\left[c_1^T A^{\sigma_1 - 1}B + c_1^T A^{\sigma_1}\left(sI - A\right)^{-1} B\right] \\ s^{-\sigma_2}\left[c_2^T A^{\sigma_2 - 1}B + c_2^T A^{\sigma_2}\left(sI - A\right)^{-1} B\right] \\ \vdots \\ s^{-\sigma_m}\left[c_m^T A^{\sigma_m - 1}B + c_m^T A^{\sigma_m}\left(sI - A\right)^{-1} B\right] \end{bmatrix}$$

$$= \begin{pmatrix} s^{-\sigma_1} & 0 & \cdots & 0 \\ 0 & s^{-\sigma_2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & s^{-\sigma_m} \end{pmatrix} \begin{bmatrix} c_1^T A^{\sigma_1 - 1}B + c_1^T A^{\sigma_1}\left(sI - A\right)^{-1} B \\ c_2^T A^{\sigma_2 - 1}B + c_2^T A^{\sigma_2}\left(sI - A\right)^{-1} B \\ \vdots \\ c_m^T A^{\sigma_m - 1}B + c_m^T A^{\sigma_m}\left(sI - A\right)^{-1} B \end{bmatrix}$$

$$= \begin{pmatrix} s^{-\sigma_1} & 0 & \cdots & 0 \\ 0 & s^{-\sigma_2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & s^{-\sigma_m} \end{pmatrix} \left[\begin{bmatrix} c_1^T A^{\sigma_1 - 1}B \\ c_2^T A^{\sigma_2 - 1}B \\ \vdots \\ c_m^T A^{\sigma_m - 1}B \end{bmatrix} + \begin{bmatrix} c_1^T A^{\sigma_1} \\ c_2^T A^{\sigma_2} \\ \vdots \\ c_m^T A^{\sigma_m} \end{bmatrix} \left(sI - A\right)^{-1} B \right]$$

$G(s)$ can be written as

$$G(s)=\begin{pmatrix} s^{-\sigma_1} & 0 & \cdots & 0 \\ 0 & s^{-\sigma_2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & s^{-\sigma_m} \end{pmatrix}\left[B^* + C^*(sI-A)^{-1}B\right],$$

where B^* and C^* are defined as

$$B^* = \begin{bmatrix} c_1^T A^{\sigma_1 - 1} B \\ c_2^T A^{\sigma_2 - 1} B \\ \vdots \\ c_m^T A^{\sigma_m - 1} B \end{bmatrix} \qquad C^* = \begin{bmatrix} c_1^T A^{\sigma_1} \\ c_2^T A^{\sigma_2} \\ \vdots \\ c_m^T A^{\sigma_m} \end{bmatrix}$$

σ_i corresponds to the relative degree of the transfer function for each row.

B* correspond to the leading coefficients of the zero polynomials.

Theorem 1 *There exists a control law* $u = -Kx + Fr$ *to decouple the system (5) if and only if the matrix*

$$B^* = \begin{pmatrix} c_1^T A^{\sigma_1 - 1} B \\ c_2^T A^{\sigma_2 - 1} B \\ \vdots \\ c_m^T A^{\sigma_m - 1} B \end{pmatrix} \tag{14}$$

is non-singular. If this is the case, let

$$C^* = \begin{bmatrix} c_1^T A^{\sigma_1} \\ c_2^T A^{\sigma_2} \\ \vdots \\ c_m^T A^{\sigma_m} \end{bmatrix}, \quad F = B^{*-1}, \qquad K = B^{*-1} C^*. \tag{15}$$

Then, the closed-loop system transfer function matrix is given by

$$H(s) = diag\left(s^{-\sigma_1},\ s^{-\sigma_2},\ \cdots,\ s^{-\sigma_m}\right),$$

which is called an integrator decoupled system.

Using B^* of (14) and C* of (15), $G(s)$ can be written as

$$G(s)=\begin{pmatrix} s^{-\sigma_1} & 0 & \cdots & 0 \\ 0 & s^{-\sigma_2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & s^{-\sigma_m} \end{pmatrix}\left[B^* + C^*(sI-A)^{-1}B\right], \tag{16}$$

where B* and C^* are

$$C^* = \begin{bmatrix} c_1^T A^{\sigma_1} \\ c_2^T A^{\sigma_2} \\ \vdots \\ c_m^T A^{\sigma_m} \end{bmatrix} \qquad B^* = \begin{bmatrix} c_1^T A^{\sigma_1 - 1} B \\ c_2^T A^{\sigma_2 - 1} B \\ \vdots \\ c_m^T A^{\sigma_m - 1} B \end{bmatrix} \tag{17}$$

It follows from Proposition 1 that

$$H(s)=G(s)\left[I+K(sI-A)^{-1}B\right]^{-1}F,$$

$$H(s)=diag\left(s^{-\sigma_1},s^{-\sigma_2},\ldots,s^{-\sigma_m}\right)\left[B^*+C^*(sI-A)^{-1}B\right]$$

$$\times\left[F^{-1}+F^{-1}K(sI-A)^{-1}B\right]^{-1}. \qquad (19)$$

Since the objective is to make

$$H(s)=diag\left(s^{-\sigma_1},s^{-\sigma_2},\ldots,s^{-\sigma_m}\right) \qquad (20)$$

We need to design K and F such that

$$\left[B^*+C^*(sI-A)^{-1}B\right]\times\left[F^{-1}+F^{-1}K(sI-A)^{-1}B\right]^{-1}=I \qquad (21)$$

$$B^*+C^*(sI-A)^{-1}B=F^{-1}+F^{-1}K(sI-A)^{-1}B$$

How to design F? $F^{-1}=B^*\Rightarrow F=B^{*-1},$

How to design K? $F^{-1}K=C^*\Rightarrow K=B^{*-1}C^*.$

and the decoupling problem is solved!

The solution is similar to the special case we considered at the beginning!

Example 1 (continued) The system:

$$\dot{x} = \begin{pmatrix} 0 & 1 \\ -1 & -2 \end{pmatrix} x + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} u,$$

$$y = \begin{pmatrix} 1 & 0.5 \\ 1 & 1 \end{pmatrix} x,$$

is required to be integrator-decoupled by state feedback. One sees that

$$G(s) = \begin{pmatrix} 1 & 0.5 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} s & -1 \\ 1 & s+2 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$= \begin{pmatrix} \dfrac{s+1.5}{(s+1)^2} & \dfrac{0.5s+1}{(s+1)^2} \\ \dfrac{1}{s+1} & \dfrac{1}{s+1} \end{pmatrix},$$

is coupled.

We have calculated before:

$$\sigma_1 = 1, \quad c_1^T B = \begin{bmatrix} 1 & 0.5 \end{bmatrix}$$

$$\sigma_2 = 1, \quad c_2^T B = \begin{bmatrix} 1 & 1 \end{bmatrix}.$$

One sees that

$$B^* = \begin{pmatrix} 1 & 0.5 \\ 1 & 1 \end{pmatrix}$$

is non-singular and the system can be decoupled. One proceeds:

$$c_1^T A^{\sigma_1} = c_1^T A = \begin{bmatrix} 1 & 0.5 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} = \begin{bmatrix} -0.5 & 0 \end{bmatrix},$$

$$c_2^T A^{\sigma_2} = c_2^T A = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} = \begin{bmatrix} -1 & -1 \end{bmatrix}.$$

so that

$$C^* = \begin{pmatrix} -0.5 & 0 \\ -1 & -1 \end{pmatrix}.$$

From (19) and (20), the control law has

$$F = B^{*-1} = \begin{bmatrix} 2 & -1 \\ -2 & 2 \end{bmatrix},$$

$$K = B^{*-1}C^{*} = \begin{pmatrix} 0 & 1 \\ -1 & -2 \end{pmatrix}.$$

The decoupling controller:

$$u = -Kx + Fr$$

The feedback system becomes

$$\dot{x} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} x + \begin{pmatrix} 2 & -1 \\ -2 & 2 \end{pmatrix} r,$$

$$y = \begin{pmatrix} 1 & 0.5 \\ 1 & 1 \end{pmatrix} x,$$

$$H(s) = \begin{pmatrix} 1 & 0.5 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} s & 0 \\ 0 & s \end{pmatrix}^{-1} \begin{pmatrix} 2 & -1 \\ -2 & 2 \end{pmatrix} = \begin{pmatrix} \frac{1}{s} & 0 \\ 0 & \frac{1}{s} \end{pmatrix}.$$

The closed loop is decoupled now. But is it stable?

No.

Can we decouple the system such that the closed loop is stable?

Decoupling with pole placement? One wishes to have the closed-loop transfer function matrix as

$$H(s)=\begin{pmatrix} \dfrac{1}{\phi_{f_1}} & 0 & \cdots & 0 \\ 0 & \dfrac{1}{\phi_{f_2}} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \dfrac{1}{\phi_{f_m}} \end{pmatrix}$$

$$=\begin{pmatrix} \dfrac{1}{s^{\sigma_1}+\gamma_{11}s^{\sigma_1-1}+\cdots+\gamma_{1\sigma_1}} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \dfrac{1}{s^{\sigma_m}+\gamma_{m1}s^{\sigma_m-1}+\cdots+\gamma_{m\sigma_m}} \end{pmatrix} \quad (23)$$

It turns out that we just need to change the way to compute C*.

Theorem 2 *When the system (5) can be decoupled by state feedback with*

$$F = \left(B^*\right)^{-1}, \qquad K = \left(B^*\right)^{-1} \begin{bmatrix} c_1^{\ T}\phi_{f1}(A) \\ c_2^{\ T}\phi_{f2}(A) \\ \vdots \\ c_m^{\ T}\phi_{fm}(A) \end{bmatrix} \tag{24}$$

where $$\phi_{f_i}(A) = A^{\sigma_i} + \gamma_{i1}A^{\sigma_{i-1}} + \cdots + \gamma_{i\sigma_i}I,$$

in which

$$\phi_{f_i}(\mathrm{s}) = s^{\sigma_i} + \gamma_{i1}s^{\sigma_{i-1}} + \cdots + \gamma_{i\sigma_i}$$

corresponds to the stable characteristic polynomial of the i-th input-output pair.

then the resultant feedback system has the transfer function given by (23).

It is obvious that when $\phi_{f_i}(\mathrm{s}) = s^{\sigma_i}$ it reduces to theorem 1.

Please refer to the lecture notes for the detailed proof.

In Example 1, the integrator decoupled system is obtained. Now, suppose that the same system is required to be decoupled to give

$$H(s) = diag\left[(s+1)^{-1}, (s+2)^{-1}\right].$$

We already have $B^* = \begin{pmatrix} 1 & 0.5 \\ 1 & 1 \end{pmatrix}$

$$F = (B^*)^{-1} = \begin{bmatrix} 2 & -1 \\ -2 & 2 \end{bmatrix}$$

So we only need to compute C**

Since $\phi_{f_1} = s+1, \qquad \phi_{f_2} = s+2,$

$$C^{**} = \begin{pmatrix} c_1^T \phi_{f_1}(A) \\ c_2^T \phi_{f_2}(A) \end{pmatrix} = \begin{pmatrix} c_1^T (A+I) \\ c_2^T (A+2I) \end{pmatrix} = \begin{pmatrix} -0.5 & 0 \\ -1 & -1 \end{pmatrix} + \begin{pmatrix} 1 & 0.5 \\ 2 & 2 \end{pmatrix} = \begin{pmatrix} 0.5 & 0.5 \\ 1 & 1 \end{pmatrix}$$

$$K = (B^*)^{-1} C^{**} = \begin{pmatrix} 2 & -1 \\ -2 & 2 \end{pmatrix} \begin{pmatrix} 0.5 & 0.5 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$$

One can verify

$$A - BK = \begin{pmatrix} 0 & 1 \\ -1 & -2 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix}.$$

So

$$H(s) = C(sI - (A-BK))^{-1} BF = diag\left[(s+1)^{-1}, (s+2)^{-1} \right]$$

§10.3 Decoupling Control by Output Feedback

So we know how to decouple the system if the state space model is given. What if we only know the transfer function G(s)? Can we still decouple the system?

It turns out the solution is even simpler. Consider the unity feedback system,

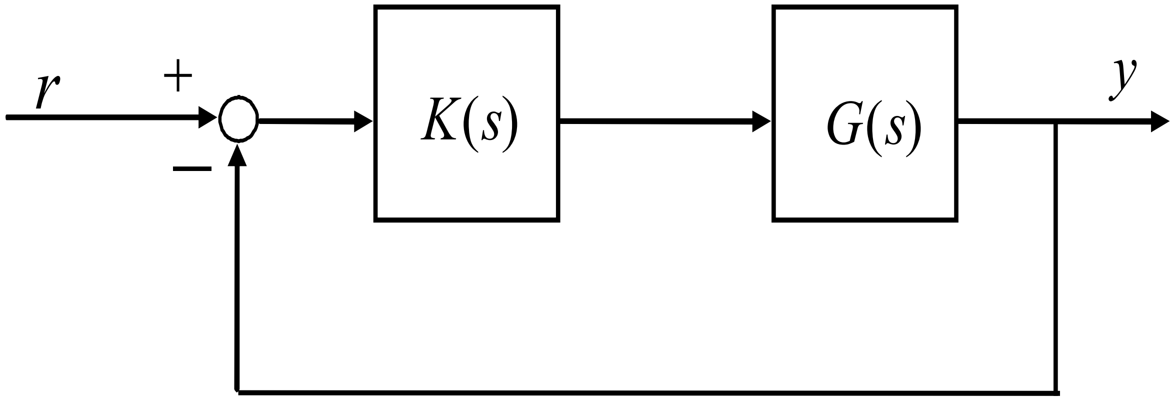


Figure 5 Unity output feedback system.

Our task here is to design $K(s)$ such that the feedback system is internally stable and the closed-loop transfer function matrix,

$$H(s)=\left[I+G(s)K(s)\right]^{-1}G(s)K(s), \tag{29}$$

is decoupled, or diagonal.

$$H(s) = \left[I + G(s)K(s) \right]^{-1} G(s)K(s)$$

$$\begin{aligned} H^{-1} &= (GK)^{-1} \left[I + GK \right] \\ &= (GK)^{-1} + I \, , \end{aligned} \qquad (30)$$

If the closed loop H is decoupled, i.e. diagonal matrix, is GK also diagonal?

It is clear that the closed loop H is decoupled iff the open loop GK is so.

Now the problem becomes how to decouple $G(s)K(s)$

Assume G(s) is nonsingular, one may come up with a simple solution like

$$K(s) = G^{-1}(s)$$

Would this work?

Let's try it on a simple SISO system:

$$G(s) = \frac{s+1}{(s-1)(s-2)}$$

$$K(s) = G^{-1}(s) = \frac{(s-1)(s-2)}{s+1}$$

$$G(s)K(s) = \frac{s+1}{(s-1)(s-2)} \times \frac{(s-1)(s-2)}{s+1} = 1$$

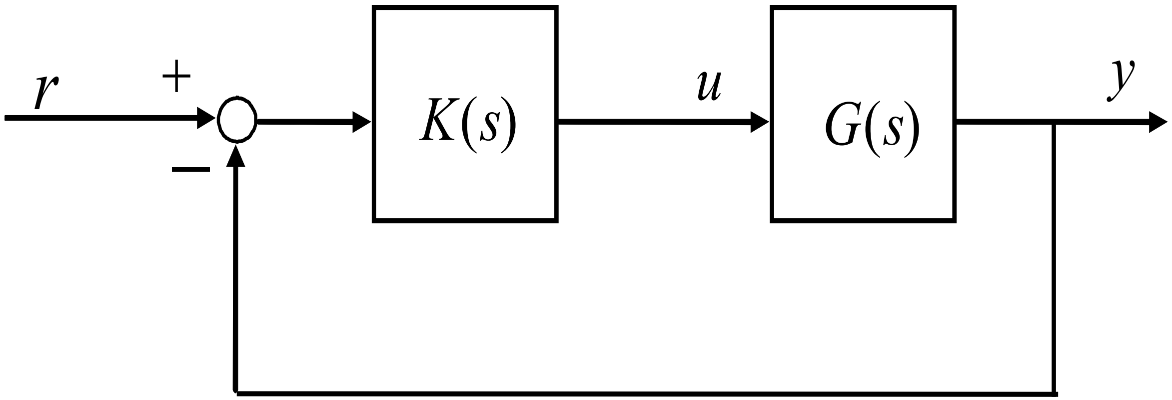
????

But it does not work because you cannot simply cancel out unstable poles and zeros in the transfer function!

There is another issue with above controller K(s), can you spot it?

K(s) is not proper.

Issues with Unstable pole-zero cancellation and stabilizability



Example 3. Let $G(s) = \dfrac{1}{s-1}$ $\quad K(s) = \dfrac{s-1}{s+2}$

$$\frac{Y(s)}{R(s)} = \frac{KG}{1+KG} = \frac{\dfrac{s-1}{s+2}\dfrac{1}{s-1}}{1+\dfrac{s-1}{s+2}\dfrac{1}{s-1}} = \frac{s-1}{(s+3)(s-1)}$$

The closed loop has an unstable pole at s=1 and is unstable.

Whenever there are unstable pole-zero cancellation, we have to be careful. It is better to avoid this complicated issue.

Therefore, we CANNOT simply design the decoupler as: $K(s) = G^{-1}(s)$

When a $K(s)$ decouples $G(s)$ with no unstable pole-zero cancellation, the decoupled loop $G(s)K(s)$ can be stabilized loop by loop, so that the resultant system is decoupled and stable.

Theorem 3. *The plant $G(s)$ is decouplable and stablilizable by output feedback if and only if there is a controller $K(s)$ such that $G(s)K(s)$ is decoupled and there is no unstable pole-zero cancellation between G(s) and K(s).*

Design procedure. Theorem 3 tells that the decoupling problem with stability by output feedback can be solved by designing $K(s)$ in two stages, i.e., $K(s)=K_d(s)K_s(s)$.

Step One: $K_d(s)$ is to make $G(s)K_d(s)$ diagonal and non-singular. In this step, $K_d(s)$ is not required to be proper. There are many ways to do this.

Step Two: Whenever such a $K_d(s)$ is found in step one, it is always possible to design a diagonal controller $K_s(s)$ to stabilize the resultant $G(s)K_d(s)$ with SISO methods or pole placement technique loop by loop so that decoupling is not affected, and to make $K(s)=K_d(s)K_s(s)$ proper and assure that there is *no unstable pole-zero cancellation between G(s) and K(s).*

It is seen from the above that the key step in decoupling control is to find $K_d(s)$. To avoid unstable pole-zero cancellation, we cannot simply let $K_d(s) = G^{-1}(s)$.

There are many ways to do it. We will show two simple ways.

Design for decoupler $K_d(s)$: Suppose that $G(s)$ is non-singular. Write

$$G(s) = \frac{N(s)}{d(s)}$$

where $d(s)$ is the least common denominator of $G(s)$. Choose

$$K_d(s) = \mathrm{adj}\big(N(s)\big)$$

so that

- $G(s)K_d(s) = \dfrac{N(s) \bullet adj(N(s))}{d(s)} = \dfrac{\det(N(s))}{d(s)} I_m$ is decoupled

******************* Revision Notes***********************

Least Common Denominator (LCD)

By definition, LCD, $d(s)$ of $\dfrac{\beta_i(s)}{\alpha_i(s)}$ $, i = 1,2,\cdots,n$ is a least multipler of $\alpha_i(s)$, i.e. $\dfrac{d(s)}{\alpha_i(s)}$ is a polynomial and thus

$$d\begin{bmatrix} \dfrac{\beta_1(s)}{\alpha_1(s)} & \dfrac{\beta_2(s)}{\alpha_2(s)} & \cdots & \dfrac{\beta_n(s)}{\alpha_n(s)} \end{bmatrix}$$

is a polynomial row, where $\alpha_i(s)$ and $\beta_i(s)$ are polynomials.

- LCD of $\dfrac{1}{s+1}$ and $\dfrac{1}{s+2}$ is $(s+1)(s+2)$.
- LCD of $\dfrac{1}{s}$ and $\dfrac{1}{s^2}$ is (s^2)

For a $n \times n$ nonsingular matrix M, one has

$$M^{-1} = \frac{adj(M)}{\det(M)} \qquad \Rightarrow M \cdot adj(M) = \det(M) \cdot I$$

$$Adj(M) = \{c_{ij}\}^T = \begin{bmatrix} c_{11} & c_{21} & c_{31} & \cdots \\ c_{12} & c_{22} & c_{32} & \\ \vdots & & & \end{bmatrix}, \quad c_{ij} : \text{cofactor of } m_{ij}$$

$$M = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix}$$

$$adj(M) = \begin{bmatrix} m_{22} & -m_{12} \\ -m_{21} & m_{11} \end{bmatrix}$$

$$M^{-1} = \frac{1}{m_{11}m_{22} - m_{12}m_{21}} \begin{bmatrix} m_{22} & -m_{12} \\ -m_{21} & m_{11} \end{bmatrix}$$

************** **End of Revisions*************

The decoupler $K_d(s)$ might not be proper. And the closed loop might not be stable. That's why we need the second step:

Design for stabilizer $K_s(s)$:

A decoupled plant has m independent SISO loops. For each SISO loop, you can use SISO methods to design a SISO controller . SISO design methods are well covered in undergraduate control courses. Alternatively, you may use the polynomial approach in Chapter 9 to design a stabilizing and/or servo controller.

- After K(s) is completed, always check that there are no unstable pole-zero cancellations between $G(s)$ and $K(s)$.

Example 4 Consider

$$G(s) = \begin{bmatrix} \dfrac{1}{s+2} & \dfrac{-2}{(s+2)} \\ \dfrac{2}{(s+1)} & \dfrac{1}{s+1} \end{bmatrix}$$

<u>What is the least common denominator (LCD)?</u>

d(s)=(s+1)(s+2)

It follows that

$$G(s) = \frac{N(s)}{d(s)} = \frac{1}{(\mathrm{s}+2)(s+1)} \begin{bmatrix} s+1 & -2(s+1) \\ 2(s+2) & s+2 \end{bmatrix}$$

and

$$\det N(s) = 5(s+1)(s+2)$$

As $\det N(s)$ and $d(s)$ have no common unstable roots, we can choose

$$K_d = \text{adj } N(s) = \begin{bmatrix} s+2 & 2(s+1) \\ -2(s+2) & s+1 \end{bmatrix}$$

and

$$GK_d = \frac{\det N}{d(s)} I_2 = \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}$$

Is $K_d(s)$ proper?   No.

So we need to design a diagonal controller

$$K_s = \begin{bmatrix} k_{s1} & 0 \\ 0 & k_{s2} \end{bmatrix}$$

to stabilize the decoupled system and make $K(s) = K_d(s)K_s(s)$ proper.

There are many possible solutions. One way is to design

$$K_s = \begin{bmatrix} \frac{1}{s+1} & 0 \\ 0 & \frac{1}{s+1} \end{bmatrix}$$

The overall controller is

$$K = K_d K_s = \begin{bmatrix} s+2 & 2(s+1) \\ -2(s+2) & s+1 \end{bmatrix} \begin{bmatrix} \frac{1}{s+1} & 0 \\ 0 & \frac{1}{s+1} \end{bmatrix} = \begin{bmatrix} \frac{s+2}{s+1} & 2 \\ \frac{-2(s+2)}{s+1} & 1 \end{bmatrix}$$

Is $K(s)$ proper now? Yes.

$$G(s)K(s) = \begin{bmatrix} \frac{1}{s+2} & \frac{-2}{s+2} \\ \frac{2}{s+1} & \frac{1}{s+1} \end{bmatrix} \begin{bmatrix} \frac{s+2}{s+1} & 2 \\ \frac{-2(s+2)}{s+1} & 1 \end{bmatrix} = \begin{bmatrix} \frac{5}{s+1} & 0 \\ 0 & \frac{5}{s+1} \end{bmatrix}$$

Are there any unstable pole-zero cancellation between G(s) and K(s)?

No.

$$G(s)K(s) = \begin{bmatrix} \frac{5}{s+1} & 0 \\ 0 & \frac{5}{s+1} \end{bmatrix}$$

Since it is decoupled system, the closed loop stability can be checked loop by loop.

For example, for the first open loop

$$\mathrm{L}(s) = \frac{5}{s+1}$$

The closed loop TF is

$$H_{cl}(s) = \frac{L(s)}{1 + L(s)} = \frac{\frac{5}{s+1}}{1 + \frac{5}{s+1}} = \frac{5}{s+6}$$

Pole s=-6 $\Rightarrow$ Stable!

Similarly, we can also check the stability of the second closed loop.

One problem with this simple method is that the order of the decoupled systems is large.

Refinement. One way to reduce the order of the decoupler is to elaborate the above method as follows. Express $G(s)$ as

$$G(s) = diag\left\{\frac{1}{d_1},\ \frac{1}{d_2},\ \cdots,\ \frac{1}{d_m}\right\}N_r(s)$$

where d_i is a least common denominator of i-th row of $G(s)$ so that $N_r(s)$ is a polynomial matrix. Choose

$$K_d = adj(N_r(s))$$

Then, if $\det(N_r)$ and $d_i(s)$ have no common unstable roots for each i, there will be no unstable pole-zero cancellations in $G(s)K_d(s)$. And

$$G(s)K_d(s) = diag\left\{\frac{\det N_r(s)}{d_1},\ \frac{\det N_r(s)}{d_2},\ \cdots,\ \frac{\det N_r(s)}{d_m}\right\}$$

Example 5. Let

$$G(s)=\begin{bmatrix}\dfrac{1}{s+2} & \dfrac{-1}{(s+1)(s+2)}\\ \dfrac{5}{s+3} & \dfrac{1}{s+3}\end{bmatrix}$$

What is the least common denominator?

For this case, d(s)=(s+1)(s+2)(s+3), the non-refined method will give a decoupled system of order 3x2=6. With the refined method, one sees that

$$G(s)=\begin{bmatrix}d_1 & 0\\ 0 & d_2\end{bmatrix}^{-1}N_r=\begin{bmatrix}(s+1)(s+2) & 0\\ 0 & s+3\end{bmatrix}^{-1}\begin{bmatrix}s+1 & -1\\ 5 & 1\end{bmatrix}$$

And $\det\left(N_r\right)=s+6$ and $d_i(s)$ have no common unstable roots for *i=1,2*. We can then take

$$K_d=adj(N_r)=\begin{bmatrix}1 & 1\\ -5 & s+1\end{bmatrix}$$

$$K_d = adj(N_r) = \begin{bmatrix} 1 & 1 \\ -5 & s+1 \end{bmatrix} \qquad GK_d = diag\left\{ \frac{s+6}{(s+1)(s+2)}, \frac{s+6}{s+3} \right\},$$

It is easy to verify that that the closed loop is stable. But there is one problem.

Is K_d proper?

No. If not proper, it may not be implemented due to causality issue.

There are many solutions. One simple solution is

$$K_s = \begin{bmatrix} \frac{1}{s+6} & 0 \\ 0 & \frac{1}{s+6} \end{bmatrix}$$

to stabilize GK_d and make the overall controller $K(s) = K_d(s)K_s(s)$ proper:

$$K = K_d K_s = \begin{bmatrix} \frac{1}{s+6} & \frac{1}{s+6} \\ \frac{-5}{s+6} & \frac{s+1}{s+6} \end{bmatrix}$$

Is K(s) proper now? Yes.

<u>Are there any unstable pole-zero cancellation between G(s) and K(s)?</u>

No.

$$GK = diag\left\{\frac{1}{(s+1)(s+2)},\ \frac{1}{s+3}\right\},$$

The resultant closed-loop system transfer function matrix is given by

$$\begin{aligned} H(s) &= \left[I + G(s)K(s)\right]^{-1} G(s)K(s) \\ &= diag\left\{\frac{1}{s^2+3s+3},\ \frac{1}{s+4}\right\}, \end{aligned}$$

and it is stable and decoupled.

Q & A…

THANK YOU !