

# EE5103 Lecture Three

## Analysis of Discrete-Time Systems

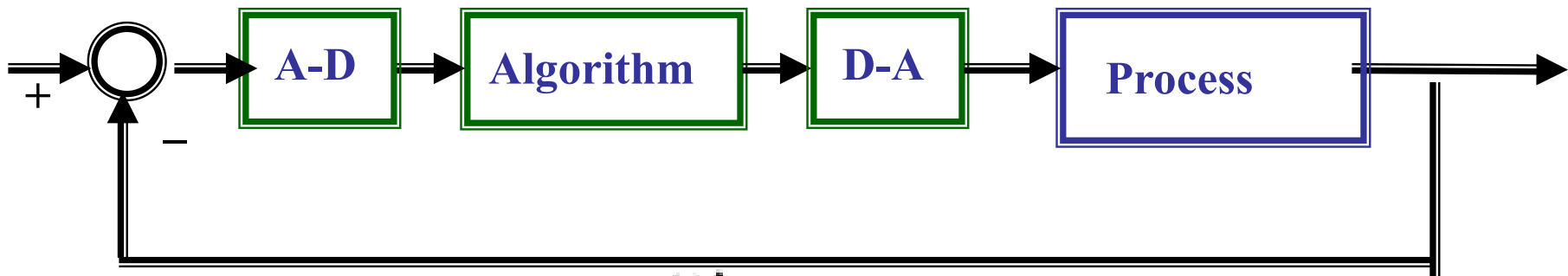
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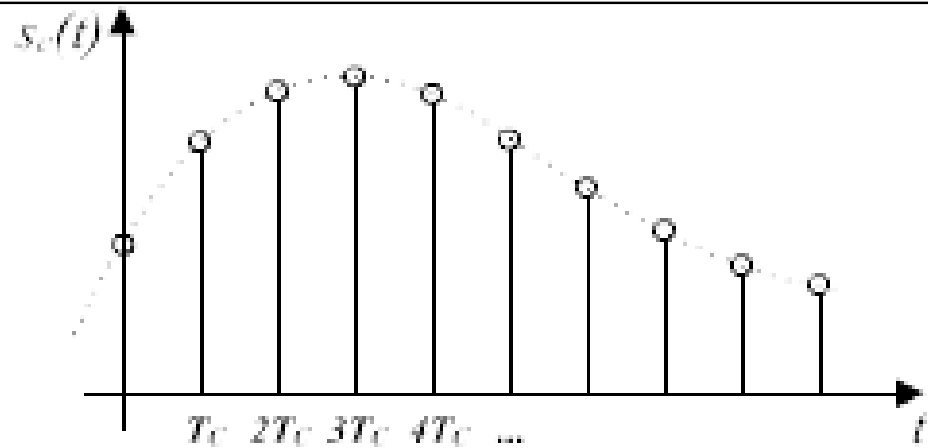
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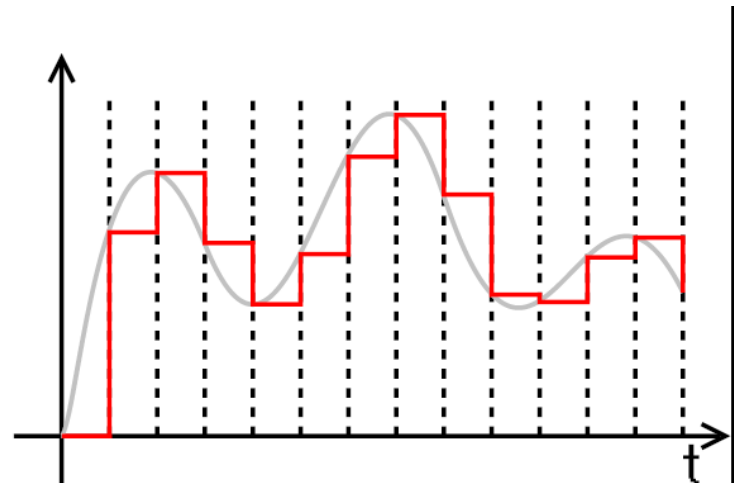
## How to do the A-D?

Uniform Sampling



## How to do the D-A?

Zero-order Hold



State-space Model

$\dot{x} = Ax + bu$       Laplace Transform

$y = cx$        $\Longrightarrow$        $sX(s) - x(0) = AX(s) + bU(s)$

$X(s) = (sI - A)^{-1} x(0) + (sI - A)^{-1} bU(s)$

Inverse Laplace Transform

$\Longrightarrow$        $x(t) = e^{At} x(0) + \int_0^t e^{A(t-\tau)} bu(\tau) d\tau$

Zero-order hold

$\Longrightarrow$        $x(k + 1) = \Phi x(k) + \Gamma u(k)$   
 $y(k) = cx(k)$

$\Phi = e^{Ah}$

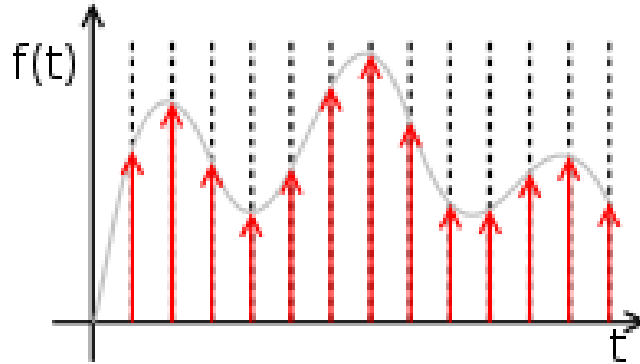
$\Gamma = (\int_0^h e^{As} ds)b$

How to calculate  $e^{At}$  ?

There are many methods. The simplest way is by Laplace Transform.

$(sI - A)^{-1} \Longrightarrow e^{At}$

## Z-transform



$$F(z) = \sum_{k=0}^{\infty} f(k)z^{-k} = f(0) + f(1)z^{-1} + f(2)z^{-2} + \cdots + f(k)z^{-k} + \cdots$$

Most important property: how to relate  $x(k+1)$  and  $x(k)$  in z-domain

$$z\{x(k+1)\} = z(X(z) - x(0)) \quad \xRightarrow{x(0)=0} \quad z\{x(k+1)\} = zX(z)$$

$$q(\text{forward} - \text{time} - \text{shift}) \Leftrightarrow z$$

$$q^{-1}(\text{backward} - \text{time} - \text{shift}) \Leftrightarrow z^{-1}$$

Let's apply z-transform to the state-space equation

$$\begin{aligned} x(k+1) &= \Phi x(k) + \Gamma u(k) \\ y(k) &= cx(k) \end{aligned} \quad \Longrightarrow \quad \begin{aligned} z(X(z) - x(0)) &= \Phi X(z) + \Gamma U(z) \\ zX(z) - \Phi X(z) &= zx(0) + \Gamma U(z) \end{aligned}$$

$$X(z) = (zI - \Phi)^{-1} zx(0) + (zI - \Phi)^{-1} \Gamma U(z)$$

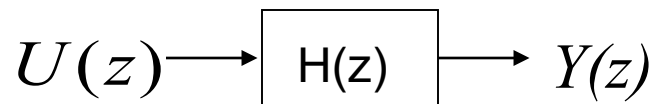
**What is the output?**

$$Y(z) = cX(z) = c(zI - \Phi)^{-1} zx(0) + c(zI - \Phi)^{-1} \Gamma U(z)$$

• If the initial conditions are zero,  $x(0)=0$

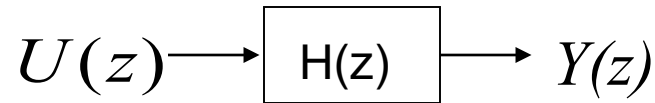
$$Y(z) = c(zI - \Phi)^{-1} \Gamma U(z) = H(z)U(z)$$

• **What is the transfer function?**



$$H(z) = c(zI - \Phi)^{-1} \Gamma = \frac{Q(z)}{P(z)}$$

$$P(z) = \det\{zI - \Phi\}$$



## What can you do with the transfer function?

- Given a transfer function, you can write down the difference equations, and vice versa

$$q(\text{forward} - \text{time} - \text{shift}) \Leftrightarrow z$$

- Calculate the output,  $Y(z)=H(z)U(z)$  when initial conditions are zero.

But in general,

$$Y(z) = c(zI - \Phi)^{-1} zx(0) + H(z)U(z)$$

zero-input response

zero-state response

- What is the zero-input response if the system is stable?

It converges to zero! So transfer functions can be used to describe the response for stable system.

•The stability can be determined by checking poles of the transfer function.

Continuous-time

$$\lambda = \sigma + j\omega$$

$\sigma < 0$                       Stable

$\sigma > 0$                       Unstable

$\sigma = 0$   
Single                      Multiple

Marginal Stable

Unstable

Discrete-time

$$\lambda = \rho e^{j\theta}$$

$\rho < 1$                       Stable

$\rho > 1$                       Unstable

$\rho = 1$   
Single                      Multiple

Marginal Stable

Unstable

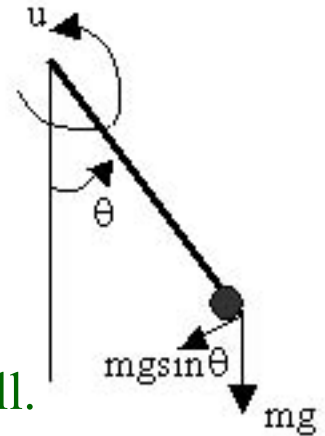
Since marginally stable system is not considered as stable system, we conclude that  
when  $\rho < 1$ , the system is stable.

- In this lecture we will discuss some key concepts for analyzing discrete-time systems.

First Concept: Stability. Why is stability so **critical** for the control system design?

- The fundamental question: what would happen if there is a small disturbance to the system?

- The robotic manipulator example



Stable: small deviation from the equilibrium position will remain small.

Asymptotic Stable: small deviation from the equilibrium position will not only remain small, but also decrease to zero.

Unstable: small deviation from the equilibrium position will result in huge deviation in the long run.

- Let's try to define this concept using more precise mathematical language.



- Stability:

Consider a system described by

$$x(k + 1) = f(x(k), k)$$

- Where is the input  $u(k)$ ?

- Stability is normally concerned with zero-input response: what would happen if there is a disturbance in the initial condition while the inputs are zero?

Let  $x_0(k)$  be the nominal solution with initial condition  $x_0(0)$ .

How do we know if the nominal solution  $x_0(k)$  is stable or not?

- What would happen if there is a small disturbance to the system?

- Would this disturbance be magnified, or attenuated?

- Therefore, we need to analyze the solution to the system with initial condition  $x(0)$ , which is different from the nominal one  $x_0(0)$ .

We are interested in the difference between these two solutions.

- Given a deviation in the initial condition  $|x(0) - x_0(0)|$ , what is the behavior of  $|x(k) - x_0(k)|$  for all  $k > 0$ ? Would it blow up or decrease?

• Given a disturbance  $|x(0)-x_0(0)|$ , how does  $|x(k)-x_0(k)|$  behave?

Equilibrium types

- Stable:

For any  $\varepsilon > 0$ ,  $|x(k)-x_0(k)| < \varepsilon$ , if  $|x(0)-x_0(0)| < \delta(\varepsilon)$ .

If the initial deviation is sufficiently small, then the resulting trajectory  $x(k)$  is always close to the nominal trajectory  $x_0(k)$  forever .

Stable



Unstable



Neutral

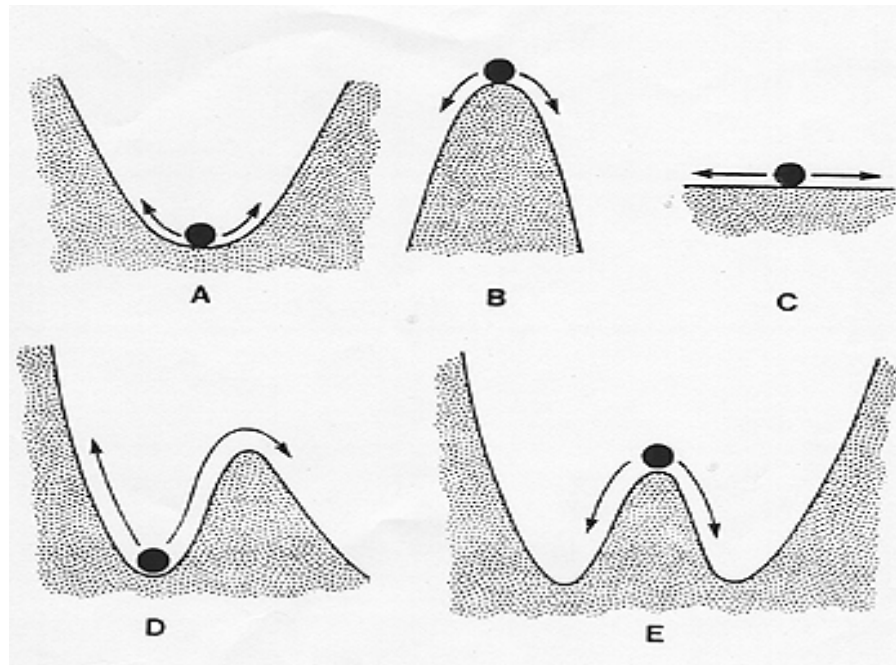


Asymptotic Stable:

The system is not only stable, the deviation will eventually go to zero,  
 $|x(k)-x_0(k)| \rightarrow 0$ .

Normally, when we say a system is stable, we mean “asymptotic stable”.

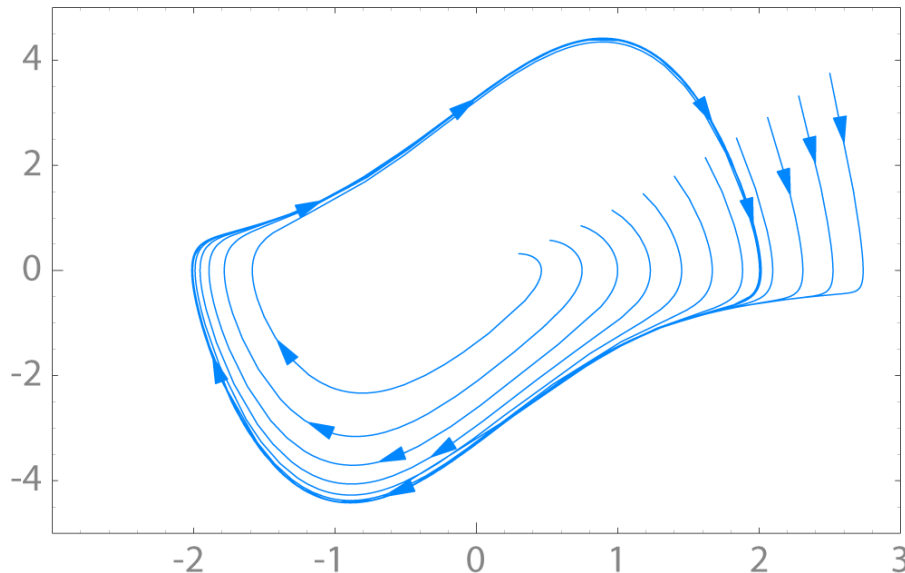
- Stability, in general, is a **local** concept.



When we talk about stability, it always refers to some nominal point --- $x_0(k)$ .

The nominal trajectory  $x_0(k)$  may be a constant like the equilibrium point.

Is it possible for  $x_0(k)$  to be time-varying signal instead of a constant?



The limit cycle!

Laminar Flow

Turbulent Flow

The biggest mystery in classical mechanics!

## How to do the stability analysis in general?

- Choose the subject: nominal trajectory:  $x_o(k)$
- Get the difference between the perturbed trajectory and the nominal one:  
 $(x(k)-x_o(k))$

Note:  $x(k+1)=f(x(k))$ ,

so  $x(1)=f(x(0))$ ,

$x(2)=f(x(1))=f(f(x(0)))$

and

$$x(k) = f(f \cdots (f(x(0)) \cdots) = f^{(k)}(x(0))$$

- You need to study

$$x(k) - x_o(k) = f^{(k)}(x(0)) - f^{(k)}(x_o(0))$$

If  $f(x)$  is a nonlinear function, it could be very complicated.

So in general, stability analysis for nonlinear system is a difficult problem.

But we can get some beautiful result for linear system.

## Linear Discrete-Time Systems

The nominal trajectory  $x_o(k)$

$$x_o(k+1) = \Phi x_o(k) \qquad x_o(0) = a_o$$

The trajectory with deviation in the initial position

$$x(k+1) = \Phi x(k) \qquad x(0) = a$$

**What is the difference between these two?**

$$\tilde{x}(k) = x(k) - x_o(k) \implies \tilde{x}(k+1) = \Phi \tilde{x}(k) \qquad \tilde{x}(0) = a - a_o$$

Easily we obtain

$$\tilde{x}(1) = \Phi \tilde{x}(0)$$

$$\tilde{x}(2) = \Phi \tilde{x}(1) = \Phi^2 \tilde{x}(0)$$

$$\tilde{x}(k) = \Phi^k \tilde{x}(0)$$

## Linear Discrete-Time Systems

$$\tilde{x}(k) = \Phi^k \tilde{x}(0)$$

Does the difference depend upon the initial condition of the nominal solution,  $x_o(0)$  ?

No. The difference only depends upon the **difference** in the initial condition!

So, if one solution  $x_o(k)$  is stable, then every other solution is also stable!

For linear, time-invariant systems, stability is thus a property of the system and not of a special solution.

The stability of linear system is a **global** concept rather than a **local** one!

$$\tilde{x}(k+1) = \Phi \tilde{x}(k) \quad \tilde{x}(0) = a - a_o$$

$$\tilde{x}(k) = \Phi^k \tilde{x}(0)$$

**Does the deviation  $\tilde{x}(k)$  decrease to zero or blow up?**

We can easily analyze this using Z-transform. You can try that by yourself.

Let's take a different approach here.

Consider the special case:

$\tilde{x}(0)$  is the eigenvector  $v$  with eigenvalue of  $\lambda$

$$\Phi v = \lambda v \quad \Longrightarrow \quad \tilde{x}(k) = \Phi^k \tilde{x}(0) = \Phi^k v = \lambda^k v$$

If

$|\lambda| < 1 \quad \lambda^k \rightarrow 0 \quad \text{It is asymptotic stable!}$

$|\lambda| > 1 \quad \lambda^k \rightarrow \infty \quad \text{It is unstable!}$

$|\lambda| = 1 \quad |\lambda^k| = 1 \quad \text{It is (marginally) stable!}$

$$\tilde{x}(k) = \Phi^k \tilde{x}(0)$$

Does the deviation  $\tilde{x}(k)$  decrease to zero or blow up?

What if  $\tilde{x}(0)$  is not the eigenvector of  $\Phi$  ?

Can you use superposition principle to find out the answer?

Assume we can express  $\tilde{x}(0)$  as the linear combination of the eigenvectors,

$$\tilde{x}(0) = a_1 v_1 + a_2 v_2 + \cdots + a_n v_n$$

Use superposition principle,

$$\tilde{x}(k) = \Phi^k (a_1 v_1 + a_2 v_2 + \cdots + a_n v_n) = a_1 \lambda_1^k v_1 + a_2 \lambda_2^k v_2 + \cdots + a_n \lambda_n^k v_n$$

This is true when  $\Phi$  is diagonalizable!

There exists a transformation P, such that

$P^{-1}\Phi P$  is the diagonal matrix

$$\begin{bmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \lambda_n \end{bmatrix}$$



What if  $\Phi$  is not diagonalizable?

For example:  $\Phi = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$

$$\Phi^2 = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix} \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} \lambda^2 & 2\lambda \\ 0 & \lambda^2 \end{bmatrix}$$

$$\Phi^k = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix} \cdots \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} \lambda^k & k\lambda^{k-1} \\ 0 & \lambda^k \end{bmatrix}$$

$\tilde{x}(k) = \Phi^k \tilde{x}(0)$  are linear combinations of  $\lambda^k$  and  $k\lambda^k$

$|\lambda| < 1 \quad k\lambda^k \rightarrow 0$  The system is asymptotic stable!

$|\lambda| > 1 \quad k\lambda^k \rightarrow \infty$  The system is unstable!

$|\lambda| = 1 \quad |k\lambda^k| \rightarrow \infty$  The system is unstable!

This corresponds to the case when the multiplicity of the pole is more than 1!

## Stability criterion for Discrete-time System

$$\lambda = \rho e^{j\theta}$$

$$\rho < 1$$

Stable

$$\rho > 1$$

Unstable

$$\rho = 1$$

Single

Multiple

Marginal Stable

Unstable

In summary,

If  $\rho < 1$  the system is stable, otherwise, not stable.

Since stability is a global property for the linear system,  
it is reasonable to state that a linear system is stable or unstable.

In general, it does not make sense to say that a nonlinear system is stable!  
Rather, we should say a specific nominal operating point or trajectory in a  
system is stable as the concept of stability is only local.

However in control engineering, we do often hear that people say some  
system is stable even when the system is nonlinear! What is the catch?

## Another type of stability: Input-Output Stability

- Bounded-Input Bounded-Output stability (BIBO Stability):



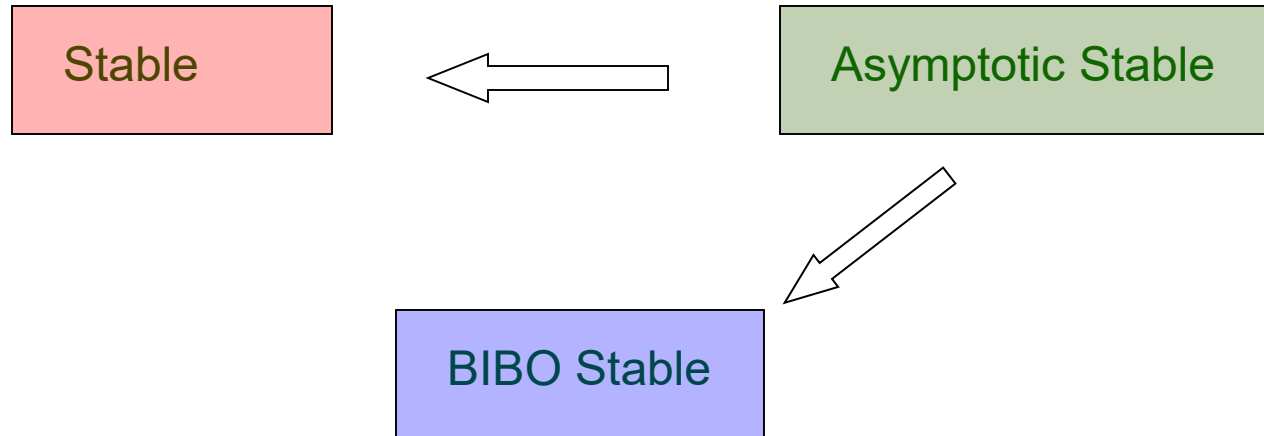
A system is defined to be bounded-input bounded output stable if a bounded input always gives a bounded output for any initial condition.

Is BIBO global or local concept?

Does it refer to any specific reference point?

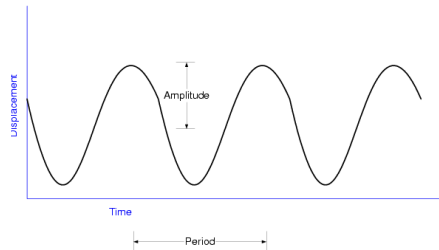
BIBO is a **global** concept. It is ok to say a nonlinear system is globally stable if it refers to the BIBO stability.

## Relation between stability concepts for linear systems:

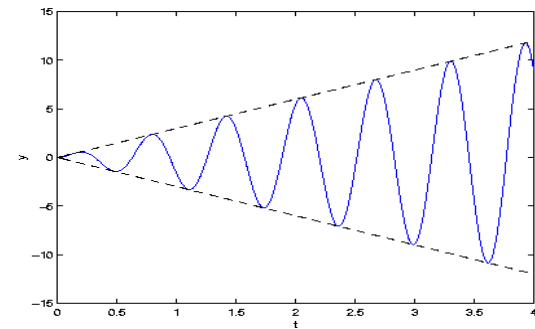


**The interesting question is:**

**If a system is marginally stable, is it also BIBO stable?**

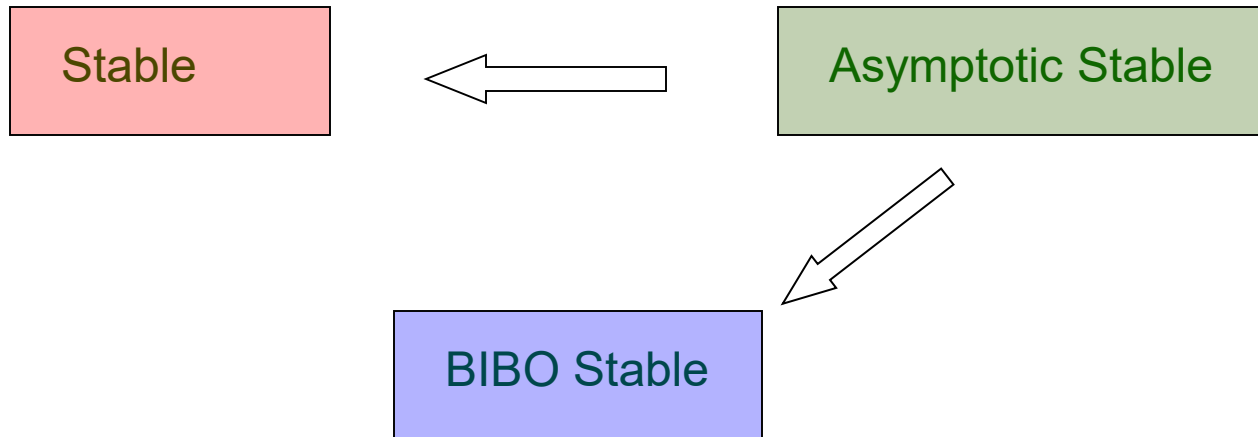


$$\frac{1}{s^2 + 1}$$



**Resonant Oscillation**

$$U(s) = \frac{1}{s^2 + 1}$$
$$Y(s) = H(s)U(s) = \frac{1}{(s^2 + 1)^2} \implies y(t) = t \sin t$$



**Does BIBO stability imply stability?**

$$x(k+1) = \begin{pmatrix} 0.5 & 0 \\ 1 & 2 \end{pmatrix} x(k) + \begin{pmatrix} 1 \\ 1 \end{pmatrix} u(k)$$

$$y(k) = (1 \quad 0)x(k) = x_1(k)$$

**What are the eigenvalues?**

2, 0.5

**Is the system stable?**

**No.**

**What is the input-output model?**

$$x_1(k+1) = 0.5x_1(k) + u(k) \quad \Longrightarrow \quad y(k+1) = 0.5y(k) + u(k)$$

**What is the transfer function?**

$$\frac{1}{z - 0.5}$$

**Is the system BIBO stable?**

It is BIBO stable. The unstable mode is not represented by the transfer function! It means that the output you are observing is bounded while other state variables (which you are not watching) might blow up.

Stability can be checked by looking at the poles of the transfer function.

• What are the poles of  $\frac{1}{z^3 + 3z^2 + 5z - 1}$ ?

• How to check the stability without solving the equation?

JURY'S STABILITY TEST:  $A(z) = a_0 z^n + a_1 z^{n-1} + \dots + a_n = 0$

Get the coefficients:

$$a_0 \quad a_1 \quad \dots \quad a_{n-1} \quad a_n$$

Reverse the order:

$$a_n \quad a_{n-1} \quad \dots \quad a_1 \quad a_0 \quad \times \frac{a_n}{a_0}$$

Eliminate the last element  $a_n$   
by elementary row operation

$$a_0^{n-1} \quad a_1^{n-1} \quad \dots \quad a_{n-1}^{n-1}$$

Repeat the process

$$a_{n-1}^{n-1} \quad a_{n-2}^{n-1} \quad \dots \quad a_0^{n-1} \quad \times \frac{a_{n-1}^{n-1}}{a_0^{n-1}}$$

Stop when there is only one element left

$$a_0^0$$

The system is stable if all the first elements

$$a_0 \quad a_0^{n-1} \quad a_0^{n-2} \quad \dots \quad a_0^0 \quad \text{are positive!}$$

Example 3.1      Stability of a second-order system

$$H(z) = \frac{1}{z^2 + a_1 z + a_2}$$

Get the coefficients:

$$1 \quad a_1 \quad a_2$$

Reverse the order:

$$a_2 \quad a_1 \quad 1 \quad \times a_2$$

Eliminate the last element

$$a_2 \quad 1 - a_2^2 \quad a_1(1 - a_2)$$

Reverse the order:

$$a_1(1 - a_2) \quad 1 - a_2^2 \quad \times \frac{a_1(1 - a_2)}{1 - a_2^2} = \frac{a_1}{1 + a_2}$$

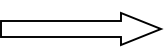
Eliminate the last element

$$a_1(1 - a_2) \quad 1 - a_2^2 - a_1(1 - a_2) \frac{a_1}{1 + a_2} = \frac{1 - a_2}{1 + a_2} ((1 + a_2)^2 - a_1^2)$$

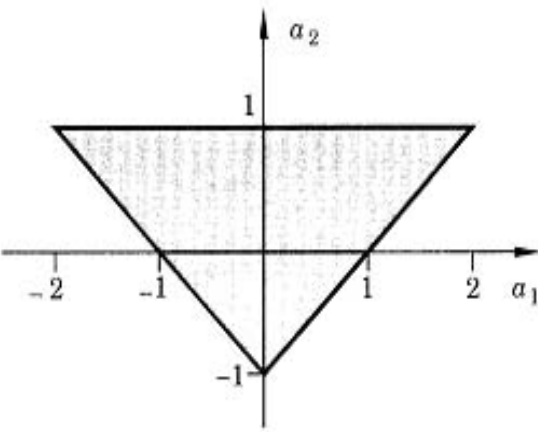
The system is stable if all the first elements

$$1 - a_2^2 > 0$$

$$\frac{1 - a_2}{1 + a_2} ((1 + a_2)^2 - a_1^2) > 0$$



$$\begin{cases} -1 < a_2 < 1 \\ a_2 > -1 + a_1 \\ a_2 > -1 - a_1 \end{cases}$$





**Break**

**State-of-the-art control systems**

**Boston Dynamics Robots**

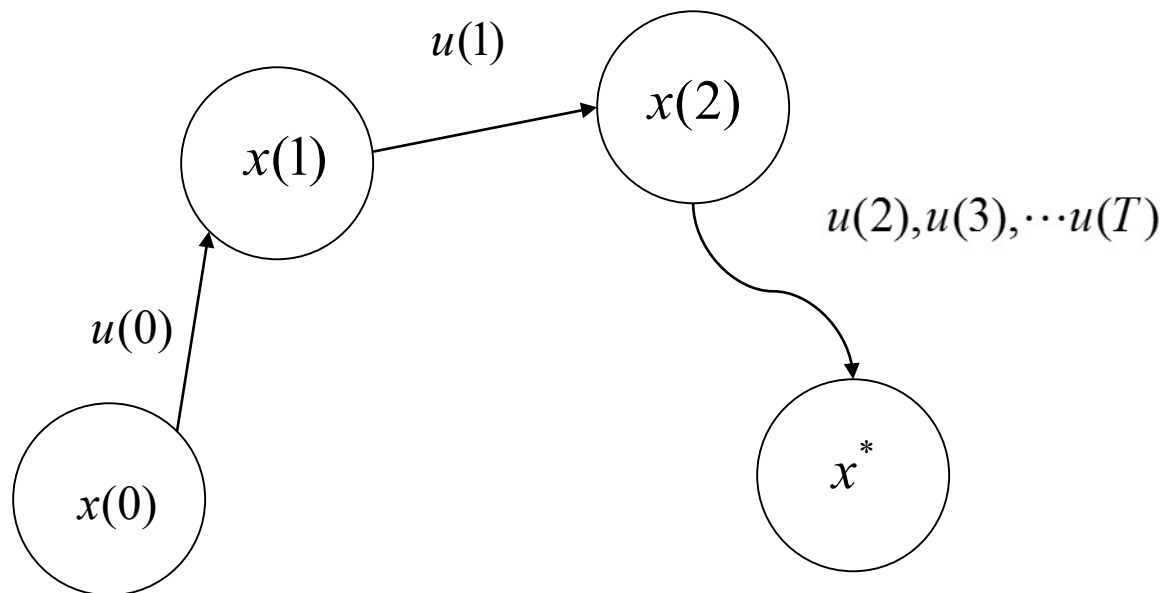
## Now we will discuss another important concept: Controllability

Whenever we want to design a control system, we'd better first check whether the system is controllable or not. Do not waste your time on uncontrollable systems.

What do we mean when we say a system is controllable?

Let's take a look at an example of controllable systems in [real world](#).

**Is it possible to steer a system from any given initial state to any other state in finite time?**



Controllable: the system can be driven from any initial state to any other destination in finite time! It does not matter how fast you can get there!

To check the controllability for nonlinear system is not easy!

Let's check the controllability for linear system:

$$x(k+1) = \Phi x(k) + \Gamma u(k) \quad x \in R^n \quad \text{n-th Order}$$

$$y(k) = cx(k) \quad y \in R, u \in R \quad \text{SISO}$$

Let's start with the initial state  $x(0)$

Then the state at the first step:  $x(1) = \Phi x(0) + \Gamma u(0)$

Then the state at the second step:

$$x(2) = \Phi x(1) + \Gamma u(1)$$

$$x(2) = \Phi(\Phi x(0) + \Gamma u(0)) + \Gamma u(1) = \Phi^2 x(0) + \Phi \Gamma u(0) + \Gamma u(1)$$

Then the state at the n-th step:

$$x(n) = \Phi x(n-1) + \Gamma u(n-1)$$

$$x(n) = \Phi(\Phi x(n-2) + \Gamma u(n-2)) + \Gamma u(n-1)$$

$$= \dots = \Phi^n x(0) + \Phi^{n-1} \Gamma u(0) + \Phi^{n-2} \Gamma u(1) + \dots + \Gamma u(n-1)$$

Let's re-write the equation into:

$$x(n) - \Phi^n x(0) = \Phi^{n-1} \Gamma u(0) + \Phi^{n-2} \Gamma u(1) + \dots + \Gamma u(n-1)$$

• Given any  $x(0)$  and any destination  $x^*$ , is it possible to find control sequence  $\{u(0), u(1), \dots, u(n-1)\}$  such that the state at  $n$ -th step,  $x(n)$ , reaches  $x^*$ ?

Take the problem as solved, and let

$$x(n) \rightarrow x^*$$

$$x^* - \Phi^n x(0) = W_c U$$

$$W_c = \begin{bmatrix} \Gamma & \Phi \Gamma & \dots & \Phi^{n-1} \Gamma \end{bmatrix}$$

$$U = \begin{bmatrix} u(n-1) \\ u(n-2) \\ \vdots \\ u(0) \end{bmatrix}$$

What is the condition on the matrix  $W_c$  for solving  $U$  for any  $x^*$ ?

Controllability matrix  $W_c$  **is nonsingular.**

## The geometrical interpretation of the controllability condition

Consider the second-order system.

Assume the initial state  $x(0)=0$ , and the final state is  $x^*$ , how to reach the final state?

Can you get there in one step?

$$x(1) = \Gamma u(0)$$

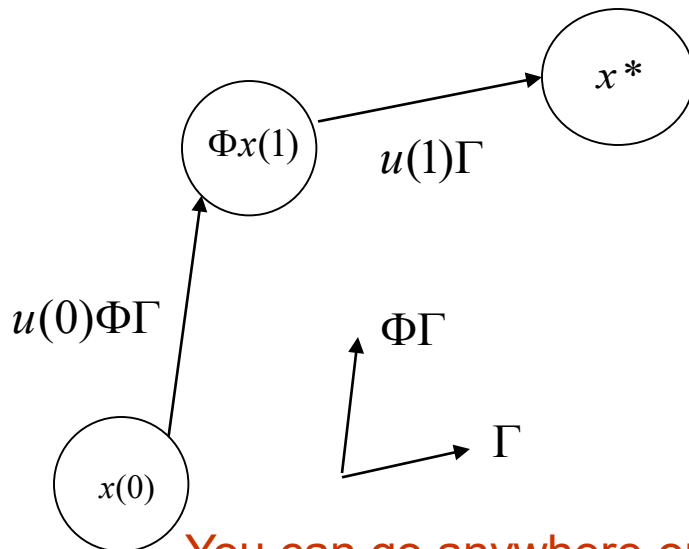
No! unless  $x^*$  is in the same direction of  $\Gamma$

So we need another step of control:

$$x(2) = \Phi x(1) + \Gamma u(1) = \Phi \Gamma u(0) + \Gamma u(1)$$

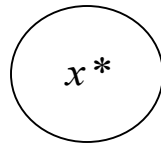
Another direction of movement is added:  $\Phi \Gamma$

$\Phi \Gamma$  has to be a new direction!



You can go anywhere on the plane if you can move with two directions!

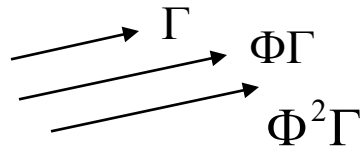
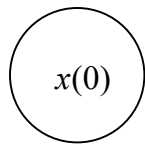
What if  $\Phi\Gamma$  is also along the direction of  $\Gamma$ ,  $\Phi\Gamma = a\Gamma$  ?



Can we reach the final destination in two steps?

A new direction of movement is needed!

Let's try another step of control!



$$x(3) = \Phi x(2) + \Gamma u(2) = \Phi^2 \Gamma u(0) + \Phi \Gamma u(1) + \Gamma u(2)$$

Another direction of movement is added!

$$\text{Is } \Phi^2\Gamma \text{ really a new direction? } \Phi^2\Gamma = \Phi(\Phi\Gamma) \xrightarrow{\Phi\Gamma = a\Gamma} \Phi^2\Gamma = a^2\Gamma$$

$\Phi^2\Gamma$  is still along the direction of  $\Gamma$

For second order system, if  $\{\Gamma, \Phi\Gamma\}$  are not independent directions, then the system is uncontrollable!

The condition that  $W_c = [\Gamma \quad \Phi\Gamma]$  is non-singular, is not only sufficient, but also necessary.

The general case: n-th order system

$\{\Gamma, \Phi\Gamma, \Phi^2\Gamma, \dots, \Phi^{n-1}\Gamma\}$  provide n directions the state can move in the state space!

If all the n directions are independent, of course, the state can go anywhere!

If the n directions are dependent, then there must be some directions where the state can not go in n steps!

Then we have to apply more steps of control, hopefully to generate new directions!

How about one more step? we have another direction  $\Phi^n\Gamma$

Is  $\Phi^n\Gamma$  an independent direction from  $\{\Gamma, \Phi\Gamma, \Phi^2\Gamma, \dots, \Phi^{n-1}\Gamma\}$  ?

No! we showed that in the second order system, how to prove it for the n-th order?

How is  $\Phi^n$  related to  $\{\Phi, \Phi^2, \dots, \Phi^{n-1}\}$  ?

One of the most important theorems in Linear Algebra

## The Cayley-Hamilton Theorem !

**The matrix satisfies its own characteristic equation!**

For any matrix  $\Phi$  its characteristic polynomial is defined as

$$\det\{zI - \Phi\} = z^n + a_1 z^{n-1} + a_2 z^{n-2} + \cdots + a_n$$

Then

$$\Phi^n + a_1 \Phi^{n-1} + a_2 \Phi^{n-2} + \cdots + a_n I = 0 \implies \Phi^n = -a_1 \Phi^{n-1} - a_2 \Phi^{n-2} + \cdots - a_n I$$

$$\Phi^n \Gamma = -a_1 \Phi^{n-1} \Gamma - a_2 \Phi^{n-2} \Gamma + \cdots - a_n \Gamma$$

So  $\Phi^n \Gamma$  is not an independent direction!

$$\{\Gamma, \Phi\Gamma, \Phi^2\Gamma, \dots, \Phi^{n-1}\Gamma\}$$

provide all the possible independent directions that the state can move!



- Is it possible to transform the system into a form which is easier to design the controller?

$$x(k+1) = \Phi x(k) + \Gamma u(k)$$

$$y(k) = cx(k)$$

- **Controllable Canonical Form**

- Assume that the system has the transfer function,

$$H(z) = \frac{b_1 z^{n-1} + b_2 z^{n-2} + \dots + b_n}{z^n + a_1 z^{n-1} + a_2 z^{n-2} + \dots + a_n}$$

$$z(k+1) = \begin{pmatrix} -a_1 & -a_2 & \dots & -a_{n-1} & -a_n \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 \end{pmatrix} z(k) + \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} u(k)$$

$$y(k) = (b_1 \dots b_n) z(k)$$

Only the first row is related to the specific system.

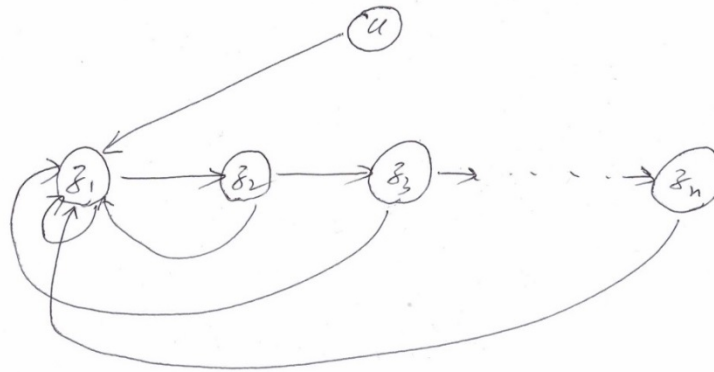
We call it the non-trivial row.

All the other rows are completely fixed for all the system! We call them trivial rows.

$$z(k+1) = \begin{pmatrix} -a_1 & -a_2 & \cdots & -a_{n-1} & -a_n \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix} z(k) + \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} u(k) \quad y(k) = (b_1 \dots b_n) z(k)$$

• Why is it easier to design the controller for this special form?

Let's look at its graphical representation:

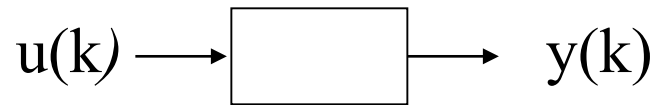


Two distinct properties:

1. The input affects only one state variable  $z_1$ .
2. The state variables are related to each other in a very special way such that if  $z_1$  are controlled properly, all the others will follow.

So we just need to focus upon how to control only one state variable, which makes the design easy!

## Observability:



What is the state  $x(k)$ ?

Can we determine the state  $x(k)$  from observations of inputs and outputs?

- This is not an easy question in general.
- Let's find out the answer for linear system with a given model.

$$x(k+1) = \Phi x(k) + \Gamma u(k)$$

$$y(k) = cx(k)$$

Assume the inputs are zero, observe the output for  $n$  steps:

$$\begin{aligned} y(0) &= cx(0) \\ y(1) &= cx(1) = c\Phi x(0) \\ y(2) &= cx(2) = c\Phi^2 x(0) \\ y(n-1) &= cx(n-1) = c\Phi^{n-1} x(0) \end{aligned} \quad \Rightarrow \quad \begin{pmatrix} C \\ C\Phi \\ \vdots \\ C\Phi^{n-1} \end{pmatrix} x(0) = \begin{pmatrix} y(0) \\ y(1) \\ \vdots \\ y(n-1) \end{pmatrix}$$

What is the condition to find out  $x(0)$  from above equation?

The state  $x(0)$  can be obtained if

The observability matrix  $W_0 = \begin{pmatrix} C \\ C\Phi \\ \vdots \\ C\Phi^{n-1} \end{pmatrix}$  is non-singular (of full rank)

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Once the initial state  $x(0)$  has been found out, can we compute other states? Yes.

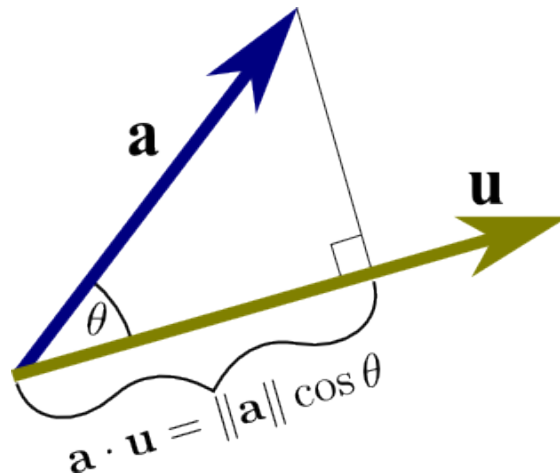
## Geometrical meaning of the Observability Matrix

Assume the inputs are zero, observe the output for the first step:

$$y(0) = cx(0)$$

$y(0)$  is the dot product of state vector  $x(0)$  and the vector  $c$ .

What is the geometrical meaning of dot product of two vectors?



$\mathbf{a} \cdot \mathbf{u}$ , is defined to be the projection of  $\mathbf{a}$  on the unit vector  $\mathbf{u}$ .

$y(0)$  can be interpreted as the projection of state  $x(0)$  on the direction  $c$ !

## Geometrical meaning of the Observability Matrix

Assume the inputs are zero, observe the output for n steps:

$$\begin{aligned} y(0) &= cx(0) \\ y(1) &= cx(1) = c\Phi x(0) \\ y(2) &= cx(2) = c\Phi^2 x(0) \\ y(n-1) &= cx(n-1) = c\Phi^{n-1} x(0) \end{aligned} \quad \Rightarrow \quad \begin{pmatrix} C \\ C\Phi \\ \vdots \\ C\Phi^{n-1} \end{pmatrix} x(0) = \begin{pmatrix} y(0) \\ y(1) \\ \vdots \\ y(n-1) \end{pmatrix}$$

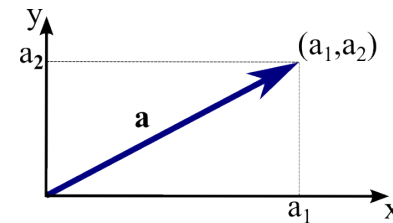
The outputs are just projections of the initial state  $x(0)$  on different directions!

### The observability matrix

$$W_0 = \begin{pmatrix} C \\ C\Phi \\ \vdots \\ C\Phi^{n-1} \end{pmatrix}$$

provides all the possible projection directions for the initial state vector!

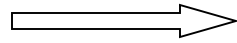
For n-dimensional vector, if you know its projections on n independent directions, can you reconstruct the vector?



If the vectors in the observability matrix can span the whole space, the information of the state can be completely recovered!

What would happen if the observability matrix  $W_o$  is singular?

$$W_o v = 0, v \neq 0$$



$$W_o(x(0) + v) = W_o(x(0))$$

$$\begin{pmatrix} C \\ C\Phi \\ \vdots \\ C\Phi^{n-1} \end{pmatrix} x(0) = \begin{pmatrix} y(0) \\ y(1) \\ \vdots \\ y(n-1) \end{pmatrix}$$

$$\begin{bmatrix} C \\ C\Phi \\ \vdots \\ C\Phi^{n-1} \end{bmatrix} (x(0) + v) = \begin{bmatrix} y(0) \\ y(1) \\ \vdots \\ y(n-1) \end{bmatrix}$$

You cannot differentiate the initial state  $x(0)$  from  $x(0)+v$  by observing the outputs!

• Can we differentiate  $x(0)$  and  $x(0)+v$  by making more observations?

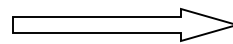
Let's try one more observation:

$$y(n) = cx(n) = c\Phi^n x(0) \quad y'(n) = cx'(n) = c\Phi^n (x(0) + v) = y(n) + c\Phi^n v$$

We already know that  $cv = 0, c\Phi v = 0, c\Phi^2 v = 0, \dots, c\Phi^{n-1} v = 0$

What is the value of  $c\Phi^n v$  ?

• The Cayley-Hamilton Theorem !



$$c\Phi^n v = 0$$

• The linear system is observable if and only if  
the observability matrix  $W_o$  is non-singular (full rank)

## •Observable Canonical Form

$$z(k+1) = \begin{pmatrix} -a_1 & 1 & 0 & \cdots & 0 \\ -a_2 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -a_{n-1} & 0 & 0 & \cdots & 1 \\ -a_n & 0 & 0 & \cdots & 0 \end{pmatrix} z(k) + \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_{n-1} \\ b_n \end{pmatrix} u(k)$$

$$y(k) = (1 \quad 0 \quad \cdots \quad 0) z(k)$$

## •Controllable Canonical Form

$$z(k+1) = \begin{pmatrix} -a_1 & -a_2 & \cdots & -a_{n-1} & -a_n \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix} z(k) + \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} u(k)$$

$$y(k) = (b_1 \quad \dots \quad b_n) z(k)$$

•The transfer function:

$$H(z) = \frac{b_1 z^{n-1} + b_2 z^{n-2} + \cdots + b_n}{z^n + a_1 z^{n-1} + a_2 z^{n-2} + \cdots + a_n}$$

Let's compare it with the controllable canonical form,

**Can you figure out the simple relationship between these two?**

The two forms are related by **transpose** operation.

- How do we check the controllability and observability given the transfer function?

$$\frac{b_1 z^{n-1} + b_2 z^{n-2} + \dots b_n}{z^n + a_1 z^{n-1} + a_2 z^{n-2} + \dots a_n}$$

### •What is Wc and Wo?

- We cannot get Wc and Wo unless we have the state-space model.
- Given a state-space model, we can find out the transfer function using z-transform.

Many control methods are designed for state-space model. Sometimes we need to convert the input-output model to state-space model.

- Realizability: how to find out the state-space model from the input-output model?

$$\frac{b_1 z^{n-1} + b_2 z^{n-2} + \dots b_n}{z^n + a_1 z^{n-1} + a_2 z^{n-2} + \dots a_n} \quad \Longrightarrow \quad \begin{aligned} x(k+1) &= \Phi x(k) + \Gamma u(k) \\ y(k) &= c x(k) \end{aligned}$$

Any state space model which can generate the same input-out model is considered as a valid realization. So there are many possible realizations.



•Let's start with the simplest case:  $y(k+1) = -a_1y(k) + b_1u(k)$

•What is the state at time k?      the natural choice:  $x(k) = y(k)$

$$x(k+1) = -a_1x(k) + b_1u(k)$$

**Rule #1: We can always choose the output  $y(k)$  as one state variable.**

•The second order system:  $y(k+1) = -a_1y(k) - a_2y(k-1) + b_1u(k) + b_2u(k-1)$

Apply rule #1: define  $x_1(k) = y(k)$

Let's compute the state at  $k+1$ ,

$$x_1(k+1) = -a_1x_1(k) - a_2y(k-1) + b_1u(k) + b_2u(k-1)$$

$$x_1(k+1) = -a_1x_1(k) + b_1u(k) - a_2y(k-1) + b_2u(k-1)$$

Compare it with the standard state-space model  $x(k+1) = \Phi x(k) + \Gamma u(k)$

•In the standard state space model, the future state only depends upon the present state and current input.  $y(k) = cx(k)$

•How to deal with all the terms in the past ?  $y(k-1), u(k-1)$

$$x_1(k+1) = -a_1x_1(k) + b_1u(k) - a_2y(k-1) + b_2u(k-1)$$

**Rule #2: Add a new state variable to summarize all the past information!**

$$x_2(k) = -a_2y(k-1) + b_2u(k-1)$$

Let's compute the state at k+1,

$$x_1(k+1) = -a_1x_1(k) + b_1u(k) + x_2(k)$$

$$x_2(k+1) = -a_2y(k) + b_2u(k) = -a_2x_1(k) + b_2u(k)$$

$$x(k+1) = \begin{pmatrix} -a_1 & 1 \\ -a_2 & 0 \end{pmatrix} x(k) + \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} u(k)$$

$$y(k) = (1 \quad 0)x(k)$$

We have learned two rules:

**Rule #1: We can always choose the output y(k) as one state variable.**

**Rule #2: Add new state variable to summarize all the past information whenever necessary!**

That's all you need to know to find out the realization!

•The third order system:

$$y(k+1) = -a_1y(k) - a_2y(k-1) - a_3y(k-2) + b_1u(k) + b_2u(k-1) + b_3u(k-2)$$

Apply rule #1: define  $x_1(k) = y(k)$

$$x_1(k+1) = -a_1x_1(k) - a_2y(k-1) - a_3y(k-2) + b_1u(k) + b_2u(k-1) + b_3u(k-2)$$

$$x_1(k+1) = -a_1x_1(k) + b_1u(k) - a_2y(k-1) - a_3y(k-2) + b_2u(k-1) + b_3u(k-2)$$

•How to deal with the terms in the past ?  $y(k-1), y(k-2), u(k-1), u(k-2)$

Apply rule #2: define a new state variable to take care of all the past information!

$$x_2(k) = -a_2y(k-1) - a_3y(k-2) + b_2u(k-1) + b_3u(k-2)$$

$$x_1(k+1) = -a_1x_1(k) + b_1u(k) + x_2(k)$$

$$x_2(k+1) = -a_2y(k) - a_3y(k-1) + b_2u(k) + b_3u(k-1)$$

$$= -a_2x_1(k) + b_2u(k) - a_3y(k-1) + b_3u(k-1)$$

**We still have some extra terms in the past. What should we do?**

Apply rule #2 again: introduce another state variable!

$$x_3(k) = -a_3 y(k-1) + b_3 u(k-1)$$

$$x_1(k+1) = -a_1 x_1(k) + x_2(k) + b_1 u(k)$$

$$x_2(k+1) = -a_2 x_1(k) + x_3(k) + b_2 u(k)$$

$$x_3(k+1) = -a_3 y(k) + b_3 u(k) = -a_3 x_1(k) + b_3 u(k)$$

$$x(k+1) = \begin{pmatrix} -a_1 & 1 & 0 \\ -a_2 & 0 & 1 \\ -a_3 & 0 & 0 \end{pmatrix} x(k) + \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} u(k)$$

$$y(k) = (1 \quad 0 \quad 0)x(k)$$

**Look at the form of the equation. Which canonical form is it in?**

**It is in the observable canonical form!**

$$H(z) = \frac{b_1 z^{n-1} + b_2 z^{n-2} + \dots + b_n}{z^n + a_1 z^{n-1} + a_2 z^{n-2} + \dots + a_n} \longrightarrow \begin{aligned} x(k+1) &= \Phi x(k) + \Gamma u(k) \\ y(k) &= c x(k) \end{aligned}$$

$$\Phi = \begin{pmatrix} -a_1 & 1 & 0 & \dots & 0 \\ -a_2 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -a_{n-1} & 0 & 0 & \dots & 1 \\ -a_n & 0 & 0 & \dots & 0 \end{pmatrix} \quad \Gamma = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} \quad c = (1 \quad 0 \quad \dots \quad 0)$$

**The observable canonical form is the natural realization of the system!**

**All the state variables are clearly defined by the input and output signals.**

$$x_1(k) = y(k)$$

$$x_2(k) = -a_2 y(k-1) - \dots - a_n y(k-n+1) + b_2 u(k-1) + \dots + b_n u(k-n+1)$$

$$x_3(k) = -a_3 y(k-1) - \dots - a_n y(k-n+2) + b_3 u(k-1) + \dots + b_n u(k-n+2)$$

$$\vdots$$

$$x_n(k) = -a_n y(k-1) + b_n u(k-1)$$

There are other ways to realize the transfer function by a state space model.  
The observable canonical form is the simplest one.

Q & A...

**THANK YOU !**

$$H(z) = \frac{b_1 z + b_2}{z^2 + a_1 z + a_2}$$

[return](#)

$G(s)$	$H(q)$ or the coefficients in $H(q)$	
$\frac{1}{s}$	$\frac{h}{q-1}$	
$\frac{1}{s^2}$	$\frac{h^2(q+1)}{2(q-1)^2}$	
$\frac{1}{s^m}$	$\frac{q-1}{q} \lim_{a \rightarrow 0} \frac{(-1)^m}{m!} \frac{\partial^m}{\partial a^m} \left( \frac{q}{q - e^{-ah}} \right)$	
$e^{-sh}$	$q^{-1}$	
$\frac{a}{s+a}$	$\frac{1 - \exp(-ah)}{q - \exp(-ah)}$	
$\frac{a}{s(s+a)}$	$b_1 = \frac{1}{a}(ah - 1 + e^{-ah})$ $a_1 = -(1 + e^{-ah})$	$b_2 = \frac{1}{a}(1 - e^{-ah} - ahe^{-ah})$ $a_2 = e^{-ah}$
$\frac{a^2}{(s+a)^2}$	$b_1 = 1 - e^{-ah}(1 + ah)$ $a_1 = -2e^{-ah}$	$b_2 = e^{-ah}(e^{-ah} + ah - 1)$ $a_2 = e^{-2ah}$
$\frac{s}{(s+a)^2}$	$\frac{(q-1)he^{-ah}}{(q - e^{-ah})^2}$	
$\frac{ab}{(s+a)(s+b)}$ $a \neq b$	$b_1 = \frac{b(1 - e^{-ah}) - a(1 - e^{-bh})}{b - a}$ $b_2 = \frac{a(1 - e^{-bh})e^{-ah} - b(1 - e^{-ah})e^{-bh}}{b - a}$ $a_1 = -(e^{-ah} + e^{-bh})$ $a_2 = e^{-(a+b)h}$	

$$H(z) = \frac{b_1 z + b_2}{z^2 + a_1 z + a_2}$$

[return](#)

$G(s)$	$H(q)$ or the coefficients in $H(q)$	
$\frac{(s+c)}{(s+a)(s+b)}$ $a \neq b$	$b_1 = \frac{e^{-bh} - e^{-ah} + (1 - e^{-bh})c/b - (1 - e^{-ah})c/a}{a - b}$	
	$b_2 = \frac{c}{ab} e^{-(a+b)h} + \frac{b-c}{b(a-b)} e^{-ah} + \frac{c-a}{a(a-b)} e^{-bh}$	
	$a_1 = -e^{-ah} - e^{-bh}$	$a_2 = e^{-(a+b)h}$
$\frac{\omega_0^2}{s^2 + 2\zeta\omega_0 s + \omega_0^2}$	$b_1 = 1 - \alpha \left( \beta + \frac{\zeta\omega_0}{\omega} \gamma \right)$	$\omega = \omega_0 \sqrt{1 - \zeta^2} \quad \zeta < 1$
	$b_2 = \alpha^2 + \alpha \left( \frac{\zeta\omega_0}{\omega} \gamma - \beta \right)$	$\alpha = e^{-\zeta\omega_0 h}$
	$a_1 = -2\alpha\beta$	$\beta = \cos(\omega h)$
	$a_2 = \alpha^2$	$\gamma = \sin(\omega h)$
$\frac{s}{s^2 + 2\zeta\omega_0 s + \omega_0^2}$	$b_1 = \frac{1}{\omega} e^{-\zeta\omega_0 h} \sin(\omega h)$	$b_2 = -b_1$
	$a_1 = -2e^{-\zeta\omega_0 h} \cos(\omega h)$	$a_2 = e^{-2\zeta\omega_0 h}$
	$\omega = \omega_0 \sqrt{1 - \zeta^2}$	
$\frac{a^2}{s^2 + a^2}$	$b_1 = 1 - \cos ah$	$b_2 = 1 - \cos ah$
	$a_1 = -2 \cos ah$	$a_2 = 1$
$\frac{s}{s^2 + a^2}$	$b_1 = \frac{1}{a} \sin ah$	$b_2 = -\frac{1}{a} \sin ah$
	$a_1 = -2 \cos ah$	$a_2 = 1$
$\frac{a}{s^2(s+a)}$	$b_1 = \frac{1-\alpha}{a^2} + h \left( \frac{h}{2} - \frac{1}{a} \right)$	$\alpha = e^{-ah}$
	$b_2 = (1-\alpha) \left( \frac{h^2}{2} - \frac{2}{a^2} \right) + \frac{h}{a} (1+\alpha)$	
	$b_3 = - \left[ \frac{1}{a^2} (\alpha - 1) + \alpha h \left( \frac{h}{2} + \frac{1}{a} \right) \right]$	
	$a_1 = -(\alpha + 2)$	$a_2 = 2\alpha + 1 \quad a_3 = -\alpha$



Entry #	Laplace Domain	Time Domain	Z Domain (t=kT)
1	1	$\delta(t)$ unit impulse	1
2	$\frac{1}{s}$	$u(t)$ unit step	$\frac{z}{z-1}$
3	$\frac{1}{s^2}$	t	$\frac{Tz}{(z-1)^2}$
4	$\frac{1}{s+a}$	$e^{-at}$	$\frac{z}{z-e^{-aT}}$
5		$b^k \quad (b = e^{-aT})$	$\frac{z}{z-b}$
6	$\frac{1}{(s+a)^2}$	$te^{-at}$	$\frac{Tze^{-aT}}{(z-e^{-aT})^2}$
7	$\frac{1}{s(s+a)}$	$\frac{1}{a}(1-e^{-at})$	$\frac{z(1-e^{-aT})}{a(z-1)(z-e^{-aT})}$
8	$\frac{b-a}{(s+a)(s+b)}$	$e^{-at} - e^{-bt}$	$\frac{z(e^{-aT} - e^{-bT})}{(z-e^{-aT})(z-e^{-bT})}$
9	$\frac{1}{s(s+a)(s+b)}$	$\frac{1}{ab} - \frac{e^{-at}}{a(b-a)} - \frac{e^{-bt}}{b(a-b)}$	
10	$\frac{1}{s(s+a)^2}$	$\frac{1}{a^2}(1-e^{-at} - ate^{-at})$	
11	$\frac{s}{(s+a)^2}$	$(1-at)e^{-at}$	
12	$\frac{b}{s^2+b^2}$	$\sin(bt)$	$\frac{z \sin(bT)}{z^2 - 2z \cos(bT) + 1}$
13	$\frac{s}{s^2+b^2}$	$\cos(bt)$	$\frac{z(z - \cos(bT))}{z^2 - 2z \cos(bT) + 1}$
14	$\frac{b}{(s+a)^2+b^2}$	$e^{-at} \sin(bt)$	$\frac{ze^{-aT} \sin(bT)}{z^2 - 2ze^{-aT} \cos(bT) + e^{-2aT}}$
15	$\frac{s+a}{(s+a)^2+b^2}$	$e^{-at} \cos(bt)$	$\frac{z^2 - ze^{-aT} \cos(bT)}{z^2 - 2ze^{-aT} \cos(bT) + e^{-2aT}}$

[return](#)