

EE5103 Lecture Four

Pole-Placement Problem

State-Space Approach

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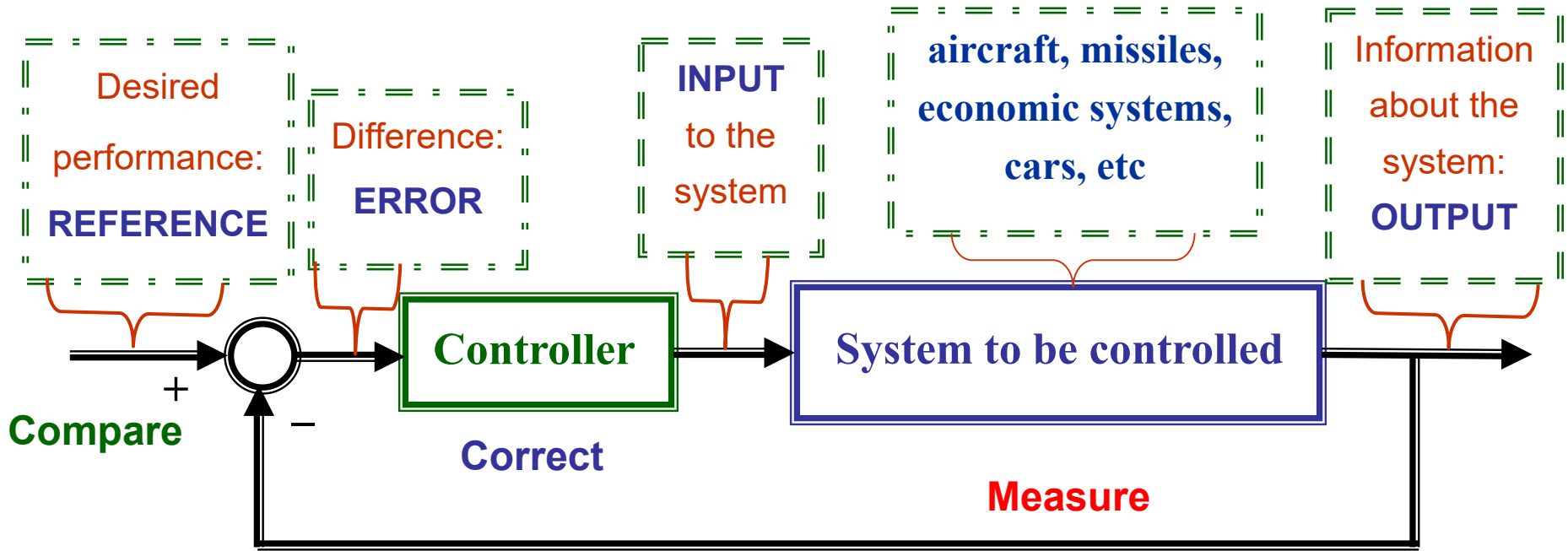
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Now we are ready to discuss controller design—the core of this module



• Feedback: Measure — Compare — Correct

Objective: To make the system **OUTPUT** and the desired **REFERENCE** as close as possible, i.e., to make the **ERROR** as small as possible.

How to specify the reference signal, or the desired output?

• Can we make the output follow arbitrary command signal?

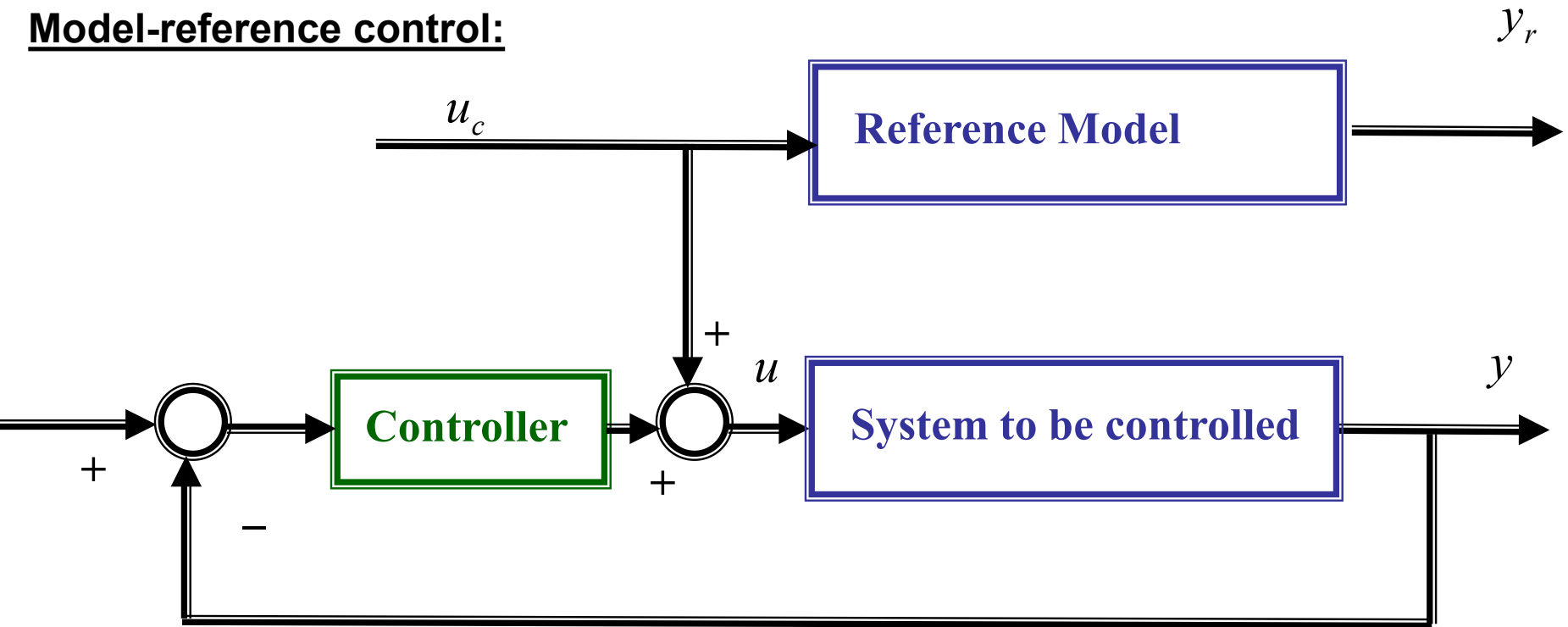
Impossible! We cannot make the car run faster than the rocket!

• The performance requirements have to be realistic!

How to meet the performance requirements?

One approach is to build a reference model first!

Model-reference control:



Objective: Since the feedback control can change the property of the system, we just need to make the closed loop model **follow the reference model** as close as possible.

If the reference model can meet all the performance requirements, then the closed loop system can do it too!

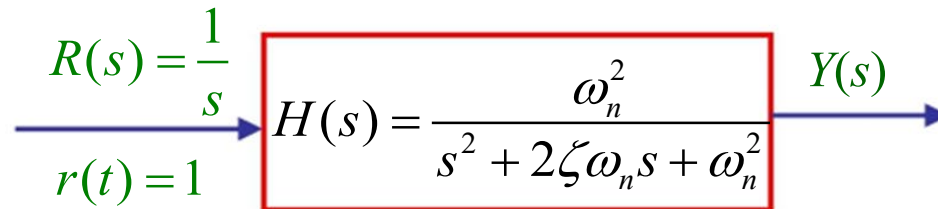
How to build the appropriate reference model in practice?

It depends upon the design specifications.

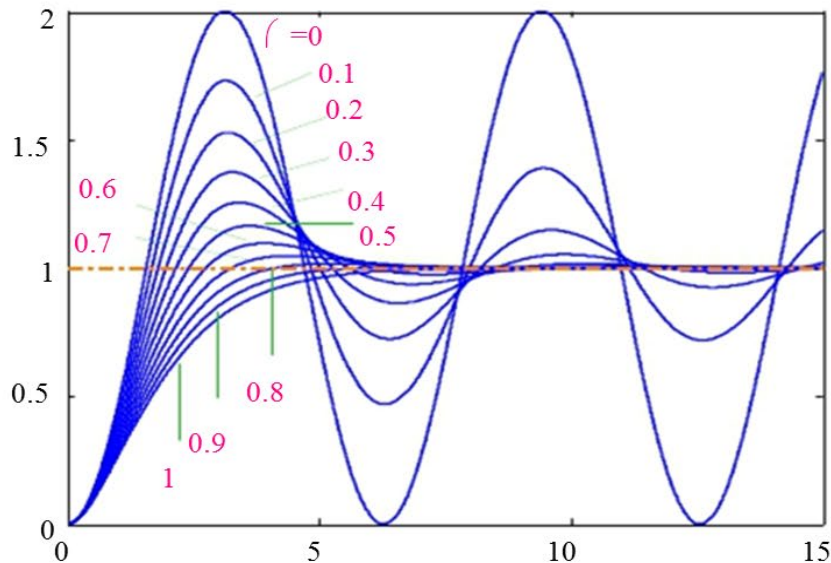
One typical way to build a reference model is to build a second order system since its behaviors are well studied.

Transient behavior of 2nd order systems

Consider the following block diagram with a standard 2nd order system under unit step

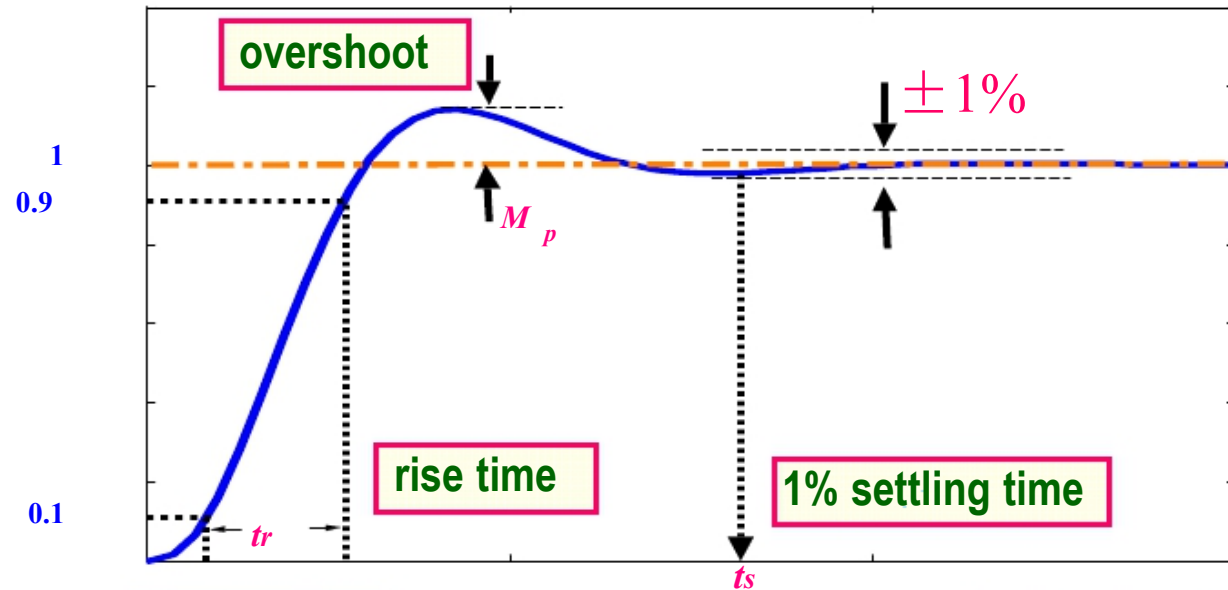


The behavior of the system is as follows:



The system behavior is fully characterized by 2 factors: the **damping ratio ζ** , and the **natural frequency ω_n** .

Settling time, overshoot and rise time — time domain specifications



$$t_r \cong \frac{1.8}{\omega_n}$$

$$M_p = e^{-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}}$$

$$t_s \cong \frac{4.6}{\zeta\omega_n}$$

$$(t_s, M_p, t_r)$$



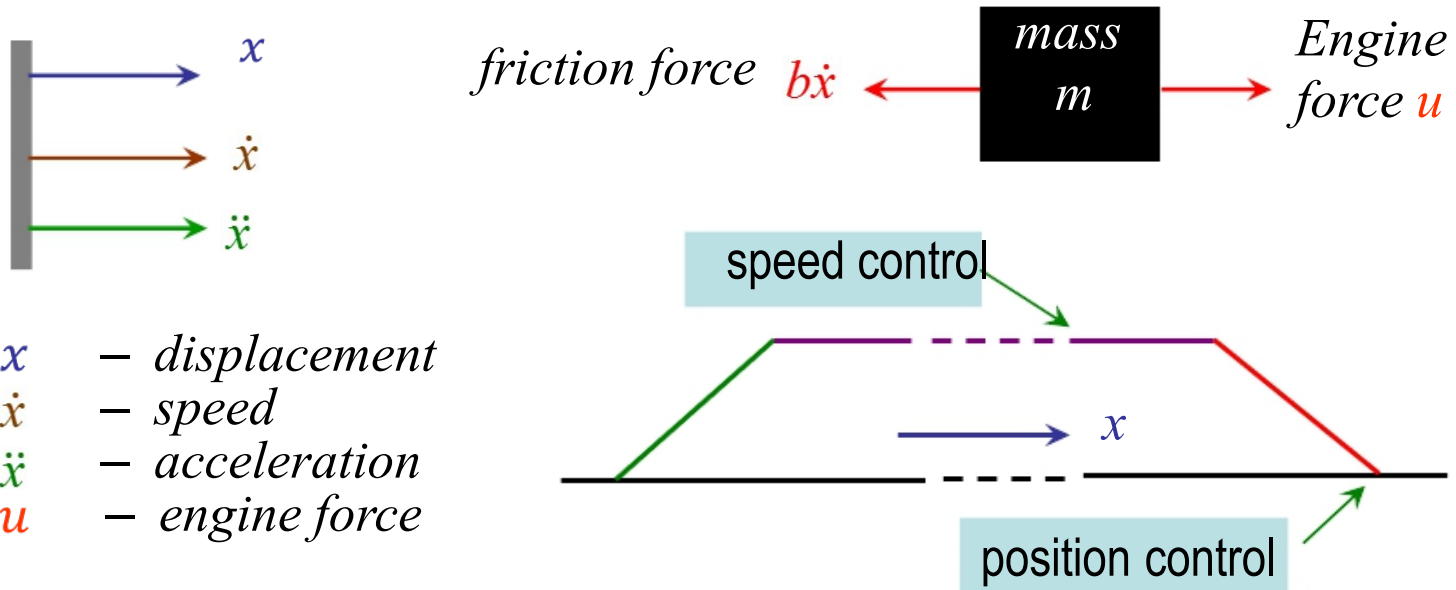
$$(\zeta, \omega_n)$$



Reference Model

Design Example: speed control system

Consider the vehicle, which has a weight $m = 1000$ kg. Assuming the average friction coefficient $b = 100$, design a speed control system such that the vehicle can reach 100 km/h from 0 km/h in 8 s with an overshoot less than 5% .



x — displacement
 \dot{x} — speed
 \ddot{x} — acceleration
 u — engine force

$$m\ddot{x} = u - b\dot{x}$$



$$m\ddot{x} + b\dot{x} = u$$



$$m\dot{y} + by = u$$



$$\frac{Y(s)}{U(s)} = \frac{1}{ms + b}$$

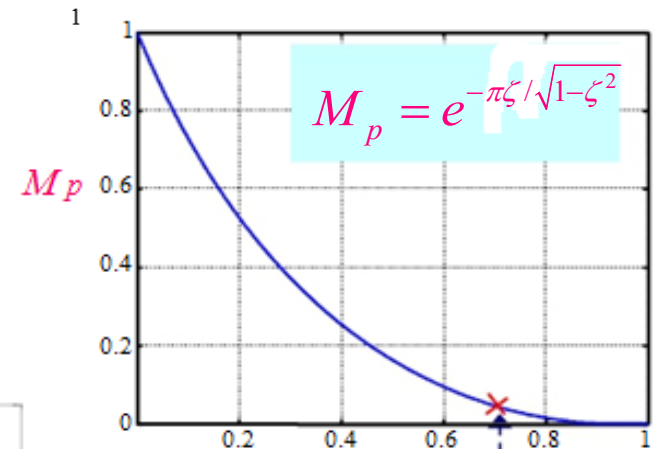
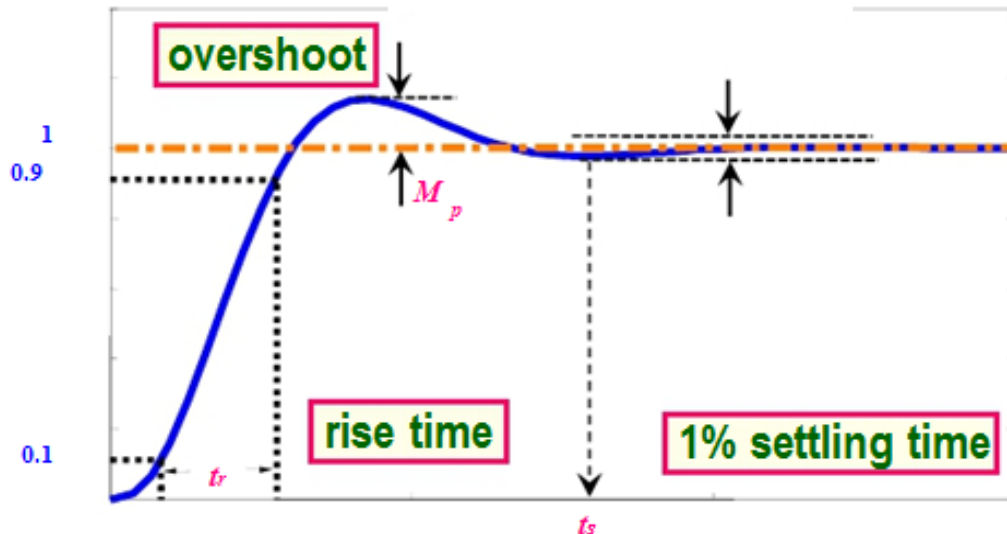
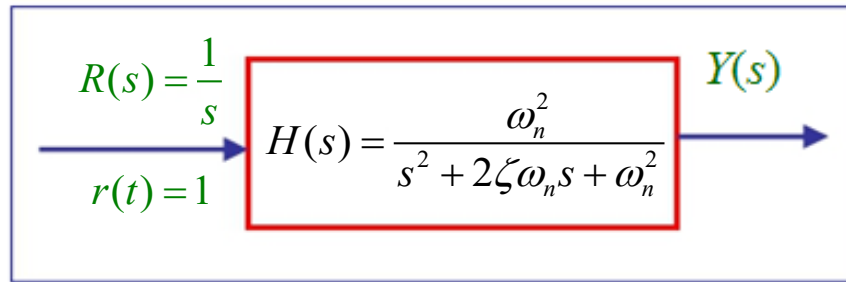
speed $y = \dot{x}$

Design the continuous-time reference model first:

First derive ξ and ω_n from the design specifications:

Overshoot: 5%

Settling Time: 8s



$$\xi \geq 0.6901 \Rightarrow \xi = 0.7$$

$$t_s \cong \frac{4.6}{\xi \omega_n} \Rightarrow \omega_n \cong \frac{4.6}{t_s \xi}$$

$$\Rightarrow \omega_n = \frac{4.6}{8 \times 0.7} = 0.82$$

Desired CLTF $\Rightarrow H_d(s) = \frac{0.67}{s^2 + 1.15s + 0.67}$

Reference Model

$$H_d(s) = \frac{0.67}{s^2 + 1.15s + 0.67}$$

Is this the reference model we try to follow for the computer control system?

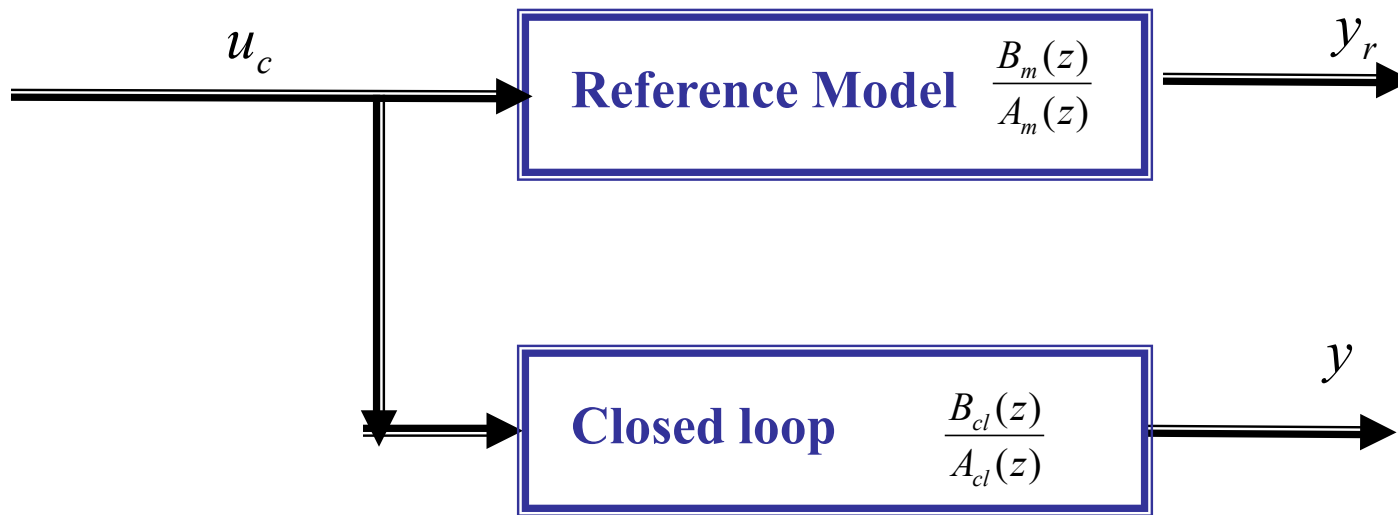
We need to convert the desired reference model from continuous time to discrete-time.

You can use the conversion [Table 2.1](#). or the MATLAB command c2d.

With sampling period $h=0.1s$

$$H_m(z) = \frac{0.003223 z + 0.003102}{z^2 - 1.885z + 0.8914}$$

Remember that the reference model in discrete time depends upon the sampling period.



Can we use feedback control to change the original system into the reference model?

• Ideal case: perfect match

$$\frac{B_{cl}(z)}{A_{cl}(z)} = \frac{B_m(z)}{A_m(z)}$$

Perfect match means that both the poles and zeros match each other!

We will try to match them in two steps:

$$A_{cl}(z) \Rightarrow A_m(z)$$

Pole Placement \rightarrow Match the Poles

$$B_{cl}(z) \Rightarrow B_m(z)$$

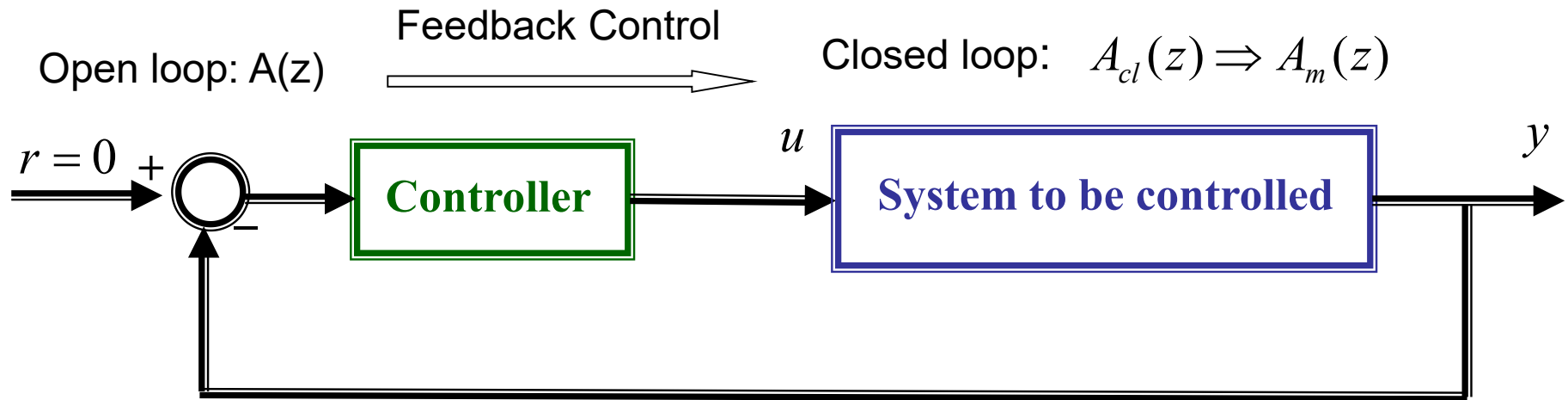
Zero Placement \rightarrow Match the zeros

Which one is more important? And Why?

Pole placement since stability is the primary requirement!

Pole Placement: Change the poles of the system to desired ones.

The desired poles are designed by the performance requirements: $A_m(z)$



- What is available for the controller design?
- The model ----state-space model v.s. input-output model
- The measurements (sensors):
State variables vs. outputs

We will start with the simplest case:

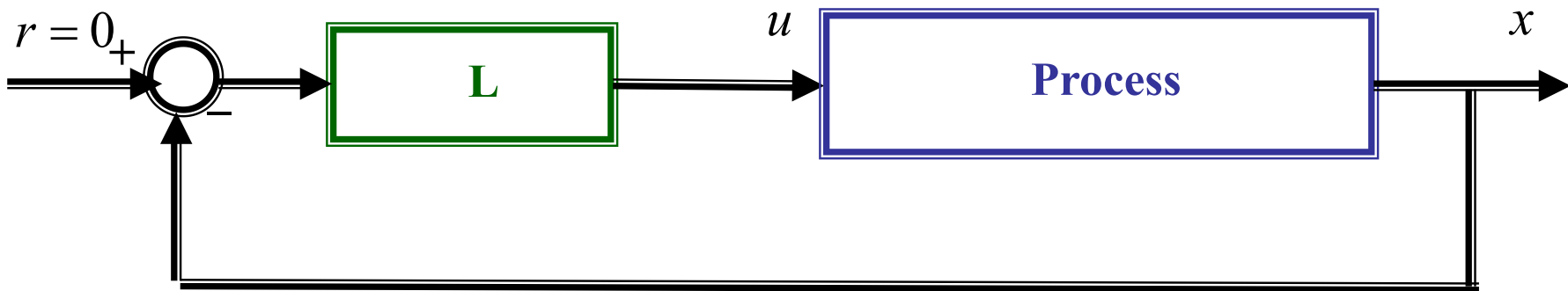
1. The state-space model is available.
2. All the state variables can be measured by various sensors.

When you do research, it is always a good strategy to attack the simplest case first, then build it up.

What's the simplest feedback-controller you have learned in the past?

Proportional Control

Similarly, we will try the simplest one first:



The objective of pole placement is stabilization, so $r=0$.

$$u(k) = Le(k) = L(0 - x(k)) = -Lx(k)$$

Is L a vector or scalar?

Since x is a vector, L has to be a row vector such that the product is a scalar!
 L corresponds to the gains for each state variable and the input is the weighted sum.

Pole placement: can we design the controller gain L such that all the poles are placed to the desired position?

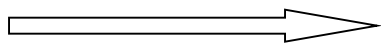
The desired poles are given by the reference model

$$A_m(z) = z^n + p_1 z^{n-1} + \cdots + p_n$$

• Simple example: Double-integrator:

$$\ddot{y}(t) = u(t)$$

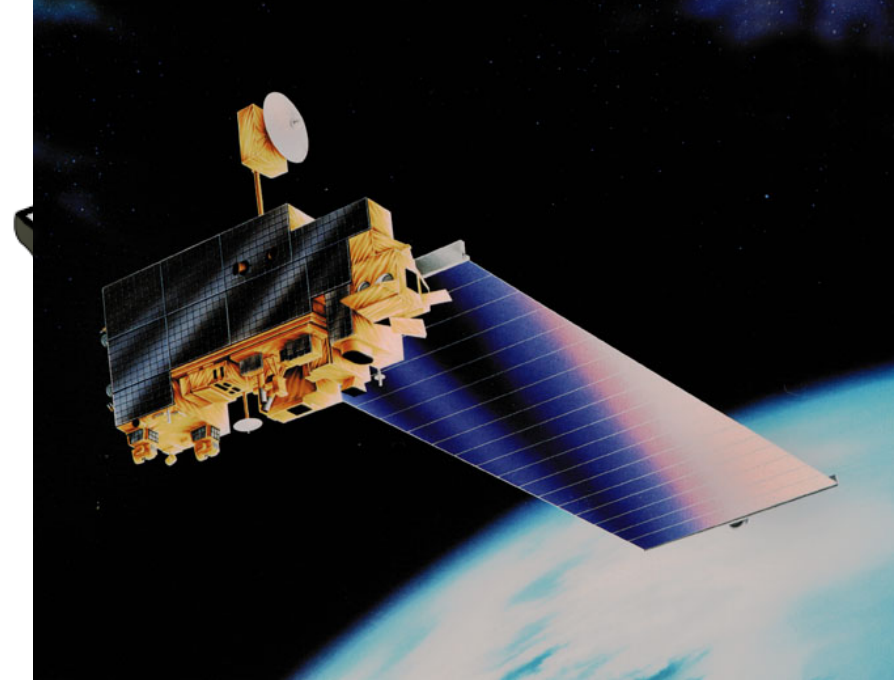
Zero-order hold



$$x(k+1) = \Phi x(k) + \Gamma u(k) = \begin{bmatrix} 1 & h \\ 0 & 1 \end{bmatrix} x(k) + \begin{bmatrix} \frac{h^2}{2} \\ h \end{bmatrix} u(k)$$

$$y(k) = \begin{bmatrix} 1 & 0 \end{bmatrix} x(k)$$

Is the system stable?



The system has two repeated pole at 1, not stable!

We need to build a feedback controller to change the behavior!
In other words, we need to place the poles to the desired positions.

Controller:

$$u(k) = -Lx(k) = -l_1 x_1(k) - l_2 x_2(k)$$

The closed-loop system:

$$x(k+1) = \Phi x(k) + \Gamma u(k) = (\Phi - \Gamma L)x(k)$$

$$\Phi \Rightarrow (\Phi - \Gamma L)$$

The system matrix has been changed by the feedback!

Therefore, the poles have been changed!

This is the whole point of feedback: change the property of the system!

$$x(k+1) = \Phi x(k) + \Gamma u(k) = \begin{bmatrix} 1 & h \\ 0 & 1 \end{bmatrix} x(k) + \begin{bmatrix} \frac{h^2}{2} \\ h \end{bmatrix} u(k)$$

$$y(k) = \begin{bmatrix} 1 & 0 \end{bmatrix} x(k)$$

Controller:

$$u(k) = -Lx(k) = -l_1 x_1(k) - l_2 x_2(k)$$

The closed-loop system:

$$x(k+1) = \Phi x(k) + \Gamma u(k) = (\Phi - \Gamma L)x(k)$$

How to design the control gain L?

$$\Gamma L = \begin{bmatrix} \frac{h^2}{2} \\ h \end{bmatrix} \begin{bmatrix} l_1 & l_2 \end{bmatrix} = \begin{bmatrix} \frac{h^2 l_1}{2} & \frac{h^2 l_2}{2} \\ h l_1 & h l_2 \end{bmatrix} \implies \Phi - \Gamma L = \begin{bmatrix} 1 - \frac{h^2 l_1}{2} & h - \frac{h^2 l_2}{2} \\ -h l_1 & 1 - h l_2 \end{bmatrix}$$

Characteristic Polynomial (C.P.) of the closed loop:

$$|zI - (\Phi - \Gamma L)| = \begin{vmatrix} z - 1 + \frac{h^2 l_1}{2} & -h + \frac{h^2 l_2}{2} \\ h l_1 & z - 1 + h l_2 \end{vmatrix} = (z - 1 + \frac{h^2 l_1}{2})(z - 1 + h l_2) - (-h + \frac{h^2 l_2}{2})h l_1$$

$$|zI - (\Phi - \Gamma L)| = z^2 + (\frac{l_1 h^2}{2} + l_2 h - 2)z + (\frac{l_1 h^2}{2} - l_2 h + 1)$$

Characteristic Polynomial (C.P.) of the closed loop:

$$|zI - (\Phi - \Gamma L)| = z^2 + \left(\frac{l_1 h^2}{2} + l_2 h - 2\right)z + \left(\frac{l_1 h^2}{2} - l_2 h + 1\right)$$

Compare it with the desired C.P. (from the reference model):

$$A_m(z) = z^2 + p_1 z + p_2$$

Let's try to match them:

$$\left(\frac{l_1 h^2}{2} + l_2 h - 2\right) = p_1$$

$$\left(\frac{l_1 h^2}{2} - l_2 h + 1\right) = p_2$$

Is it possible to solve L from above equations?

$$l_1 = \frac{1 + p_1 + p_2}{h^2}$$

$$l_2 = \frac{3 + p_1 - p_2}{2h}$$

Is it possible to design the controller to place the poles at any desired locations?

Yes. The poles can be placed anywhere you like,
thanks to the power of feedback!

Can we do this for any other linear systems?

• Controllable Canonical Form: $z(k+1) = \tilde{\Phi}z(k) + \tilde{\Gamma}u(k)$

$$\tilde{\Phi} = \begin{bmatrix} -a_1 & -a_2 & \cdots & -a_n \\ 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 1 & 0 \end{bmatrix} \quad \tilde{\Gamma} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad H(z) = \frac{b_1 z^{n-1} + b_2 z^{n-2} + \cdots + b_n}{z^n + a_1 z^{n-1} + a_2 z^{n-2} + \cdots + a_n}$$

What is the C.P. of the canonical form?

Take the first row (the non-trivial row)!

$$A(z) = z^n + a_1 z^{n-1} + \cdots + a_n$$

State-Feedback Controller:

$$u(k) = -\tilde{L}z(k)$$

The closed-loop system: $z(k+1) = \tilde{\Phi}z(k) + \tilde{\Gamma}u(k) = (\tilde{\Phi} - \tilde{\Gamma}\tilde{L})z(k)$

$$\tilde{\Gamma}\tilde{L} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \begin{bmatrix} \tilde{l}_1 & \tilde{l}_2 & \cdots & \tilde{l}_n \end{bmatrix} = \begin{bmatrix} \tilde{l}_1 & \tilde{l}_2 & \cdots & \tilde{l}_n \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}$$

$$\tilde{\Phi} - \tilde{\Gamma}\tilde{L} = \begin{bmatrix} -(a_1 + \tilde{l}_1) & -(a_2 + \tilde{l}_2) & \cdots & -(a_n + \tilde{l}_n) \\ 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 1 & 0 \end{bmatrix}$$

State-Feedback Controller: $u(k) = -\tilde{L}z(k)$

The closed-loop system: $z(k+1) = \tilde{\Phi}z(k) + \tilde{\Gamma}u(k) = (\tilde{\Phi} - \tilde{\Gamma}\tilde{L})z(k)$

$$\tilde{\Phi} = \begin{bmatrix} -a_1 & -a_2 & \cdots & -a_n \\ 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 1 & 0 \end{bmatrix} \Rightarrow \tilde{\Phi} - \tilde{\Gamma}\tilde{L} = \begin{bmatrix} -(a_1 + \tilde{l}_1) & -(a_2 + \tilde{l}_2) & \cdots & -(a_n + \tilde{l}_n) \\ 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 1 & 0 \end{bmatrix}$$

Let's take a closer look at the system matrix after feedback.

Look at the first row, the non-trivial row.

Does feedback controller change the non-trivial row?

Yes. Every element of non-trivial row can be tuned independently!

How about the trivial rows? Does feedback affect them?

No! The trivial rows keep the same!

The input does not change other state variables directly,
which simplifies the controller design!

Is $\tilde{\Phi} - \tilde{\Gamma}\tilde{L}$ still in the controllable canonical form?

Yes.

State-Feedback Controller:

$$u(k) = -\tilde{L}z(k)$$

The closed-loop system:

$$z(k+1) = \tilde{\Phi}z(k) + \tilde{\Gamma}u(k) = (\tilde{\Phi} - \tilde{\Gamma}\tilde{L})z(k)$$

$$\tilde{\Phi} = \begin{bmatrix} -a_1 & -a_2 & \cdots & -a_n \\ 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 1 & 0 \end{bmatrix} \Rightarrow \tilde{\Phi} - \tilde{\Gamma}\tilde{L} = \begin{bmatrix} -(a_1 + \tilde{l}_1) & -(a_2 + \tilde{l}_2) & \cdots & -(a_n + \tilde{l}_n) \\ 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 1 & 0 \end{bmatrix}$$

How do we get the C.P. of the closed loop: $\tilde{\Phi} - \tilde{\Gamma}\tilde{L}$?

Take the non-trivial row!

$$A_{cl}(z) = z^n + (a_1 + \tilde{l}_1)z^{n-1} + \cdots + (a_n + \tilde{l}_n)$$

Compare it with the desired C.P.:

$$A_m(z) = z^n + p_1z^{n-1} + \cdots + p_n$$

Can you match them by designing the control gains?

$$\tilde{L} = [p_1 - a_1 \quad p_2 - a_2 \quad \cdots \quad p_n - a_n]$$

The gains are just the differences between the coefficients of the C.P.s

$$u(k) = -\tilde{L}z(k) = -[p_1 - a_1 \quad p_2 - a_2 \quad \cdots \quad p_n - a_n]z(k)$$

So we have obtained a simple solution for controllable canonical form!

•How about systems in other forms?

$$x(k+1) = \Phi x(k) + \Gamma u(k)$$

Can we transform any linear system into controllable canonical form?

What is the condition?

•If the system is controllable:

$$\begin{array}{lll} x(k+1) = \Phi x(k) + \Gamma u(k) & z(k) = Tx(k) & z(k+1) = \tilde{\Phi} z(k) + \tilde{\Gamma} u(k) \\ y(k) = cx(k) & \Longrightarrow & y(k) = \tilde{c} z(k) \end{array}$$

Design the state-feedback controller for the controllable canonical form first:

$$\begin{aligned} \tilde{L} &= [p_1 - a_1 \quad p_2 - a_2 \quad \cdots \quad p_n - a_n] \\ u(k) &= -\tilde{L} z(k) = -[p_1 - a_1 \quad p_2 - a_2 \quad \cdots \quad p_n - a_n] z(k) \end{aligned}$$

$$z(k) = Tx(k) \quad \Longrightarrow \quad u(k) = -\tilde{L} Tx(k)$$

It can be shown by Cayley-Hamilton Theorem:

$$u(k) = -Lx(k)$$

$$L = [0 \quad \cdots \quad 0 \quad 1] W_c^{-1} A_m(\Phi)$$

Ackermann's formula

Algebraic problem:

Given pair $\{\Phi, \Gamma\}$, find L , such that $(\Phi - \Gamma L)$ has any desired eigenvalues.

Choose L such that the C.P. of $(\Phi - \Gamma L)$ is any desired one: $A_m(z)$

Example: double-integrator

$$x(k+1) = \begin{bmatrix} 1 & h \\ 0 & 1 \end{bmatrix} x(k) + \begin{bmatrix} \frac{h^2}{2} \\ h \end{bmatrix} u(k)$$

$$W_c = \begin{bmatrix} \frac{h^2}{2} & \frac{3h^2}{2} \\ \frac{h}{h} & \frac{h}{h} \end{bmatrix}$$

Is it controllable?

Yes.

Applying Ackermann's formula

$$L = [0 \quad \dots \quad 0 \quad 1] W_c^{-1} A_m(\Phi)$$

$$W_c^{-1} = \begin{bmatrix} -\frac{1}{h^2} & \frac{1.5}{h} \\ \frac{1}{h^2} & -\frac{0.5}{h} \end{bmatrix}$$

$$A_m(\Phi) = \Phi^2 + p_1 \Phi + p_2 I = \begin{bmatrix} 1 + p_1 + p_2 & 2h + p_1 h \\ 0 & 1 + p_1 + p_2 \end{bmatrix}$$

$$L = [0 \quad 1] W_c^{-1} A_m(\Phi) = \begin{bmatrix} \frac{1 + p_1 + p_2}{h^2} & \frac{3 + p_1 - p_2}{2h} \end{bmatrix}$$

Compared to the first method by matching $|zI - (\Phi - \Gamma L)| = A_m(z)$

which one do you prefer?

• The special case: Deadbeat control

Can we drive the state to zero in n steps?

System: $x(k+1) = \Phi x(k) + \Gamma u(k)$

State Feedback Control: $u(k) = -Lx(k)$

Closed Loop System: $x(k+1) = \Phi x(k) + \Gamma u(k) = (\Phi - \Gamma L)x(k)$

The C.P. of the closed loop matrix: $\Phi - \Gamma L$ $A_{cl}(z) = z^n + p_1 z^{n-1} + \dots + p_n$

Let all the poles be placed at the origin: $A_{cl}(z) = z^n$

What is the value of $A_{cl}(z) \big|_{z=\Phi-\Gamma L} = (\Phi - \Gamma L)^n = ???$

Cayley-Hamilton theorem $\implies (\Phi - \Gamma L)^n = 0$

If the controller is implemented at time k:

$$x(k+1) = (\Phi - \Gamma L)x(k)$$

$$x(k+2) = (\Phi - \Gamma L)x(k+1) = (\Phi - \Gamma L)^2 x(k)$$

$$\implies x(k+n) = (\Phi - \Gamma L)^n x(k) = 0$$

Deadbeat!

But there is no free lunch! Deadbeat controller is rarely used in practice because the energy consumption is too high.

• But what if the system is not controllable?

• Example:
$$x(k+1) = \begin{bmatrix} a & 1 \\ 0 & b \end{bmatrix} x(k) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(k)$$

The controllability matrix
$$W_c = \begin{bmatrix} 1 & a \\ 0 & 0 \end{bmatrix}$$

Can we use Ackermann's Formula?

No.

Can we still use Proportional (state-feedback) control?

Of course.

Let's see what would happen if we use state-feedback control

$$u(k) = -l_1 x_1(k) - l_2 x_2(k)$$

$$x(k+1) = \begin{bmatrix} a-l_1 & 1-l_2 \\ 0 & b \end{bmatrix} x(k)$$

What are the poles of the closed loop?

$a - l_1$ and b .

What are the poles of the original system?

a and b .

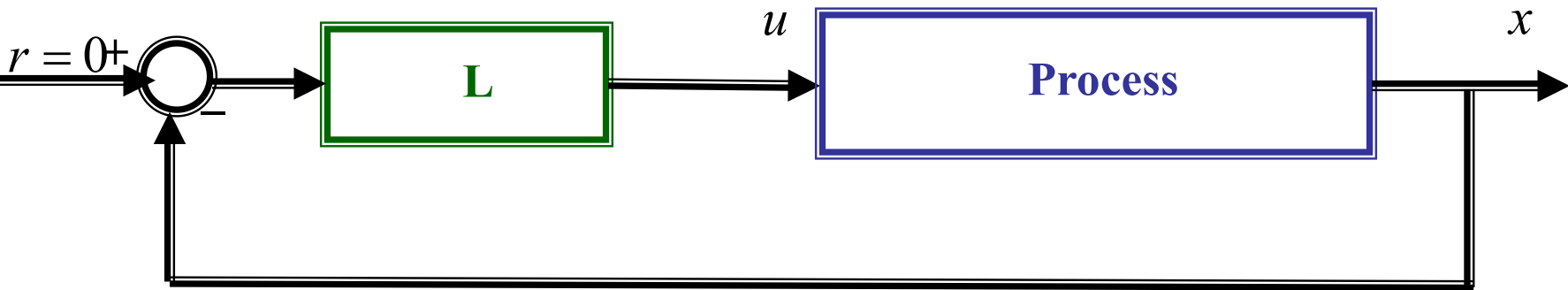
Can you design L such that the poles can be placed at any desired positions?

No. The pole $z=a$ can be changed, but the pole $z=b$ can not be changed!

If $|b| < 1$, this uncontrollable mode is stable, we call such kind of system stabilizable.

If $|b| > 1$, since this mode is unstable, there's nothing we can do to control this system unless we make major changes to the systems.

Proportional Control --- State Feedback Control



A very simple controller:

$$u(k) = -Lx(k)$$

Closed Loop System:

$$x(k+1) = \Phi x(k) + \Gamma u(k) = (\Phi - \Gamma L)x(k)$$

If the system is controllable, the poles can be placed anywhere:

$$A_m(z) = z^n + p_1 z^{n-1} + \cdots + p_n$$

- If the system is uncontrollable, then some of the poles cannot be moved!
- If the uncontrollable poles are stable, then we can still control the system to some degree.
- If the uncontrollable poles are unstable, then we have to make major changes to the original system such that the system becomes controllable!

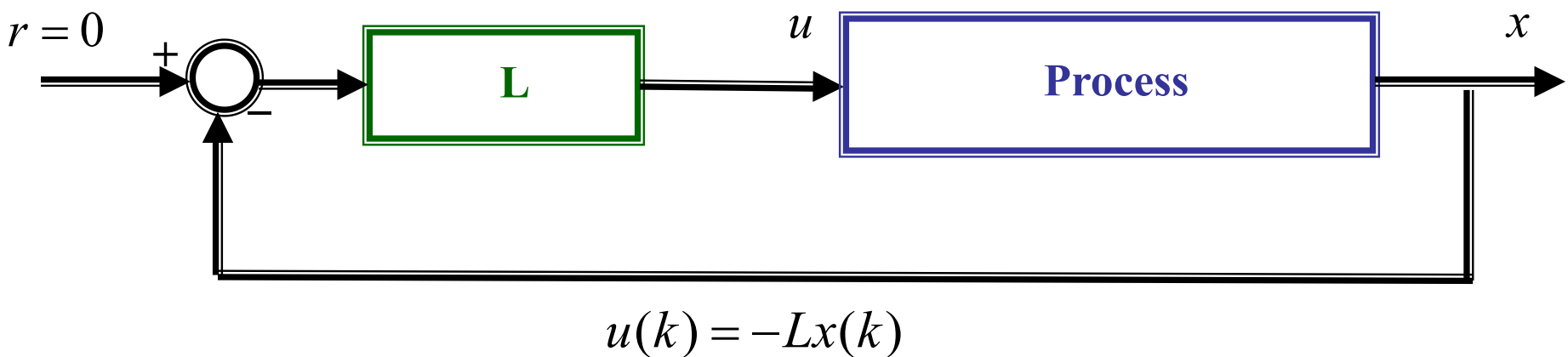
Can you suggest a simple way to change the controllability?

Think out of the box, and choose another more effective way to control!

- But there is one serious limitation of this type of controller in terms of implementation. How many sensors does it need for a n-th order system?

Break

Robotic car control systems



In many practical problems, we have only one sensor to measure the outputs. Only $y(k)$ is available, not all the state variables, $x(k)$. How to solve the control problem?

We either build a new control system, or modify the existing technology!

Which way is simpler?

• If we still want to use the proportional controller, then how to get the missing $x(k)$?

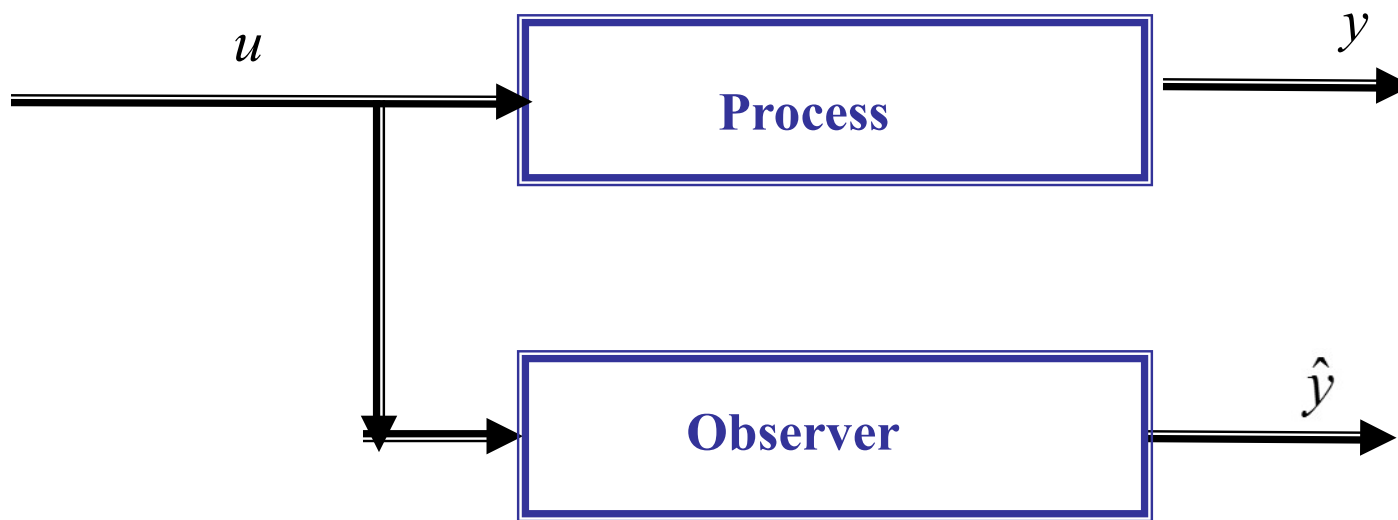
• Is it possible to estimate $x(k)$?

If yes, then we just replace the state vector with its estimate,

$$u(k) = -L\hat{x}(k)$$

• How to estimate $x(k)$ using the inputs and outputs?

• We need to build observer to estimate the state variables.



- Since the state-space model is known:

$$x(k+1) = \Phi x(k) + \Gamma u(k)$$

$$y(k) = cx(k)$$

A natural (naïve) observer:

(an exact copy of the original model) :

$$\hat{x}(k+1) = \Phi \hat{x}(k) + \Gamma u(k)$$

$$\hat{y}(k) = c\hat{x}(k)$$

Would the estimation error go to zero?

$$e(k) = x(k) - \hat{x}(k) \implies$$

$$x(k+1) - \hat{x}(k+1) = \Phi(x(k) - \hat{x}(k))$$

$$e(k+1) = \Phi e(k)$$

The matrix describing the error dynamics is the same as that of the system.

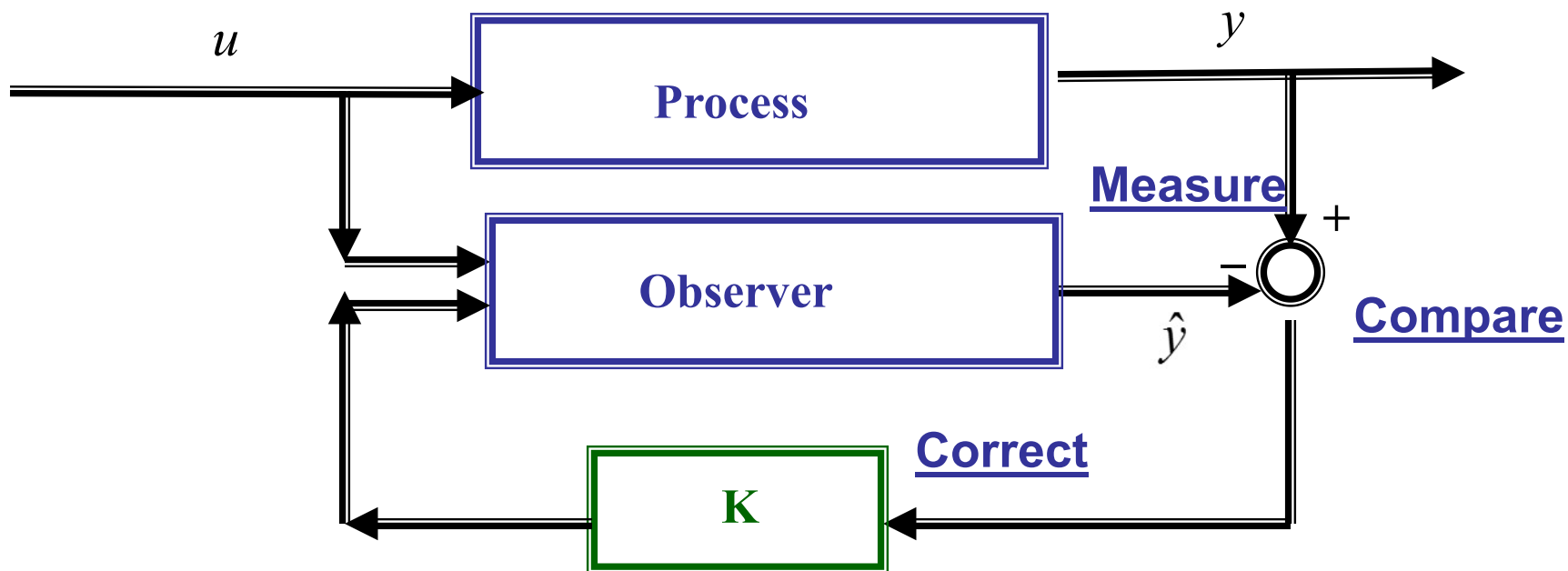
When will the estimation error go to zero?

Only if the system is stable.

• Does this naïve observer depend upon the actual outputs of the system?

• Are there any feedbacks to the naïve observer?

No!



- What is the desired output of the observer? $\hat{y} \rightarrow y$
- Can we use feedback mechanism to improve the observer?
- Feedback: Measure — Compare — Correct

$$\hat{x}(k+1) = \Phi \hat{x}(k) + \Gamma u(k) + K(y(k) - \hat{y}(k))$$

$$\hat{y}(k) = c\hat{x}(k)$$

$$x(k+1) - \hat{x}(k+1) = \Phi(x(k) - \hat{x}(k)) - K(y(k) - \hat{y}(k))$$

$$x(k+1) = \Phi x(k) + \Gamma u(k)$$

$$y(k) = cx(k)$$

$$e(k) = x(k) - \hat{x}(k)$$

$$e(k+1) = \Phi e(k) - Kce(k) = (\Phi - Kc)e(k)$$

The feedback has changed the error dynamics!

$$\Phi \Rightarrow (\Phi - Kc)$$

Can we design K such that $\Phi - Kc$ is stable?

$$e(k+1) = \Phi e(k) - Kce(k) = (\Phi - Kc)e(k)$$

Given pair $\{\Phi, c\}$, can we choose K such that $(\Phi - Kc)$ has any desired C.P. $A_o(z)$?

But we have already solved a very similar problem:

Given pair $\{\Phi, \Gamma\}$, find L , such that $(\Phi - \Gamma L)$ has any desired C.P. $A_m(z)$

Ackemann's formula

$$L = [0 \quad \dots \quad 0 \quad 1] W_c^{-1} A_m(\Phi)$$

It is possible to apply this formula again. But there is one more obstacle to clear!

$$(\Phi - Kc) \quad \Leftrightarrow \quad (\Phi - \Gamma L)$$

Is it possible to make the design parameter K appear on the right side?

$$(\Phi - Kc)^T = \Phi^T - c^T K^T$$

Are the eigenvalues the same for A and A^T ?

Yes.

$$\Phi \Rightarrow \Phi^T$$

$$\Gamma \Rightarrow c^T$$

$$L \Rightarrow K^T$$

$$W_c = \begin{bmatrix} c^T & \Phi^T c^T & \dots & (\Phi^{n-1})^T c^T \end{bmatrix} = \begin{bmatrix} c \\ c\Phi \\ \vdots \\ c\Phi^{n-1} \end{bmatrix}^T = W_o^T$$

$$K^T = [0 \quad \dots \quad 0 \quad 1] (W_o^T)^{-1} A_o(\Phi^T)$$

$$K = A_o(\Phi) (W_o)^{-1} [0 \quad \dots \quad 0 \quad 1]^T$$

The system has to be observable!

• Is it possible to make the estimation perfect in finite steps?

• **From the error dynamics:** $e(k+1) = \Phi e(k) - Kce(k) = (\Phi - Kc)e(k)$
 $e(k+2) = (\Phi - Kc)e(k+1) = (\Phi - Kc)^2 e(k)$
 $e(k+n) = (\Phi - Kc)^n e(k)$

• Can we make $(\Phi - Kc)^n = 0$?

How to choose the desired C.P. , $A_o(z)$ for $(\Phi - Kc)$?

$$A_o(z) = z^n \quad \Longrightarrow \quad (\Phi - Kc)^n = 0$$

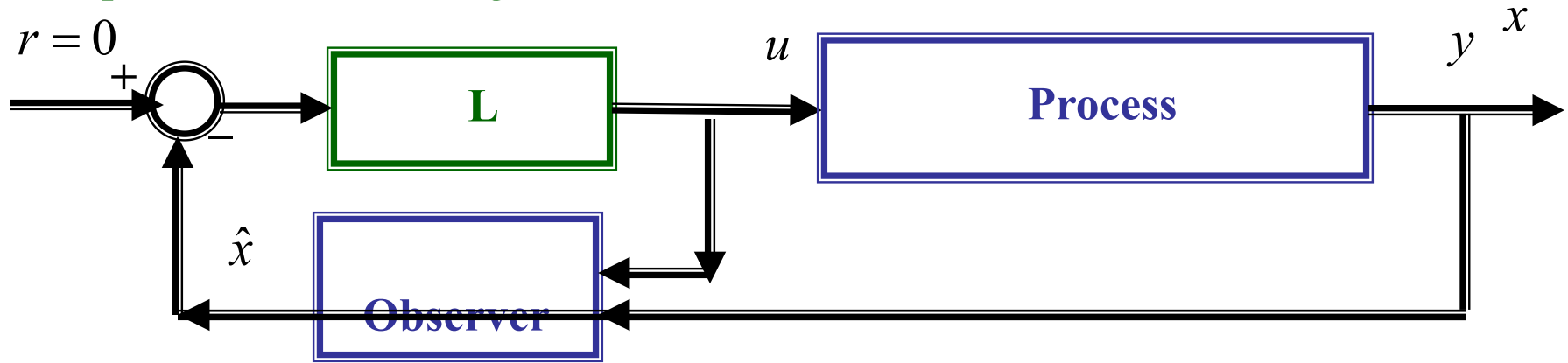
Then the observation error goes to zero at most n steps.

Dead-beat Observer!

But again there is no free lunch. The Dead-beat observer is rarely used in practice either because it is sensitive to noise!

Output-Feedback Controller

- Step 1: Controller Design: $u(k) = -Lx(k)$



Choose L such that the closed loop matrix $(\Phi - \Gamma L)$ has desired C.P. $A_m(z)$

L can be calculated by Ackermann's formula

$$L = [0 \quad \cdots \quad 0 \quad 1] W_c^{-1} A_m(\Phi)$$

L can also be calculated by matching

$$|zI - (\Phi - \Gamma L)| = A_m(z)$$

- Step 2: Design observer to estimate $x(k)$

$$\begin{aligned} \hat{x}(k+1) &= \Phi \hat{x}(k) + \Gamma u(k) + K(y(k) - \hat{y}(k)) \\ \hat{y}(k) &= c \hat{x}(k) \end{aligned} \quad \Longrightarrow \quad e(k+1) = (\Phi - Kc)e(k)$$

Choose K such that the observer matrix $(\Phi - Kc)$ has desired C.P. $A_o(z)$

- K can be calculated by Ackermann's formula

$$K = A_o(\Phi)(W_o)^{-1} [0 \quad \cdots \quad 1]^T$$

Step 3: Output feedback control $u(k) = -L\hat{x}(k)$

So far, we did not consider the effect of the disturbance in our model.

$$x(k+1) = \Phi x(k) + \Gamma u(k)$$

$$y(k) = cx(k)$$

But disturbance may occur at any moment and everywhere!

Load Disturbances:
Most often encountered

Wind gusts on an antenna

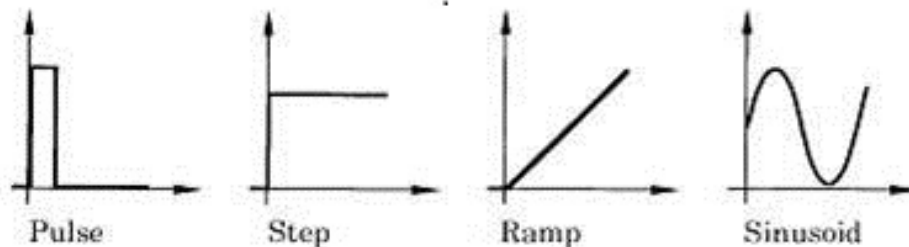
Waves on a ship

Load on a motor

Variations in temperature

Measurement noises, errors

Simple models:



Real world examples:

Ship hit by bomb.

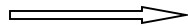
Ship hit by wave.

Disturbance Rejection Problem



$$x(k+1) = \Phi x(k) + \Gamma u(k)$$

$$y(k) = cx(k)$$



$$x(k+1) = \Phi x(k) + \Phi_{x\omega} \omega(k) + \Gamma u(k)$$

$$\omega(k+1) = \Phi_{\omega} \omega(k)$$

$$y(k) = cx(k)$$

Can we just simply ignore the disturbance?

$$u(k) = -Lx(k)$$

$$x(k+1) = \Phi x(k) + \Phi_{x\omega} \omega(k) + \Gamma u(k) = (\Phi - \Gamma L)x(k) + \Phi_{x\omega} \omega(k)$$

• Would the state $x(k)$ decrease to zero when the disturbance is **present**?

• The disturbance cannot be ignored.

$$x(k+1) = \Phi x(k) + \Phi_{x\omega} \omega(k) + \Gamma u(k)$$

$$\omega(k+1) = \Phi_{\omega} \omega(k)$$

$$y(k) = cx(k)$$

What's the state of the system?

If we know $x(k)$ and $u(k)$, can we predict $x(k+1)$?

No.

What do we need to predict the future value of x , $x(k+1)$?

$$x(k), \omega(k), u(k)$$

Should we include the disturbance term, $\omega(k)$

as a part of the complete state, if we do not consider it as input?

Yes!

Augmented state vector,

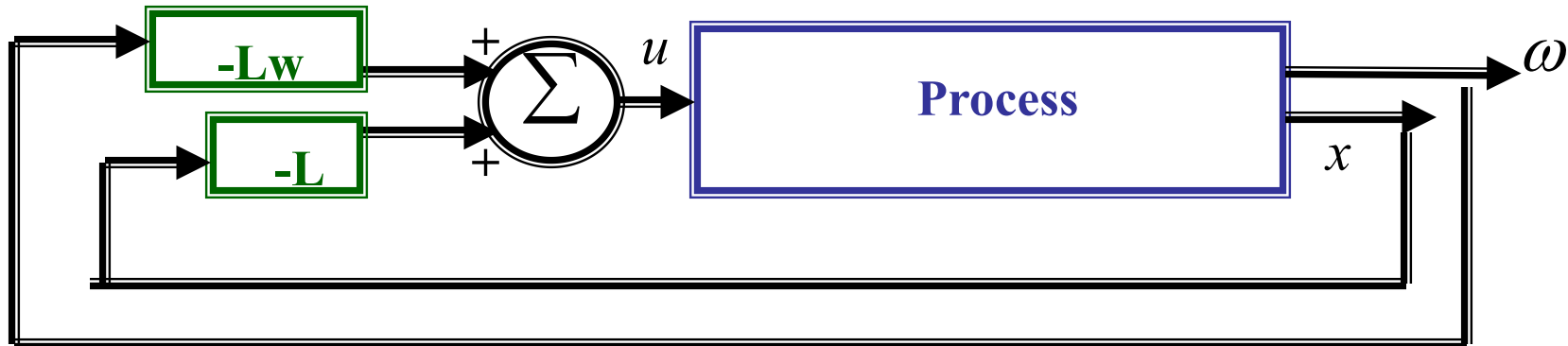
$$z(k) = \begin{bmatrix} x(k) \\ \omega(k) \end{bmatrix}$$

$$z(k+1) = \begin{bmatrix} x(k+1) \\ \omega(k+1) \end{bmatrix} = \begin{bmatrix} \Phi & \Phi_{x\omega} \\ 0 & \Phi_{\omega} \end{bmatrix} z(k) + \begin{bmatrix} \Gamma \\ 0 \end{bmatrix} u(k)$$

$$y(k) = \begin{bmatrix} c & 0 \end{bmatrix} z(k)$$

Is it in the standard form of state space model now? Yes!

- Step 1: Assume the state vector $z(k)$ is available, design state-feedback controller



$$u(k) = -L_c z(k) = -Lx(k) - L_\omega \omega(k)$$

We assume that the disturbance is accessible, is this assumption realistic?

No.

When you try to solve a complicated problem, you always start with the simplest case, even if it is unreal! This is the most important trick in doing good research work!

The closed loop system becomes

$$\begin{bmatrix} x(k+1) \\ \omega(k+1) \end{bmatrix} = \begin{bmatrix} \Phi & \Phi_{x\omega} \\ 0 & \Phi_\omega \end{bmatrix} \begin{bmatrix} x(k) \\ \omega(k) \end{bmatrix} - \begin{bmatrix} \Gamma \\ 0 \end{bmatrix} \begin{bmatrix} L & L_\omega \end{bmatrix} \begin{bmatrix} x(k) \\ \omega(k) \end{bmatrix} = \begin{bmatrix} \Phi - \Gamma L & \Phi_{x\omega} - \Gamma L_\omega \\ 0 & \Phi_\omega \end{bmatrix} \begin{bmatrix} x(k) \\ \omega(k) \end{bmatrix}$$

$$y(k) = \begin{bmatrix} c & 0 \end{bmatrix} z(k)$$

How to choose L?

Choose L such that $\Phi - \Gamma L$ has desired poles of $\text{Am}(z)$ as before.

$$x(k+1) = (\Phi - \Gamma L)x(k) + (\Phi_{x\omega} - \Gamma L_{\omega})\omega(k)$$

$$\omega(k+1) = \Phi_{\omega}\omega(k)$$

$$y(k) = cx(k)$$

•But how to choose L_{ω} ?

Choose L_{ω} to reduce the effect of the disturbance.

Ideal Case: Choose L_{ω} such that $\Phi_{x\omega} - \Gamma L_{\omega} = 0$

the disturbance can be completely cancelled out.

What if it is impossible to make $\Phi_{x\omega} - \Gamma L_{\omega} = 0$?

We need to analyze the transfer function from the disturbance to the output

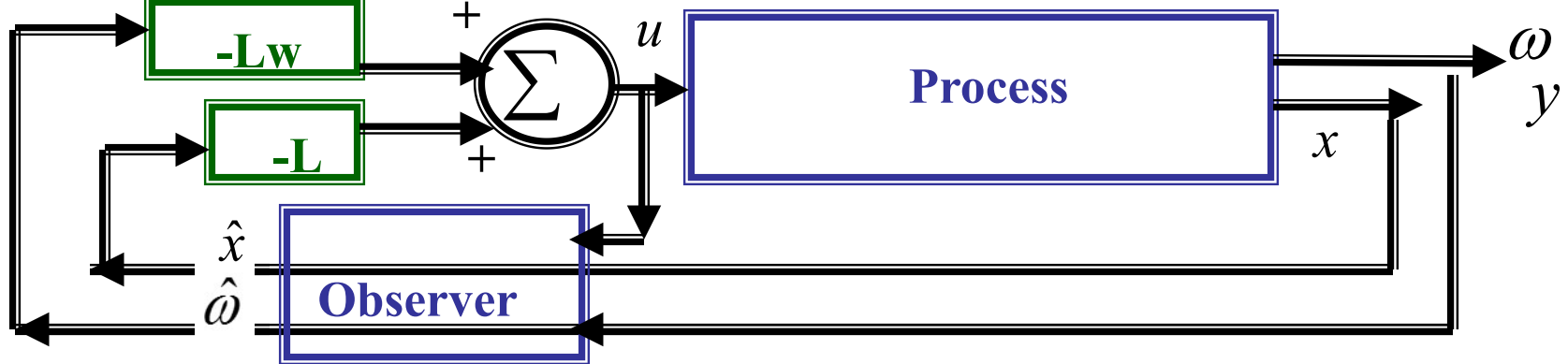
$$zX(z) = (\Phi - \Gamma L)X(z) + (\Phi_{x\omega} - \Gamma L_{\omega})\omega(z) \quad X(z) = (zI - (\Phi - \Gamma L))^{-1}(\Phi_{x\omega} - \Gamma L_{\omega})\omega(z)$$

$$Y(z) = cX(z) = c(zI - (\Phi - \Gamma L))^{-1}(\Phi_{x\omega} - \Gamma L_{\omega})\omega(z) = H_{\omega}(z)\omega(z)$$

$$\omega(z) \longrightarrow \boxed{H_{\omega}(z)} \longrightarrow Y(z)$$

If the disturbance is constant, is it possible to make its output ZERO?

$$H_{\omega}(1) = 0$$



Now we can make the realistic assumption: both x and w are unknown!

What should we do? Have we already solved the case when x is not accessible?

Step 2: Design the observer to estimate both the state x and disturbance as disturbance can be treated as part of the state!

$$z(k+1) = \begin{bmatrix} x(k+1) \\ \omega(k+1) \end{bmatrix} = \begin{bmatrix} \Phi & \Phi_{x\omega} \\ 0 & \Phi_{\omega} \end{bmatrix} z(k) + \begin{bmatrix} \Gamma \\ 0 \end{bmatrix} u(k)$$

$$y(k) = [c \quad 0] z(k)$$

$$\hat{z}(k+1) = \begin{bmatrix} \Phi & \Phi_{x\omega} \\ 0 & \Phi_{\omega} \end{bmatrix} \hat{z}(k) + \begin{bmatrix} \Gamma \\ 0 \end{bmatrix} u(k) + K(y(k) - \hat{y}(k))$$

$$\hat{y}(k) = [c \quad 0] \hat{z}(k)$$

K can be chosen such that the C.P. of $\begin{bmatrix} \Phi & \Phi_{x\omega} \\ 0 & \Phi_{\omega} \end{bmatrix} - K[c \quad 0]$ matches any desired one.

Step 3: $u(k) = -L\hat{x}(k) - L_{\omega}\hat{\omega}(k)$

•Example: Double integrator

$$x(k+1) = \begin{bmatrix} 1 & h \\ 0 & 1 \end{bmatrix} x(k) + \begin{bmatrix} \frac{h^2}{2} \\ h \end{bmatrix} (u(k) + \omega(k))$$

$$\omega(k+1) = \omega(k)$$

$$y(k) = \begin{bmatrix} 1 & 0 \end{bmatrix} x(k)$$

•Step 1: $u(k) = -Lx(k) - L_w \omega(k)$

•How to choose L? Choose L to place the poles.

•How to choose L_w?

$$L_w = 1$$

$$u(k) = -Lx(k) - \omega(k)$$

•Step 2: Observer $z(k) = \begin{bmatrix} x(k) \\ \omega(k) \end{bmatrix}$

$$\begin{bmatrix} x(k+1) \\ \omega(k+1) \end{bmatrix} = \begin{bmatrix} 1 & h & \frac{h^2}{2} \\ 0 & 1 & h \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x(k) \\ \omega(k) \end{bmatrix} + \begin{bmatrix} \frac{h^2}{2} \\ h \\ 0 \end{bmatrix} u(k)$$

$$y(k) = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x(k) \\ \omega(k) \end{bmatrix}$$

$$\begin{bmatrix} \hat{x}(k+1) \\ \hat{\omega}(k+1) \end{bmatrix} = \begin{bmatrix} 1 & h & \frac{h^2}{2} \\ 0 & 1 & h \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \hat{x}(k) \\ \hat{\omega}(k) \end{bmatrix} + \begin{bmatrix} \frac{h^2}{2} \\ h \\ 0 \end{bmatrix} u(k) + K(y(k) - \hat{y}(k))$$

$$\hat{y}(k) = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \hat{x}(k) \\ \hat{\omega}(k) \end{bmatrix}$$

choose K to place the poles for observer error dynamics:

$$e(k+1) = (\Phi - Kc)e(k)$$

•Step 3: $u(k) = -L\hat{x}(k) - \hat{\omega}(k)$

Limitation of the pure state feedback control system

Open Loop System
(original system)

$$x(k+1) = \Phi x(k) + \Gamma u(k)$$
$$y(k) = c x(k)$$

Pure state feedback controller:

$$u(k) = -Lx(k)$$

Closed Loop System:

$$x(k+1) = \Phi x(k) + \Gamma u(k) = (\Phi - \Gamma L)x(k)$$
$$y(k) = c x(k)$$

What is the steady state output when the closed loop is stable?

$$x(k) \rightarrow 0, \Rightarrow y(k) \rightarrow 0$$

The output will go to zero if only feedback control is used!

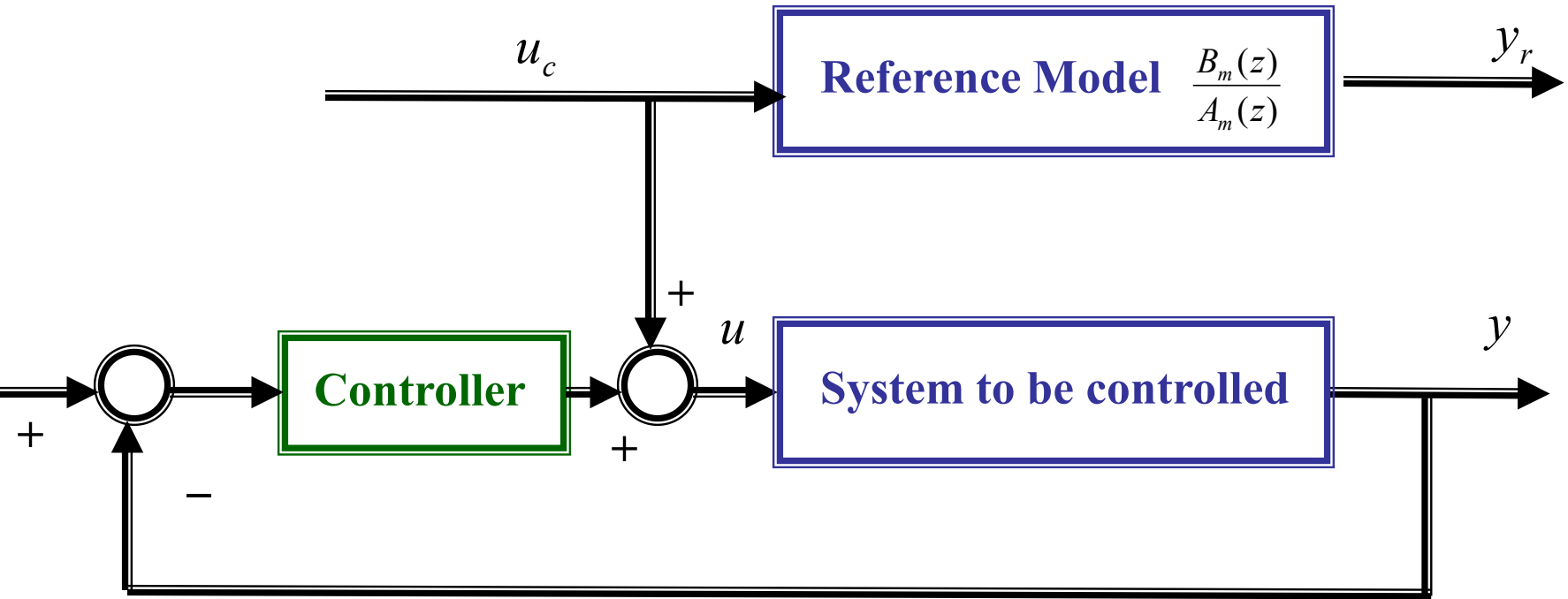
The purpose of feedback is to change the internal property of the system, like the poles.

If the objective is to make the output follow command signal rather than zero, we need to add feedforward control signal to the system:

$$u(k) = -Lx(k) + u_{ff}(k)$$

Pole Placement (Stabilization)

Tracking (following the reference model)



Once the poles are matched, we need to consider how to match the zeros!

In the output feedback control $u(k) = -L\hat{x}(k)$

The controller is purely feedback without any information from the command signal, u_c

Now it is time to add the influence of u_c

such that we can make the T.F. from u_c to the output y match the reference model

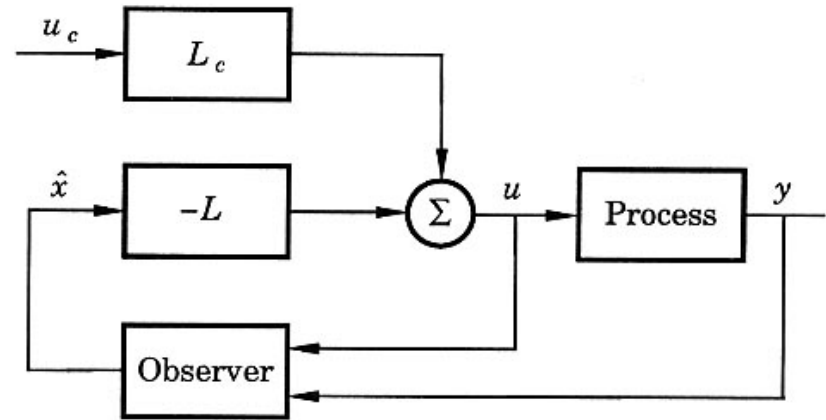
$$u(k) = -L\hat{x}(k) + u_{ff}(k)$$

Feedback+Feedforward

A Naive (simple) Approach: (Proportional Control)

$$u_{ff}(k) = L_c u_c(k)$$

$$u(k) = -L\hat{x}(k) + u_{ff}(k) = -L\hat{x}(k) + L_c u_c(k)$$



The closed-loop system:

$$x(k+1) = \Phi x(k) + \Gamma u(k)$$

$$y(k) = cx(k)$$

$$\hat{x}(k+1) = \Phi \hat{x}(k) + \Gamma u(k) + K(y(k) - \hat{y}(k))$$

$$\hat{y}(k) = c\hat{x}(k)$$

$$u(k) = -L\hat{x}(k) + L_c u_c(k)$$

$$e(k) = x(k) - \hat{x}(k)$$

$$x(k+1) = (\Phi - \Gamma L)x(k) + \Gamma L e(k) + \Gamma L_c u_c(k)$$

$$e(k+1) = (\Phi - Kc)e(k)$$

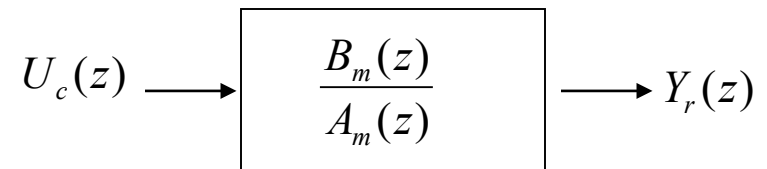
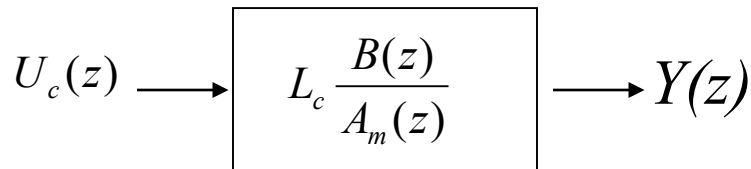
$$y(k) = cx(k)$$

Let's derive the transfer function from u_c to y

$$zX(z) = (\Phi - \Gamma L)X(z) + \Gamma L_c U_c(z) \implies X(z) = (zI - (\Phi - \Gamma L))^{-1} \Gamma L_c U_c(z)$$

$$Y(z) = cX(z) = c(zI - (\Phi - \Gamma L))^{-1} \Gamma L_c U_c(z) \implies H_{cl}(z) = c(zI - \Phi + \Gamma L)^{-1} \Gamma L_c = L_c \frac{B(z)}{A_m(z)}$$

The pure state feedback control does not change the zeros!



Now let's try to choose L_c such that

$$L_c \frac{B(z)}{A_m(z)} = \frac{B_m(z)}{A_m(z)} \quad \Longrightarrow \quad L_c = \frac{B_m(z)}{B(z)}$$

But L_c is assumed to be only a proportional gain in the controller!

What should we do?

We have to increase the freedom of the controller.

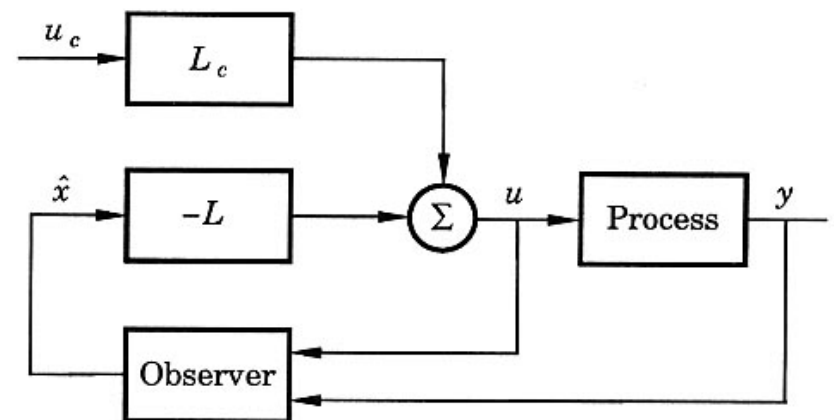
Where to find the extra freedom?

Can we change the feedback controller?

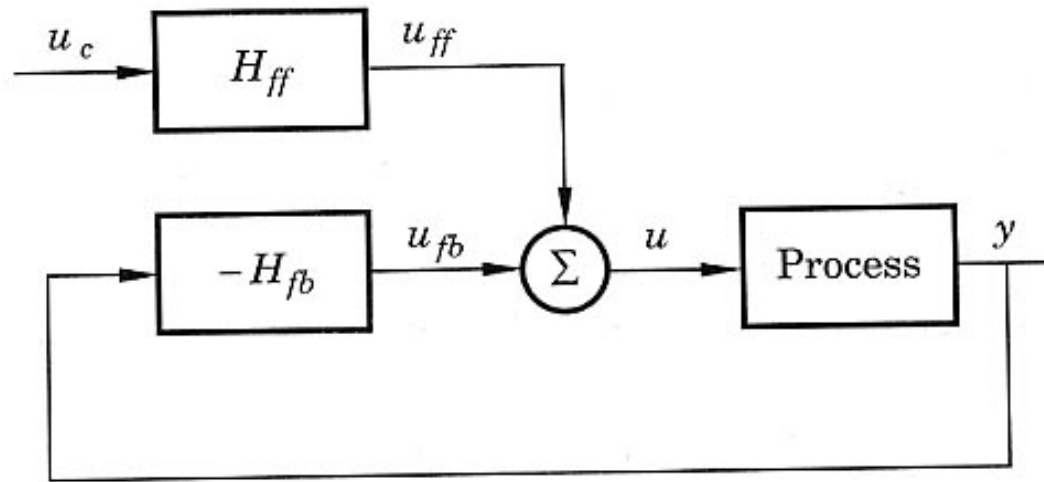
No, we do not want to change the poles any more!

- Therefore, we have to design L_c with a dynamic system rather than a constant gain.

- Just let $L_c = \frac{B_m(z)}{B(z)}$



Two-Degree-of-Freedom Controller



$$u = u_{fb} + u_{ff}$$

Pole placement

Zero placement

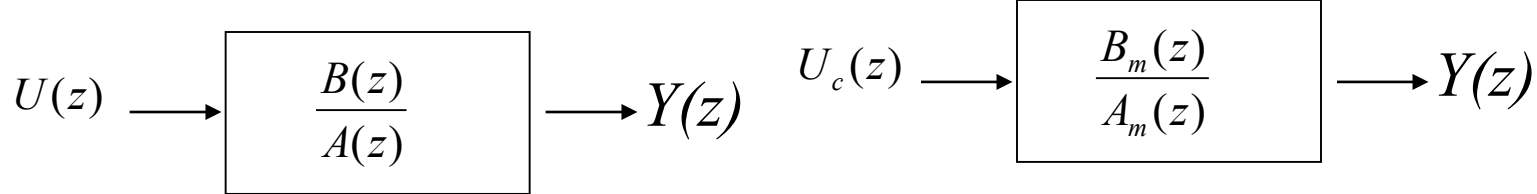
$$H_{ff} = L_c = \frac{B_m(z)}{B(z)}$$

$$\frac{Y(z)}{U_c(z)} = H_{cl}(z) = L_c \frac{B(z)}{A_m(z)} = \frac{B_m(z)}{B(z)} \frac{B(z)}{A_m(z)} = \frac{B_m(z)}{A_m(z)} = H_m(z)$$

Why should we cancel out B(z) in this case?

Both B(z) are operating on u_{ff} !

So finally we can match the reference model perfectly!



But can we solve the tracking problem completely for any linear system?

What are the conditions?

Controllable? Yes.

Observable? Yes.

Is the open loop (original system) required to be stable? poles of $A(z)$ No.

Let's check whether all the signals are bounded.

Is the output y bounded?

$$\frac{Y}{U_c} = \frac{B_m}{A_m} \quad \text{Yes.}$$

Is the input u bounded?

We need to get T.F. from command signal to u .

$$\frac{Y}{U_c} = \frac{B_m}{A_m}$$

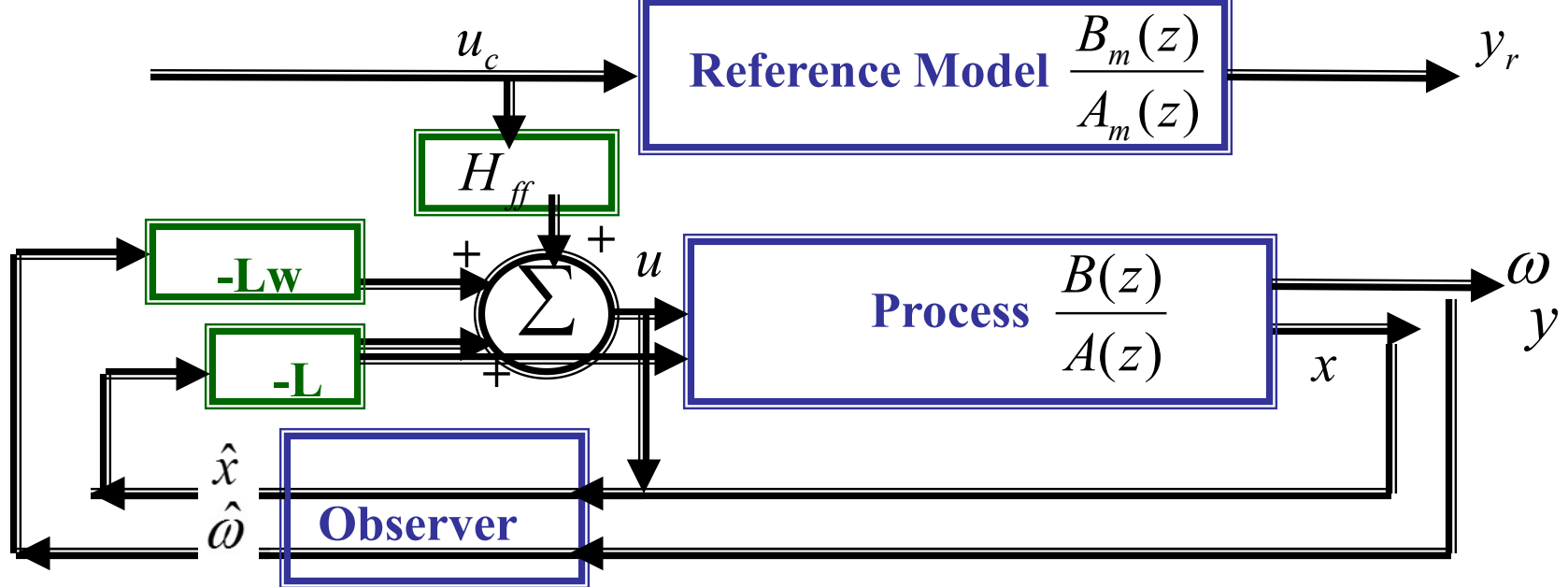
$$\frac{Y}{U} = \frac{B}{A}$$

$$\frac{U}{U_c} = \frac{B_m A}{A_m B}$$

What is the condition to assure the input is also bounded?

The system is required to have a stable inverse $B(z)$ for perfect tracking!

Is this condition required for pole placement (stabilization) only?



How to choose L ?

Pole Placement for the controller: $|zI - (\Phi - \Gamma L)| = A_m(z)$

How to choose Lw ?

Problem Dependent

How to design the observer?

Pole Placement for observer: $|zI - (\Phi - Kc)| = A_o(z)$

How to choose the Feedforward T.F.?

$$H_{ff} = \frac{B_m(z)}{B(z)}$$

Zero Placement

B(z) has to be stable for perfect tracking!

Q & A...

THANK YOU !

$$H(z) = \frac{b_1 z + b_2}{z^2 + a_1 z + a_2}$$

$G(s)$	$H(q)$ or the coefficients in $H(q)$	
$\frac{1}{s}$	$\frac{h}{q-1}$	
$\frac{1}{s^2}$	$\frac{h^2(q+1)}{2(q-1)^2}$	
$\frac{1}{s^m}$	$\frac{q-1}{q} \lim_{a \rightarrow 0} \frac{(-1)^m}{m!} \frac{\partial^m}{\partial a^m} \left(\frac{q}{q - e^{-ah}} \right)$	
e^{-sh}	q^{-1}	
$\frac{a}{s+a}$	$\frac{1 - \exp(-ah)}{q - \exp(-ah)}$	
$\frac{a}{s(s+a)}$	$b_1 = \frac{1}{a}(ah - 1 + e^{-ah})$ $a_1 = -(1 + e^{-ah})$	$b_2 = \frac{1}{a}(1 - e^{-ah} - ahe^{-ah})$ $a_2 = e^{-ah}$
$\frac{a^2}{(s+a)^2}$	$b_1 = 1 - e^{-ah}(1 + ah)$ $a_1 = -2e^{-ah}$	$b_2 = e^{-ah}(e^{-ah} + ah - 1)$ $a_2 = e^{-2ah}$
$\frac{s}{(s+a)^2}$	$\frac{(q-1)he^{-ah}}{(q - e^{-ah})^2}$	
$\frac{ab}{(s+a)(s+b)}$ $a \neq b$	$b_1 = \frac{b(1 - e^{-ah}) - a(1 - e^{-bh})}{b-a}$ $b_2 = \frac{a(1 - e^{-bh})e^{-ah} - b(1 - e^{-ah})e^{-bh}}{b-a}$ $a_1 = -(e^{-ah} + e^{-bh})$ $a_2 = e^{-(a+b)h}$	

$$H(z) = \frac{b_1 z + b_2}{z^2 + a_1 z + a_2}$$

[return](#)

$G(s)$	$H(q)$ or the coefficients in $H(q)$	
$\frac{(s+c)}{(s+a)(s+b)}$ $a \neq b$	$b_1 = \frac{e^{-bh} - e^{-ah} + (1 - e^{-bh})c/b - (1 - e^{-ah})c/a}{a - b}$	
	$b_2 = \frac{c}{ab} e^{-(a+b)h} + \frac{b-c}{b(a-b)} e^{-ah} + \frac{c-a}{a(a-b)} e^{-bh}$	
	$a_1 = -e^{-ah} - e^{-bh}$	$a_2 = e^{-(a+b)h}$
$\frac{\omega_0^2}{s^2 + 2\zeta\omega_0 s + \omega_0^2}$	$b_1 = 1 - \alpha \left(\beta + \frac{\zeta\omega_0}{\omega} \gamma \right)$	$\omega = \omega_0 \sqrt{1 - \zeta^2} \quad \zeta < 1$
	$b_2 = \alpha^2 + \alpha \left(\frac{\zeta\omega_0}{\omega} \gamma - \beta \right)$	$\alpha = e^{-\zeta\omega_0 h}$
	$a_1 = -2\alpha\beta$	$\beta = \cos(\omega h)$
	$a_2 = \alpha^2$	$\gamma = \sin(\omega h)$
$\frac{s}{s^2 + 2\zeta\omega_0 s + \omega_0^2}$	$b_1 = \frac{1}{\omega} e^{-\zeta\omega_0 h} \sin(\omega h)$	$b_2 = -b_1$
	$a_1 = -2e^{-\zeta\omega_0 h} \cos(\omega h)$	$a_2 = e^{-2\zeta\omega_0 h}$
	$\omega = \omega_0 \sqrt{1 - \zeta^2}$	
$\frac{a^2}{s^2 + a^2}$	$b_1 = 1 - \cos ah$	$b_2 = 1 - \cos ah$
	$a_1 = -2 \cos ah$	$a_2 = 1$
$\frac{s}{s^2 + a^2}$	$b_1 = \frac{1}{a} \sin ah$	$b_2 = -\frac{1}{a} \sin ah$
	$a_1 = -2 \cos ah$	$a_2 = 1$
$\frac{a}{s^2(s+a)}$	$b_1 = \frac{1-\alpha}{a^2} + h \left(\frac{h}{2} - \frac{1}{a} \right)$	$\alpha = e^{-ah}$
	$b_2 = (1-\alpha) \left(\frac{h^2}{2} - \frac{2}{a^2} \right) + \frac{h}{a} (1+\alpha)$	
	$b_3 = - \left[\frac{1}{a^2} (\alpha - 1) + \alpha h \left(\frac{h}{2} + \frac{1}{a} \right) \right]$	
	$a_1 = -(\alpha + 2)$	$a_2 = 2\alpha + 1 \quad a_3 = -\alpha$