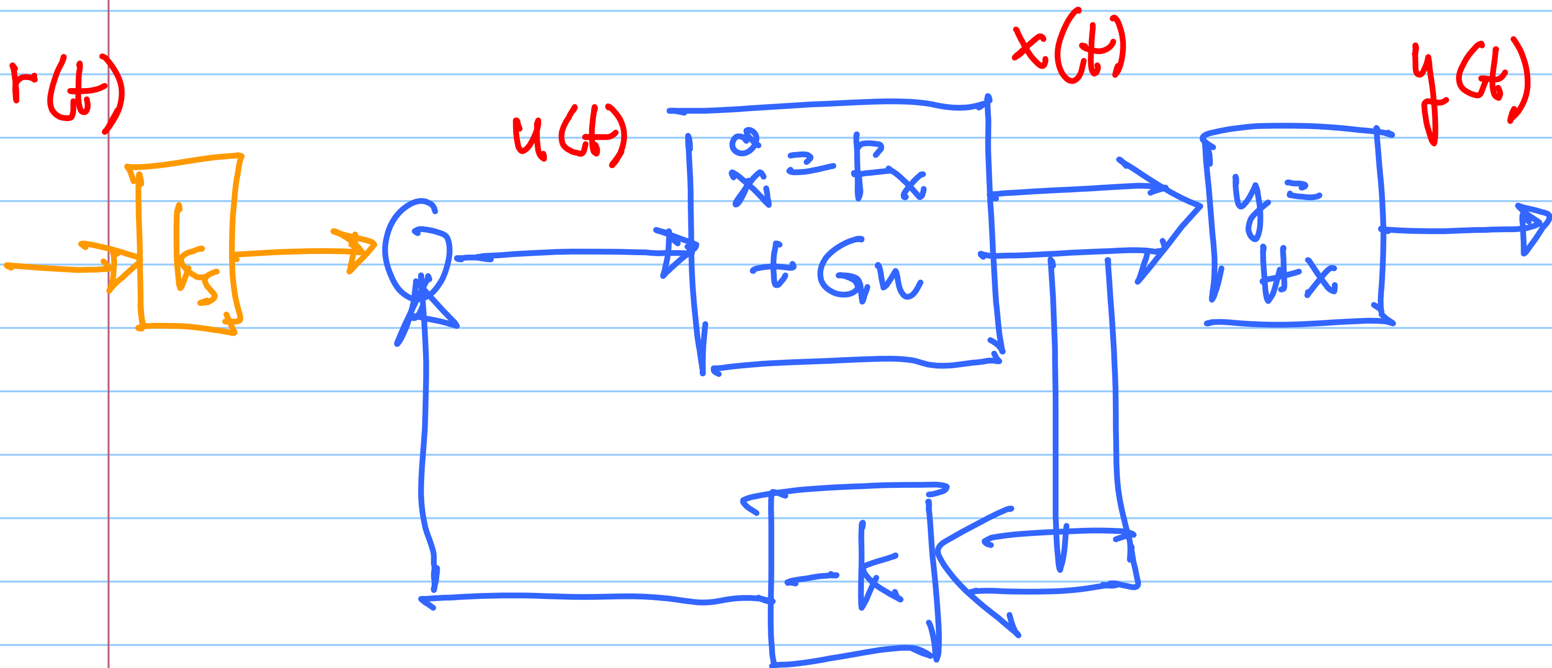
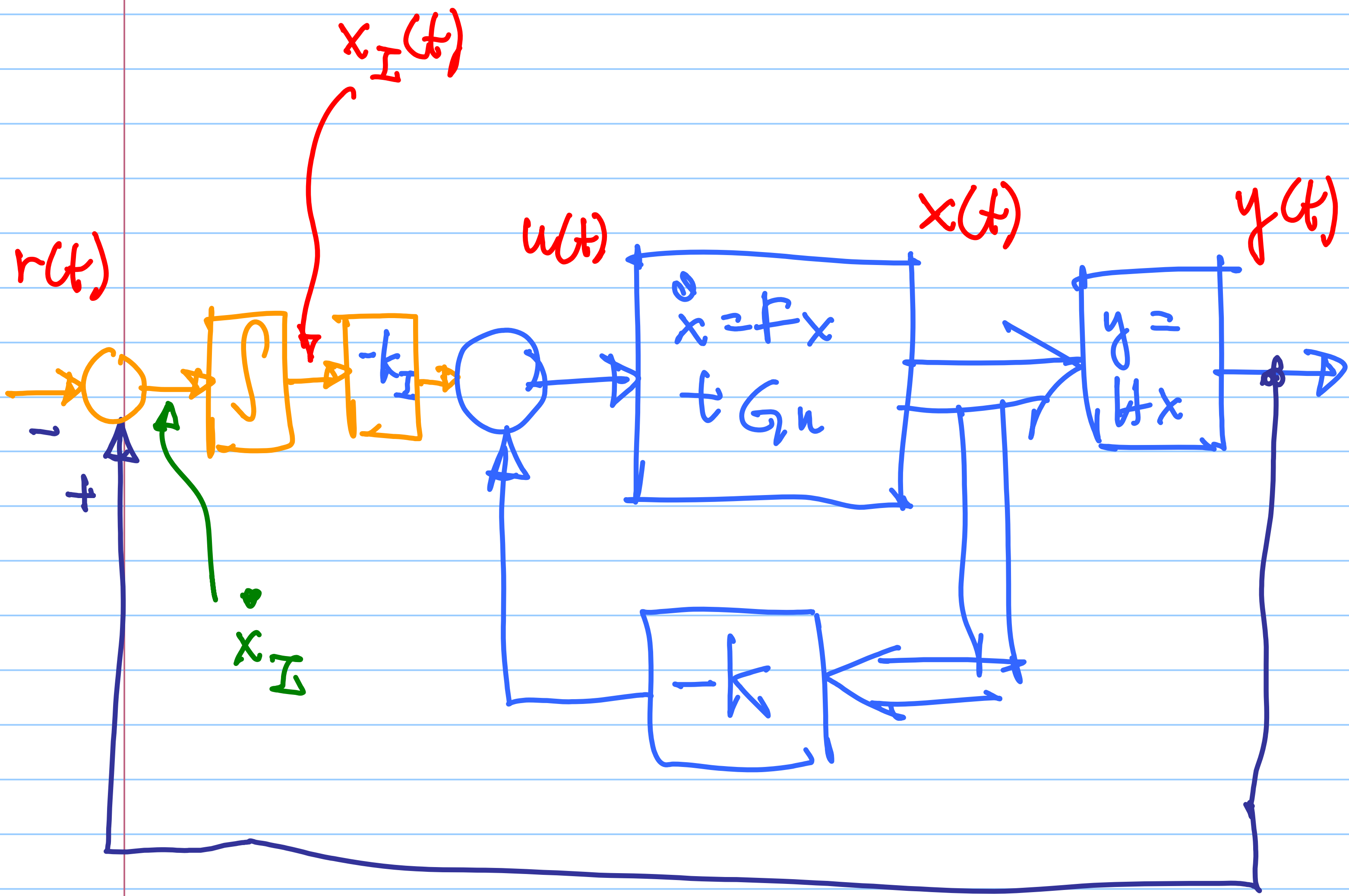
6a

Pre-Class

Our capability up to this point:





$x_I(t)$
$r(t)$
$u(t)$
$x(t)$
$y(t)$
$-$
$+$
$\int$
$-k_I$
$\dot{x} = Fx + Gu$
$y = Hx$
$-k$
$\dot{x}_I$

Next :

What happens if $x(t)$ is not fully measurable?

Going to the Lecture Notes ....

The approach is to estimate/observe the unmeasurable $x(t)$ with the "estimated/observed" signal $\hat{x}(t)$

Simple approach is to consider firstly : "Open-Loop Estimator/Observer"

... in the notes, shown clearly that this simple approach does not work well.

A much more proper approach is to use the

"Closed-Loop Estimator/Observer"

$$\dot{x} = Fx + Gu$$

$$y = Hx \quad \text{——} \quad (5.1)$$

Not all the state-variables can be measured.

So, we set up the "Closed-Loop Estimator/Observer" given by:

$$\dot{\hat{x}} = F\hat{x} + Gu + L(y - H\hat{x})$$

How does this work?

Thus, consider the "estimation error"

$$\tilde{x}(t) \triangleq x(t) - \hat{x}(t)$$

For this, we have:

$$\dot{\tilde{x}} = \dot{x} - \dot{\hat{x}}$$

$$= \left\{ Fx + Gu \right\} - \left\{ F\hat{x} + Gu + L\left(\overset{Hx}{y} - H\hat{x}\right) \right\}$$

$$\vdots$$

i.e. $$\dot{\tilde{x}} = \left\{ F - LH \right\} \tilde{x} \qquad (5.5)$$

Here, we need to choose L

so that the eigenvalues of

$$\{F - LH\}$$

are sufficiently fast and

stable.

This is equivalent to requiring

the eigenvalues of

$$\{F - LH\}^T = \{F^T - H^T L^T\}$$

to be sufficiently fast and

stable.

Compare with

$$\{F - GK\} \quad (n \times n) \text{ matrix}$$

where we chose an $\alpha_c(s)$!!

Typical "Real" System

$$\dot{x} = Fx + Gu$$

$$y = Hx$$

Example could be the d.c. motor in CA3

(I) Run some tests/calibrations to obtain F, G and H

(II) If all state variables are measurable, then we can construct a controller

$$u = -Kx + k_s r$$

(a) Choose desired

$$\alpha_c(s) = s^n + \alpha_1 s^{n-1} + \dots + \alpha_n$$

using

- Prototype Response Tables
- SRL
- LQR

(b) Calculate K using Ackermann's formula.

also can use

$$\dot{x}_I = y - r$$

$$u = -Kx - k_I x_I$$

$$= -\begin{bmatrix} K & k_I \end{bmatrix} \begin{bmatrix} x \\ x_I \end{bmatrix}$$

III If state-variables are not all measurable, then have to construct an Estimator/Observer first.

i.e. construct

$$\dot{\hat{x}} = F\hat{x} + Gu + L(y - H\hat{x})$$

Here, $\{F, G, H\}$ already obtained by calibration experiments. And Equation above will be implemented in real-time, e.g. using LabView software.

(a) For observer/estimator gain L, first choose desired

$$\bar{\alpha}_e(s) = s^n + \bar{\alpha}_1 s^{n-1} + \dots + \bar{\alpha}_n$$

using
- Prototype Response Tables
- SRL
- LQR

typically remembering to choose

bandwidth of $\bar{\alpha}_e(s)$ to be about 3 to 5 times faster than $\alpha_c(s)$.

(2) Calculate L using Ackermann's formula.

$y = Hx$

i.e. we have

$$\dot{\hat{x}} = F\hat{x} + Gu + L(y - H\hat{x})$$

$$u = -K\hat{x} + k_s r$$

Summary

$$\dot{x} = Fx + Gu$$
$$y = Hx$$

I State-feedback

$$u = -Kx$$

leading to closed-loop

$$\dot{x} = (F - GK)x$$
$$y = Hx$$

Closed-loop poles are freely assignable iff

$\mathcal{C}(F, G)$ is of full rank.

Recall:

$$\mathcal{C}(F, G) \triangleq \left[G \,\vdots\, FG \,\vdots\, F^2 G \,\vdots\, \cdots \,\vdots\, F^{n-1} G \right]$$

II Estimator

$$\dot{\hat{x}} = F\hat{x} + Gu + L(y - H\hat{x})$$

leading to estimator closed-loop

$$\tilde{x} \triangleq \hat{x} - x$$

$$\dot{\tilde{x}} = (F - LH)\tilde{x}$$

Estimator closed-loop poles are freely assignable iff

$\mathcal{O}(H, F)$ is of full rank.

Recall:

$$\mathcal{O}(H,F) \triangleq \begin{bmatrix} H \\ \hdashline HF \\ \hdashline HF^2 \\ \hdashline \vdots \\ \hdashline HF^{n-1} \end{bmatrix}$$

Summary

$$\boxed{u = -Kx + k_s r}$$

if all s-v. are measurable.

$$\dot{x} = Fx + Gu$$
$$y = Hx$$

Complete estimator-controller:

$$\dot{\hat{x}} = (F - GK - LH)\hat{x} + Ly$$
$$u = -K\hat{x} + k_s r$$

- State-fb gain K chosen so that

$$\alpha_c(s) = \det\left[sI - (F - GK)\right]$$

are the closed-loop state-fb poles.

- Estimator gain L chosen so that

$$\alpha_e(s) = \det\left[sI - (F - LH)\right]$$

are the closed-loop estimator poles.

$\longrightarrow$ Reduced-Order Observer...

Combined State-Feedback and State Estimator/Observer

$$\dot{x} = Fx + Gu$$
$$y = Hx \qquad \longleftarrow \quad (7.1)$$

Not all s.v. are measurable.

thus, use:

$$\dot{\hat{x}} = F\hat{x} + Gu + L(y - H\hat{x}) \qquad \text{—— } (7.2a)$$

$$u = -K\hat{x} \qquad \text{—— } (7.2b)$$

Note that in above:

(I) K is chosen so that

$$\det\left\{ sI - [F - GK] \right\} = \alpha_c(s) \qquad \text{—— } (7.5a)$$

(II) L is chosen so that

$$\det\left\{sI - [F - LH]\right\} = \alpha_0(s) \quad \text{--- (7.5b)}$$

In the above, the <u>complete</u> system is described by:

$$\dot{x} = Fx + Gu \quad \text{--- (7.6a)}$$

$$= Fx + G\{-K\hat{x}\}$$

$$\dot{\hat{x}} = F\hat{x} + Gu + L(y - H\hat{x})$$

$$= F\hat{x} - GK\hat{x} + L(Hx - H\hat{x})$$

$$\text{--- (7.6b)}$$

and this is a complete description of the system with the full equivalent state-variable description from (7.6a) and (7.6b) with :

$$\begin{bmatrix} \dot{x} \\ \dot{\hat{x}} \end{bmatrix} = \left[\begin{array}{c|c} F & -GK \\ \hline LH & F-GK \\ & -LH \end{array}\right] \begin{bmatrix} x \\ \hat{x} \end{bmatrix} \quad \text{—— (7.7)}$$

Thus, the above is <u>one</u> possible complete description of the system.

Consider next, the use of

$$\tilde{x} = x - \hat{x}$$

i.e. $\hat{x} = x - \tilde{x}$

then (7.6a) becomes:

$$\dot{x} = Fx - GK(x - \hat{x}) \quad \text{--- (7.8a)}$$

and (7.6b) becomes:

$$\dot{\hat{x}} = \{F - GK\}\hat{x} + LH\{x - \hat{x}\}$$

$$= \{F - GK\}(x - \tilde{x}) + LH\tilde{x} \quad \text{--- (7.8b)}$$

And also note from earlier:

$$\dot{\tilde{x}} = \{F - LH\}\tilde{x} \quad \text{--- (7.8c)}$$

Expand (7-8a) to:

$$\dot{x} = (F - GK)x - GK\tilde{x} \quad \text{(7-9a)}$$

$$\dot{\tilde{x}} = (F - LH)\tilde{x} \quad \text{(7-9b)}$$

The equivalent complete overall system is thus:

$$\begin{bmatrix} \dot{x} \\ \dot{\tilde{x}} \end{bmatrix} = \underbrace{\left[\begin{array}{c|c} F-GK & -GK \\ \hline 0_{n\times n} & F-LH \end{array}\right]}_{F_{composite}} \begin{bmatrix} x \\ \tilde{x} \end{bmatrix}$$

The eigenvalues of this complete $(2n \times 2n)$ system is given by:

$$\det\left\{ sI_{2n\times 2n} - F_{composite} \right\} = 0$$

$$\det\left\{ \left[\begin{array}{c|c} sI_{n\times n} - \{F - GK\} & GK \\ \hline 0_{n\times n} & sI_{n\times n} - \{F - LH\} \end{array}\right] \right\} = 0$$

This simplifies to:

$$\det\left\{ sI_{n\times n} - \left[F - GK \right] \right\} \cdot \det\left\{ sI_{n\times n} - \left[F - LH \right] \right\} = 0$$

i.e. $\alpha_c(s) \cdot \alpha_o(s) = 0$

This is the so-called famous "Kalman" Separation Principle!!

$$\begin{bmatrix} x \\ \tilde{x} \end{bmatrix} = \underbrace{\left[\begin{array}{c|c} \quad & \quad \\ \hline \quad & \quad \end{array}\right]}_{\text{Transformation}} \begin{bmatrix} x \\ \hat{x} \end{bmatrix}$$