EE5103 Computer Control Systems: Homework #1 Solution

Q1 Solution

a) Applying Laplace transform we can get:

$$L(sI(s)-i(0))+RI(s)+\frac{1}{Cs}I(s)=E_i(s) \tag{1.1}$$

$$\frac{1}{C}\frac{1}{s}I(s)=E_o(s) \tag{1.2}$$

Assuming the initial condition is zero, then we have

$$\frac{1}{Cs}\frac{E_i(s)}{Ls+R+\frac{1}{Cs}}=E_o(s) \tag{1.3}$$

Then the transfer function is as follows:

$$G(s)=\frac{Y(s)}{U(s)}=\frac{E_o(s)}{E_i(s)}=\frac{1}{LCs^2+RCs+1}=\frac{1}{0.5s^2+s+1} \tag{1.4}$$

b) Since $u=e_i, y=e_o, x_2=\dot{e}_o, x_1=e_o$

$$\frac{1}{C}\int idt=e_o \Rightarrow x_2=\dot{e}_o=\frac{1}{C}i \Rightarrow \frac{di}{dt}=C\dot{x}_2 \tag{1.5}$$

$$L\frac{di}{dt}+Ri+\frac{1}{C}\int idt=e_i \Rightarrow LC\dot{x}_2+RCx_2+x_1=u \tag{1.6}$$

Thus, the above equations yield

$$\dot{x}_1=\dot{e}_o=x_2 \tag{1.7}$$

$$\dot{x}_2=-\frac{1}{LC}x_1-\frac{R}{L}x_2+\frac{1}{LC}u \tag{1.8}$$

$$y=e_o=x_1 \tag{1.9}$$

The state space model is

$$\begin{aligned}\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix}&=\begin{bmatrix}0 & 1\\ -\frac{1}{LC} & -\frac{R}{L}\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}+\begin{bmatrix}0\\ \frac{1}{LC}\end{bmatrix}u=\begin{bmatrix}0 & 1\\ -2 & -2\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}+\begin{bmatrix}0\\ 2\end{bmatrix}u\\ y&=\begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}\end{aligned} \tag{1.10}$$

c) Denote the above state space model as

$$\begin{aligned}x(k+1)&=\Phi x(k)+\Gamma u(k)\\ y(k)&=Cx(k)\end{aligned} \tag{1.11}$$

Then we have

$$\Phi = e^{Ah} = e^{A} \tag{1.12}$$

where $A = \begin{bmatrix} 0 & 1 \\ -2 & -2 \end{bmatrix}$ is the state transition matrix in (1.10). To find e^{A}, we can use the Laplace transform. Let $f(t) = e^{At}$, and its Laplace transform is

$$F(s) = \mathcal{L}[f(t)] = (sI - A)^{-1} = \frac{1}{s^2 + 2s + 2}\begin{bmatrix} s+2 & 1 \\ -2 & s \end{bmatrix} \tag{1.13}$$

Here we need to rearrange equation (1.13) into standard forms of the inverse Laplace transform given at the end of Lecture Two slides, as follows: (note that $s^2 + 2s + 2 = (s+1)^2 + 1$ and we should utilize the two formulas of row 14 and 15)

$$F(s) = \frac{1}{s^2 + 2s + 2}\begin{bmatrix} s+2 & 1 \\ -2 & s \end{bmatrix} = \begin{bmatrix} \frac{s+1}{(s+1)^2+1} + \frac{1}{(s+1)^2+1} & \frac{1}{(s+1)^2+1} \\ -2\frac{1}{(s+1)^2+1} & \frac{s+1}{(s+1)^2+1} - \frac{1}{(s+1)^2+1} \end{bmatrix} \tag{1.14}$$

Now it is straightforward to derive the inverse Laplace transform

$$f(t) = \mathcal{L}^{-1}[F(s)] = \begin{bmatrix} e^{-t}\cos(t) + e^{-t}\sin(t) & e^{-t}\sin(t) \\ -2e^{-t}\sin(t) & e^{-t}\cos(t) - e^{-t}\sin(t) \end{bmatrix} \tag{1.15}$$

Let $t = h = 1$, we have

$$\Phi = e^{A} = f(1) = \begin{bmatrix} 0.5083 & 0.3096 \\ -0.6191 & -0.1108 \end{bmatrix} \tag{1.16}$$

To compute the input matrix Γ, generally we have two approaches:

- If A is nonsingular, then make use of the fact $\frac{de^{A\tau}}{d\tau} = Ae^{A\tau}$ and we have

$$\begin{aligned}
\Gamma &= \int_0^h e^{A\tau} d\tau B = A^{-1} e^{A\tau} \Big|_{\tau=0}^{1} B \\
&= A^{-1}(e^{A} - I)B \\
&= \begin{bmatrix} 0 & 1 \\ -2 & -2 \end{bmatrix}^{-1} \left(\begin{bmatrix} 0.5083 & 0.3096 \\ -0.6191 & -0.1108 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} 0 \\ 2 \end{bmatrix} \\
&= \begin{bmatrix} 0.4917 \\ 0.6191 \end{bmatrix}
\end{aligned} \tag{1.17}$$

- A more general method is to compute the element-wise integration even when A is singular

$$\Gamma = \int_0^h e^{A\tau} d\tau\, B = \begin{bmatrix} \int_0^h e^{-\tau}\left(\sin\tau + \cos\tau\right) d\tau & \int_0^h e^{-\tau} \sin\tau\, d\tau \\ -2\int_0^h e^{-\tau} \sin\tau\, d\tau & \int_0^h e^{-\tau} (\cos\tau - \sin\tau) d\tau \end{bmatrix} \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$
$$= \begin{bmatrix} 2\int_0^1 e^{-\tau} \sin\tau\, d\tau \\ 2\int_0^1 e^{-\tau} (\cos\tau - \sin\tau) d\tau \end{bmatrix} = \begin{bmatrix} -e^{-t}(\sin t + \cos t) \\ 2e^{-t}\sin t \end{bmatrix}\Bigg|_0^1 = \begin{bmatrix} 0.4917 \\ 0.6191 \end{bmatrix} \tag{1.18}$$

Thus, the state space model for sampled discrete-time system is[1]

$$x(k+1) = \begin{bmatrix} 0.5083 & 0.3096 \\ -0.6191 & -0.1108 \end{bmatrix} x(k) + \begin{bmatrix} 0.4917 \\ 0.6191 \end{bmatrix} u(k)$$
$$y(k) = \begin{bmatrix} 1 & 0 \end{bmatrix} x(k) \tag{1.19}$$

d) Applying Z transform on discrete state space model (1.11) yields

$$zX(z) - zx(0) = \Phi X(z) + \Gamma U(z)$$
$$Y(z) = CX(z) \tag{1.20}$$

Then we can get

$$Y(z) = CX(z) = C(zI - \Phi)^{-1}\Gamma U(z) + zC(zI - \Phi)^{-1} x(0) \tag{1.21}$$

Assuming zero initial conditions we can get the z transfer function as

$$\text{H(z)} = \frac{Y(z)}{U(z)} = C(zI - \Phi)^{-1}\Gamma = \frac{0.4917\, z + 0.2462}{z^2 - 0.3975z + 0.1354} \tag{1.22}$$

Therefore, we can get

$$z^2 Y(z) - 0.3975zY(z) + 0.1354Y(z) = 0.4917zU(z) + 0.2462U(z)$$
$$\Rightarrow y(k+2) - 0.3975y(k+1) + 0.1354y(k) = 0.4917u(k+1) + 0.2462u(k) \tag{1.23}$$

The input-output model is

$$y(k+1) = 0.3975y(k) - 0.1354y(k-1) + 0.4917u(k) + 0.2462u(k-1) \tag{1.24}$$

e) We know that

$$x(0) = \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} y(0) \\ \dot{y}(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \tag{1.25}$$

$$U(z) = Z[u(k)] = \frac{z}{z-1} \tag{1.26}$$

[1] There may be some roundoff errors during computation. Solutions that are close enough will also be accepted. For example, $\Gamma = [0.4916, 0.6191]^T$ is considered to be right as well.

Substituting $x(0), \mathrm{U(z)}$ into equation (1.21) yields

$$
\begin{aligned}
Y(z) &= C(zI-\Phi)^{-1}\left(\frac{z}{z-1}\begin{bmatrix}0.4917\\0.6191\end{bmatrix}+z\begin{bmatrix}1\\0\end{bmatrix}\right)\\
&=\begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}z-0.5083 & -0.3096\\ 0.6191 & z+0.1108\end{bmatrix}^{-1}\begin{bmatrix}\dfrac{z^2-0.5083z}{z-1}\\ \dfrac{0.6191z}{z-1}\end{bmatrix} \qquad (1.27)\\
&=\frac{z^3-0.0879z^2-0.1742z}{z^3-1.3975z^2+0.5329z-0.1354}
\end{aligned}
$$

(*After you get the z-transform of* $Y(z)$ *as above, you will get 1 point for this question*.)

To get the output sequence $y(k)$, we need to apply the inverse z transform to (1.27). After some inspection, it turns out that we can transform $Y(z)$ into the following form

$$
Y(z)=\frac{z}{z-1}+\frac{e^{-1}\sin(1)z}{z^2-2e^{-1}\cos(1)z+e^{-2}}. \qquad (1.28)
$$

Then, by checking the z transform table, the output sequence is

$$
y(k)=1+e^{-k}\sin(k) \quad (k\geq 0). \qquad (1.29)
$$

Q2 Solution

a) Poles: $s^2(s+1)=0 \Rightarrow s_1=0, s_2=0, s_3=-1$.

Since there are two identical poles s_1 and s_2 with a real part of 0, the system is unstable.

For the inverse system, its pole will be the zero of this original system, i.e., $s_0=1$. Therefore, the inverse system is unstable.

(Note: for a system $G(s)=\frac{N(s)}{D(s)}$, by *inverse* we mean the system $G'(s)=\frac{D(s)}{N(s)}$.)

b) Mapping the above poles onto z-plane, we have poles for the discrete sampled system,

$$
\begin{aligned}
z_1 &= e^{s_1h}=1 \Rightarrow |z_1|=1\\
z_1 &= e^{s_2h}=1 \Rightarrow |z_2|=1 \qquad (2.1)\\
z_3 &= e^{s_3h}=e^{-h} \Rightarrow |z_2|=e^{-h}<e^0=1
\end{aligned}
$$

Since there are multiple poles with magnitude 1, the sampled system is still unstable.

c) Since there is no direct mapping for zeros between s-domain and z-domain, we have to derive the z-transfer function first.

$$G(s)=\frac{s-1}{s^2(s+1)}=\frac{2}{s}-\frac{1}{s^2}-\frac{2}{s+1} \tag{2.2}$$

From the z transform table, we can find the sampled system is

$$\begin{aligned} G(z) &= 2\frac{h}{z-1}-\frac{h^2(z+1)}{2(z-1)^2}-2\frac{1-e^{-h}}{z-e^{-h}} \\ &= \frac{[4e^{-h}-(h-2)^2]z^2+[e^{-h}(h^2-4h-8)-(h^2+4h-8)]z+e^{-h}(h+2)^2-4}{2(z-1)^2(z-e^{-h})} \end{aligned} \tag{2.3}$$

Now we need to find the zeros of the samples system (2.3), that is, the poles of the inverse sampled system, to check the stability of the inverse sampled system. There are two cases to be considered depending on whether the coefficient of the 2nd-order term is zero.

- If $4e^{-h}-(h-2)^2$ is zero, there will only be one zero for (2.3). To find such h, we can resort to the MATLAB function *fzero* and get the solution to be $h = 2.5569$. Inserting h into (2.3), we can get the single zero as $z = -0.2471$. Since this zero lies inside the unit circle, this special sampling period $h = 2.5569$ can give a stable inverse.
- If $4e^{-h}-(h-2)^2$ is not zero, then two zeros can be found for (2.3). The MATLAB function *roots* can be used to get the roots for this 2nd-order polynomial. We vary h in range $[0.01, 5]$ and plot the two zeros shown in Figure 1. As can be seen, the magnitude of z_1 is always bigger than 1. There exists $h \to 0$, $z_1 \to 1$ and $h \to \infty$, $z_1 \to -1$. Therefore, z_1 is always a zero outside the unit circle and the inverse system is unstable.

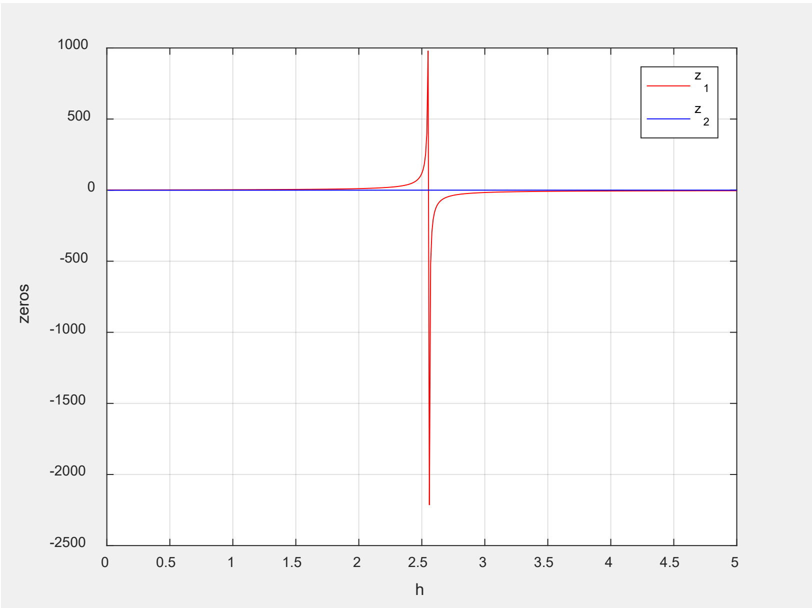

Figure 1 Zeros of the sampled system of case 2

For your reference, the MATLAB code of this question is given below.

```
%% case 1
fun = @(h) 4*exp(-h) - (h-2)^2;
x = fzero(fun, 3);
%% case 2
N = 500;
h = linspace(0.01, 5, N);
P = [4*exp(-h) - (h-2).^2;
    exp(-h).*(h.^2-4*h-8) - (h.^2+4*h-8);
    exp(-h).*(h+2).^2 - 4];
% each column of P is a group of three coefficients
Z = NaN(2, N);
for ii = 1:N
    Z(:,ii) = roots(P(:,ii));
end
% plot: each column of Z is the two zeros
figure;
plot(h, Z(1,:), 'r', h, Z(2,:), 'b');
legend('z_1', 'z_2');
xlabel('h'); ylabel('zeros');
grid on;
```

To sum up the above two cases, we can see that it is possible to get a stable inverse by choosing $h = 2.5569$. For all other sampling time h's, the inverse system is unstable.

As can be seen, sampling will not change the stability of the system due to the pole mapping relation between continuous-time and discrete-time domain. However, there is no simple relation between the zeros. Further, even the original system has no zeros, after sampling, it may have zeros.

Q3 Solution

a) The state transition matrix, input matrix and output matrix are

$$\Phi = \begin{bmatrix} 1 & 3 \\ 2 & 0 \end{bmatrix}, \Gamma = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, C = \begin{bmatrix} 1 & 0 \end{bmatrix} \tag{3.1}$$

The characteristic equation and the poles are

$$\det(\lambda I - \Phi) = \lambda^2 - \lambda - 6 = 0 \Rightarrow \lambda_1 = 3, \lambda_2 = -2 \tag{3.2}$$

Since the poles are NOT in the unit circle of the z-plane, the system is unstable.

Controllability matrix is

$$W_c = \begin{bmatrix} \Gamma & \Phi\Gamma \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ 1 & 2 \end{bmatrix} \tag{3.3}$$

Since W_c is nonsingular, the system is controllable.

The observability matrix is

$$W_o = \begin{bmatrix} C \\ C\Phi \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 3 \end{bmatrix} \tag{3.4}$$

Since W_o is nonsingular, the system is observable.

b) The z transfer function is

$$\begin{aligned} H(z) &= C(zI-\Phi)^{-1}\Gamma \\ &= \frac{z+3}{(z-3)(z+2)} = \frac{z+3}{z^2-z-6} \end{aligned} \tag{3.5}$$

Since $H(z) = \dfrac{Y(z)}{U(z)}$, we can get

$$z^2Y(z) - zY(z) - 6Y(z) = zU(z) + 3U(z) \tag{3.6}$$

Applying inverse z transform yields

$$y(k+2) - y(k+1) - 6y(k) = u(k+1) + 3u(\mathrm{k}),\ \mathrm{k} = 0,1,2,\cdots \tag{3.7}$$

That is,

$$y(k+1) = y(k) + 6y(k-1) + u(k) + 3u(k-1), \quad k \ge 1 \tag{3.8}$$

c) Applying z transform to the controller signal gives

$$U(z) = Z[\mathrm{u(k)}] = K(U_c(z) - Y(z)) \tag{3.9}$$

Since we have known $Y(z) = H(z)U(z)$, then it follows that

$$Y(z) = KH(z)(U_c(z) - Y(z)) \tag{3.10}$$

Thus, it can be derived that

$$\frac{Y(z)}{U_c(z)} = \frac{KH(z)}{1+KH(z)} = \frac{K(z+3)}{z^2+(K-1)z+3K-6} \tag{3.11}$$

(Note: for this question, you can also work it out by first writing the input-output relation from $u_c(k)$ to $y(k)$ and then applying Laplace transform.)

d) The characteristic polynomial is $z^2+(K-1)z+3K-6$. Then Jury's test can be listed in a table as follows.

ID	Operation	Result		
(1)	Get coefficients	1	K-1	3K-6
(2)	Reverse	3K-6	K-1	1
(3)	(1)-(2)*(3K-6)	(-3K+7)(3K-5)	(K-1)(7-3K)	
(4)	Reverse	(K-1)(7-3K)	(-3K+7)(3K-5)	
(5)	(3) - (4)* $\frac{(K-1)(7-3K)}{(-3K+7)(3K-5)}$	$\frac{(7-3K)(-K+2)(K-3)}{(3K-5)}$		

According to Jury's test, if the system is stable, the first element of all the **odd** rows should be positive (in blue), i.e.,

$$\begin{cases} (-3K+7)(3K-5) > 0 \\ \dfrac{(7\text{-}3K)(2-K)(K-3)}{3K-5} > 0 \end{cases} \tag{3.12}$$

From the first inequality in (3.12), we can get $\frac{5}{3} < K < \frac{7}{3}$.

Then, on the basis of $\frac{5}{3} < K < \frac{7}{3}$, for the second inequality in (3.12), it must satisfy $(2-K)(K-3)>0$, that is, $2 < K < 3$.

Therefore, the final answer is

$$\begin{cases} \frac{5}{3} < K < \frac{7}{3} \\ 2 < K < 3 \end{cases} \Rightarrow \quad 2 < K < \frac{7}{3} \tag{3.13}$$

e) The z transform for $u_c(k)$, a unit step input, is

$$U_c(z) = \frac{z}{z-1} \tag{3.14}$$

And the output Y(z) is got according to equation (3.11)

$$Y(z) = \frac{K(z+3)}{z^2+(K-1)z+3K-6}\frac{z}{z-1} \tag{3.15}$$

Assuming the system is stable, according to the final value theorem, the stable state value of $y(k)$ is

$$\lim_{k\to\infty} y(k) = \lim_{z\to1}(z-1)Y(z) = \lim_{z\to1}\frac{K(z+3)}{z^2+(K-1)z+3K-6} = \frac{2K}{2K-3} \tag{3.16}$$

Thus, the steady-state error is

$$\lim_{k\to\infty}(u_c(k)-y(k)) = 1-\frac{2K}{2K-3} = \frac{-3}{2K-3} \tag{3.17}$$

Q4 Solution

a) Applying z transform by assuming zero initial conditions can yield

$$zY(z) = 3Y(z) - 2z^{-1}Y(z) + z^{-1}U(z) + 2z^{-2}U(z) \tag{4.1}$$

Then, the transfer function is derived to be

$$H(z) = \frac{Y(z)}{U(z)} = \frac{z^{-1}+2z^{-2}}{z-3+2z^{-1}} = \frac{z+2}{z^3-3z^2+2z} = \frac{z+2}{z(z-1)(z-2)} \tag{4.2}$$

Poles: $z(z-1)(z-2)=0 \Rightarrow z_1=0, z_2=1, z_3=2$

Since z_3 is outside the unit circle, the system is unstable.

Zeros: $z+2=0 \Rightarrow z=-2$.

The single zero is also outside the unit circle. Thus, the inverse system is also unstable.

b) Observable but uncontrollable?

Since the given system $H(z)$ is coprime, i.e., no common zeros/poles, any minimal realization (3rd-order) of $H(z)$ must be both controllable and observable[2]. Therefore, to answer question b) and c), we have to try possible non-minimal realizations of $H(z)$, that is, a realization of order > 3.

To write down a possible non-minimal realization, we can first add a common zero/pole pair to the original transfer function $H(z)$ to get $H'(z)$. Next, the canonical forms of $H'(z)$ can be figured out. However, remember to check whether such form is a truly non-minimal realization of the original system $H(z)$ at the end.

For example, if we let

$$H'(z) = \frac{(z+2)(z+1)}{z(z-1)(z-2)(z+1)} = \frac{z^2+3z+2}{z^4-2z^3-z^2+2z}. \tag{4.3}$$

whose observable canonical form is

$$\Phi = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ -2 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \Gamma = \begin{bmatrix} 0 \\ 1 \\ 3 \\ 2 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}. \tag{4.4}$$

The corresponding observability and controllability matrix of (4.4) are

[2] Chen, Chi-Tsong. Linear system theory and design. Oxford University Press, Inc., 1995, Chapter 7 Minimal Realizations, Page 187.

$$W_o = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 5 & 2 & 1 & 0 \\ 10 & 5 & 2 & 1 \end{bmatrix}, \quad W_c = \begin{bmatrix} 0 & 1 & 5 & 13 \\ 1 & 3 & 3 & 3 \\ 3 & 2 & -2 & -10 \\ 2 & 0 & 0 & 0 \end{bmatrix}, \tag{4.5}$$

and their ranks are $rank(W_o) = 4$ and $rank(W_c) = 3$ respectively. However, is (4.4) a true realization of $H(z)$ in (4.2)? To verify it, we write down the difference equation of (4.4) as

$$\begin{aligned} x_1(k+1) &= 2x_1(k) + x_2(k) \\ x_2(k+1) &= x_1(k) + x_3(k) + u(k) \\ x_3(k+1) &= -2x_1(k) + x_4(k) + 3u(k) \\ x_4(k+1) &= 0 + 2u(k) \end{aligned} \tag{4.6}$$

Then, for the input-output model we have

$$\begin{aligned} & y(k+1) \\ &= x_1(k+1) \\ &= 2x_1(k) + x_2(k) \\ &= 2x_1(k) + x_1(k-1) + x_3(k-1) + u(k-1) \\ &= 2x_1(k) + x_1(k-1) - 2x_1(k-2) + x_4(k-2) + 3u(k-2) + u(k-1) \\ &= 2x_1(k) + x_1(k-1) - 2x_1(k-2) + 2u(k-3) + 3u(k-2) + u(k-1) \\ &= 2y(k) + y(k-1) - 2y(k-2) + 2u(k-3) + 3u(k-2) + u(k-1) \end{aligned} \tag{4.7}$$

The corresponding transfer function is exactly $H'(z)$ in (4.3). In other words, the realization (4.4) is a realization of the 4th-order $H'(z)$ instead of the original 3rd-order system $H(z)$. In fact, you cannot obtain a non-minimal realization of $H(z)$ which is observable. Since the original system $H(z)$ is indeed only 3rd-order, if you have more than 3 state variables, you cannot observe them all. You may try to add other common poles/zeros to get a new $H'(z)$ and test whether you can get an observable non-minimal realization. The result should remain the same.

Therefore, it is **impossible** to realize the system such that it is observable but not controllable.

(*Note: for this question, if you just give the final conclusion, you get 1 mark; if you present justifications for your conclusion with some example or theoretical proof, you get the full marks.*)

c) Controllable but unobservable?

Similarly, the controllable canonical form of $H'(z)$ in (4.3) is

$$\Phi = \begin{bmatrix} 2 & 1 & -2 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad \Gamma = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 1 & 3 & 2 \end{bmatrix} \tag{4.8}$$

and the associated observability and controllability matrix are

$$W_o = \begin{bmatrix} 0 & 1 & 3 & 2 \\ 1 & 3 & 2 & 0 \\ 5 & 3 & -2 & 0 \\ 13 & 3 & -10 & 0 \end{bmatrix}, \quad W_c = \begin{bmatrix} 1 & 2 & 5 & 10 \\ 0 & 1 & 2 & 5 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \tag{4.9}$$

whose ranks are $rank(W_o) = 3$ and $rank(W_c) = 4$ respectively. The input-output model of the realization (4.8) is

$$\begin{aligned} x_1(k+1) &= 2x_1(k) + x_2(k) - 2x_3(k) + u(k) \\ x_2(k+1) &= x_1(k) \\ x_3(k+1) &= x_2(k) \\ x_4(k+1) &= x_3(k) \\ y(k) &= x_2(k) + 3x_3(k) + 2x_4(k) \end{aligned} \tag{4.10}$$

which further leads to

$$\begin{aligned} & y(k+1) \\ & = x_2(k+1) + 3x_3(k+1) + 2x_4(k+1) \\ & = x_1(k) + 3x_2(k) + 2x_3(k). \end{aligned} \tag{4.11}$$

Now combining (4.11) and the first three equations in (4.10) we can see that the state variable $x_4(k)$ can be excluded from the subsequent computations, indicating that this is actually a 3rd-order system. Following (4.11), we continue our derivation to get the input-output model for the potential realization in (4.8), which is

$$\begin{aligned} & y(k+1) \\ & = x_1(k) + 3x_2(k) + 2x_3(k) \\ & = 2x_1(k-1) + x_2(k-1) - 2x_3(k-1) + u(k-1) + 3x_1(k-1) + 2x_2(k-1) \\ & = 3\left(x_1(k-1) + 3x_2(k-1) + 2x_3(k-1)\right) + 2x_1(k-1) - 6x_2(k-1) - 8x_3(k-1) + u(k-1) \\ & = 3y(k) + 2x_1(k-1) - 6x_2(k-1) - 8x_3(k-1) + u(k-1) \\ & = 3y(k) + 2\left(2x_1(k-2) + x_2(k-2) - 2x_3(k-2) + u(k-2)\right) - 6x_1(k-2) - 8x_2(k-2) + u(k-1) \\ & = 3y(k) - 2\left(x_1(k-2) + 3x_2(k-2) + 2x_3(k-2)\right) + 2u(k-2) + u(k-1) \\ & = 3y(k) - 2y(k-1) + u(k-1) + 2u(k-2). \end{aligned} \tag{4.12}$$

As can be seen, equation (4.12) is exactly the input-output model of the original system H(z) in (4.2). That is to say, this is a right non-minimal realization of the given original system.

Therefore, it is **possible** to realize the system such that it is controllable but not observable.

(*Note: for this question, you may of course give other non-minimal realizations as long as you can prove that it is a true realization of the original system and draw the same conclusion. You will get only 1 mark if only the final conclusion is given.*)

d) Both controllable and observable?

This question is simple since the given system is a coprime one. That is, any minimal realization of it will be both controllable and observable. For example, the controllable canonical form of the minimal realization is given as

$$\Phi = \begin{bmatrix} 3 & -2 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \quad \Gamma = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad C = \begin{bmatrix} 0 & 1 & 2 \end{bmatrix}. \tag{4.13}$$

Check the observability of the state space model (4.13). The observability matrix is

$$W_o = \begin{bmatrix} C \\ C\Phi \\ C\Phi^2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 0 \\ 5 & -2 & 0 \end{bmatrix}. \tag{4.14}$$

Since W_o has full rank, the system is observable. You can also write down the observable canonical form and verify that it is also controllable. Since the original system has no common poles/zeros, the above minimal realization must be a true realization of the original system, which you may verify like what we have done in question c).

Therefore, it is **possible** to realize the system such that it is both controllable and observable.

(*Please note that for a coprime transfer function, i.e., one with no common zero/poles, its minimal realization is always both controllable and observable. Thus, if you only test the minimal realizations in question b) or c) and draw the conclusion of NO, you will get zero marks since it implies that you may lack an understanding of the basic principles.)*