EE5110 Segment 3

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Q1

a) Calculus of variations

$The\ input\ signal\ is:$

$$u = -\frac{a}{2}y + \frac{\dot{y}}{2}$$

The function $J(y, u)$ changes to:

$$J(y) = \int_0^\infty [y^2 + (-\frac{a}{2}y + \frac{\dot{y}}{2})^2]dt$$

$$= \int_0^\infty [(1 + \frac{a^2}{4})y^2 + \frac{1}{4}\dot{y}^2]dt - \frac{a}{2}\int_0^\infty y\dot{y}dt$$

$\because$

$$\int_0^\infty y\dot{y}dt = y^2|_0^\infty - \int_0^\infty y\dot{y}dt$$

$$\int_0^\infty y\dot{y}dt = \frac{1}{2}[y^2(\infty) - y^2(0)]$$

$\therefore$

$$J(y) = \int_0^\infty [(\frac{a^2}{4} + 1)y^2 + \frac{1}{4}\dot{y}^2]dt - \frac{a}{4}y^2(\infty) + \frac{a}{4}c^2$$

$Let\ z(t)\ denote\ any\ function\ of\ t\ with\ the\ property\ that\ J(z)\ exists.\ Take\ \varepsilon\ to\ be\ a$

$$J(y_0 + \varepsilon z) = \int_0^\infty [(\frac{a^2}{4} + 1)(y_0 + \varepsilon z)^2 + \frac{1}{4}(\dot{y}_0 + \varepsilon\dot{z})^2]dt - \frac{a}{4}y^2(\infty) + \frac{a}{4}c^2$$

$The\ J(y_0 + \varepsilon z)\ must\ have\ an\ absolute\ minimum\ at\ \varepsilon = 0$

$$\frac{d}{d\varepsilon}J(y_0 + \varepsilon z)|_{\varepsilon=0} = 0$$

$$J(y_0 + \varepsilon z) = \int_0^\infty [(\frac{a^2}{4} + 1)y_0^2 + \frac{1}{4}\dot{y}_0^2]dt + 2\varepsilon\int_0^\infty [(\frac{a^2}{4} + 1)(y_0 z) + \frac{a^2}{4}\dot{y}_0\dot{z}]dt$$

$$+\varepsilon^2\int_0^\infty [(\frac{a^2}{4} + 1)z^2 + \frac{a^2}{4}\dot{z}^2]dt - \frac{a^2}{4}y^2(\infty) + \frac{a^2}{4}c^2$$

$We\ see\ then\ that\ the\ variational\ condition\ derived\ is:$

$$\int_0^\infty [(\frac{a^2}{4} + 1)y_0^2 + \frac{1}{4}\dot{y}_0^2]dt = 0$$

$$2\varepsilon\int_0^\infty [(\frac{a^2}{4} + 1)(y_0 z) + \frac{a^2}{4}\dot{y}_0\dot{z}]dt = 0$$

$$int_0^\infty [(\frac{a^2}{4} + 1)z^2 + \frac{a^2}{4}\dot{z}^2]dt - \frac{a^2}{4}y^2(\infty) + \frac{a^2}{4}c^2 = 0$$

$Since\ y_0 + \varepsilon\ z\ is\ an\ admissible\ function\ satisfies\ the\ initial\ condition:$

$$y_0(0) + \varepsilon z(0) = c$$

$We\ see\ that\ z(0) = 0.$

$Since\ the\ left-hand\ side\ must\ be\ zero\ for\ all\ admissible\ z,\ we\ suspect\ that$

$$(\frac{a^2}{4} + 1)y_0 - \frac{1}{4}\ddot{y}_0 = 0$$

$First,\ we\ use\ T\ to\ replace\ \infty,\ \dot{y_0}(T) = 0.\ And\ we\ obtain\ no\ condition\ on\ \dot{y_0}(0).\ W\epsilon$

$$y_0(0) = c,\ \dot{y}_o(T) = 0$$

$The\ general\ solution\ of\ the\ differential\ equation\ is:$

$$y = c_1 e^{\sqrt{4+a^2}\cdot t} + c_2 e^{-\sqrt{4+a^2}\cdot t}$$

$Using\ the\ boundary\ conditions,\ we\ have\ the\ two\ equations\ to\ determine\ the\ coeffi$

$$c = c_1 + c_2$$

$$0 = c_1 e^{\sqrt{4+a^2}\cdot T} - c_2 e^{-\sqrt{4+a^2}\cdot T}$$

$Solving,\ we\ obtain\ the\ expression:$

$$y_o(t) = c(\frac{e^{\sqrt{4+a^2}(t-T)} + e^{-\sqrt{4+a^2}(t-T)}}{e^{-\sqrt{4+a^2}\cdot T} + e^{\sqrt{4+a^2}\cdot T}}) = c\frac{cosh(\sqrt{4+a^2}(t-T))}{cosh(\sqrt{4+a^2}\cdot T)}$$

$Let\ T \to \infty,\ We\ have$

$$y_o(t) = c(\frac{e^{\sqrt{4+a^2}(t-T)} + e^{-\sqrt{4+a^2}(t-T)}}{e^{-\sqrt{4+a^2}\cdot T} + e^{\sqrt{4+a^2}\cdot T}}) = c(\frac{e^{\sqrt{4+a^2}(t-2T)} + e^{-\sqrt{4+a^2}\cdot t}}{e^{-2\sqrt{4+a^2}\cdot T} + 1})$$

$$\dot{y}_o(t) = c\sqrt{4+a^2}\cdot(\frac{e^{\sqrt{4+a^2}(t-T)} - e^{-\sqrt{4+a^2}(t-T)}}{e^{-\sqrt{4+a^2}\cdot T} + e^{\sqrt{4+a^2}\cdot T}}) = c\sqrt{4+a^2}\cdot(\frac{e^{\sqrt{4+a^2}(t-2T)} - e^{-\sqrt{4+a}}}{e^{-2\sqrt{4+a^2}\cdot T} + 1}$$

$$\dot{y}_o = -\sqrt{4+a^2}\cdot y_o(t)$$

$So\ we\ have\ the\ control\ laws$

$$u(t) = -\frac{a}{2}y(t) + \frac{1}{2}\dot{y}(t) = -\frac{1}{2}(a + \sqrt{4+a^2})y(t)$$

b) Dynamic programming

$Optimal\ Value\ function:$

$$V(c,T) = \min_y J(y)$$

$$J(y) = \int_0^\Delta + \int_\Delta^T = (c^2 + u^2)\Delta + V(c + (ac+2u)\Delta, T-\Delta) + O(\Delta^2)$$

$We\ can\ use\ Taylor\ series\ to\ relate\ V(c + (ac+2u)\Delta, T-\Delta)\ with\ V(c,T),\ J(y)\ wi$

$$V(c,T) = \min_u[(c^2+u^2)\Delta + V(c,T) + \frac{\partial V}{\partial c}(ac+2u)\Delta - \frac{\partial V}{\partial T}\Delta + O(\Delta^2)]$$

$Ignoring\ the\ higher\ order\ terms\ of\ \Delta,\ we\ have$

$$\frac{\partial V}{\partial T} = \min_u[(c^2+u^2) + \frac{\partial V}{\partial c}(ac+2u)]$$

$When\ T \to \infty,\ V(c,T)\ becomes\ V(c),$

$$V(c) = \min_u[(c^2+u^2)\Delta + V(c + (ac+2u)\Delta)] + O(\Delta^2)$$

$$0 = \min_u[(c^2+u^2) + \dot{V}(c)(ac+2u)]$$

$Take\ the\ derivative\ respect\ to\ u\ gives\ 2u + 2\dot{V}(c)a = 0,\ so$

$$u = -\dot{V}(c)$$
$$0 = (c^2 + (\dot{V}(c))^2) + \dot{V}(c)(ac - 2\dot{V}(c))$$
$$\dot{V}^2(c) - ac\dot{V}(c) + c^2 = 0$$

$So\ we\ have\ two\ possibilities,\ with\ the\ condition\ V(0) = 0,\ we\ can\ obtain\ two\ poss$

$$V(c) = \frac{a + \sqrt{a^2 + 4}}{2}$$
$$V(c) = \frac{a - \sqrt{a^2 + 4}}{2}$$

$Since\ V(c) \geqslant 0,\ we\ see\ that\ V(c) = \frac{a+\sqrt{a^2+4}}{2},\ the\ optimal\ value\ can\ be\ easily\ obtai$

$$u = -\frac{a + \sqrt{a^2 + 4}}{2}c$$

$Since\ y(0) = c,\ so\ we\ have\ u(0) = \frac{a+\sqrt{a^2+4}}{2}\ y(0).\ At\ any\ time\ t,\ we\ will\ have\ the\ c$

$$u(t) = -\frac{a + \sqrt{a^2 + 4}}{2}y(t)$$

Q2

$We\ write\ the\ optimal\ value\ function\ as$
$$V_N(c) = \min_{u_n} J_N(y, u)$$
$After\ u(0)\ is\ chosen,\ the\ new\ state\ of\ the\ system\ is\ y(1) = 2c + au(0),\ The\ cost\ f$
$$c^2 + u^2(0) + \sum_{n=1}^{N}(y^2(n) + u^2(n))$$
$The\ long\ term\ cost\ can\ be\ expressed\ as\ optimal\ value\ starting\ from\ 2c + au(0)\ wit$
$$\sum_{n=1}^{N}(y^2(n) + u^2(n)) = V_{N-1}(ac + 2u(0))$$
$Then$
$$V_N(c) = \min_{u(0)}[c^2 + u^2(0) + V_{N-1}(ac + 2u(0))]$$
$For\ the\ continuous\ case\ we\ have V(c, T) = c^2 r(T)$

$It\ is\ reasonable\ to\ guess\ that$

$$V_N(c) = c^2 r_N$$
$$c^2 r_N = \min_{u(0)}[c^2 + u^2(0) + (ac + 2u(0))^2 r_{N-1}]$$
$The\ value\ of\ u(0)\ that\ minimizes\ is\ readily\ obtained\ by\ differentiation$
$$2u(0) + 2a(2a + au(0))r_{N-1} = 0$$
$$u(0) = -\frac{2acr_{N-1}}{1 + 4r_{N-1}}$$
$Using\ this\ value,\ we\ obtain\ the\ recurrence\ relation$
$$r_N = 1 + \frac{a^2 r_{N-1}}{1 + 4r_{N-1}}$$

$At\ each\ time\ t = k,\ the\ input\ control\ is$

$$u(k) = -\frac{2acr_{N-k-1}y(k)}{1 + 4r_{N-k-1}}$$

$When\ N \to \infty,\ let\ r = \lim_{N \to \infty} r_N,\ then\ r\ is\ the\ positive\ root\ of\ the\ quadratic\ equatic$

$$r = \frac{(a^2 + 3) + \sqrt{(a^2 + 3)(4 + 4r^2)}}{8}$$

$The\ control\ signal\ will\ change\ to:$

$$\lim_{N \to \infty} u(0) = -\frac{2acr}{1 + 4r}$$

$We\ see\ that$

$$V(c) = \min_{u(n)} \sum_{n=0}^{\infty} (y^2(n) + u^2(n))$$

$$V(c) = \min_{u(0)} [c^2 + u^2(0) + V(ac + 2u(0))]$$

$$V(c) = rc^2$$

$$\therefore\ for\ the\ inftyite\ time\ process,\ the\ optimal\ feedback\ controller\ is:$$

$$u(k) = -\frac{2ary(k)}{1 + 4r}$$

Q3

$Assume\ that\ the\ lifeguard\ will\ run\ to\ (a,0),\ and\ then\ swim\ to\ the\ swimmer.\ The\ p$
$/s.\ The\ optimal\ function\ can\ be\ expressed\ as:$

$J(a) = \min_a[(\frac{\sqrt{a^2+10^2}}{v_1})^2 + (\frac{\sqrt{(20-a)^2+(-10)^2}}{v_2})^2] = \min_a[\frac{a^2+10^2}{v_1^2} + \frac{(20-a)}{v}$

$Take\ the\ derivative\ respect\ to\ a,\ we\ get$

$$\frac{29a^2}{100} - 10a + 129 = 0$$

$$a = \frac{10}{0.58} = 17.24$$

$So,\ the\ shortest\ time\ path\ is\ that\ lifeguard\ run\ to\ (18.823,0)\ and\ then\ swim\ to\ the$

$$t_{min} = \frac{\sqrt{a^2+10^2}}{v_1} + \frac{\sqrt{(20-a)^2+(-20)^2}}{v_2} = 9.17s$$

In [1]:
```
import numpy as np
np.sqrt(17.24**2+100)/5+np.sqrt((20-17.24)**2+100)/2
```

Out[1]: 9.173008873571147

Q4

First, we can put all attractions and hotel in the xyplane and sort them by x coordi $(i \leqslant j)$ *is the shortest closed curve which contain* $p_1, p_2, \ldots, p_j$. *This path goes fro* $V_{n,n}$ *is what we want in this topic.*

Assume that the length of $V_{i,j}$ *is* $l(i,j)$, *the distance between* p_i *and* p_j *is* $dist(i,j) =$

In the path $V_{i,j}$, p_i *is in the path* $p_i \rightarrow p_1$, p_j *is in the path* $p_1 \rightarrow p_j$. *Now, let's tal*

(1) $i < j-1$

Because p_{j-1} *is on the right side of* p_i, *so* p_{j-1} *is in the path* $p_1 \rightarrow p_j$. *Besides,* p_{j-} *so it connect to* p_j *directly. We can get*

$$l(i,j) = l(i,j-1) + dist(j-1,j)$$

(2) $i = j-1$

In this case, p_{j-1} *is* p_i, *so* p_{j-1} *is in the path* $p_i \rightarrow p_1$. *Any point from* $p_1, p_2, \ldots, p$. *Assume that point is* $p_k (1 \leqslant k \leqslant j-2)$. *We need to chose an appropriate point* p_k

$$l(i,j) = \min_{1 \leqslant k \leqslant j-2} [l(k,j-1) + dist(k,j)]$$

(3) $i = j$

This only happens when $i = j = n$. *In this case,* p_{n-1} *connect to* p_n, *we can get :*

$$l(n,n) = l(n-1,n) + dist(n-1,n)$$

In conclusion the optimal function is :

$$l(i,j) = \begin{cases} l(i,j-1) + dist(j-1,j), & i < j-1 \\ \min 1 \leqslant k \leqslant j-2[l(k,j-1) + dist(k,j)], & i = j-1 \\ l(n-1,n) + dist(n-1,n), & i = j = n \end{cases}$$

This function is what we want.