# Chapter 11 State Estimation

## §11.1 Introduction

**Why is state feedback used?**

It is used for improvement of performance and robustness such as

(i) Pole placement

(ii) Optimal control

(iii) Decoupling

(iv) Servo control

(v) H-infinity control

(vi) …

## What are the problems with state feedback?

(i) Not all state variables are measurable; and
(ii) Possible sensitivity to uncertainties.

Solution to (i) is state estimation, which is the topic of this chapter.

Solution to (ii) is robust control design, which will be addressed in EE5102 multivariable control system.

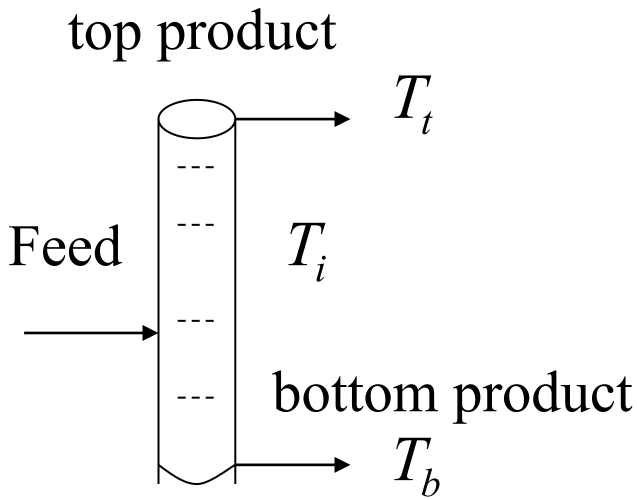
# Why state estimation?

- The state space approach uses state feedback:

$$u = -Kx + Fr\ .$$

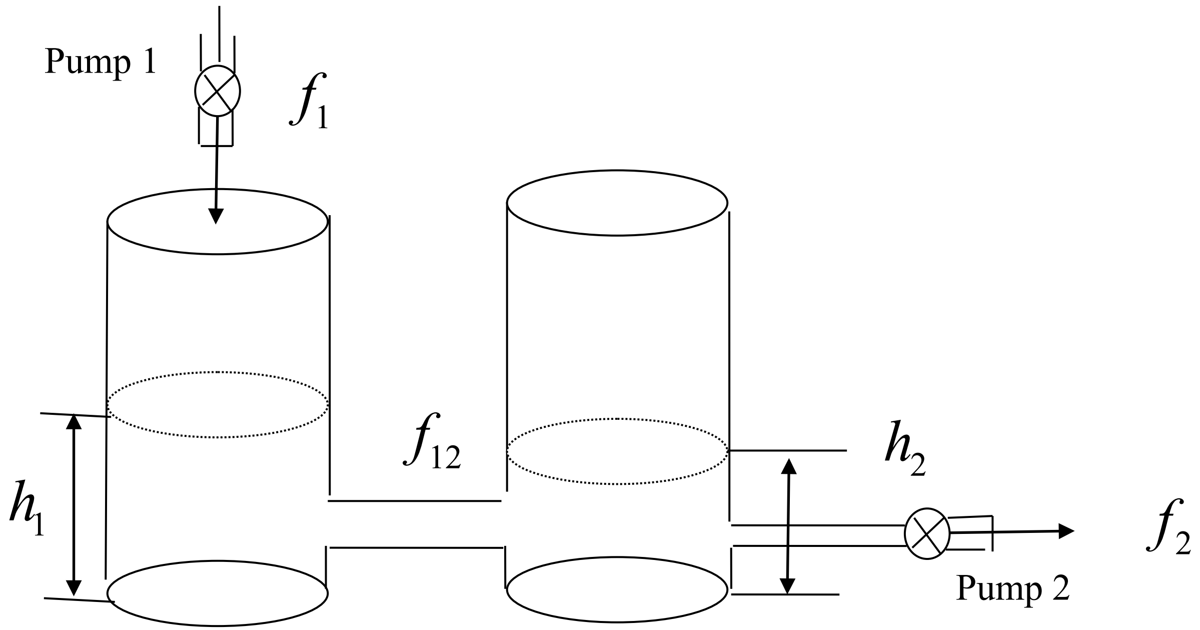
- But usually, $x$ is not available as all state variables are not measurable.
- Only $y$ is measurable.
- To implement state feedback without $x(t)$ necessitates use of state estimation.

For instance, a full state space model for a Distillation Column will have each plate temperature $T_i$ as a state variable. Not all $T_i$ are measurable. Only $T_b$ and $T_t$ are usually measured.



# An Industrial Motivation: a couple-tank level estimation

***Process***: Consider a coupled-tank (Goodwin G. C. etc., Control system design, Prentice Hall, USA, 2002) as follows




Water flows into the first tank through pump 1 at a rate of $f_1$

$f_{12}$, affecting both $h_1$ and $h_2$. Water flows out of tank 2 at a rate of

$f_2$ controlled by pump 2. The challenge is to build an observer to estimate the height of liquid in tank 1 from measurement of the height of liquid in tank 2 and the flows $f_1$ and $f_2$.

***Model***: The height of liquid in tank 1 can be described by equation

$$\frac{dh_1(t)}{dt} = \frac{1}{A}\left(f_1(t) - f_{12}(t)\right).$$

Similarly, $h_2(t)$ is described by

$$\frac{dh_2(t)}{dt} = \frac{1}{A}\left(f_{12}(t) - f_2(t)\right).$$

The flow between the two tanks can be approximated by the free-fall velocity for the difference in height between tanks:

$$f_{12}(t) = \sqrt{2g\left(h_1(t) - h_2(t)\right)}.$$

Now, if we measure the liquid height in the tanks in % (where 0% is empty and 100% is full), we can convert the flow rates into equivalent in % per second (where $f_1(t)$ is the equivalent flow into tank 1 and $f_2(t)$ is the equivalent flow out of tank 2). The model for the system is then

$$\begin{bmatrix} \dot{h}_1(t) \\ \dot{h}_2(t) \end{bmatrix} = \begin{bmatrix} -K\sqrt{h_1(t) - h_2(t)} + f_1(t) \\ K\sqrt{h_1(t) - h_2(t)} - f_2(t) \end{bmatrix},$$

where $K = \dfrac{\sqrt{2g}}{A} = 0.26$. This nonlinear model can be linearized around a nominal steady-state, $\left(H_1, H_2\right)$,

$$h_1(t) = H_1 + \Delta h_1(t),$$
$$h_2(t) = H_2 + \Delta h_2(t).$$

Now, we have

$$\begin{bmatrix} \Delta\dot{h}_1 \\ \Delta\dot{h}_2 \end{bmatrix} = \begin{bmatrix} \left.\dfrac{-0.13}{\sqrt{h_1-h_2}}\right|_{\substack{h_1=H_1,\\ h_2=H_2.}} \Delta h_1 + \left.\dfrac{0.13}{\sqrt{h_1-h_2}}\right|_{\substack{h_1=H_1,\\ h_2=H_2.}} \Delta h_2 + \Delta f_1 \\ \left.\dfrac{0.13}{\sqrt{h_1-h_2}}\right|_{\substack{h_1=H_1,\\ h_2=H_2.}} \Delta h_1 + \left.\dfrac{-0.13}{\sqrt{h_1-h_2}}\right|_{\substack{h_1=H_1,\\ h_2=H_2.}} \Delta h_2 - \Delta f_2 \end{bmatrix}$$

This yields the following linear model:

$$\begin{bmatrix} \Delta\dot{h}_1 \\ \Delta\dot{h}_2 \end{bmatrix} = \begin{bmatrix} -k & k \\ k & -k \end{bmatrix} \begin{bmatrix} \Delta h_1 \\ \Delta h_2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \Delta f_1 \\ \Delta f_2 \end{bmatrix},$$

where $k = \dfrac{0.13}{\sqrt{H_1 - H_2}}$. If we let

$$x = \begin{bmatrix} \Delta h_1 \\ \Delta h_2 \end{bmatrix},$$

$$u = \begin{bmatrix} \Delta f_1 \\ \Delta f_2 \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix},$$

the process model can be expressed as

$$\dot{x} = Ax + Bu,$$
$$y = Cx + Du,$$

where $A = \begin{bmatrix} -k & k \\ k & -k \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, $C = \begin{bmatrix} 0 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} 0 & 0 \end{bmatrix}$. If we assume that the operating point is at $H_1 = 50\%$ and $H_2 = 34\%$, then $k = 0.325$.

***Task***: With $h_2(t)$ measured, estimate $h_1(t)$.

**State estimation problem**: Let the system be

$$\dot{x} = Ax + Bu,$$
$$y = Cx + Du. \tag{1}$$

With knowledge of $y(t)$ and $u(t)$, one wants to estimate $x(t)$.

The questions arising are

Q1: Is it possible to estimate *x(t)?*

Q2: How to estimate *x(t)?*

# Possible Estimation Schemes

One way of finding $x(t_1)$ is

$$y(t) = Cx(t) + Du(t),$$

$$\begin{aligned}\dot{y}(t) &= C\dot{x}(t) + D\dot{u}(t) \\ &= CAx(t) + CBu(t) + D\dot{u}(t),\end{aligned}$$

$$\vdots$$

$$\frac{d^{n-1}y(t)}{dt^{n-1}} = CA^{n-1}x(t) + \cdots + D\frac{d^{n-1}u(t)}{dt^{n-1}}$$

$$\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} x(t_1) = \begin{bmatrix} y(t_1) - Du(t_1) \\ \dot{y}(t_1) - CBu(t_1) - D\dot{u}(t_1) \\ \\ \vdots \end{bmatrix}$$

If the system is observable, then it is possible to find $x(t_1)$ by solving these linear equations.

But the measurement of output $y$ inevitably has noise and differentiation of a noisy signal will cause large errors.

Another possible way to estimate the state is to simulate a model with the complete knowledge of *A, B*, and *C*, and construct a so-called open-loop estimator as shown in Figure 1. If one chooses $x_e(0) = x(0)$ , then, $\hat{x}(t) = x(t)$ for all $t$>0. But one does not know the initial state.

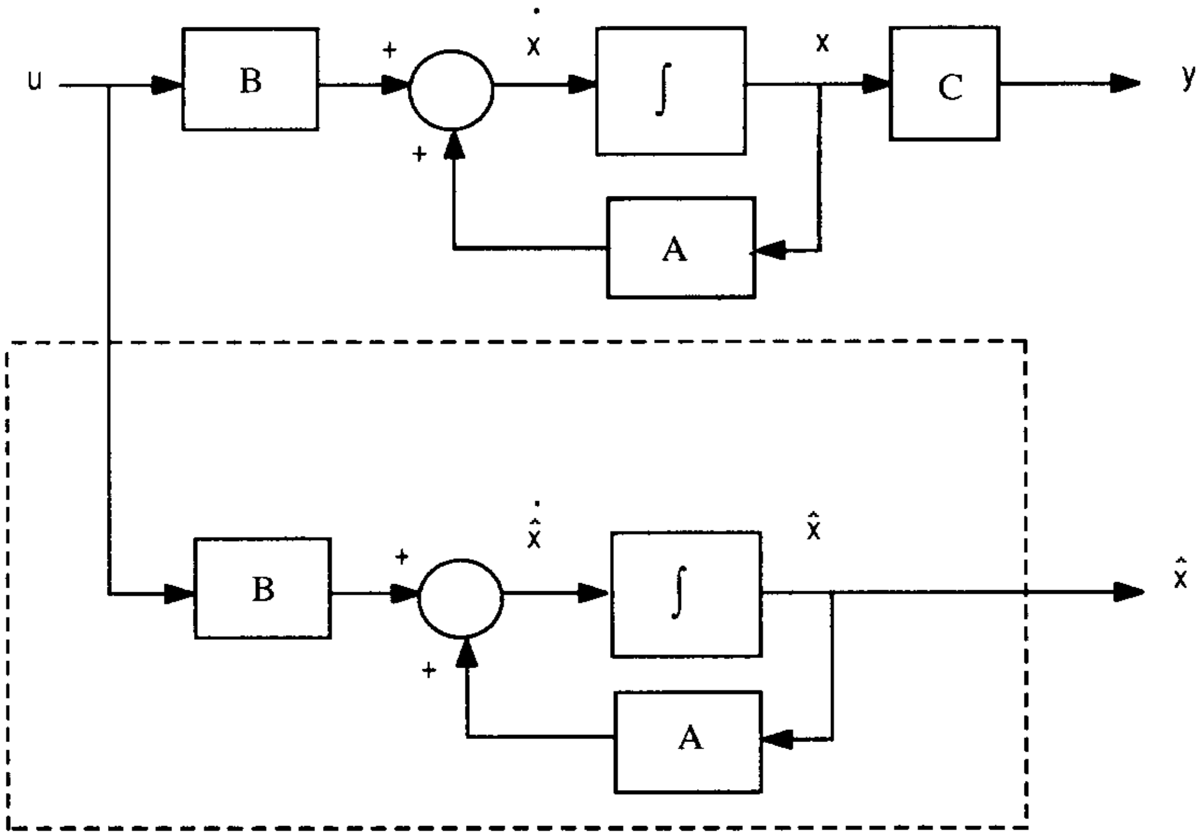



**Figure 1**   An open-loop estimator.

Let the state estimate error be

$$\tilde{x}(t) = x(t) - \hat{x}(t). \tag{2}$$

It follows that the dynamic behavior of $\tilde{x}(t)$ is governed by

$$\dot{\tilde{x}}(t) = \dot{x}(t) - \dot{\hat{x}}(t) = AX + Bu - (A\hat{x} + Bu) = A\tilde{x}(t) \tag{3}$$

and its solution is given by

$$\tilde{x}(t) = e^{At}\tilde{x}(0), t \geq 0. \tag{4}$$

Note that

- No derivatives of $y$ are used: good.
- But, we do not have the initial state $x(0)$. Instant determination of the state: $\hat{x}(t) = x(t),$ is impossible. It only enables asymptotic estimation of the state: $\hat{x}(\infty) = x(\infty),$ if A is stable.
- But the error dynamics is fixed by $A$, and thus not adjustable.
- The error would increase exponentially for unstable matrix $A$ and $\tilde{x}(0) \neq 0$.

# §11.2 Full-order Observers

To overcome these shortcomings, the estimator, shown in Fig. 2, is a closed-loop estimator which was first introduced by Luenberger (1964).

**Where does it come and how it work?**

## Observations:

- $x(t)$ unknown $\rightarrow$ $x(0)$ unknown $\rightarrow$ state estimation error is expected!
- The only thing to do is to have some correction function for the state estimation error, and the correction carries on till, hopefully, the error settles to zero in the end.
- The state estimation error is NOT available, too, while the error between the measured output $y(t)$ and predicted output $\hat{y}(t) = C\hat{x}(t)$ from state estimate can be obtained.
- Thus, one may correct and reduce the error in the estimation $\hat{x}$ by some feedback based on the observed error between the measured output $y(t)$ and predicted output $\hat{y}(t) = C\hat{x}(t)$.

# Scheme

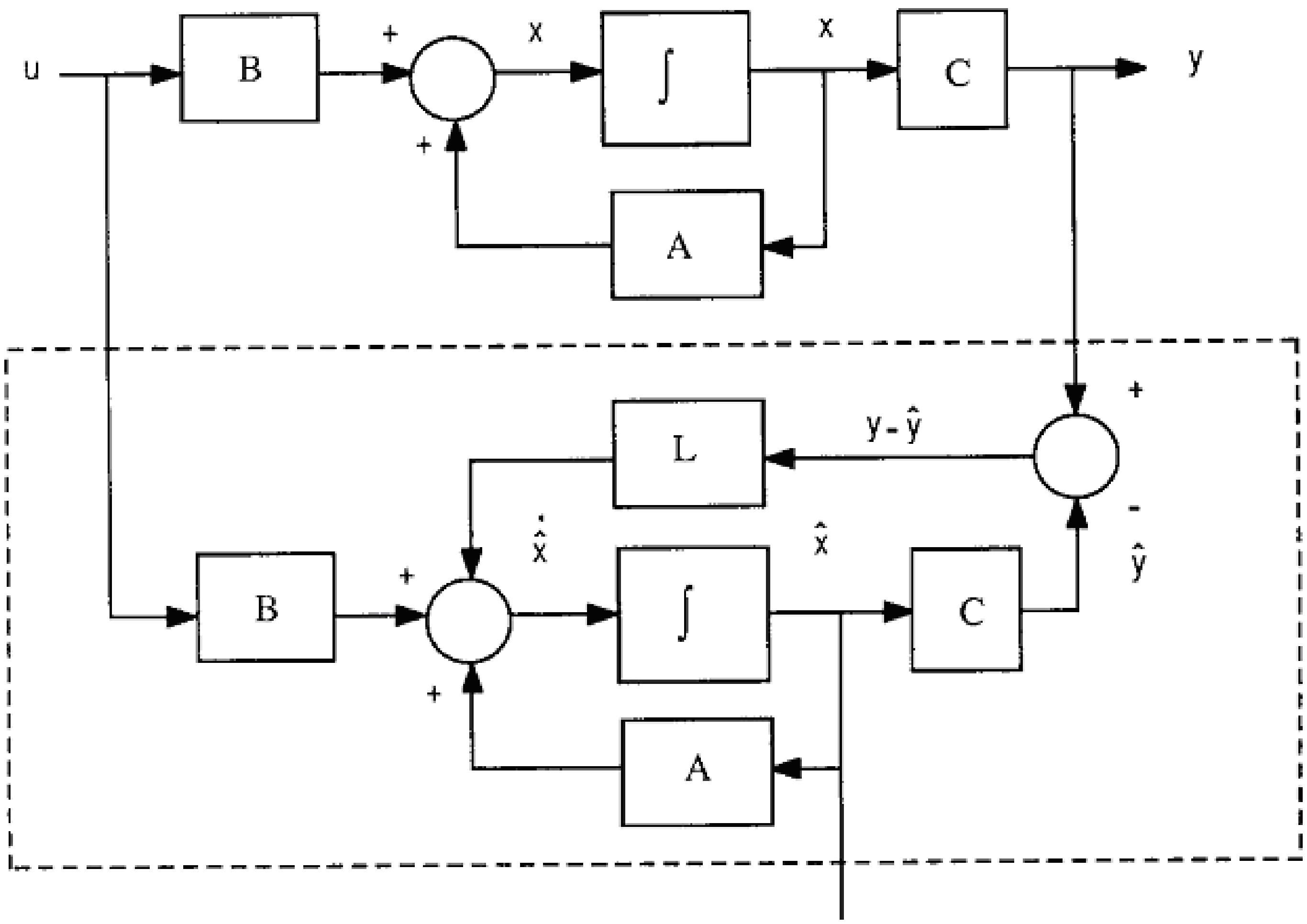



**Figure 2**   A closed-loop estimator.

## Objectives:

- $\hat{x}(t) \to x(t)$ or $x(t) - \hat{x}(t) \xrightarrow[t\to\infty]{} 0$, or the observer is stable.
- The rate of error convergence is adjustable.

## Analysis

Let the system be

$$\dot{x} = Ax + Bu,$$

$$y = Cx.$$

If $D \neq 0$, one simply uses $z = y - Du$ in place of $y$ in what follows.

Consider an estimator:

$$\dot{\hat{x}}(t) = A\hat{x}(t) + Bu(t) + L[y - C\hat{x}], \tag{5}$$

where $\hat{x}(t)$ denotes the estimate of $x(t)$.

Let the estimation error in the state be

$$\tilde{x} = x - \hat{x}.$$

Then, it follows that

$$\dot{\tilde{x}} = \dot{x} - \dot{\hat{x}} = Ax + Bu - \{A\hat{x} + Bu + L(y - C\hat{x})\}$$

$$= A(x - \hat{x}) - L(Cx - C\hat{x})$$

$$= A(x - \hat{x}) - LC(x - \hat{x})$$

$$\dot{\tilde{x}} = (A - LC)\tilde{x}, \qquad \tilde{x}(0) = x(0) - \hat{x}(0). \tag{6}$$

If we choose $L$ such that $(A - LC) = A_1$ is stable, we have

$$\dot{\tilde{x}} = (A - LC)\tilde{x} = A_1\tilde{x},$$

$$\tilde{x}(t) = e^{A_1 t}\tilde{x}(0).$$

Clearly $\tilde{x}(t) \to 0$ as $t \to \infty$. Thus the estimated state $\hat{x}(t)$ will 'track' $x(t)$ asymptotically.

Note that

- no derivatives of $y$ are used;
- the error dynamics is adjustable by $L$; and
- the error could decay to zero even for unstable matrix $A$ and $\tilde{x}(0) \neq 0$, as long as $(A-LC) = A_1$ is stable.

**Then, when can $(A-LC) = A_1$ be made stable and have arbitrary eigenvalues?**

For the observer problem, write its error characteristic polynomial:

$$\det\left[sI-\left(A-LC\right)\right] = \det\left[sI-\left(A-LC\right)\right]^T$$

$$= \det\left[sI-\left(A-LC\right)^T\right]$$

$$= \det[sI-\left(A^T-C^TL^T\right)]$$

$$= \det\left[sI-(\tilde{A}-\tilde{B}\tilde{K})\right]$$

where $\tilde{A} = A^T$, $\tilde{B} = C^T$, and $\tilde{K} = L^T$.

- By duality, $(\tilde{A}, \tilde{B}) = (A^T, C^T)$ is controllable iff the pair *(A, C)* is observable.

- By the pole placement theorem, $L$ can be found to place the roots of $|sI-(A-LC)|=0$ arbitrarily, or the observer can have any desired eigenvalues if the pair *(A, C)* is observable.

## Design

The observer design is to find a suitable observer gain matrix $L$. One may use the pole placement algorithms to adjust the 'rate of convergence' of $\hat{x}(t)$ to $x(t)$. The choice of observer poles is a trade-off between speed and constraints imposed by noise, saturation and nonlineararity. A simple guideline is to place observer poles 3-5 times faster than control poles. A modern design is to use loop transfer function recovery/H_infinity techniques, which is out of scope of this course.

A design procedure is summarized as follows.

(i) Choose the observer poles 3-5 times faster than control poles;

(ii) Use a pole placement algorithm to obtain *L;*

(iii) Implement the observer in (5).

**Example 1** Design an observer for the plant,

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u,$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}.$$

Test $\{A, C\}$ for observability:

$$\Theta_o = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

which is nonsingular, and the system is observable. One forms

$$A_1 = A - LC = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} l_1 \\ l_2 \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix}$$
$$= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} l_1 & 0 \\ l_2 & 0 \end{bmatrix}$$
$$= \begin{bmatrix} -l_1 & 1 \\ -l_2 & 0 \end{bmatrix}.$$

The observer poles are the roots of

$$\left|\lambda I - A_1\right| = \begin{vmatrix} \lambda + l_1 & -1 \\ l_2 & \lambda \end{vmatrix} = \lambda^2 + \lambda l_1 + l_2 = 0.$$

Choose the observer poles at $\lambda_1 = \lambda_2 = -5$ and this choice gives $l_1 = 10$ and $l_2 = 25$. The designed observer is given by

$$\begin{bmatrix} \dot{\xi}_1(t) \\ \dot{\xi}_2(t) \end{bmatrix} = \begin{bmatrix} -10 & 1 \\ -25 & 0 \end{bmatrix} \begin{bmatrix} \xi_1(t) \\ \xi_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u + \begin{bmatrix} 10 \\ 25 \end{bmatrix} y.$$

The system and observer are shown in Figure 3.

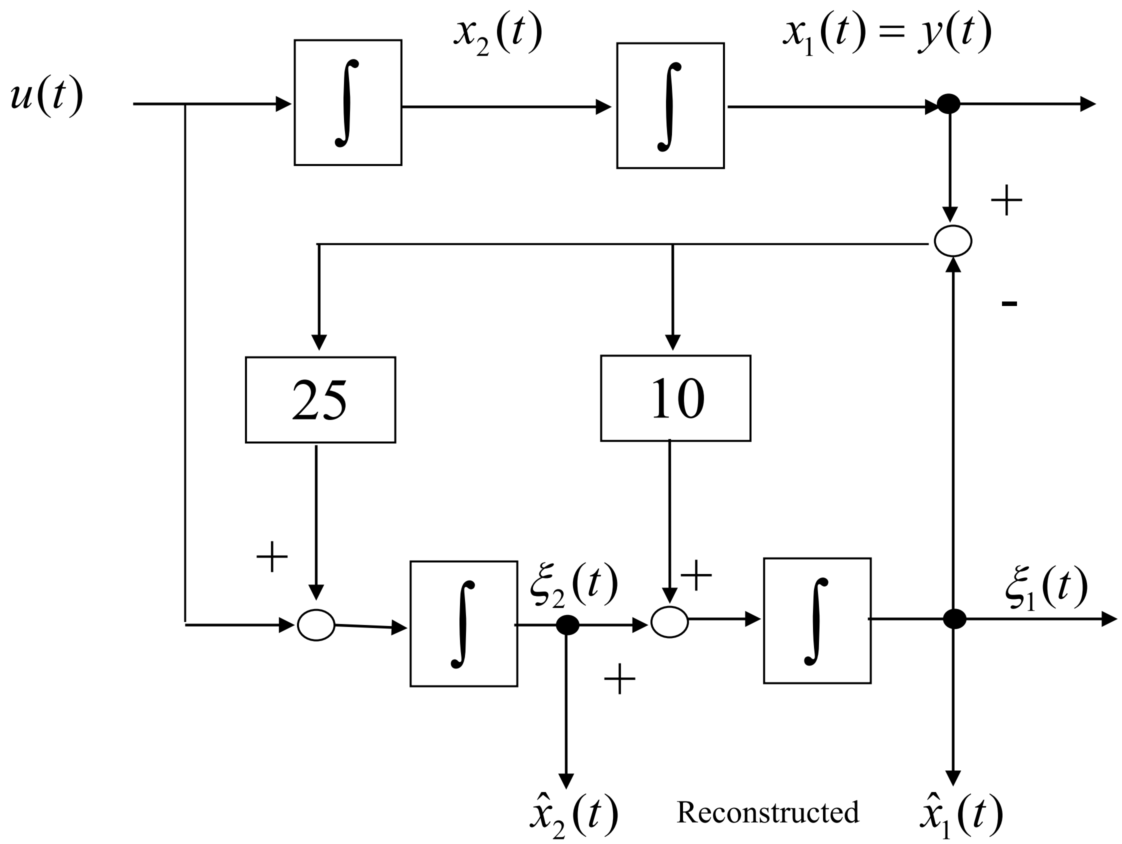



**Figure 3** Observer design for Example 1.

Note that in this case the observer is redundant in the sense that $x_1(t)$ is already available as $y(t)$ and so there is no need to 'reconstruct' it. In fact, only $x_2(t)$ needs to be reconstructed. This suggests that this observer only needs to be a first-order system, instead of the second-order one that was just designed. It is this observation that leads to the investigation of lower-order observers.

## An Industrial Application: the coupled-tank level estimation revisited

**Model:** The linearized model of the coupled-tank system at the operating point of $H_1 = 50\%, H_2 = 34\%$ is

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -0.325 & 0.325 \\ 0.325 & -0.325 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix},$$

$$y = x_2 = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

**Observer design**: The system is both controllable and observable. Let

$$L = \begin{bmatrix} l_1 \\ l_2 \end{bmatrix},\ A_1 = A - LC = \begin{bmatrix} -0.325 & 0.325 - l_1 \\ 0.325 & -0.325 - l_2 \end{bmatrix}.$$

The characteristic polynomial of the observer is readily seen to be

$$\det(\lambda I - (A - LC)) = \lambda^2 + (0.65 + l_2)\lambda + 0.325\left(l_2 + l_1\right).$$

We can choose the observer poles; that choice gives us values for $l_1$ and $l_2$. If we want two poles at $\lambda = -1$, then $l_2 = 1.35$ and $l_1 = 1.727$. The designed observer is given by

$$\dot{\hat{x}}(t) = \begin{bmatrix} -0.325 & -1.402 \\ 0.325 & -1.675 \end{bmatrix} \hat{x}(t) + \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} u(t) + \begin{bmatrix} 1.727 \\ 1.35 \end{bmatrix} y(t).$$

The estimator performance is shown in Figure 4 for the initial state of the process at $x(0)=\begin{bmatrix}0.5 & 0.34\end{bmatrix}^T$ and that of the observer set at $\hat{x}(0)=\begin{bmatrix}0 & 0\end{bmatrix}^T$.

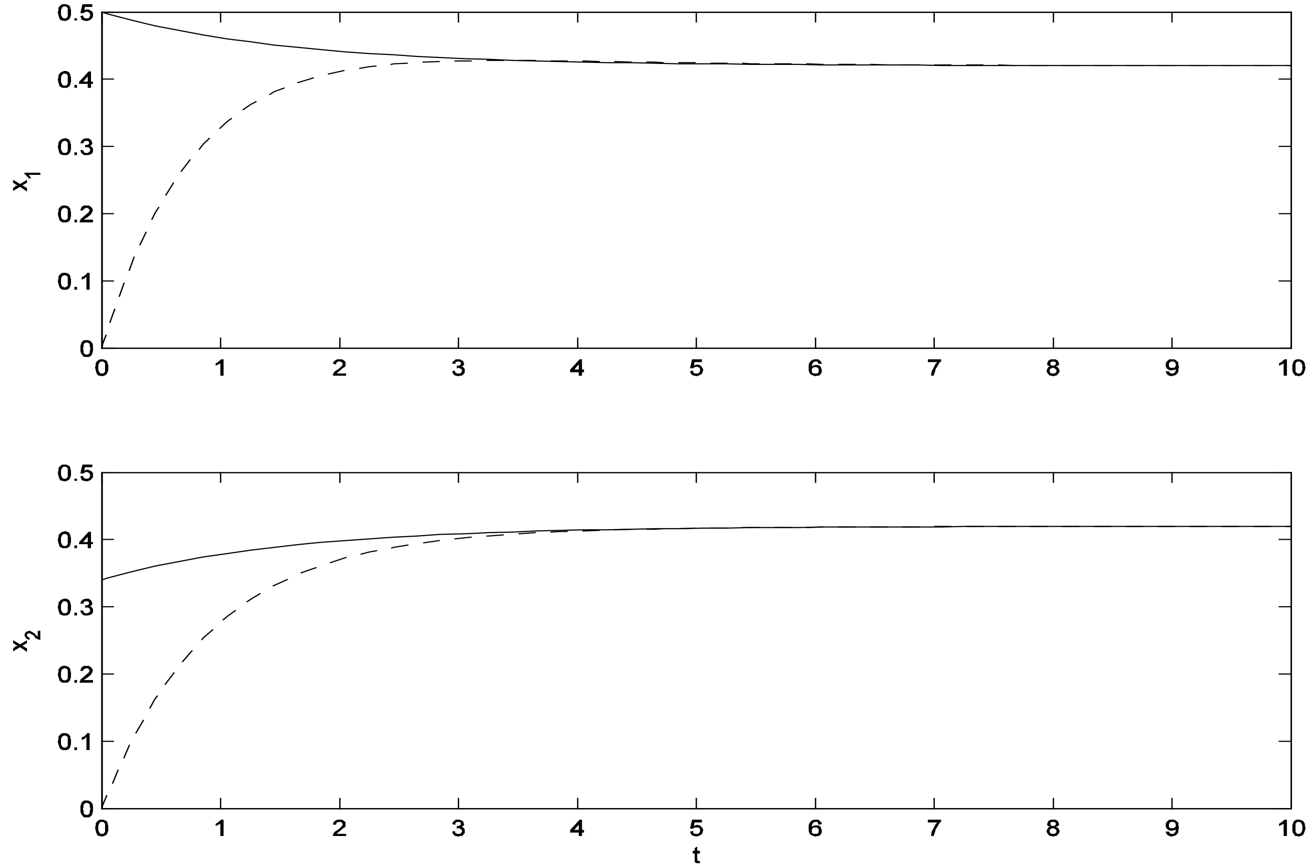



**Figure 4** Observer performace for Tanks.
(solid-actual state variables; dash-estimated state variables)

# §11.3 Full Order Observer/Controller Combination

**What happens when the state feedback control is implemented with $\hat{x}$ instead of $x$ ?**

It should be noted that the controller design assumes all the state variables are available, but the actual implementation uses only the estimated $\hat{x}(t)$. What is the overall performance of the resulting system with the observer compared with that with the true state feedback?

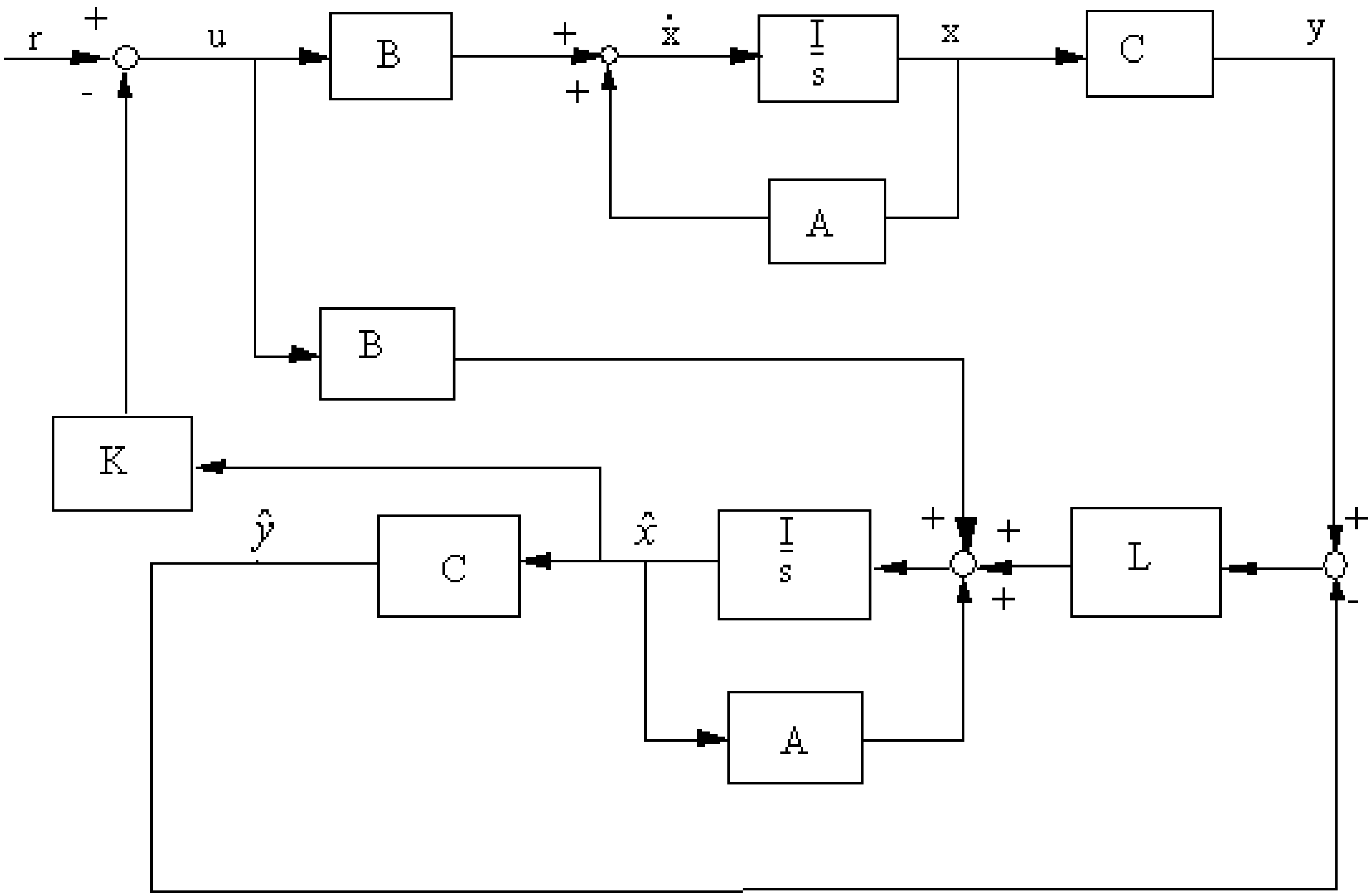



**Figure 5** System of plant/observer/controller.

The observer control system has the plant:

$$\dot{x} = Ax + Bu$$
$$y = Cx$$

the observer:

$$\dot{\hat{x}} = A\hat{x} + Bu + L\left[y - C\hat{x}\right],$$

and the control law:

$$u = (r - K\hat{x}).$$

It follows that

$$\begin{aligned}\dot{x} &= Ax + B\left[r - K(x - \tilde{x})\right] \\ &= (A - BK)x + BK\tilde{x} + Br\end{aligned} \tag{7}$$

Also, we have the estimation error dynamics:

$$\dot{\tilde{x}} = (A - LC)\tilde{x} \tag{8}$$

The composite system with (7) and (8) is united as

$$\begin{bmatrix} \dot{x} \\ \dot{\tilde{x}} \end{bmatrix} = \begin{bmatrix} A - BK & BK \\ 0 & A - LC \end{bmatrix} \begin{bmatrix} x \\ \tilde{x} \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} r, \tag{9}$$

$$y = Cx.$$

The Laplace transform gives

$$\begin{aligned} X(s) &= (sI - A + BK)^{-1} x(0) \\ &\quad + (sI - A + BK)^{-1} BK (sI - A + LC)^{-1} \tilde{x}(0) \\ &\quad + (sI - A + BK)^{-1} BR(s), \\ \tilde{X}(s) &= (sI - A + LC)^{-1} \tilde{x}(0), \end{aligned}$$

and

$$Y(s) = CX(s)$$

One concludes that:

(i) The closed-loop poles are the eigenvalues of ($A$-$BK$) and ($A$-$LC$). (Separation Principle);

(ii) $\tilde{x}$ is uncontrollable.

(iii) The transfer function matrix relating $X(s)$ (or $Y(s)$) to $R(s)$ is the same as obtained equivalently by using $x$ as feedback.

(iv) There would have been no difference on performance at all if $\tilde{x}(0)$ were zero! Otherwise, the error in the initial state estimation, $\tilde{x}(0)$, propagates through the feedback loop until its effect goes to zero asymptotically. The error in $x$ with and without an observer is

$$X_E(s) = (sI - A + BK)^{-1} BK(sI - A + LC)^{-1} \tilde{x}_0 \, .$$

**Example 2.** To see the difference between the systems with $x$ and $\hat{x}$, Let the plant be

$$\dot{x} = 2x + 3u,$$

$$y = x,$$

with $r = 0$ and $x(0) = 1$.

<u>The case of true state feedback:</u> Suppose

$$u = -Kx ,$$

where $K$=2 will move the closed loop pole to – 4: $A - bK = 2 - 3 \times 2 = -4$.

Then the closed-loop is described by

$$\dot{x} = 2x + 3(-2x) = -4x,$$
$$\dot{y} = -4y,\ y(0) = 1.$$

It yields the transient:

$$y(t) = e^{-4t} y(0) = e^{-4t}. \tag{10}$$

Case of estimated state feedback: Use the observer:

$$\dot{\hat{x}} = 2\hat{x} + 3u + L(y - \hat{y}),$$

where $L = 10$ will place the observer pole at -8. The control law becomes

$$u = -2\hat{x}.$$

Let $\hat{x}(0) = 0$. Then, $\tilde{x} = x(0) - \hat{x}(0) = 1$. The difference between the above two cases is

$$\begin{aligned} Y_E(s) &= C(sI - A + BK)^{-1} BK(sI - A + LC)^{-1} \tilde{x}(0) \\ &= 1 \cdot (s - 2 + 3\times 2)^{-1} \cdot 3\times 2 \cdot (s - 2 + 10\times 1)^{-1} \cdot 1 \\ &= \frac{6}{(s+4)(s+8)} = \frac{1.5}{s+4} - \frac{1.5}{s+8}. \end{aligned}$$

We have

$$y_E(t) = 1.5e^{-4t} - 1.5e^{-8t}. \qquad (11)$$

So, the output with the observer is

$$y(t) = (10) + (11) = 2.5e^{-4t} - 1.5e^{-8t}.$$

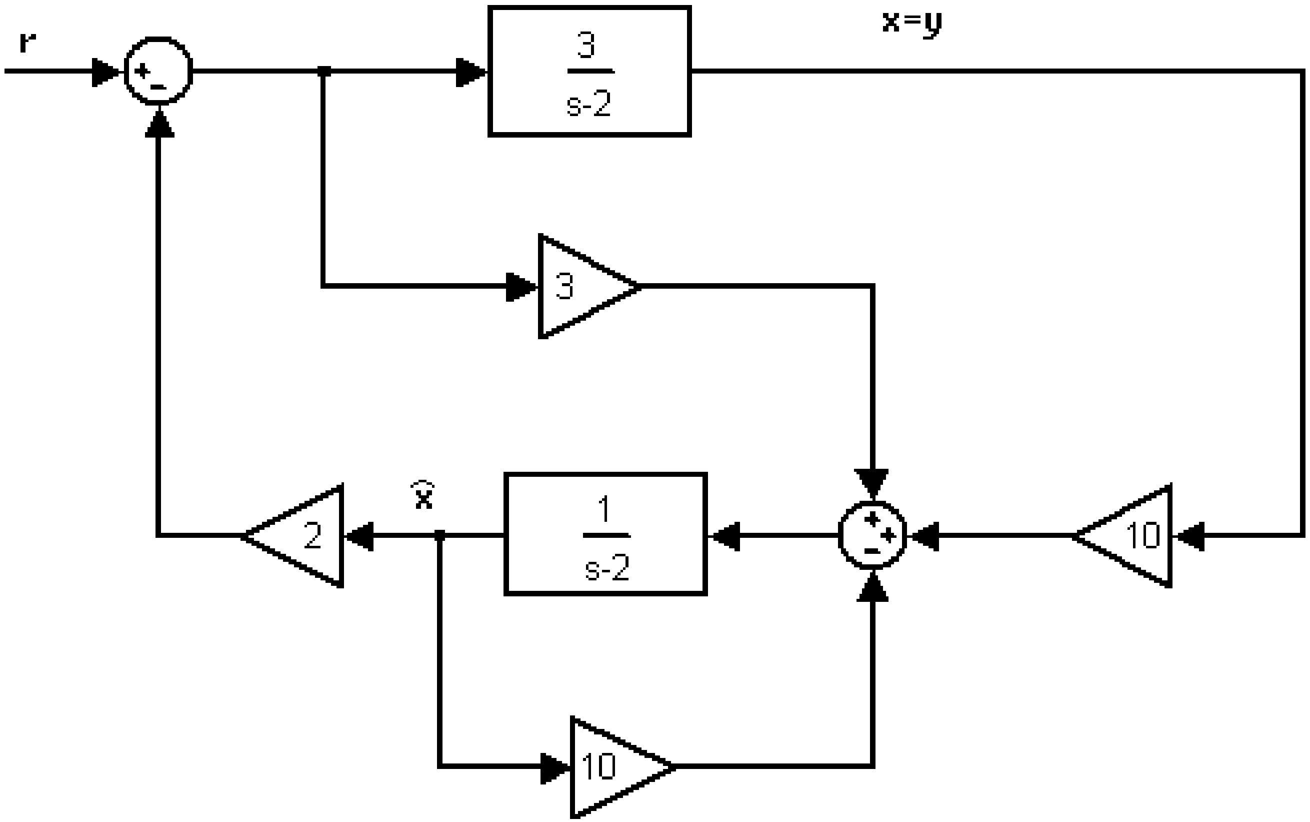



**Figure 6** Oserver control system for Example 2.

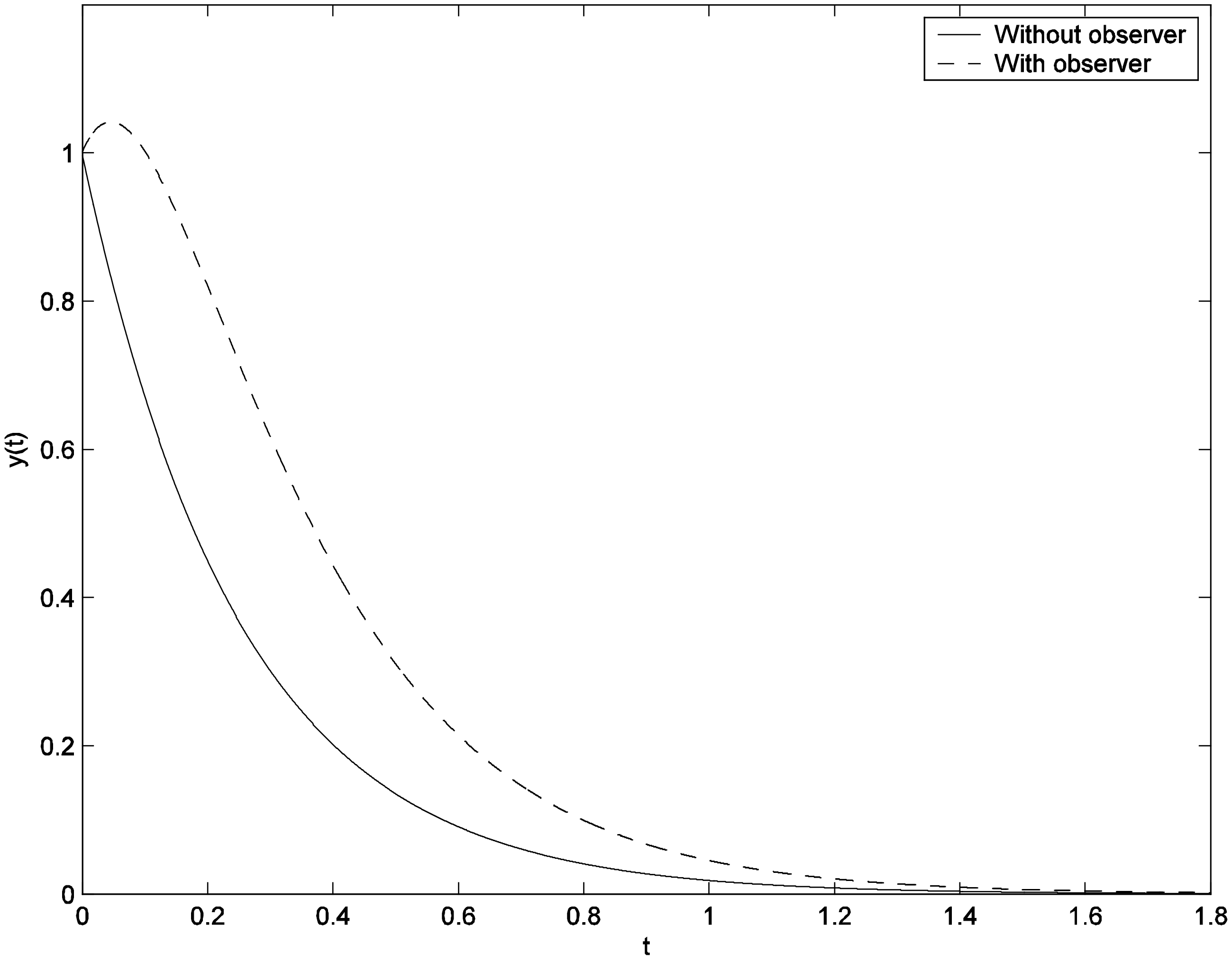



**Figure 7** Control performance with and without observer.

# §11.4 Reduced-Order Observers

**Idea:** Look at $y = Cx$, where $y$ is $m\times 1$, $x$ $n\times 1$ and $C$ $m\times n$. Suppose that $C$ has rank of $m$. We want to take advantage of the $m$ state variables that are available through $y$ and construct an observer of order $n-m$, lower than $n$.

Let $\xi = Tx$, where $T$ is $(n-m)\times n$ with $(n-m)$ rows that are linearly independent of $C$. This is one of the keys to the design of a reduced-order.

Combining these two equations results in

$$\begin{bmatrix} y \\ \xi \end{bmatrix} = \begin{bmatrix} C \\ T \end{bmatrix} x, \quad \begin{bmatrix} C \\ T \end{bmatrix} \text{ is of } n\times n \text{ and non-singular.} \tag{12}$$

The state can be obtained by inverting (12) as

$$x = \begin{bmatrix} C \\ T \end{bmatrix}^{-1} \begin{bmatrix} y \\ \xi \end{bmatrix}. \tag{13}$$

This equation says that $x$ is linear combination of plant's output and the observer's output.

In general, for a $n$th-order system with $m$ outputs, an observer with $n$-$m$ outputs is needed. Since all of the state variables of the observer are really outputs of that observer, this is the same as saying that an observer of order $n$-$m$ is needed. This is a direct consequence of (12) and (13).

**Construction**: For the system:

$$\dot{x} = Ax + Bu,$$

$$y = Cx,$$

construct an observer as

$$\dot{\xi} = D\xi + Eu + Gy, \tag{14}$$

such that

$$\xi \to Tx \text{ as } t \text{ goes to infinity.} \tag{15}$$

The purpose of the $D$ matrix in (14) is to determine how rapidly $\xi$ will approach $Tx$.

Equation (15) requires that the output of the observer eventually approach a linear combination of the state variables. This equation plays the role of an error equation in this formulation. The goal is to get $\xi$ to approach $Tx$ as t goes to infinity, that is,

$$[\xi - Tx] \to 0 \quad \text{As } t \to \infty.$$

To find this error's dynamics, look at $\frac{d}{dt}[\xi - Tx] = \dot{\xi} - T\dot{x}$, where

$$T\dot{x} = TAx + TBu,$$

so that

$$\dot{\xi} - T\dot{x} = D\xi + Eu + Gy - TAx - TBu. \tag{16}$$

We add and subtract $DTx$ to produce

$$\begin{aligned} \dot{\xi} - T\dot{x} &= D\xi - DTx + DTx - TAx + GCx + Eu - TBu \\ \frac{d}{dt}[\xi - Tx] &= D[\xi - Tx] + (DT - TA + GC)x + (E - TB)u \end{aligned} \tag{17}$$

Equation (17) is now a differential equation in the error, $\xi - Tx$, and this error will go to zero as $t$ goes to infinity, if the eigenvalues of $D$ have negative real parts and

$$DT - TA + GC = 0,$$

$$E - TB = 0.$$

Finally, one needs to reconstruct the state variables from the plant's output and the observer's output based on (13). But note that $\xi(t) = Tx(t)$ does not hold till $t = \infty$. Thus calculated from (13) is not the precise state but its estimate:

$$\hat{x} = \begin{bmatrix} C \\ T \end{bmatrix}^{-1} \begin{bmatrix} y \\ \xi \end{bmatrix}. \tag{18}$$

**Design**: Determine *T, D, G* and *E* as follows.

(i) Find the constraints on $T$ such that $\begin{bmatrix} C \\ T \end{bmatrix}$ is non-singular;

(ii) Choose D such that its eigenvalues have negative real parts, or desired decay rates;

(iii) Solve $DT - TA + GC = 0$ for *T* and *G*; and

(iv) Calculate $E = TB$.

**Example 3** Design a first-order observer for the plant described in Example 1:

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u,$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}.$$

Only $x_2(t)$ needs to be reconstructed since $x_1(t) = y(t)$. The observer is of first-order and given by

$$\dot{\xi} = d\xi + eu + gy$$

where $d, e$ and $g$ are scalars.

$T = \begin{bmatrix} t_1 & t_2 \end{bmatrix}$ is such that the matrix $\begin{bmatrix} C \\ T \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ t_1 & t_2 \end{bmatrix}$ must be nonsingular.

This is the case iff $t_2 \neq 0$.

Choose $d = -3$.

Look at $dT - TA + gC = 0$, or

$$
\begin{array}{ccccc}
TA & - & dT & = & gC \\
\begin{bmatrix} t_1 & t_2 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} & + & 3\begin{bmatrix} t_1 & t_2 \end{bmatrix} & = & g\begin{bmatrix} 1 & 0 \end{bmatrix} \\
\begin{bmatrix} 0 & t_1 \end{bmatrix} & + & \begin{bmatrix} 3t_1 & 3t_2 \end{bmatrix} & = & \begin{bmatrix} g & 0 \end{bmatrix}
\end{array}
$$

$$g = 3t_1 \ , \ t_1 + 3t_2 = 0 .$$

Try $g = 1$, and solve for $t_1$ and $t_2$ and check the rank of $\begin{bmatrix} 1 & 0 \\ t_1 & t_2 \end{bmatrix}$. This yields $t_1 = \frac{1}{3}$, $t_2 = -\frac{1}{9}$ and the matrix in question will have full rank.

One then calculates

$$e=TB=\begin{bmatrix} t_1 & t_2 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = t_2 = -\frac{1}{9}.$$

The observer is thus determined as

$$\dot{\xi} = -3\xi - \frac{1}{9}u + y.$$

One then reconstructs the state variables from the plant's output and the observer's output from (18) as

$$\begin{bmatrix} \hat{x}_1(t) \\ \hat{x}_2(t) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \frac{1}{3} & -\frac{1}{9} \end{bmatrix}^{-1} \begin{bmatrix} y(t) \\ \xi(t) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 3 & -9 \end{bmatrix} \begin{bmatrix} y(t) \\ \xi(t) \end{bmatrix}$$

or

$$\hat{x}_1(t) = y(t) = x_1,$$
$$\hat{x}_2(t) = 3y(t) - 9\xi(t).$$

The plant and the first-order observer are shown in Figure 8.

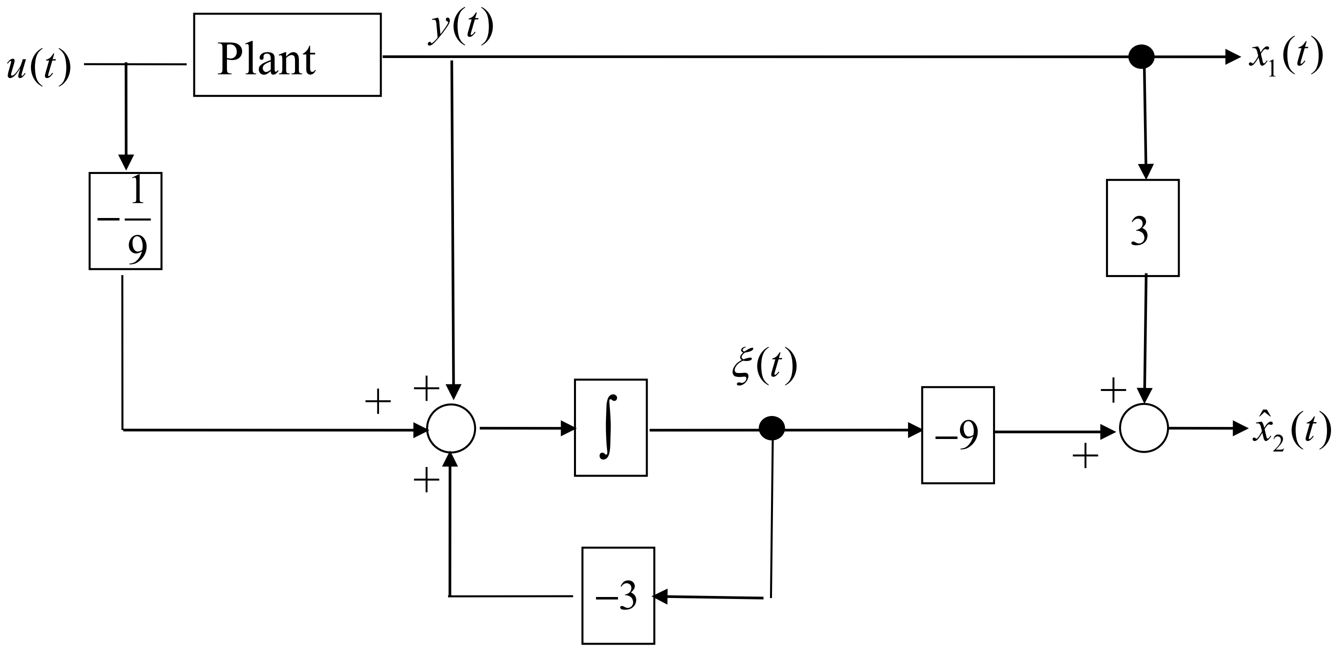



**Figure 8** Reduced-order observer for Example 3.

**Example 4** Design an observer of minimal order for the system with

$$A=\begin{bmatrix}0&1&0\\0&0&1\\1&0&0\end{bmatrix},\ \mathrm{B}=\begin{bmatrix}0\\1\\0\end{bmatrix},\ \mathrm{C}=\begin{bmatrix}1&0&0\\1&1&0\end{bmatrix}.$$

A check of $\begin{bmatrix}C^T & A^TC^T & (A^T)^2C^T\end{bmatrix}$ reveals that the system is observable.

Since the dimenstion of $y$ is 2, we can reconstruct all three state variables of this system by a first-order observer,

$$\dot{\xi}=d\xi+eu+Gy$$

where $d$ and $e$ are scalars and $G=\begin{bmatrix}g_1 & g_2\end{bmatrix}$.

Let $T=\begin{bmatrix}t_1 & t_2 & t_3\end{bmatrix}$. We have

$$\begin{bmatrix}y\\\xi\end{bmatrix}=\begin{bmatrix}C\\T\end{bmatrix}x=\begin{bmatrix}1&0&0\\1&1&0\\t_1&t_2&t_3\end{bmatrix}x, \tag{19}$$

the rank of $\begin{bmatrix} C \\ T \end{bmatrix}$ must be 3, or

$$\det\begin{bmatrix} C \\ T \end{bmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ t_1 & t_2 & t_3 \end{vmatrix} = t_3 \neq 0.$$

Choose the eigenvalue of the observer as $d = -2$.

The main design equation is

$$TA - dT = GC, \tag{20}$$

$$\begin{bmatrix} t_1 & t_2 & t_3 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} - d\begin{bmatrix} t_1 & t_2 & t_3 \end{bmatrix} = \begin{bmatrix} g_1 & g_2 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix}$$

or

$$\begin{bmatrix} t_3 & t_1 & t_2 \end{bmatrix} + 2\begin{bmatrix} t_1 & t_2 & t_3 \end{bmatrix} = \begin{bmatrix} g_1 + g_2 & g_2 & 0 \end{bmatrix}.$$

This results in

$$
\begin{aligned}
&t_3 + 2t_1 = g_1 + g_2, \\
&t_1 + 2t_2 = g_2, \\
&t_2 + 2t_3 = 0,\ t_3 \neq 0.
\end{aligned}
\tag{21}
$$

Choosing $g_1 = 10$ and $g_2 = 1$, the solution is $t_1 = 5$, $t_2 = -2$, and $t_3 = 1$, which satisfies $t_3 \neq 0$.

One then gets

$$
e = TB = \begin{bmatrix} 5 & -2 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = -2.
$$

The observer becomes

$$
\dot{\xi} = -2\xi - 2u + \begin{bmatrix} 10 & 1 \end{bmatrix} y.
$$

The reconstructed state variables are now formed from a linear combination of the system output and the observer output as

$$\hat{x} = \begin{bmatrix} C \\ T \end{bmatrix}^{-1} \begin{bmatrix} y \\ \xi(t) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 5 & -2 & 1 \end{bmatrix}^{-1} \begin{bmatrix} y \\ \xi(t) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -7 & 2 & 1 \end{bmatrix} \begin{bmatrix} y \\ \xi(t) \end{bmatrix},$$

or

$$\begin{aligned} \hat{x}_1(t) &= y_1(t) = x_1(t) \\ \hat{x}_2(t) &= -y_1(t) + y_2(t) = x_2(t) \\ \hat{x}_3(t) &= -7y_1(t) + 2y_2(t) + \xi(t) \end{aligned} \quad .$$

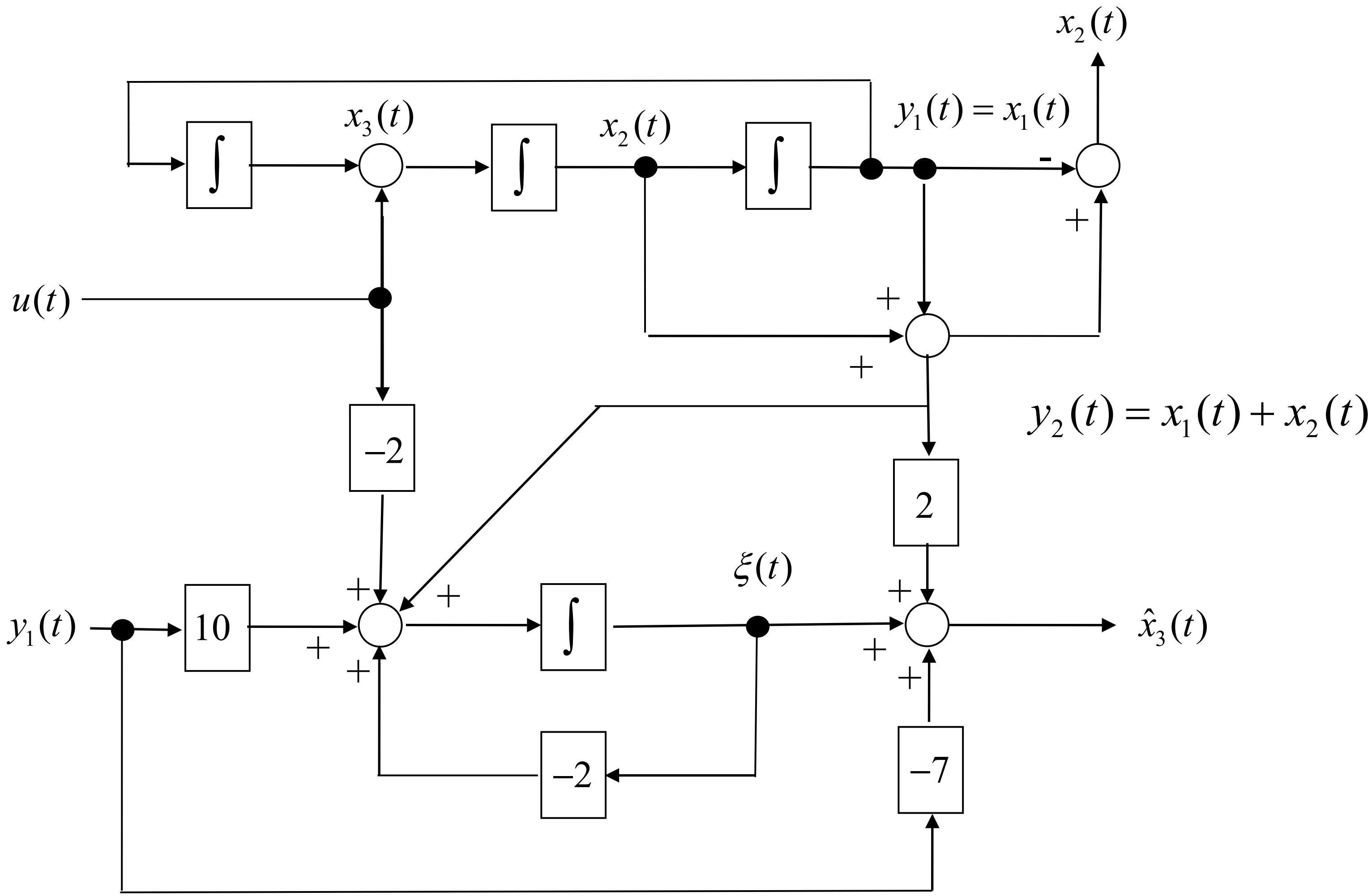



**Figure 9** Reduced-order observer for Example 4.

**Relationship to full order observer**: For now, assume that we make no use of outputs in (12). Then, $T$ will be of $n \times n$ and can be taken as identity matrix $I$. This will lead to the full-order observer discussed before:

- $\xi = Tx = x$ produces:

$$x = T^{-1}\xi = \xi . \tag{22}$$

- The design equations:

$$DT - TA + GC = 0$$

$$E - TB = 0$$

become

$$\begin{aligned} D &= A - GC \\ E &= B \end{aligned} . \tag{23}$$

The eigenvalues of $D$ should have negative real parts. Since $A$ and $C$ are known and observable, the matrix $G$ can be used to specify the eigenvalues of $D$.

- The observer:

$$\dot{\xi} = D\xi + Eu + Gy$$
$$\xi = Tx$$

becomes

$$\dot{\xi} = (A - GC)\xi + Bu + Gy,$$
$$\xi = x.$$