EE4302 ADVANCED CONTROL SYSTEM
TUTORIAL SOLUTION

1(a)

$$
\begin{aligned}
\dot{x}_1 &= x_2 = 0 \\
\dot{x}_2 &= -\sin x_1 - 0.5x_2 = 0 \\
\sin x_1 &= 0 \\
x_1 &= 0, \pm\pi, \pm 2\pi \ldots
\end{aligned}
$$

Hence equilibrium points are $(0,0)$, $(\pm\pi, 0)$, $(\pm 2\pi, 0)$ ...

1(b)

$$
\begin{aligned}
\dot{x}_1 &= x_2 = g \\
\dot{x}_2 &= -\sin x_1 - 0.5x_2 + u = h
\end{aligned}
$$

$$\begin{bmatrix} \Delta\dot{x}_1 \\ \Delta\dot{x}_2 \end{bmatrix} = \begin{bmatrix} \frac{dg}{dx_1} & \frac{dg}{dx_2} \\ \frac{dh}{dx_1} & \frac{dh}{dx_2} \end{bmatrix} \begin{bmatrix} \Delta x_1 \\ \Delta x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \Delta u$$

$$\frac{dg}{dx_1} = 0$$

$$\frac{dg}{dx_2} = 1$$

$$\frac{dh}{dx_1} = -\cos x_1$$

$$\frac{dh}{dx_2} = 0.5$$

$$\frac{dg}{dh} = 1$$

$$\begin{bmatrix} \Delta\dot{x}_1 \\ \Delta\dot{x}_2 \end{bmatrix} = A \begin{bmatrix} \Delta x_1 \\ \Delta x_2 \end{bmatrix} + B\Delta u$$

$$\Delta y = C \begin{bmatrix} \Delta x_1 \\ \Delta x_2 \end{bmatrix}$$

Choose equilibrium $x_1 = x_2 = 0$ gives

$$A = \begin{bmatrix} 0 & 1 \\ -1 & -0.5 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

and

$$\frac{\Delta Y(s)}{\Delta U(s)} = C(sI - A)^{-1}B$$

$$= \frac{1}{s^2 + 0.5s + 1}$$

Q1c,d

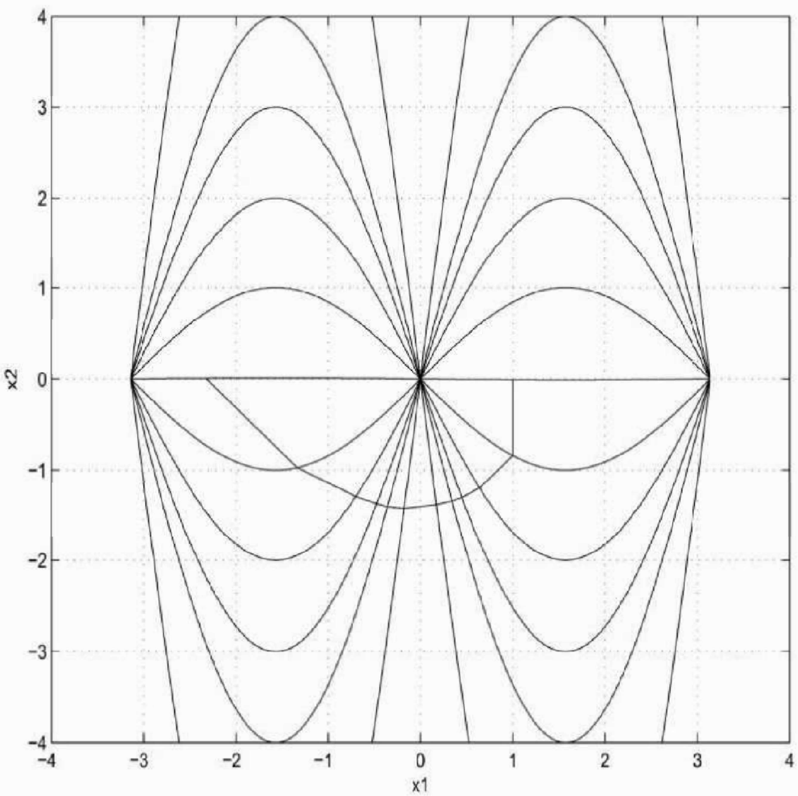
$$\begin{aligned}\frac{dx_2}{dx_1} &= \frac{\sin x_1}{x_2} = a \\ x_2 &= -\frac{\sin x_1}{a}\end{aligned}$$

Q1e

x_1	$\dot{x}_1$	$\frac{\Delta x_1}{\lvert\dot{x}_1\rvert} = \frac{0.5}{\lvert\dot{x}_1\rvert} = \Delta t$	t
1 to 0.5	-0.6	0.8	0.8
0.5 to 0	-1.3	0.4	1.2
0 to -0.5	-1.4	0.4	1.6
-0.5 to -1	-1.3	0.4	2.0
-1 to -1.5	-1.1	0.5	2.5
-1.5 to -2	-0.6	0.8	3.3
-2 to -2.3	-0.2	2.5	5.8

Plot

t	0	0.8	1.2	1.6	2.0	2.5	3.3	5.8
x_1	1	0.5	0	-0.5	-1	-1.5	-2	-2.3



4
3
2
1
0
−1
−2
−3
−4
x2
−4
−3
−2
−1
0
1
2
3
4
x1

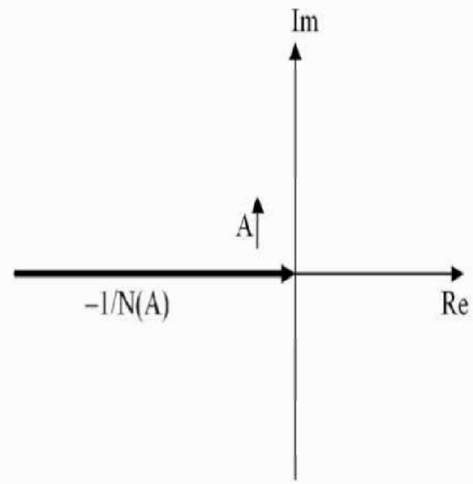
2(a)

$$
\begin{aligned}
N(A) &= \frac{2}{\pi A}\int_0^\pi A^5 \sin^6\theta d\theta \\
&= \frac{2A^4}{\pi}\int_0^\pi \sin^6\theta d\theta \\
&= \frac{2A^4}{\pi}\int_0^\pi \frac{1}{8}(1-3\cos 2\theta + 3\cos^2 2\theta - \cos^3 2\theta)d\theta \\
&= \frac{2A^4}{\pi}\int_0^\pi \frac{1}{8}(1-3\cos 2\theta + \frac{3}{2} + \frac{3}{2}\cos 4\theta - \cos^3 2\theta)d\theta
\end{aligned}
$$

The term $\cos 2\theta$, $\cos^3 2\theta$ and $\cos 4\theta$ integrate to zero over 0 to π.

$$
N(A) = \frac{2A^4}{\pi}\int_0^\pi \frac{5}{16}d\theta = \frac{2A^4}{\pi} \times \frac{5\pi}{16} = \frac{5A^4}{8}
$$

2(b)



2(c)

$$
\begin{aligned}
-\frac{1}{N(A)} &= G(j\omega) \\
-\frac{8}{5A^4} &= \frac{1-j\omega}{j\omega(j\omega+1)} \\
\text{Im part } 8\omega^2 - j8\omega &= 5A^4\omega
\end{aligned}
$$

$$
\begin{aligned}
A &= 1.125 \\
\text{Re part } 8\omega^2 &= 5A^4 \\
\omega &= 1
\end{aligned}
$$

4.

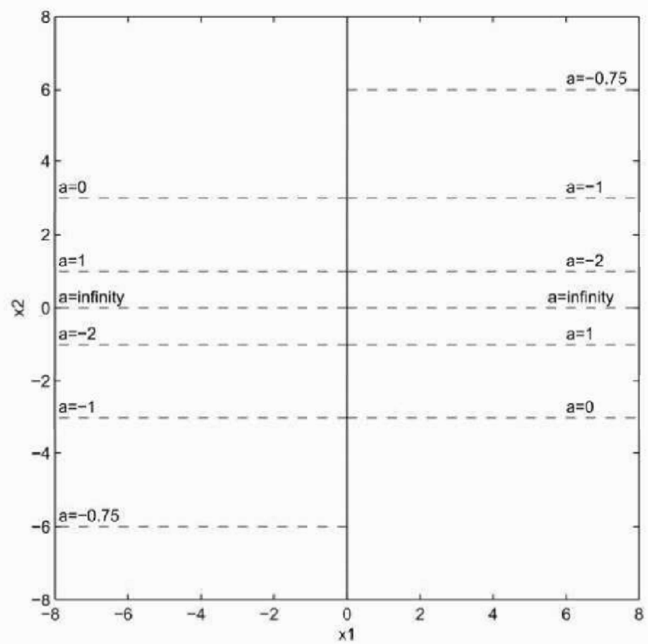
$$
\begin{aligned}
\frac{Y(s)}{U(s)} &= \frac{3}{s(2s+1)} \\
2s^2Y(s) + sY(s) &= 3U(s) \\
2\ddot{y} + \dot{y} &= 3u \\
e &= -y \\
-2\ddot{e} - \dot{e} &= 3\text{sgn}(e) \\
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -\frac{1}{2}x_2 - \frac{3}{2}\text{sgn}(x_1) \\
\frac{dx_2}{dx_1} &= -0.5 - \frac{1.5\text{sgn}(x_1)}{x_2} = \alpha \\
x_2 &= -\frac{1.5\text{sgn}(x_1)}{0.5 + \alpha}
\end{aligned}
$$

$x_1 > 0$

α	-0.75	-1	-2	0	1	∞
x_2	6	3	1	-3	-1	0

$x_1 < 0$

α	-0.75	-1	-2	0	1	∞
x_2	-6	-3	-1	3	1	0



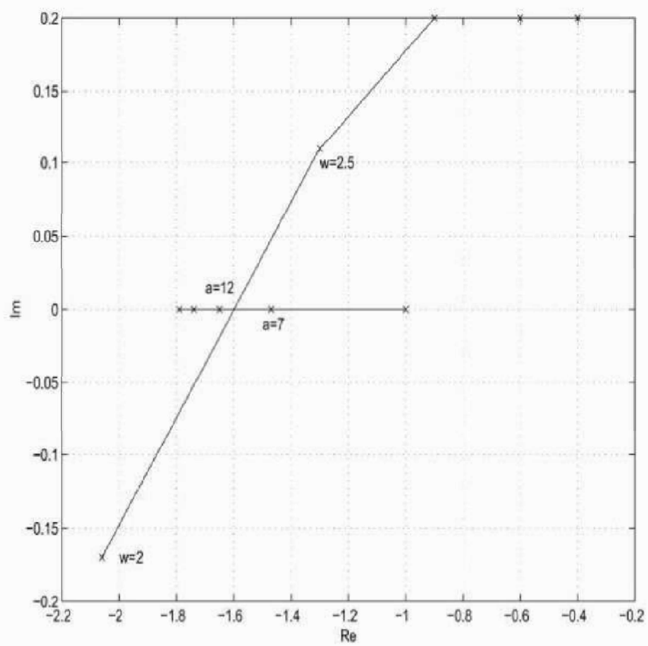
a=−0.75
a=0
a=−1
a=1
a=−2
a=infinity
a=infinity
a=−2
a=1
a=−1
a=0
a=−0.75
x2
x1
8
6
4
2
0
−2
−4
−6
−8
−8
−6
−4
−2
0
2
4
6
8

5.

$$
\begin{aligned}
N(a)G(j\omega) &= -1 \\
G(j\omega) &= -\frac{1}{N(a)}
\end{aligned}
$$

ω	$G(j\omega)$
2	$-2 - j0.17$
2.5	$-1.3 + j0.1$
3	$-0.9 + j0.2$
3.5	$-0.6 + j0.2$
4	$-0.4 + j0.2$

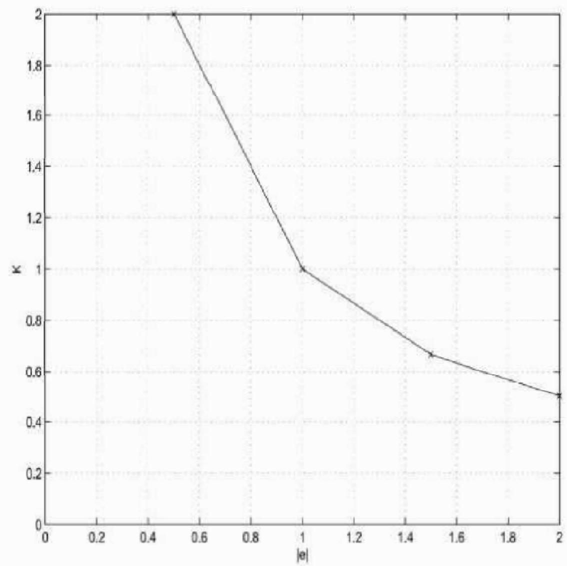
a	2	7	12	17	22
$-\frac{1}{N(a)}$	-1	-1.47	-1.65	-1.74	-1.79



Solution–13

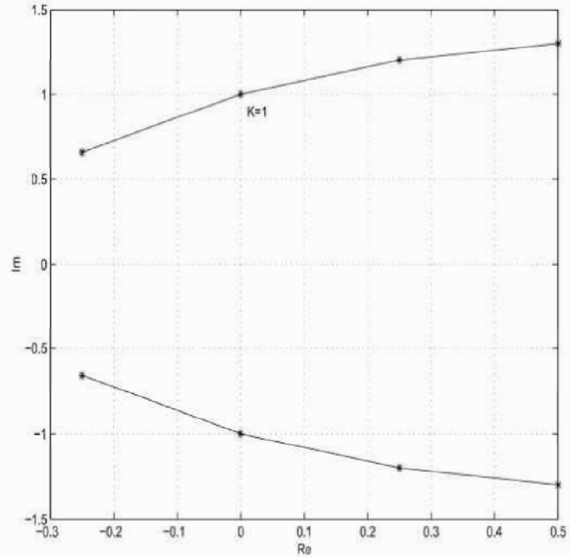
6(a)

$\lvert e \rvert$	0.5	1	1.5	2
K	2	1	$\frac{2}{3}$	$\frac{1}{2}$



6(b)

$$
\begin{aligned}
1+\frac{K(1-s)}{s(s+1)} &= 0\\
s^2+s(1-K)+K &= 0\\
s &= \frac{-1+K\pm\sqrt{(1-K)^2-4K}}{2}\\
&= \frac{-1+K\pm\sqrt{K^2-6K+1}}{2}
\end{aligned}
$$



K	s
0.5	$-0.25 \pm j0.66$
1	$\pm j$
1.5	$0.25 \pm j1.2$
2	$0.5 \pm j1.3$

6(c) $K_u = 1$ $T_u = 2\pi$

6(d)

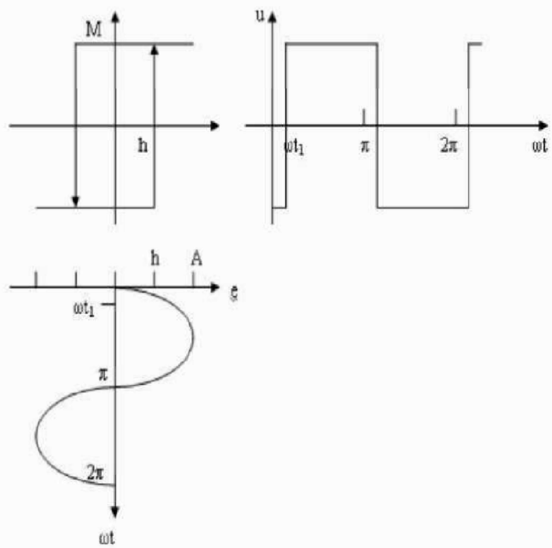
$$
\begin{aligned}
u(t) &= \begin{cases} 1 & 0 < \omega t < \pi \\ -1 & \pi < \omega t < 2\pi \end{cases} \\
b_1 &= \frac{4}{\pi} \int_0^{\frac{\pi}{2}} \sin(\omega t) d(\omega t) \\
&= \frac{4}{\pi} \\
N(A) &= \frac{4}{\pi a}
\end{aligned}
$$

6(e)

$$
\begin{aligned}
NG &= -1 \\
\frac{4}{\pi a} \frac{1 - j\omega}{j\omega(j\omega + 1)} &= -1
\end{aligned}
$$

$$
\begin{aligned}
4 - j4\omega &= \pi a \omega^2 - j\pi a \omega \\
4 &= \pi a \omega^2 \\
4\omega &= \pi a \omega \\
a &= \frac{4}{\pi} \\
\omega &= \frac{4}{\pi} \times \frac{\pi}{4} = 1 \\
K_u &= N(A) = \frac{4}{\pi a} \\
T_u &= 2\pi
\end{aligned}
$$

6(f)



$$A\sin(\omega t_1) = h$$

$$\omega t_1 = \sin^{-1}(\frac{h}{A})$$

$$\cos(\omega t_1) = \frac{\sqrt{A^2 - h^2}}{A}$$

$$
\begin{aligned}
a_1 &= \frac{2}{\pi}\int_{\omega t_1}^{\pi+\omega t_1} M\cos(\omega t)d\omega t \\
&= \frac{2M}{\pi}(\sin(\omega t))|_{\omega t_1}^{\pi+\omega t_1} \\
&= -\frac{4Mh}{\pi A} \\
b_1 &= \frac{2}{\pi}\int_{\omega t_1}^{\pi+\omega t_1} M\sin(\omega t)d\omega t \\
&= \frac{2M}{\pi}(-\cos(\omega t))|_{\omega t_1}^{\pi+\omega t_1} \\
&= \frac{4M}{\pi A}\sqrt{A^2-h^2} \\
N(A) &= \frac{1}{A}(b_1+ja_1) = \frac{4M}{\pi A}(\frac{\sqrt{A^2-h^2}}{A} - j\frac{h}{A})
\end{aligned}
$$

9(a)

$$
\begin{aligned}
\dot{x} &= 0.5x + u \quad 0 \le t \le t_\sigma \\
sX(s) - x(0) &= 0.5X(s) + U(s) \\
X(s) &= \frac{x(0)}{s-0.5} + \frac{1}{s-0.5}U(s) \\
\sigma &= -x \\
u &= -M = -2 \\
X(s) &= \frac{1}{s-0.5} - \frac{2}{s(s-0.5)} \\
x(t) &= e^{0.5t} + 4(1 - e^{0.5t}) \\
y(t) &= x(t) = 4 - 3e^{0.5t} \\
\sigma &= -x = 0 \\
4 - 3e^{-0.5t_\sigma} &= 0 \\
3e^{0.5t_\sigma} &= 4
\end{aligned}
$$

$$
\begin{aligned}
0.5t_\sigma &= \ln\frac{4}{3} \\
t_\sigma &= 0.58 \\
y(t) &= 0 \text{ for } t > t_\sigma
\end{aligned}
$$

9(b)

$$
\begin{aligned}
\dot{x} &= ax + bu + d \\
y &= x \\
\sigma &= -x \\
u &= M\mathrm{sign}(\sigma) \\
V &= \frac{\sigma^2}{2b} \\
\dot{V} &= \frac{\sigma}{b}\dot{\sigma} = \frac{\sigma}{b}(-\dot{x}) = \frac{\sigma}{b}(-ax - bu - d)
\end{aligned}
$$

$$
\begin{aligned}
&= \sigma\left(\frac{-ax-d}{b} - u\right) \\
&= \sigma\left(\frac{-ax-d}{b} - M\mathrm{sign}(\sigma)\right)
\end{aligned}
$$

$$
\text{For } \dot{V} < 0 \Rightarrow M > g^{+} = \frac{a^{+}x^{+} + d^{+}}{b}
$$

10(a) From lecture notes

$$
\begin{aligned}
u(t) &= -\frac{p^T f}{p^T g} - \frac{k}{p^T g}\mathrm{sign}(\sigma(x)) \\
&= -2x_1 - 0.5\mathrm{sign}(\sigma(x))
\end{aligned}
$$

where

$$
\begin{aligned}
k &= 0.5 \\
p^T &= \begin{bmatrix} 1 & 1 \end{bmatrix} \\
f &= \begin{bmatrix} x_1 \\ x_1 \end{bmatrix} \\
g &= \begin{bmatrix} 1 \\ 0 \end{bmatrix}
\end{aligned}
$$

10(b) Substituting for $u(t)$ gives

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} (-2x_1 - 0.5\text{sign}(\sigma))$$

$$= \begin{bmatrix} -x_1 - 0.5\text{sign}(\sigma) \\ x_1 \end{bmatrix}$$

At

$$\begin{aligned} t &= 0 \\ x_1(0) &= 1.5; \\ x_2(0) &= 0 => \sigma > 0 \end{aligned}$$

$$\begin{aligned} sX_1(s) - x_1(0) &= -X_1(s) - \frac{0.5}{s} \\ x_1(t) &= 2e^{-t} - 0.5 \end{aligned}$$

$$
\begin{aligned}
x_2(t) &= \int x_1 dt = -1.5e^{-t} - 0.5(t + e^{-t}) + c \\
x_2(0) &= 0 => c = 2 \\
x_2(t) &= -2e^{-t} - 0.5t + 2
\end{aligned}
$$

When we reach the sliding line

$$\sigma = x_1(t) + x_2(t) = 0$$

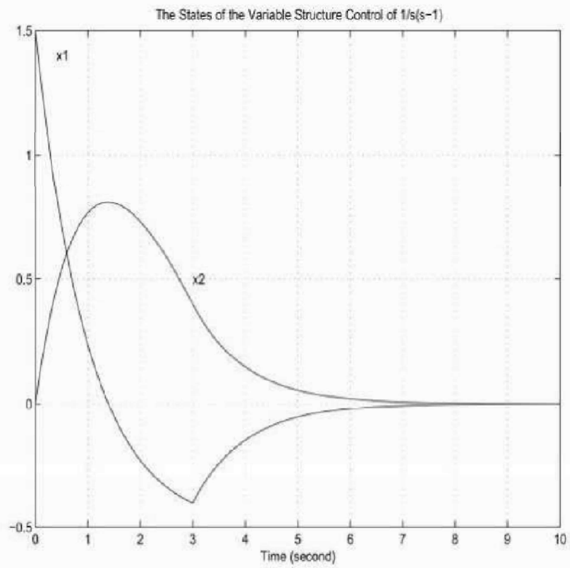
giving $t = 3$.

For $t \geq 3$, we are on the sliding line

$$\begin{bmatrix} \dot{x}_1(t+3) \\ \dot{x}_2(t+3) \end{bmatrix} = \begin{bmatrix} -x_1(t+3) \\ x_1(t+3) \end{bmatrix}$$

$$
\begin{aligned}
se^{3s}X_1(s) - x_1(3) &= -e^{3s}X_1(s) \\
X_1(s) &= \frac{x_1(3)e^{-3s}}{s+1} = \frac{(2e^{-3} - 0.5)e^{-3s}}{s+1} = \frac{-0.4e^{-3s}}{s+1}
\end{aligned}
$$

$$
\begin{aligned}
x_1(t) &= -0.4e^{-(t-3)} \\
x_2(t) &= \int x_1(t)dt = 0.4e^{-(t-3)} + c \\
x_2(3) &= -2e^{-3} - 0.5(3) + 2 = 0.4 => c = 0 \\
x_2(t) &= 0.4e^{-(t-3)}
\end{aligned}
$$



10(c) Phase trajectory

For $0 \leq t < 3$

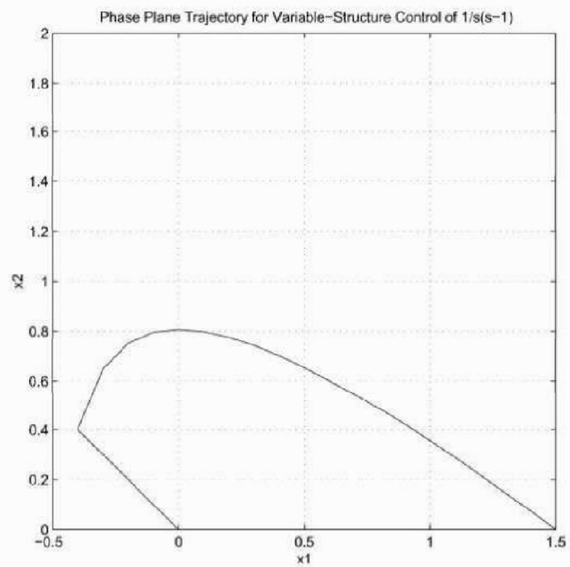
$$
\begin{aligned}
x_1(t) &= 2e^{-t} - 0.5 \\
t &= \ln\frac{2}{x_1 + 0.5}
\end{aligned}
$$

Substitute into

$$
\begin{aligned}
x_2(t) &= -2e^{-t} - 0.5t + 2 \\
&= 1.5 - x_1 - 0.5\ln\frac{2}{x_1 + 0.5}
\end{aligned}
$$

For $t > 3$, we are on the sliding line

$$x_1 + x_2 = 0$$


Phase Plane Trajectory for Variable−Structure Control of 1/s(s−1)
x2
x1
2
1.8
1.6
1.4
1.2
1
0.8
0.6
0.4
0.2
0
−0.5
0
0.5
1
1.5

11(a)

$$
\begin{aligned}
\ddot{y} &= -1.5\dot{y}^2 \cos 3y + u \\
\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &= \begin{bmatrix} \dot{y} \\ y \end{bmatrix} \\
\ddot{y} &= -1.5\dot{y}^2 \cos 3y + u \\
\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} &= \begin{bmatrix} -1.5x_1^2 \cos 3x_2 \\ x_1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u \\
&= f + gu \\
y &= \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\
\sigma &= \dot{y} + 20y = x_1 + 20x_2 \\
V &= \frac{\sigma^2}{2}
\end{aligned}
$$

$$
\begin{aligned}
\frac{dV}{dt} &= \sigma\dot{\sigma} = \sigma \begin{bmatrix} 1 & 20 \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} \\
&= \sigma \left(\begin{bmatrix} 1 & 20 \end{bmatrix} f + \begin{bmatrix} 1 & 20 \end{bmatrix} g u \right) \\
u &= -\frac{\begin{bmatrix} 1 & 20 \end{bmatrix} f}{\begin{bmatrix} 1 & 20 \end{bmatrix} g} - \frac{k \mathrm{sign}(\sigma)}{\begin{bmatrix} 1 & 20 \end{bmatrix} g} \\
&= 1.5 x_1^2 \cos 3x_2 - 20x_1 - k \mathrm{sign}(\sigma) \\
\frac{dV}{dt} &= -k \mathrm{sign}(\sigma) \\
u &= 1.5\dot{y}^2 \cos 3y - 20\dot{y} - 0.1 \mathrm{sign}(\dot{y} + 20y)
\end{aligned}
$$

(b)Use

$$\dot{\sigma} = -k$$

At

$$y = \dot{y} = 1$$

gives

$$\begin{aligned}
\sigma &= 21 \\
\int_{21}^{0} d\sigma &= -\int_{0}^{t} k dt \\
-21 &= -kt \\
t &= \frac{21}{k} \\
&= \frac{21}{0.1} \\
&= 210 \text{ seconds}
\end{aligned}$$

12(a)

$$\begin{aligned} A\dot{h} &= u - av \\ mgh &= \frac{1}{2}mv^2 \\ v &= \sqrt{2gh} \\ \dot{h} &= \frac{1}{A}(u - a\sqrt{2gh}) \end{aligned}$$

12(b)

$$\begin{aligned} \frac{dh}{dt} &= -\frac{a}{A}\sqrt{2gh} \\ \int_{h_0}^{0} \frac{1}{\sqrt{h}} dh &= -\frac{a}{A}\sqrt{2g} \int_0^T dt \\ 2\sqrt{h}\Big|_{h_0}^{0} &= -\frac{a}{A}\sqrt{2g}t\Big|_0^T \end{aligned}$$

$$T = \frac{2A}{a\sqrt{2g}}\sqrt{h_0}$$

12(c)

$$\begin{aligned}\dot{h} &= \frac{u}{A} - \frac{a}{A}\sqrt{2gh} = f \\ f &= 0 \\ \bar{u} &= a\sqrt{2g\bar{h}}\end{aligned}$$

12(d)

$$\begin{aligned}\dot{h} &= f + \left.\frac{df}{dh}\right|_{\bar{h},\bar{u}} \Delta h + \left.\frac{df}{du}\right|_{\bar{h},\bar{u}} \Delta u \\ \Delta\dot{h} &= -\frac{a\sqrt{2g\bar{h}}}{2A\bar{h}}\Delta h + \frac{1}{A}\Delta u\end{aligned}$$

$$\frac{\Delta H(s)}{\Delta U(s)} = \frac{1/A}{s + \frac{a\sqrt{2g\bar{h}}}{2A\bar{h}}}$$

$$\frac{\Delta H(s)}{\Delta U(s)} = \frac{1/A}{s + \frac{\bar{u}}{2A\bar{h}}} = \frac{\beta}{s+\alpha}$$

13(a)

Let the tank be controlled by the proportional plus integral controller given as

$$G_c(s) = K_c\left(1 + \frac{1}{sT_i}\right)$$

From 12(d),

$$G_p(s) = \frac{\Delta H}{\Delta U} = \frac{1/A}{s + \frac{\bar{u}}{2A\bar{h}}} = \frac{\beta}{s+\alpha}$$

The closed-loop poles

$$
\begin{aligned}
1 + G_c(s)G_p(s) = 1 + \frac{\beta K_c(sT_i + 1)}{(s+\alpha)sT_i} &= 0 \\
s^2 + s(\alpha + \beta K_c) + \frac{\beta K_c}{T_i} &= 0
\end{aligned}
$$

Compare with the characteristic equation

$$
s^2 + s2\zeta\omega_n + \omega_n^2 = 0
$$

gives

$$
\begin{aligned}
K_c &= \frac{2\zeta\omega_n - \alpha}{\beta} \\
T_i &= \frac{2\zeta\omega_n - \alpha}{\omega_n^2}
\end{aligned}
$$