Torque Derivation in AC machines using Space Vectors

Energy balance (only as reference)

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AC Machine Fundamentals

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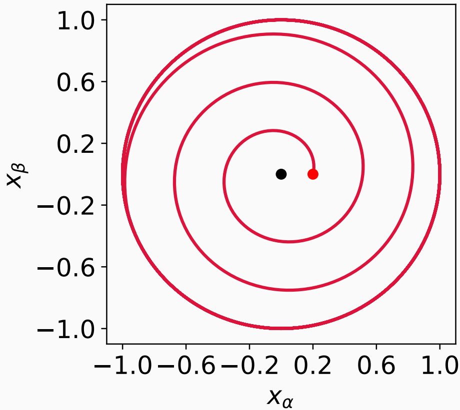


Space Vectors

Space Vectors: Describes rotating field quantities produced by time-varying currents i

AC machine consist of electromagnetic quantities that are distributed in space (means the value of the quantity varies with the angular position on the air gap periphery), or varies in time (means the value of the quantity at any position on the air gap periphery varies with time). Both these quantities interact to produce torque. Hence, in order to describe this interaction, we need a method of modelling that incorporates both the space dependent and time dependent quantities. Hence, we use space vectors for the purpose.

Note

What are space vectors? Space vectors are vector quantities (i.e. they have various attributes and not just the magnitude). A space vector quantity is a function of angular position and time.

Space Vectors i

Definition

A transformation of a 3 phase AC machine rotating filed into an equivalent 2 phase-quadrature axis machine producing the rotating field of equal magnitude.

A 3 phase winding excited by 3 phase current system will produce a resultant sinusoidal mmf in the machine. The cause of this mmf can be attributed to a current space vector defined as

$$\vec{I}_s = \frac{2}{3}\left(I_U(t) + \vec{a}I_V(t) + \vec{a}^2 I_W(t)\right) \tag{1}$$

$$\vec{I}_{s,0} = \frac{1}{3}\left(I_U(t) + I_V(t) + I_W(t)\right) \tag{2}$$

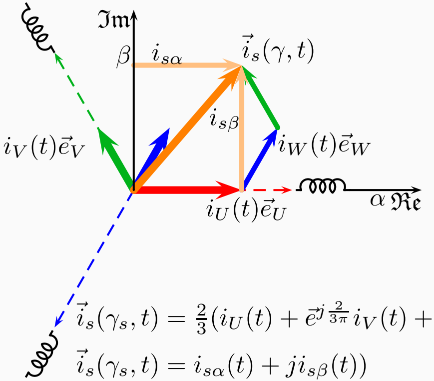
where

$$I_U(t) = \hat{I}_U(t) cos(\omega_s t) \tag{3}$$

$$I_V(t) = \hat{I}_V(t) cos(\omega_s t - 2\pi/3) \tag{4}$$

$$I_W(t) = \hat{I}_W(t) cos(\omega_s t - 4\pi/3) \tag{5}$$

$$\vec{a} = e^{j\frac{2\pi}{3}} \tag{6}$$

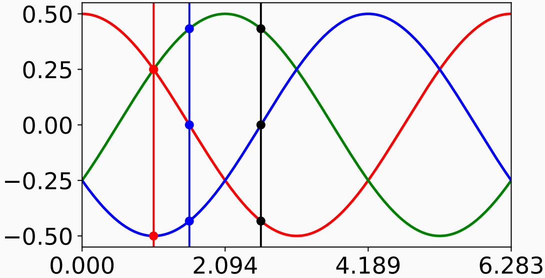


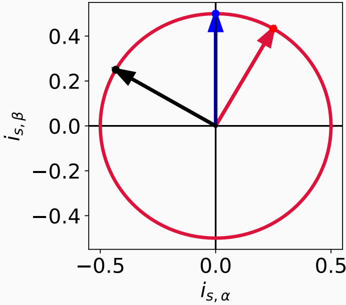
$$\vec{i}_s(\gamma_s, t) = \tfrac{2}{3}(i_U(t) + \vec{e}^{j\frac{2}{3\pi}} i_V(t) + \vec{e}^{j\frac{4}{3\pi}} i_W(t))$$

$$\vec{i}_s(\gamma_s, t) = i_{s\alpha}(t) + j i_{s\beta}(t))$$

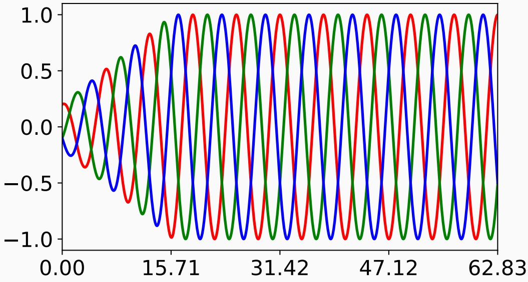
It is seen that a space vector is a complex quantity by definition. Therefore the trajectory of the space vector describes the instantaneous value of the electromagnetic variable in a complex plane.

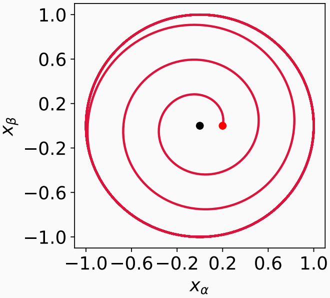
Defining Space vectors: Visually





Space Vectors can describe dynamics due to time-varying phase variables





Curve is the trajectory followed by the tip of the space vector

Space Vector: Flux Linkages

The stator flux linkages produced by the sinusoidal resultant mmf can be expressed as a space vector resulting from the individual phase flux linkages as

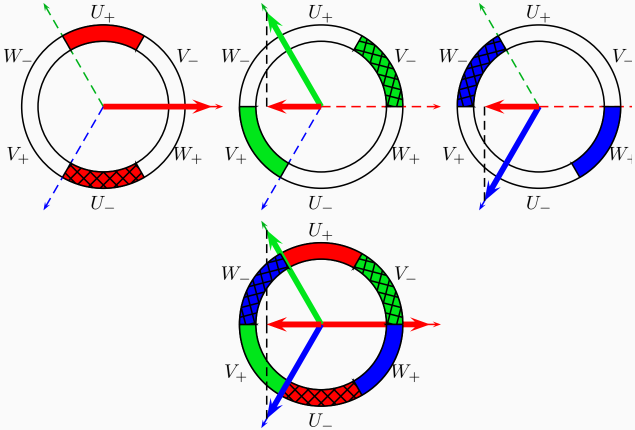
$$\vec{\Psi}_s = \frac{2}{3}\big(\Psi_a(t) + \vec{a}\Psi_b(t) + \vec{a}^2\Psi_c(t)\big) \tag{7}$$

and the relation between the two can be expressed in terms of a resultant inductance as

$$\vec{\Psi}_s = L_s\vec{I}_s \tag{8}$$

Flux linkages and magnetizing inductance

Flux coupling phase U.



Flux linking phase U i

Total flux linkage of winding U is

$$\Psi_U = w_U[\phi_{UU} + \phi_{UV} + \phi_{UW}]$$
$$\Psi_U = w_U \left[\frac{L_m I_U(t)}{w_U} - \frac{1}{2}\frac{L_m I_V(t)}{w_V} - \frac{1}{2}\frac{L_m I_W(t)}{w_W} \right]$$

Since $w_U = w_V = w_W$, and $I_U(t) = \sqrt{2} I_m cos(\omega_s t)$,
$I_V(t) = \sqrt{2} I_m cos(\omega_s t - 120)$ and $I_W(t) = \sqrt{2} I_m cos(\omega_s t - 240)$

Flux linking phase U ii

we get

$$\begin{aligned}
\Psi_U =& L_m[\sqrt{2}I_m cos(\omega_s t) - \frac{1}{2}\sqrt{2}I_m cos(\omega_s t - 120) - \frac{1}{2}\sqrt{2}I_m cos(\omega_s t - 240)] \\
\Psi_U =& L_m\sqrt{2}I_m[cos(\omega_s t) - \frac{1}{2}(cos(\omega_s t)(-\frac{1}{2}) - sin(\omega_s t)(0.866)+ \\
& cos(\omega_s t)(-\frac{1}{2}) - sin(\omega_s t)(-0.866))] \\
\Psi_U =& \frac{3}{2}L_m\sqrt{2}I_m cos(\omega_s t)
\end{aligned}$$

Hence the total inductance of winding U is

$$\begin{aligned}
L_U &= \Psi_U/\sqrt{2}I_m cos(\omega_s t) \\
L_U &= \frac{3}{2}L_m
\end{aligned} \tag{9}$$

Flux linking phase U iii

Similarly

$$
\begin{aligned}
L_V &= \frac{3}{2}L_m \\
L_W &= \frac{3}{2}L_m
\end{aligned}
$$

Flux linkage for rotating field

The three phase mutual inductance is $L_h = \frac{3}{2} L_m$. Every winding has some leakage flux and this flux is associated only with that winding and not with any other winding. Hence the leakage flux space vector is easily found as

$$\vec{\Psi}_{s\sigma} = L_{s\sigma} \vec{I}_s \tag{10}$$

Hence the flux linkage space vector is the sum of the two. Therefore the three phase self inductance is defined as

$$L_s = L_h + Ls\sigma \tag{11}$$

Voltage space vector from 3 phase flux linkage

Since the induced voltage is given by

$$V_i = \frac{d\Psi}{dt} \tag{12}$$

we can also talk about a voltage space vector as

$$\vec{V}_s = \frac{2}{3}\left(V_a(t) + \vec{a}V_b(t) + \vec{a}^2 V_c(t)\right) \tag{13}$$

Space Vector equation of AC Machine

$$\vec{V}_s = \vec{I}_s R_s + \frac{d\vec{\Psi}_s}{dt}$$

$$\vec{V}_r = \vec{I}_r R_r - j\Omega\vec{\Psi}_r + \frac{d\vec{\Psi}_r}{dt}$$

$$\vec{\Psi}_s = L_s \vec{I}_s + L_h \vec{I}_r$$

$$\vec{\Psi}_r = L_h \vec{I}_s + L_r \vec{I}_r$$

Power in terms of Space Vectors

The instantaneous power is given as

$$P = \frac{3}{2}\Re\{\vec{V}_s^* \vec{I}_s\} \tag{14}$$

where $\vec{V}_s^*$ is the complex conjugate of the voltage space vector. To prove this we will write

$$P = \frac{3}{2}\Re\left\{\frac{2}{3}\left[V_a + \vec{a}^* V_b + \vec{a}^{*2} V_c\right]\frac{2}{3}[I_a + \vec{a} I_b + \vec{a}^2 I_c\right]\right\} \tag{15}$$

$$P = \frac{2}{3}\Re\{V_a I_a + V_b I_b + V_c I_c + \tag{16}$$

$$\ldots \vec{a}^2(V_a I_c + V_b I_a + V_c I_b) + \vec{a}(V_a I_b + V_b I_c + V_c I_a)\} \tag{17}$$

Since

$$\vec{a}^* = \vec{a}^{-1} = \vec{a}^2$$

$$\vec{a}^{*2} = \vec{a}^{-2} = \vec{a}$$

$$\Re\{\vec{a}\} = \Re\{\vec{a}^2\} = -\frac{1}{2}$$

Power definition

Hence

$$P = \frac{2}{3}\left\{V_aI_a + V_bI_b + V_cI_c - \frac{1}{2}\left[V_a(I_b + I_c) + V_b(I_a + I_c) + V_c(I_a + I_b)\right]\right\}$$

Since for a three phase three wire system $I_a + I_b + I_c = 0$ we get, $I_b + I_c = -I_a$ and so on, hence

$$P = \frac{2}{3}\left\{V_aI_a + V_bI_b + V_cI_c + \frac{1}{2}\left[V_aI_a + V_bI_b + V_cI_c\right]\right\}$$

We get

Definition

$P = V_aI_a + V_bI_b + V_cI_c$

Instantaneous Torque using energy balance in electro-mechanical system i

An electric motor is an electro-magentic-mechanical system. The energy balance equation for the electric motor is written in differential form showing the relationship between small changes in the energy of the system as

$$\underbrace{dW_{elec}}_{\textbf{change in electrical energy}} = \underbrace{dW_{loss}}_{\textbf{Change in losses}} + \underbrace{dW_{field}}_{\textbf{change in stored energy in magnetic field}} + \underbrace{dW_{mech}}_{\textbf{virtual work}}$$

$$dW_{elec} = \frac{3}{2}\mathfrak{Re}\left[\vec{V}_s^* \vec{I}_s\right] dt$$

Energy Balance in AC motor

we can substitute the value of the space vectors for voltage and current to get the Torque

$$
\begin{aligned}
dW_{elec} &= \frac{3}{2}\mathfrak{Re}\left[\vec{V}_s^*\vec{I}_s\right]dt \\
&= \frac{3}{2}\mathfrak{Re}\left[\left(\vec{I}_s^* R_s + \frac{d\vec{\Psi}_s^*}{dt} - j\Omega\vec{\Psi}_s^*\right)\vec{I}_s\right]dt \\
&= \frac{3}{2}\vec{I}_s^2 R_s dt + \frac{3}{2}\mathfrak{Re}\left[\left(\frac{d\vec{\Psi}_s^*}{dt} - j\Omega\vec{\Psi}_s^*\right)\vec{I}_s\right]dt
\end{aligned}
$$

The above equations are written in the rotor coordinate system.

Change in electrical energy stator and rotor

If we assume that the rotor is also provided with external excitation, we can add another term similar to the stator terms based on $\frac{3}{2}\mathfrak{Re}[\vec{V}_r^* \vec{I}_r]$, to get the total change in electrical energy as

$$dW_{elec} = \frac{3}{2}\left(\vec{I}_s^2 R_s + \vec{I}_r^2 R_r\right) dt + \frac{3}{2}\mathfrak{Re}\left[\left(\frac{d\vec{\Psi}_s^*}{dt} - j\Omega d\vec{\Psi}_s^*\right)\vec{I}_s + \frac{d\vec{\Psi}_r^*}{dt}\vec{I}_r\right] dt$$

Change in stored magnetic energy

The magnetic energy stored in stator and rotor fields is independent of the actual rotor angular velocity and depend only on the values of voltage and currents. Hence it is easier to calculate the stored energy at standstill ($\Omega = 0$).

$$dW_{field} = \frac{3}{2}\mathfrak{Re}\left(\frac{d\vec{\Psi}_s^*}{dt}\vec{I}_s + \frac{d\vec{\Psi}_r^*}{dt}\vec{I}_r\right)dt$$

Change in Mechanical energy

We will denote Torque as M and a virtual angular displacement by $d\gamma$, the differential mechanical energy or virtual work can be written as

$$dW_{mech} = Md\gamma = M\Omega dt$$

Energy Balance

This would be equated to the energy balance equation to give

$$dW_{elec} = dW_{loss} + dW_{field} + dW_{mech}$$

$$\frac{3}{2}\left(\vec{I}_s^2 R_s + \vec{I}_r^2 R_r\right) dt + \frac{3}{2}\mathfrak{Re}\left[\left(\frac{d\vec{\Psi}_s^*}{dt} - j\Omega d\vec{\Psi}_s^*\right)\vec{I}_s + \frac{d\vec{\Psi}_r^*}{dt}\vec{I}_r\right] dt =$$

$$dW_{loss}$$

$$+ dW_{field}$$

$$+ M\Omega dt$$

Which is

$$\frac{3}{2}\left(\vec{I}_s^2 R_s + \vec{I}_r^2 R_r\right) dt + \frac{3}{2}\mathfrak{Re}\left[\left(\frac{d\vec{\Psi}_s^*}{dt}\vec{I}_s + \frac{d\vec{\Psi}_r^*}{dt}\vec{I}_r\right) + -j\Omega d\vec{\Psi}_s^*\vec{I}_s\right] dt =$$

$$dW_{loss}$$

$$+ dW_{field}$$

$$+ M\Omega dt$$

Change in Mechanical Energy

Which gives us

$$M\Omega dt = \frac{3}{2}\mathfrak{Re}\left[-j\Omega\vec{\Psi}_s^*\vec{I}_s\right]dt \tag{18}$$

$$M = \frac{3}{2}\mathfrak{Re}\left[-j\vec{\Psi}_s^*\vec{I}_s\right] \tag{19}$$

$$M = \frac{3}{2}\mathfrak{Im}\left[\vec{\Psi}_s^*\vec{I}_s\right] \tag{20}$$

$$M = \frac{3}{2}\mathfrak{Im}\left[\left(\Psi_\alpha - j\Psi_\beta\right)\left(I_\alpha + jI_\beta\right)\right] \tag{21}$$

$$M = \frac{3}{2}\left(\Psi_\alpha I_\beta - \Psi_\beta I_\alpha\right) \tag{22}$$

Instantaneous Torque

Definition

The instantaneous electromagnetic torque is given by

$$M_e = \frac{3}{2}\mathfrak{Im}\left\{\vec{\Psi}_s^*\vec{I}_s\right\}$$